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THE RICCI FLOW ON THE SPHERE WITH MARKED POINTS[1]

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Abstract

The Ricci flow on the 2-sphere with marked points is shown to converge in all three stable, semi-stable, and unstable cases. In the stable case, the flow was known to converge without any reparametrization, and a new proof of this fact is given. The semi-stable and unstable cases are new, and it is shown that the flow converges in the Gromov–Hausdorff topology to a limiting metric space which is also a 2-sphere, but with different marked points and, hence, a different complex structure. The limiting metric is the unique conical constant curvature metric in the semi-stable case, and the unique conical shrinking gradient Ricci soliton metric in the unstable case.

1. Introduction

A central theme in geometry is to characterize geometric structures by canonical metrics. The Uniformization Theorem achieves this for smooth compact Riemann surfaces. In higher dimensions, a well-known conjecture of Yau [**63**], broadly stated, is that the existence of a canonical metric should be equivalent to a suitable notion of stability in geometric invariant theory. When the structure is not stable, it is expected that a canonical metric should still exist, albeit with singularities or on an adjacent structure. A model scenario is that of holomorphic vector bundles $E \to M$ over a compact Kähler manifold M. When E is stable, a Hermitian–Einstein metric will exist, by the celebrated theorem of Donaldson–Uhlenbeck–Yau [**21**, **58**]. When E is unstable, the Yang–Mills flow will converge instead to a Yang–Mills connection on the double dual of the Harder–Narasimhan–Seshadri filtration of E. This last statement was conjectured by Bando and Siu [**1**]. It was proved in $\dim M = 2$ by Daskalopoulos and Wentworth [**17**], and very recently in general by Jacob [**24**], Sibley [**41**], and Sibley and Wentworth [**42**],

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building on the ideas of Donaldson [21], Uhlenbeck and Yau [58], and Uhlenbeck [57].

When we pass from holomorphic vector bundles to complex manifolds, the Yang–Mills flow is replaced by the Ricci flow. The major questions are then to determine the metric, the complex structure, and the singularities which would emerge from its long-time limit. This question is significantly more difficult than in the Yang–Mills case, because the equations are more non-linear, and the group of diffeomorphisms is more subtle than the group of gauge transformations. The case of positive curvature has proved in particular to be quite challenging, and there are still few reasonably complete results. In particular, the convergence of the flow is closely related to the geometric stability of the underlying Kähler manifold, in accordance with the principles of Yau's program [63]. Two very recent important advances are the work of Tian–Zhu [53] in 3 dimensions, and the work of Chen–Wang [12] in all dimensions, on the convergence of the flow to a soliton with mild singularities. However, it is not known whether the solutions can be extended across the singularities, and what would be the limiting spaces and metrics.

The purpose of the present paper is to provide the complete analysis of the Kähler–Ricci flow in a situation where the flow may cause the complex structure to jump.[1] For this, we need to consider a geometric situation where there may be instability. The simplest is the case of Riemann surfaces with conic singularities. Before describing our results on the Kähler–Ricci flow on Riemann surfaces with singularities, we begin by discussing the existing body of works on Kähler–Einstein metrics and Kähler–Ricci solitons on such surfaces.

First, we need the notion of metrics with conical singularities on a Kähler manifold. In [62], Yau considers complex Monge–Ampère equations with a singular right hand side as a generalization of his solution to the Calabi conjecture and he derives various important a priori estimates. In particular, Yau obtains local regularity for solutions of complex Monge–Ampère equations with conical singularities. The general Schauder estimates for equations of background conical Kähler metrics are established more recently by Donaldson in [20]. The analysis is considerably easier in the case of Riemann surfaces. Let M be a compact Riemann surface, and p a given point on M. A metric g on M is said to have a conical singularity at p if it can be expressed as

$$g = e^{f(z)}|z|^{-2\beta}|dz|^2 \tag{1.1}$$

near p, with $f(z)$ a bounded function. Here z is a local holomorphic coordinate centered at p, and $\beta \in (0,1)$ is a constant. The constant β is sometimes referred to as the weight of g at p, and the cone angle of g at p is $2(1-\beta)\pi$. To lighten the notation, we denote by g both the

[1] Please also see [49, 60] for some examples from the smooth Kähler–Ricci flow.

metric and the corresponding Kähler form $\frac{\sqrt{-1}}{2\pi}e^{f(z)}|z|^{-2\beta}|dz|^2$, when which is intended is clear from the context. Kähler metrics with conical singularities have been extensively studied.

More generally, we consider a compact Riemann surface M with given points $p_1, \cdots, p_k$, and weights β_j associated to each point p_j. We denote by β the divisor $\beta = \sum_{j=1}^k \beta_j[p_j]$, and refer to the data (M, β) as a pair. Throughout this paper, we always assume that $\beta_j \in (0, 1)$ for $j = 1, ..., k$. If g is a C^2 metric on $M \setminus \beta$, the Ricci curvature can be defined on $M \setminus \beta$ by the usual formula

$$Ric(g) = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log g.$$

For simplicity, we identify each Kähler metric with its associated Kähler form. If, in addition, g admits a conical singularity with weight β_j at each point p_j, then the Ricci curvature $\mathbf{Ric}(g)$ can be defined on the whole surface M as a current

$$\mathbf{Ric}(g) = Ric(g) + \sum_{j=1}^k \beta_j[p_j],$$

where $[p_j]$ is the Dirac measure at p_j. This is because, in two dimensions, the contribution $i\partial\bar{\partial}f$ of the conformal factor f in (1.1) cannot include a singular measure if f is bounded and C^2 away from the point p. We restrict ourselves to metrics g with conical singularities whose Ricci current $\mathbf{Ric}(g)$ is still in $c_1(M)$, i.e., the same Chern class as the Ricci curvature of smooth metrics on M. This means that

$$\chi(M) = \int_M \mathbf{Ric}(g) = \int_{M\setminus\beta} Ric(g) + \sum_{j=1}^k \beta_j,$$

and, hence,

$$\int_{M\setminus\beta} Ric(g) = \chi(M) - \sum_{j=1}^k \beta_j \equiv \chi(M, \beta).$$

where the second equality just defines the Euler characteristic $\chi(M, \beta)$ of the pair (M, β).

The metric g with conical singularities is said to have constant Ricci curvature if it satisfies $Ric(g) = \mu\, g$ on $M \setminus \beta$ for some constant μ. In view of the requirement that $\mathbf{Ric}(g)$ is still in the same Chern class $c_1(M)$, it follows that the constant μ must satisfy the constraint $\chi(M, \beta) = \mu \int_M g$. If we normalize the metric g so that $\int_M g = 2$, the equation of constant Ricci curvature becomes

$$Ric(g) = \frac{1}{2}\chi(M, \beta)g \tag{1.2}$$

on $M \setminus \beta$. The metrics with conical singularities and constant Ricci curvature on Riemann surfaces (M, β) with weights have been extensively studied by Troyanov [56], McOwen [30], Luo–Tian [28]. When the pair (M, β) has Euler characteristic $\chi(M, \beta) \leq 0$, it has been shown in [56] that it always admits a conical metric with constant Ricci curvature, and that such a metric is unique up to scaling. Clearly, $\chi(M, \beta)$ can be strictly positive only when

$$M = \mathbb{S}^2, \qquad \sum_{j=1}^{k} \beta_j < 2. \tag{1.3}$$

Henceforth, we shall make these assumptions. In this case, there are, indeed, in general obstructions for the existence of metrics with conical singularities and constant Ricci curvature. More specifically, the following is known:

• When $k = 1$, equation (1.2) does not admit a solution. Instead, one can construct a unique rotationally symmetric compact shrinking soliton $g \in c_1(\mathbb{S}^2)$ (the tear drop) [61, 4, 37].

• When $k = 2$, if $\beta_1 = \beta_2$, there exists a unique rotationally symmetric solution of equation (1.2) (the football) [13, 4, 37].

• When $k = 2$ and $\beta_1 \neq \beta_2$, equation (1.2) does not admit a solution. Instead, one can construct a unique rotationally symmetric compact shrinking soliton g [61, 4, 37].

• When $k \geq 3$, there does not exist any holomorphic vector field on $\mathbb{S}^2$ fixing $p_1, ..., p_k$ since any holomorphic vector field can at most vanish at 2 distinct points. Then the equation (1.2) admits a unique solution if and only if [56, 28]

$$2 \max_j \beta_j < \sum_{j=1}^{k} \beta_j.$$

Next, we describe what is known about the Ricci flow on the sphere $\mathbb{S}^2$. The case without marked points and conic singularities has been completely settled by the work of Hamilton [23] and Chow [13]. The orbifold shrinking gradient solitons on $\mathbb{S}^2$ have been classified by Wu [61]. The Ricci flow for metrics with conical singularities on Riemann surfaces was first studied by Yin [64, 65], who provided an important analytic framework as well as a proof of the long-time existence, and convergence of the flow when $\chi(M, \beta) \leq 0$. Another approach to existence results for the Ricci flow for metrics with conic singularities was given by Mazzeo, Rubinstein and Sesum [29], using an extensive machinery of polyhomogeneous expansions, conormal distributions and b-spaces. They also prove the convergence of the flow on any stable pair $(\mathbb{S}^2, \beta)$. When the pair $(\mathbb{S}^2, \beta)$ is not stable, they argue for some notion of "geometric" convergence (see Theorem 1.3 as well as section §5, and especially Proposition 5.3 in [29]). The fact is that the case of $(\mathbb{S}^2, \beta)$ not stable presents some significant new difficulties which were absent in the stable case and which cannot be bypassed. On one hand, the existence and global structure of the limiting space have to be determined, and it is far from evident a priori that the limiting space is another pair $(\mathbb{S}^2, \beta_\infty)$, or even that its singular set is closed. In fact, all conical singularities except the main one may converge to a single limiting conical point, and, thus, the injectivity radius will converge generically to 0, preventing any application of Hamilton's compactness theorem. Furthermore, for the possible convergence to a soliton, one must allow for reparametrizations, thus, ruling out techniques based solely on multiplier ideal sheaves.

The case of pairs $(\mathbb{S}^2, \beta)$ is of particular importance as it is the most basic example of spaces that can exhibit all three types of geometric structures, namely stable, semi-stable, and unstable structures, and where the phenomenon of the jumping of complex structures in the limit can take place. It is also a good model case of the Ricci flow on complex manifolds with singularities. We shall provide a complete understanding of the long-time behavior of the Ricci flow in this case. Besides its own interest, such an understanding should be valuable in the development of any program to produce canonical metrics on singular Kähler manifolds by the Ricci flow.

We state now precisely our results. Fix the following metric g_β on $\mathbb{S}^2$ with conic singularities at the points $\{p_1, \cdots, p_k\}$ with weights $\beta_1, \cdots, \beta_k \in (0, 1)$,

$$g_\beta = \prod_j \left(\frac{1 + |z|^2}{|z - p_j|^2} \right)^{\beta_j} g_{FS}, \tag{1.4}$$

where g_{FS} is the Fubini–Study metric. Explicitly $g_{FS} = \frac{\sqrt{-1}}{2\pi} \frac{|dz|^2}{(1+|z|^2)^2}$. Note that $\sigma_j = (z - p_j)^2 \frac{\partial}{\partial z}$ is the defining section for $[2p_j]$, and that $\int_{\mathbb{S}^2} g_\beta = 2$, which means that $g_\beta \in c_1(\mathbb{S}^2)$. We consider the conical Ricci flow given by

$$\frac{\partial g(t)}{\partial t} = -Ric(g) + \frac{1}{2}\chi(\mathbb{S}^2, \beta) g(t), \qquad g(0) = g_0 \tag{1.5}$$

on $\mathbb{S}^2 \setminus \beta$, where g_0 is a "regular metric", i.e., a metric of the form

$$g_0 = e^{u_0} g_\beta, \qquad u_0 \in C^\infty(\mathbb{S}^2), \qquad \int_{\mathbb{S}^2} g_0 = 2. \tag{1.6}$$

Theorem 1.1. *Consider the conical Ricci flow with initial metric g_0 satisfying the conditions (1.6). Then there exists a solution $g(t) = g_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi(t)$ satisfying*

(1) *$\varphi(t) \in PSH(\mathbb{S}^2, g_{FS}) \cap L^\infty(\mathbb{S}^2)$ for each $t \in [0, \infty)$. $\varphi \in C^\infty(\mathbb{S}^2 \setminus \beta \times [0, \infty))$.*

(2) *If we set $g(t) = e^{u(t)} g_\beta$, then $u(t) \in L^\infty(\mathbb{S}^2)$ for each $t \in [0, \infty)$ and $u \in C^\infty(\mathbb{S}^2 \setminus \beta \times [0, \infty))$.*

(3) *For any $T > 0$, there exists $C > 0$ such*

$$|u|_{L^\infty(\mathbb{S}^2\times[0,T])} \leq C.$$

(4) *For any $t > 0$, there is a unique function $v \in C^\infty(\mathbb{S}^2 \setminus \beta) \cap L^\infty(\mathbb{S}^2)$ satisfying*

$$Ric(g) = \frac{1}{2}\chi(\mathbb{S}^2, \beta) g - \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} v \quad \text{on } \mathbb{S}^2 \setminus \beta, \qquad \int_{\mathbb{S}^2} e^{-v} g = 1.$$

This function is called the Ricci potential of $g(t)$.

(5) *For any $t_0 > 0$ and any $k \in \mathbf{Z}^+ \cup \{0\}$, there exists $C(k, t_0) > 0$ such that*

$$\sup_{(\mathbb{S}^2\setminus\beta)\times[t_0,\infty)} (|\Delta^k R|^2 + |\nabla \Delta^k R|) \leq C_{k,t_0},$$

where R is the curvature of $g(t)$, and the operators Δ, ∇ are defined with respect to $g(t)$.

One can show that $u(t)$ is actually continuous on $\mathbb{S}^2 \times [0, \infty)$ because $\Delta_{g(t)} u(t)$ and $\frac{\partial u}{\partial t}$ are bounded on $\mathbb{S}^2 \times [0, T]$ for any $T > 0$ ($\Delta_{g(t)} u(t) = -R(t) + R(g_\beta)$ is bounded and $\frac{\partial}{\partial t} u$ is equivalent to $\Delta_{g(t)} u(t)$ on any closed time interval). Of particular importance to us is the following property of the spaces $(\mathbb{S}^2 \setminus \beta, g(t))$:

Theorem 1.2. *Let (X_t, d_t) be the metric completion of $(\mathbb{S}^2 \setminus \beta, g(t))$. Then*

(1) *(X_t, d_t) is a compact metric space homeomorphic to $\mathbb{S}^2$ for any $t \in [0, \infty)$, with uniformly bounded diameters and curvature,*

$$\sup_{t\in[0,\infty)} diam(X_t, d_t) \leq C, \quad \sup_{\mathbb{S}^2\setminus\beta\times[0,\infty)} |R| \leq C, \tag{1.7}$$

for some $C > 0$.

(2) *(X_t, d_t) is a continuous family of compact metric spaces in the Gromov–Hausdorff topology for all $t \in [0, \infty)$.*

To describe the convergence of the flow, we introduce the following terminology:

Definition 1.1. Let $(\mathbb{S}^2, \beta)$ be a sphere with marked points, with $k \geq 3$. We shall say that

- $(\mathbb{S}^2, \beta)$ is stable if $\sum_{i=1}^k \beta_i \geq 2$ or $2\max_j \beta_j < \sum_{i=1}^k \beta_i$.
- $(\mathbb{S}^2, \beta)$ is semi-stable if $\sum_{i=1}^k \beta_i < 2$ and $2\max_j \beta_j = \sum_{i=1}^k \beta_i$.
- $(\mathbb{S}^2, \beta)$ is unstable if $\sum_{i=1}^k \beta_i < 2$ and $2\max_j \beta_j > \sum_{i=1}^k \beta_i$.

Without loss of generality, we assume that

$$0 < \beta_1 \leq \beta_2 \leq ... \leq \beta_k = \beta_{max}.$$

We have then the following main result of the paper.

Theorem 1.3. *Let $(\mathbb{S}^2, \beta)$ be a sphere with k marked points, $\beta = \sum_{j=1}^k \beta_j[p_j]$, $\sum_{j=1}^k \beta_j < 2$ and $k \geq 3$. Consider the Ricci flow $g(t)$ with an initial metric g_0 satisfying the condition (1.6).*

(1) *If $(\mathbb{S}^2, \beta)$ is stable, then the flow converges in the Gromov–Hausdorff topology and in $C^\infty(\mathbb{S}^2 \setminus \beta)$ to the unique conical constant curvature metric $g_\infty \in c_1(\mathbb{S}^2)$ on $(\mathbb{S}^2, \beta)$.*

(2) *If $(\mathbb{S}^2, \beta)$ is semi-stable, then the flow converges in the Gromov–Hausdorff topology to the unique conical constant curvature metric g_∞ on a pair $(\mathbb{S}^2, \beta_\infty)$, where the divisor β_∞ is given by*

$$\beta_\infty = \beta_{max}[p_\infty] + \beta_{max}[q_\infty],$$

with $\beta_{max} \equiv \beta_k$. The convergence is smooth on $\mathbb{S}^2 \setminus \{p_\infty, q_\infty\}$. In particular, p_k converges in Gromov–Hausdorff distance to one of the two points p_∞ and q_∞, while $p_1, ..., p_{k-1}$ converge to the other.

(3) *If $(\mathbb{S}^2, \beta)$ is unstable, then the flow converges in the Gromov–Hausdorff topology to the unique rotationally invariant shrinking soliton g_∞ on the pair $(\mathbb{S}^2, \beta_\infty)$, where the divisor β_∞ is given by*

$$\beta_\infty = \beta_k[p_\infty] + \beta_k'[q_\infty],$$

with $\beta_k' \equiv \sum_{j<k} \beta_j$. The convergence is smooth on $\mathbb{S}^2 \setminus \{p_\infty, q_\infty\}$. In particular, p_k converges in Gromov–Hausdorff distance to the point p_∞ while $p_1, ..., p_{k-1}$ converge to q_∞.

We remark that the stable case (1) in Theorem 1.3 is proved in [**29**]. In the unstable case (3), we also show that $p_1, ..., p_{k-1}$ converge to q_∞ in distance exponentially fast as $t \to \infty$ (c.f. Lemma 7.11). We conjecture that in the semi-stable case (2) in Theorem 1.3, the convergence of $p_1, ..., p_{k-1}$ to q_∞ is in distance with polynomial decay.

The present paper is a substantially revised and improved version of the paper with the same title which was posted as ArXiv:14.07.1118. A first significant improvement is the proof of a conjecture left open in the original paper, on the convergence in the unstable case of the conic singularities $p_1, \cdots, p_{k-1}$ to q_∞. The second significant improvement is a self-contained proof of the long-time existence and Perelman monotonicity for the conical Ricci flow. The original paper ArXiv:14.07.1118 had relied on the results of Yin [**64**, **65**], by quoting directly his long-time existence results in suitable Schauder spaces, and by using his methods for the proof of Perelman monotonicity. Besides being self-contained, the proof in the present paper of the long-time existence of the conical Ricci flow and Perelman monotonicity applies to a more familiar class of initial metrics g_0 (see Theorem 1.1) than metrics defined in the more subtle Schauder spaces that Yin [**64**, **65**] had introduced specifically for the heat equation on surfaces with conic singularities. The main idea is to use smooth approximations of conical metrics, and make use of traditional techniques for the twisted Kähler–Ricci flow as in [16] and [45, 46]. It is likely that the idea will also be useful in higher dimensions.

The paper is organized as follows. In Section §2, we establish the long-time existence of the conical Ricci flow, together with basic estimates for the conformal factor, the Kähler potential, and the Ricci potential. As a consequence, we obtain the essential fact that the scalar curvature and the diameter are uniformly bounded along the flow. Since in two dimensions, the scalar curvature determines the full Riemannian curvature, we are in a situation similar to that considered in [32], with the key additional complication that the manifolds are not compact, and the injectivity radius not bounded from below. In Section §3, we establish Perelman monotonicity for the conical Ricci flow. This requires some delicate arguments of integration by parts. Section §4 is devoted to the analysis of the long-time behavior in the stable case. In this case, the functional F_β is proper, and we adapt the arguments in the smooth case to show the convergence of the flow. The Perelman monotonicity for the conical Ricci flow is only needed for this section, but not for the remaining sections on the Gromov–Hausdorff convergence in the semistable and unstable cases. Section §5 is devoted to the proof of the sequential convergence of the flow in the Gromov–Hausdorff sense, by combining Cheeger–Colding theory, the partial C^0 estimate, and Hamilton's entropy. The limit is a sphere with marked points, equipped with either a metric of constant curvature or a shrinking gradient soliton. The semi-stable case is treated in Section §6: we derive a weak lower bound for the functional F_β which allows us, nevertheless, to show the existence of a sequence of times along which the Ricci flow converges to a metric of constant curvature. The unstable case is treated in Section §7. We show that the limiting metric cannot have constant curvature, and, hence, it must be a soliton with rotational symmetry. We further prove that the conical Ricci flow must uniformly converge to a unique shrinking Ricci soliton with two conical points and such a soliton metric does not depend on the choice of initial metric.

Some basic facts about the α-invariant and the F-functional for pairs $(\mathbb{S}^2, \beta)$ are summarized in the appendices.

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2. Long-time existence for the conical Ricci flow

In this section, we establish some general properties of the conical Ricci flow, including the existence of long-time solutions $g(t)$, and the identification of the metric completion of the space $(\mathbb{S}^2 \setminus \beta)$ for each time t. The main idea is to use a smooth approximation of conical metrics. This bypasses the problem of solving partial differential equations on open manifolds. The key quantities such as the metric $g(t)$, the conformal factor $u(t)$, the Kähler potential $\varphi(t)$, and the Ricci potential $v(t)$ are obtained by smooth convergence over compact subsets of $\mathbb{S}^2 \setminus \beta$. Nevertheless, we can establish $L^\infty(\mathbb{S}^2 \setminus \beta)$ bounds for u, ∇u, φ, $\nabla\varphi$, v, $\nabla\varphi$ on any finite time interval $[0, T]$.

2.1. A smooth approximation of the initial metric. Let g_0 be the initial conical Kähler metric satisfying (1.6), and consider the following approximating smooth Kähler metrics for $\epsilon > 0$,

$$g_{0,\epsilon} = e^{u_0 + a_\epsilon} \prod_j \left(\frac{1 + |z|^2}{|z - p_j|^2 + \epsilon} \right)^{\beta_j} g_{FS}, \tag{2.1}$$

where a_ϵ is the normalizing constant so that $\int g_{0,\epsilon} = 2$. Then

$$2e^{-a_\epsilon} = \int_{\mathbb{S}^2} e^{u_0} \prod_j \left(\frac{(1 + |z|^2)}{|z - p_j|^2 + \epsilon} \right)^{\beta_j} g_{FS},$$

and a_ϵ is uniformly bounded for $\epsilon \in (0, 1)$ with $a_\epsilon \to a_0 = 0$ as $\epsilon \to 0$. Clearly,

$$g_{0,\epsilon} \geq C g_{FS}, \tag{2.2}$$

for some constant $C > 0$ independent of ϵ. Define the $(1, 1)$-form θ_ϵ by

$$\theta_\epsilon = \sum_j \beta_j \frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log(|z - p_j|^2 + \epsilon). \tag{2.3}$$

Then the Ricci curvature $Ric(\omega_{0,\epsilon})$ of $\omega_{0,\epsilon}$ is given by

$$Ric(g_{0,\epsilon}) = \frac{1}{2}\chi(\mathbb{S}^2, \beta) g_{FS} + \theta_\epsilon - \frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} u_0. \tag{2.4}$$

Let ϕ_ϵ be the Kähler potential of the approximating metric $g_{0,\epsilon}$, i.e.,

$$g_{0,\epsilon} = g_{FS} + \frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \phi_\epsilon, \qquad \sup_{\mathbb{S}^2} \phi_\epsilon = 0. \tag{2.5}$$

The function ϕ_ϵ is then the unique solution, with the normalization indicated, of the following smooth Laplace equation on $\mathbb{S}^2$,

$$\Delta_{g_{FS}} \phi_\epsilon = e^{u_0 + a_\epsilon} \prod_j \left(\frac{(1 + |z|^2)}{|z - p_j|^2 + \epsilon} \right)^{\beta_j} - 1. \tag{2.6}$$

Lemma 2.1. *There exists a constant $C > 0$ so that, for all $\epsilon \in (0, 1)$, we have*

$$\|\phi_\epsilon\|_{L^\infty} + \|\nabla\phi_\epsilon\|_{g_{0,\epsilon}} \leq C. \tag{2.7}$$

Furthermore, for any $k > 0$ and $K \subset\subset \mathbb{S}^2 \setminus \{p\}$, there exists $C_{k,K}$ such that

$$\|\phi_\epsilon\|_{C^k(K)} \leq C_{k,K} \quad \text{for all } \epsilon \in (0, 1). \tag{2.8}$$

Proof. The right hand side of the equation (2.6) is uniformly bounded for all $\epsilon \in (0,1)$ in $L^p(\mathbb{S}^2, g_{FS})$ for some fixed $p > 1$. The L^∞ estimate then follows immediately from the Laplacian estimates and Sobolev embedding.

We now prove the estimates for $\|\nabla\phi_\epsilon\|_{g_{0,\epsilon}}$, using the techniques for combining gradient estimates and Yau's Schwarz lemma first used in [**45**, **46**]. For the remainder of the proof, we denote by just Δ and ∇ the Laplacian and gradient with respect to the metric $g_{0,\epsilon}$. We also suppress the subindex ϵ in ϕ_ϵ. Using the equation $\Delta\phi = 1 - \mathrm{tr}_g g_{FS}$, we find

$$\begin{aligned} \Delta|\nabla\phi|^2 &= |\nabla\nabla\phi|^2 + |\nabla\bar{\nabla}\phi|^2 + R(g)|\nabla\phi|^2 - 2\mathrm{Re}\langle\nabla\mathrm{tr}_g(g_{FS}), \nabla\phi\rangle \\ &\geq |\nabla\nabla\phi|^2 + |\nabla\bar{\nabla}\phi|^2 - 2|\nabla\mathrm{tr}_g(g_{FS})|\,|\nabla\phi| - C_1|\nabla\phi|^2, \end{aligned} \tag{2.9}$$

since $R(g)$ is bounded from below by a constant $-C_1$ independent of ϵ. Next, let A be a constant so large that $A + \phi > 1$ on $\mathbb{S}^2$. A straightforward computation gives

$$\Delta\left(\frac{|\nabla\phi|^2}{A+\phi}\right) = \frac{\Delta|\nabla\phi|^2}{A+\phi} - 2\mathrm{Re}\left\langle\frac{\nabla\phi}{A+\phi}, \nabla\left(\frac{|\nabla\phi|^2}{A+\phi}\right)\right\rangle - \frac{\Delta\phi|\nabla\phi|^2}{(A+\phi)^2}. \tag{2.10}$$

On the other hand, we can also write

$$\begin{aligned} -\mathrm{Re}\left\langle\frac{\nabla\phi}{A+\phi}, \nabla\left(\frac{|\nabla\phi|^2}{A+\phi}\right)\right\rangle &= \frac{|\nabla\phi|^4}{(A+\phi)^3} - 2\mathrm{Re}\left\langle\frac{\nabla\phi}{A+\phi}.\frac{\nabla|\nabla\phi|^2}{A+\phi}\right\rangle \\ &\geq \frac{|\nabla\phi|^4}{(A+\phi)^3} - \frac{|\nabla\phi|^2(|\nabla\nabla\phi| + |\nabla\bar{\nabla}\phi|)}{(A+\phi)^2} \\ &\geq \frac{1}{2}\frac{|\nabla\phi|^4}{(A+\phi)^3} - \frac{1}{2}\frac{(|\nabla\nabla\phi| + |\nabla\bar{\nabla}\phi|)^2}{A+\phi}. \end{aligned} \tag{2.11}$$

Fix $0 < \delta < \frac{1}{4}$. The above inequalities imply

$$\begin{aligned} \Delta\left(\frac{|\nabla\phi|^2}{A+\phi}\right) &\geq \delta\,\frac{|\nabla\phi|^4}{(A+\phi)^3} g_{FS} - 2(1-\delta)\mathrm{Re}\left\langle\frac{\nabla\phi}{A+\phi}, \nabla\left(\frac{|\nabla\phi|^2}{A+\phi}\right)\right\rangle \\ &\quad -\frac{1}{A+\phi}\left(\frac{|\nabla\phi|^2\Delta\phi}{A+\phi} + |\nabla\mathrm{tr}_g(g_{FS})|\,|\nabla\phi| + C_1|\nabla\phi|^2\right). \end{aligned} \tag{2.12}$$

The troublesome term is $|\nabla\mathrm{tr}_g(g_{FS})|\,|\nabla\phi|$ on the right hand side. To handle it, write

$$\Delta\log\mathrm{tr}_g(g_{FS}) = \frac{\Delta\mathrm{tr}_g(g_{FS})}{\mathrm{tr}_g(g_{FS})} - \frac{|\nabla\mathrm{tr}_g(g_{FS})|^2}{(\mathrm{tr}_g(g_{FS}))^2}. \tag{2.13}$$

The left hand side can be recognized as

$$-\Delta\log g + \mathrm{tr}_g(g_{FS})\,\Delta_{FS}\log g_{FS} = R(g) - \mathrm{tr}_g(g_{FS})R(g_{FS}) \geq -C_2, \tag{2.14}$$

where C_2 is a constant independent of ϵ. Thus, we have

$$\Delta \mathrm{tr}_g(g_{FS}) \geq \frac{|\nabla \mathrm{tr}_g(g_{FS})|^2}{\mathrm{tr}_g(g_{FS})} - C_2 \mathrm{tr}_g(g_{FS}) \geq \frac{|\nabla \mathrm{tr}_g(g_{FS})|^2}{\mathrm{tr}_g(g_{FS})} - C_3. \tag{2.15}$$

Set then

$$H = \frac{|\nabla\phi|^2}{A+\phi} + M\,\mathrm{tr}_g(g_{FS}). \tag{2.16}$$

We have

$$\begin{aligned}
\Delta H &= \Delta\Big(\frac{|\nabla\phi|^2}{A+\phi}\Big) + M\,\Delta \mathrm{tr}_g(g_{FS}) \\
&\geq \delta\,\frac{|\nabla\phi|^4}{(A+\phi)^3} + M\,\frac{|\nabla \mathrm{tr}_g(g_{FS})|^2}{\mathrm{tr}_g(g_{FS})} - 2(1-\delta)\mathrm{Re}\langle\frac{\nabla\phi}{A+\phi}, \nabla H\rangle - MC_3 \\
&\quad + 2M(1-\delta)\mathrm{Re}\langle\frac{\nabla\phi}{A+\phi}, \nabla \mathrm{tr}_g(g_{FS})\rangle \\
&\quad - \left(\frac{|\nabla\phi|^2\Delta\phi}{(A+\phi)^2} + \frac{|\nabla \mathrm{tr}_g(g_{FS})|\,|\nabla\phi| + C_1|\nabla\phi|^2}{A+\phi}\right).
\end{aligned}$$

The terms involving $|\nabla \mathrm{tr}_g(g_{FS})|$ can be grouped into

$$\begin{aligned}
M\,\frac{|\nabla \mathrm{tr}_g(g_{FS})|^2}{\mathrm{tr}_g(g_{FS})} &+ 2M(1-\delta)\mathrm{Re}\langle\frac{\nabla\phi}{A+\phi}, \nabla \mathrm{tr}_g(g_{FS})\rangle \\
&- \frac{|\nabla \mathrm{tr}_g(g_{FS})|\,|\nabla\phi|}{A+\phi} \\
&\geq \frac{M}{2}\,\frac{|\nabla \mathrm{tr}_g(g_{FS})|^2}{\mathrm{tr}_g(g_{FS})} - C_4\frac{|\nabla\phi|^2}{(A+\phi)^2},
\end{aligned}$$

if M is large enough, and the constant C_4 is allowed to depend on M. Since $\Delta\phi$ is bounded, we obtain in this manner

$$\Delta H \geq \delta\,\frac{|\nabla\phi|^4}{(A+\phi)^3} - 2(1-\delta)\mathrm{Re}\langle\frac{\nabla\phi}{A+\phi}, \nabla H\rangle - C_5\frac{|\nabla\phi|^2}{A+\phi}. \tag{2.17}$$

It follows now from the maximum principle, applied to the function H, that the function $|\nabla\phi|$ is uniformly bounded in ϵ.

Finally, the estimates for $\|\phi_\epsilon\|_{C^k(K)}$ are standard local estimates in linear elliptic theory. q.e.d.

2.2. A smooth approximation for the Ricci flow. We consider now the approximating Ricci flow

$$\frac{\partial g_\epsilon}{\partial t} = -Ric(g_\epsilon) + \frac{1}{2}\chi(\mathbb{S}^2,\beta)g_\epsilon + \theta_\epsilon, \qquad g_\epsilon|_{t=0} = g_{0,\epsilon}. \tag{2.18}$$

It is an example of a twisted Ricci flow with a positive $(1,1)$-form θ_ϵ, as considered in [**16**]. When $\epsilon = 0$, it coincides with the Ricci flow. If we let $g_\epsilon = g_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi_\epsilon$, then the corresponding flow for φ_ϵ is given by the parabolic Monge–Ampère equation,

$$\frac{\partial\varphi_\epsilon}{\partial t} = \log\frac{g_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi_\epsilon}{g_{FS}} + \frac{1}{2}\chi(\mathbb{S}^2,\beta)\varphi_\epsilon + \sum_j \beta_j \log\frac{|z-p_j|^2+\epsilon}{1+|z|^2},$$

$$\varphi_\epsilon|_{t=0} = \phi_\epsilon. \tag{2.19}$$

Lemma 2.2. *The following hold.*
(1) *For any $T>0$, there exists $C>0$ such that for all $\epsilon\in(0,1)$,*

$$\|\varphi_\epsilon\|_{L^\infty(\mathbb{S}^2\times[0,T])} \le C.$$

(2) *For any $T>0$, there exists $C=C(T)>0$ such that for all $\epsilon\in(0,1)$*

$$\sup_{\mathbb{S}^2\times[0,T]} |u_\epsilon| \le C,$$

where $g_\epsilon(t) = e^{u_\epsilon(t)}\prod_j\left(\frac{(1+|z|^2)}{|z-p_j|^2+\epsilon}\right)^{\beta_j} g_{FS}$.
(3) *For any $K\subset\subset \mathbb{S}^2\backslash\beta$, $T>0$ and $k>0$, there exists $C=C_{K,T,k}>0$,*

$$\|\varphi_\epsilon\|_{C^k(K\times[0,T])} \le C.$$

Proof. The proposition follows from general results from [**45**]. However, we include a proof for completeness.

Let $H=\varphi_\epsilon(t)-\phi_\epsilon$. Then

$$(\frac{\partial}{\partial t} - \Delta_{g_\epsilon})H = \log\frac{g_{0,\epsilon} + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}H}{g_{0,\epsilon}} + H,\ \ H|_{t=0}=0.$$

Then the maximum principle immediately implies that for any $T>0$, H is uniformly bounded on $\mathbb{S}^2\times[0,T]$ for all $\epsilon\in(0,1)$. Since ϕ_ϵ is uniformly bounded, this proves (1).

Next, $\frac{\partial\varphi_\epsilon}{\partial t}$ satisfies the equation

$$(\frac{\partial}{\partial t} - \Delta_\epsilon)\frac{\partial\varphi_\epsilon}{\partial t} = \frac{1}{2}\chi(\mathbb{S}^2,\beta)\frac{\partial\varphi_\epsilon}{\partial t}.$$

The initial value

$$\frac{\partial\varphi_\epsilon}{\partial t}|_{t=0} = \log\frac{g_{0,\epsilon}}{g_{FS}} + \frac{1}{2}\chi(\mathbb{S}^2,\beta)\phi_\epsilon + \frac{1}{2}\sum_j\beta_j\log\frac{|z-p_j|^2+\epsilon}{1+|z|^2}$$

is uniformly bounded for all $\epsilon\in(0,1)$ because ϕ_ϵ is uniformly bounded. The maximum principle immediately implies that for any $T>0$, there exists $C>0$ such that for all $\epsilon\in(0,1)$,

$$|\frac{\partial\varphi_\epsilon}{\partial t}| \le C. \tag{2.20}$$

On the other hand, the defining relation for the conformal factor u can be rewritten as

$$u_\epsilon + a_\epsilon = \log \frac{g_\epsilon}{g_{FS}} + \sum_j \beta_j \log \frac{|z-p_j|^2+\epsilon}{1+|z|^2}. \tag{2.21}$$

Comparing this with the equation (2.19), we find

$$\frac{\partial \varphi_\epsilon}{\partial t} = u_\epsilon + a_\epsilon + \frac{1}{2}\chi(\mathbb{S}^2,\beta)\varphi_\epsilon. \tag{2.22}$$

Thus, the uniform boundedness of u_ϵ follows from that of φ_ϵ and $\partial\varphi_\epsilon/\partial t$, proving (2).

Finally, from the estimates for $\frac{\partial\varphi_\epsilon}{\partial t}$, for any $K \subset\subset \mathbb{S}^2 \setminus \beta$ and $T>0$, there exists $C>0$ such that for all $\epsilon \in (0,1)$,

$$\|\Delta_0 \varphi_\epsilon\|_{L^\infty(K\times[0,T])} \leq C,$$

where Δ_0 is the Laplacian operator with respect to g_{FS}. The higher order estimates follow immediately from the standard linear parabolic theory. The proof of the lemma is complete. q.e.d.

2.3. Perelman monotonicity for the approximating flow. We consider Perelman's W-functional for the twisted Ricci flow as introduced in [**16**]

$$W_\theta(g,f,\tau) = \int_X e^{-f}\tau^{-n}(\tau(R - tr_g(\theta) + |\nabla f|^2) + f)dV_g,$$

and the corresponding twisted μ-functional,

$$\mu_\theta(g,\tau) = \inf_{f\in C^\infty(X),\ (2\tau)^{-n}\int_X e^{-f}dV_g=1} W_\theta(g,f,\tau).$$

It is proved in [**16**] that μ_θ is increasing along the Ricci flow, and that Perelman's estimates can be exactly reproduced for any fixed smooth positive twisted form θ. In our situation, $X=\mathbb{S}^2$, $\tau = 1/2$, $\theta=\theta_\epsilon$. Thus,

Lemma 2.3. *Let $g_\epsilon(t)$ be the solution of the twisted Ricci flow (2.18). There exists $C>0$, such that for all $\epsilon \in (0,1)$,*

$$\mu_{\theta_\epsilon}(g_\epsilon(t), 1/2) \geq -C.$$

Proof. By its definition, the initial $g_{0,\epsilon}$ has uniformly bounded diameter for $\epsilon\in(0,1)$ and its curvature is uniformly bound below in view of (2.4). Also the volume of $g_{0,\epsilon}$ is uniformly bounded above and below away from 0 for all $\epsilon \in (0,1)$. Hence, the Sobolev and Log Sobolev constants are uniformly bounded. Together they give a uniform lower bound for the μ_{θ_ϵ}-functional at the initial time. The lemma follows from the monotonicity of μ_{θ_ϵ}. q.e.d.

We define the twisted Ricci potential v_ϵ by

$$v_\epsilon = \frac{\partial \varphi_\epsilon}{\partial t} + c_\epsilon(t), \qquad \int_{\mathbb{S}^2} e^{-v_\epsilon} g_\epsilon(t) = 1. \tag{2.23}$$

The function v_ϵ is the twisted Ricci potential in the sense that

$$Ric(g_\epsilon) = \frac{1}{2}\chi(\mathbb{S}^2, \beta) g_\epsilon + tr_{g_\epsilon}(\theta_\epsilon) - \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} v_\epsilon.$$

Lemma 2.4. *There exists $C > 0$ such that for all $\epsilon \in (0, 1)$,*

$$\left(\sup_{\mathbb{S}^2}(|v_\epsilon| + |\nabla v_\epsilon|_{g_\epsilon} + |\Delta_{g_\epsilon} v_\epsilon|)\right)|_{t=0} \leq C.$$

Proof. The bound for v_ϵ follows from the bound for $\partial\varphi_\epsilon/\partial t$ established in (2.20). To establish the bound for $|\nabla v_\epsilon|_{g_\epsilon}$, we note that

$$v_\epsilon|_{t=0} + c_\epsilon(0) = u_0 + a_\epsilon + \frac{1}{2}\chi(\mathbb{S}^2, \beta)\phi_\epsilon.$$

Then

$$(|\nabla v_\epsilon|_{g_\epsilon})|_{t=0} \leq |\nabla u_0|_{g_{0,\epsilon}} + \frac{1}{2}\chi(\mathbb{S}^2, \beta)|\nabla \phi_\epsilon|_{g_{0,\epsilon}} \leq C$$

by Lemma 2.1 because g_{FS} is bounded above by a positive multiple of $g_{0,\epsilon}$. Finally, let R_ϵ be the twisted scalar curvature defined by

$$R_\epsilon = R(g_\epsilon) - \mathrm{tr}_{g_\epsilon}\theta_\epsilon. \tag{2.24}$$

At $t = 0$, the explicit formula for $R(g_{0,\epsilon})$ given in (2.4) shows that $|R_\epsilon| \leq C$. Since $\Delta_{g_\epsilon} v_\epsilon = R_\epsilon - \frac{1}{2}\chi(\mathbb{S}^2, \beta)$, the desired estimate follows. q.e.d.

Lemma 2.5. *There exists $C > 0$ such that for all $\epsilon \in (0, 1)$ and for all $t \in [0, \infty)$*

$$\sup_{\mathbb{S}^2}(|v_\epsilon| + |\nabla v_\epsilon|_{g_\epsilon} + |\Delta_{g_\epsilon} v_\epsilon|) \leq C,$$

$$diam(\mathbb{S}^2, g_\epsilon(t)) \leq C, \ -C \leq R_\epsilon \leq C. \tag{2.25}$$

Proof. The same argument of Perelman for the Fano Kähler–Ricci flow [**39**] have been shown to generalize to the twisted Fano Kähler–Ricci flow by [**16**]. In particular, the bounds only depend on the lower bound of the μ-functional at the initial time as well as the bounds of $|v_\epsilon|$, $|\nabla v_\epsilon|_{g_\epsilon}$, $\Delta_{g_\epsilon} v_\epsilon$ at the initial time (see the proof of Lemma 2.2 in [**35**] for such bounds of a family of the Kähler–Ricci flow). Since the μ_{θ_ϵ}-functional is uniformly bounded from below for all $\epsilon \in (0, 1)$, the resulting bounds for v_ϵ, ∇v_ϵ, and $\Delta_{g_\epsilon} v_\epsilon$ are all uniformly bounded in $\epsilon \in (0, 1)$ and $t \in [0, \infty)$ from Lemma 2.4. q.e.d.

Lemma 2.6. *For any $k > 0$, $t_0 > 0$ there exists $C_{k,t_0} > 0$ such that for all $\epsilon \in (0, 1)$,*

$$\sup_{\mathbb{S}^2 \times [t_0, \infty)} (|\Delta^k_{g_\epsilon} R_\epsilon| + |\nabla_{g_\epsilon} \Delta^k_{g_\epsilon} R_\epsilon|_{g_\epsilon}) \leq C_{k,t_0}.$$

Proof. We consider the evolution of $|\nabla R_\epsilon|^2$. There exist $A, B > 0$, such that

$$\frac{\partial |\nabla R_\epsilon|^2_{g_\epsilon}}{\partial t} \leq \Delta_{g_\epsilon} |\nabla R_\epsilon|^2 - |\nabla_{g_\epsilon} \nabla_{g_\epsilon} R_\epsilon|^2_{g_\epsilon} - |\nabla_{g_\epsilon} \bar{\nabla}_{g_\epsilon} R_\epsilon|^2_{g_\epsilon} + A(R_\epsilon^2 + B)|\nabla R_\epsilon|^2_{g_\epsilon}.$$

Let $H = t|\nabla R_\epsilon|^2_{g_\epsilon} - 2MR_\epsilon^2$ for sufficiently large $M > 0$. We have

$$\frac{\partial H}{\partial t} \leq \Delta_{g_\epsilon} H - M|\nabla_{g_\epsilon} R_\epsilon|^2_{g_\epsilon} + R_\epsilon^3 + C,$$

because R_ϵ is uniformly bounded. The maximum principle immediately gives a uniform bound for H and, thus, $t|\nabla_{g_\epsilon} R_\epsilon|^2$ on $\mathbb{S}^2 \times [0, \infty)$.

We now pick any $\delta > 0$. Let

$$G = (t - \delta)\Delta_{g_\epsilon} R_\epsilon + N|\nabla_{g_\epsilon} R_\epsilon|^2_{g_\epsilon}$$

on $\mathbb{S}^2 \times [\delta, \infty)$. Then there exists $C > 0$ so that

$$\frac{\partial G}{\partial t} \leq \Delta_{g_\epsilon} G - G + C$$

on $\mathbb{S}^2 \times [\delta, \infty)$. By the maximum principle, G is uniformly bounded above on $\mathbb{S}^2 \times [\delta, \infty)$ and so is $(t - \delta)\Delta_{g_\epsilon} R_\epsilon$. The lower bound of $(t - \delta)\Delta_{g_\epsilon} R_\epsilon$ is obtained by applying the maximum principle to $(t - \delta)\Delta_{g_\epsilon} R_\epsilon + N|\nabla_{g_\epsilon} R_\epsilon|^2_{g_\epsilon}$.

The proof of the lemma is completed by repeating the procedure applied to $|\nabla_{g_\epsilon} \Delta^k R_\epsilon|$ and $(\Delta_{g_\epsilon})^{k+1} R_\epsilon$. q.e.d.

We noticed after we finished writing our paper that the conical Kähler–Ricci flow was treated similarly in [**27**] by applying approximation of twisted Kähler–Ricci flow with results in [**16**]. The argument in our case is simpler and the curvature estimates are stronger, due to the low dimension.

2.4. Proof of Theorem 1.1 and Theorem 1.2. Now we can take limits and let $g(t) = \lim_{\epsilon \to 0} g_\epsilon(t)$ after passing to a convergent subsequence. Theorem 1.1 follows immediately. We note also that the same limiting process shows that the curvature of the metrics along the conical Ricci flow is uniformly bounded on $\mathbb{S}^2 \setminus \beta$.

Next, we establish Theorem 1.2.

Part 1 follows from well-known arguments in Riemannian geometry because the conformal factors are uniformly bounded and converge smoothly away from finitely many points, and the sectional curvature is uniformly bounded below. More precisely, we fix t. Then any sequence of points $\{x_j\}$ in $\mathbb{S}^2 \setminus \beta$ converging to a point $p \notin \mathbb{S}^2 \setminus \beta$ with respect to $g(t)$ must also converge in distance with respect to $\prod_j \left(\frac{|z|^2+1}{|z-p_j|^2}\right)^{\beta_j} g_{FS}$ because $u(t)$ is bounded. But such a sequence of points must converge to a unique conical point $p \in \mathbb{S}^2$. This implies that (X_t, d_t) is homeomorphic to $\mathbb{S}^2$.

For fixed t, we consider the sequence $\{(\mathbb{S}^2, g_{1/j}(t))\}_j$. Then the diameter of $(\mathbb{S}^2, g_{1/j}(t))$ is uniformly bounded below, the volume is always 2 and the curvature is uniformly bounded below. Applying the Cheeger–Colding theory for degeneration of non-collapsed Riemmanian manifolds with Ricci lower bound after passing to a sequence, $(\mathbb{S}^2, g_{1/j}(t))$ converges to a compact metric length space (Y, d_Y). By the smooth convergence of $g_{1/j}(t)$ to $g(t)$ on $\mathbb{S}^2 \setminus \beta$, (Y, d_Y) must coincide with (X_t, d_t), the metric completion of $(\mathbb{S}^2 \setminus \beta, g(t))$ by the almost geodesic convexity results of Cheeger–Colding theory. In particular, the diameter of (X_t, d_t) is uniformly bounded above for all $t \in [0, \infty)$. The uniform curvature bound follows immediately from the uniform bound for R_ϵ for all $\epsilon \in (0, 1)$ and the smooth convergence of $g_\epsilon(t)$ to $g(t)$ on $\mathbb{S}^2 \setminus \beta$. One might also apply the Alexanderov theory for Riemann surfaces with sectional curvature bounded below.

For Part 2, it suffices to prove the family of metric spaces $(\mathbb{S}^2, g(t))$ is continuous at some $t_0 > 0$. For any $\epsilon > 0$, there exists an open neighborhood U of β such that $diam(U, g(t)) < \epsilon/2$ for $t \in [0, 2t_0]$ because $g(t)$ is uniformly equivalent to g_{FS} on a fixed closed time interval. There exists $\delta > 0$ such that for $t \in [t_0 - \delta, t_0 + \delta]$,

$$d_{GH}((\mathbb{S}^2 \setminus U, g(t)), (\mathbb{S}^2 \setminus U, g(t_0))) < \epsilon/2,$$

because $g(t)$ converges smoothly to $g(t_0)$ on $\mathbb{S}^2 \setminus \beta$ as $t \to t_0$. By the triangle inequality, for all $t \in [t_0 - \delta, t_0 + \delta]$,

$$d_{GH}((\mathbb{S}^2, g(t)), (\mathbb{S}^2, g(t_0))) < \epsilon.$$

We also obtain the following approximation using the solution $g_\epsilon(t)$ of the approximating twisted Ricci flow.

Corollary 2.1. *There exists $K > 0$ such that for all $t \in [0, \infty)$ and $\delta > 0$, there exists a smooth metric $g_{t,\delta} \in c_1(\mathbb{S}^2)$ such that*

$$R(g_{t,\delta}) > -K, \ d_{GH}((\mathbb{S}^2, g(t)), (\mathbb{S}^2, g_{t,\delta})) < \delta.$$

We conclude this section with two important gradient bounds:

Lemma 2.7. *For each $T > 0$, there exists a constant C_T so that for all $t \in [0, T]$,*

$$\|\nabla \varphi\|_{L^\infty} + \|\nabla u\|_{L^\infty} \le C_T. \tag{2.26}$$

Proof. We shall show that

$$\|\nabla \varphi_\epsilon\|_{L^\infty} \le C_T. \tag{2.27}$$

This will imply the desired bound for $\|\nabla \varphi\|_{L^\infty}$, since $\nabla \varphi_\epsilon$ and g_ϵ converges uniformly to $\nabla \varphi$ and $g(t)$ on compact subsets of $\mathbb{S}^2 \setminus \beta$.

We apply the same argument as in Lemma 2.1 for the proof of the bound for $|\nabla \phi_\epsilon|_{g_{0,\epsilon}}$. Notice that

$$\Delta_\epsilon \varphi_\epsilon = 1 - \mathrm{tr}_{g_\epsilon}(g_{FS})$$

is uniformly bounded on $[0,T]$ for all $\epsilon \in (0,1)$. By a straightforward calculation as in the proof of Lemma 2.1, we have

$$\begin{aligned}\Delta_\epsilon |\nabla \varphi_\epsilon|^2_{g_\epsilon} &= |\nabla_\epsilon \nabla_\epsilon \varphi_\epsilon|^2_{g_\epsilon} + |\nabla_\epsilon \bar{\nabla}_\epsilon \varphi_\epsilon|^2_{g_\epsilon} - 2\mathrm{Re}\langle \nabla tr_{g_\epsilon}(g_{FS}), \nabla \varphi_\epsilon\rangle_{g_\epsilon} \\ &\quad + R(g_\epsilon)|\nabla \varphi_\epsilon|^2_{g_\epsilon} \\ &\geq |\nabla_\epsilon \nabla_\epsilon \varphi_\epsilon|^2_{g_\epsilon} + |\nabla_\epsilon \bar{\nabla}_\epsilon \varphi_\epsilon|^2_{g_\epsilon} - 2\mathrm{Re}\langle \nabla \mathrm{tr}_{g_\epsilon}(g_{FS}), \nabla \varphi_\epsilon\rangle_{g_\epsilon} \\ &\quad - C|\nabla \varphi_\epsilon|^2_{g_\epsilon}\end{aligned}$$

since R_ϵ is uniformly bounded. Similarly, arguing as in the proof of the inequality (2.15), and using the fact that $\mathrm{tr}_{g_\epsilon}(g_{FS})$ is uniformly bounded on $[0,T]$, we have

$$\Delta_\epsilon \mathrm{tr}_{g_\epsilon}(g_{FS}) \geq \frac{|\nabla \mathrm{tr}_{g_\epsilon}(g_{FS})|^2_{g_\epsilon}}{\mathrm{tr}_{g_\epsilon}(g_{FS})} - C_T,$$

for some uniform $C_T > 0$. Then we apply the maximum principle to

$$H = \frac{|\nabla \varphi_\epsilon|^2_{g_\epsilon}}{\varphi_\epsilon + A + B} + B\ \mathrm{tr}_{g_\epsilon}(g_{FS}),$$

where $A = \sup_{\epsilon\in(0.1)} \sup_{\mathbb{S}^2\times[0,T]} |\varphi_\epsilon|$ and B is some fixed sufficiently large constant. Then H is uniformly bounded on $[0,T]$ because

$$\Delta_\epsilon H \geq \frac{\delta|\nabla\varphi_\epsilon|^4}{2(\varphi_\epsilon + A + B)} + \frac{2-2\delta}{\varphi_\epsilon + A + B} Re\langle \nabla H, \bar{\nabla}\varphi_\epsilon\rangle_{g_\epsilon} - C_\delta,$$

for some fixed sufficiently small $\delta > 0$. The desired bound for $|\nabla\varphi_\epsilon|$ follows. Finally, in view of the equations (2.22) and (2.23), we have

$$v_\epsilon - c_\epsilon(t) = u_\epsilon + a_\epsilon + \frac{1}{2}\chi(\mathbb{S}^2, \beta). \tag{2.28}$$

Thus, the uniform bound for $|\nabla\varphi_\epsilon|$ implies a uniform bound for $|\nabla u_\epsilon|$, and, hence, for $|\nabla u|$, by taking limits over compact subsets of $\mathbb{S}^2 \setminus \beta$ as $\epsilon \to 0$. This completes the proof for the lemma. q.e.d.

3. Perelman monotonicity for the conical Ricci flow

In this section, we introduce the W-functional and establish Perelman monotonicity for the conical Ricci flow itself. This cannot be obtained by taking the limits of the W-functionals for the approximating Ricci flows, because the convergence of the approximating flows is only over compact subsets of $\mathbb{S}^2\setminus\beta$ and the limiting W-functional is different from the W-functional for the conical Ricci flow.

Consider the unnormalized conical Ricci flow on $(\mathbb{S}^2, \beta)$,

$$\frac{\partial g}{\partial t} = -2Ric(g),\ g|_{t=0} = g_0 = e^{u_0} g_\beta, \tag{3.1}$$

for some $u_0 \in C^\infty(\mathbb{S}^2)$. Suppose $g(t)$ solves the conical Ricci flow on $[0,T)$. We define

$$W(g,f,\tau) = \int_{\mathbb{S}^2\setminus\beta} (\tau(R+|\nabla f|^2)+f-2)\frac{e^{-f}}{4\pi\tau}\, g, \qquad \tau>0, \tag{3.2}$$

We define

$$\mu(g,\tau) = \inf_{\int_{\mathbb{S}^2} \frac{e^{-f}}{4\pi\tau} g=1, f\in C^\infty(\mathbb{S}^2)} W(g,f,\tau). \tag{3.3}$$

From its definition, $g(t)$ has bounded curvature and it is equivalent to g_β. Then the log Sobolev constant for $g(t)$ is bounded and, hence, $\mu(g(t),\tau) > -\infty$ for all $\tau>0$.

3.1. A maximum principle. The next lemma is a maximum principle which holds thanks to the existence of a geometric barrier.

Lemma 3.1. *Let $g=g(t)$ be a C^0 metric on $(\mathbb{S}^2\setminus\beta)\times[0,T]$. Let $f\in C^2((\mathbb{S}^2\setminus\beta)\times[0,T])\cap L^\infty(\mathbb{S}^2\times[0,T])$ satisfy the following differential inequality*

$$\partial_t f \geq \Delta_g f + b(x,t) f, \tag{3.4}$$

where Δ_g is the Laplacian with respect to g, and $b(x,t)$ is a bounded function.

(1) *If $f(x,0)\geq 0$ for all $x\in\mathbb{S}^2\setminus\beta$, then $f(x,t)\geq 0$ in $(\mathbb{S}^2\setminus\beta)\times[0,T]$;*

(2) *If $f(x,0)\geq\delta>0$ for all $x\in\mathbb{S}^2\setminus\beta$ and some constant $\delta>0$, then $f(x,t)\geq \delta e^{-t\|b\|_{L^\infty}}$ on $(\mathbb{S}^2,\beta)\times[0,T]$. In particular, $f(x,t)>0$ on $(\mathbb{S}^2\setminus\beta)\times[0,T]$.*

Proof. We prove (1) first. Replacing f by fe^{-At} for some large positive constant A, we can replace $b(x,t)$ by $b(x,t)-A$, and, hence, assume that b is strictly less than $-B$, for any fixed positive constant B. Let $\tilde f = f - \epsilon \log\prod_{j=1}^k |s_j|_h^2$ where s_j is a holomorphic section of $K_{\mathbb{S}^2}^{-1}$ with divisor $[2p_j]$, normalized so that $|s_j|_h^2 \leq \frac{1}{2}$, with h a smooth metric on $K_{\mathbb{S}^2}^{-1}$. The function $\tilde f$ satisfies the differential inequality

$$\partial_t \tilde f \geq \Delta_g \tilde f + b\tilde f + \epsilon(k\, R_h + b\log\prod_{j=1}^k |s_j|_h^2), \tag{3.5}$$

and, thus, since R_h is the contraction of the curvature of h with g and, hence, bounded,

$$\partial_t \tilde f \geq \Delta_g \tilde f + b\tilde f, \tag{3.6}$$

if we choose B to be sufficiently large. Since $\tilde f\to+\infty$ near each of the conical singularities p_j, it must attain its minimum somewhere in $(\mathbb{S}^2\setminus\beta)\times[0,T]$. Assume that this minimum is strictly negative, and let

$t_0 > 0$ be the first time when it is achieved, at some point $x_0 \in \mathbb{S}^2 \setminus \beta$. The above differential inequality would imply that

$$\partial_t \tilde{f}(x_0, t_0) \geq b(x_0, t_0)\tilde{f}(x_0, t_0) > 0. \tag{3.7}$$

But this would imply in turn that $\tilde{f}$ must have attained values strictly lower than $\tilde{f}(x_0, t_0)$, which is a contradiction. Thus, the minimum of $\tilde{f}$ must be non-negative. Letting $\epsilon \to 0$, it follows that f is non-negative, and (1) is proved.

Next, we prove (2). This time, we set $\tilde{f} = f - \epsilon e^{-At}$. Then the function $\tilde{f}$ satisfies the differential inequality

$$\partial_t \tilde{f} \geq \Delta_g \tilde{f} + b\tilde{f} + \epsilon(A + b)e^{-At} \geq \Delta_g \tilde{f} + b\tilde{f}, \tag{3.8}$$

for $A = \|b\|_{L^\infty}$. In view of Part (1), we have $\tilde{f} \geq 0$ for all $t \in [0, T]$, if $\tilde{f} \geq 0$ at $t = 0$. Thus, we choose $\epsilon = \delta$, and obtain the bound $f(x, t) \geq \delta e^{-t\|b\|_{L^\infty}}$, as claimed. The proof of Lemma 3.1 is complete. q.e.d.

3.2. Regularity of the coupled system. Suppose $g(t)$ is a solution of the conical Ricci flow on $[0, T)$. We would like to show that $\mu(g(t), T-t)$ is increasing along the Ricci flow, just as in the case of smooth manifolds. For this we fix T, and consider the following coupled system of equations

$$\frac{\partial g}{\partial t}(t) = -2\, Ric(g(t)), \qquad \frac{\partial F(t)}{\partial \tau} = \Delta_t F - R(t)F, \qquad F|_{t=t_0} = F_0, \tag{3.9}$$

with $\tau = T - t$, $t_0 \in (0, T)$, $F_0 \in C^\infty(\mathbb{S}^2)$ and $F > 0$.

Lemma 3.2. *There exists a unique solution $F(t)$ solving the linear equation (3.9) satisfying*

(1) $F \in L^\infty(\mathbb{S}^2 \times [0, t_0])$ *and* $F \in C^\infty(\mathbb{S}^2 \setminus \beta \times [0, t_0])$.
(2) $\inf_{\mathbb{S}^2 \times [0,t_0]} F > 0$.
(3) *For any* $k \in \mathbf{Z}^+ \cup \{0\}$ *and* $0 < \delta < t_0$,

$$\sup_{\mathbb{S}^2 \times [\delta, t_0]} \big(|\nabla F| + |\Delta F| + (t_0 - t)^2 |\Delta^2 F|\big) < \infty. \tag{3.10}$$

Proof. We recall the approximating twisted Ricci flow in Section 2. Here we consider the unnormalized flow

$$\frac{\partial g_\epsilon}{\partial t} = -2Ric(g_\epsilon) + 2\theta_\epsilon,$$

which only differs from the normalized flow by a reparametrization. The form θ_ϵ is the same as in (2.3). Then we consider the linear equation,

$$-\frac{\partial F_\epsilon}{\partial t} = \Delta_\epsilon F_\epsilon - R_\epsilon F_\epsilon, \;\; F_\epsilon|_{t=t_0} = F_0,$$

where $R_\epsilon = R(g_\epsilon(t)) - \theta_\epsilon$. Obviously, there exists a unique $F_\epsilon \in C^\infty(\mathbb{S}^2 \times [0, t_0])$ solving the above equation. Since R_ϵ is uniformly bounded on

$\mathbb{S}^2 \times [0, t_0]$ for all $\epsilon \in (0,1)$, the maximum principle implies that there exists $C > 0$ such that for all $\epsilon \in (0,1)$,

$$\sup_{\mathbb{S}^2 \times [0,t_0]} |F_\epsilon| \leq C.$$

By the maximum principle, there also exists $c > 0$ such that for all $\epsilon \in (0,1)$,

$$\inf_{\mathbb{S}^2 \times [0,t_0]} F_\epsilon > c.$$

The evolution for $|\nabla F_\epsilon|^2_{g_\epsilon}$ is given by

$$\begin{aligned} -\frac{\partial}{\partial t}|\nabla F_\epsilon|^2_{g_\epsilon} &= \Delta_\epsilon |\nabla F_\epsilon|^2_{g_\epsilon} - |\nabla_\epsilon \nabla_\epsilon F_\epsilon|^2 - |\bar{\nabla}_\epsilon \nabla_\epsilon F_\epsilon|^2_{g_\epsilon} - 2R_\epsilon |\nabla F_\epsilon|^2_{g_\epsilon} \\ &\quad + 2\mathrm{Re}\nabla_\epsilon (R_\epsilon F_\epsilon) \cdot \bar{\nabla}_\epsilon F_\epsilon - \mathrm{tr}_{g_\epsilon}(\theta_\epsilon) |\nabla F_\epsilon|^2_{g_\epsilon} \\ &\leq \Delta_\epsilon |\nabla F_\epsilon|^2_{g_\epsilon} - |\nabla_\epsilon \nabla_\epsilon F_\epsilon|^2 - |\bar{\nabla}_\epsilon \nabla_\epsilon F_\epsilon|^2_{g_\epsilon} + C|\nabla F_\epsilon|^2_{g_\epsilon}, \end{aligned}$$

for some uniform $C > 0$ which only depends on δ because $|\nabla R_\epsilon|^2_{g_\epsilon}$ is uniformly bounded on $[\delta, t_0]$. On the other hand, the evolution for $\dot{F}_\epsilon = \frac{\partial F_\epsilon}{\partial t}$ is given by

$$-\frac{\partial \dot{F}_\epsilon}{\partial t} = \Delta_\epsilon \dot{F}_\epsilon - \dot{R}_\epsilon F_\epsilon - R_\epsilon \dot{F}_\epsilon - 2R_\epsilon \Delta_\epsilon F_\epsilon \leq -C\dot{F}_\epsilon,$$

for some uniform $C > 0$ which only depends on δ because $\Delta_\epsilon R_\epsilon$ is uniformly bounded on $[\delta, t_0]$. We can use then the same arguments as before for the twisted scalar curvature. First we notice that

$$\left(-\frac{\partial}{\partial t} - \Delta_\epsilon\right)(\Delta_\epsilon F_\epsilon)^2 \leq -|\nabla \Delta_\epsilon F_\epsilon|^2_{g_\epsilon} + C,$$

$$\left(-\frac{\partial}{\partial t} - \Delta_\epsilon\right)|\nabla \Delta_\epsilon F_\epsilon|^2_{g_\epsilon} \leq -|\nabla_\epsilon \nabla_\epsilon \Delta_\epsilon F_\epsilon|^2_{g_\epsilon} - |\bar{\nabla}_\epsilon \nabla_\epsilon \Delta_\epsilon F_\epsilon|^2_{g_\epsilon} + C|\nabla \Delta_\epsilon F_\epsilon|^2_{g_\epsilon},$$

$$\left(-\frac{\partial}{\partial t} - \Delta_\epsilon\right)\left(\frac{\partial^2}{\partial t^2}F_\epsilon\right) \leq C\left|\frac{\partial^2}{\partial t^2}F_\epsilon\right| + C,$$

for some uniform constant C on $[\delta, t_0]$. Then applying the maximum principle to

$$H = (t_0 - t)|\nabla \Delta_\epsilon F_\epsilon|^2_{g_\epsilon} + A(\Delta_\epsilon F_\epsilon)^2,$$

and

$$G_+ = (t_0 - t)^2 (\Delta_\epsilon)^2 F_\epsilon + A(t_0 - t)|\nabla \Delta_\epsilon F_\epsilon|^2_{g_\epsilon} + A^2 (\Delta_\epsilon F_\epsilon)^2,$$

and

$$G_= - (t_0 - t)^2 (\Delta_\epsilon)^2 F_\epsilon + A(t - t_0)|\nabla \Delta_\epsilon F_\epsilon|^2_{g_\epsilon} + A^2 (\Delta_\epsilon F_\epsilon)^2,$$

for sufficiently large $A > 0$, we can show that there exists $C > 0$ such that for all $\epsilon \in (0,1)$,

$$\sup_{\mathbb{S}^2 \times [\delta, t_0]} (H + G_+ + G_-) \leq C.$$

The desired solution F can now be obtained as the limit of $F_{1/j}$ as $j \to \infty$ after passing to a convergent subsequence. The lemma follows from the uniform estimates we obtained above because F is smooth away from β by the standard linear parabolic theory. q.e.d.

3.3. Continuity of the W-functional along the coupled system.

Lemma 3.3. *Let (g, F) be the solution of the coupled system (3.9). Then $W(g, f, \tau)$ is continuous on $(0, t_0]$.*

Proof. Let ϕ_j be a sequence of cut-off functions $0 \le \phi_j \le 1$ on $\mathbb{S}^2$ such that each ϕ_j vanishes in a neighborhood of β and for any compact set K in $\mathbb{S}^2 \setminus \beta$, ϕ_j uniformly tends to 1. We can further assume that

$$\sup_{t\in[0,t_0]} \int_{\mathbb{S}^2} |\nabla \phi_j|^2 g \to 0.$$

We first prove the continuity on $(0, t_0)$. We fix any time interval $[\delta, t_0] \subset (0, t_0]$. Then for any $\epsilon > 0$, there exists $j > 0$ such that for all $t \in [\delta, t_0]$,

$$\left| W(g, f, \tau) - \int_{\mathbb{S}^2\setminus\beta} \phi_j(\tau(R + |\nabla f|^2) + f - 2)\frac{e^{-f}}{4\pi\tau} g \right| < \epsilon.$$

This is because R, f and $|\nabla f|$ are all uniformly bounded on $[\delta, t_0]$.

However, all the data are smooth on $\mathbb{S}^2 \setminus \beta \times [0, t_0]$, therefore, $\int_{\mathbb{S}^2\setminus\beta} \phi_j(\tau(R+|\nabla f|^2)+f-2)\frac{e^{-f}}{4\pi\tau} g$ is continuous. Then for any $t' \in [\delta, t_0]$, there exists $\delta_1 > 0$ such that

$$-\epsilon < \left(\int_{\mathbb{S}^2\setminus\beta} \phi_j(\tau(R + |\nabla f|^2) + f - 2)\frac{e^{-f}}{4\pi\tau} g \right)\Bigg|_{t'}^{t} < \epsilon,$$

for all $t \in (t' - \delta_1, t' + \delta_1) \cap [\delta, t_0]$ and so

$$|W(g(t), f(t), t_0 - t) - W(g(t'), f(t'), t_0 - t')| < 2\epsilon.$$

This proves the lemma. q.e.d.

Lemma 3.4. *Let (g, F) be the solution of the coupled system (3.9). Then for any $t \in (0, t_0]$,*

$$\mu(g(t), T - t) \le W(g(t), f(t), T - t).$$

Proof. We fix $t \in (0, t_0]$ Let

$$F_j = (1 + \epsilon_j)\left(\phi_j F(t) + (1 - \phi_j)\right) \in C^\infty(\mathbb{S}^2), \ \int_{\mathbb{S}^2} F_j g = 1.$$

Since $F(t)$ is bounded and positive, F_j will be uniformly bounded above and below from 0. In particular, $\epsilon_j \to 0$. Let $F_j = (4\pi\tau)^{n/2} e^{-f_j}$. Then a straightforward calculation shows that

$$\mu(g(t), T-t) \leq \lim_{j\to\infty} W(g(t), f_j(t), T-t) = W(g(t), f(t), T-t).$$

q.e.d.

3.4. Monotonicity of $\mu(g,\tau)$. We can complete now the proof of the monotonicity of the function $\mu(g,\tau)$ along the Ricci flow exactly as in Perelman's original arguments (c.f. [**55**]). Thus, set

$$v = \left(\tau(2\Delta f - |\nabla f|^2 + R) + f - 2\right)\frac{e^{-f}}{4\pi\tau},\ F = (4\pi\tau)^{-1}e^{-f}. \tag{3.11}$$

We need the following lemmas on integration by parts:

Lemma 3.5. *Let $g = e^u g_\beta$ with $u \in C^\infty(\mathbb{S}^2 \setminus \beta) \cap L^\infty(\mathbb{S}^2)$. If $f \in C^\infty(\mathbb{S}^2 \setminus \beta) \cap L^\infty(\mathbb{S}^2)$, and*

$$\inf_{\mathbb{S}^2\setminus\beta} \Delta f > -\infty, \tag{3.12}$$

then $|\nabla f| \in L^2(\mathbb{S}^2, g)$ and

$$\int_{\mathbb{S}^2} (\Delta f) g = 0. \tag{3.13}$$

Proof. Since $\Delta f \geq -C$ on $\mathbb{S}^2 \setminus \beta$ for some $C > 0$, f is quasi-subharmonic on $\mathbb{S}^2 \setminus \beta$, i.e.,

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial} f + Cg \geq 0,$$

where $g = g_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi(t)$ is the Kähler current associated to $g(t)$ and it has bounded local potentials. On the other hand, since $f \in L^\infty(\mathbb{S}^2)$ and β is a set of finite points (hence, pluripolar), f can be uniquely extended to a bounded quasi-subharmonic function on $\mathbb{S}^2$. The Chern–Levine–Nirenberg inequality immediately implies that Δf is integrable and integration by parts is valid for bounded subharmonic functions

$$\int_{\mathbb{S}^2} |\nabla f|^2 g = -\int_{\mathbb{S}^2} f(\Delta f) g < \infty,\ \int_{\mathbb{S}^2} (\Delta f) g = 0.$$

q.e.d.

Lemma 3.6. *Let F be the unique bounded solution of the coupled system (3.9). Then for any $t \in (0, t_0)$,*

$$\inf_{\mathbb{S}^2\setminus\beta} \Delta\left(\frac{|\nabla F|^2}{F}\right) > -\infty.$$

Proof. A straightforward calculation shows that

$$\begin{aligned}\Delta\left(\frac{|\nabla F|^2}{F}\right) &= \frac{|\nabla\nabla F|^2+|\nabla\bar{\nabla}F|^2+2Re\langle\nabla\Delta F,\bar{\nabla}F\rangle}{F}\\ &-2Re\left(\frac{g^{z\bar{z}}g^{z\bar{z}}(\nabla_z\nabla_z F)\nabla_{\bar{z}}F\nabla_{\bar{z}}F}{F}\right)\\ &-3\frac{|\nabla F|^2\Delta F}{F^2}+2\frac{|\nabla F|^4}{F^3}+\frac{R|\nabla F|^2}{F}.\end{aligned}$$

Since F, F^{-1}, ∇F, ΔF, $\nabla\Delta F$ are bounded, the above formula implies that there exists $C>0$ such that

$$\Delta\left(\frac{|\nabla F|^2}{F}\right) > -C. \qquad \text{q.e.d.}$$

The next lemma follows immediately from the preceding two, because F, $|\nabla F|$ are bounded:

Lemma 3.7. *Let F be the unique bounded solution of the coupled system. Then for any $t\in(0,t_0)$,*

$$\int_{\mathbb{S}^2}\Delta\left(\frac{|\nabla F|^2}{F}\right)g=0.$$

We return to the study of the W-functional.

Lemma 3.8. *Suppose F is the solution of the coupled system (3.9). Then for any $\tau=T-t$ with $t\in(0,t_0)$,*

$$W(g,f,\tau)=\int_{\mathbb{S}^2} vg(t). \tag{3.14}$$

Proof. Since F, $|\nabla F|$ and ΔF are all bounded for any fixed $t\in(0,t_0)$, the lemma follows directly from integration by parts.

Now the same calculation on the smooth part $\mathbb{S}^2\setminus\beta$ as in Perelman's original arguments gives

$$\Box^* v=-2\tau|Ric(g)+Hess(f)-(2\tau)^{-1}g|^2\frac{e^{-f}}{4\pi\tau}, \tag{3.15}$$

where $\Box^*=-\frac{\partial}{\partial t}-\Delta_t+R(t)$. It follows that

$$\frac{\partial W}{\partial t}(g,f,\tau)=\frac{\partial}{\partial t}\int_{\mathbb{S}^2} vg(t)=\int_{\mathbb{S}^2}(-\Box^* v-\Delta_t v)g(t). \tag{3.16}$$

q.e.d.

Lemma 3.9. *Suppose F is the solution of the coupled system (3.9). Then for any $\tau=T-t$ with $t\in(0,t_0)$,*

$$\int_{\mathbb{S}^2}(\Delta_t v)g(t)=0,$$

for all $t\in(0,t_0)$.

Proof. Indeed, v can be rewritten in terms of $F = \frac{e^{-f}}{4\pi\tau}$ as

$$v = -2\tau\Delta F + \tau\frac{|\nabla F|^2}{F} + (\tau R + f - 2)F. \tag{3.17}$$

As a consequence of Lemma 3.2, Lemma 3.6, Lemma 3.7 and Lemma 3.8, we have

$$\int_{\mathbb{S}^2} \Delta_t v\, g(t) = 0. \tag{3.18}$$

This completes the proof of the lemma. q.e.d.

An immediate corollary is

Lemma 3.10. *For $\tau \in (0, t_0)$, we have*

$$\frac{\partial W}{\partial t}(g, f, \tau) = \int_{\mathbb{S}^2} |Ric(g) + Hess(f) - (2\tau)^{-1}g|^2 \frac{e^{-f}}{4\pi\tau} g(t) \geq 0.$$

Furthermore, $W(g, f, \tau)$ is increasing on $[0, t_0]$.

Theorem 3.1. *$\mu(g, T - t)$ is increasing along the conical Ricci flow for $t \in [0, T)$.*

Proof. By Lemma 3.10 and Lemma 3.4, we have for any $0 < t_1 < t_2 < T$,

$$\mu(g(t_2), T - t_2) \geq \mu(g(t_1), T - t_1).$$

We now prove by contradiction that for any $0 \leq t < T$,

$$\mu(g(t), T - t) \geq \mu(g(0), T).$$

Suppose there exists $c > 0$ and $t_0 > 0$ such that for all $t \in (0, t_0)$,

$$\mu(g(t), T - t) < \mu(g(0), T) - c.$$

We solve the coupled system $(g(t), F(t))$ on $(0, t_0]$ with

$$F(t_0) \in C^\infty(\mathbb{S}^2),\ W(g(t_0), f(t_0), T - t_0)) < \mu(g(0), T) - c.$$

Then immediately, there exists $C > 0$ such that for all $t \in (0, t_0]$,

$$\sup_{\mathbb{S}^2 \times (0, t_0]} |f(t)| \leq C,\ \sup_{t \in (0, t_0]} \int_{\mathbb{S}^2} |\nabla f(t)|^2 g(t) \leq C.$$

Let $b(t) \in \mathbb{R}$ be the function of time defined by

$$(4\pi T)^{-1} \int_{\mathbb{S}^2} e^{-f(t) - b(t)} g(0) = 1.$$

Then $\lim_{t \to 0} b(t) = 0$ because $g(t)$ smooth on $\mathbb{S}^2 \setminus \beta \times [0, t_0]$ and $g(t)$ is uniformly equivalent to $g(0)$ for all $t \in [0, t_0]$. Let $\tilde{f}(t) = f(t) + b(t)$. Then

$$\begin{aligned}
&\left| \int_{\mathbb{S}^2} R(0) e^{-\tilde{f}(t)} g(0) - \int_{\mathbb{S}^2} R(t) e^{-f(t)} g(t) \right| \\
\leq\ & \int_{\mathbb{S}^2} |R(0) - R(t)| e^{-\tilde{f}(t)} g(0) + \int_{\mathbb{S}^2} |R(t)| e^{-f(t)} |e^{u(t)} - e^{u_0}| g_\beta \to 0,
\end{aligned}$$

and

$$\begin{aligned} & \left| \int_{\mathbb{S}^2} |\nabla \tilde{f}(t)|^2 e^{-\tilde{f}(t)} g(0) - \int_{\mathbb{S}^2} |\nabla f(t)|^2 e^{-f(t)} g(t) \right| \\ \leq & \int_{\mathbb{S}^2} |\nabla f(t)|^2 e^{-\tilde{f}(t)} |e^{-b(t)+u_0} - e^{u(t)}| g(t) \to 0, \end{aligned}$$

since $R(t)$, $u(t)$ and $f(t)$ are both uniformly bounded and converge smoothly away from β when $t \to 0$. The above estimates imply that

$$\begin{aligned} \mu(g(0),T) & \leq \liminf_{t\to 0} W(g(0), \tilde{f}(t), T) \leq \inf_{(0,t_0]} W(g(t), f(t), T-t) \\ & < \mu(g(0),T) - c, \end{aligned}$$

which is a contradiction. q.e.d.

4. The stable case

In this section, we prove the convergence of the conical Ricci flow on $(\mathbb{S}^2, \beta)$ in the stable case. This case is the easiest, and it does not require the more sophisticated machinery of the other cases, because the convergence is just the smooth convergence of the Kähler potentials, without any need for reparametrizations. The following is a more precise version of Part 1, Theorem 1.3:

Lemma 4.1. *If $2\beta_{max} < \sum_{i=1}^k \beta_i$, then for any regular initial metric $g_0 = e^{u_0} g_\beta \in c_1(\mathbb{S}^2)$, the conical Ricci flow converges to the unique constant curvature metric $g_\infty \in c_1(\mathbb{S}^2)$ on $(\mathbb{S}^2, \beta)$ in the sense that the potentials φ are uniformly bounded in some Schauder space $C^\alpha(\mathbb{S}^2)$ for some $\alpha > 0$, and they converge in C^∞ on any compact subset of $\mathbb{S}^2 \setminus \beta$. Furthermore, $(\mathbb{S}^2, g(t))$ converges in Gromov–Hausdorff topology to the unique constant curvature metric g_∞ on $(\mathbb{S}^2, \beta)$.*

Proof. The proof is an adaptation of the methods in [**33**] and [**34**], exploiting the properness of the functional F_β (see Appendix B for a brief review of F_β) and the fact, established in the previous section, that the Ricci potential v is uniformly bounded along the conical Ricci flow.

First, we note as in [**34**] that if we express the metrics $g(t)$ along the Ricci flow as

$$g(t) = g_{FS} + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi(t), \tag{4.1}$$

then $v = \dot{\varphi}(t) + c(t)$, where $c(t)$ is a constant depending only on the time t. Arguing as in [**34**], we see that the constant $c(t)$ can be made uniformly bounded in t by choosing suitably the arbitrary constant in the definition of $\varphi(0)$. The integrations by parts in the argument are justified because v is bounded and $|\nabla v|$ is bounded. With this choice of normalization, we have then

$$\sup_t \|\dot\varphi\|_{C^0} < \infty. \tag{4.2}$$

This estimate together with the properness of F_β can now be shown to imply the following key estimate for the gradient

$$\int_{\mathbb{S}^2} \frac{\sqrt{-1}}{2\pi} \partial\varphi \wedge \bar\partial\varphi \leq C, \tag{4.3}$$

and for the average of φ along the flow

$$\left| \int_{\mathbb{S}^2} \varphi\, g_{FS} \right| \leq C. \tag{4.4}$$

To see this, we begin by noting, as in [**33**], that the functional F_β and the functional F^0_β defined by

$$F^0_\beta(\varphi) = \frac{\sqrt{-1}}{8\pi} \int_{\mathbb{S}^2} \partial\varphi \wedge \bar\partial\varphi - \frac{1}{2} \int_{\mathbb{S}^2} \varphi\, g_{FS} \tag{4.5}$$

are comparable along the flow,

$$|F_\beta(\varphi) - F^0_\beta(\varphi)| \leq C. \tag{4.6}$$

This is because their difference satisfies

$$\begin{aligned} 2e^{-\|\dot\varphi\|_{C^0}} &\leq \int_{\mathbb{S}^2} e^{-\frac{1}{2}\chi(\mathbb{S}^2,\beta)\varphi} \prod_{i=1}^{k} |\sigma_i|^{-\beta_i}_{g_{FS}} g_{FS} \\ &= \int_{\mathbb{S}^2} e^{-\dot\varphi} \left(g_\beta + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi\right) \leq 2e^{\|\dot\varphi\|_{C^0}}, \end{aligned}$$

which is uniformly bounded since $\|\dot\varphi\|_{C^0}$ is uniformly bounded.

Next, a straightforward calculation shows that F_β is decreasing along the Kähler–Ricci flow, and, hence, using the properness of F_β,

$$F_\beta(\varphi(0)) \geq F_\beta(\varphi) \geq C_1 \int_{\mathbb{S}^2} \frac{\sqrt{-1}}{2\pi} \partial\varphi \wedge \bar\partial\varphi - C_2, \tag{4.7}$$

which shows that

$$\int_{\mathbb{S}^2} \frac{\sqrt{-1}}{2\pi} \partial\varphi \wedge \bar\partial\varphi \leq C_3, \tag{4.8}$$

for all t, which is equation (4.3). It follows also that $|F_\beta(\varphi)|$ is uniformly bounded, and, hence, that $|F^0_\beta(\varphi)|$ is uniformly bounded. Since we already know that (4.3) holds, the estimate (4.4) follows at once.

We can now apply the Trudinger inequality on compact Riemannian manifolds of dimension 2: there exist constants $C > 0$ and $\kappa > 0$ so that for any $p > 0$,

$$\int_{\mathbb{S}^2} e^{p|\varphi|} g_{FS} \leq C \exp\left(\kappa p^2 \int_{\mathbb{S}^2} \frac{\sqrt{-1}}{2\pi} \partial\varphi \wedge \bar\partial\varphi + p \left| \frac{1}{2} \int_{\mathbb{S}^2} \varphi\, g_{FS} \right| \right). \tag{4.9}$$

We deduce that for any p,

$$\sup_t \int_{\mathbb{S}^2} e^{p|\varphi|} g_{FS} < \infty. \tag{4.10}$$

Next we rewrite the equation for the flow as

$$g_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi = g_{FS} e^{\dot{\varphi}} e^{-\frac{1}{2}\chi(\mathbb{S}^2,\beta)\varphi} \prod_{i=1}^{k} |\sigma_i|_{g_{FS}}^{-\beta_i}. \tag{4.11}$$

Since $e^{\frac{1}{2}\chi(\mathbb{S}^2,\beta)|\varphi|}$ is in L^p for any $p < \infty$, we can apply Hölder's inequality and find that the right hand side of the above equation is in L^p for some $p > 1$. By the standard $W^{2,p}$ estimate for elliptic linear PDE, we can conclude that

$$\varphi - \frac{1}{2}\int_{\mathbb{S}^2} \varphi g_{FS} \tag{4.12}$$

is uniformly bounded in $W^{2,p}(\mathbb{S}^2) \subset C^\alpha(\mathbb{S}^2)$ for $\alpha = 2 - \frac{2}{p}$. In view of (4.4), we can conclude that φ is uniformly bounded in $C^\alpha(\mathbb{S}^2)$.

We can then apply the parabolic versions ([**34**]) of the standard estimates of Aubin and Yau for the second derivatives, and of Calabi for the third order derivatives, modified by $\epsilon \log \prod_{i=1}^k |\sigma_i|_{FS}^2$, to obtain the uniform boundedness of the potentials φ in $C^\infty(K)$ for any compact subset $K \subset\subset \mathbb{S}^2 \setminus \beta$.

Applying the arguments in [**36**] (c.f. Lemma 6.1), one can show that $\|\dot{\varphi}(t)\|_{L^\infty} \to 0$ as $t \to \infty$. This implies the convergence of the potentials φ in C^∞ on compact subsets of $\mathbb{S}^2 \setminus \beta$ for a subsequence $\varphi(t_j)$. Since the limit is unique, the convergence along subsequences implies the convergence of the whole flow. The Gromov–Hausdorff convergence follows immediately because $\|\dot{\varphi}\|_{C^\alpha(\mathbb{S}^2, g_{FS})}$ and $\|\varphi\|_{C^\alpha(\mathbb{S}^2, g_{FS})}$ are uniformly bounded for all $t > 0$ for some fixed $\alpha > 0$ and the standard sphere metric g_{FS}. The proof of the lemma is complete. q.e.d.

5. Sequential convergence of the conical Ricci flow

The sequential convergence for the conical Ricci flow in both the semi-stable and unstable cases can be established as follows.

Lemma 5.1. *After passing to a subsequence, for any sequence $t_j \to \infty$, the spaces $(\mathbb{S}^2, g(t_j))$ converge to a compact length metric space (X, d) satisfying the following properties:*

(1) *X is homeomorphic to $\mathbb{S}^2$;*

(2) *the singular set D is a finite set of isolated points;*

(3) *$(X \setminus D, d)$ is a smooth surface equipped with a smooth Riemannian metric with volume 2.*

In particular, the convergence is smooth on $X \setminus D$.

Proof. We first apply Corollary 2.1 for $g(t_j)$ so that we can find smooth metrics g_j for all j such that

$$R(g_j) \geq -K, \ d_{GH}((\mathbb{S}^2, g(t_j)), (\mathbb{S}^2, g_j)) \leq j^{-1},$$

where K is the constant in Corollary 2.1. In particular, $g_j \in c_1(\mathbb{S}^2)$ and the diameter of $(\mathbb{S}^2, g_j)$ is uniformly bounded. We can now directly apply Cheeger–Colding theory and obtain a Gromov–Hausdorff limit (X, d), a compact metric length space, after taking a convergent subsequence of $(\mathbb{S}^2, g_j)$. Without loss of generality and by passing to a convergent subsequence, we can assume that $(\mathbb{S}^2, g(t_j))$ converges to (X, d) in Gromov–Hausdorff topology.

We would like to show that the singular set D of (X, d) is finite and that, in fact, it coincides with the set of limiting points of the conical points along the sequence $(\mathbb{S}^2, g(t_j))$. Let D' be the set of all the limiting points of the conical points. Obviously, D' must be finite. Furthermore, $D' \subset D$ because by (1) the volume comparison, (2) the uniform lower bound of the curvature, and (3) the fact that the angles of all the conical points are less than $2\pi(1 - \beta_k)$, there exists $\delta > 0$ such that, for any conical p_j, the geodesic ball $B_{g(t)}(p_j, r)$ has volume less than $(1 - \delta)B_E(0, r)$ for all $t \geq 0$ and $r \in (0, 1]$, where $B_E(0, r)$ is the Euclidian ball of radius r.

We will show that $D = D'$. Suppose $P \in X \setminus D'$, then there exist a sequence of points $P_j \in (\mathbb{S}^2, g(t_j))$ converging to P in Gromov–Hausdorff sense. Since D' is finite, we can assume that the distance from P to D' is bounded from below by $2r > 0$. Therefore, the distance from P_j to the set of all conical points in $(\mathbb{S}^2, g(t_j))$ is bounded from below by r for sufficiently large j. We then consider the sequence of balls $B_{g(t_j)}(P_j, r)$, which do not contain any conical point. Since the curvature of $g(t_j)$ is uniformly bounded on $B_{g(t_j)}(P_j, r)$ and one has uniformly nonlocal κ-collapsing for all $(\mathbb{S}^2, g(t_j))$, the injectivity radius of any point in $B_{g(t_j)}(P_j, r/A)$ is uniformly bounded below by applying Klingenberg's lemma (see Section 8.4 in [**55**]) for a fixed sufficiently large $A > 0$. Therefore, $B_{g(t_j)}(P_j, r/A)$ converges in $C^{1,\alpha}$ after passing to a convergent subsequence by Gromov's compactness theorem. This implies that P must be a regular point of X and so $D = D'$.

We can now establish the partial C^0-estimates as in [**22**]. Of course, one has to make a slight modification and this is essentially the case studied in [**10**, **52**]. However, in our situation, the singular set is much simpler since there is no singular set of Hausdorff codimension greater than 2 and, thus, each tangent cone of (X, d) is a flat metric cone on $\mathbb{C}$. This implies that each tangent cone is good, i.e., one can construct appropriate cut-off functions. Hence, one can immediately obtain the partial C^0-estimates for the evolving metric $g(t)$. More precisely, there exist $\epsilon > 0$ and $N > 0$ such that for all $t \geq 0$ and any $p \in \mathbb{S}^2$, there exists $\sigma \in H^0(\mathbb{S}^2, K_{\mathbb{S}^2}^{-N})$ satisfying

$$\left(|\sigma|^2(g(t))^N\right)(p) \geq \epsilon, \ \int_{\mathbb{S}^2} |\sigma|^2(g(t))^{N+1} = 2.$$

Here $g(t)$ is the volume form of $g(t)$ and so it is a hermitian metric on $K_{\mathbb{S}^2}^{-1}$. Suppose p_∞ is a singular point in (X, d). Then any tangent cone at p_∞ must be a metric cone $\mathbb{C}_\gamma$ on $\mathbb{C}$ with a cone metric $g_\gamma = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}|z|^{2\gamma}$ for some $\gamma \in (0,1]$ with 0 being p_∞. The trivial line bundle on $\mathbb{C}_\gamma$ is equipped with the hermitian metric $e^{-|z|^{2\gamma}}$. Let $0 \leq F \leq 1$ be the standard smooth cut-off function on $[0,\infty)$ with $F = 1$ on $[0, 1/2]$ and $F = 0$ on $[1,\infty)$. We then let

$$\rho_\epsilon = F\left(\frac{\eta_\epsilon}{\log \epsilon}\right), \ \eta_\epsilon = \max(\log |z|^2, 2\log \epsilon).$$

Then one can show by straightforward calculations that

$$\int_{\mathbb{C}} |\nabla \rho_\epsilon|^2 g_\gamma < C(-\log \epsilon)^{-1},$$

for some fixed constant $C > 0$ uniform in $\epsilon \in (0,1]$. Obviously for any $K \subset\subset \mathbb{C}^*$ and $\delta > 0$, there exists sufficiently small $\epsilon > 0$ such that $\rho_\epsilon = 1$ on K, $\operatorname{supp} \rho_\epsilon \subset\subset \mathbb{C}^*$. Using the construction of ρ_ϵ, one can prove the partial C^0-estimate as in [**22**].

We can now make use of the arguments of [**22**]: the existence of the above sections $\sigma(g(t))$ for any t implies that the surfaces $(\mathbb{S}^2, g(t))$ can be uniformly imbedded into some $\mathbf{CP}^N$, separating points, and that the limit of their images must be a normal variety. Since this normal variety is a projective degeneration of $\mathbb{S}^2$, it must be $\mathbb{S}^2$.

From Shi's local estimates [**40**], namely that uniform bounds for the curvature along the Ricci flow on any bounded domains implies similar bounds for the derivatives of the curvature on smaller domains, the convergence on $X \setminus D$ is smooth and the limiting metric is a smooth metric on $X \setminus D$. The lemma is then proved. q.e.d.

Xiaochun Rong has pointed out to us that one can apply Perelman's stability theorem for Alexandrov spaces instead of the partial C^0-estimate to show that X is homeomorphic to $\mathbb{S}^2$. But what was established above via the partial C^0 estimate is slightly stronger: the image in $\mathbf{P}^N$ is a smooth $\mathbf{P}^1$, excluding the possibility of singular rational curve which is also homeomorphic to $\mathbb{S}^2$, e.g., a rational curve with cuspidal singularities.

To identify the metric on the limiting space, we make use next of Hamilton's entropy functional [**23**], defined for metrics with $\inf_{\mathbb{S}^2} R > 0$ by

$$N = \int_{\mathbb{S}^2} R \log R\, g. \tag{5.1}$$

The assumption $\inf_{\mathbb{S}^2} R_0 > 0$ can be removed by a trick of Chow [**13**], which still works in exactly the same way in the case of the sphere with marked points (c.f. Section 8.2, Chapter 5 [**15**]), after replacing R by $R - s$, where s is defined by $\frac{\partial s}{\partial t} = s(s - \frac{1}{2}\chi(\mathbb{S}^2, \beta))$ with $s(0) < \inf_{\mathbb{S}^2} R_0$. Thus, we can, henceforth, assume that $\inf_{\mathbb{S}^2} R > 0$.

Lemma 5.2. *Let v be the Ricci potential defined as in (2.22). Along the conical Ricci flow, we have*

$$\frac{\partial N}{\partial t} = -\int_{\mathbb{S}^2} \frac{|\nabla R + R\nabla v|^2}{R} g - 2\int_{\mathbb{S}^2} |\nabla\nabla v - \frac{1}{2}(\Delta v) g|^2 g, \tag{5.2}$$

if $\inf_{\mathbb{S}^2} R_0 > 0$.

Proof. From the flow equation for R and the maximum principle Lemma 3.1, it follows that $\inf_{\mathbb{S}^2} R > 0$ for all t if $\inf_{\mathbb{S}^2} R_0 > 0$. Thus, the entropy functional N is well-defined for all time. It suffices now to apply the same arguments as in Hamilton [**23**]. The integration by parts which are required are justified in the lemmas which we state below. q.e.d.

Lemma 5.3. *Let $g = e^u g_\beta$ be a conical metric with u, ∇u, and Δu all bounded, where Δ is the Laplacian with respect to g. Then*

$$\int_{\mathbb{S}^2} |\nabla^2 u|^2 g = \int_{\mathbb{S}^2} (\Delta u)^2 g + \int_{\mathbb{S}^2} R^i_j \nabla^i u \nabla_j u \; g.$$

In particular, $\int_{\mathbb{S}^2} |\nabla^2 u|^2 g < \infty$.

Proof. Let ρ_ϵ be a family of cut-off functions ρ_ϵ, $0 \le \rho_\epsilon \le 1$, with the following properties: for any $\epsilon > 0$ and any $K \subset\subset \mathbb{S}^2 \backslash \beta$, $\rho_\epsilon \in C_0^\infty(\mathbb{S}^2 \backslash \beta)$, $\rho_\epsilon = 1$ on K, and $\int_{\mathbb{S}^2} |\nabla \rho_\epsilon|^2 g_\beta < \epsilon$.

$$\begin{aligned}
\int_{\mathbb{S}^2} \rho_\epsilon^2 \nabla^i \nabla^j u \nabla_i \nabla_j u &= -2\int_{\mathbb{S}^2} \rho_\epsilon \nabla^i \rho_\epsilon \nabla^j u \nabla_i \nabla_j u - \int_{\mathbb{S}^2} \rho_\epsilon^2 \; \nabla^j u \nabla^i \nabla_j \nabla_i u \\
&= -2\int_{\mathbb{S}^2} \rho_\epsilon \nabla^i \rho_\epsilon \nabla^j u \nabla_i \nabla_j u - \int_{\mathbb{S}^2} \rho_\epsilon^2 \nabla^j u \nabla_j \nabla^i \nabla_i u \\
&\quad + \int_{\mathbb{S}^2} \rho_\epsilon^2 R^i_j \nabla^j u \nabla_i u \\
&= \int_{\mathbb{S}^2} \rho_\epsilon^2 (\Delta u)^2 + \int_{\mathbb{S}^2} \rho_\epsilon^2 R^i_j \nabla^j u \nabla_i u \\
&\quad - 2\int_{\mathbb{S}^2} \rho_\epsilon \nabla^i \rho_\epsilon \nabla^j u \nabla_i \nabla_j u + 2\int_{\mathbb{S}^2} \rho_\epsilon \nabla_j \rho_\epsilon \nabla^j u \Delta u.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left| \int_{\mathbb{S}^2} \rho_\epsilon^2 |\nabla^2 u|^2 - \int_{\mathbb{S}^2} \rho_\epsilon^2 (\Delta u)^2 - \int_{\mathbb{S}^2} \rho_\epsilon^2 R^i_j \nabla^j u \nabla_i u \right| \\
&\quad \le 2\left| \int_{\mathbb{S}^2} \rho_\epsilon \nabla^i \rho_\epsilon \nabla^j u \nabla_i \nabla_j u \right| + 2\left| \int_{\mathbb{S}^2} \rho_\epsilon \nabla_j \rho_\epsilon \nabla^j u \Delta u \right|
\end{aligned}$$

$$\leq 2\left(\int_{\mathbb{S}^2}|\nabla u|^2|\nabla\rho_\epsilon|^2\right)^{1/2}\left(\int_{\mathbb{S}^2}\rho_\epsilon^2|\nabla^2 u|^2\right)^{1/2}$$
$$+ 2\left(\int_{\mathbb{S}^2}|\nabla\rho_\epsilon|^2\right)^{1/2}\left(\int_{\mathbb{S}^2}|\nabla u||\nabla u|\right)^{1/2}$$
$$\leq 2\left(\int_{\mathbb{S}^2}|\nabla u|^2|\nabla\rho_\epsilon|^2\right)^{1/2}$$
$$\times\left(\int_{\mathbb{S}^2}\rho_\epsilon^2|\nabla^2 u|^2 - \int_{\mathbb{S}^2}\rho_\epsilon^2(\Delta u)^2 - \int_{\mathbb{S}^2}\rho_\epsilon^2 R^i_j\nabla^j u\nabla_i u\right)^{1/2}$$
$$+ C\left(\int_{\mathbb{S}^2}|\nabla u|^2|\nabla\rho_\epsilon|^2\right)^{1/2} + 2\left(\int_{\mathbb{S}^2}|\nabla\rho_\epsilon|^2\right)^{1/2}\left(\int_{\mathbb{S}^2}|\nabla u||\nabla u|\right)^{1/2},$$

for some uniform constant $C > 0$ since $|\nabla u|$, Δu and R are bounded. Then the proposition is proved by letting $\epsilon \to 0$. q.e.d.

More generally, we have the following proposition, which can be proved in exactly the same way:

Lemma 5.4. *Let g be a conical metric $g = e^u g_\beta$ as in the preceding lemma. Suppose $f \in C^\infty(\mathbb{S}^2 \setminus \beta) \cap L^\infty(\mathbb{S}^2)$, with $|\nabla f|$, Δf bounded, and $\nabla(\Delta f) \in L^2$. Then*

$$\int_{\mathbb{S}^2}|\nabla^2 f|^2 g = \int_{\mathbb{S}^2}(\Delta f)^2 g + \int_{\mathbb{S}^2} R^i_j\nabla^i f\nabla_j f\ g,$$

where Δ is the Laplacian with respect to g. In particular,

$$\int_{\mathbb{S}^2}|\nabla^2 f|^2 g < \infty.$$

Since $R\log R$ is bounded from below, the entropy N is bounded from below, and Lemma 5.2 implies immediately

Lemma 5.5. *If $\inf_{\mathbb{S}^2} R_0 > 0$, then*

$$\lim_{t\to\infty}\int_{s=t}^{t+1}\int_{\mathbb{S}^2}|\nabla\nabla v - \frac{1}{2}(\Delta v)g|^2 g\ ds = 0. \tag{5.3}$$

The following lemma will help establish the limiting soliton equation of the conical Ricci flow. It should be well-known in complex analysis, but we include the proof since we cannot find exact references.

Lemma 5.6. *Let $f(z,\bar z)$ be a smooth real-valued harmonic function on the punctured disc $\mathbb{B}^* \subset \mathbb{C}$. Then*

$$f(z,\bar z) = Re(F(z)) + c\log|z|^2,$$

where $F(z)$ is a holomorphic function on $\mathbb{B}^*$ and $c \in \mathbb{R}$. In particular, if $e^f \in L^1(\mathbb{B})$, then F extends to a holomorphic function on $\mathbb{B}$.

Proof. Let h be the exponential map from the left plane $\{w \in \mathbb{C} \mid Re(w) < 0\}$ to $\mathbb{B}^*$. Then $u(w) = h^* f(w) = f(e^w)$ is also a harmonic function satisfying $u(w + 2\pi\sqrt{-1}) = u(w)$. Since the left plane is simply connected, there exists a complex conjugate $v(w)$ for $u(w)$. In particular, $\nabla v(w + 2\pi\sqrt{-1}) = \nabla v(w)$ for all w and we can define a holomorphic function G satisfying

$$G(w) = u(w) + \sqrt{-1}v(w) - cw,\ G(w + 2\pi i\sqrt{-1}) = G(w),$$

for some $c \in \mathbb{R}$. Hence,

$$f(z, \bar{z}) = Re(G(\log z)) + c\log|z|,$$

and $F(z) = G(\log z)$ is obviously a holomorphic function on $\mathbb{B}^*$ because $G(w + 2\pi\sqrt{-1}) = G(w)$.

If $e^f \in L^1(\mathbb{B})$, then

$$\int_{\mathbb{B}} e^f = \int_{\mathbb{B}} |z|^{2c}|e^{F/2}|^2 < \infty.$$

Hence, $z^m e^{F/2}$ is a holomorphic function on $\mathbb{B}$ for some sufficiently large $m \in \mathbb{Z}^+$ and this implies that F cannot have a singularity at 0. The proof of the lemma is complete. q.e.d.

Lemma 5.7. *Let $t_j \to \infty$. Then by passing to a subsequence, $(\mathbb{S}^2, g(t_j))$ converges in Gromov–Hausdorff topology to one of the following:*

(1) *a conical metric space $(\mathbb{S}^2, \beta_\infty)$ of constant curvature $1-\frac{1}{2}\sum_{i=1}^k \beta_i$,*

(2) *a rotationally symmetric conical shrinking gradient Ricci soliton on $(\mathbb{S}^2, \beta_\infty)$ with $\beta_\infty = \beta_{p_\infty}[p_\infty] + \beta_{q_\infty}[q_\infty]$ with $0 \le \beta_{q_\infty} < \beta_{p_\infty} < 1$.*

Proof. Using Lemma 5.1, we see that the flow converges smoothly on $X \setminus D$. Since $R = \Delta v + \frac{1}{2}\chi(\mathbb{S}^2, \beta)$ and v is uniformly bounded in C^0, it follows from the standard estimates for the Laplace equation that $v(t_j)$ is locally bounded in $C^{2,\alpha}$ for all j near any limiting point in $X \setminus D$, where D is the singular set of X. After passing to a subsequence, we can assume that $(\mathbb{S}^2, g(t_j))$ converges in Gromov–Hausdorff topology to $(\mathbb{S}^2, d)$ equipped with a smooth Riemannian metric g_∞ on $\mathbb{S}^2 \setminus D$, where D is the singular set of $(\mathbb{S}^2, d)$. Then $v(t_j)$ converges smoothly on $\mathbb{S}^2 \setminus D$ to v_∞ satisfying on $X \setminus D$

$$R(g_\infty) = \frac{1}{2}\chi(\mathbb{S}^2, \beta) + \Delta_\infty v_\infty. \tag{5.4}$$

Furthermore, from the curvature bounds and injectivity radius bounds, for any domain $\mathcal{K} \subset\subset \mathbb{S}^2 \setminus D$, we can apply the local version of Hamilton's compactness theorem for the Ricci flow, i.e., there exist domains $\mathcal{K}_j \subset\subset \mathbb{S}^2 \setminus D$ and diffeomorphisms $\Phi_j : \mathcal{K} \to \mathcal{K}_j$, such that the Ricci flow $g(t_j + t)$ for $t \in [0, 1]$ converges to a smooth family of Riemannian metrics $g_\infty(t)$ for $t\in[0,1]$ on $\mathcal{K}$ satisfying the Ricci flow

$$\frac{\partial g_\infty(t)}{\partial t} = -Ric(g_\infty(t)) + \frac{1}{2}\chi(\mathbb{S}^2,\beta)g_\infty(t),\ g_\infty(0)=g_\infty.$$

In particular, $v(t_j+t)$ converges to $v_\infty(t)$ smoothly on $\mathcal{K}\times[0,1]$ after reparametrization.

We claim that on $\mathbb{S}^2\setminus D$, we have

$$\nabla^2_\infty v_\infty = \frac{1}{2}(\Delta v_\infty)g_\infty.$$

Otherwise, there exists a domain $\mathcal{K}\subset\subset \mathbb{S}^2\setminus D$ such that

$$\inf_{\mathcal{K}} |\nabla^2_\infty v_\infty - \frac{1}{2}(\Delta v_\infty)g_\infty|^2_{g_\infty} > 0.$$

Then there exists some $\delta\in(0,1)$ such that

$$\inf_{\mathcal{K}\times[0,\delta]} |\nabla^2_{g_\infty(t)} v_\infty(t) - \frac{1}{2}(\Delta_{g_\infty(t)} v_\infty(t))g_\infty(t)|^2_{g_\infty(t)} > 0,$$

in particular,

$$\int_0^1\int_{\mathcal{K}} |\nabla_{g_\infty(t)}\nabla_{g_\infty(t)} v_\infty(t) - \frac{1}{2}(\Delta_\infty v_\infty(t))g_\infty(t)|_{g_\infty(t)} g_\infty(t) > 0.$$

From the smooth convergence of the Ricci flow $g(t_j+t)$, this then implies that

$$\liminf_{j\to\infty}\int_{t=t_j}^{t_j+1}\int_{\mathbb{S}^2} |\nabla^2 v - \frac{1}{2}(\Delta v)g|^2_g\ g\ dt > 0.$$

Contradiction by Lemma 5.5.

Hence, (g_∞, v_∞) is a shrinking gradient soliton on $X\setminus D$. In particular, $X_\infty = \uparrow\overline{\partial} v_\infty$ is a holomorphic vector field on $X\setminus D$. From the partial C^0-estimate, g_∞ extends to a Kähler current with bounded local potentials. Since v_∞ is bounded in $W^{1,2}(\mathbb{S}^2)$ with respect to a fixed smooth metric on $\mathbb{S}^2$, X_∞ must extend to a holomorphic vector field on $\mathbb{S}^2$.

We consider the following two cases:

1) X_∞ is trivial.

In this case, v_∞ is a constant and the limiting metric is a constant curvature metric on $\mathbb{S}^2\setminus D$. Suppose that $D=\{P_1,...,P_l\}$. We choose holomorphic sections $\sigma_i\in H^0(\mathbb{S}^2,-K_{\mathbb{S}^2})$ such that σ_i vanishes at P_i of order 2. Let $g_{FS}\in c_1(\mathbb{S}^2)$ be the standard smooth sphere metric on $\mathbb{S}^2$. From the partial C^0 estimate and the fact that the limiting metric has constant curvature $1-\frac{1}{2}\sum_{i=1}^k\beta_i$ on $X\setminus D$, the limiting equation must be of the following form

$$\begin{aligned} g_\infty &= g_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi_\infty \\ &= e^{-(1-\frac{1}{2}\sum_{i=1}^k\beta_i)\varphi_\infty}(g_{FS})^{1-\frac{1}{2}\sum_{i=1}^k\beta_i}|\sigma_1|^{-\sum_{i=1}^k\beta_i}e^F, \end{aligned}$$

for some bounded potential φ_∞ and some smooth harmonic function F on $\mathbb{S}^2 \setminus D$. Lemma 5.6 implies that the preceding equation can be rewritten in the following form

$$g_\infty = g_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi_\infty = \frac{e^{-(1-\frac{1}{2}\sum_{i=1}^k \beta_i)\varphi_\infty}}{\prod_{i=1}^l |\sigma_i|_{g_{FS}}^{2\gamma_i}} g_{FS}.$$

This implies that g_∞ is a conical constant curvature metric. The fact that the cone angle of each conical point is less than 2π implies that $\gamma_i \in (0, 1/2)$, and $\sum_{i=1}^l 2\gamma_i = \sum_{i=1}^k \beta_i$, from the Gauss–Bonnet formula.

2) X_∞ is nontrivial.

Each nontrivial holomorphic vector field on $\mathbb{S}^2$ can vanish at two distinct points at most and the imaginary part of X is a Killing vector field induced from an S^1-action. This implies that D can have at most two points fixed by X_∞ and the limiting soliton metric g_∞ must be rotationally symmetric. By the same argument as for the case of $X_\infty = 0$ or directly by solving an ODE equation, we see that the limiting metric must be a conical shrinking gradient Ricci soliton metric on $\mathbb{S}^2$ with 0, 1 or 2 conical points. We denote $(\mathbb{S}^2, \beta_\infty)$ the limiting conical sphere. In particular, $\beta_\infty \neq 0$, by the Gauss–Bonnet formula.

Combining the above, Lemma 5.7 is proved. q.e.d.

6. The semi-stable case: $2\beta_{max} = \sum_{i=1}^k \beta_i$

The goal of this section is to obtain a sequence converging to a conical constant curvature metric space along the conical Ricci flow on a semi-stable pair $(\mathbb{S}^2, \beta)$.

The first step in the proof is to establish the following lower bound for the conical functional $F_\beta(\varphi)$: let $g(t) = g_\beta + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi(t)$ be the solution of Ricci flow (1.5). Then for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that for all $t \in [0, \infty)$ and $\varphi = \varphi(t)$,

$$F_\beta(\varphi) \geq -\epsilon \frac{\sqrt{-1}}{2\pi} \int_{\mathbb{S}^2} \partial\varphi \wedge \overline{\partial}\varphi - C_\epsilon. \tag{6.1}$$

To do this, we introduce for each $\epsilon > 0$ the following approximation $F_{\beta,\epsilon}(\varphi)$ of the functional $F_\beta(\varphi)$ of (B.2),

$$\begin{aligned} F_{\beta,\epsilon}(\varphi) &= \frac{\sqrt{-1}}{8\pi}\int_{\mathbb{S}^2} \partial\varphi \wedge \overline{\partial}\varphi - \frac{1}{2}\int_{\mathbb{S}^2} \varphi g_\beta \\ &\quad - \frac{1}{\frac{1}{2}\chi(\mathbb{S}^2, \beta) - \epsilon} \log \int_{\mathbb{S}^2} e^{-(\frac{1}{2}\chi(\mathbb{S}^2,\beta)-\epsilon)\varphi + h_\beta} g_\beta. \end{aligned}$$

We claim that for any $\epsilon \in (0, \frac{1}{2}\chi(\mathbb{S}^2,\beta))$, the functional $F_{\beta,\epsilon}(\varphi)$ is bounded from below,

$$F_{\beta,\epsilon}(\varphi) \geq -C_\epsilon, \tag{6.2}$$

for all $\varphi \in PSH(\mathbb{S}^2, g_\beta) \cap L^\infty(\mathbb{S}^2)$. This can be shown by the following argument. First by Corollary A.1, the conical alpha invariant is equal to $\frac{1}{2}$ for the semi-stable pair $(\mathbb{S}^2,\beta)$. By the interpolation method in [**47**], the Euler–Lagrange equation for the functional $F_{\beta,\epsilon}(\varphi)$ is a Monge–Ampère equation which can be solved by the method of continuity for these values of ϵ (see [**47**]). By the results of [**3**], the corresponding functional $F_{\beta,\epsilon}(\varphi)$ must be bounded from below by a constant for these values of ϵ.

The lower bound for the functional $F_{\beta,\epsilon}$ implies the following lower bound for the functional F_β,

$$\begin{aligned}
F_\beta(\varphi) &\geq \frac{\sqrt{-1}}{8\pi}\int_{\mathbb{S}^2}\partial\varphi\wedge\bar\partial\varphi - \frac{1}{2}\int_{\mathbb{S}^2}\varphi g_\beta \\
&\quad -\frac{2}{\chi(\mathbb{S}^2,\beta)}(\log\int_{\mathbb{S}^2} e^{-(\frac{1}{2}\chi(\mathbb{S}^2,\beta)-\epsilon)\varphi}g_\beta - \epsilon\mathrm{inf}_{\mathbb{S}^2}\varphi) \\
&\geq \frac{\chi(\mathbb{S}^2,\beta)-2\epsilon}{\chi(\mathbb{S}^2,\beta)}F_{\beta,\epsilon}(\varphi) \\
&\quad +\frac{2\epsilon}{\chi(\mathbb{S}^2,\beta)}(\frac{\sqrt{-1}}{8\pi}\int_{\mathbb{S}^2}\partial\varphi\wedge\bar\partial\varphi - \frac{1}{2}\int_{\mathbb{S}^2}\varphi g_\beta + \mathrm{inf}_{\mathbb{S}^2}\varphi) \\
&\geq \frac{2\epsilon}{\chi(\mathbb{S}^2,\beta)}(\frac{\sqrt{-1}}{8\pi}\int_{\mathbb{S}^2}\partial\varphi\wedge\bar\partial\varphi - \frac{1}{2}\int_{\mathbb{S}^2}\varphi g_\beta + \mathrm{inf}_{\mathbb{S}^2}\varphi) - C_\epsilon. \qquad (6.3)
\end{aligned}$$

By Theorem 1.2 and the remark at the beginning of Section §6, we know that the curvature and the diameter along the conical Ricci flow are uniformly bounded. We let $g_\beta = \hat g + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\psi$ and $g(t) = \hat g + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\psi + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi(t)$, where $\hat g \in [g_\beta]$ is a smooth Kähler metric and ψ is a fixed continuous function in $PSH(\mathbb{S}^2,\hat g)$. Let g_j be the approximating smooth Kähler metrics for $g(t)$ for a fixed t after applying Corollary 2.1. Then there exists $C > 0$ such that $g_j = \hat g + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}(\psi+\varphi_j)$ satisfy

$$Ric(g_j) \geq -Cg_j,\ diam_{g_j}(\mathbb{S}^2) \leq C,\ |\varphi_j - \varphi(t)|_{L^\infty(\mathbb{S}^2)} \to 0,$$

for all $j > 0$. Then the Green's functions G_j for g_j are uniformly bounded below for all j and so

$$\begin{aligned}
-\inf_{\mathbb{S}^2}(\psi+\varphi_j) &\leq -\frac{1}{2}\int_{\mathbb{S}^2}(\psi+\varphi_j)(\hat g + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}(\psi+\varphi_j)) + K \\
&\leq \int_{\mathbb{S}^2}\frac{\sqrt{-1}}{4\pi}\partial\varphi_j\wedge\overline{\partial}\varphi_j + \frac{1}{2}\int_{\mathbb{S}^2}\varphi_j g_\beta + K'',
\end{aligned}$$

for fixed K, $K' > 0$ because $\psi \in PSH(\mathbb{S}^2, \hat{g}) \cap L^\infty(\mathbb{S}^2)$. Hence, by letting $j \to \infty$, we have

$$- \inf_{\mathbb{S}^2} \varphi \le \int_{\mathbb{S}^2} \frac{\sqrt{-1}}{4\pi} \partial\varphi \wedge \overline{\partial}\varphi + \frac{1}{2}\int_{\mathbb{S}^2} \varphi g_\beta + K'', \tag{6.4}$$

since φ_j converges to φ in L^∞, where K'' only depends on K and not on t. Substituting this inequality into (6.3) gives the desired inequality (after a renaming of ϵ).

We remark that in the last step, we can avoid using the lower bound of the Green's function for the evolving metrics. It suffices to approximate the evolving metrics $g(t)$ and φ by smooth metrics and potentials so that the estimate (6.4) holds uniformly for the approximation.

The estimate (6.1) is slightly weak, since the ideal bound for the F_β functional should be a uniform bound from below by a constant. However, (6.1) suffices for our purpose, which is to show that the curvature converges to a constant:

Lemma 6.1. *There exists a sequence $t_j \to \infty$ such that the scalar curvature $R(t_j)$ converges uniformly to $1 - \beta_{max}$ along the conical Ricci flow.*

Proof. First, a straightforward calculation shows that

$$\frac{\partial}{\partial t} F_\beta(\varphi_t) = -\int_{\mathbb{S}^2} v(1 - e^{-v}) g \le 0,$$

where v is the Ricci potential defined in Theorem 1.1. Next, we claim that

$$\inf_{t \in [T, \infty)} \int_{\mathbb{S}^2} v(1 - e^{-v}) g = 0, \tag{6.5}$$

for any $T \ge 0$. We prove the claim by contradiction. If not, then $\inf_{[T', \infty)} \int_{\mathbb{S}^2} v(1 - e^{-v}) g > \delta$ for some fixed $\delta > 0$ and some sufficiently large $T' > 0$. This implies that

$$F_\beta(\varphi_t) \le -\delta t + C_1,$$

for some fixed $C_1 > 0$. On the other hand, by the estimate for v established in Lemma 2.5 and the fact that $\dot{\varphi}(t) = v$, $\varphi(t)$ has at worst linear growth in t modulo a bounded time-dependent constant. Thus,

$$\int_{\mathbb{S}^2} \sqrt{-1}\partial\varphi(t) \wedge \overline{\partial}\varphi(t) = \int_{\mathbb{S}^2} \varphi(t)(g_\beta - g(t)) \le At + C_2,$$

for some fixed $A, C_2 > 0$, where g_β and $g(t)$ are the Kähler forms associated to g_β and $g(t)$. Therefore,

$$\begin{aligned} F_\beta(\varphi_t) + \epsilon \int_{\mathbb{S}^2} \sqrt{-1}\partial\varphi_t \wedge \overline{\partial}\varphi_t &\le -\delta t + \epsilon \int_{\mathbb{S}^2} \sqrt{-1}\partial\varphi_t \wedge \overline{\partial}\varphi_t + C_1 \\ &\le -(\delta - A\epsilon)t + C_3 \to -\infty \end{aligned}$$

as $t \to \infty$ if we choose $\epsilon > 0$ sufficiently small. This would contradict the estimate (6.1).

Since $|\nabla v|$ is uniformly bounded for $t \in [0, \infty)$, and we have κ-nonlocal collapsing along the flow, the equality (6.5) implies that there exists a sequence $t_j \to \infty$ with $\lim_{j\to\infty} \sup_{\mathbb{S}^2} |v(t_j)| = 0$. Furthermore, we have

$$\lim_{j\to\infty} \sup_{(z,t)\in\mathbb{S}^2\times[t_j,t_j+1]} |v(z,t)| = 0,$$

since $\frac{\partial v}{\partial t} = \Delta v + v$ and v is uniformly bounded.

Finally, one can apply the smoothing techniques in [**36**] in the conical setting and show that there exist a sequence $t'_j \in [t_j, t_j+1] \to \infty$ such that

$$\lim_{j\to\infty} \sup_{\mathbb{S}^2} (|\nabla v(t'_j)| + |R(t'_j) - \frac{1}{2}\chi(\mathbb{S}^2, \beta)|) = 0.$$

This requires only the application of the maximum principle over $[t_j, t_j+1]$ combined with a family of barrier functions as in Lemma 3.1, which is justified by the regularity of v. For example, for the bound on $|\nabla v|$, we would apply the maximum principle to $e^{-2t}(v^2 + t|\nabla v|^2) - \epsilon \log |\sigma|_g^2$, and for Δv, to $e^{-(t-1)}(|\nabla v|^2 + (t-1)\Delta v) - \epsilon|\sigma|_g^2$, and let $\epsilon \to 0$. This completes the proof of Lemma 6.1. q.e.d.

By Lemma 5.7, after passing to a subsequence, we can assume that $(\mathbb{S}^2, g(t_j))$ in Lemma 6.1 converges to a limiting space (X, d).

Lemma 6.2. *The limiting space* (X, d) *has the following properties:*

(1) *The singular set* D *consists of two points* p_∞, q_∞ *with weights* β_{max}*,*

(2) *the conical Ricci flow converges in Gromov–Hausdorff topology to* $((\mathbb{S}^2, \beta_\infty), g_\infty)$ *with* $\beta_\infty = \beta_{max}[p_\infty] + \beta_{max}[q_\infty]$*, and* $g_\infty \in c_1(\mathbb{S}^2)$ *is the unique conical metric with constant curvature* $1 - \beta_{max}$*,*

(3) *the convergence is in* C^∞ *on* $\mathbb{S}^2 \setminus \{p_\infty, q_\infty\}$*.*

Proof. By Lemma 5.7 and Lemma 6.1, $(\mathbb{S}^2, g(t_j))$ converges to a conical metric of constant curvature $1 - \beta_{max}$. The angle of p_k is $2(1-\beta_{max})\pi$, which is the smallest angle. The point p_k will converge to a limiting point p_∞ in the limiting space (X, d) along any convergent subsequence in the Gromov–Hausdorff topology. Applying the convergence results from the theory of Cheeger–Colding and volume comparison, we obtain

$$\frac{Vol(B_d(p_\infty, r))}{Vol(B^{1-\beta_{max}}(r))} = \lim_{j\to\infty} \frac{Vol(B_{g(t_j)}(p_k, r))}{Vol(B^{1-\beta_{max}}(r))}$$

$$\leq \lim_{j\to\infty}\lim_{r\to 0}\left(\frac{Vol(B_{g(t_j)}(p_k,r))}{Vol(B^{\inf_{\mathbb{S}^2} R(g(t_j))}(r))}\right)\left(\frac{Vol(B^{\inf_{\mathbb{S}^2} R(g(t_j))}(r))}{Vol(B^{1-\beta_{max}}(r))}\right)$$
$$= (1-\beta_{max}),$$

because the curvature tends to $1-\beta_{max}$ uniformly as $j\to\infty$, where $B^H(r)$ is the metric ball of radius r on $\mathbb{S}^2$ of constant curvature H for $H>0$. Therefore, p_∞ must be a singular point on X by volume comparison. In particular, p_∞ is a conical point with cone angle at most $2(1-\beta_{max})\pi$.

Applying Troyanov's stability condition [**56**] for the existence of constant curvature combined with $R=1-\beta_{max}$ on $\mathbb{S}^2$, we can conclude that there can be only another conical point q_∞ with the same cone angle as p_∞, otherwise, the curvature must be strictly less than $1-\beta_{max}$.

Since the constant curvature metric in $c_1(\mathbb{S}^2)$ with two conical points of cone angle $2\pi(1-\beta_k)$ is unique, the flow must converge to the same limiting space for any convergent subsequence. This completes the proof of Lemma 6.2. q.e.d.

By the uniqueness of conical constant curvature metric on $(\mathbb{S}^2,\beta_\infty)$, the limiting metric g_∞ must be rotationally symmetric. We now want to show that the conical points $p_1,p_2,...,p_{k-1}$ will merge into one point in the limiting space. This would complete the proof of Part 2 of Theorem 1.3.

Lemma 6.3. *Let $\mathcal{A}=\{p_1,p_2,...,p_{k-1}\}$. Then the diameter of $\mathcal{A}$ with respect to $g(t_j)$ converges to 0 as $t_j\to\infty$. Furthermore, $\mathcal{A}$ converges to a conical point in the limiting space.*

Proof. First we pick p_k and let p_∞ be the limiting point of p_k along the flow. It suffices to show that $\liminf_{j\to\infty} dist_{g(t_j)}(p_k,\mathcal{A})>0$ by Lemma 6.2. This is because there is only one conical point q_∞ other than p_∞, and the limit of each p_i must be a singular point by the volume comparison, hence, $\mathcal{A}$ must converge to q_∞.

We will prove the proposition by contradiction. Suppose that a subset $\mathcal{A}'$ of $\mathcal{A}$ converges to p_∞ instead of q_∞, say $q_1,...,q_l$ with weights $\beta_{q_1},...,\beta_{q_l}$. We know that the limiting space is a football of constant curvature metric. Let $2L$ be the distance from p_∞ and q_∞ on (X,d). Then $B_{g(t_j)}(p_k,L)$ converges to the half football $B_{g_\infty}(p_\infty,L)$ in Gromov–Hausdorff topology, furthermore, the convergence is smooth on $B_{g_\infty}(p_\infty,L)\setminus B_{g_\infty}(p_\infty,L/2)$.

Let $K=\mathbb{S}^2\setminus\{\overline{B_{g_\infty}(p_\infty,L/4)}\cup\overline{B_{g_\infty}(q_\infty,L/4)}\}$. Then for any $\epsilon>0$, there exists $T>0$ such that for $t_j>T$, there exists a diffeomorphism

$$\sigma_{t_j}: K\to K(t_j)\subset\mathbb{S}^2\setminus\beta,$$

such that

$$\|\sigma_{t_j}^* g(t_j)-g_\infty\|_{C^2(K,g_\infty)}<\epsilon.$$

Let η be a smooth cut-off function on $(\mathbb{S}^2 \setminus \beta_\infty, g_\infty)$ such that $0 \le \eta(z) \le 1$ on $\mathbb{S}^2$ with $\eta = 0$ on $\mathbb{S}^2 \setminus B_{g_\infty}(q_\infty, L/2)$ and $\eta = 1$ on $\mathbb{S}^2 \setminus B_{g_\infty}(p_\infty, L/2)$. Let $\tilde\sigma_{t_j}$ be a smooth diffeomorphism of $\mathbb{S}^2$ which is a smooth extension of σ_{t_j} to $\mathbb{S}^2 \setminus \overline{B_{g_\infty}(q_\infty, L/4)}$. We then define a conical metric $\tilde g(t)$ by

$$\tilde g(t_j) = \eta g_\infty + (1-\eta)\ \tilde\sigma^*_{t_j} g(t_j). \tag{6.6}$$

Obviously, $\tilde g(t_j) = g_\infty$ on $B_{g_\infty}(q_\infty, d/2)$ and $\tilde g(t_j) = \tilde\sigma^*_{t_j} g(t_j)$ on $B_{g_\infty}(p_\infty, d/2)$.

Since $g(t_j)$ converges to g_∞ on K, $\tilde g(t_j)$ converges to g_∞ on K smoothly as $t_j \to \infty$. This implies that $R(\tilde g(t_j))$ converge to $1 - \frac{1}{2}\sum_{i=1}^k \beta_i$ in $L^\infty(\mathbb{S}^2)$, and the total volume of $\tilde g(t_j)$ converges to 2, i.e., $\lim_{t\to\infty} \int_{\mathbb{S}^2} \tilde g(t_j) = 2$ because $B_{g(t_j)}(p_k, d/2)$ converges to $B_{g_\infty}(p_\infty, d/2)$ in Gromov–Hausdorff topology as well as in measure. Therefore,

$$\lim_{t\to\infty} \int_{\mathbb{S}^2} R(\tilde g(t_j))\tilde g(t) = 2 - \sum_{i=1}^k \beta_i. \tag{6.7}$$

On the other hand, by Gauss–Bonnet formula,

$$\int_{\mathbb{S}^2} R(\tilde g(t_j))\tilde g(t) = 2 - \beta_k - \sum_{i=1}^l \beta_{q_i} > 2 - \sum_{i=1}^k \beta_i. \tag{6.8}$$

Equations (6.7) and (6.8) lead to contradiction by choosing t sufficiently large. q.e.d.

This lemma illustrates why one cannot apply a local version of Hamilton's compactness theorem to the local C^∞-convergence as in Proposition 5.3 in [**29**]. This is because from the partial C^0-estimates, the gauge transformations come from the $\mathbb{C}^*$-action, and all points but p_k will converge to a single limiting conical point. Thus, the injectivity radius will always tend to 0 for generic points on $\mathbb{S}^2 \setminus \beta$.

7. The unstable case: $2\beta_{max} > \sum_{i=1}^k \beta_i$

In this section, we will show that in the unstable case, the conical Ricci flow must converge to the unique shrinking gradient Ricci soliton. Such a soliton metric is rotationally symmetric and does not depend on the initial conical metric.

7.1. Uniform convergence to a rotationally symmetric soliton. First we show that if $(\mathbb{S}^2, \beta)$ is unstable, then any sequential limit cannot be a conical constant curvature metric space.

Lemma 7.1. *Suppose $(\mathbb{S}^2, \beta_\infty)$ is a sequential limit of the conical Ricci flow (1.2) on an unstable pair $(\mathbb{S}^2, \beta)$. Then $\beta_\infty = \beta_{p_\infty}[p_\infty] + \beta_{q_\infty}[q_\infty]$ with $0 \le \beta_{q_\infty} < \beta_{p_\infty} < 1$. Therefore, the limiting soliton metric cannot be a constant curvature metric.*

Proof. Suppose $(\mathbb{S}^2, g(t_j))$ converges to a limiting conical shrinking gradient Ricci soliton $((\mathbb{S}^2, \beta_\infty), g_\infty)$. By the volume comparison, p_k must converge to a limiting conical point, say p_∞, such that the cone angle of g_∞ at p_∞ must be at most $2\pi(1-\beta_k)$ by volume comparison as the curvature is uniformly bounded. On the other hand, by the boundedness of R and smooth convergence of R on $\mathbb{S}^2 \setminus D$, we have

$$\int_{\mathbb{S}^2} R(g_\infty) g_\infty = 2 - \sum_{i=1}^{k} \beta_i.$$

Suppose $\beta_\infty = \beta_{p_\infty}[p_\infty] + \sum_{i=1}^{l} \beta_{q_i}[q_i]$ with distinct points $p_\infty, q_1, ..., q_l \in D$. Then $\beta_{p_\infty} \geq \beta_k$ and by the Gauss–Bonnet formula, we have

$$\beta_{p_\infty} + \sum_{i=1}^{l} \beta_{q_i} = \sum_{i=1}^{k} \beta_k,$$

and so

$$\sum_{i=1}^{l} \beta_{q_i} \leq \sum_{i=1}^{k-1} \beta_i < \beta_k \leq \beta_{p_\infty}.$$

This contradicts Troyanov's stability condition and so $(\mathbb{S}^2, \beta_\infty)$ does not admit a conical constant curvature metric. q.e.d.

Lemma 7.2. *Assume that* $(\mathbb{S}^2, \beta_\infty, g_{sol,\beta_\infty})$ *is the limit of a sequence* $(\mathbb{S}^2, \beta, g(t_m))$ *along the conical Ricci flow as* $m \to \infty$. *Then* $(\mathbb{S}^2, \beta_\infty, g_{sol,\beta_\infty})$ *is a rotationally symmetric conical gradient shrinking Ricci soliton. Furthermore,*

$$p_i \to p_\infty,\ i \in I,\ p_j \to q_\infty,\ j \in J,$$

and

$$\beta_{p_\infty} = \sum_{i\in I} \beta_i,\ \beta_{q_\infty} = \sum_{j \in J} \beta_j,$$

for some $I \subset \{1, 2, ..., k\}$ *and* $J = \{1, 2, ..., k\} \setminus I$.

Proof. By Lemma 7.1, the limiting metric cannot have constant curvature and there are at most two distinct conical points. Then by the classification of nontrivial shrinking solitons with conical singularities on $\mathbb{S}^2$, the limiting soliton metric must be always rotationally symmetric.

Without loss of generality, we can assume that for some $I \subset \{1, 2, ..., k\}$ and $J = \{1, 2, ..., k\} \setminus I$,

$$p_i \to p_\infty,\ p_j \to q_\infty,$$

for $i \in I$ and $j \in J$. It suffices to show that the weights at p_∞ and q_∞ satisfy

$$\beta_{p_\infty} = \sum_{i\in I} \beta_i,\ \beta_{q_\infty} = \sum_{j \in J} \beta_j.$$

This can be shown by the same arguments in the proof of Lemma 6.3

by gluing and by the Gauss–Bonnet formula, because the curvature is uniformly bounded and converges uniformly away from p_∞ and q_∞. q.e.d.

Lemma 7.3. *Let $\mathcal{S}$ be the set of all conical shrinking gradient Ricci solitons $(\mathbb{S}^2, \beta_\infty, g_{sol,\beta_\infty})$ which arise as sequential limits for the Ricci flow. Then $\mathcal{S}$ is a finite set.*

Proof. For fixed β_∞, the conical shrinking soliton $(\mathbb{S}^2, \beta_\infty, g_{sol,\beta_\infty})$ is unique. The corollary immediately follows from Lemma 7.2 and the fact that there are only finitely many combinations of $I \sqcup J = \{1, 2, ..., k\}$. q.e.d.

Lemma 7.4. *Let $g(t)$ be the solution of the Ricci flow. Then $(\mathbb{S}^2, \beta, g(t))$ converges uniformly in Gromov–Hausdorff topology to a shrinking gradient conical Ricci soliton $(\mathbb{S}^2, \beta_\infty, g_{sol,\beta_\infty})$ for $t \to \infty$.*

Proof. The proof is by contradiction. Suppose not. Then there exist two sequences of solutions for the conical Ricci flow $g(t_l)$ and $g(t_l')$ converging to two distinct conical shrinking solitons $(\mathbb{S}^2, \beta', g'_{sol})$ and $(\mathbb{S}^2, \beta'', g''_{sol})$ in Gromov–Hausdorff topology, as $l \to \infty$. In particular, there exists $D > 0$ such that

$$d_{GH}((\mathbb{S}^2, g'), (\mathbb{S}^2, g'')) = 2D,$$

and $L > 0$ such that for all $l > L$,

$$d_{GH}((\mathbb{S}^2, g(t_l)), (\mathbb{S}^2, g(t_l'))) > D.$$

Without loss of generality, we can assume $t_l' > t_l$ for each l. Then we consider the function

$$f_l(s) = d_{GH}((\mathbb{S}^2, g((1-s)t_l + s\ t_l')), (\mathbb{S}^2, g(t_l))),\ s \in [0,1].$$

Since the conformal factor of $g(t)$ with respect to $g(0)$ is continuous, $f(s)$ is a continuous function with $f_l(0) = 0$ and $f_l(1) > D$.

Therefore, for any $d \in [0, D]$, there exist a sequence $g(t_{l,d})$ such that

$$d_{GH}((\mathbb{S}^2, g(t_{l,d})), (\mathbb{S}^2, g(t_l))) = d.$$

After passing to a subsequence, $(\mathbb{S}^2, g(t_{l,d}))$ converges to a conical shrinking soliton $(\mathbb{S}^2, \beta_d, g_{sol,d})$ satisfying

$$d_{GH}((\mathbb{S}^2, g_{sol,d}), (\mathbb{S}^2, g'_{sol})) = d.$$

This implies that there are infinitely many distinct limiting conical shrinking solitons from the conical Ricci flow. This contradicts Lemma 7.3. q.e.d.

We remark that Lemma 7.4 does not prove the limiting soliton is independent of the choice of initial metrics. We will prove such a strong uniqueness result in §7.3.

7.2. Rotationally symmetric solitons on $\mathbb{S}^2$. In the previous section, we prove that in the unstable case, the conical Ricci flow must converge to a unique shrinking Ricci soliton metric g_{sol} on $(\mathbb{S}^2, \beta_\infty)$ with

$$\beta_\infty = \beta_{p_\infty}[p_\infty] + \beta_{q_\infty}[q_\infty],\ \beta_{p_\infty} + \beta_{q_\infty} = \sum_{j=1}^{k} \beta_j,\ 0 \le \beta_{q_\infty} < \beta_{p_\infty}.$$

Therefore, g_{sol} must be a rotationally symmetric soliton metric possibly depending on the choice of the initial metric. Toric Kähler–Ricci soliton metrics with conical singularities are completely classified in [**19**] on compact toric manifolds, generalizing the work of [**59**].

We consider $\mathbb{C}^*$ in S^2 with holomorphic coordinates $z = e^{\frac{\rho}{2}+\sqrt{-1}\theta}$, $\rho \in (-\infty, \infty)$, $\theta \in [0, 2\pi)$. Then a rotationally conical soliton metric g_{sol} on S^2 with conical points β_∞ can be identified on $\mathbb{C}^*$ as

$$g_{sol} = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi,$$

for some smooth function $\varphi(z) = u(\rho)$. The function $u = u(\rho)$ defined for $\rho \in (-\infty, \infty)$ satisfies the following properties from the results in [**19**].

1) $u' > 0$, $u'' > 0$.
2) $\lim_{\rho\to-\infty} u'(\rho) = -1$, $\lim_{\rho\to\infty} u'(\rho) = 1$.
3) $u_0(e^{(1-\beta_{q_\infty})\rho}) = u(\rho) + \rho$ and $u_\infty(e^{-(1-\beta_{p_\infty})\rho}) = u(\rho) - \rho \in$ are both smooth functions on $[0, \infty)$.
4) The soliton equation can be expressed as
$$u'' = e^{-\overline{R}u - cu' + \overline{R}\tau\rho},$$
where
$$\overline{R} = 1 - \frac{\beta_{p_\infty} + \beta_{q_\infty}}{2} = 1 - \frac{\sum_{j=1}^k \beta_j}{2}$$
is the average of the curvature and the constants c and τ are defined by
$$\tau = \frac{\beta_{p_\infty} - \beta_{q_\infty}}{2 - \beta_{p_\infty} - \beta_{q_\infty}} = \frac{\int_{-1}^1 x e^{cx} dx}{\int_{-1}^1 e^{cx} dx}.$$

The following lemma immediately follows from the definition of τ and c, and the fact that $\beta_{p_\infty} > \beta_{q_\infty} \ge 0$.

Lemma 7.5. *$\tau \in (0, 1)$ and $c > 0$.*

The following lemma shows that the curvature near the conical point with greater weight β_{p_∞} is smaller than $\overline{R}$ and the curvature near the conical point with smaller weight β_{q_∞} is greater than $\overline{R}$.

Lemma 7.6. *Let g_{sol} be the limiting soliton metric on $(\mathbb{S}^2, \beta_\infty)$. Then there exist $\epsilon > 0$ and $r > 0$ such that*

$$R(g_{sol}) < \overline{R} - 4\epsilon,\ \text{on } B_{g_{sol}}(p_\infty, 2r),$$

and

$$R(g_{sol}) > \overline{R} + 4\epsilon, \text{ on } B_{g_{sol}}(q_\infty, 2r).$$

Proof. By the calculations in [**5**] (c.f. [**48**]), the scalar curvature R of g_{sol} is given by

$$R = \frac{v''}{u''},\ v = -\log u''.$$

Then straightforward calculations show that

$$R - \overline{R} = -c\overline{R}(u' - \tau) - c^2 u''$$

by making use of the soliton equation. From the soliton equation, $\lim_{|\rho|\to\infty} u''(\rho) = 0$ because $\tau \in (-1, 1)$. Therefore,

$$\lim_{\rho\to\infty}(R(\rho) - \overline{R}) = -c\overline{R}(1-\tau) < 0,\ \lim_{\rho\to-\infty}(R(\rho) - \overline{R}) = c\overline{R}(1+\tau) > 0,$$

and the lemma immediately follows. q.e.d.

The rest of the section is devoted to calculate the μ-functional for all shrinking gradient Ricci solitons on $\mathbb{S}^2$ with conical singularities. We define the following normalized W-functional

$$W(g,f) = \int_{\mathbb{S}^2}\left(\frac{1}{2-\sum_{i=1}^k \beta_i}(R + |\nabla f|^2) + f\right)e^{-f}g,\ \int_{\mathbb{S}^2} e^{-f}g = 2,$$

with $g \in c_1(\mathbb{S}^2)$. Note that the singular time for the unnormalized conical Ricci flow on $(\mathbb{S}^2, \beta)$ is

$$\tau = (2 - \sum_{i=1}^k \beta_i)^{-1},$$

and so

$$W(g,f) = 2W\left(g, f + \log\frac{2-\sum_{i=1}^k\beta_i}{2\pi}, (2-\sum_{i=1}^k \beta_i)^{-1}\right)$$
$$+ 4 - 2\log\frac{2-\sum_{i=1}^k\beta_i}{2\pi},$$

with

$$(4\pi\tau)^{-1}\int_{\mathbb{S}^2} e^{-\left(f+\log\frac{2-\sum_{i=1}^k\beta_i}{2\pi}\right)}g = 1.$$

As usual, we define

$$\mu(g) = \inf_f W(g,f),\ \int_{\mathbb{S}^2} e^{-f}g = 2,$$

where the infimum is taken over functions f satisfying the conditions in the definition of μ (c.f (2.1)).

Suppose $((\mathbb{S}^2,\beta),g_{sol})$ is a gradient shrinking soliton with $g_{sol}\in c_1(\mathbb{S}^2)$. Then g_{sol} is rotationally symmetric and satisfies

$$R(g_{sol})=(1-\frac{1}{2}(\beta_p+\beta_q))+\Delta_{g_{sol}}\theta_{sol},\ \nabla^2_{g_{sol}}\theta_{sol}=\frac{1}{2}(\Delta_{g_{sol}}\theta_{sol})g_{sol},$$

and

$$\int_{\mathbb{S}^2}e^{\theta_{sol}}g_{sol}=2,$$

for a unique θ_{sol}.

Lemma 7.7. *Suppose $((\mathbb{S}^2,\beta),g_{sol})$ is a gradient shrinking soliton. Then*

$$W(g_{sol},-\theta_{sol})=1-\int_{\mathbb{S}^2}\theta_{sol}e^{\theta_{sol}}g_{sol}. \tag{7.1}$$

Proof. Following the same argument as in Hamilton [**23**], one can show that

$$R-\Delta\theta_{sol}=(1-\frac{1}{2}\sum_{i=1}^k\beta_i),\ R+|\nabla\theta_{sol}|^2=-(1-\frac{1}{2}\sum_{i=1}^k\beta_i)(\theta_{sol}+C).$$

Integrating by parts, we obtain

$$C=1+\frac{1}{2}\int_{\mathbb{S}^2}\theta_{sol}e^{\theta_{sol}}g_{sol},$$

and the lemma immediately follows. q.e.d.

We now compare $W(g_{sol},-\theta_{sol})$ for different markings, and establish a monotonicity formula for different conical shrinking soliton metrics.

Lemma 7.8. *Let $(\mathbb{S}^2,\beta)$ and $(\mathbb{S}^2,\beta')$ be two conical spheres with $\beta=\beta_p[p]+\beta_q[q]$, $\beta'=\beta'_{p'}[p']+\beta'_{q'}[q']$, $\beta_p,\beta_q,\beta_{p'},\beta_{q'}\in[0,1)$. Let $g_{sol,\beta},g_{sol,\beta'}\in c_1(\mathbb{S}^2)$ be the shrinking gradient soliton metrics on $(\mathbb{S}^2,\beta)$ and $(\mathbb{S}^2,\beta')$. If $\beta_p+\beta_q=\beta'_{p'}+\beta'_{q'}$ and $|\beta_p-\beta_q|<|\beta'_{p'}-\beta'_{q'}|$, then*

$$W(g_{sol,\beta},-\theta_{sol,\beta})>W(g_{sol,\beta'},-\theta_{sol,\beta'}).$$

Proof. It suffices to calculate the integral $\int_{\mathbb{S}^2}\theta_{sol}e^{\theta_{sol}}g_{sol}$. We can apply the calculations in [**19**], since the soliton metrics are toric. The polytope associated to $(\mathbb{S}^2,c_1(\mathbb{S}^2))$ is $P=[-1,1]$ with defining functions $l_0(x)=1-x\geq 0$ and $l_\infty(x)=1+x\geq 0$. The soliton equation for g_{sol} is given by

$$Ric(g_{sol})=(1-\frac{1}{2}(\beta_0+\beta_\infty))g_{sol}+L_\xi g_{sol}+\beta_0[D_0]+\beta_\infty[D_\infty],$$

where D_0, D_∞ are the two points fixed by the torus action, ξ is a holomorphic vector field. We let $\eta=|\beta_0-\beta_\infty|$. By Theorem 1.1 in

[19], one can solve the above equation if and only if

$$\begin{aligned}&\beta_0[D_0]+\beta_\infty[D_\infty]\\&\quad=(1-(1-\frac{1}{2}(\beta_0+\beta_\infty)l_0(\tau)))[D_0]\\&\qquad+(1-(1-\frac{1}{2}(\beta_0+\beta_\infty)l_\infty(\tau)))[D_\infty].\end{aligned}$$

Immediately one has

$$\tau=\frac{\beta_\infty-\beta_0}{2-\beta_0-\beta_\infty}.$$

Obviously, $|\tau|$ is an increasing function in η since $\beta_0+\beta_\infty$ is fixed. Then one can uniquely solve c from the following equation

$$\tau=\frac{\int_{-1}^1 xe^{cx}dx}{\int_{-1}^1 e^{cx}dx}.$$

In particular,

$$\tau'(c)=\left(\int_{-1}^1 e^{cx}dx\right)^{-2}\left(\int_{-1}^1 x^2e^{cx}dx\int_{-1}^1 e^{cx}dx-\left(\int_{-1}^1 xe^{cx}dx\right)^2\right)>0.$$

Therefore, $|c|$ is an increasing function in $|\tau|$. From [59, 19],

$$\theta_{sol}=\log\frac{2e^{cx}}{\int_{-1}^1 e^{cx}dx},$$

and

$$F(c)=\int_{\mathbb{S}^2}\theta_{sol}e^{\theta_{sol}}g_{sol}=A^{-1}\int_{-1}^1(cx-\log A)e^{cx}dx,\ A=\frac{1}{2}\int_{-1}^1 e^{cx}dx.$$

Straightforward calculations show that

$$F'(c)=\frac{c}{2A^2}\left(\int_{-1}^1 x^2e^{cx}dx\int_{-1}^1 e^{cx}dx-\left(\int_{-1}^1 xe^{cx}dx\right)^2\right).$$

Therefore, $F'(c)>0$ if $c>0$ and $F'(c)<0$ if $c<0$, and it immediately implies that $\int_{\mathbb{S}^2}\theta_{sol}e^{\theta_{sol}}g_{sol}$ is strictly increasing in terms of η. This completes the proof of the lemma. q.e.d.

Let $(\mathbb{S}^2,\beta)$ be the sphere with marked points $\beta=\sum_{i=1}^k\beta_i[p_i]$, and $\beta_k>\sum_{i<k}\beta_i$. We let $I\sqcup J=\{1,2,...,k\}$ be a division of $\{1,2,...,k\}$ and define a sphere with new marked points $(\mathbb{S}^2,\beta_{I,J})$ by

$$\beta_{I,J}=\sum_{i\in l}\beta_i[p]+\sum_{j\in J}\beta_j[q].$$

Then we can order the finite set $\{\mu(g_{sol,\beta_{I,J}},-\theta_{g_{sol,\beta_{I,J}}})\}_{I,J}$ by

$$\mu_1>\mu_2\geq\mu_3\geq...\geq\mu_N,$$

for some N and $\mu_1 = W(g_{sol,\beta_{I,J}}, -\theta_{g_{sol,\beta_{I,J}}})$ with $I = \{k\}$ and $J = \{1, 2, ..., k-1\}$. Theorem 1.3 would imply that the conical Ricci flow in the unstable case will always converge to the conical Ricci soliton with the highest μ-energy among the finite set μ_1.

7.3. Uniqueness of the limiting soliton metrics. Now we prove the unstable case of Theorem 1.3 by showing that the limiting soliton metric does not depend on the choice of initial metric. In particular, the limiting metric has the highest μ-energy μ_1 among $\mu_1, ..., \mu_k$. In order to complete the proof of Theorem 1.3, it suffices to show that p_k converges to p_∞ and $p_1, ..., p_{k-1}$ converge to q_∞ by Lemma 7.2.

We consider the conical Ricci flow $g(t)$ on the unstable pair $(\mathbb{S}^2, \beta)$. Let g_{sol} be the limiting nontrivial soliton metric on $(\mathbb{S}^2, \beta_\infty)$ with $\beta_\infty = \beta_{p_\infty}[p_\infty] + \beta_{q_\infty}[q_\infty]$, $0 \leq \beta_{q_\infty} < \beta_{p_\infty}$.

Lemma 7.9. *There exist $t_0 > 0$ and $r_0 > 0$ such that for all $t \geq t_0$ and $r \leq r_0$,*

$$R(t) < \overline{R} - 2\epsilon, \text{ on } B_{g(t)}(p_k, r),$$

and

$$\{p_1, p_2, ..., p_{k-1}\} \cap \overline{B_{g(t_0)}(p_k, r_0)} = \phi.$$

Proof. First of all, by the Cheeger–Colding theory, $(B_{g(t)}(p_k, 2r), g(t))$ converges in Gromov–Hausdorff topology to $(B_{g_{sol}}(p_\infty, 2r), g_{sol})$. Furthermore, the convergence is smooth on $B_{g_{sol}}(p_\infty, 2r) \setminus \{p_\infty\}$ by Hamilton's compactness theorem.

We consider $U = B_{g_{sol}}(p_\infty, 2r) \setminus B_{g_{sol}}(p_\infty, \delta)$ for some sufficiently small $\delta > 0$ to be determined later. Then $g(t)$ converges smoothly and uniformly to g_{sol} on U. In particular, there exist $t_0 > 0$ and $r_0 > 0$ such that for all $t \geq t_0$ and $r \leq r_0$, we have on $B_{g(t)}(p_k, r) \setminus B_{g(t)}(p_k, 2\delta)$,

$$R(t) < \overline{R} - 3\epsilon,$$

from Lemma 7.6. Furthermore, we can always assume that $\{p_1, p_2, ..., p_{k-1}\} \cap \overline{B_{g(t_0)}(p_k, r_0)} = \phi$ after choosing a smaller r_0.

On the other hand, there exists $K > 0$, such that for all $t \geq t_0$,

$$\sup_{S^2} |\nabla R|_{g(t)} \leq K.$$

Therefore, for all $t \geq t_0$, we have

$$\sup_{B_{g(t)}(p_k, 2\delta)} R(t) \leq \overline{R} - 3\epsilon + 2\delta K < \overline{R} - 2\epsilon,$$

if we choose

$$\delta < \min(\frac{\epsilon}{2K}, \frac{r}{4}).$$

The lemma then immediately follows. q.e.d.

We now will prove a monotonicity result for geodesic balls centered at p_k.

Lemma 7.10. *For any* $t_0 \le t_1 \le t_2$, $0 < r < r_0$, *we have*

$$\overline{B_{g(t_2)}(p_k, r)} \ \subset \ \overline{B_{g(t_1)}(p_k, r)},$$

where t_0 *and* r_0 *are chosen as in Lemma 7.9.*

Proof. For any $t_1 > t_0$ and $0 < r < r_0$, we define

$$\mathcal{S} = \{t > t_1 \mid B_{g(t)}(p_k, r) \cap \left(S^2 \setminus \overline{B_{g(t_1)}(p_k, r)}\right) \neq \phi\}.$$

We claim that

$$\mathcal{S} = \phi.$$

If $\mathcal{S}$ is not empty, then we define

$$T = \inf \mathcal{S} \in [t_1, \infty).$$

By the definition of T and continuity of $(S^2, g(t))$ in Gromov–Hausdorff topology,

$$B_{g(T)}(p_k, r) \subset \overline{B_{g(t_1)}(p_k, r)}.$$

Since the curvature R is Lipschitz both in space and time, there exists $\delta > 0$ such that

$$\sup_{\overline{B_{g(T)}(p_k,r)} \times [T, T+\delta]} R(x,t) < \overline{R} - \epsilon.$$

The Ricci flow implies that

$$\frac{\partial g}{\partial t} = -Ric + \overline{R} g > \epsilon g,$$

and so

$$g(t) \ge e^{\epsilon t} g(T), \ T \le t \le T + \delta.$$

This implies that for $T \le t \le T + \delta$, the metric is monotonically increasing on $[T, T+\delta]$ and so

$$B_{g(t)}(p_k, r) \subset \overline{B_{g(T)}(p_k, r)}.$$

Then it follows that for $t \in [T, T+\delta]$,

$$B_{g(t)}(p_k, r) \subset \overline{\Omega}, \text{ or } B_{g(t)}(p_k, r) \cap \left(S^2 \setminus \overline{\Omega}\right) = \phi,$$

which contradicts the definition of T.

The above claim that $\mathcal{S} = \phi$ immediately implies that

$$\overline{B_{g(t_2)}(p_k, r)} \ \subset \ \overline{B_{g(t_1)}(p_k, r)},$$

for all $t_2 \ge t_1 \ge t_0$. This completes the proof of Lemma 7.10. q.e.d.

We can now complete the proof of Theorem 1.3 for the unstable case.

Corollary 7.1. *p_k converges to p_∞ and $p_1, ..., p_{k-1}$ converge to q_∞ in Gromov–Hausdorff distance as $t \to \infty$. Furthermore, the limiting soliton g_{sol} is the unique rotationally symmetric shrinking soliton metric on (S^2, g_∞) with*

$$\beta_{p_\infty} = \beta_{p_k},\ \beta_{q_\infty} = \sum_{j=1}^{k-1} \beta_j.$$

Proof. We first note that the limiting soliton can have at most two singularities and the limiting point of any conical point must be a singular point. Then the first statement immediately follows from Lemma 7.10. The second statement follows by Lemma 7.2. q.e.d.

Lemma 7.11. *Let*

$$D_{p_1,...,p_{k-1}}(t) = \max_{1\le i<j\le k-1} d_{g(t)}(p_i, p_j),$$

where $d_{g(t)}(p, q)$ is the geodesic distance between p and q in the metric space $(\mathbb{S}^2, g(t))$. Then $D_{p_1,....,p_{k-1}}(t)$ converges to 0 exponentially fast as $t \to \infty$.

Proof. Note that $p_1, ..., p_{k-1}$ converge to q_∞ by Corollary 7.1. By the same argument used in Lemma 7.9, there exist $\epsilon > 0$, $t_0 > 0$ and $r > 0$ such that for $t \ge t_0$, we have on $B_{g(t)}(p_1, r)$,

$$R(t) > \overline{R} + 2\epsilon,\ p_1, ..., p_{k-1} \in B_{g(t)}(p_1, r).$$

Then we apply similar argument in the proof of Lemma 7.10 to show that $\overline{B_{g(t)}(p_1, r)}$ is increasing in time for $t \ge t_0$. Then on the fixed domain $B_{g(t_0)}(p_1, r)$ and for $t \ge t_0$,

$$R(t) > \overline{R} + 2\epsilon,\ g(t) \le e^{-\epsilon t} g(t_0).$$

Therefore, there exists $C > 0$ such that for all $t \ge t_0$,

$$Diam_{g(t)}(B_{g(t_0)}(p_1, r)) \le Ce^{-\frac{\epsilon t}{2}},$$

and the lemma follows immediately since $p_1, ..., p_{k-1} \in B_{g(t_0)}(p_1, r)$. q.e.d.

Appendix A. The α-invariant for $(\mathbb{S}^2, \beta)$

The α-invariant [**51**] can be readily extended to the case of the sphere with marked points. We define

$$\alpha(\mathbb{S}^2, \beta) = \sup \alpha, \tag{A.1}$$

where α satisfies the condition

$$\sup_{\varphi\in PSH(\mathbb{S}^2, g_{FS})\cap C^\infty(\mathbb{S}^2)} \int_{\mathbb{S}^2} e^{-\frac{1}{2}\alpha\chi(\mathbb{S}^2,\beta)(\varphi - \sup_{\mathbb{S}^2}\varphi)} \prod_{i=1}^{k} |\sigma_i|^{-\beta_i} (g_{FS})^{\frac{1}{2}\chi(\mathbb{S}^2,\beta)} < \infty,$$

where $\sigma_i \in H^0(\mathbb{S}^2, K_{\mathbb{S}^2}^{-1})$ vanishes at p_i of order 2, $i = 1, ..., k$.

The following lemma is due to Berman [2]. We reproduce the short proof below, for the convenience of the reader:

Lemma A.1. *Assume that* $\sum_{j=1}^{k}\beta_j < 2$. *Then*

$$\alpha(\mathbb{S}^2,\beta) = \frac{1-\beta_k}{\chi(\mathbb{S}^2,\beta)}. \tag{A.2}$$

Proof. By a theorem of Demailly [8] in the smooth case, and extended to the conical case by Berman [2], the α-invariant $\alpha(\mathbb{S}^2,\beta)$ is equal to the log canonical threshold $Lct(\mathbb{S}^2,\beta)$, which is defined as follows. Let h be any hermitian metric on $K_{\mathbb{S}^2}^{-1}$. Then

$$Lct(\mathbb{S}^2,\beta) = \sup\alpha, \tag{A.3}$$

where α satisfies the condition

$$\int_{\mathbb{S}^2} |\sigma|^{-\frac{\alpha}{m}\chi(\mathbb{S}^2,\beta)} \prod_{i=1}^{k} |\sigma_i|^{-\beta_i} h^{\frac{1}{2}\chi(\mathbb{S}^2,\beta)(1-\alpha)} < \infty, \tag{A.4}$$

for any $m \in \mathbb{Z}^+$, $\sigma \in H^0(\mathbb{S}^2, K_{\mathbb{S}^2}^{-m})$.

Since β_k is the largest among β_i, we can calculate the integral near p_k. Without loss of generality we may assume $p_k = 0$. Then for any $0 \le l \le 2m$ there is a $\sigma \in H^0(\mathbb{S}^2, K_{\mathbb{S}^2}^{-m})$ which admits an expansion $\sigma = z^\ell f(z)$ near 0 for some holomorphic function $f(z)$ satisfying $f(0) = 1$. Then we have

$$\begin{aligned} &\int_{\mathbb{S}^2} |\sigma|^{-\frac{\alpha}{m}\chi(\mathbb{S}^2,\beta)} \prod_{i=1}^{k} |\sigma_i|^{-\beta_i} \\ = \;& \sqrt{-1}\int_{|z|\le 1} |z|^{-2\frac{\alpha\ell}{m}\chi(\mathbb{S}^2,\beta)} |z|^{-2\beta_k} dz\wedge d\bar z + O(1) \\ \le \;& \sqrt{-1}\int_{|z|\le 1} |z|^{-2\alpha\chi(\mathbb{S}^2,\beta)-2\beta_k} dz\wedge d\bar z + O(1), \end{aligned}$$

which is finite for any $\alpha < \frac{1-\beta_k}{\chi(\mathbb{S}^2,\beta)}$. The equality follows easily by applying a test holomorphic section σ in the anti-pluricanonical system. The proof is complete. q.e.d.

The threshold for the conical alpha invariant of (S^2,β) is $\frac{1}{2}$.

Corollary A.1. *Assume that* $\sum_{j=1}^{k}\beta_j < 2$. *If* $\beta_k < \sum_{j=1}^{k-1}\beta_j$,

$$\alpha(\mathbb{S}^2,\beta) > \frac{1}{2}.$$

If $\beta_k = \sum_{j=1}^{k-1}\beta_j$,

$$\alpha(\mathbb{S}^2,\beta) = \frac{1}{2}.$$

Appendix B. The F functional for pairs $(\mathbb{S}^2, \beta)$

It is well-known in the smooth case that the equation of constant scalar curvature admits a variational formulation. As for the α-invariant, this can be readily extended to the case of Riemann surfaces with marked points. We can write the corresponding functional in two different ways, depending on whether we use the Fubini–Study metric g_{FS} or the metric g_β with conical singularities as reference metric.

With g_{FS} as reference metric, we set for $\varphi \in PSH(\mathbb{S}^2, g_{FS}) \cap L^\infty(\mathbb{S}^2)$,

$$\begin{aligned} F_\beta(\varphi) &= \frac{\sqrt{-1}}{8\pi}\int_{\mathbb{S}^2} \partial\varphi \wedge \bar\partial\varphi - \frac{1}{2}\int_{\mathbb{S}^2} \varphi \, g_{FS} \\ &\quad - \frac{2}{\chi(\mathbb{S}^2,\beta)} \log\left(\int_{\mathbb{S}^2} e^{-\frac{1}{2}\chi(\mathbb{S}^2,\beta)\varphi} \prod_{i=1}^k |\sigma_i|^{-\beta_i} g_{FS}^{\frac{1}{2}\chi(\mathbb{S}^2,\beta)} \right), \end{aligned} \tag{B.1}$$

while with g_β as reference metric, we set

$$\begin{aligned} F_\beta(\varphi) & \\ = \frac{\sqrt{-1}}{8\pi}\int \partial\varphi \wedge \bar\partial\varphi &- \frac{1}{2}\int \varphi g_\beta - \frac{2}{\chi(\mathbb{S}^2,\beta)} \log\left(\int_{\mathbb{S}^2} e^{-\frac{1}{2}\chi(\mathbb{S}^2,\beta)\varphi + h_\beta} g_\beta\right). \end{aligned} \tag{B.2}$$

In view of the fact that the potentials are always bounded with bounded Dirichlet energy, integration by parts is justified and the two formulations of the F_β functional can be verified to agree. The Euler–Lagrange equation for F_β is exactly the equation for the stationary points of the flow

$$\frac{\partial\psi}{\partial t}(t) = \log\left(\frac{g_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\psi}{g_{FS}}\right) + \frac{1}{2}\chi(\mathbb{S}^2,\beta)\psi + \frac{1}{2}\sum_{j=1}^k \beta_j \log \frac{|\sigma_j|^2}{g_{FS}} . \tag{B.3}$$

It is a special case of the functional F_β defined in [**47**] for paired Fano manifolds and it satisfies the co-cycle condition.

We shall need the following simple property of F_β, which is a straightforward adaptation of the similar property established in the smooth case in [**43**]:

Lemma B.1. *Let $(\mathbb{S}^2, \beta)$ be a pair with $\sum_{j=1}^k \beta_j < 2$. If $\alpha(\mathbb{S}^2, \beta) > 1/2$, then there exists $\epsilon > 0$ and $C_\epsilon > 0$ such that for all $\varphi \in PSH(\mathbb{S}^2, g_{FS}) \cap L^\infty(\mathbb{S}^2)$,*

$$F_\beta(\varphi) \geq \epsilon \frac{\sqrt{-1}}{2\pi}\int_{\mathbb{S}^2} \partial\varphi \wedge \bar\partial\varphi - C_\epsilon. \tag{B.4}$$

In particular, the equation (B.3) is solvable.

We remark that when $(\mathbb{S}^2, \beta)$ is not stable, the functional F_β is not bounded below and so $(\mathbb{S}^2, \beta)$ does not admit a constant curvature metric. Combined with Lemma A.1, this can provide a complex geometric proof for the criterion of Troyanov and Luo–Tian as suggested in [**56**, **28**].

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