J. DIFFERENTIAL GEOMETRY
114 (2020) 41-115

HODGE-THEORETIC MIRROR SYMMETRY FOR TORIC STACKS

Tom Coates, Alessio Corti, Hiroshi Iritani & Hsian-Hua Tseng

Abstract

Using the mirror theorem [15], we give a Landau–Ginzburg mirror description for the big equivariant quantum cohomology of toric Deligne–Mumford stacks. More precisely, we prove that the big equivariant quantum D-module of a toric Deligne–Mumford stack is isomorphic to the Saito structure associated to the mirror Landau–Ginzburg potential. We give a Gelfand-Kapranov-Zelevinsky (GKZ) style presentation of the quantum D-module, and a combinatorial description of quantum cohomology as a quantum Stanley–Reisner ring. We establish the convergence of the mirror isomorphism and of quantum cohomology in the big and equivariant setting.

Contents

1. Introduction 42
2. Toric stacks 46
   2.1. Definition 46
   2.2. Torus action and divisor sequence 48
   2.3. Inertia stack, Box and orbifold cohomology 48
   2.4. Refined fan sequence 50
   2.5. Mori cone and associated monoids 51
3. Toric Gromov–Witten theory 52
   3.1. Gromov–Witten invariants 53
   3.2. Quantum cohomology and the quantum connection 53
   3.3. Galois symmetry 56
   3.4. Givental cone 57
   3.5. Mirror theorem 58
4. The Gauss–Manin system and the mirror isomorphism 59
   4.1. Landau–Ginzburg model 59
   4.2. An unfolding of the Landau–Ginzburg potential 61
   4.3. Galois action on the Landau–Ginzburg model 62

Received October 16, 2016.

4.4. The Gauss–Manin system 64
4.5. Solution and freeness 68
4.6. Mirror isomorphism 73
5. Presentations of the quantum D-module and quantum cohomology ring 77
5.1. The fan D-module 77
5.2. GKZ-style presentation 80
5.3. Quantum cohomology ring 83
5.4. Examples 85
6. The higher residue pairing and the Poincaré pairing match 88
6.1. Critical points 88
6.2. Higher residue pairing via asymptotic expansion 92
6.3. The pairings match 96
7. Convergence 99
7.1. Result 100
7.2. Estimates for the Gauss–Manin connection 102
7.3. Gauge fixing 105
7.4. Proof of Theorem 7.2 and Corollary 7.3 109
References 110

1. Introduction

This paper is the last in a series of papers [**14**, **15**] that study the genus-zero Gromov–Witten theory of toric Deligne–Mumford stacks. Let $\mathfrak{X}$ be a toric Deligne–Mumford stack, or toric stack for short, that satisfies a mild semi-projectivity hypothesis (spelled out below). In [**15**] we proved a mirror theorem that says that a certain hypergeometric function, called the I-function, lies on the Givental cone for $\mathfrak{X}$. This determines all genus-zero Gromov–Witten invariants of $\mathfrak{X}$. The present paper builds on this mirror theorem to establish Hodge-theoretic mirror symmetry for toric stacks in a very general setting – without assuming that $\mathfrak{X}$ is compact, or imposing any positivity condition on $c_1(\mathfrak{X})$. We prove that the big and equivariant quantum cohomology D-module of $\mathfrak{X}$ can be described as the Saito structure of the Landau–Ginzburg model mirror to $\mathfrak{X}$.

It has been proposed by Givental [**32**] (see also [**46**]) that the mirror of a toric manifold X is a Landau–Ginzburg model, or more precisely, a Laurent polynomial function $F = F(x_1, \dots, x_n)$ with Newton polytope equal to the fan polytope of X. In particular, Givental [**37**] showed that, for weak Fano toric manifolds X, oscillatory integrals $\int e^{F/z} \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}$ give solutions of the small quantum cohomology D-module of X. His result also implies that the quantum cohomology ring of X is isomorphic to the Jacobian ring of F, via an isomorphism which matches the Poincaré pairing with the residue pairing. Givental-style mirror symmetry has been extended to big quantum cohomology by Barannikov, Douai–Sabbah, and Mann [**3**, **27**, **60**]; this compares the Frobenius manifold structure [**28**] defined by the big quantum cohomology of X with K. Saito's flat structure [**71**, **72**] associated to a miniversal unfolding of F.

Let us briefly review our main construction. Let $\mathfrak{X}$ be a toric Deligne–Mumford stack with semi-projective coarse moduli space. (This means that the coarse moduli space is projective over affine and contains a torus-fixed point.) We introduce an unfolding $F(x;y)$ of Givental's Landau–Ginzburg potential by choosing a finite subset G in the fan lattice $\mathbf{N}$:

$$F(x;y) = \sum_{i=1}^{m} y_i Q^{\lambda(b_i)} x^{b_i} + \sum_{\mathbf{k}\in G} y_{\mathbf{k}} Q^{\lambda(\mathbf{k})} x^{\mathbf{k}},$$

where $b_1, \ldots, b_m$ are generators of one-dimensional cones of the stacky fan of $\mathfrak{X}$, Q is the Novikov variable, $\lambda(b_i)$ and $\lambda(\mathbf{k}) \in H_2(X,\mathbb{Q})$ are certain curve classes, $x \in \mathrm{Hom}(\mathbf{N}, \mathbb{C}^\times)$ is a torus co-ordinate, and y_i, $y_{\mathbf{k}}$ are deformation parameters. See §4.2 for details. Generalizing the construction in [**52**] to stacks, we introduce a *formal* and *logarithmic* Landau–Ginzburg model (see §4.4)

$$\begin{array}{ccc} \widehat{\mathcal{Y}} & \xrightarrow{F(x;y)} & \mathbb{C} \\ \downarrow & & \\ \widehat{\mathcal{M}} & & \end{array}$$

where $\widehat{\mathcal{Y}} \to \widehat{\mathcal{M}}$ is a degenerating family of affine toric varieties over the base $\widehat{\mathcal{M}} = \operatorname{Spf} \mathbb{C}[\![\mathbf{\Lambda}_+]\!] \times \operatorname{Spf} \mathbb{C}[\![y]\!]$ with $\mathbb{C}[\![\mathbf{\Lambda}_+]\!]$ the Novikov ring (i.e. the completed semigroup ring of the monoid $\mathbf{\Lambda}_+ \subset H_2(X,\mathbb{Q})$ of effective curves). The spaces $\widehat{\mathcal{Y}}$ and $\widehat{\mathcal{M}}$ have natural log structures defined by their toric boundaries. We then consider the logarithmic twisted de Rham complex

$$\left(\Omega^\bullet_{\widehat{\mathcal{Y}}/\widehat{\mathcal{M}}}\{z\}, zd + dF\wedge\right)$$

and define the Gauss–Manin system $\mathrm{GM}(F)$ to be the top cohomology of this complex. In the equivariant case, we consider the potential $F_\chi = F - \sum_{i=1}^n \chi_i \log x_i$ in place of F, where χ_i are torus-equivariant parameters. The equivariant Gauss–Manin system $\mathrm{GM}(F_\chi)$ is equipped with the Gauss–Manin connection ∇, the grading operator Gr^{B} and the higher residue pairing $P\colon \mathrm{GM}(F_\chi) \times \mathrm{GM}(F_\chi) \to S_{\mathbb{T}}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]$. We call the quadruple

$$\left(\mathrm{GM}(F_\chi), \nabla, \mathrm{Gr}^{\mathrm{B}}, P\right) \tag{1.1}$$

the *Saito structure* associated with the Landau–Ginzburg model.

Theorem 1.1 (see Theorems 4.28, 6.11 for the details)**.** *The Saito structure* (1.1) *is isomorphic to the big and equivariant quantum connection of the toric stack $\mathfrak{X}$ together with the Poincaré pairing, under the identification of the base spaces given by a mirror map.*

One of the important aspects in our construction is that the Landau–Ginzburg model is partially compactified across the large radius limit point $Q = 0$; the Gauss–Manin connection then has logarithmic singularities at $Q = 0$. The choice of a partial compactification is subtle when $\mathfrak{X}$ is a toric stack, rather than a toric manifold, because the family $\widehat{\mathcal{Y}} \to \widehat{\mathcal{M}}$ then carries an additional *Galois symmetry* of $\operatorname{Pic}^{\mathrm{st}}(\mathfrak{X}) := \operatorname{Pic}(\mathfrak{X})/\operatorname{Pic}(X)$, where X is the coarse moduli space of $\mathfrak{X}$. The Galois symmetry has stabilizers along the compactifying divisor, and the quotient family $\widehat{\mathcal{Y}}/\operatorname{Pic}^{\mathrm{st}}(\mathfrak{X}) \to \widehat{\mathcal{M}}/\operatorname{Pic}^{\mathrm{st}}(\mathfrak{X})$ gives a partial compactification of the traditional mirror family[1] . Our construction gives a generalization of the work of de Gregorio–Mann [**22**] who studied the Jacobian ring at the limit $Q = 0$ for mirrors of weighted projective spaces (see also [**23**] for a related partial compactification). The new ingredient for us is the *refined fan sequence* (2.6) for stacky fans.

Our results yield a combinatorial description for the quantum D-module of toric stacks which is closely related to the better-behaved GKZ system of Borisov–Horja [**6**]. We introduce a *fan D-module* for a stacky fan (Definition 5.1) and show that the quantum D-module of the corresponding toric stack $\mathfrak{X}$ is isomorphic to the (Q, y)-adic completion of the fan D-module (see Theorem 5.6). By taking the semiclassical limit $z \to 0$ of the D-module, we obtain a quantum Stanley–Reisner description of the big and equivariant quantum cohomology of $\mathfrak{X}$ as follows, generalizing the previous works [**4**, **5**, **30**, **34**, **37**, **41**, **48**, **50**, **62**]:

Theorem 1.2 (see Theorem 5.13 for the details)**.** *The big and equivariant quantum cohomology of $\mathfrak{X}$ is isomorphic to the Jacobian ring of F_χ under the identification of parameters given by the mirror map. The latter ring is isomorphic to the space $\widehat{\bigoplus}_{\mathbf{k}\in\mathbf{N}\cap|\Sigma|} \mathbb{C}[\![\mathbf{\Lambda}_+]\!][\![y]\!]\mathbb{1}_{\mathbf{k}}$ equipped with the following product and $H^*_{\mathbb{T}}(\mathrm{pt}, \mathbb{C})$-module structure:*

$$\mathbb{1}_{\mathbf{k}} \star \mathbb{1}_{\boldsymbol{\ell}} = Q^{d(\mathbf{k},\boldsymbol{\ell})}\mathbb{1}_{\mathbf{k}+\boldsymbol{\ell}}, \qquad \chi = \sum_{i=1}^{m} (\chi \cdot b_i) y_i \mathbb{1}_{b_i} + \sum_{\boldsymbol{\ell}\in G} (\chi \cdot \boldsymbol{\ell}) y_{\boldsymbol{\ell}} \mathbb{1}_{\boldsymbol{\ell}}$$

with $\chi \in H^2_{\mathbb{T}}(\mathrm{pt}) = \mathbf{N}^\star \otimes \mathbb{C}$. See (2.5) *for the definition of $d(\mathbf{k}, \boldsymbol{\ell}) \in H_2(X, \mathbb{Q})$.*

In the last part of the paper, we discuss the convergence of the mirror map and the mirror isomorphism in Theorem 1.1. Beyond the weak

[1]The correct mirror family should be thought of as a formal stack $[\widehat{\mathcal{Y}}/\operatorname{Pic}^{\mathrm{st}}(\mathfrak{X})] \to [\widehat{\mathcal{M}}/\operatorname{Pic}^{\mathrm{st}}(\mathfrak{X})]$.

Fano case or small quantum cohomology, it was known [**48**] that the mirror isomorphism is not fully analytic: it is only defined over formal power series in z in general. We prove a partial analyticity result for the mirror isomorphism which in turn shows that the big and equivariant quantum cohomology itself is convergent and analytic.

Theorem 1.3 (see Theorem 7.2, Corollary 7.3 for the details)**.** *The mirror map and the isomorphism in Theorem 1.1 satisfy the following:*

(1) *The mirror map is analytic in* (Q, y, χ).
(2) *With respect to a basis of the Gauss–Manin system* $\mathrm{GM}(F_\chi)$ *formed by polynomial differential forms, the mirror isomorphism is a formal power series in* z *with coefficients in analytic functions in* (Q, y, χ).

It follows that the structure constants of the big and equivariant quantum cohomology of semi-projective toric stacks are analytic functions in their arguments (τ, χ).

The proof uses mirror symmetry in an essential way. We combine the fact that the formal asymptotic expansions of oscillatory integrals are Gevrey series of order 1 with a gauge fixing result from [**48**, Proposition 4.8].

Remark 1.4. Hodge theoretic mirror symmetry for toric varieties or stacks has been studied by many people. We explain how our results fit with this earlier work.

(1) In singularity theory, our Gauss–Manin system has been studied for isolated hypersurface singularities under the name of *Brieskorn lattice*. K. Saito [**72**] and M. Saito [**73**] constructed flat (Frobenius manifold) structures on the base of miniversal deformations of isolated singularities. This was generalized to global singularities by Sabbah, Barannikov, and Douai–Sabbah [**3**, **26**, **27**, **70**], and applications to mirror symmetry are discussed there. See also Mann [**60**].

(2) More recently, Reichelt–Sevenheck [**68**, **69**] constructed nc-Hodge structures, which roughly speaking correspond to the Saito structures here, for mirrors of weak Fano toric manifolds and discussed their relation to the GKZ system. They also used log structures on the mirror to define the twisted de Rham complex; more precisely, they put log structures along the toric boundary of each fiber of the mirror family $\mathcal{Y} \to \mathcal{M}$, but not along $Q = 0$. They described a logarithmic extension of the mirror D-module across $Q = 0$ using a GKZ-style presentation. See also T. Mochizuki [**63**].

(3) Mirror symmetry for non-weak-Fano toric manifolds and its convergence were analysed by Iritani [**48**]. Fukaya–Oh–Ohta–Ono [**30**] gave a Jacobian description of the quantum cohomology of general toric manifolds using Lagrangian Floer theory (see Chan–Lau–Leung–Tseng [**9**] for

an explicit computation in the weak Fano case). Gross [43] constructed mirrors of the big quantum cohomology of $\mathbb{P}^2$ by counting tropical discs.

(4) A mirror theorem for weighted projective spaces was proved by Coates–Corti–Lee–Tseng [16] and was generalized to toric stacks in our previous work [15]; see Cheong–Ciocan-Fontanine–Kim [12] for a more general result. Based on these works, Landau–Ginzburg mirror symmetry for small quantum D-modules was described by Iritani [50] for weak Fano toric stacks, and by Douai–Mann [25] for weighted projective spaces. Iritani [50] also described the natural integral structure on the mirror in terms of the Gamma class (see also [57]); a missing piece in the present work is the identification of the integral (or rational, or real) structure for mirrors of general toric stacks.

(5) González–Woodward [41] used gauged Gromov–Witten theory and quantum Kirwan maps to give a Jacobian description for quantum cohomology of toric stacks.

(6) After we finished a draft of this paper, we learned that Mann–Reichelt [61] studied closely related logarithmic degenerations of mirrors of weak-Fano toric orbifolds along $Q = 0$. They used an extended version of the refined fan sequence in the case where $\mathbf{N}$ has no torsion (see [61, equation 2.17]) and obtained a logarithmic extension of the mirror D-module via a GKZ-style presentation (see [61, Definition 4.9, Theorem 6.6] and Remark 5.11).

Remark 1.5. It should be possible to construct the mirror map and the mirror isomorphism for toric stacks via the Seidel representation [74] and shift operators [7], as [40, 52] did for toric manifolds.

Acknowledgements. We thank Thomas Reichelt for his interest in our paper and helpful comments. A.C. was supported in part by EPSRC grant EP/E022162/1. T.C. was supported in part by a Royal Society University Research Fellowship, ERC Starting Investigator Grant number 240123, and the Leverhulme Trust. H.I. was supported in part by EPSRC grant EP/E022162/1 and JSPS Kakenhi Grant Number 17H06127, 16K05127, 16H06337, 16H06335, 25400069, 26610008, 23224002, 25400104, 22740042. H.-H.T. was supported in part by NSF grant DMS-1506551 and a Simons Foundation Collaboration Grant.

2. Toric stacks

In this section, we establish notation for toric Deligne–Mumford stacks (toric stacks for short) in the sense of Borisov, Chen and Smith [5]. For the basics on toric stacks or varieties, we refer the reader to [5, 21, 29, 53, 54].

2.1. Definition. A *stacky fan* [5] is a triple $\mathbf{\Sigma} = (\mathbf{N}, \Sigma, \beta)$ consisting of

- a finitely generated abelian group $\mathbf{N}$ of rank n;

- a rational simplicial fan Σ in the vector space $\mathbf{N}_{\mathbb{R}} = \mathbf{N} \otimes_{\mathbb{Z}} \mathbb{R}$;
- a homomorphism $\beta \colon \mathbb{Z}^m \to \mathbf{N}$ such that $\{\mathbb{R}_{\geq 0} b_1, \dots, \mathbb{R}_{\geq 0} b_m\}$ is the set of one-dimensional cones of Σ, where $b_i = \beta(e_i)$ is the image of the ith basis vector $e_i \in \mathbb{Z}^m$.

Abusing notation, we shall identify a cone σ of Σ with the subset $\{i : b_i \in \sigma\}$ of $\{1, \dots, m\}$. For instance, we write $I \in \Sigma$ for $I \subset \{1, \dots, m\}$ if the cone spanned by $\{b_i : i \in I\}$ belongs to Σ, and we write $i \in \sigma$ for a cone $\sigma \in \Sigma$ if $b_i \in \sigma$. Define

$$\mathcal{U}_\Sigma := \mathbb{C}^m \setminus \bigcup_{\{1,\dots,m\} \setminus I \notin \Sigma} \mathbb{C}^I,$$

where $\mathbb{C}^I = \{(Z_1, \dots, Z_m) \in \mathbb{C}^m : Z_i = 0 \text{ for } i \notin I\}$. Define the group $\mathbb{G}$ by

$$\mathbb{G} := H^{-1}(\mathrm{Cone}(\beta) \otimes^{\mathbb{L}} \mathbb{C}^\times).$$

This is isomorphic to the product of the algebraic torus $(\mathbb{C}^\times)^{m-n}$ and a finite group. The group $\mathbb{G}$ acts on $\mathbb{C}^m$ via the connecting homomorphism $H^{-1}(\mathrm{Cone}(\beta) \otimes^{\mathbb{L}} \mathbb{C}^\times) \to H^0(\mathbb{Z}^m \otimes \mathbb{C}^\times) = (\mathbb{C}^\times)^m$. A *toric Deligne–Mumford stack* $\mathfrak{X}$ associated to the stacky fan $\mathbf{\Sigma}$ is defined [**5**] to be the quotient stack

$$\mathfrak{X} := [\mathcal{U}_\Sigma / \mathbb{G}].$$

We assume that

- Σ contains a cone of maximal dimension $n = \dim \mathbf{N}_{\mathbb{R}}$;
- the support $|\Sigma|$ of the fan Σ is convex;
- the fan admits a strictly convex piecewise linear function $f \colon |\Sigma| \to \mathbb{R}$ which is linear on each cone.

These assumptions are equivalent to the condition that the coarse moduli space X of $\mathfrak{X}$ is semi-projective [**21**], that is, X is projective over an affine variety and has a torus fixed point. We set

$$\mathbb{L} := \mathrm{Ker}(\beta), \qquad \mathbf{M} := \mathrm{Hom}(\mathbf{N}, \mathbb{Z}).$$

By definition, $\mathbb{L}$ is the lattice of relations among $b_1, \dots, b_m$. The *fan sequence* is the exact sequence:

$$0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^m \xrightarrow{\ \beta\ } \mathbf{N} \tag{2.1}$$

and the *divisor sequence* is its Gale dual [**5**]:

$$0 \longrightarrow \mathbf{M} \xrightarrow{\ \beta^\star\ } (\mathbb{Z}^m)^\star \xrightarrow{\ D\ } \mathbb{L}^\vee, \tag{2.2}$$

where $\mathbb{L}^\vee := H^1(\mathrm{Cone}(\beta)^\star) \cong \mathrm{Hom}(\mathbb{G}, \mathbb{C}^\times)$ and the map $D \colon (\mathbb{Z}^m)^\star \to \mathbb{L}^\vee$ is induced by the natural map $(\mathbb{Z}^m)^\star \to (\mathrm{Cone}(\beta)[-1])^\star = \mathrm{Cone}(\beta)^\star[1]$. The ordinary dual $\mathbb{L}^\star = \mathrm{Hom}(\mathbb{L}, \mathbb{Z})$ can be identified with the torsion-free part $\mathbb{L}^\vee / (\mathbb{L}^\vee)_{\mathrm{tor}}$ of $\mathbb{L}^\vee$ and the torsion part of $\mathbb{L}^\vee$ is given by $(\mathbb{L}^\vee)_{\mathrm{tor}} = \mathrm{Hom}(\mathrm{Cok}(\beta), \mathbb{C}^\times)$. Note that $\mathrm{Cok}(\beta) \cong \pi_0(\mathbb{G})$ is isomorphic to the orbifold fundamental group of $\mathfrak{X}$. The torsion part $\mathbf{N}_{\mathrm{tor}}$ of $\mathbf{N}$ is isomorphic to the generic stabilizer $\mathrm{Ker}(\mathbb{G} \to (\mathbb{C}^\times)^m)$ of $\mathfrak{X}$.

We write $u_i = e_i^\star \in (\mathbb{Z}^m)^\star$ for the ith basis vector and $D_i := D(u_i)$ for the image of u_i by D.

Notation 2.1. By the subscripts $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, we mean the tensor products with $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ (over $\mathbb{Z}$), e.g. $\mathbf{N}_\mathbb{R} = \mathbf{N} \otimes \mathbb{R}$, $\mathbb{L}_\mathbb{Q} = \mathbb{L} \otimes \mathbb{Q}$. For an element $\mathbf{k} \in \mathbf{N}$, we denote by $\overline{\mathbf{k}}$ the image of $\mathbf{k}$ in $\mathbf{N}_\mathbb{Q}$ (or in $\mathbf{N}_\mathbb{R}$). By abuse of notation, we write $\mathbf{N} \cap |\Sigma| := \{\mathbf{k} \in \mathbf{N} : \overline{\mathbf{k}} \in |\Sigma|\}$.

2.2. Torus action and divisor sequence. The $(\mathbb{C}^\times)^m$-action on $\mathcal{U}_\Sigma \subset \mathbb{C}^m$ naturally induces the action of the Picard stack $\mathfrak{T} = [(\mathbb{C}^\times)^m/\mathbb{G}]$ on $\mathfrak{X}$ [**29**]. A line bundle on $\mathfrak{X}$ corresponds to a $\mathbb{G}$-equivariant line bundle on $\mathcal{U}_\Sigma$, which is determined by a character of $\mathbb{G}$. Similarly, a $\mathfrak{T}$-equivariant line bundle on $\mathfrak{X}$ corresponds to a $(\mathbb{C}^\times)^m$-equivariant line bundle on $\mathcal{U}_\Sigma$, which is determined by a character of $(\mathbb{C}^\times)^m$. Therefore we have the following natural identifications:

$$\begin{aligned}\operatorname{Pic}(\mathfrak{X}) &\cong \operatorname{Hom}(\mathbb{G}, \mathbb{C}^\times) = \mathbb{L}^\vee,\\ \operatorname{Pic}^{\mathfrak{T}}(\mathfrak{X}) &\cong \operatorname{Hom}((\mathbb{C}^\times)^m, \mathbb{C}^\times) = (\mathbb{Z}^m)^\star.\end{aligned}$$

The natural map $\operatorname{Pic}^{\mathfrak{T}}(\mathfrak{X}) \to \operatorname{Pic}(\mathfrak{X})$ can be identified with the map $D\colon (\mathbb{Z}^m)^\star \to \mathbb{L}^\vee$ in the divisor sequence (2.2). Let $\mathbb{T} := (\mathbb{C}^\times)^m / \operatorname{Im}(\mathbb{G} \to (\mathbb{C}^\times)^m) \cong \mathbf{N} \otimes \mathbb{C}^\times$ be the coarse moduli space[2] of $\mathfrak{T}$. The torus $\mathbb{T}$ acts on the coarse moduli space X of $\mathfrak{X}$. Taking first Chern classes of (equivariant) line bundles, we obtain the following canonical identifications over $\mathbb{Q}$:

$$\begin{aligned}H^2(X, \mathbb{Q}) &\cong \mathbb{L}^\star_\mathbb{Q},\\ H^2_\mathbb{T}(X, \mathbb{Q}) &\cong (\mathbb{Q}^m)^\star,\\ H^2_\mathbb{T}(\mathrm{pt}, \mathbb{Q}) &\cong \mathbf{M}_\mathbb{Q}\end{aligned} \tag{2.3}$$

such that the divisor sequence (2.2) over $\mathbb{Q}$ is identified with

$$0 \longrightarrow H^2_\mathbb{T}(\mathrm{pt}, \mathbb{Q}) \longrightarrow H^2_\mathbb{T}(X, \mathbb{Q}) \longrightarrow H^2(X, \mathbb{Q}) \longrightarrow 0.$$

Via the identification (2.3), we regard $u_i = e_i^\star \in (\mathbb{Q}^m)^\star$, $D_i \in \mathbb{L}^\star_\mathbb{Q}$ as (equivariant or non-equivariant) cohomology classes. These are the (equivariant or non-equivariant) Poincaré duals of the toric divisor $[\{Z_i = 0\}/\mathbb{G}] \subset [\mathcal{U}_\Sigma/\mathbb{G}]$, where Z_i is the ith co-ordinate on $\mathbb{C}^m$.

2.3. Inertia stack, Box and orbifold cohomology. Recall that the *inertia stack* $I\mathfrak{X}$ is defined to be the fiber product $\mathfrak{X} \times_{\Delta, \mathfrak{X}\times\mathfrak{X}, \Delta} \mathfrak{X}$ of the diagonal morphisms $\Delta\colon \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$. A point of $I\mathfrak{X}$ is given by a pair (x, g) of a point $x \in \mathfrak{X}$ and an automorphism $g \in \operatorname{Aut}(x)$. There is a map $\operatorname{inv}\colon I\mathfrak{X} \to I\mathfrak{X}$ that sends (x, g) to (x, g^{-1}). Borisov, Chen and

[2] There exists an exact sequence of Picard stacks: $1 \to BN_{\mathrm{tor}} \to \mathfrak{T} \to \mathbb{T} \to 1$ and this sequence splits: $\mathfrak{T} \cong BN_{\mathrm{tor}} \times \mathbb{T}$; see [**29**, Proposition 2.5]. Thus the $\mathbb{T}$-action on X lifts to a $\mathbb{T}$-action on $\mathfrak{X}$.

Smith [5, Lemma 4.6] showed that connected components of the inertia stack $I\mathfrak{X}$ are indexed by the set Box:

$$\mathrm{Box} := \bigcup_{\sigma\in\Sigma} \mathrm{Box}(\sigma),$$

$$\mathrm{Box}(\sigma) := \left\{ v \in \mathbf{N} \;\middle|\; \overline{v} \in \sigma,\ \overline{v} = \sum_{i\in\sigma} c_i \overline{b}_i,\ c_i \in [0,1) \right\}.$$

Let $\mathfrak{X}_v$ denote the component of $I\mathfrak{X}$ corresponding to $v \in \mathrm{Box}$; then we have $I\mathfrak{X} = \bigsqcup_{v\in\mathrm{Box}} \mathfrak{X}_v$. The component $\mathfrak{X}_v$ is isomorphic to the closed toric substack of $\mathfrak{X}$ associated with the minimal cone $\sigma(v) \in \Sigma$ containing $\overline{v} \in \mathbf{N}_{\mathbb{R}}$. See the proof of Lemma 4.7 below for the description of the stabilizer $g_v \in \mathbb{G}$ along $\mathfrak{X}_v$.

Notation 2.2. We introduce a function $\Psi\colon \mathbf{N} \cap |\Sigma| \to (\mathbb{Q}_{\geq 0})^m$ by

$$\Psi(\mathbf{k}) = (\Psi_i(\mathbf{k}))_{1\leq i\leq m}, \qquad \Psi_i(\mathbf{k}) := \begin{cases} c_i & i \in \sigma; \\ 0 & i \in \{1,\dots,m\} \setminus \sigma, \end{cases}$$

where $\mathbf{k} \in \mathbf{N}\cap|\Sigma|$, $\sigma \in \Sigma$ is the minimal cone containing $\overline{\mathbf{k}}$ and we write $\overline{\mathbf{k}} = \sum_{i\in\sigma} c_i \overline{b}_i$. The age function $|\cdot|\colon \mathbf{N} \cap |\Sigma| \to \mathbb{Q}_{\geq 0}$ is defined to be $|\mathbf{k}| = \sum_{i=1}^m \Psi_i(\mathbf{k})$.

Let X_v denote the coarse moduli space of $\mathfrak{X}_v$. The *orbifold cohomology group* [11] of $\mathfrak{X}$ is defined to be

$$H^*_{\mathrm{CR}}(\mathfrak{X}) := \bigoplus_{v\in\mathrm{Box}} H^{*-2|v|}(X_v, \mathbb{C}).$$

The $\mathbb{T}$-*equivariant orbifold cohomology group* is defined similarly:

$$H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) := \bigoplus_{v\in\mathrm{Box}} H^{*-2|v|}_{\mathbb{T}}(X_v, \mathbb{C}).$$

Chen and Ruan [11] introduced a super-commutative product structure on orbifold cohomology, called the *Chen–Ruan cup product.* For toric stacks, the Chen–Ruan product is commutative. The orbifold cohomology ring of the toric stack $\mathfrak{X}$ has been computed by Borisov–Chen–Smith [5] in the complete case, Jiang–Tseng [55] in the semi-projective case, and by Liu [58] in the equivariant case. The $\mathbb{T}$-equivariant orbifold cohomology ring is:

$$H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \cong \bigoplus_{\mathbf{k}\in\mathbf{N}\cap|\Sigma|} \mathbb{C}\phi_{\mathbf{k}},$$

where the product structure is given by:

$$\phi_{\mathbf{k}_1}\cdot\phi_{\mathbf{k}_2} = \begin{cases} \phi_{\mathbf{k}_1+\mathbf{k}_2} & \text{if } \overline{\mathbf{k}}_1, \overline{\mathbf{k}}_2 \text{ lie in the same cone of } \Sigma; \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

and the $R_{\mathbb{T}} := H^*_{\mathbb{T}}(\mathrm{pt}, \mathbb{C}) = \mathrm{Sym}^*(\mathbf{M}_{\mathbb{C}})$-module structure is given by $\chi \mapsto \sum_{i=1}^m (\chi \cdot b_i)\phi_{b_i}$ for $\chi \in \mathbf{M}_{\mathbb{C}}$. The non-equivariant orbifold cohomology ring is the quotient of $H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})$ by the ideal generated by equivariant parameters $\chi \in \mathbf{M}_{\mathbb{C}}$, i.e.

$$H^*_{\mathrm{CR}}(\mathfrak{X}) \cong \frac{\bigoplus_{\mathbf{k}\in\mathbf{N}\cap|\Sigma|} \mathbb{C}\phi_{\mathbf{k}}}{\langle \sum_{i=1}^m (\chi \cdot b_i)\phi_{b_i} : \chi \in \mathbf{M}_{\mathbb{C}}\rangle}.$$

For a box element $v \in \mathrm{Box}$, ϕ_v represents the identity class $\mathbf{1}_v \in H^0_{\mathbb{T}}(X_v)$ supported on the component X_v. The element ϕ_{b_i}, $1 \le i \le m$, represents the class u_i (or D_i in the non-equivariant case) of a toric divisor; see §2.2. In particular, we have

$$\phi_{\mathbf{k}} = \left(\prod_{i=1}^m u_i^{\lfloor \Psi_i(\mathbf{k})\rfloor}\right)\mathbf{1}_v$$

for $v = \mathbf{k} - \sum_{i=1}^m \lfloor \Psi_i(\mathbf{k})\rfloor b_i \in \mathrm{Box}$.

Remark 2.3. Since the odd cohomology of X_v vanishes, the Serre spectral sequence for $X_v \times_{\mathbb{T}} E\mathbb{T} \to B\mathbb{T}$ degenerates over $\mathbb{Q}$ at the E_2 term, and we find that $H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})$ is a free module over $R_{\mathbb{T}} = H^*_{\mathbb{T}}(\mathrm{pt})$ of rank $\dim H^*_{\mathrm{CR}}(\mathfrak{X})$.

2.4. Refined fan sequence. We introduce an overlattice $\mathbf{\Lambda}$ of $\mathbb{L} \cong \mathrm{Hom}(\mathrm{Pic}(\mathfrak{X}), \mathbb{Z})$ which contains all curve classes in $\mathfrak{X}$. This overlattice $\mathbf{\Lambda}$ fits into a refined version of the fan sequence (2.1). Let $\mathbb{O}$ be the subgroup of $\mathbb{Q}^m \oplus \mathbf{N}$ given by

$$\mathbb{O} := \sum_{\mathbf{k}\in\mathbf{N}\cap|\Sigma|} \mathbb{Z}(\Psi(\mathbf{k}), \mathbf{k}).$$

Note that $\mathbb{O}$ is a subgroup of $\{(\lambda, \mathbf{k}) \in \mathbb{Q}^m \oplus \mathbf{N} : \beta(\lambda) = \overline{\mathbf{k}}\}$. We define:

$$\mathbf{\Lambda} := \{(\lambda, 0) \in \mathbb{O}\} \subset \mathbb{Q}^m.$$

The lattice $\mathbf{\Lambda}$ is a subgroup of $\mathbb{L}_{\mathbb{Q}} \subset \mathbb{Q}^m$, and contains $\mathbb{L}$. For $\mathbf{k}_1, \mathbf{k}_2 \in \mathbf{N} \cap |\Sigma|$, define an element $d(\mathbf{k}_1, \mathbf{k}_2) \in \mathbb{Q}^m$ by

$$d(\mathbf{k}_1, \mathbf{k}_2) := \Psi(\mathbf{k}_1) + \Psi(\mathbf{k}_2) - \Psi(\mathbf{k}_1 + \mathbf{k}_2). \tag{2.5}$$

It is easy to see that $d(\mathbf{k}_1, \mathbf{k}_2) \in \mathbf{\Lambda}$, and moreover that $\mathbf{\Lambda}$ is generated by these classes:

$$\mathbf{\Lambda} = \sum_{\mathbf{k}_1,\mathbf{k}_2\in\mathbf{N}\cap|\Sigma|} \mathbb{Z}d(\mathbf{k}_1, \mathbf{k}_2).$$

We obtain the following extension of the fan sequence (2.1):

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{L} \arrow[r] \arrow[d, hook] & \mathbb{Z}^m \arrow[r] \arrow[d, hook] & \mathbf{N} \arrow[d, equal] & \\ 0 \arrow[r] & \mathbf{\Lambda} \arrow[r] & \mathbb{O} \arrow[r] & \mathbf{N} \arrow[r] & 0 \end{tikzcd} \tag{2.6}$$

where the map $\mathbb{O} \to \mathbf{N}$ is projection to the second factor and the map $\mathbb{Z}^m \to \mathbb{O}$ is given by sending $e_i \in \mathbb{Z}^m$ to $(e_i, b_i) \in \mathbb{O}$. We call the exact sequence in the second row the *refined fan sequence*. Note that the torsion part $\mathbb{O}_{\mathrm{tor}}$ of $\mathbb{O}$ is isomorphic to $\mathbf{N}_{\mathrm{tor}}$ under projection, and hence the refined fan sequence splits.

Example 2.4. The fan sequence and the refined fan sequence for the toric stack $\mathfrak{X} = \mathbb{P}(1,1,2)$ are:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "{}^t(1\ 1\ 2)"] \arrow[d, hook] & \mathbb{Z}^3 \arrow[r, "(b_1\ b_2\ b_3)"] \arrow[d, hook] & \mathbb{Z}^2 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \frac{1}{2}\mathbb{Z} \arrow[r] & \mathbb{Z}^3 + \mathbb{Z}\begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} \arrow[r] & \mathbb{Z}^2 \arrow[r] & 0 \end{tikzcd}$$

where $\mathbf{N} = \mathbb{Z}^2$, $\mathbb{L} = \mathbb{Z}$, $\boldsymbol{\Lambda} = \frac{1}{2}\mathbb{Z}$, $b_1 = {}^t(1,0)$, $b_2 = {}^t(-1,2)$, $b_3 = {}^t(0,-1)$ and $\mathbb{O} = \mathbb{Z}^3 + \mathbb{Z}\,{}^t(1/2, 1/2, 0)$.

Example 2.5. The refined fan sequence for the toric stack $\mathfrak{X} = B\mu_2$ is

$$0 \longrightarrow 0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0,$$

where $\mathbf{N} = \mathbb{Z}/2\mathbb{Z}$, $\mathbb{O} = \mathbb{Z}/2\mathbb{Z}$ and $\boldsymbol{\Lambda} = 0$.

Remark 2.6. The set of degrees in $H_2(X, \mathbb{Q}) \cong \mathbb{L}_{\mathbb{Q}}$ of stable maps to $\mathfrak{X}$ is generated by representable toric morphisms $f \colon \mathbb{P}^1_{r_1,r_2} \to \mathfrak{X}$, where $\mathbb{P}^1_{r_1,r_2}$ denotes the one-dimensional toric stack with coarse moduli space equal to $\mathbb{P}^1$, isotropy groups μ_{r_1}, μ_{r_2} at 0, ∞ respectively, and no other isotropy groups. These toric morphisms are classified in [**15**, §3.5]. It is easy to see that the degrees $l(c, \sigma, j)$ of such toric morphisms given in [**15**, Definition 12, Remark 13] can be written as $d(\mathbf{k}_1, \mathbf{k}_2)$ for some $\mathbf{k}_1, \mathbf{k}_2 \in \mathbf{N} \cap |\Sigma|$ lying in two maximal cones meeting along a codimension one face. Therefore $\boldsymbol{\Lambda}$ contains all homology classes of stable maps. We will prove in Lemma 4.8(2) below that $\boldsymbol{\Lambda}$ is the dual lattice of the Picard group of the coarse moduli space. On the other hand, $\mathbb{O}$ should correspond to the group generated by classes in $H_2(X, L; \mathbb{Q}) \cong \mathbb{Q}^m$ of orbi-discs with boundaries in a Lagrangian torus orbit $L \subset X$. The notation $\mathbb{O}$ is intended to mean degrees of "open" curves.

2.5. Mori cone and associated monoids. For a cone $\sigma \in \Sigma$, we define the following cones:

$$\begin{aligned} \widetilde{C}_\sigma &:= \{\lambda \in \mathbb{R}^m : \beta(\lambda) \in \sigma, \lambda_i \geq 0 \text{ for } i \notin \sigma\}, \\ C_\sigma &:= \mathbb{L}_{\mathbb{R}} \cap \widetilde{C}_\sigma = \{\lambda \in \mathbb{L}_{\mathbb{R}} : D_i \cdot \lambda \geq 0 \text{ for } i \notin \sigma\}. \end{aligned} \tag{2.7}$$

Note that $D_i \cdot \lambda$ is the ith component of $\lambda \in \mathbb{L}_{\mathbb{R}}$ regarded as an element of $\mathbb{R}^m$. We also define

$$\mathrm{OE}(\mathfrak{X}) := \sum_{\sigma \in \Sigma} \widetilde{C}_\sigma \subset \mathbb{R}^m,$$
$$\mathrm{NE}(\mathfrak{X}) := \sum_{\sigma \in \Sigma} C_\sigma \subset \mathbb{L}_{\mathbb{R}}.$$

Under the identification $\mathbb{L}_{\mathbb{R}} \cong H_2(X, \mathbb{R})$ from (2.3), $\mathrm{NE}(\mathfrak{X})$ corresponds to the *Mori cone*, i.e. the cone generated by effective curves. The cone $\mathrm{OE}(\mathfrak{X})$ should be its open analogue. We define:

$$\mathbf{\Lambda}_+ := \mathbf{\Lambda} \cap \mathrm{NE}(\mathfrak{X}),$$
$$\mathbb{O}_+ := \{(\lambda, \mathbf{k}) \in \mathbb{O} : \lambda \in \mathrm{OE}(\mathfrak{X})\}.$$

Lemma 2.7. *Projection to the second factor defines a map $\mathbb{O}_+ \to \mathbf{N} \cap |\Sigma|$. The fiber of this map at $\mathbf{k} \in \mathbf{N} \cap |\Sigma|$ equals $(\Psi(\mathbf{k}), \mathbf{k}) + \mathbf{\Lambda}_+$. In particular, the fiber at $0 \in \mathbf{N} \cap |\Sigma|$, which is $\mathbf{\Lambda} \cap \mathrm{OE}(\mathfrak{X})$, equals $\mathbf{\Lambda}_+$.*

Proof. It is clear from the definition that the projection to the second factor of an element in $\mathbb{O}_+$ lies in $\mathbf{N} \cap |\Sigma|$. Also it is clear that $(\Psi(\mathbf{k}), \mathbf{k}) + \mathbf{\Lambda}_+$ is contained in the fiber at $\mathbf{k} \in \mathbf{N} \cap |\Sigma|$. Let $(\lambda, \mathbf{k})$ lie in $\mathbb{O}_+$. We have $(\lambda, \mathbf{k}) = (\lambda - \Psi(\mathbf{k}), 0) + (\Psi(\mathbf{k}), \mathbf{k})$. We want to show that $\lambda - \Psi(\mathbf{k}) \in \mathbf{\Lambda}_+$. It suffices to show that $\lambda - \Psi(\mathbf{k}) \in \mathrm{NE}(\mathfrak{X})$, since it is clear that $\lambda - \Psi(\mathbf{k})$ lies in $\mathbf{\Lambda}$. By assumption there exist $\lambda_\sigma \in \widetilde{C}_\sigma$ for $\sigma \in \Sigma$ such that $\lambda = \sum_{\sigma \in \Sigma} \lambda_\sigma$. We have that $\overline{\mathbf{k}}_\sigma = \beta(\lambda_\sigma) \in \sigma$ and $\overline{\mathbf{k}} = \beta(\lambda)$. Then

$$\lambda - \Psi(\mathbf{k}) = \sum_{\sigma \in \Sigma} (\lambda_\sigma - \Psi(\overline{\mathbf{k}}_\sigma)) + \sum_{\sigma \in \Sigma} \Psi(\overline{\mathbf{k}}_\sigma) - \Psi(\mathbf{k}).$$

Here $\lambda_\sigma - \Psi(\overline{\mathbf{k}}_\sigma)$ lies in C_σ and $\sum_{\sigma \in \Sigma} \Psi(\overline{\mathbf{k}}_\sigma) - \Psi(\mathbf{k})$ lies in C_τ for a cone τ containing $\overline{\mathbf{k}}$. It follows that $\lambda - \Psi(\mathbf{k})$ lies in $\sum_{\sigma \in \Sigma} C_\sigma = \mathrm{NE}(\mathfrak{X})$.

q.e.d.

Remark 2.8. The group ring $\mathbb{C}[\mathbb{O}_+]$ of $\mathbb{O}_+$ can be viewed as an equivariant and orbifold generalization of Batyrev's quantum ring [**4**, **34**]. See Theorem 5.13 below for its relation to quantum cohomology.

3. Toric Gromov–Witten theory

In this section, we review Gromov–Witten invariants, quantum cohomology, the quantum connection, and the Givental cone. Most of the arguments apply to semi-projective smooth Deligne–Mumford stacks equipped with $\mathbb{T}$-action; we restrict ourselves, however, to the toric Deligne–Mumford stacks $\mathfrak{X}$ from §2. We also recall the main result of our previous paper [**15**].

3.1. Gromov–Witten invariants. Gromov–Witten theory for symplectic orbifolds or smooth Deligne–Mumford stacks has been developed by Chen–Ruan [**10**] and Abramovich–Graber–Vistoli [**1**]. We refer the reader to [**1**, **10**, **75**] for a detailed discussion. For $l \geq 0$ and $d \in H_2(X, \mathbb{Z})$, we denote by $\mathfrak{X}_{0,l,d}$ the moduli stack of genus-zero twisted stable maps to $\mathfrak{X}$ of degree d (this is denoted by $\mathcal{K}_{0,l}(\mathfrak{X}, d)$ in [**1**]). Note that $\mathfrak{X}_{0,l,d}$ is empty if $l \leq 2$ and $d = 0$. There are evaluation maps $\mathrm{ev}_i \colon \mathfrak{X}_{0,l,d} \to \overline{I}\mathfrak{X}$, $i = 1, \ldots, l$, to the rigidified cyclotomic inertia stack $\overline{I}\mathfrak{X}$ [**1**, §3.4]. For cohomology classes $\alpha_1, \ldots, \alpha_l \in H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})$ and non-negative integers $k_1, \ldots, k_l$, *equivariant Gromov–Witten invariants* are defined to be the $\mathbb{T}$-equivariant integrals

$$\left\langle \alpha_1 \psi^{k_1}, \ldots, \alpha_l \psi^{k_l} \right\rangle_{0,l,d} = \int^{\mathbb{T}}_{[\mathfrak{X}_{0,l,d}]^{\mathrm{vir}}} \prod_{i=1}^{l} \psi_i^{k_i} \, \mathrm{ev}_i^\star(\alpha_i),$$

where $[\mathfrak{X}_{0,l,d}]^{\mathrm{vir}}$ is the (equivariant) virtual fundamental class [**1**, **75**] and ψ_i is the first Chern class of the ith universal cotangent line bundle over $\mathfrak{X}_{0,l,d}$. We note that:

- since the underlying complex analytic spaces of the rigidified inertia stack and the inertia stack are the same, we can pull back the cohomology classes α_i via ev_i;
- when $\mathfrak{X}$ is non-compact, we can define the right-hand side by the Atiyah–Bott-style virtual localization formula [**42**, **58**]. In this case the integral takes values in the fraction field $S_{\mathbb{T}} := \mathrm{Frac}(R_{\mathbb{T}})$ of $R_{\mathbb{T}} = H^*_{\mathbb{T}}(\mathrm{pt}, \mathbb{C})$.

A special case of Gromov–Witten invariants yields the *orbifold Poincaré pairing*: it is defined by

$$(\alpha, \beta) := \langle 1, \alpha, \beta \rangle_{0,3,0} = \int_{I\mathfrak{X}} \alpha \cup \mathrm{inv}^\star \beta,$$

where the map inv was defined in §2.3. The pairing takes values in $R_{\mathbb{T}}$ or in $S_{\mathbb{T}}$, depending on whether $\mathfrak{X}$ is compact or non-compact.

3.2. Quantum cohomology and the quantum connection. Recall the lattice $\mathbf{\Lambda} \subset H_2(X, \mathbb{Q})$ and the monoid $\mathbf{\Lambda}_+$ of curve classes from §§2.4–2.5. The quantum cohomology of $\mathfrak{X}$ is defined over the Novikov ring $\mathbb{C}[\![\mathbf{\Lambda}_+]\!]$, which is the completion of the group ring $\mathbb{C}[\mathbf{\Lambda}_+]$. We write $Q^d \in \mathbb{C}[\![\mathbf{\Lambda}_+]\!]$ for the element corresponding to $d \in \mathbf{\Lambda}_+$. The *big equivariant quantum product* $\star$ is a formal family of commutative ring structures parametrized by $\tau \in H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})$, defined by

$$(\alpha \star \beta, \gamma) = \sum_{l=0}^{\infty} \frac{1}{l!} \langle \alpha, \beta, \gamma, \tau, \ldots, \tau \rangle_{0,l+3,d} \, Q^d, \tag{3.1}$$

where $\alpha, \beta, \gamma \in H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})$. Since the orbifold Poincaré pairing is non-degenerate, this uniquely defines $\alpha \star \beta$. Choose a homogeneous basis

$\{T_i\}$ of $H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})$ over $R_{\mathbb{T}}$ and write $\tau = \sum_i \tau^i T_i$. We regard $\{\tau^i\}$ as co-ordinates on $H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})$. For a ring K, we write $K[\![\tau]\!] = K[\![\{\tau^i\}]\!]$ for the ring of formal power series in $\{\tau^i\}$. The big equivariant quantum product defines a commutative ring structure on

$$H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \otimes_{R_{\mathbb{T}}} R_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![\tau]\!].$$

Note that we do not need the fraction field $S_{\mathbb{T}}$ even when $\mathfrak{X}$ is non-compact. This is because the evaluation map $\mathrm{ev}_i \colon \mathfrak{X}_{0,l,d} \to \overline{I}\mathfrak{X}$ is always proper, and we can define the quantum product in terms of the push-forward along the ev_3. The properness of ev_i is ensured by the fact that $\mathfrak{X}$ is semi-projective. In particular, the non-equivariant quantum product is defined on

$$H^*_{\mathrm{CR}}(\mathfrak{X}) \otimes \mathbb{C}[\![\mathbf{\Lambda}_+]\!][\![\tau]\!]$$

as the non-equivariant limit. We also note that the Chen–Ruan cup product is the limit of the quantum product as $Q \to 0$ and $\tau \to 0$.

The quantum connection is a pencil of flat connections with pencil parameter z. For $\xi \in \mathbb{L}^\star_{\mathbb{C}} \cong H^2(X,\mathbb{C})$ and a ring K, we write $\xi Q\frac{\partial}{\partial Q}$ for the derivation of $K[\![\mathbf{\Lambda}_+]\!]$ such that $\xi Q\frac{\partial}{\partial Q} \cdot Q^d = (\xi \cdot d)Q^d$. We also fix a splitting $\mathbb{L}^\star_{\mathbb{C}} \to (\mathbb{C}^m)^\star \cong H^2_{\mathbb{T}}(X,\mathbb{C})$ (over $\mathbb{C}$) of the composition of the divisor sequence (2.2) with projection to the free part $\mathbb{L}^\vee \to \mathbb{L}^\star$, and write $\hat{\xi} \in H^2_{\mathbb{T}}(X,\mathbb{C})$ for the lift of $\xi \in \mathbb{L}^\star_{\mathbb{C}}$ with respect to the splitting. The quantum connection is defined by

$$\begin{aligned} \nabla_{\xi Q \frac{\partial}{\partial Q}} &= \xi Q \frac{\partial}{\partial Q} + \frac{1}{z}(\hat{\xi}\star), \\ \nabla_{\frac{\partial}{\partial \tau^i}} &= \frac{\partial}{\partial \tau^i} + \frac{1}{z}(T_i \star) \end{aligned} \tag{3.2}$$

for $\xi \in H^2(X,\mathbb{C})$. These operators define maps

$$H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \otimes_{R_{\mathbb{T}}} R_{\mathbb{T}}[z][\![\mathbf{\Lambda}_+]\!][\![\tau]\!] \to z^{-1} H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \otimes_{R_{\mathbb{T}}} R_{\mathbb{T}}[z][\![\mathbf{\Lambda}_+]\!][\![\tau]\!].$$

The quantum connection is flat, i.e. $[\nabla_{\vec{v}}, \nabla_{\vec{w}}] = \nabla_{[\vec{v},\vec{w}]}$. Quantum cohomology has a grading structure. Define the *Euler vector field* by

$$\mathcal{E}^{\mathrm{A}} = c_1(\mathfrak{X}) Q \frac{\partial}{\partial Q} + \sum_i \left(1 - \tfrac{1}{2}\deg T_i\right) \tau^i \frac{\partial}{\partial \tau^i} + \sum_{i=1}^n \chi_i \frac{\partial}{\partial \chi_i}, \tag{3.3}$$

where $c_1(\mathfrak{X}) = D_1 + \cdots + D_m \in \mathbb{L}^\star$ is the first Chern class of $T\mathfrak{X}$, $\deg T_i$ means the age-shifted degree of T_i, and $(\chi_1, \ldots, \chi_n)$ is a basis of $\mathbf{M}_{\mathbb{Q}} = H^2_{\mathbb{T}}(\mathrm{pt},\mathbb{Q})$ (so that $R_{\mathbb{T}} = \mathbb{C}[\chi_1,\ldots,\chi_n]$). $\mathcal{E}^{\mathrm{A}}$ is a derivation of $R_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![\tau]\!]$. We define $\mathrm{Gr}_0 \in \mathrm{End}_{\mathbb{C}}(H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}))$ by

$$\mathrm{Gr}_0(\alpha) = \frac{1}{2}(\deg \alpha)\alpha \tag{3.4}$$

for a homogeneous element $\alpha \in H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})$. We note that Gr_0 is not linear over $H^*_{\mathbb{T}}(\mathrm{pt},\mathbb{C}) = R_{\mathbb{T}}$. The grading on $H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \otimes_{R_{\mathbb{T}}} R_{\mathbb{T}}[z][\![\boldsymbol{\Lambda}_+]\!][\![\tau]\!]$ is defined by the operator

$$\mathrm{Gr}^{\mathrm{A}} := z\frac{\partial}{\partial z} + \mathcal{E}^{\mathrm{A}} + \mathrm{Gr}_0,$$

where $z\frac{\partial}{\partial z} + \mathcal{E}^{\mathrm{A}}$ acts on the coefficient ring $R_{\mathbb{T}}[z][\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$ and Gr_0 acts on $H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})$. That is, we have

$$\mathrm{Gr}^{\mathrm{A}}(c\alpha) = \left(\left(z\frac{\partial}{\partial z} + \mathcal{E}^{\mathrm{A}}\right) c\right) \alpha + c\, \mathrm{Gr}_0\, \alpha \tag{3.5}$$

for $c \in R_{\mathbb{T}}[z][\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$ and $\alpha \in H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})$. The grading structure is compatible with the quantum connection in the sense that

$$\left[\nabla_{\xi Q\frac{\partial}{\partial Q}}, \mathrm{Gr}^{\mathrm{A}}\right] = 0, \quad \left[\nabla_{\frac{\partial}{\partial \tau^i}}, \mathrm{Gr}^{\mathrm{A}}\right] = (1 - \tfrac{1}{2}\deg T_i)\nabla_{\frac{\partial}{\partial \tau^i}}.$$

There is a canonical fundamental solution for the quantum connection. Define $M(\tau, z) \in \mathrm{End}_{R_{\mathbb{T}}}(H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})) \otimes R_{\mathbb{T}}(\!(z^{-1})\!)[\![\boldsymbol{\Lambda}_+]\!][\![\tau]\!]$ by

$$M(\tau, z)\alpha = \alpha + \sum_{d \in \boldsymbol{\Lambda}_+} \sum_{l=0}^{\infty} \sum_i \frac{Q^d}{l!} \left\langle \alpha, \tau, \ldots, \tau, \frac{T_i}{z - \psi} \right\rangle_{0,l+2,d} T^i,$$

where $\{T^i\}$ is the basis of $H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \otimes_{R_{\mathbb{T}}} S_{\mathbb{T}}$ dual to $\{T_i\}$ with respect to the orbifold Poincaré pairing, so that $(T_i, T^j) = \delta_i^j$.

Proposition 3.1. *The fundamental solution $M(\tau, z)$ satisfies the following differential equations:*

$$\begin{aligned} M(\tau, z)\nabla_{\frac{\partial}{\partial \tau^i}}\alpha &= \frac{\partial}{\partial \tau^i} M(\tau, z)\alpha, \\ M(\tau, z)\nabla_{\xi Q\frac{\partial}{\partial Q}}\alpha &= \left(\xi Q\frac{\partial}{\partial Q} + z^{-1}\hat{\xi}\right) M(\tau, z)\alpha, \\ M(\tau, z)(\mathrm{Gr}^{\mathrm{A}}\,\alpha) &= \mathrm{Gr}^{\mathrm{A}}(M(\tau, z)\alpha) \end{aligned}$$

for $\alpha \in H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \otimes_{R_{\mathbb{T}}} R_{\mathbb{T}}[z][\![\boldsymbol{\Lambda}_+]\!][\![\tau]\!]$. Moreover, $M(\tau, z)$ preserves the Poincaré pairing in the sense that $(M(\tau, -z)\alpha, M(\tau, z)\beta) = (\alpha, \beta)$ for all α, $\beta \in H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})$.

Proof. These properties are well-known; see [**35**, Corollary 6.7], [**65**, Proposition 2], [**36**, §1], [**19**, Proposition 2.4]. The first equation follows from the topological recursion relations as explained in [**65**], the second equation follows by combining the first one with the divisor equation (see e.g. [**19**, §2.6]), and the last one follows from the degree axiom for Gromov–Witten invariants. That $M(\tau, z)$ preserves the Poincaré pairing is shown in [**36**, §1], [**19**, Proposition 2.4]. q.e.d.

We define the J-function to be

$$\begin{aligned} J(\tau, z) &= zM(\tau, z)\mathbf{1} \\ &= z\mathbf{1} + \tau + \sum_{\substack{d \in \mathbf{\Lambda}_+, l \geq 0 \\ (d,l) \neq (0,0)}} \frac{Q^d}{l!} \left\langle \tau, \dots, \tau, \frac{T^i}{z - \psi} \right\rangle_{0,l+1,d} T_i, \end{aligned} \tag{3.6}$$

where $\mathbf{1}$ is the identity class supported on the non-twisted sector $\mathfrak{X} \subset I\mathfrak{X}$.

Remark 3.2. In the non-equivariant theory, we can introduce the connection in the z-direction by the formula:

$$\nabla_{z\frac{\partial}{\partial z}} = \mathrm{Gr} - \nabla_{\mathcal{E}^{\mathrm{A}}} - \frac{n}{2}. \tag{3.7}$$

3.3. Galois symmetry. We introduce the Galois symmetry of the equivariant quantum connection. This is an adaptation of [**50**, Proposition 2.3] to our setting. The *age* of a line bundle $L \to \mathfrak{X}$ along the twisted sector $\mathfrak{X}_v \subset I\mathfrak{X}$, $v \in \mathrm{Box}$, is defined to be the rational number $f = \mathrm{age}_v(L) \in [0, 1)$ such that the stabilizer g_v along $\mathfrak{X}_v$ acts on fibers of $L|_{\mathfrak{X}_v}$ by $\exp(2\pi\sqrt{-1}f)$. Recall from §2.2 that $\mathrm{Pic}(\mathfrak{X}) \cong \mathbb{L}^\vee$. For $\xi \in \mathbb{L}^\vee$, we write L_ξ for the line bundle corresponding to ξ. We define a linear map $g_0(\xi)\colon H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \to H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})$ by

$$g_0(\xi)\left(\bigoplus_{v \in \mathrm{Box}} \tau_v\right) = \bigoplus_{v \in \mathrm{Box}} e^{2\pi\sqrt{-1}\,\mathrm{age}_v(L_\xi)}\tau_v$$

with $\tau_v \in H^*_{\mathbb{T}}(X_v, \mathbb{C})$. This map preserves the orbifold Poincaré pairing. Let $g(\xi)^*\colon \mathbb{C}[\![\mathbf{\Lambda}_+]\!][\![\tau]\!] \to \mathbb{C}[\![\mathbf{\Lambda}_+]\!][\![\tau]\!]$ denote the action on the variables (Q, τ) given as the pull-back of the cohomology parameter τ by $g_0(\xi)$ and

$$g(\xi)^* Q^d = e^{-2\pi\sqrt{-1}\xi\cdot d} Q^d$$

with $d \in \mathbf{\Lambda}_+$. This defines a morphism $g(\xi)\colon \mathrm{Spf}\,\mathbb{C}[\![\mathbf{\Lambda}_+]\!][\![\tau]\!] \to \mathrm{Spf}\,\mathbb{C}[\![\mathbf{\Lambda}_+]\!][\![\tau]\!]$ which induces a map $g_0(\xi)$ on cohomology at $Q = 0$. We call the maps $g_0(\xi)$, $g(\xi)^*$ the *Galois action* of $\xi \in \mathbb{L}^\vee$. Note that the Galois action descends to the action of the "stacky" Picard group (see Lemma 4.8(2))

$$\mathrm{Pic}^{\mathrm{st}}(\mathfrak{X}) := \mathrm{Pic}(\mathfrak{X}) / \mathrm{Pic}(X).$$

The following proposition can be proved using an argument similar to [**50**, Proposition 2.3].

Proposition 3.3. *The quantum connection is equivariant under the Galois action in the sense that the map* $g_0(\xi)\colon H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \to H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})$

intertwines the quantum connection ∇ with $g(\xi)^*\nabla$ for $\xi \in \mathbb{L}^\vee$:

$$g_0(\xi) \circ \nabla_{\eta Q \frac{\partial}{\partial Q}} \circ g_0(\xi)^{-1} = (g(\xi)^*\nabla)_{\eta Q \frac{\partial}{\partial Q}} = \eta Q \frac{\partial}{\partial Q} + \frac{1}{z} g(\xi)^*(\hat{\eta}\star),$$
$$g_0(\xi) \circ \nabla_{\frac{\partial}{\partial \tau^i}} \circ g_0(\xi)^{-1} = (g(\xi)^*\nabla)_{\frac{\partial}{\partial \tau^i}} = \frac{\partial}{\partial \tau^i} + \frac{1}{z} g(\xi)^*((g_0(\xi)T_i)\star),$$

where $\eta \in \mathbb{L}^\star_{\mathbb{C}}$. Moreover, the fundamental solution $M(\tau, z)$ in Proposition 3.1 satisfies $g(\xi)^*M(\tau,z) = g_0(\xi)M(\tau,z)g_0(\xi)^{-1}$.

3.4. Givental cone. Givental's symplectic vector space [**15**, **38**, **39**] in equivariant Gromov–Witten theory is

$$\mathcal{H} = H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \otimes_{R_\mathbb{T}} S_\mathbb{T}(\!(z^{-1})\!)[\![\boldsymbol{\Lambda}_+]\!]$$

equipped with the symplectic form:

$$\Omega(f,g) = -\operatorname{Res}_{z=\infty}(f(-z), g(z))dz.$$

The space $\mathcal{H}$ has a standard polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where

$$\begin{aligned}\mathcal{H}_+ &= H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \otimes S_\mathbb{T}[z][\![\boldsymbol{\Lambda}_+]\!],\\ \mathcal{H}_- &= z^{-1}H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \otimes S_\mathbb{T}[\![z^{-1}]\!][\![\boldsymbol{\Lambda}_+]\!]\end{aligned}$$

are Ω-isotropic subspaces. We can identify $\mathcal{H}$ with the contangent bundle $T^\star\mathcal{H}_+$ via this polarization. The equivariant Givental cone $\mathcal{L}_\mathfrak{X} \subset \mathcal{H}$ is defined as the graph of the differential of the genus-zero descendant potential, defined in the formal neighbourhood of $-z\mathbf{1} \in \mathcal{H}_+$ [**15**, **39**]. Equivalently, we can describe it as the set of points in $\mathcal{H}$ of the form:

$$-z\mathbf{1} + \mathbf{t}(z) + \sum_{l=0}^{\infty} \sum_{d \in \boldsymbol{\Lambda}_+} \sum_i \frac{Q^d}{l!} \left\langle \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi), \frac{T_i}{-z-\psi} \right\rangle_{0,l+1,d} T^i, \tag{3.8}$$

where $\{T_i\}$, $\{T^i\}$ are the mutually dual bases of $H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \otimes_{R_\mathbb{T}} S_\mathbb{T}$ with respect to the orbifold Poincaré pairing, i.e. $(T_i, T^j) = \delta_i^j$, and $\mathbf{t}(z) \in \mathcal{H}_+$. For example, the J-function $J(\tau, -z)$ (3.6) is a point on $\mathcal{L}_\mathfrak{X}$. A more precise definition of the notion of points on $\mathcal{L}_\mathfrak{X}$ is as follows. Let $t = (t_1, \ldots, t_N)$ be an arbitrary set of formal variables. An $S_\mathbb{T}[\![\boldsymbol{\Lambda}_+]\!][\![t]\!]$-*valued point* on $\mathcal{L}_\mathfrak{X}$ is a point of the form (3.8) with

$$\mathbf{t}(z) \in \mathcal{H}_+[\![t]\!], \quad \mathbf{t}(z)|_{Q=0,t=0} = 0.$$

The Givental cone is a cone – i.e. it is invariant under dilation in $\mathcal{H}$, – and has very special geometric properties, which are sometimes referred to as being "over-ruled". We refer the reader to [**39**] or [**13**, Appendix B] for details. In this paper, we need the following fact.

Proposition 3.4 ([**39**], [**13**, Proposition B.4])**.** *The tangent space of $\mathcal{L}_\mathfrak{X}$ at an $S_\mathbb{T}[\![\boldsymbol{\Lambda}_+]\!][\![t]\!]$-valued point is spanned over $S_\mathbb{T}[z][\![\boldsymbol{\Lambda}_+]\!][\![t]\!]$ by the* *derivatives of the J-function:*

$$\partial_{\tau^i} J(\tau, -z) = M(\tau, -z) T_i$$

for some $\tau \in H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \otimes_{R_\mathbb{T}} S_\mathbb{T}[\![\boldsymbol{\Lambda}_+]\!][\![t]\!]$ *with* $\tau|_{Q=t=0} = 0$.

3.5. Mirror theorem. We next review the mirror theorem from [**15**]. We fix a finite subset $G \subset \mathbf{N} \cap |\Sigma|$ in this section.

Definition 3.5. Let $\mathbb{K}_0^G$ denote the set of $\lambda = (\lambda_i, \lambda_\mathbf{k} : 1 \le i \le m, \mathbf{k} \in G) \in \mathbb{Q}^m \times \mathbb{Z}^G$ such that $\sum_{i=1}^m \lambda_i \bar{b}_i + \sum_{\mathbf{k} \in G} \lambda_\mathbf{k} \overline{\mathbf{k}} = 0$ and that $\{1 \le i \le m : \lambda_i \notin \mathbb{Z}\} \in \Sigma$. For $\lambda \in \mathbb{K}_0^G$, we define

$$v(\lambda) := \sum_{i=1}^m \lceil \lambda_i \rceil b_i + \sum_{\mathbf{k} \in G} \lambda_\mathbf{k} \mathbf{k}.$$

Since $\overline{v(\lambda)} = \sum_{i=1}^m \langle -\lambda_i \rangle b_i$, $v(\lambda)$ belongs to Box. We also set

$$d(\lambda) := (\lambda_1, \dots, \lambda_m) + \sum_{\mathbf{k} \in G} \lambda_\mathbf{k} \Psi(\mathbf{k}) \in \mathbb{L}_\mathbb{Q} \subset \mathbb{Q}^m,$$

where Ψ is given in Notation 2.2.

Lemma 3.6. *For $\lambda \in \mathbb{K}_0^G$, $d(\lambda) \in \boldsymbol{\Lambda}$ (see §2.4 for $\boldsymbol{\Lambda}$).*

Proof. Using $\Psi(\overline{v(\lambda)}) = (\langle -\lambda_1 \rangle, \dots, \langle -\lambda_m \rangle)$, we deduce

$$d(\lambda) := (\lceil \lambda_1 \rceil, \dots, \lceil \lambda_m \rceil) + \sum_{\mathbf{k} \in G} \lambda_\mathbf{k} \Psi(\mathbf{k}) - \Psi(v(\lambda)).$$

We also have $\sum_{i=1}^m \lceil \lambda_i \rceil b_i + \sum_{\mathbf{k} \in G} \lambda_\mathbf{k} \mathbf{k} - v(\lambda) = 0$. Consequently, we have

$$(d(\lambda), 0) = \sum_{i=1}^m \lceil \lambda_i \rceil (\Psi(b_i), b_i) + \sum_{\mathbf{k} \in G} \lambda_\mathbf{k} (\Psi(\mathbf{k}), \mathbf{k}) - (\Psi(v(\lambda)), v(\lambda))$$

in $\mathbb{Q}^m \times \mathbf{N}$. The right-hand side belongs to $\mathbb{O}$, and thus $d(\lambda) \in \boldsymbol{\Lambda}$.

q.e.d.

Definition 3.7 ([**15**])**.** Let $G \subset \mathbf{N} \cap |\Sigma|$ be a finite subset. The *G-extended I-function* is the cohomology-valued power series

$$I(Q, \mathfrak{y}, t, z) = z e^{\sum_{i=1}^m t_i u_i / z} \times \sum_{\lambda \in \mathbb{K}_0^G} Q^{d(\lambda)} e^{t \cdot d(\lambda)} \mathfrak{y}^\lambda \left(\prod_{i \in \{1, \dots, m\} \cup G} \frac{\prod_{c \le 0, \langle c \rangle = \langle \lambda_i \rangle} u_i + cz}{\prod_{c \le \lambda_i, \langle c \rangle = \langle \lambda_i \rangle} u_i + cz} \right) \mathbf{1}_{v(\lambda)},$$

where

- $t = (t_1, \dots, t_m)$ and $\mathfrak{y} = (\mathfrak{y}_\mathbf{k} : \mathbf{k} \in G)$ are parameters;
- $t \cdot d(\lambda) := \sum_{i=1}^m t_i (D_i \cdot d(\lambda))$ and $\mathfrak{y}^\lambda := \prod_{\mathbf{k} \in G} \mathfrak{y}_\mathbf{k}^{\lambda_\mathbf{k}}$;
- for $1 \le i \le m$, u_i is the equivariant Poincaré dual of a toric divisor in §2.2, and for $i \in G$ we set $u_i := 0$;
- $\mathbf{1}_{v(\lambda)}$ is the identity class supported on the twisted sector $\mathfrak{X}_{v(\lambda)}$.

The G-extended I-function belongs to $H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})\otimes_{R_{\mathbb{T}}} R_{\mathbb{T}}(\!(z^{-1})\!)[\![\mathbf{\Lambda}_+]\!][\![t,\mathfrak{y}]\!]$ – see Lemma 4.18 for a proof of this in a more general setting.

Theorem 3.8 ([**15**, Theorem 31])**.** *The G-extended I-function $I(Q,\mathfrak{y},t,z)$ is an $S_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![\mathfrak{y},t]\!]$-valued point on the equivariant Givental cone $\mathcal{L}_{\mathfrak{X}}$ of the toric Deligne–Mumford stack $\mathfrak{X}$.*

4. The Gauss–Manin system and the mirror isomorphism

In this section, we introduce a (partially compactified) Landau–Ginzburg model that corresponds under mirror symmetry to the toric stack $\mathfrak{X}$, and show that the quantum connection for $\mathfrak{X}$ is isomorphic to the Gauss–Manin system associated with the Landau–Ginzburg potential. The construction closely follows the one in [**52**] for toric manifolds.

4.1. Landau–Ginzburg model. Recall the refined fan sequence (2.6). Applying the exact functor $\mathrm{Hom}(-,\mathbb{C}^\times)$ to it, we obtain

$$1 \longrightarrow \mathrm{Hom}(\mathbf{N},\mathbb{C}^\times) \longrightarrow \mathrm{Hom}(\mathbb{O},\mathbb{C}^\times) \longrightarrow \mathrm{Hom}(\mathbf{\Lambda},\mathbb{C}^\times) \longrightarrow 1.$$

Note that $\mathrm{Hom}(\mathbf{N},\mathbb{C}^\times)$ is a disjoint union of $|\mathbf{N}_{\mathrm{tor}}|$ copies of the algebraic torus $(\mathbb{C}^\times)^n$. The *uncompactified Landau–Ginzburg model* is given by the smooth family of algebraic varieties $\mathrm{Hom}(\mathbb{O},\mathbb{C}^\times)\to\mathrm{Hom}(\mathbf{\Lambda},\mathbb{C}^\times)$ equipped with the Landau–Ginzburg potential $f\colon \mathrm{Hom}(\mathbb{O},\mathbb{C}^\times)\to\mathbb{C}$:

$$f = w_1+\cdots+w_m,$$

where $w_i\in\mathbb{C}[\mathbb{O}]$ is a function on $\mathrm{Hom}(\mathbb{O},\mathbb{C}^\times)$ given by the evaluation at $(e_i,b_i)\in\mathbb{O}$.

Next we introduce a partial compactification of the above construction using the cones and monoids from §2.5. The *partially compactified Landau–Ginzburg model* is given by the flat family of algebraic varieties:

$$\operatorname{Spec}\mathbb{C}[\mathbb{O}_+]\to\operatorname{Spec}\mathbb{C}[\mathbf{\Lambda}_+]$$

induced by the inclusion $\mathbf{\Lambda}_+\to\mathbb{O}_+$ of monoids, equipped with the potential function $f\colon \operatorname{Spec}\mathbb{C}[\mathbb{O}_+]\to\mathbb{C}$ as above (since $(e_i,b_i)\in\mathbb{O}_+$, w_i extends to a function on $\operatorname{Spec}\mathbb{C}[\mathbb{O}_+]$). When we refer to the Landau–Ginzburg model, we will mean the partially compactified one unless otherwise stated.

We introduce a co-ordinate system on the Landau–Ginzburg model. We write $w^{(\lambda,\mathbf{k})}\in\mathbb{C}[\mathbb{O}_+]$ for the element corresponding to $(\lambda,\mathbf{k})\in\mathbb{O}_+$. Define functions w_i, $w_{\mathbf{k}}$ for $\mathbf{k}\in\mathbf{N}\cap|\Sigma|$, and $Q^\lambda\in\mathbb{C}[\mathbb{O}_+]$ for $\lambda\in\mathbf{\Lambda}_+$ by

$$w_i := w^{(e_i,b_i)},\quad w_{\mathbf{k}} := w^{(\Psi(\mathbf{k}),\mathbf{k})},\quad Q^\lambda := w^{(\lambda,0)}.$$

See Notation 2.2 for Ψ. We have that $w_i = w_{b_i}$ and $w_{\mathbf{k}} = (\prod_{i=1}^m w_i^{\lfloor\Psi_i(\mathbf{k})\rfloor})w_v$ for $v=\mathbf{k}-\sum_{i=1}^m\lfloor\Psi_i(\mathbf{k})\rfloor b_i\in\mathrm{Box}$. By Lemma 2.7, we have

$$\mathbb{C}[\mathbb{O}_+] = \bigoplus_{\mathbf{k}\in\mathbf{N}\cap|\Sigma|}\mathbb{C}[\mathbf{\Lambda}_+]w_{\mathbf{k}}. \tag{4.1}$$

We choose a splitting $\varsigma\colon \mathbf{N} \to \mathbb{O}$ of the refined fan sequence (2.6). Using the splitting ς, we let $x^{\mathbf{k}} \in \mathbb{C}[\mathbb{O}]$ with $\mathbf{k} \in \mathbf{N} \cap |\Sigma|$ denote the element corresponding to $\varsigma(\mathbf{k}) \in \mathbb{O}$. (Note that $\varsigma(\mathbf{k})$ may not lie in $\mathbb{O}_+$.) Then we have:

$$w_{\mathbf{k}} = Q^{\lambda(\mathbf{k})} x^{\mathbf{k}}, \qquad w_i = Q^{\lambda(b_i)} x^{b_i}$$

with $\mathbf{k} \in \mathbf{N} \cap |\Sigma|$, $\lambda(\mathbf{k}) := (\Psi(\mathbf{k}), \mathbf{k}) - \varsigma(\mathbf{k}) \in \mathbf{\Lambda}$. Finally, by choosing an isomorphism $\mathbf{N} \cong \mathbb{Z}^n \times \mathbf{N}_{\mathrm{tor}}$, we write $x^{\mathbf{k}} = x_1^{k_1} \cdots x_n^{k_n} x^{\zeta}$ for $\mathbf{k} = (k_1, \ldots, k_n, \zeta) \in \mathbf{N}$.

Remark 4.1. For a maximal cone $\sigma_0 \in \Sigma$, we can define a splitting $\varsigma\colon \mathbf{N} \to \mathbb{O}$ by the requirement that $\varsigma(b_i) = (e_i, b_i)$ for all $i \in \sigma_0$. For this choice of ς, $\lambda(\mathbf{k}) = (\Psi(\mathbf{k}), \mathbf{k}) - \varsigma(\mathbf{k})$ lies in $\mathbf{\Lambda}_+$.

With this choice of co-ordinates, we define the *equivariant Landau–Ginzburg potential* to be the multi-valued function on $\operatorname{Spec} \mathbb{C}[\mathbb{O}_+]$:

$$f_\chi = w_1 + \cdots + w_m - \sum_{i=1}^{n} \chi_i \log x_i,$$

where $\chi = (\chi_1, \ldots, \chi_n) \in \mathrm{Lie}(\mathbb{T}) = \mathbf{N}_{\mathbb{C}} \cong \mathbb{C}^n$ are equivariant parameters. (We can also view χ_i, $1 \le i \le n$, as a basis of $\mathbf{M} = \mathbf{N}^\star$.)

Remark 4.2. The equivariant potential f_χ depends on the choice of splittings ς and $\mathbf{N}/\mathbf{N}_{\mathrm{tor}} \to \mathbf{N}$. If we choose different splittings then the equivariant potential is shifted by a term of the form $\sum_{i=1}^n \chi_i (\log Q^{d_i} + \log w^{(0,\zeta_i)})$ for some $d_i \in \mathbf{\Lambda}$, $\zeta_i \in \mathbf{N}_{\mathrm{tor}}$.

Example 4.3. Recall from Example 2.4 that the fan sequence and refined fan sequence for the toric stack $\mathfrak{X} = \mathbb{P}(1,1,2)$ are:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "{}^t(1\ 1\ 2)"] \arrow[d, hook] & \mathbb{Z}^3 \arrow[r, "(b_1\ b_2\ b_3)"] \arrow[d, hook] & \mathbb{Z}^2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \frac{1}{2}\mathbb{Z} \arrow[r] & \mathbb{Z}^3 + \mathbb{Z}\begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} \arrow[r] & \mathbb{Z}^2 \arrow[r] & 0
\end{tikzcd}$$

where $\mathbf{N} = \mathbb{Z}^2$, $\mathbb{L} = \mathbb{Z}$, $\mathbf{\Lambda} = \frac{1}{2}\mathbb{Z}$, $b_1 = {}^t(1,0)$, $b_2 = {}^t(-1,2)$, $b_3 = {}^t(0,-1)$ and $\mathbb{O} = \mathbb{Z}^3 + \mathbb{Z}\,{}^t(1/2, 1/2, 0)$. When we construct a mirror Landau–Ginzburg model using the (unrefined) fan sequence as considered in [**32**, **50**], we obtain a family

$$\mathbb{C}^3 = \operatorname{Spec} \mathbb{C}[\mathbb{Z}^3_{\ge 0}] \to \mathbb{C} = \operatorname{Spec} \mathbb{C}[\mathbb{L}_+], \qquad (w_1, w_2, w_3) \mapsto Q = w_1 w_2 w_3^2$$

equipped with the potential $f = w_1 + w_2 + w_3$, where we set $\mathbb{L}_+ := \mathbb{L} \cap \mathrm{NE}(\mathfrak{X}) \cong \mathbb{Z}_{\ge 0}$. When we pull back this family via the map $\operatorname{Spec} \mathbb{C}[\mathbf{\Lambda}_+] \to \operatorname{Spec} \mathbb{C}[\mathbb{L}_+]$, $t \mapsto Q = t^2$, we obtain the family

$$\{(w_1, w_2, w_3, t) \in \mathbb{C}^4 : w_1 w_2 w_3^2 = t^2\} \to \mathbb{C}, \quad (w_1, w_2, w_3, t) \mapsto t$$

with non-normal total space. The Landau–Ginzburg model (based on the refined fan sequence) is given by the normalization of this: it is

$$\{(w_1, w_2, w_3, u) \in \mathbb{C}^4 : w_1 w_2 = u^2\} \to \mathbb{C}, \quad (w_1, w_2, w_3, u) \mapsto u w_3,$$

where $t = u w_3$. This is the same as the family constructed by de Gregorio–Mann [**22**].

Example 4.4. Recall from Example 2.5 that the refined fan sequence for $\mathfrak{X} = B\mu_2$ is

$$0 \longrightarrow 0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0,$$

where $\mathbf{N} = \mathbb{Z}/2\mathbb{Z}$, $\mathbb{O} = \mathbb{Z}/2\mathbb{Z}$ and $\mathbf{\Lambda} = 0$. Thus the Landau–Ginzburg model is the identity map from the two-point set to itself, equipped with the zero potential.

4.2. An unfolding of the Landau–Ginzburg potential. We consider an unfolding of the Landau–Ginzburg potential given by choosing a finite set $G \subset \mathbf{N} \cap |\Sigma|$. We assume that G is disjoint from $\{b_1, \ldots, b_m\}$ and set $S := \{b_1, \ldots, b_m\} \cup G$. Introduce co-ordinates $y_{\mathbf{k}}$ for each $\mathbf{k} \in S$ and set $y = \{y_{\mathbf{k}} : \mathbf{k} \in S\}$; we sometimes write $y_i = y_{b_i}$ for $1 \le i \le m$. Define

$$\begin{aligned} F(x; y) &:= \sum_{\mathbf{k} \in S} y_{\mathbf{k}} w_{\mathbf{k}} = \sum_{i=1}^{m} y_i w_i + \sum_{\mathbf{k} \in G} y_{\mathbf{k}} w_{\mathbf{k}} = \sum_{\mathbf{k} \in S} y_{\mathbf{k}} Q^{\lambda(\mathbf{k})} x^{\mathbf{k}}, \\ F_\chi(x; y) &:= F(x; y) - \sum_{i=1}^{n} \chi_i \log x_i. \end{aligned}$$

We call $F(x;y)$, $F_\chi(x;y)$ the *G-unfolded Landau–Ginzburg potentials*. The unfolding $F(x;y)$ is an element of $\mathbb{C}[\mathbb{O}_+][y]$. Under the specialization

$$\begin{aligned} y_i &= 1 && \text{for } 1 \le i \le m, \\ y_{\mathbf{k}} &= 0 && \text{for } \mathbf{k} \in G \end{aligned} \tag{4.2}$$

the potentials $F(x;y)$, $F_\chi(x;y)$ become, respectively, the original ones $f(x)$, $f_\chi(x)$. We refer to the shifted origin (4.2) of y as y^*.

Remark 4.5. The deformation parameters $y_1, \ldots, y_m$ for $F(x;y)$ are redundant in the sense that those deformations can be reduced to a deformation along $Q \in \operatorname{Spec} \mathbb{C}[\mathbf{\Lambda}_+]$ via a rescaling of the variables x. It is, however, convenient to keep $y_1, \ldots, y_m$ as deformation parameters when we use $\mathbb{C}[\mathbf{\Lambda}_+]$ as a ground ring. See also Remark 5.4.

Remark 4.6. The paper [**52**] introduced infinitely many deformation parameters $y_{\mathbf{k}}$ for all $\mathbf{k} \in \mathbf{N} \cap |\Sigma|$; each $\mathbf{k}$ corresponds to a basis element $\phi_{\mathbf{k}}$ for $H^*_{\mathbb{T}}(X)$ over $\mathbb{C}$. This gives a natural identification between the deformation space of F and equivariant cohomology. On the other hand, in the present paper, we restrict to finitely many deformation terms; this is compensated for by working over the ground ring $R_{\mathbb{T}} = H^*_{\mathbb{T}}(\mathrm{pt}, \mathbb{C})$.

4.3. Galois action on the Landau–Ginzburg model. Similarly to the Galois symmetry in quantum cohomology (§3.3), we can define an action of $\mathrm{Pic}^{\mathrm{st}}(\mathfrak{X}) = \mathrm{Pic}(\mathfrak{X})/\mathrm{Pic}(X)$ on the Landau–Ginzburg model.

Recall that $\mathrm{Pic}(\mathfrak{X}) \cong \mathbb{L}^\vee$ and introduce a bilinear pairing

$$\mathrm{age}\colon \mathbb{L}^\vee \times \mathbb{O} \to \mathbb{Q}/\mathbb{Z} \cong [0,1) \cap \mathbb{Q}$$

as follows. We recall an explicit description of $\mathbb{L}^\vee = H^1(\mathrm{Cone}(\beta)^\star)$ from [5, §2]. Choose a free resolution $0 \to K \xrightarrow{\iota} F \to \mathbf{N} \to 0$ of $\mathbf{N}$ and a lift $\tilde{\beta}\colon \mathbb{Z}^m \to F$ of β. Then $\mathrm{Cone}(\beta)$ is quasi-isomorphic to the complex $\iota \oplus \tilde{\beta}\colon K \oplus \mathbb{Z}^m \to F$ of free modules, and

$$\mathbb{L}^\vee = \mathrm{Cok}\left(\iota^\star \oplus \tilde{\beta}^\star\colon F^\star \to K^\star \oplus (\mathbb{Z}^m)^\star\right).$$

Let $\xi \in \mathbb{L}^\vee$ and $(\lambda, \mathbf{k}) \in \mathbb{O}$ be given. We choose a lift $\tilde{\xi} \in K^\star \oplus (\mathbb{Z}^m)^\star$ of ξ and a lift $\tilde{\mathbf{k}} \in F$ of $\mathbf{k} \in \mathbf{N}$. Then $\tilde{\mathbf{k}} - \tilde{\beta}(\lambda) \in F_{\mathbb{Q}}$ lies in the kernel of $F_{\mathbb{Q}} \to \mathbf{N}_{\mathbb{Q}}$ and thus lies in $K_{\mathbb{Q}}$. Hence we obtain the element $(\tilde{\mathbf{k}} - \tilde{\beta}(\lambda), \lambda)$ of $K_{\mathbb{Q}} \oplus \mathbb{Q}^m$. We define

$$\mathrm{age}(\xi, (\lambda, \mathbf{k})) := \left(\tilde{\xi} \cdot (\tilde{\mathbf{k}} - \tilde{\beta}(\lambda), \lambda) \mod \mathbb{Z}\right) \in \mathbb{Q}/\mathbb{Z}.$$

It is easy to see that this is independent of the choices made and defines a bilinear form.

Lemma 4.7. (1) *Let L_ξ denote the line bundle corresponding to $\xi \in \mathbb{L}^\vee$. For $v \in \mathrm{Box}$, $\mathrm{age}(\xi, (\Psi(v), v))$ equals the age $\mathrm{age}_v(L_\xi)$ of the line bundle L_ξ along the sector $\mathfrak{X}_v$.*

(2) $\mathrm{age}(\xi, (\lambda, \mathbf{k})) = 0$ *for all $\xi \in \mathbb{L}^\vee$ if and only if $(\lambda, \mathbf{k}) \in \mathbb{O}$ lies in the image of the inclusion $\mathbb{Z}^m \to \mathbb{O}$, $e_i \mapsto (e_i, b_i)$.*

(3) *We have* $\mathrm{age}(\xi, (\lambda, \mathbf{k})) = 0$ *for all $(\lambda, \mathbf{k}) \in \mathbb{O}$, if and only if the line bundle L_ξ corresponding to $\xi \in \mathbb{L}^\vee$ is the pull-back of a line bundle on the coarse moduli space X.*

Proof. (1) Recall from §2.2 that the line bundle L_ξ is given by the quotient of $\mathcal{U}_\Sigma \times \mathbb{C}$ by the $\mathbb{G}$-action $g \cdot (Z, v) = (g \cdot Z, \xi(g)v)$, where we regard $\xi \in \mathbb{L}^\vee \cong \mathrm{Hom}(\mathbb{G}, \mathbb{C}^\times)$ as a character of $\mathbb{G}$. On the other hand, the stabilizer $g_v \in \mathbb{G}$ associated with $v \in \mathrm{Box}$ is defined as follows (see [5, Lemma 4.6]). We use the notation in the paragraph preceding this lemma. Choose a lift $\tilde{v} \in F$ of v. Then $\tilde{v} - \tilde{\beta}(\Psi(v)) \in F_{\mathbb{Q}}$ lies in the kernel of $F_{\mathbb{Q}} \to \mathbf{N}_{\mathbb{Q}}$, and thus $(\tilde{v} - \tilde{\beta}(\Psi(v)), \Psi(v))$ lies in $K_{\mathbb{Q}} \oplus \mathbb{Q}^m$. The stabilizer $g_v \in \mathbb{G} = H^{-1}(\mathrm{Cone}(\beta) \otimes^{\mathbb{L}} \mathbb{C}^\times)$ is then given by

$$g_v = e^{2\pi\sqrt{-1}\left(\tilde{v} - \tilde{\beta}(\Psi(v)), \Psi(v)\right)} \in \mathrm{Ker}\left((K \oplus \mathbb{Z}^m) \otimes \mathbb{C}^\times \to F \otimes \mathbb{C}^\times\right).$$

Therefore g_v acts on fibers of L_ξ along $\mathfrak{X}_v$ by $\exp(2\pi\sqrt{-1}\,\mathrm{age}(\xi, (\Psi(v), v)))$, and part (1) follows.

(2) Suppose that $\mathrm{age}(\xi, (\lambda, \mathbf{k})) = 0$ for all $\xi \in \mathbb{L}^\vee$. With notation as above, we have that $(\tilde{\mathbf{k}} - \tilde{\beta}(\lambda), \lambda) \in K \oplus \mathbb{Z}^m$. This implies that $\lambda \in \mathbb{Z}^m$ and $\beta(\lambda) = \mathbf{k}$. Thus $(\lambda, \mathbf{k}) = \sum_{i=1}^m \lambda_i(e_i, b_i)$ lies in the image of $\mathbb{Z}^m$.

(3) Note that $\mathbb{O}$ is generated by $(\Psi(v), v)$ with $v \in \mathrm{Box}$ and (e_i, b_i), $i = 1, \ldots, m$. Therefore, by parts (1) and (2), we have that $\mathrm{age}(\xi, (\lambda, \mathbf{k})) = 0$ for all $(\lambda, \mathbf{k}) \in \mathbb{O}$ if and only if the age of L_ξ along each twisted sector $\mathfrak{X}_v$ is 0. This happens if and only if L_ξ is the pull-back of a line bundle from the coarse moduli space. q.e.d.

The above lemma says that the bilinear pairing $\mathrm{age}(\cdot, \cdot)$ descends to a perfect pairing $\mathrm{age}\colon \mathrm{Pic}^{\mathrm{st}}(\mathfrak{X}) \times \mathbb{O}/\mathbb{Z}^m \to \mathbb{Q}/\mathbb{Z}$. We define the action of $\mathrm{Pic}^{\mathrm{st}}(\mathfrak{X})$ on $\mathbb{C}[\mathbb{O}_+]$ by

$$\xi \cdot w^{(\lambda,\mathbf{k})} = e^{2\pi\sqrt{-1}\,\mathrm{age}(\xi,(\lambda,\mathbf{k}))} w^{(\lambda,\mathbf{k})}$$

for $\xi \in \mathrm{Pic}^{\mathrm{st}}(\mathfrak{X})$ and $(\lambda, \mathbf{k}) \in \mathbb{O}_+$. This defines the $\mathrm{Pic}^{\mathrm{st}}(\mathfrak{X})$-action on the total space $\mathrm{Spec}\,\mathbb{C}[\mathbb{O}_+]$. The group $\mathrm{Pic}^{\mathrm{st}}(\mathfrak{X})$ also acts on the unfolding parameters $y = \{y_i, y_{\mathbf{k}} : 1 \le i \le m, \mathbf{k} \in G\}$ of the G-unfolded Landau–Ginzburg potential by

$$\xi \cdot y_i = y_i, \qquad \xi \cdot y_{\mathbf{k}} = e^{-2\pi\sqrt{-1}\,\mathrm{age}(\xi,(\Psi(\mathbf{k}),\mathbf{k}))} y_{\mathbf{k}}.$$

The G-unfolded potential $F(x; y)$ is invariant under the $\mathrm{Pic}^{\mathrm{st}}(\mathfrak{X})$-action. The equivariant potential $F_\chi(x; y)$ is not $\mathrm{Pic}^{\mathrm{st}}(\mathfrak{X})$-invariant, but the $\mathrm{Pic}^{\mathrm{st}}(\mathfrak{X})$-action shifts $F_\chi(x; y)$ only by a (constant) linear form in $\chi_1, \ldots, \chi_n$, hence its derivative in x is $\mathrm{Pic}^{\mathrm{st}}(\mathfrak{X})$-invariant.

Lemma 4.8. (1) *For $\xi \in \mathbb{L}^\vee$ and $\lambda \in \mathbf{\Lambda}$, we have* $\mathrm{age}(\xi, \lambda) \equiv \xi \cdot \lambda \mod \mathbb{Z}$, *where we regard λ as an element of $\mathbb{O}$ by the inclusion $\mathbf{\Lambda} \subset \mathbb{O}$. In particular, $\xi \cdot Q^\lambda$ equals the Galois action $(g(\xi)^*)^{-1} Q^\lambda$ for the Novikov variables defined in §3.3.*

(2) *For $\xi \in \mathrm{Pic}(X)$ and $\lambda \in \mathbf{\Lambda}$, we have $\xi \cdot \lambda \in \mathbb{Z}$. Moreover, we have* $\mathrm{Pic}(X) \cong \mathbf{\Lambda}^\star$.

Proof. With notation as above, we have the exact sequence $0 \to \mathbb{L} \to K \oplus \mathbb{Z}^m \xrightarrow{\iota \oplus \tilde{\beta}} F$, where the map $\mathbb{L} \to K \oplus \mathbb{Z}^m$ is given by $\lambda \mapsto (-\tilde{\beta}(\lambda), \lambda)$. Thus the natural pairing between $\mathbb{L}^\vee = \mathrm{Cok}(F^\star \to K^\star \oplus (\mathbb{Z}^m)^\star)$ and $\mathbb{L}_\mathbb{Q}$ is given by $\xi \cdot \lambda = \tilde{\xi} \cdot (-\tilde{\beta}(\lambda), \lambda)$ where $\lambda \in \mathbb{L}_\mathbb{Q}$ and $\tilde{\xi} \in K^\star \oplus (\mathbb{Z}^m)^\star$ is a representative of $\xi \in \mathbb{L}^\vee$. Part (1) follows from this and the definition of $\mathrm{age}(\xi, \lambda)$. The first statement of part (2) follows from part (1) and Lemma 4.7(3). This gives a natural map $\mathrm{Pic}(X) \to \mathbf{\Lambda}^\star$, which is injective since $\mathrm{Pic}(X)$ has no torsion [**21**, Proposition 4.2.5]. To show the second statement, we embed both $\mathrm{Pic}(X)$ and $\mathbf{\Lambda}^\star$ into $\mathrm{Pic}(\mathfrak{X}) \cong \mathbb{L}^\vee$;

the natural map $\mathbf{\Lambda}^\star \to \mathbb{L}^\vee$ is given by taking the Gale dual of (2.6):

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{M} \arrow[r] & (\mathbb{Z}^m)^\star \arrow[r] & \mathbb{L}^\vee & \\ 0 \arrow[r] & \mathbf{M} \arrow[r] \arrow[u, equal] & \mathbb{O}^\star \arrow[r] \arrow[u, hook] & \mathbf{\Lambda}^\star \arrow[r] \arrow[u, hook] & 0 \end{tikzcd}$$

Via these embeddings, we have $\mathrm{Pic}(X) \subset \mathbf{\Lambda}^\star \subset \mathbb{L}^\vee$. To see the converse inclusion $\mathbf{\Lambda}^\star \subset \mathrm{Pic}(X)$ it suffices, in view of Lemma 4.7(3), to show that an element $\xi \in \mathbf{\Lambda}^\star \subset \mathbb{L}^\vee$ satisfies $\mathrm{age}(\xi, (\lambda, \mathbf{k})) = 0$ for all $(\lambda, \mathbf{k}) \in \mathbb{O}$. Note that $\xi \in \mathbf{\Lambda}^\star$ comes from an element of $\mathbb{O}^\star$ in the above diagram. By the definition of the age pairing, it follows easily that $\mathrm{age}(\xi, (\lambda, \mathbf{k})) = 0$. The conclusion follows. q.e.d.

4.4. The Gauss–Manin system. In this section we fix a subset $G \subset \mathbf{N} \cap |\Sigma|$ disjoint from $\{b_1, \ldots, b_m\}$, and construct a Gauss–Manin system associated to the G-unfolded Landau–Ginzburg potential (§4.2). The Gauss–Manin system constitutes, together with the higher residue pairing introduced in §6, the *Saito structure* of the Landau–Ginzburg model.

4.4.1. Definition. We consider a formal completion of the total space $\mathrm{Spec}\,\mathbb{C}[\mathbb{O}_+]$ along the fiber at $\{Q = 0\} \in \mathrm{Spec}\,\mathbb{C}[\mathbf{\Lambda}_+]$. Let K be a ring, and let $\mathfrak{m}$ denote the ideal of $K[\mathbf{\Lambda}_+]$ generated by Q^λ with $\lambda \in \mathbf{\Lambda}_+ \setminus \{0\}$. Let $\tilde{\mathfrak{m}} \subset K[\mathbb{O}_+]$ be the ideal generated by $\mathfrak{m}$. We set

$$K[\![\mathbf{\Lambda}_+]\!] := \text{the completion of } K[\mathbf{\Lambda}_+] \text{ with respect to the } \mathfrak{m}\text{-adic topology},$$
$$K\{\mathbb{O}_+\} := \text{the completion of } K[\mathbb{O}_+] \text{ with respect to the } \tilde{\mathfrak{m}}\text{-adic topology}.$$

Note that we have (cf. (4.1))

$$K\{\mathbb{O}_+\} = \widehat{\bigoplus_{\mathbf{k} \in \mathbf{N} \cap |\Sigma|}} K[\![\mathbf{\Lambda}_+]\!] w_{\mathbf{k}}, \tag{4.3}$$

where the right-hand side is the completed direct sum with respect to the $\mathfrak{m}$-adic topology. We also write

$$\begin{aligned} K[\![y]\!] &= K[\![y_1 - 1, \ldots, y_m - 1, \{y_{\mathbf{k}} : \mathbf{k} \in G\}]\!], \\ K[\chi] &= K[\chi_1, \ldots, \chi_n]. \end{aligned} \tag{4.4}$$

Note that we use the completion at the shifted origin (4.2) of y. In this section we consider the *formal* Landau–Ginzburg model given by the G-unfolded potential $F(x; y)$

$$\begin{tikzcd} \widehat{\mathcal{Y}} \arrow[r, "F(x;y)"] \arrow[d] & \mathbb{C} \\ \widehat{\mathcal{M}} & \end{tikzcd}$$

with $\widehat{\mathcal{Y}} = \operatorname{Spf} \mathbb{C}\{\mathbb{O}_+\}[\![y]\!]$ and $\widehat{\mathcal{M}} = \operatorname{Spf} \mathbb{C}[\![\mathbf{\Lambda}_+]\!][\![y]\!]$. We regard $\widehat{\mathcal{Y}}$, $\widehat{\mathcal{M}}$ as formal log schemes whose log structures are given by their toric boundaries (see e.g. [44, Ch. 3]); the family $\widehat{\mathcal{Y}} \to \widehat{\mathcal{M}}$ is then log smooth. The Gauss–Manin system is given as the top cohomology of the logarithmic twisted de Rham complex $(\Omega^\bullet_{\widehat{\mathcal{Y}}/\widehat{\mathcal{M}}}\{z\}, zd + dF\wedge)$ where

$$\Omega^k_{\widehat{\mathcal{Y}}/\widehat{\mathcal{M}}}\{z\} = \bigoplus_{i_1<\cdots<i_k} \mathbb{C}[z]\{\mathbb{O}_+\}[\![y]\!] \frac{dx_{i_1}\cdots dx_{i_k}}{x_{i_1}\cdots x_{i_k}}.$$

In what follows, we give a more concrete definition. Introduce the relative differential forms[3] $\omega := |\mathbf{N}_{\mathrm{tor}}|^{-1} \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$ and $\omega_i := \iota_{x_i \frac{\partial}{\partial x_i}} \omega$.

Definition 4.9. (1) The *equivariant Gauss–Manin system* $\mathrm{GM}(F_\chi)$ is defined to be the cokernel of the map

$$zd + dF_\chi \wedge \colon \bigoplus_{i=1}^n \mathbb{C}[z]\{\mathbb{O}_+\}[\![y]\!][\chi]\omega_i \to \mathbb{C}[z]\{\mathbb{O}_+\}[\![y]\!][\chi]\omega,$$

where d denotes the differential in the variable x which is linear over $\mathbb{C}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!][\chi]$; we define $dw_{\mathbf{k}} = \sum_{i=1}^n k_i w^{\mathbf{k}} \frac{dx_i}{x_i}$; and the map $zd + dF_\chi\wedge$ is given explicitly by

$$(zd + dF_\chi\wedge) w_{\mathbf{k}}\omega_i = \left(zk_i + \sum_{\ell \in S} l_i y_{\boldsymbol{\ell}} w_{\boldsymbol{\ell}} - \chi_i \right) w_{\mathbf{k}}\omega, \tag{4.5}$$

where k_i, l_i denote the ith components of $\overline{\mathbf{k}}$, $\overline{\boldsymbol{\ell}} \in \mathbf{N}_{\mathbb{Q}} \cong \mathbb{Q}^n$ respectively.

(2) The *non-equivariant Gauss–Manin system* $\mathrm{GM}(F)$ is defined to be the cokernel of the map

$$zd + dF\wedge \colon \bigoplus_{i=1}^n \mathbb{C}[z]\{\mathbb{O}_+\}[\![y]\!]\omega_i \to \mathbb{C}[z]\{\mathbb{O}_+\}[\![y]\!]\omega.$$

Clearly we have $\mathrm{GM}(F) \cong \mathrm{GM}(F_\chi)/\sum_{i=1}^n \chi_i \,\mathrm{GM}(F_\chi)$.

Remark 4.10. The quantity (4.5) gives a relation in $\mathrm{GM}(F_\chi)$ which can be viewed as defining the action of χ_i. In fact, it is easy to check that

$$\chi_i \colon w_{\mathbf{k}}\omega \mapsto \left(zk_i + \sum_{\ell\in S} l_i y_{\boldsymbol{\ell}} w_{\boldsymbol{\ell}} \right) w_{\mathbf{k}}\omega, \quad 1 \le i \le n \tag{4.6}$$

defines commuting $\mathbb{C}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]$-module endomorphisms χ_i on $\mathbb{C}[z]\{\mathbb{O}_+\}[\![y]\!]\omega$, and that there is a canonical isomorphism:

$$\mathrm{GM}(F_\chi) \cong \mathbb{C}[z]\{\mathbb{O}_+\}[\![y]\!]\omega \cong \widehat{\bigoplus_{\mathbf{k}\in \mathbf{N}\cap|\Sigma|}} \mathbb{C}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!] w_{\mathbf{k}}\omega. \tag{4.7}$$

[3]The volume form ω is normalized so that the integral against the maximal compact subgroup of $\mathrm{Hom}(\mathbf{N}, \mathbb{C}^\times)$ equals $(2\pi\sqrt{-1})^n$. The factor $|\mathbf{N}_{\mathrm{tor}}|^{-1}$ plays a role in §6, when we show that the pairings match.

It is also easy to see that, under the action (4.6), $\mathrm{GM}(F_\chi) = \mathbb{C}[z]\{\mathbb{O}_+\}[\![y]\!]$ is a module over $R_{\mathbb{T}}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]$, where $R_{\mathbb{T}} = H^*_{\mathbb{T}}(\mathrm{pt}, \mathbb{C}) = \mathbb{C}[\chi]$.

4.4.2. The Gauss–Manin connection and the grading operator. The equivariant Gauss–Manin system is equipped with a natural flat connection ∇, called the *Gauss–Manin connection*, and a *grading operator*. For $\xi \in \mathbb{L}^\star_{\mathbb{C}}$, let $\xi Q\frac{\partial}{\partial Q}$ be the derivation of $K[\![\mathbf{\Lambda}_+]\!]$ such that $\xi Q\frac{\partial}{\partial Q} \cdot Q^\lambda = (\xi \cdot \lambda)Q^\lambda$. For a vector field $\vec{v}$ in the parameters (Q, y), the connection operator $\nabla_{\vec{v}} \colon \mathrm{GM}(F_\chi) \to z^{-1}\,\mathrm{GM}(F_\chi)$ is defined by the formula:

$$\nabla_{\vec{v}} = \partial_{\vec{v}} + z^{-1}(\vec{v}F_\chi),$$

where we use the splitting from §4.1 and think of elements in $\mathrm{GM}(F_\chi) \cong \mathbb{C}[z]\{\mathbb{O}_+\}[\![y]\!]$ as functions in (z, x, Q, y). More precisely, we define the actions of $\nabla_{\xi Q\frac{\partial}{\partial Q}}$, $\nabla_{\frac{\partial}{\partial y_{\boldsymbol{\ell}}}}$ on the topological $\mathbb{C}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]$-basis $w_{\mathbf{k}}\omega$ of $\mathrm{GM}(F_\chi)$ – see (4.7) – by

$$\begin{aligned}
\nabla_{\xi Q\frac{\partial}{\partial Q}} w_{\mathbf{k}}\omega &:= (\xi \cdot \lambda(\mathbf{k}))w_{\mathbf{k}}\omega + \frac{1}{z}\left(\sum_{\boldsymbol{\ell}\in S}(\xi\cdot\lambda(\boldsymbol{\ell}))y_{\boldsymbol{\ell}}w_{\boldsymbol{\ell}}\right)w_{\mathbf{k}}\omega, \\
\nabla_{\frac{\partial}{\partial y_{\boldsymbol{\ell}}}} w_{\mathbf{k}}\omega &:= \frac{1}{z}w_{\boldsymbol{\ell}}w_{\mathbf{k}}\omega = \frac{1}{z}Q^{d(\mathbf{k},\boldsymbol{\ell})}w_{\boldsymbol{\ell}+\mathbf{k}}\omega,
\end{aligned} \tag{4.8}$$

where $\mathbf{k} \in \mathbf{N} \cap |\Sigma|$, $\boldsymbol{\ell} \in S$, $\xi \in \mathbb{L}^\star_{\mathbb{C}}$, and extend them (continuously) to the whole of $\mathrm{GM}(F_\chi)$ by the Leibniz rule. Recall that $\lambda(\boldsymbol{\ell}) \in \mathbf{\Lambda}$ was introduced in §4.1 by choosing a splitting $\varsigma \colon \mathbf{N} \to \mathbb{O}$ and that $d(\mathbf{k}, \boldsymbol{\ell})$ was defined in (2.5). Define the *Euler vector field* by

$$\mathcal{E}^{\mathrm{B}} = c_1(\mathfrak{X})Q\frac{\partial}{\partial Q} + \sum_{\mathbf{k}\in S}(1 - |\mathbf{k}|)y_{\mathbf{k}}\frac{\partial}{\partial y_{\mathbf{k}}} + \sum_{i=1}^{n}\chi_i\frac{\partial}{\partial \chi_i}, \tag{4.9}$$

where $c_1(\mathfrak{X}) := D_1 + \cdots + D_m \in \mathbb{L}^\star$ and $|\mathbf{k}| = \sum_{i=1}^m \Psi_i(\mathbf{k})$ is the age function from Notation 2.2. Define the grading operator $\mathrm{Gr}^{\mathrm{B}} \colon \mathrm{GM}(F_\chi) \to \mathrm{GM}(F_\chi)$ by requiring that

$$\begin{aligned}
\mathrm{Gr}^{\mathrm{B}}(w_{\mathbf{k}}\omega) &= |\mathbf{k}|w_{\mathbf{k}}\omega, \\
\mathrm{Gr}^{\mathrm{B}}(c\Omega) &= \left(\left(z\tfrac{\partial}{\partial z} + \mathcal{E}^{\mathrm{B}}\right)c\right)\Omega + c\,\mathrm{Gr}^{\mathrm{B}}(\Omega)
\end{aligned} \tag{4.10}$$

for $c = c(z, Q, y) \in \mathbb{C}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]$ and $\Omega \in \mathrm{GM}(F_\chi)$. The following proposition is an immediate consequence of the definitions.

Proposition 4.11. *The connection operators $\nabla_{\xi Q\frac{\partial}{\partial Q}}$ and $\nabla_{\frac{\partial}{\partial y_{\mathbf{k}}}}$ are linear over $R_{\mathbb{T}}$, that is, they commute with the action (4.6) of equivariant* *parameters χ_i. Moreover:*

$$\begin{aligned}
&\Big[\nabla_{\xi Q\frac{\partial}{\partial Q}}, \nabla_{\eta Q\frac{\partial}{\partial Q}}\Big] = \Big[\nabla_{\xi Q\frac{\partial}{\partial Q}}, \nabla_{\frac{\partial}{\partial y_{\mathbf{k}}}}\Big] = \Big[\nabla_{\frac{\partial}{\partial y_{\mathbf{k}}}}, \nabla_{\frac{\partial}{\partial y_{\boldsymbol{\ell}}}}\Big] = 0,\\
&\Big[\nabla_{\xi Q\frac{\partial}{\partial Q}}, \mathrm{Gr}^{\mathrm{B}}\Big] = 0, \quad \Big[\nabla_{\frac{\partial}{\partial y_{\mathbf{k}}}}, \mathrm{Gr}^{\mathrm{B}}\Big] = \nabla_{\left[\frac{\partial}{\partial y_{\mathbf{k}}}, \mathcal{E}^{\mathrm{B}}\right]} = \nabla_{(1-|\mathbf{k}|)\frac{\partial}{\partial y_{\mathbf{k}}}},\\
&\mathrm{Gr}^{\mathrm{B}}(c\Omega) = \left(\left(z\frac{\partial}{\partial z} + \mathcal{E}^{\mathrm{B}}\right) c\right)\Omega + c\,\mathrm{Gr}^{\mathrm{B}}(\Omega),
\end{aligned}$$

where ξ, $\eta \in \mathbb{L}^\star_{\mathbb{C}}$; $\mathbf{k}$, $\boldsymbol{\ell} \in S$; $c = c(z,\chi,Q,y) \in R_{\mathbb{T}}[z][\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$; and $\Omega \in \mathrm{GM}(F_\chi)$.

Remark 4.12. We can also work with a different Euler vector field $\widetilde{\mathcal{E}}^{\mathrm{B}}$ and grading operator $\widetilde{\mathrm{Gr}}^{\mathrm{B}}$ defined by

$$\widetilde{\mathcal{E}}^{\mathrm{B}} = \sum_{\mathbf{k}\in S} y_{\mathbf{k}} \frac{\partial}{\partial y_{\mathbf{k}}} + \sum_{i=1}^n \chi_i \frac{\partial}{\partial \chi_i}, \qquad \widetilde{\mathrm{Gr}}^{\mathrm{B}}(w_{\mathbf{k}}\omega) = 0$$

together with $\widetilde{\mathrm{Gr}}^{\mathrm{B}}(c\Omega) = \left(\left(z\frac{\partial}{\partial z} + \widetilde{\mathcal{E}}^{\mathrm{B}}\right)c\right)\Omega + c\widetilde{\mathrm{Gr}}^{\mathrm{B}}(\Omega)$ for $c \in \mathbb{C}[z][\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$, $\Omega \in \mathrm{GM}(F_\chi)$. This choice of Euler vector field and the grading operator was made[4] in [**52**]. It has the advantage that they are linear over $\mathbb{C}[\![\boldsymbol{\Lambda}_+]\!]$, whereas the current choice has the advantage that the origin $Q = 0$, $y = y^*$ – where y^* is the shifted origin in (4.2) – is a fixed point of the Euler flow. The two choices are, however, essentially equivalent. We can easily check that

$$\mathrm{Gr}^{\mathrm{B}} - \widetilde{\mathrm{Gr}}^{\mathrm{B}} = \nabla_{\mathcal{E}^{\mathrm{B}} - \widetilde{\mathcal{E}}^{\mathrm{B}}} + \frac{\kappa}{z}$$

where $\kappa \in \mathbf{M}_{\mathbb{Q}}$ is the equivariant parameter defined by $\kappa = (e_1^\star + \cdots + e_m^\star) \circ \overline{\varsigma} \colon \mathbf{N} \to \mathbb{Q}$, and $\varsigma(\mathbf{k}) = (\overline{\varsigma}(\mathbf{k}), \mathbf{k})$ with $\overline{\varsigma} \colon \mathbf{N} \to \mathbb{Q}^m$.

Remark 4.13. By Remark 4.2, a different choice of the splitting ς shifts $F_\chi(x;y)$ only by a quantity which does not depend on x, and thus the Gauss–Manin system $\mathrm{GM}(F_\chi)$ is independent of the choice of splitting as an $R_{\mathbb{T}}[z][\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$-module. This shift of $F_\chi(x;y)$, however, depends on Q. Therefore the Gauss–Manin connection $\nabla_{\xi Q\frac{\partial}{\partial Q}}$ in the Q-direction depends on the choice of ς; the difference of the connections $\nabla_{\xi Q\frac{\partial}{\partial Q}}$ corresponding to two splittings is the multiplication by $\frac{1}{z}\sum_{i=1}^n \chi_i(\xi \cdot d_i)$ for some $d_i \in \boldsymbol{\Lambda}$.

Remark 4.14. The connection operators and grading operators descend to the *non-equivariant* Gauss–Manin system. In the non-equivari-

[4] The Euler vector field in [**52**] does not contain the term $\sum_{i=1}^n \chi_i \frac{\partial}{\partial \chi_i}$ since the χ_i-direction is contained as part of the (infinite-dimensional) y-directions in the setting of [**52**].

ant case, the operator $\nabla_{\xi Q \frac{\partial}{\partial Q}}$ does not depend on the choice of splitting ς. Also we can introduce the connection $\nabla_{z\frac{\partial}{\partial z}}$ in the z-direction by the formula:

$$\nabla_{z\frac{\partial}{\partial z}} = \mathrm{Gr}^{\mathrm{B}} - \nabla_{\mathcal{E}^{\mathrm{B}}} - \frac{n}{2}.$$

Compare with the quantum connection in the z-direction (3.7).

4.4.3. Galois symmetry. Since dF_χ is invariant under the Galois action in §4.3, the $\mathrm{Pic}^{\mathrm{st}}(\mathfrak{X})$-action on $\mathbb{C}[z]\{\mathbb{O}_+\}[\![y]\!]$ induces a $\mathrm{Pic}^{\mathrm{st}}(\mathfrak{X})$-action on the equivariant Gauss–Manin system $\mathrm{GM}(F_\chi)$ such that ω is $\mathrm{Pic}^{\mathrm{st}}(\mathfrak{X})$-invariant. It is easy to check that the Gauss–Manin connection satisfies

$$\begin{aligned}\nabla_{\eta Q\frac{\partial}{\partial Q}}(\xi\cdot\Omega) &= \xi\cdot\nabla_{\eta Q\frac{\partial}{\partial Q}}\Omega,\\ \nabla_{\frac{\partial}{\partial y_{\boldsymbol{\ell}}}}(\xi\cdot\Omega) &= e^{-2\pi\sqrt{-1}\,\mathrm{age}(\xi,(\Psi(\mathbf{k}),\mathbf{k}))}\xi\cdot\nabla_{\frac{\partial}{\partial y_{\boldsymbol{\ell}}}}\Omega,\end{aligned}$$

where $\xi\in\mathrm{Pic}^{\mathrm{st}}(\mathfrak{X})$, $\eta\in\mathbb{L}^\star_{\mathbb{C}}$, $\boldsymbol{\ell}\in S$, $\Omega\in\mathrm{GM}(F_\chi)$. Moreover, the Galois action commutes with the grading operator Gr^{B}.

4.5. Solution and freeness.

We next describe a cohomology-valued solution (localization map) to the equivariant Gauss–Manin system and show that the equivariant Gauss–Manin system is free over $R_{\mathbb{T}}[z][\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$.

Definition 4.15 (cf. Definition 3.5)**.** For $\mathbf{k}\in\mathbf{N}\cap|\Sigma|$, let $\mathbb{K}^G_{\mathbf{k}}$ denote the set of $\lambda=(\lambda_i,\lambda_{\boldsymbol{\ell}}:1\le i\le m,\boldsymbol{\ell}\in G)\in\mathbb{Q}^m\times\mathbb{Z}^G$ such that $\overline{\mathbf{k}}+\sum_{i=1}^m\lambda_i\overline{b}_i+\sum_{\boldsymbol{\ell}\in G}\lambda_{\boldsymbol{\ell}}\overline{\boldsymbol{\ell}}=0$ and that $\{1\le i\le m:\lambda_i\notin\mathbb{Z}\}\in\Sigma$. For $\lambda\in\mathbb{K}^G_{\mathbf{k}}$, we define

$$v(\lambda):=\mathbf{k}+\sum_{i=1}^m\lceil\lambda_i\rceil b_i+\sum_{\boldsymbol{\ell}\in G}\lambda_{\boldsymbol{\ell}}\boldsymbol{\ell}.$$

Since $\overline{v(\lambda)}=\sum_{i=1}^m\langle-\lambda_i\rangle b_i$, $v(\lambda)$ belongs to Box. We also set

$$d(\lambda):=\Psi(\mathbf{k})+(\lambda_1,\dots,\lambda_m)+\sum_{\boldsymbol{\ell}\in G}\lambda_{\boldsymbol{\ell}}\Psi(\boldsymbol{\ell})\in\mathbb{L}_{\mathbb{Q}}\subset\mathbb{Q}^m,$$

where Ψ is given in Notation 2.2.

The proof of the following lemma is similar to Lemma 3.6, and is omitted.

Lemma 4.16. *For* $\lambda\in\mathbb{K}^G_{\mathbf{k}}$, $d(\lambda)\in\boldsymbol{\Lambda}$.

Definition 4.17 (cf. Definition 3.7)**.** The *localization map* Loc

$$\mathrm{Loc}\colon\ \mathrm{GM}(F_\chi)\to H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})\otimes_{R_{\mathbb{T}}}R_{\mathbb{T}}(\!(z^{-1})\!)[\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$$

is a $\mathbb{C}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]$-linear map defined by

$$\begin{aligned}(4.11)\quad \mathrm{Loc}(w_{\mathbf{k}}\omega) &= e^{\sum_{i=1}^m u_i \log y_i/z} \\ &\times \sum_{\lambda\in\mathbb{K}_{\mathbf{k}}^G} Q^{d(\lambda)} y^\lambda \left(\prod_{i\in\{1,\dots,m\}\cup G} \frac{\prod_{c\le 0, \langle c\rangle=\langle\lambda_i\rangle} u_i + cz}{\prod_{c\le \lambda_i, \langle c\rangle=\langle\lambda_i\rangle} u_i + cz}\right) \mathbf{1}_{v(\lambda)},\end{aligned}$$

where

- $y^\lambda := \prod_{i=1}^m y_i^{\lambda_i} \prod_{\mathbf{k}\in G} y_{\mathbf{k}}^{\lambda_{\mathbf{k}}}$;
- for $1\le i\le m$, u_i is the equivariant Poincaré dual of a toric divisor in §2.2; for $i\in G$, we set $u_i := 0$;
- $\mathbf{1}_{v(\lambda)}$ is the identity class supported on the twisted sector $\mathfrak{X}_{v(\lambda)}$.

For $1\le i\le m$, λ_i is a (possibly negative) rational number, and both $\log y_i$ and $y_i^{\lambda_i}$ should be expanded in Taylor series at $y_i = 1$ – see (4.4).

The following lemma shows that the localization map takes values in $H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})\otimes_{R_{\mathbb{T}}} R_{\mathbb{T}}(\!(z^{-1})\!)[\![\mathbf{\Lambda}_+]\!][\![y]\!]$.

Lemma 4.18. *Take $\lambda\in\mathbb{K}_{\mathbf{k}}^G$. If either $\lambda_{\boldsymbol{\ell}}<0$ for some $\boldsymbol{\ell}\in G$ or $d(\lambda)\notin\bigcup_{\sigma\in\Sigma} C_\sigma$, then the summand corresponding to λ in the left-hand side of (4.11) vanishes (see (2.7) for C_σ).*

Proof. If $\lambda_{\boldsymbol{\ell}}<0$ for some $\boldsymbol{\ell}\in G$, then the summand contains a factor $\prod_{\lambda_{\boldsymbol{\ell}}<c\le 0} cz = 0$ and thus vanishes. Suppose that $\lambda_{\boldsymbol{\ell}}\ge 0$ for all $\boldsymbol{\ell}\in G$ and that $d(\lambda)\notin\bigcup_{\sigma\in\Sigma} C_\sigma$. Let $\sigma_0\in\Sigma$ be the minimal cone containing $v(\lambda)$. Note that $\sigma_0 = \{1\le i\le m : \lambda_i\notin\mathbb{Z}\}$. By assumption we have $I := \{1\le i\le m : D_i\cdot d(\lambda)<0\}\notin\Sigma$. Set $J := \{1\le i\le m : \lambda_i\in\mathbb{Z}_{<0}\}$. Since $D_i\cdot d(\lambda) = \Psi_i(\mathbf{k}) + \lambda_i + \sum_{\boldsymbol{\ell}\in G}\lambda_{\boldsymbol{\ell}}\Psi_i(\boldsymbol{\ell})$, we have

$$I\subset J\cup\sigma_0.$$

This implies that $J\cup\sigma_0\notin\Sigma$. The summand corresponding to λ contains a factor $\prod_{i\in J} u_i$. But $(\prod_{i\in J} u_i)\mathbf{1}_{v(\lambda)} = 0$ since

$$\bigcap_{i\in J}\{Z_i = 0\}\cap\mathfrak{X}_{v(\lambda)} = \bigcap_{i\in J\cup\sigma_0}\{Z_i=0\} = \varnothing$$

by the definition of $\mathcal{U}_\Sigma$ in §2.1. The conclusion follows. q.e.d.

Remark 4.19. $\mathrm{Loc}(\omega)$ is essentially equivalent to the G-extended I-function in Definition 3.7. This will be explained in the next section.

Remark 4.20. The name "localization map" stems from Givental's heuristic argument [**32**] involving S^1-equivariant Floer theory. In fact, we can compute Loc via equivariant localization on polynomial loop spaces, generalizing the method in [**16**, **33**, **47**].

Remark 4.21. Let $S_{\mathbb{T}\times\mathbb{C}^\times}$ denote the fraction field of $H^*_{T\times\mathbb{C}^\times}(\mathrm{pt}) = R_{\mathbb{T}}[z]$, where z is regarded as the equivariant parameter for $\mathbb{C}^\times$. The localization map takes values also in $H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})\otimes_{R_{\mathbb{T}}} S_{\mathbb{T}\times\mathbb{C}^\times}[\![\mathbf{\Lambda}_+]\!][\![y]\!]$. This fact will be used in §6.3 below.

We show that the localization map gives a solution to the Gauss–Manin system. Recall that we introduced a splitting $\varsigma\colon \mathbf{N}\to\mathbb{O}$ in §4.1. We write $\varsigma(\mathbf{k}) = (\overline{\varsigma}(\mathbf{k}),\mathbf{k})$ for $\mathbf{k}\in\mathbf{N}$. The map $\overline{\varsigma}(\mathbf{k})\colon \mathbf{N}_{\mathbb{Q}}\to\mathbb{Q}^m$ defines a splitting of the fan sequence (2.1) over $\mathbb{Q}$, and thus induces a splitting $\mathbb{L}^\star_{\mathbb{C}}\to(\mathbb{C}^m)^\star$, $\xi\mapsto\hat{\xi}$ of the divisor sequence (2.2) such that $\hat{\xi}$ vanishes on the image of $\overline{\varsigma}$.

Proposition 4.22. *The localization map is linear over $R_{\mathbb{T}}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]$ and satisfies the following differential equations:*

$$\begin{aligned}
\mathrm{Loc}(\nabla_{\xi Q\frac{\partial}{\partial Q}}\Omega) &= \left(\xi Q\frac{\partial}{\partial Q}+\frac{1}{z}\hat{\xi}\right)\mathrm{Loc}(\Omega), && \xi\in\mathbb{L}^\star_{\mathbb{C}},\\
\mathrm{Loc}(\nabla_{\frac{\partial}{\partial y_{\boldsymbol{\ell}}}}\Omega) &= \frac{\partial}{\partial y_{\boldsymbol{\ell}}}\mathrm{Loc}(\Omega), && \boldsymbol{\ell}\in S,\\
\mathrm{Loc}(\mathrm{Gr}^{\mathrm{B}}\,\Omega) &= \left(z\frac{\partial}{\partial z}+\mathcal{E}^{\mathrm{B}}+\mathrm{Gr}_0\right)\mathrm{Loc}(\Omega),
\end{aligned}$$

where $\Omega\in\mathrm{GM}(F_\chi)$, $\hat{\xi}\in(\mathbb{C}^m)^\star\cong H^2_{\mathbb{T}}(X,\mathbb{C})$ is the lift of ξ described above, and $\mathrm{Gr}_0\in\mathrm{End}_{\mathbb{C}}(H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}))$ is the grading operator in (3.4). We note that $z\frac{\partial}{\partial z}+\mathcal{E}^{\mathrm{B}}$ acts on the coefficient ring $R_{\mathbb{T}}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]$ and Gr_0 acts on $H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})$, cf. (3.5).

Proof. It suffices to establish these formulas for $\Omega = w_{\mathbf{k}}\omega$. For $\lambda\in\mathbb{Q}^m\times\mathbb{Z}^G$, we set

$$\Box_\lambda = \prod_{i\in\{1,\dots,m\}\cup G}\frac{\prod_{c\le 0,\langle c\rangle=\langle\lambda_i\rangle}u_i+cz}{\prod_{c\le\lambda_i,\langle c\rangle=\langle\lambda_i\rangle}u_i+cz}.$$

For notational convenience, write $\lambda_{b_i}=\lambda_i$ for the ith component of $\lambda\in\mathbb{Q}^m\times\mathbb{Z}^G$, $1\le i\le m$, and think of λ as an element of $\mathbb{Q}^S$. Hence $\mathbb{K}^G_{\mathbf{k}}$ is regarded as a subgroup of $\mathbb{Q}^S$. We also set $u_{b_i}=u_i$ for $1\le i\le m$. For $\boldsymbol{\ell}\in S$, we have the natural identification

$$\mathbb{K}^G_{\mathbf{k}+\boldsymbol{\ell}}\cong\mathbb{K}^G_{\mathbf{k}},\qquad \lambda\mapsto\lambda'=\lambda+e_{\boldsymbol{\ell}}.$$

Under this identification, we can easily check that

$$d(\lambda')=d(\lambda)+d(\mathbf{k},\boldsymbol{\ell}),\quad v(\lambda')=v(\lambda),\quad \Box_\lambda=(u_{\boldsymbol{\ell}}+\lambda'_{\boldsymbol{\ell}}z)\Box_{\lambda'}.$$

Using this, for $\boldsymbol{\ell} \in S$, $\mathbf{k} \in \mathbf{N} \cap |\Sigma|$, we have

$$\begin{aligned}
\operatorname{Loc}(\nabla_{\frac{\partial}{\partial y_{\boldsymbol{\ell}}}} w_{\mathbf{k}} \omega) &= \operatorname{Loc}(z^{-1} Q^{d(\boldsymbol{\ell},\mathbf{k})} w_{\boldsymbol{\ell}+\mathbf{k}}) \\
&= z^{-1} Q^{d(\boldsymbol{\ell},\mathbf{k})} e^{\sum_{i=1}^m u_i \log y_i/z} \sum_{\lambda \in \mathbb{K}^G_{\mathbf{k}+\boldsymbol{\ell}}} Q^{d(\lambda)} y^\lambda \square_\lambda \mathbf{1}_{v(\lambda)} \\
&= z^{-1} e^{\sum_{i=1}^m u_i \log y_i/z} \sum_{\lambda' \in \mathbb{K}^G_{\mathbf{k}}} (u_{\boldsymbol{\ell}} + \lambda'_{\boldsymbol{\ell}} z) y_{\boldsymbol{\ell}}^{-1} Q^{d(\lambda')} y^{\lambda'} \square_{\lambda'} \mathbf{1}_{v(\lambda')} \\
&= \frac{\partial}{\partial y_{\boldsymbol{\ell}}} \operatorname{Loc}(w_{\mathbf{k}} \omega).
\end{aligned} \tag{4.12}$$

For $\xi \in \mathbb{L}^\star_{\mathbb{C}}$ and $\mathbf{k} \in \mathbf{N} \cap |\Sigma|$,

$$\begin{aligned}
&\operatorname{Loc}(\nabla_{\xi Q \frac{\partial}{\partial Q}} w_{\mathbf{k}} \omega) \\
&= \operatorname{Loc}\left((\xi \cdot \lambda(\mathbf{k})) w_{\mathbf{k}} \omega + z^{-1} \textstyle\sum_{\boldsymbol{\ell} \in S} (\xi \cdot \lambda(\boldsymbol{\ell})) y_{\boldsymbol{\ell}} \nabla_{\frac{\partial}{\partial y_{\boldsymbol{\ell}}}} w_{\mathbf{k}} \omega \right) \\
&= \left((\xi \cdot \lambda(\mathbf{k})) + \textstyle\sum_{\boldsymbol{\ell} \in S} (\xi \cdot \lambda(\boldsymbol{\ell})) y_{\boldsymbol{\ell}} \frac{\partial}{\partial y_{\boldsymbol{\ell}}} \right) \operatorname{Loc}(w_{\mathbf{k}} \omega),
\end{aligned} \tag{4.13}$$

where we used the previous calculation (4.12) in the second line. By the choice of the lift $\hat{\xi}$, we have for $\lambda \in \mathbb{K}^G_{\mathbf{k}}$,

$$\xi \cdot \lambda(\mathbf{k}) + \sum_{\boldsymbol{\ell} \in S} (\xi \cdot \lambda(\boldsymbol{\ell})) \lambda_{\boldsymbol{\ell}} = \hat{\xi} \cdot \Psi(\mathbf{k}) + \sum_{\boldsymbol{\ell} \in S} (\hat{\xi} \cdot \Psi(\boldsymbol{\ell})) \lambda_{\boldsymbol{\ell}} = \xi \cdot d(\lambda),$$

$$\sum_{\boldsymbol{\ell} \in S} (\xi \cdot \lambda(\boldsymbol{\ell})) u_{\boldsymbol{\ell}} = \sum_{i=1}^m (\hat{\xi} \cdot \Psi(b_i)) u_i = \hat{\xi}.$$

This implies, for $\lambda \in \mathbb{K}^G_{\mathbf{k}}$,

$$\begin{aligned}
&\left(\xi \cdot \lambda(\mathbf{k}) + \sum_{\boldsymbol{\ell} \in S} (\xi \cdot \lambda(\boldsymbol{\ell})) y_{\boldsymbol{\ell}} \frac{\partial}{\partial y_{\boldsymbol{\ell}}} \right) y^{u/z} Q^{d(\lambda)} y^\lambda \\
&\qquad = \left(\xi Q \frac{\partial}{\partial Q} + \frac{1}{z} \hat{\xi} \right) y^{u/z} Q^{d(\lambda)} y^\lambda,
\end{aligned}$$

where we write $y^{u/z} = e^{\sum_{i=1}^m u_i \log y_i/z}$. Therefore (4.13) equals $(\xi Q \frac{\partial}{\partial Q} + z^{-1} \hat{\xi}) \operatorname{Loc}(w_{\mathbf{k}} \omega)$ as required. Note that we have for $\lambda \in \mathbb{K}^G_{\mathbf{k}}$,

$$\left(z \frac{\partial}{\partial z} + \operatorname{Gr}_0 \right) \square_\lambda = -\left(\sum_{\boldsymbol{\ell} \in S} \lceil \lambda_{\boldsymbol{\ell}} \rceil \right) \square_\lambda, \quad \left(z \frac{\partial}{\partial z} + \mathcal{E}^{\mathrm{B}} + \operatorname{Gr}_0 \right) y^{u/z} = 0,$$

$$\mathcal{E}^{\mathrm{B}}(Q^{d(\lambda)} y^\lambda) = \left(|\mathbf{k}| + \sum_{\boldsymbol{\ell} \in S} \lambda_{\boldsymbol{\ell}} \right) Q^{d(\lambda)} y^\lambda, \quad \operatorname{Gr}_0 \mathbf{1}_{v(\lambda)} = \left(\sum_{\boldsymbol{\ell} \in S} \langle -\lambda_{\boldsymbol{\ell}} \rangle \right) \mathbf{1}_{v(\lambda)}.$$

These formulas imply that $(z \frac{\partial}{\partial z} + \mathcal{E}^{\mathrm{B}} + \operatorname{Gr}_0) \operatorname{Loc}(w_{\mathbf{k}} \omega) = |\mathbf{k}| \operatorname{Loc}(w_{\mathbf{k}} \omega) = \operatorname{Loc}(\operatorname{Gr}^{\mathrm{B}}(w_{\mathbf{k}} \omega))$. Finally we check that Loc is linear over $R_{\mathbb{T}}$. We have, for $1 \le i \le n$,

$$\begin{aligned}\mathrm{Loc}(\chi_i \cdot w_{\mathbf{k}}\omega) &= zk_i \,\mathrm{Loc}(w_{\mathbf{k}}\omega) + \sum_{\boldsymbol{\ell}\in S} l_i y_{\boldsymbol{\ell}} \,\mathrm{Loc}(z\nabla_{\frac{\partial}{\partial y_{\boldsymbol{\ell}}}} w_{\mathbf{k}}\omega) \\ &= z\left(k_i + \sum_{\boldsymbol{\ell}\in S} l_i y_{\boldsymbol{\ell}} \frac{\partial}{\partial y_{\boldsymbol{\ell}}}\right) \mathrm{Loc}(w_{\mathbf{k}}\omega),\end{aligned}$$

where we again used (4.12). Since

$$z\left(k_i + \sum_{\boldsymbol{\ell}\in S} l_i y_{\boldsymbol{\ell}} \frac{\partial}{\partial y_{\boldsymbol{\ell}}}\right) y^{u/z} y^{\lambda} = \chi_i y^{u/z} y^{\lambda}$$

for $\lambda \in \mathbb{K}^G_{\mathbf{k}}$, the conclusion follows. q.e.d.

Proposition 4.23. *The localization map is equivariant with respect to the Galois action in §3.3 and §4.3, that is, we have*

$$\mathrm{Loc}(\xi \cdot \Omega) = g_0(\xi)\left(\xi \cdot \mathrm{Loc}(\Omega)\right),$$

where $\xi \in \mathrm{Pic}^{\mathrm{st}}(\mathfrak{X})$, $\Omega \in \mathrm{GM}(F_\chi)$, *and* ξ *on the right-hand side acts on the coefficient ring* $R_{\mathbb{T}}(\!(z^{-1})\!)[\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$.

Proof. By the definitions of $\mathrm{Loc}(w_{\mathbf{k}}\omega)$ and the Galois action, it suffices to check that

$$\mathrm{age}(\xi, (\Psi(\mathbf{k}), \mathbf{k})) \equiv \mathrm{age}(\xi, d(\lambda)) - \sum_{\boldsymbol{\ell}\in G} \lambda_{\boldsymbol{\ell}}\,\mathrm{age}(\xi, \boldsymbol{\ell}) + \mathrm{age}_{v(\mathbf{k})}(L_\xi) \mod \mathbb{Z}.$$

Since $\mathrm{age}_{v(\mathbf{k})}(L_\xi) \equiv \mathrm{age}(\xi, (\Psi(v(\mathbf{k})), v(\mathbf{k})))$ by Lemma 4.7(1), this reduces to showing that

$$(\Psi(\mathbf{k}), \mathbf{k}) \equiv d(\lambda) - \sum_{\boldsymbol{\ell}\in G} \lambda_{\boldsymbol{\ell}} \boldsymbol{\ell} + (\Psi(v(\mathbf{k})), v(\mathbf{k})) \mod \mathbb{Z}^m$$

by Lemma 4.7(2). This is straightforward. q.e.d.

Let $\mathrm{Loc}^{(0)}$ denote the restriction of Loc to $Q = 0$, $y = y^*$ (where y^* is the shifted origin from equation 4.2). Write $\mathfrak{m}^G$ for the ideal of $\mathbb{C}[\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$ generated by $\mathfrak{m}$, $y_i - 1$ $(1 \le i \le m)$ and $y_{\boldsymbol{\ell}}$ $(\boldsymbol{\ell} \in G)$. Then $\mathrm{Loc}^{(0)}$ defines a map:

$$\begin{aligned}\mathrm{Loc}^{(0)}\colon \mathrm{GM}(F_\chi)/\mathfrak{m}^G\,\mathrm{GM}(F_\chi) \cong \bigoplus_{\mathbf{k}\in \mathbf{N}\cap|\Sigma|} \mathbb{C}[z] w_{\mathbf{k}}\omega \\ \longrightarrow H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \otimes_{R_{\mathbb{T}}} R_{\mathbb{T}}(\!(z^{-1})\!).\end{aligned}$$

Proposition 4.24. *The image of* $\mathrm{Loc}^{(0)}$ *is* $H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \otimes_{R_{\mathbb{T}}} R_{\mathbb{T}}[z]$ *and* $\mathrm{Loc}^{(0)}$ *is an isomorphism onto its image.*

Proof. Only the summand with $\lambda = (-\Psi(\mathbf{k}), 0) \in \mathbb{Q}^m \times \mathbb{Z}^G$ in (4.11) contributes to $\mathrm{Loc}^{(0)}(w_{\mathbf{k}}\omega)$. For $\mathbf{k} \in \mathbf{N} \cap |\Sigma|$, we have

$$\begin{aligned}\mathrm{Loc}^{(0)}(w_{\mathbf{k}}\omega) &= \left(\prod_{-\Psi_i(\mathbf{k}) < c \leq 0, \langle c \rangle = \langle -\Psi_i(\mathbf{k}) \rangle} (u_i + cz) \right) \mathbf{1}_{\mathbf{k} - \sum_{i=1}^m \lfloor \Psi_i(\mathbf{k}) \rfloor b_i} \\ &= \phi_{\mathbf{k}} + O(z),\end{aligned}$$

where recall that $\{\phi_{\mathbf{k}} : \mathbf{k} \in \mathbf{N} \cap |\Sigma|\}$ is a $\mathbb{C}$-basis of $H^*_{\mathbb{T}}(\mathfrak{X})$ (see §2.3). Since $\mathrm{Loc}^{(0)}(w_{\mathbf{k}}\omega)$ is homogeneous of $(z\frac{\partial}{\partial z} + \mathrm{Gr}_0)$-degree $|\mathbf{k}|$, the conclusion follows. q.e.d.

Remark 4.25. The map $\mathrm{Loc}^{(0)}$ determines a "limit opposite subspace" at the large radius limit $Q = 0, y = y^*$ of the Gauss–Manin system, in the sense of [**73**], [**20**, Definition 2.10], [**18**, Definition 4.103]. This together with the limit primitive section ω at $Q = 0$, $y = y^*$ determines a Frobenius manifold structure corresponding to quantum cohomology.

Theorem 4.26. *The equivariant Gauss–Manin system* $\mathrm{GM}(F_\chi)$ *is a free module over* $R_{\mathbb{T}}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]$ *of rank* $\dim H^*_{\mathrm{CR}}(\mathfrak{X})$*. Moreover, we can choose an* $R_{\mathbb{T}}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]$*-basis* $\Omega_1, \dots, \Omega_N$ *of* $\mathrm{GM}(F_\chi)$ *which is homogeneous with respect to the* Gr^{B}*-grading.*

Proof. Using Proposition 4.24, we can choose Gr^{B}-homogeneous elements $\Omega_1, \dots, \Omega_N$ in $\mathrm{GM}(F_\chi)$ such that $\mathrm{Loc}^{(0)}(\Omega_1), \dots, \mathrm{Loc}^{(0)}(\Omega_N)$ is an $R_{\mathbb{T}}[z]$-basis of $H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \otimes_{R_{\mathbb{T}}} R_{\mathbb{T}}[z]$, where $N = \dim H^*_{\mathrm{CR}}(\mathfrak{X})$. We claim that $\Omega_1, \dots, \Omega_N$ give a free $R_{\mathbb{T}}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]$-basis of $\mathrm{GM}(F_\chi)$. That $\Omega_1, \dots, \Omega_N$ generate $\mathrm{GM}(F_\chi)$ is implied by the following well-known fact. Let K be a commutative ring with an ideal $\mathfrak{m} \subset K$ such that K is complete with respect the $\mathfrak{m}$-adic topology. If a K-module M is Hausdorff with respect to the $\mathfrak{m}$-adic topology ($\bigcap_p \mathfrak{m}^p M = \{0\}$) and $\Omega_1, \dots, \Omega_N \in M$ generate $M/\mathfrak{m}M$ over $K/\mathfrak{m}$, then $\Omega_1, \dots, \Omega_N$ generate M over K. See for instance [**76**, Corollary 2, §3, Ch.VIII]. That $\Omega_1, \dots, \Omega_N$ are linearly independent over $R_{\mathbb{T}}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]$ follows from the fact that $\mathrm{Loc}(\Omega_1), \dots, \mathrm{Loc}(\Omega_N)$ are linearly independent in $H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \otimes_{R_{\mathbb{T}}} R_{\mathbb{T}}(\!(z^{-1})\!)[\![\mathbf{\Lambda}_+]\!][\![y]\!]$. q.e.d.

Corollary 4.27. *The non-equivariant Gauss–Manin system* $\mathrm{GM}(F)$ *is a free module over* $\mathbb{C}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]$ *of rank* $\dim H^*_{\mathrm{CR}}(\mathfrak{X})$*.*

4.6. Mirror isomorphism. In this section we prove the following theorem.

Theorem 4.28. *There is a mirror map*

$$\tau = \tau(y) \in H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \otimes_{R_{\mathbb{T}}} R_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![y]\!]$$

with $\tau(y^*)|_{Q=0} = 0$ *(see* (4.2) *for* y^**) and an* $R_{\mathbb{T}}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]$*-linear mirror isomorphism*

$$\Theta\colon \operatorname{GM}(F_\chi) \xrightarrow{\cong} H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \otimes_{R_{\mathbb{T}}} R_{\mathbb{T}}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]$$

with $\Theta|_{Q=0,y=y^*} = \operatorname{Loc}^{(0)}$ *such that the following holds.*

1) Θ *intertwines the Gauss–Manin connection with the pull-back* $\tau^*\nabla$ *of the quantum connection by* τ*;*
2) *the Euler vector fields* (3.3), (4.9) *correspond under* τ*:* $\tau_*\mathcal{E}^{\mathrm{B}} = \mathcal{E}^{\mathrm{A}}$*;*
3) Θ *intertwines the grading operators:* $\Theta \circ \mathrm{Gr}^{\mathrm{B}} = (z\frac{\partial}{\partial z} + \mathcal{E}^{\mathrm{B}} + \mathrm{Gr}_0) \circ \Theta$*;*
4) τ *and* Θ *intertwine the Galois symmetries:* $\tau(y) = g_0(\xi)(\xi \cdot \tau(y))$ *and* $\Theta(\xi \cdot \Omega) = g_0(\xi)(\xi \cdot \Theta(\Omega))$ *for* $\xi \in \operatorname{Pic}^{\mathrm{st}}(\mathfrak{X})$*, where* ξ *on the right-hand side acts on the coefficient ring* $\mathbb{C}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]$*, and* $g_0(\xi) \in \operatorname{End}_{R_{\mathbb{T}}}(H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}))$ *is defined in §3.3.*

Remark 4.29. Recall that we chose a splitting $\mathbb{L}^\star_{\mathbb{C}} \to (\mathbb{C}^m)^\star$ when defining the quantum connection in the Q-direction (see §3.2) and we chose a splitting $\varsigma\colon \mathbf{N} \to \mathbb{O}$ when defining the Landau–Ginzburg potential F_χ (see §4.1). In the above theorem, it is assumed that these two choices are compatible in the sense explained before Proposition 4.22.

Remark 4.30. The construction of the isomorphism Θ in the proof below gives the following commutative diagram:

$$\begin{tikzcd}
\operatorname{GM}(F_\chi) \arrow[rr, "\Theta"] \arrow[dr, "\mathrm{Loc}"'] & & H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \otimes_{R_{\mathbb{T}}} R_{\mathbb{T}}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!] \arrow[dl, "M(\tau(y),z)"] \\
& \mathcal{H}[\![y]\!] &
\end{tikzcd}$$

where $\mathcal{H} = H^*_{\mathrm{CR},T}(\mathfrak{X}) \otimes_{R_{\mathbb{T}}} S_{\mathbb{T}}(\!(z^{-1})\!)[\![\mathbf{\Lambda}_+]\!]$ is Givental's symplectic space from §3.4 and $M(\tau, z)$ is the fundamental solution in Proposition 3.1. Note that the images of Loc and $M(\tau(y), z)$ lie in the smaller space $H^*_{\mathrm{CR},T}(\mathfrak{X}) \otimes_{R_{\mathbb{T}}} R_{\mathbb{T}}(\!(z^{-1})\!)[\![\mathbf{\Lambda}_+]\!][\![y]\!]$.

Remark 4.31. The Galois action $\xi \cdot (-)$ on $\mathbb{C}[\![\mathbf{\Lambda}_+]\!][\![y]\!]$ corresponds, under the mirror map, to the pull-back of functions by $g(\xi)^{-1}$: $\operatorname{Spf} \mathbb{C}[\![\mathbf{\Lambda}_+]\!][\![\tau]\!] \to \operatorname{Spf} \mathbb{C}[\![\mathbf{\Lambda}_+]\!][\![\tau]\!]$. We have already seen in Lemma 4.8(1) that the Galois actions on the Novikov variables Q are the same on both sides.

First we explain the relationship between the I-function and the localization map. Let $I(Q, \mathfrak{y}, t, z)$ denote the G-extended I-function from Definition 3.7. For convenience, we denote by $\overline{\mathcal{L}}_{\mathfrak{X}} = \mathcal{L}_{\mathfrak{X}}|_{z \to -z}$ the Givental cone with the sign of z flipped.

Lemma 4.32. *The functions $z\,\mathrm{Loc}(\omega)$ and $I(Q,\mathfrak{y},t,z)$ coincide under the change of variables*

$$t_i = \log y_i, \qquad \mathfrak{y}_{\boldsymbol{\ell}} = y_{\boldsymbol{\ell}} \prod_{i=1}^{m} y_i^{-\Psi_i(\boldsymbol{\ell})}.$$

Therefore $z\,\mathrm{Loc}(\omega)$ defines an $S_{\mathbb{T}}[\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$-valued point on the equivariant Givental cone $\overline{\mathcal{L}}_{\mathfrak{X}}$. Moreover, $\mathrm{Loc}(w_{\mathbf{k}}\omega)$ lies in the tangent space of $\overline{\mathcal{L}}_{\mathfrak{X}}$ at this point for every $\mathbf{k} \in \mathbf{N} \cap |\Sigma|$.

Proof. The first statement follows from a straightforward computation. We show that $\mathrm{Loc}(w_{\mathbf{k}}\omega)$ lies in the tangent space at $z\,\mathrm{Loc}(\omega)$. If $\mathbf{k} \in S$, Proposition 4.22 gives that $\mathrm{Loc}(w_{\mathbf{k}}\omega) = \frac{\partial}{\partial y_{\mathbf{k}}} z\,\mathrm{Loc}(\omega)$, and thus $\mathrm{Loc}(w_{\mathbf{k}}\omega)$ is a tangent vector. Suppose $\mathbf{k} \notin S$. We consider the $(G \cup \{\mathbf{k}\})$-extended version of the equivariant Gauss–Manin system $\mathrm{GM}'(F_\chi)$ and the localization map Loc'. It is obvious that $\mathrm{GM}'(F_\chi)|_{y_{\mathbf{k}}=0} \cong \mathrm{GM}(F_\chi)$ and that $\mathrm{Loc}'|_{y_{\mathbf{k}}=0} = \mathrm{Loc}$. Hence we obtain

$$\mathrm{Loc}(w_{\mathbf{k}}\omega) = \mathrm{Loc}'(w_{\mathbf{k}}\omega)\Big|_{y_{\mathbf{k}}=0} = \frac{\partial}{\partial y_{\mathbf{k}}} z\,\mathrm{Loc}'(\omega)\Big|_{y_{\mathbf{k}}=0}.$$

This, together with the fact that $z\,\mathrm{Loc}'(\omega)$ lies in $\overline{\mathcal{L}}_{\mathfrak{X}}$, implies the lemma. q.e.d.

The above lemma implies that for every $\Omega \in \mathrm{GM}(F_\chi)$, the vector $\mathrm{Loc}(\Omega)$ lies in the tangent space at $z\,\mathrm{Loc}(\omega) \in \overline{\mathcal{L}}_{\mathfrak{X}}$. By Theorem 4.26, we can take a Gr^{B}-homogeneous $R_{\mathbb{T}}[\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$-basis $\Omega_1, \dots, \Omega_N$ of $\mathrm{GM}(F_\chi)$. Since $\mathrm{Loc}(\Omega_i)$ is a tangent vector, Proposition 3.4 implies that there exist $\tau(y) \in H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \otimes_{R_{\mathbb{T}}} S_{\mathbb{T}}[\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$ and $\Upsilon_i(y,z) \in H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \otimes_{R_{\mathbb{T}}} S_{\mathbb{T}}[z][\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$ such that $\tau(y^*)|_{Q=0} = 0$ and

$$\mathrm{Loc}(\Omega_i) = M(\tau(y),z)\Upsilon_i(y,z) \qquad 1 \le i \le N. \tag{4.14}$$

This $\tau(y)$ gives the mirror map. First we claim that $\tau(y)$ and $\Upsilon_i(y,z)$ are defined over $R_{\mathbb{T}}$, i.e. they are elements of $H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \otimes_{R_{\mathbb{T}}} R_{\mathbb{T}}[z][\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$. As noted in [**17**, **45**], we can view the relation (4.14) as a Birkhoff factorization [**67**] of the matrix $(\mathrm{Loc}(\Omega_1), \dots, \mathrm{Loc}(\Omega_N))$:
(4.15)

$$\begin{pmatrix} | & & | \\ \mathrm{Loc}(\Omega_1) & \cdots & \mathrm{Loc}(\Omega_N) \\ | & & | \end{pmatrix} = M(\tau(y),z) \begin{pmatrix} | & & | \\ \Upsilon_1(y,z) & \cdots & \Upsilon_N(y,z) \\ | & & | \end{pmatrix}.$$

Here we regard both sides as elements of the loop group $\mathrm{LGL}_N(\mathbb{C})$ with loop parameter z; note that $M(\tau(y),z) = I + O(z^{-1})$ and that $(\Upsilon_1(y,z), \dots, \Upsilon_N(y,z))$ does not contain negative powers of z. The left-hand side belongs to

$$\mathrm{End}_{R_{\mathbb{T}}}(H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})) \otimes_{R_{\mathbb{T}}} R_{\mathbb{T}}(\!(z^{-1})\!)[\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$$

and is invertible (over $R_{\mathbb{T}}$) at $Q=0$ and $y=y^*$, by the choice of $\Omega_1,\dots,\Omega_N$. Therefore the Birkhoff factorization can be performed over $R_{\mathbb{T}}$ uniquely and recursively in powers of $y-y^*$ and Q. This shows that $\Upsilon_i(y,z)$ and $M(\tau(y),z)$ are defined over $R_{\mathbb{T}}$. Moreover, the asymptotics $J(\tau(y),z)=zM(\tau(y),z)\mathbf{1}=z\mathbf{1}+\tau(y)+O(z^{-1})$ shows that $\tau(y)$ is also defined over $R_{\mathbb{T}}$.

We define an $R_{\mathbb{T}}[\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$-linear isomorphism Θ by

$$\Theta(\Omega_i):=\Upsilon_i(y,z).$$

That Θ intertwines the quantum connection follows from the fact that Loc and $M(\tau,z)$ are solutions (see Propositions 3.1, 4.22). By differentiating (4.14) by $\xi Q\frac{\partial}{\partial Q}+z^{-1}\hat{\xi}$ and $\frac{\partial}{\partial y_{\boldsymbol{\ell}}}$ (with $\boldsymbol{\ell}\in S$), we obtain

$$\begin{aligned}\mathrm{Loc}(\nabla_{\xi Q\frac{\partial}{\partial Q}}\Omega_i)&=M(\tau(y),z)\nabla_{\xi Q\frac{\partial}{\partial Q}}\Upsilon_i(y,z),\\ \mathrm{Loc}(\nabla_{\frac{\partial}{\partial y_{\boldsymbol{\ell}}}}\Omega_i)&=M(\tau(y),z)(\tau^*\nabla)_{\frac{\partial}{\partial y_{\boldsymbol{\ell}}}}\Upsilon_i(y,z),\end{aligned}$$

where note that $(\tau^*\nabla)_{\frac{\partial}{\partial y_{\boldsymbol{\ell}}}}=\frac{\partial}{\partial y_{\boldsymbol{\ell}}}+z^{-1}(\frac{\partial\tau}{\partial y_{\boldsymbol{\ell}}}\star)$. This implies that $\Theta(\nabla_{\xi Q\frac{\partial}{\partial Q}}\Omega_i)=\nabla_{\xi Q\frac{\partial}{\partial Q}}\Upsilon_i(y,z)$ and $\Theta(\nabla_{\frac{\partial}{\partial y_{\boldsymbol{\ell}}}}\Omega_i)=(\tau^*\nabla)_{\frac{\partial}{\partial y_{\boldsymbol{\ell}}}}\Theta(\Omega_i)$. Thus Θ intertwines the flat connections.

Next we show that Θ preserves grading. By the choice of Ω_i, $T_i:=\mathrm{Loc}(\Omega_i)|_{Q=0,y=y^*,z=0}$ with $1\le i\le N$ form a homogeneous basis of $H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})$ over $R_{\mathbb{T}}$. Expanding in the basis $\{T_i\}$, we regard (4.15) as an equation in matrices whose entries lie in $R_{\mathbb{T}}(\!(z^{-1})\!)[\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$. Proposition 4.22 implies that the matrix $L:=(\mathrm{Loc}(\Omega_1),\dots,\mathrm{Loc}(\Omega_N))$ satisfies the homogeneity equation:

$$\widehat{\mathcal{E}}L+[\mathrm{gr}_0,L]=0,$$

where $\widehat{\mathcal{E}}=z\frac{\partial}{\partial z}+\mathcal{E}^{\mathrm{B}}$ and $\mathrm{gr}_0=\mathrm{diag}(|1|,|2|,\dots,|N|)$ with $|i|=\frac{1}{2}\deg T_i$. This implies the following equation for the matrices $M=M(\tau(y),z)$ and $Y=(\Upsilon_1(y,z),\dots,\Upsilon_N(y,z))$:

$$M^{-1}(\widehat{\mathcal{E}}M+[\mathrm{gr}_0,M])=-(\widehat{\mathcal{E}}Y+[\mathrm{gr}_0,Y])Y^{-1}.$$

Since $M=I+O(z^{-1})$ and Y does not contain negative powers of z, both sides have to vanish. Therefore we have

$$\begin{aligned}\left(z\frac{\partial}{\partial z}+\mathcal{E}^{\mathrm{B}}+\mathrm{Gr}_0\right)M(\tau(y),z)\mathbf{1}&=0,\\ \left(z\frac{\partial}{\partial z}+\mathcal{E}^{\mathrm{B}}+\mathrm{Gr}_0\right)\Upsilon_i(y,z)&=|i|\Upsilon_i(y,z).\end{aligned}$$

The second equation implies that $\Theta\circ\mathrm{Gr}^{\mathrm{B}}=(z\frac{\partial}{\partial z}+\mathcal{E}^{\mathrm{B}}+\mathrm{Gr}_0)\circ\Theta$. The first equation gives

$$(\mathcal{E}^{\mathrm{B}}+\mathrm{Gr}_0)\tau(y)=\tau(y)$$

which is equivalent to $\tau_*\mathcal{E}^{\mathrm{B}}=\mathcal{E}^{\mathrm{A}}$.

Finally we show that τ and Θ intertwine the Galois actions. We may assume that the basis $\Omega_1, \ldots, \Omega_N$ of $\mathrm{GM}(F_\chi)$ consists of simultaneous eigenvectors for the $\mathrm{Pic}^{\mathrm{st}}(\mathfrak{X})$-action. The relation $\mathrm{Loc}(\Omega_i) = M(\tau(y), z)\Theta(\Omega_i)$ implies

$$\begin{aligned} M(\tau(y), z)\Theta(\xi \cdot \Omega_i) &= \mathrm{Loc}(\xi \cdot \Omega_i) = g_0(\xi)(\xi \cdot \mathrm{Loc}(\Omega_i)) \\ &= g_0(\xi)(\xi \cdot M(\tau(y), z))(\xi \cdot \Theta(\Omega_i)) \\ &= \left(\xi \cdot (g(\xi)^* M(\tau, z))|_{\tau=\tau(y)}\right) g_0(\xi)(\xi \cdot \Theta(\Omega_i)), \end{aligned}$$

where we used Proposition 4.23 in the first line and Proposition 3.3 in the third line. From uniqueness of Birkhoff factorization, we conclude that

$$M(\tau(y), z) = \xi \cdot (g(\xi)^* M(\tau, z))|_{\tau=\tau(y)}, \quad \Theta(\xi \cdot \Omega_i) = g_0(\xi)(\xi \cdot \Theta(\Omega_i)).$$

It follows that $\Theta(\xi \cdot \Omega) = g_0(\xi)(\xi \cdot \Theta(\Omega))$ for all $\Omega \in \mathrm{GM}(F_\chi)$. Using $M(\tau, z)\mathbf{1} = \mathbf{1} + z^{-1}\tau + O(z^{-2})$, we find that $\tau(y) = g_0(\xi)(\xi \cdot \tau(y))$. The theorem is proved.

5. Presentations of the quantum D-module and quantum cohomology ring

In this section, we recast the construction of the equivariant Gauss–Manin system in combinatorial terms, and give presentations of the quantum D-module and the quantum cohomology ring of $\mathfrak{X}$.

5.1. The fan D-module. Recall that the equivariant Gauss–Manin system has a topological basis $\{w_{\mathbf{k}}\omega : \mathbf{k} \in \mathbf{N} \cap |\Sigma|\}$ over $\mathbb{C}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]$ – see (4.7). In this section, replacing $w_{\mathbf{k}}\omega$ with the symbol $\mathbb{1}_{\mathbf{k}}$, we define the *fan D-module* in an abstract and combinatorial way. This is closely related to the *better-behaved GKZ system* of Borisov and Horja [**6**], also known as the *multi-GKZ system* [**51**].

Recall the following notation from §2 and §4.2:

- the stacky fan $\mathbf{\Sigma} = (\mathbf{N}, \Sigma, \beta)$ (see §2.1);
- the map $\beta\colon \mathbb{Z}^m \to \mathbf{N}$ sending e_i to b_i;
- the fan sequence $0 \to \mathbb{L} \to \mathbb{Z}^m \xrightarrow{\beta} \mathbf{N}$ from (2.1);
- the divisor sequence $0 \to \mathbf{M} \to (\mathbb{Z}^m)^\star \to \mathbb{L}^\vee$ from (2.2); tensored with $\mathbb{Q}$, this yields $0 \to H^2_{\mathbb{T}}(\mathrm{pt}, \mathbb{Q}) \to H^2_{\mathbb{T}}(X, \mathbb{Q}) \to H^2(X, \mathbb{Q}) \to 0$;
- the monoid and lattice $\mathbf{\Lambda}_+ \subset \mathbf{\Lambda} \subset \mathbb{L}_{\mathbb{Q}}$ (see §2.4–§2.5); we write Q^d, $d \in \mathbf{\Lambda}$, for the element in the group ring $\mathbb{C}[\mathbf{\Lambda}]$ that corresponds to d;
- the finite subset $G \subset \mathbf{N} \cap |\Sigma|$ disjoint from $\{b_1, \ldots, b_m\}$; set $S := G \cup \{b_1, \ldots, b_m\}$;
- deformation parameters $y_{\mathbf{k}}$ with $\mathbf{k} \in S$; we also write $y_i := y_{b_i}$ for $1 \le i \le m$.

To avoid the use of a splitting of the divisor sequence over $\mathbb{C}$ (see §3.2), we introduce a vector field $z\vartheta_\rho$ corresponding to an equivariant cohomology class $\rho \in H^2_{\mathbb{T}}(X, \mathbb{C}) \cong (\mathbb{C}^m)^\star$. We *require* that $z\vartheta_\rho$ acts on the parameters Q^d, $y_{\mathbf{k}}$ as

$$z\vartheta_\rho \cdot Q^d = z(\overline{\rho} \cdot d)Q^d, \qquad z\vartheta_\rho \cdot y_{\mathbf{k}} = 0,$$

where $d \in \mathbf{\Lambda}$, $\mathbf{k} \in S$, and $\overline{\rho} \in \mathbb{L}^\star_{\mathbb{C}}$ is the image of ρ under $(\mathbb{C}^m)^\star \to \mathbb{L}^\star_{\mathbb{C}}$. We identify an equivariant parameter $\chi \in \mathbf{M}_{\mathbb{C}}$ with the vector field $z\vartheta_\chi$ via the inclusion $\mathbf{M}_{\mathbb{C}} \subset (\mathbb{C}^m)^\star$, i.e. $\chi = z\vartheta_\chi$. By choosing a splitting of the divisor sequence, we can write

$$z\vartheta_\rho = \chi^\rho + z\overline{\rho}Q\frac{\partial}{\partial Q}, \tag{5.1}$$

where $\rho \in H^2_{\mathbb{T}}(X, \mathbb{C}) \cong (\mathbb{C}^m)^\star$ corresponds to $(\chi^\rho, \overline{\rho}) \in \mathbf{M}_{\mathbb{C}} \oplus \mathbb{L}^\star_{\mathbb{C}}$ under the splitting. Set

$$z\vartheta_i := z\vartheta_{u_i} \qquad 1 \leq i \leq m,$$

where $u_i \in H^2_{\mathbb{T}}(X, \mathbb{C})$ corresponds to the ith standard basis element $e_i^\star \in (\mathbb{C}^m)^\star$. We write $K[y]$ for the polynomial ring in the variables $\{y_{\mathbf{k}} : \mathbf{k} \in S\}$ over a ring K, and write $\partial_{\mathbf{k}} := \partial/\partial y_{\mathbf{k}}$ for the partial derivative in $y_{\mathbf{k}}$. We also write $\partial_i := \partial_{b_i} = \partial/\partial y_i$.

Definition 5.1. The *fan D-module* is the $\mathbb{C}[z][\mathbf{\Lambda}_+][y]$-module

$$\mathcal{M}(\mathbf{\Sigma}, G) := \bigoplus_{\mathbf{k} \in \mathbf{N} \cap |\Sigma|} \mathbb{C}[z][\mathbf{\Lambda}_+][y]\mathbb{1}_{\mathbf{k}}$$

equipped with the action of differential operators $z\vartheta_1, \ldots, z\vartheta_m$, $z\partial_{\boldsymbol{\ell}}$ (with $\boldsymbol{\ell} \in S$) as follows:

$$\begin{aligned} z\vartheta_i \cdot \mathbb{1}_{\mathbf{k}} &:= z\Psi_i(\mathbf{k})\mathbb{1}_{\mathbf{k}} + \sum_{\boldsymbol{\ell} \in S} \Psi_i(\boldsymbol{\ell}) y_{\boldsymbol{\ell}} Q^{d(\mathbf{k}, \boldsymbol{\ell})} \mathbb{1}_{\mathbf{k}+\boldsymbol{\ell}}, \\ z\partial_{\boldsymbol{\ell}} \cdot \mathbb{1}_{\mathbf{k}} &:= Q^{d(\mathbf{k}, \boldsymbol{\ell})} \mathbb{1}_{\mathbf{k}+\boldsymbol{\ell}}, \end{aligned}$$

where $\mathbf{k} \in \mathbf{N} \cap |\Sigma|$, $\Psi \colon \mathbf{N} \cap |\Sigma| \to (\mathbb{Q}_{\geq 0})^m$ is defined in Notation 2.2, and $d(\mathbf{k}, \boldsymbol{\ell}) \in \mathbf{\Lambda}_+$ is defined in (2.5). These actions on the basis $\mathbb{1}_{\mathbf{k}}$ are extended to $\mathcal{M}(\mathbf{\Sigma}, G)$ by the standard Leibnitz rule. Equivariant parameters $\chi \in \mathbf{M}_{\mathbb{C}}$ act via the identification $\chi = z\vartheta_\chi$ as follows:

$$\chi \cdot \mathbb{1}_{\mathbf{k}} := z\vartheta_\chi \cdot \mathbb{1}_{\mathbf{k}} = z(\chi \cdot \mathbf{k})\mathbb{1}_{\mathbf{k}} + \sum_{\boldsymbol{\ell} \in S} (\chi \cdot \boldsymbol{\ell}) y_{\boldsymbol{\ell}} Q^{d(\mathbf{k}, \boldsymbol{\ell})} \mathbb{1}_{\mathbf{k}+\boldsymbol{\ell}}.$$

Remark 5.2. The fan D-module arises from the Gauss–Manin connection in §4.4. The generator $\mathbb{1}_{\mathbf{k}}$ of $\mathcal{M}(\mathbf{\Sigma}, G)$ corresponds to $w_{\mathbf{k}}\omega \in \mathrm{GM}(F_\chi)$ and the actions of $z\vartheta_\rho$, $z\partial_{\mathbf{k}}$ correspond respectively to the Gauss–Manin connections $\chi^\rho + z\nabla_{\overline{\rho}Q\frac{\partial}{\partial Q}}$, $z\nabla_{\frac{\partial}{\partial y_{\mathbf{k}}}}$ once we choose a splitting as in (5.1) – compare the above definition with (4.6) and (4.8). It is easy to check that the actions of $z\vartheta_i$, $z\partial_{\mathbf{k}}$ commute with each other.

Remark 5.3. The fan D-module is graded with respect to the following (complex) grading:

$$\deg \mathbb{1}_{\mathbf{k}} = |\mathbf{k}|, \quad \deg z = \deg \chi = 1, \quad \deg y_{\mathbf{k}} = 1 - |\mathbf{k}|,$$
$$\deg Q^d = c_1(\mathfrak{X}) \cdot d, \quad \deg z\vartheta_\rho = \deg z\partial_{\mathbf{k}} = 1,$$

where $|\mathbf{k}| = \sum_{i=1}^m \Psi_i(\mathbf{k})$. This corresponds to the grading operator Gr^{B} – see (4.10) – on the Gauss–Manin system.

Remark 5.4 (reduced fan D-module)**.** As noted in Remark 4.5, there is redundancy among the parameters $(Q, y_1, \dots, y_m, \{y_{\mathbf{k}} : \mathbf{k} \in G\})$. Defining new parameters and basis as

$$\overline{Q}^d := \left(\prod_{i=1}^m y_i^{D_i \cdot d}\right) Q^d, \quad \overline{y}_{\boldsymbol{\ell}} := \frac{y_{\boldsymbol{\ell}}}{\prod_{i=1}^m y_i^{\Psi_i(\boldsymbol{\ell})}}, \quad \overline{\mathbb{1}}_{\mathbf{k}} := \left(\prod_{i=1}^m y_i^{\Psi_i(\mathbf{k})}\right) \mathbb{1}_{\mathbf{k}},$$

where $d \in \mathbf{\Lambda}_+$, $\boldsymbol{\ell} \in G$, and $\mathbf{k} \in \mathbf{N} \cap |\Sigma|$, we can remove $y_1, \dots, y_m$ from the presentation of the D-module. Indeed, the vector fields corresponding to these new parameters $(\overline{Q}, \overline{y}_{\boldsymbol{\ell}})$ are given by

$$z\overline{\vartheta}_\rho = z\vartheta_\rho, \quad z\overline{\partial}_{\boldsymbol{\ell}} = \left(\prod_{i=1}^m y_i^{\Psi_i(\boldsymbol{\ell})}\right) z\partial_{\boldsymbol{\ell}}$$

with $\rho \in (\mathbb{C}^m)^\star$, $\boldsymbol{\ell} \in G$, and we have

$$z\overline{\vartheta}_i \cdot \overline{\mathbb{1}}_{\mathbf{k}} = z\Psi_i(\mathbf{k})\overline{\mathbb{1}}_{\mathbf{k}} + \overline{Q}^{d(\mathbf{k}, b_i)}\overline{\mathbb{1}}_{\mathbf{k}+b_i} + \sum_{\boldsymbol{\ell} \in G} \overline{y}_{\boldsymbol{\ell}} \overline{Q}^{d(\mathbf{k}, \boldsymbol{\ell})}\overline{\mathbb{1}}_{\mathbf{k}+\boldsymbol{\ell}},$$
$$z\overline{\partial}_{\boldsymbol{\ell}} \cdot \overline{\mathbb{1}}_{\mathbf{k}} = \overline{Q}^{d(\mathbf{k}, \boldsymbol{\ell})}\overline{\mathbb{1}}_{\mathbf{k}+\boldsymbol{\ell}}$$

with $1 \le i \le m$ and $\boldsymbol{\ell} \in G$. We also have $zy_i\partial_i \cdot \overline{\mathbb{1}}_{\mathbf{k}} = (z\overline{\vartheta}_i - \sum_{\boldsymbol{\ell} \in G} \Psi_i(\boldsymbol{\ell})\overline{y}_{\boldsymbol{\ell}}\overline{\partial}_{\boldsymbol{\ell}})\overline{\mathbb{1}}_{\mathbf{k}}$ for $\partial_i := \partial_{b_i} = \partial/\partial y_i$. This implies that the fan D-module descends to the space of the parameters $(\overline{Q}, \{\overline{y}_{\boldsymbol{\ell}} : \boldsymbol{\ell} \in G\})$. We can alternatively get this reduction by setting $y_1 = \cdots = y_m = 1$ in the definition of $z\vartheta_i$, $z\partial_{\boldsymbol{\ell}}$. We call $\mathcal{M}_{\mathrm{red}}(\mathbf{\Sigma}, G) := \mathcal{M}(\mathbf{\Sigma}, G)/(y_1 - 1, \dots, y_m - 1)$ the *reduced fan D-module.*

Definition 5.5. Let $\mathfrak{m}$ denote the ideal of $\mathbb{C}[z][\mathbf{\Lambda}_+][y]$ generated by Q^d with $d \in \mathbf{\Lambda}_+ \setminus \{0\}$, $y_1 - 1, \dots, y_m - 1$ and $y_{\mathbf{k}}$ with $\mathbf{k} \in G$. We define the *completed fan D-module* to be the $\mathfrak{m}$-adic completion of $\mathcal{M}(\mathbf{\Sigma}, G)$:

$$\widehat{\mathcal{M}}(\mathbf{\Sigma}, G) := \varprojlim_k \mathcal{M}(\mathbf{\Sigma}, G)/\mathfrak{m}^k \mathcal{M}(\mathbf{\Sigma}, G) = \widehat{\bigoplus_{\mathbf{k} \in \mathbf{N} \cap |\Sigma|}} \mathbb{C}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]\mathbb{1}_{\mathbf{k}},$$

where we used the convention (4.4) for the ring of power series in $\{y_{\mathbf{k}} : \mathbf{k} \in S\}$. This naturally becomes a module over $R_{\mathbb{T}}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]$, where $R_{\mathbb{T}} = \mathrm{Sym}^*(\mathbf{M}_{\mathbb{C}})$, and is equipped with the action of $z\vartheta_1, \dots, z\vartheta_m$ and $z\partial_{\mathbf{k}}$ with $\mathbf{k} \in S$. We can similarly define the *completed reduced fan D-module* $\widehat{\mathcal{M}}_{\mathrm{red}}(\mathbf{\Sigma}, G)$.

It is clear from the definition that the completed fan D-module is isomorphic to the equivariant Gauss–Manin system. The *equivariant quantum D-module* of $\mathfrak{X}$ is defined to be the module $H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \otimes_{R_{\mathbb{T}}} R_{\mathbb{T}}[z][\![\mathbf{\Lambda}_+]\!][\![\tau]\!]$ equipped with the action of the quantum connection ∇ multiplied by z – see (3.2). Note that the quantum connection $z\nabla_{\xi Q \frac{\partial}{\partial Q}}$ in the Q-direction and equivariant parameter χ together give the action of $H^2_{\mathbb{T}}(X,\mathbb{C}) \cong (\mathbb{C}^m)^\star$ similarly to (5.1); the action of $H^2_{\mathbb{T}}(X,\mathbb{C})$ is canonical in the sense that it does not depend on the choice of a splitting. Theorem 4.28 gives the following:

Theorem 5.6. *The completed fan D-module $\widehat{\mathcal{M}}(\mathbf{\Sigma}, G)$ is isomorphic to the pull-back of the equivariant quantum D-module of the toric stack $\mathfrak{X}$ by the mirror map $\tau = \tau(y)$ from Theorem 4.28. The isomorphism here preserves the grading.*

5.2. GKZ-style presentation. We give a Gelfand–Kapranov–Zelevinsky-style presentation [**31**] of the (completed) fan D-module. The following two propositions follow immediately from the definition.

Proposition 5.7. *We have the relations $\mathcal{R}_{i,\mathbf{k}} = \mathcal{P}_{d_1,d_2;a,a';\mathbf{k}_1,\mathbf{k}_2} = 0$ in the fan D-module, where*

$$\mathcal{R}_{i,\mathbf{k}} := z\vartheta_i \cdot \mathbb{1}_{\mathbf{k}} - \left(z\Psi_i(k) + zy_i\partial_i + \sum_{\boldsymbol{\ell}\in G} zy_{\boldsymbol{\ell}}\partial_{\boldsymbol{\ell}} \right) \mathbb{1}_{\mathbf{k}},$$

$$\mathcal{P}_{d_1,d_2;a,a';\mathbf{k}_1,\mathbf{k}_2} := Q^{d_1} \prod_{\boldsymbol{\ell}\in S} (z\partial_{\boldsymbol{\ell}})^{a_{\boldsymbol{\ell}}} \mathbb{1}_{\mathbf{k}_1} - Q^{d_2} \prod_{\boldsymbol{\ell}\in S} (z\partial_{\boldsymbol{\ell}})^{a'_{\boldsymbol{\ell}}} \mathbb{1}_{\mathbf{k}_2}.$$

Here $1 \le i \le m$, $\mathbf{k} \in \mathbf{N} \cap |\Sigma|$, and $(d_1, d_2, a, a', \mathbf{k}_1, \mathbf{k}_2)$ ranges over all $d_1, d_2 \in \mathbf{\Lambda}_+$, $a, a' \in (\mathbb{Z}_{\ge 0})^S$, $\mathbf{k}_1, \mathbf{k}_2 \in \mathbf{N} \cap |\Sigma|$ such that $\mathbf{k}_1 + \sum_{\boldsymbol{\ell}\in S} a_{\boldsymbol{\ell}}\boldsymbol{\ell} = \mathbf{k}_2 + \sum_{\boldsymbol{\ell}\in S} a'_{\boldsymbol{\ell}}\boldsymbol{\ell}$ in $\mathbf{N}$ and that $d_1 + \sum_{\boldsymbol{\ell}\in S} a_{\boldsymbol{\ell}}\Psi(\boldsymbol{\ell}) + \Psi(\mathbf{k}_1) = d_2 + \sum_{\boldsymbol{\ell}\in S} a'_{\boldsymbol{\ell}}\Psi(\boldsymbol{\ell}) + \Psi(\mathbf{k}_2)$ in $\mathbb{Q}^m$.

Proposition 5.8 (reduced case)**.** *In the reduced fan D-module in Remark 5.4, we have the relations $\mathcal{P}'_{d_1,d_2;a,a';\mathbf{k}_1,\mathbf{k}_2} = 0$ with*

$$\begin{aligned}
&\mathcal{P}'_{d_1,d_2;a,a';\mathbf{k}_1,\mathbf{k}_2} \\
&\quad := Q^{d_1} \prod_{\boldsymbol{\ell}\in G} (z\partial_{\boldsymbol{\ell}})^{a_{\boldsymbol{\ell}}} \cdot \prod_{i=1}^{m} \prod_{\nu=0}^{a_i-1} \left(z\vartheta_i - z(\nu + \Psi_i(\mathbf{k}_1)) - \sum_{\boldsymbol{\ell}\in G} zy_{\boldsymbol{\ell}}\partial_{\boldsymbol{\ell}} \right) \mathbb{1}_{\mathbf{k}_1} \\
&\quad - Q^{d_2} \prod_{\boldsymbol{\ell}\in G} (z\partial_{\boldsymbol{\ell}})^{a'_{\boldsymbol{\ell}}} \cdot \prod_{i=1}^{m} \prod_{\nu=0}^{a'_i-1} \left(z\vartheta_i - z(\nu + \Psi_i(\mathbf{k}_2)) - \sum_{\boldsymbol{\ell}\in G} zy_{\boldsymbol{\ell}}\partial_{\boldsymbol{\ell}} \right) \mathbb{1}_{\mathbf{k}_2},
\end{aligned}$$

where $(d_1, d_2, a, a', \mathbf{k}_1, \mathbf{k}_2)$ ranges over the same set as in Proposition 5.7.

These relations give a presentation of the (reduced) fan D-module. We set

$$\mathcal{D} = \mathbb{C}[z][\mathbf{\Lambda}_+][y]\,\langle z\vartheta_1, \dots, z\vartheta_m, \{z\partial_{\boldsymbol{\ell}}\}_{\boldsymbol{\ell}\in S}\rangle\,,$$
$$\mathcal{D}' = \mathbb{C}[z][\mathbf{\Lambda}_+][\{y_{\boldsymbol{\ell}}\}_{\boldsymbol{\ell}\in G}]\,\langle z\vartheta_1, \dots, z\vartheta_m, \{z\partial_{\boldsymbol{\ell}}\}_{\boldsymbol{\ell}\in G}\rangle\,,$$

where the standard commutation relations $[z\vartheta_i, Q^d] = z(D_i \cdot d)Q^d$, $[z\partial_{\boldsymbol{\ell}}, y_{\mathbf{k}}] = z\delta_{\mathbf{k},\boldsymbol{\ell}}$, $[z\vartheta_i, y_{\mathbf{k}}] = [z\partial_{\mathbf{k}}, Q^d] = [z\vartheta_i, z\vartheta_j] = [z\vartheta_i, z\partial_{\boldsymbol{\ell}}] = [z\partial_{\boldsymbol{\ell}}, z\partial_{\mathbf{k}}] = 0$ are implicitly imposed. The fan D-module $\mathcal{M}(\mathbf{\Sigma}, G)$ (resp. $\mathcal{M}_{\mathrm{red}}(\mathbf{\Sigma}, G)$) is a $\mathcal{D}$-module (resp. $\mathcal{D}'$-module).

Theorem 5.9. *Let $\mathbf{k}_1, \dots, \mathbf{k}_s$ be elements of $\mathbf{N} \cap |\Sigma|$ such that for every maximal cone $\sigma \in \Sigma(n)$ we have*

$$\mathbf{N} \cap \sigma = \bigcup_{1\le i\le s:\overline{\mathbf{k}_i}\in\sigma} \left(\mathbf{k}_i + \sum_{\boldsymbol{\ell}\in S:\overline{\boldsymbol{\ell}}\in\sigma} \mathbb{Z}_{\ge 0}\boldsymbol{\ell}\right).$$

Then we have the following.

1. *As a $\mathcal{D}$-module, the fan D-module $\mathcal{M}(\mathbf{\Sigma}, G)$ is generated by $\mathbb{1}_{\mathbf{k}_1}, \dots, \mathbb{1}_{\mathbf{k}_s}$. All the relations among $\mathbb{1}_{\mathbf{k}_1}, \dots, \mathbb{1}_{\mathbf{k}_s}$ are generated by $\mathcal{R}_{\mathbf{k}_j,i}$, $\mathcal{P}_{d_1,d_2;a,a';\mathbf{k}_j,\mathbf{k}_l}$ with $1 \le j, l \le s$ in Proposition 5.7.*
2. *As a $\mathcal{D}'$-module, the reduced fan D-module $\mathcal{M}_{\mathrm{red}}(\mathbf{\Sigma}, G)$ is generated by $\mathbb{1}_{\mathbf{k}_1}, \dots, \mathbb{1}_{\mathbf{k}_s}$. All the relations among $\mathbb{1}_{\mathbf{k}_1}, \dots, \mathbb{1}_{\mathbf{k}_s}$ are generated by $\mathcal{P}'_{d_1,d_2;a,a';\mathbf{k}_j,\mathbf{k}_l}$ with $1 \le j, l \le s$ in Proposition 5.8.*

Proof. We give a proof of part (1). The proof of part (2) is similar and is left to the reader. Our assumption on $\mathbf{k}_1, \dots, \mathbf{k}_s$ implies that for any $\mathbf{k} \in \mathbf{N} \cap \sigma$, there exist $1 \le i \le s$ and $\boldsymbol{\ell}_1, \dots, \boldsymbol{\ell}_t \in S$ such that $\mathbf{k} = \mathbf{k}_i + \boldsymbol{\ell}_1 + \cdots + \boldsymbol{\ell}_t$ and that $\overline{\mathbf{k}}_i, \overline{\boldsymbol{\ell}}_1, \dots, \overline{\boldsymbol{\ell}}_t \in \sigma$. Then we have $\mathbb{1}_{\mathbf{k}} = z\partial_{\boldsymbol{\ell}_1} \cdots z\partial_{\boldsymbol{\ell}_t}\mathbb{1}_{\mathbf{k}_i}$. Therefore $\mathcal{M}(\mathbf{\Sigma}, G)$ is generated by $\mathbb{1}_{\mathbf{k}_1}, \dots, \mathbb{1}_{\mathbf{k}_s}$ as a $\mathcal{D}$-module. Suppose that we have a relation $\sum_{j=1}^s f_j(z, Q, y, z\vartheta, z\partial)\mathbb{1}_{\mathbf{k}_j} = 0$ with $f_j \in \mathcal{D}$ in the fan D-module. Modulo the relations $\mathcal{R}_{\mathbf{k}_j,i}$, this relation can be reduced to a relation which does not involve $z\vartheta_i$. Thus we may assume that f_j does not contain $z\vartheta_1, \dots, z\vartheta_m$. Then we can expand

$$f_j = \sum_{a\in(\mathbb{Z}_{\ge 0})^S} f_{j,a}(z, Q, y) \prod_{\boldsymbol{\ell}\in S} (z\partial_{\boldsymbol{\ell}})^{a_{\boldsymbol{\ell}}}$$

for some $f_{j,a} \in \mathbb{C}[z][\mathbf{\Lambda}_+][y]$. The relation implies that

$$\sum_{(j,a):\mathbf{k}_j+\sum_{\boldsymbol{\ell}\in S} a_{\boldsymbol{\ell}}\boldsymbol{\ell}=\mathbf{k}} f_{j,a}(z, Q, y) Q^{\Psi(\mathbf{k}_j)+\sum_{\boldsymbol{\ell}\in S} a_{\boldsymbol{\ell}}\Psi(\boldsymbol{\ell})-\Psi(\mathbf{k})} = 0 \tag{5.2}$$

for every $\mathbf{k} \in \mathbf{N} \cap |\Sigma|$. The relation $\sum_{j=1}^s f_j\mathbb{1}_{\mathbf{k}_j} = 0$ is the sum of the relations

$$\sum_{(j,a):\mathbf{k}_j+\sum_{\boldsymbol{\ell}\in S} a_{\boldsymbol{\ell}}\boldsymbol{\ell}=\mathbf{k}} f_{j,a}(z, Q, y) \prod_{\boldsymbol{\ell}\in S} (z\partial_{\boldsymbol{\ell}})^{a_{\boldsymbol{\ell}}} \cdot \mathbb{1}_{\mathbf{k}_j} = 0.$$

One can easily see that each of these is generated by $\mathcal{P}_{d_1,d_2;a,a';\mathbf{k}_j,\mathbf{k}_l}$'s over $\mathbb{C}[z][y]$ by using (5.2). q.e.d.

The completed fan D-module is described in terms of the closure of relations in the $\mathfrak{m}$-adic topology. We introduce the following rings of differential operators:

$$\widehat{\mathcal{D}} = \mathbb{C}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]\langle z\vartheta_1, \ldots, z\vartheta_m, \{z\partial_{\boldsymbol{\ell}}\}_{\boldsymbol{\ell}\in S}\rangle,$$
$$\widehat{\mathcal{D}}' = \mathbb{C}[z][\![\mathbf{\Lambda}_+]\!][\![\{y_{\boldsymbol{\ell}}\}_{\boldsymbol{\ell}\in G}]\!]\langle z\vartheta_1, \ldots, z\vartheta_m, \{z\partial_{\boldsymbol{\ell}}\}_{\boldsymbol{\ell}\in G}\rangle,$$

where we use the convention (4.4) for the ring of power series in $\{y_{\boldsymbol{\ell}} : \boldsymbol{\ell} \in S\}$. We define the topology on $\widehat{\mathcal{D}}$ (resp. $\widehat{\mathcal{D}}'$) by the decreasing $\mathbb{C}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]$-submodules $\mathfrak{m}^p\widehat{\mathcal{D}}$ (resp. $\mathbb{C}[z][\![\mathbf{\Lambda}_+]\!][\![\{y\}_{\boldsymbol{\ell}\in G}]\!]$-submodules $\mathfrak{m}'^p\widehat{\mathcal{D}}'$), where $\mathfrak{m} \subset \mathbb{C}[z][\mathbf{\Lambda}_+][y]$ is as in Definition 5.5 and $\mathfrak{m}' := \mathfrak{m} \cap \mathbb{C}[z][\mathbf{\Lambda}_+][\{y_{\boldsymbol{\ell}}\}_{\boldsymbol{\ell}\in G}]$. Note that $\mathfrak{m}^p\widehat{\mathcal{D}}$ and $\mathfrak{m}'^p\widehat{\mathcal{D}}'$ are only right ideals. Note also that $\widehat{\mathcal{D}}$ and $\widehat{\mathcal{D}}'$ are not complete with respect to their topologies.

Theorem 5.10. *Let $\mathbf{k}_1, \ldots, \mathbf{k}_s \in \mathbf{N} \cap |\Sigma|$ be elements satisfying the condition in Theorem 5.9. Then we have the following.*

(1) *The completed fan D-module $\widehat{\mathcal{M}}(\mathbf{\Sigma}, G)$ has the following presentation as a $\widehat{\mathcal{D}}$-module:*

$$\widehat{\mathcal{M}}(\mathbf{\Sigma}, G) \cong \bigoplus_{j=1}^{s} \widehat{\mathcal{D}}\mathbb{1}_{\mathbf{k}_j} \Big/ \overline{\mathfrak{I}},$$

where $\mathfrak{I}$ is the left $\widehat{\mathcal{D}}$-submodule of $\bigoplus_{j=1}^{s} \widehat{\mathcal{D}}\mathbb{1}_{\mathbf{k}_j}$ generated by $\mathcal{R}_{i,\mathbf{k}_j}$, $\mathcal{P}_{d_1,d_2;a,a';\mathbf{k}_j,\mathbf{k}_l}$ in Proposition 5.7 with $1 \le j, l \le s$, and $\overline{\mathfrak{I}}$ is the closure[5] *of $\mathfrak{I}$ in $\bigoplus_{j=1}^{s} \widehat{\mathcal{D}}\mathbb{1}_{\mathbf{k}_j}$.*

(2) *The completed reduced fan D-module $\widehat{\mathcal{M}}_{\mathrm{red}}(\mathbf{\Sigma}, G)$ has the following presentation as a $\widehat{\mathcal{D}}'$-module:*

$$\widehat{\mathcal{M}}_{\mathrm{red}}(\mathbf{\Sigma}, G) \cong \bigoplus_{j=1}^{s} \widehat{\mathcal{D}}'\mathbb{1}_{\mathbf{k}_j} \Big/ \overline{\mathfrak{I}'},$$

where $\mathfrak{I}'$ is the left $\widehat{\mathcal{D}}'$-submodule of $\bigoplus_{j=1}^{s} \widehat{\mathcal{D}}'\mathbb{1}_{\mathbf{k}_j}$ generated by $\mathcal{P}'_{d_1,d_2;a,a';\mathbf{k}_j,\mathbf{k}_l}$ in Proposition 5.8 with $1 \le j, l \le s$, and $\overline{\mathfrak{I}'}$ is the closure of $\mathfrak{I}'$ in $\bigoplus_{j=1}^{s} \widehat{\mathcal{D}}'\mathbb{1}_{\mathbf{k}_j}$.

Proof. We only give a proof of part (1). The proof of part (2) is similar. The fact that $\mathbb{1}_{\mathbf{k}_1}, \ldots, \mathbb{1}_{\mathbf{k}_s}$ generate $\widehat{\mathcal{M}}(\mathbf{\Sigma}, G)$ as a $\widehat{\mathcal{D}}$-module follows from the discussion in the proof of Theorem 5.9. It is easy to show that elements of $\overline{\mathfrak{I}}$ are relations in $\widehat{\mathcal{M}}(\mathbf{\Sigma}, G)$. Suppose we have a relation $\sum_{j=1}^{s} f_j(z, Q, y, z\vartheta, z\partial)\mathbb{1}_{\mathbf{k}_j} = 0$ in $\widehat{\mathcal{M}}(\mathbf{\Sigma}, G)$ for some $f_j \in \widehat{\mathcal{D}}$.

[5] One can check that the closure of a left $\widehat{\mathcal{D}}$-submodule in $\bigoplus_{j=1}^{s} \widehat{\mathcal{D}}\mathbb{1}_{\mathbf{k}_j}$ becomes a left $\widehat{\mathcal{D}}$-submodule using the fact that $z\vartheta_i\mathfrak{m}^p\widehat{\mathcal{D}} \subset \mathfrak{m}^p\widehat{\mathcal{D}}$ and $z\partial_{\boldsymbol{\ell}}\mathfrak{m}^p\widehat{\mathcal{D}} \subset \mathfrak{m}^{p-1}\widehat{\mathcal{D}}$.

For each $p \in \mathbb{Z}_{\geq 0}$ we can write $f_j = f_j^{(p)} + r_j^{(p)}$ with $f_j^{(p)} \in \mathcal{D}$ and $r_j^{(p)} \in \mathfrak{m}^p \widehat{\mathcal{D}}$. Then we have that

$$x := \sum_{j=1}^{s} f_j^{(p)} \mathbb{1}_{\mathbf{k}_j} = -\sum_{j=1}^{s} r_j^{(p)} \mathbb{1}_{\mathbf{k}_j}$$

belongs to $\mathfrak{m}^p \mathcal{M}(\boldsymbol{\Sigma}, G) = \bigoplus_{\mathbf{k} \in \mathbf{N} \cap |\Sigma|} \mathfrak{m}^p \mathbb{1}_{\mathbf{k}}$. By the surjectivity of $\bigoplus_{j=1}^{s} \mathfrak{m}^p \mathcal{D} \mathbb{1}_{\mathbf{k}_j} \to \mathfrak{m}^p \mathcal{M}(\boldsymbol{\Sigma}, G)$, we can write $x = \sum_{j=1}^{s} g_j^{(p)} \mathbb{1}_{\mathbf{k}_j}$ for some $g_j^{(p)} \in \mathfrak{m}^p \mathcal{D}$. We now have that $\sum_{j=1}^{s} (f_j^{(p)} - g_j^{(p)}) \mathbb{1}_{\mathbf{k}_j} = 0$, and thus $h^{(p)} := \bigoplus_{j=1}^{s} (f_j^{(p)} - g_j^{(p)}) \mathbb{1}_{\mathbf{k}_j}$ belongs to $\mathfrak{I}$ by Theorem 5.9. Since $h^{(p)}$ converges to $\bigoplus_{j=1}^{s} f_j \mathbb{1}_{\mathbf{k}_j}$ as $p \to \infty$, we have that $\bigoplus_{j=1}^{s} f_j \mathbb{1}_{\mathbf{k}_j} \in \overline{\mathfrak{I}}$.

q.e.d.

Remark 5.11. When $\mathfrak{X}$ is a toric manifold, $\mathcal{M}(\boldsymbol{\Sigma}, G)$, $\widehat{\mathcal{M}}(\boldsymbol{\Sigma}, G)$ and their reduced versions are generated by $\mathbb{1}_0$. The relations $\mathcal{P}_{d_1, d_2; a, a'; 0, 0}$ then define the standard GKZ system [**31**] and also appear as relations in the quantum D-module [**32**, Theorem 1]. (For general $\mathfrak{X}$, these D-modules are generated by $\mathbb{1}_0$ for a sufficiently large G.) The closure of the GKZ ideal appeared in [**48**, Proposition 5.4] for compact toric manifolds. A closely related presentation has been discussed in [**50**, §4.2], [**2**, §5.2], [**61**, Theorem 6.6] for compact toric stacks. The relations $\mathcal{P}'_{d_1, d_2; a, a'; 0, 0}$ for $\mathbb{1}_0$ in the reduced fan D-module also appeared in [**61**, Lemma 4.7].

Remark 5.12. The same result as Theorem 5.10 holds, with the same proof, when we replace $\widehat{\mathcal{D}}$ and $\widehat{\mathcal{D}}'$ respectively with

$$\begin{aligned} & R_{\mathbb{T}}[z][\![\boldsymbol{\Lambda}_+]\!][\![y]\!] \left\langle z\vartheta_1, \ldots, z\vartheta_m, \{z\partial_{\boldsymbol{\ell}}\}_{\boldsymbol{\ell} \in S} \right\rangle \\ \text{and} \quad & R_{\mathbb{T}}[z][\![\boldsymbol{\Lambda}_+]\!][\![\{y_{\boldsymbol{\ell}}\}_{\boldsymbol{\ell} \in G}]\!] \left\langle z\vartheta_1, \ldots, z\vartheta_m, \{z\partial_{\boldsymbol{\ell}}\}_{\boldsymbol{\ell} \in G} \right\rangle, \end{aligned}$$

where the relations $\chi = z\vartheta_\chi$ with $\chi \in \mathbf{M}_{\mathbb{C}}$ are implicitly imposed.

5.3. Quantum cohomology ring. We next give a quantum Stanley–Reisner (or Batyrev-style) description of the quantum cohomology algebra of $\mathfrak{X}$.

Theorem 5.13. *Let $\tau = \tau(y)$ be the mirror map from Theorem 4.28. The pull-back of the big and equivariant quantum cohomology ring $\left(H^*_{\mathrm{CR}, \mathbb{T}}(\mathfrak{X}) \otimes_{R_{\mathbb{T}}} R_{\mathbb{T}}[\![\boldsymbol{\Lambda}_+]\!][\![y]\!], \tau^*(\star)\right)$ by τ is isomorphic to any one of the following rings as an $R_{\mathbb{T}}[\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$-algebra:*

(a) *the Jacobian ring of the equivariant Landau–Ginzburg potential F_χ,*

$$\operatorname{Jac}(F_\chi) := \mathbb{C}\{\mathbb{O}_+\}[\![y]\!][\chi] \Big/ \left\langle x_i \frac{\partial F_\chi}{\partial x_i} : 1 \leq i \leq n \right\rangle \cong \mathbb{C}\{\mathbb{O}_+\}[\![y]\!];$$

(b) *the vector space* $\widehat{\bigoplus}_{\mathbf{k}\in\mathbf{N}\cap|\Sigma|}\mathbb{C}[\![\mathbf{\Lambda}_+]\!][\![y]\!]\mathbb{1}_{\mathbf{k}}$ *equipped with the following product and the* $R_{\mathbb{T}}$*-module structure:*

$$\mathbb{1}_{\mathbf{k}}\star\mathbb{1}_{\boldsymbol{\ell}}=Q^{d(\mathbf{k},\boldsymbol{\ell})}\mathbb{1}_{\mathbf{k}+\boldsymbol{\ell}},\qquad \chi=\sum_{\boldsymbol{\ell}\in S}(\chi\cdot\boldsymbol{\ell})y_{\boldsymbol{\ell}}\mathbb{1}_{\boldsymbol{\ell}},$$

where $\widehat{\bigoplus}$ *denotes the completed direct sum with respect to the* $\mathfrak{m}$*-adic topology (see Definition 5.5),* $\chi\in\mathbf{M}_{\mathbb{C}}$*, and* $d(\mathbf{k},\boldsymbol{\ell})$ *is defined in* (2.5)*.*

Remark 5.14. In part (a) above, we follow the notation from §4.1–§4.2 and define co-ordinates $x_1,\dots,x_n$ by choosing an isomorphism $\mathbf{N}\cong\mathbb{Z}^n\times\mathbf{N}_{\mathrm{tor}}$; we have $x_i\frac{\partial F_\chi}{\partial x_i}=\sum_{\mathbf{k}\in S}y_{\mathbf{k}}k_iw_{\mathbf{k}}-\chi_i$ with k_i being the ith component of $\overline{\mathbf{k}}\in\mathbf{N}/\mathbf{N}_{\mathrm{tor}}\cong\mathbb{Z}^n$ and χ_i being the ith basis of $\mathbf{M}\cong\mathbb{Z}^n$.

Remark 5.15. Note that the presentation in part (b) yields the description of the Chen–Ruan cup product in §2.3 at the classical limit $Q=0$, $y=y^*$ – see (4.2) for y^*.

Remark 5.16. A presentation of the quantum cohomology of Fano toric manifolds was originally found by Givental [**34**] and Batyrev [**4**], and generalizations to arbitrary toric manifolds were discussed in [**8**, **30**, **37**, **48**, **62**]. The Jacobian description of the quantum cohomology of toric stacks, in the non-equivariant case, was also given in [**41**, **50**].

Proof of Theorem 5.13. We remark that for an extension $G_1\subset G_2$ of the finite set G, the corresponding mirror maps and mirror isomorphisms in Theorem 4.28 are related by restriction. Therefore we can define the partial derivative $(\partial\tau/\partial y_{\mathbf{k}})\in H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})\otimes_{R_{\mathbb{T}}}R_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![y]\!]$ for every $\mathbf{k}\in\mathbf{N}\cap|\Sigma|$ by adding $\mathbf{k}$ to G (if necessary) and then restricting to $y_{\mathbf{k}}=0$ if $\mathbf{k}\notin S$. We claim that the $\mathbb{C}[\![\mathbf{\Lambda}_+]\!][\![y]\!]$-module homomorphism sending $w_{\mathbf{k}}\in\mathrm{Jac}(F_\chi)$ to $(\partial\tau/\partial y_{\mathbf{k}})$ gives[6] the desired isomorphism in part (a). The mirror isomorphism Θ induces the isomorphism of $R_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![y]\!]$-modules:

$$\mathrm{Jac}(F_\chi)\cdot\omega\cong\mathrm{GM}(F_\chi)/z\,\mathrm{GM}(F_\chi)\xrightarrow{\Theta|_{z=0}}H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})\otimes_{R_{\mathbb{T}}}R_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![y]\!].$$

Again by adding $\mathbf{k}$ to G if necessary, we can show that this intertwines the action of $w_{\mathbf{k}}$ with the quantum multiplication by $\partial\tau/\partial y_{\mathbf{k}}$ (since Θ intertwines the Gauss–Manin connection $z\nabla_{\partial/\partial y_{\mathbf{k}}}$ with the quantum connection $z(\tau^*\nabla)_{\partial/\partial y_{\mathbf{k}}}$). Part (a) follows. Part (b) follows easily from part (a) by noting that $\mathbb{C}\{\mathbb{O}_+\}[\![y]\!]$ is isomorphic to $\widehat{\bigoplus}_{\mathbf{k}\in\mathbf{N}\cap|\Sigma|}\mathbb{C}[\![\mathbf{\Lambda}_+]\!][\![y]\!]$ – see (4.3). q.e.d.

[6] This is well-defined, since $\{w_{\mathbf{k}}\}_{\mathbf{k}\in\mathbf{N}\cap|\Sigma|}$ is a topological $\mathbb{C}[\![\mathbf{\Lambda}_+]\!][\![y]\!]$-basis of $\mathrm{Jac}(F_\chi)$.

5.4. Examples. We give several examples of the fan D-module.

5.4.1. The case where $\mathfrak{X} = \mathbb{P}^1$ and $G = \varnothing$. The stacky fan is given by $\mathbf{N} = \mathbb{Z}$, $b_1 = 1$, $b_2 = -1$. We have $\mathbb{L} = \mathbf{\Lambda} = \{(l,l) \in \mathbb{Z}^2 \mid l \in \mathbb{Z}\} \subset \mathbb{Z}^2$. Let $Q \in \mathbb{C}[\mathbf{\Lambda}]$ correspond to the positive generator $(1,1) \in \mathbf{\Lambda}_+$ and let $\chi \in \mathbf{M} = \mathrm{Hom}(\mathbf{N}, \mathbb{Z}) \cong \mathbb{Z}$ be the standard generator. We have the identification

$$\chi = z\vartheta_1 - z\vartheta_2.$$

By choosing $\chi_1, \chi_2 \in \mathbf{M}_{\mathbb{C}}$ with $\chi_1 - \chi_2 = \chi$, we can also write

$$z\vartheta_1 = zQ\frac{\partial}{\partial Q} + \chi_1, \quad z\vartheta_2 = zQ\frac{\partial}{\partial Q} + \chi_2.$$

The action of these operators on the reduced fan D-module $\mathcal{M}_{\mathrm{red}}(\mathbf{\Sigma}_{\mathbb{P}^1})$ is shown in Figure 1.

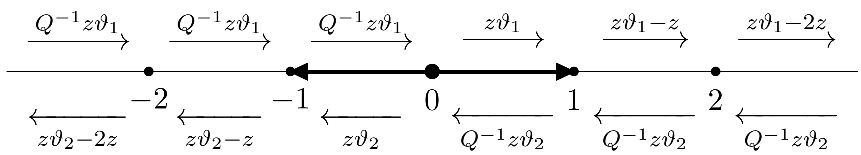

Figure 1. $\mathcal{M}_{\mathrm{red}}(\mathbf{\Sigma}_{\mathbb{P}^1})$.

From the relation $(z\vartheta_1 z\vartheta_2 - Q)\mathbb{1}_0 = 0$, we obtain the presentation:

$$\mathcal{M}_{\mathrm{red}}(\mathbf{\Sigma}_{\mathbb{P}^1}) \cong \mathbb{C}[z, Q]\langle z\vartheta_1, z\vartheta_2\rangle / \langle z\vartheta_1 z\vartheta_2 - Q\rangle.$$

The $\mathbb{C}[\chi, z, Q]$-basis $\mathbb{1}_0, \mathbb{1}_1$ of $\mathcal{M}_{\mathrm{red}}(\mathbf{\Sigma}_{\mathbb{P}^1})$ corresponds to the basis $\{1, u_1\} \subset H^*_{\mathbb{T}}(\mathbb{P}^1)$ under the mirror isomorphism. (In this case, we do not need the completion.)

5.4.2. The case where $\mathfrak{X} = \mathbb{P}(1,2)$ and $G \neq \varnothing$. The stacky fan is given by $\mathbf{N} = \mathbb{Z}$, $b_1 = 2$, $b_2 = -1$. We choose $G = \{1\}$. Then $\mathbb{L} = \{(l, 2l) : l \in \mathbb{Z}\} \subset \mathbf{\Lambda} = \frac{1}{2}\mathbb{L}$. Let $Q \in \mathbb{C}[\mathbf{\Lambda}_+]$ be the variable corresponding to $(1,2) \in \mathbf{\Lambda}_+$ and let $y = y_1$ be the variable corresponding to $1 \in G$. We write χ for the standard generator of $\mathbf{M} = \mathrm{Hom}(\mathbf{N}, \mathbb{Z}) \cong \mathbb{Z}$. We have

$$2z\vartheta_1 - z\vartheta_2 = \chi. \tag{5.3}$$

The actions of $z\vartheta_1, z\vartheta_2, z\partial_y$ on the reduced fan D-module $\mathcal{M}_{\mathrm{red}}(\mathbf{\Sigma}_{\mathbb{P}(1,2)}, \{1\})$ are shown in Figure 2.

Equation (5.3) gives relations among consecutive 4 basis elements, for example:

$$0 = (2z\vartheta_1 - z\vartheta_2 - \chi)\mathbb{1}_0 = -\mathbb{1}_{-1} - \chi\mathbb{1}_0 + y\mathbb{1}_1 + 2\mathbb{1}_2,$$

$$0 = (2z\vartheta_1 - z\vartheta_2 - \chi)\mathbb{1}_{-1} = -\mathbb{1}_{-2} - (\chi + z)\mathbb{1}_{-1} + yQ^{\frac{1}{2}}\mathbb{1}_0 + 2Q^{\frac{1}{2}}\mathbb{1}_1.$$

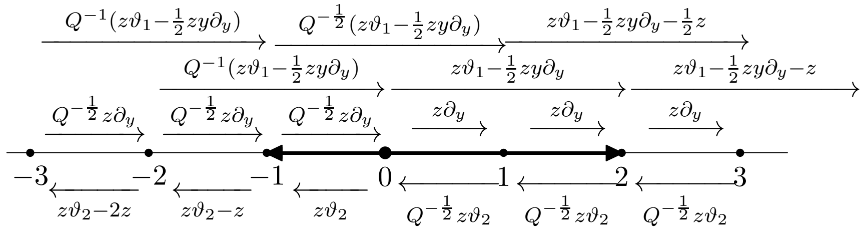

Figure 2. $\mathcal{M}_{\mathrm{red}}(\mathbf{\Sigma}_{\mathbb{P}(1,2)}, \{1\})$.

In fact, $\{\mathbb{1}_0, \mathbb{1}_{-1}, \mathbb{1}_1\}$ gives a free $\mathbb{C}[z, \chi, Q^{\frac{1}{2}}, y]$-basis of $\mathcal{M}_{\mathrm{red}}(\mathbf{\Sigma}_{\mathbb{P}(1,2)}, \{1\})$. The actions of $z\vartheta_2$ and $z\partial_y$ in this basis are represented by the following matrices:

$$\begin{pmatrix} 0 & yQ^{\frac{1}{2}} & Q^{\frac{1}{2}} \\ 1 & -\chi & 0 \\ 0 & 2Q^{\frac{1}{2}} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & Q^{\frac{1}{2}} & \frac{1}{2}\chi \\ 0 & 0 & \frac{1}{2} \\ 1 & 0 & -\frac{1}{2}y \end{pmatrix}.$$

The basis $\{\mathbb{1}_0, \mathbb{1}_{-1}, \mathbb{1}_1\}$ corresponds to $\{1, u_2, \mathbf{1}_1\} \subset H^*_{\mathrm{CR},\mathbb{T}}(\mathbb{P}(1,2))$ under the mirror isomorphism, where $\mathbf{1}_1$ is the twisted sector supported on $\mathbb{P}(2)$, and the above matrices represent the quantum multiplication by u_2 and $\mathbf{1}_1$. Here the completion is (again) unnecessary and the mirror map is given by

$$\tau(y) = y\mathbf{1}_1.$$

The reduced fan D-module is generated by $\mathbb{1}_0$ and defined by the following relations:

$$\begin{aligned} R_1 &= \left(z\vartheta_1 - \frac{1}{2}zy\partial_y\right) z\vartheta_2(z\vartheta_2 - z) - Q, \\ R_2 &= z\partial_y z\vartheta_2 - Q^{\frac{1}{2}}, \\ R_3 &= z\vartheta_1 - \frac{1}{2}zy\partial_y - (z\partial_y)^2, \\ R_4 &= Q^{\frac{1}{2}}z\partial_y - \left(z\vartheta_1 - \frac{1}{2}zy\partial_y\right) z\vartheta_2. \end{aligned}$$

5.4.3. The case where $\mathfrak{X} = [\mathbb{C}^2/\mu_2]$ and $G \neq \varnothing$. The stacky fan is given by $\mathbf{N} = \mathbb{Z}^2$, $b_1 = (0,1)$, $b_2 = (2,1)$. We choose $G = \{b_3\}$ with $b_3 = (1,1)$. We have $\mathbf{\Lambda} = 0$ (all curves in $\mathfrak{X}$ are constant). Let y be the variable corresponding to $b_3 \in G$ and let χ_1, χ_2 be the basis of $\mathbf{M}_{\mathbb{C}}$ dual to b_1, b_2. We have

$$\chi_1 = z\vartheta_1, \quad \chi_2 = z\vartheta_2.$$

We have the following relation in $\mathcal{M}_{\mathrm{red}}(\mathbf{\Sigma}_{[\mathbb{C}^2/\mu_2]}, G)$, illustrated in Figure 3:

$$(z\partial_y)^2\mathbb{1}_0 = \left(z\vartheta_1 - \frac{1}{2}zy\partial_y\right)\left(z\vartheta_2 - \frac{1}{2}zy\partial_y\right)\mathbb{1}_0.$$

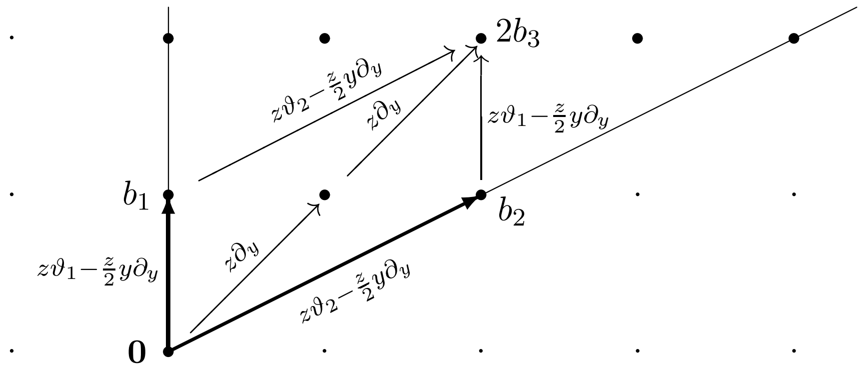

Figure 3. $\mathcal{M}_{\mathrm{red}}(\mathbf{\Sigma}_{[\mathbb{C}^2/\mu_2]}, \{(1,1)\})$.

Thus, if we invert $4 - y^2$,

$$\mathbb{1}_{2b_3} = (z\partial_y)^2 \mathbb{1}_0 = \frac{4\chi_1\chi_2}{4-y^2}\mathbb{1}_0 + \frac{(z - 2(\chi_1+\chi_2))y}{4-y^2}\mathbb{1}_{b_3}.$$

The elements $\mathbb{1}_0$, $\mathbb{1}_{b_3}$ generate $\mathcal{M}_{\mathrm{red}}(\mathbf{\Sigma}_{[\mathbb{C}^2/\mathbb{Z}_2]}, G)[(4-y^2)^{-1}]$ freely over $\mathbb{C}[z, \chi_1, \chi_2, y, (4-y^2)^{-1}]$. In an analytic neighbourhood of $y = 0$, we define

$$\hat{\mathbb{1}}_{b_3} := \sqrt{1 - (y^2/4)}\,\mathbb{1}_{b_3}$$

and make the co-ordinate change $y = 2\sin(\theta/2)$. In the basis $\{\mathbb{1}_0, \hat{\mathbb{1}}_{b_3}\}$, the action of $z(\partial/\partial\theta)$ is represented by the following z-independent matrix:

$$\begin{bmatrix} 0 & \chi_1\chi_2 \\ 1 & -(\chi_1+\chi_2)\sin(\frac{\theta}{2}) \end{bmatrix}.$$

The basis $\{\mathbb{1}_0, \hat{\mathbb{1}}_{b_3}\}$ corresponds to $\{1, \mathbf{1}_{b_3}\} \subset H^2_{\mathrm{CR},\mathbb{T}}([\mathbb{C}^2/\mu_2])$ under the mirror isomorphism, and the mirror map is given by $\tau(y) = \theta\mathbf{1}_{b_3}$. The above matrix gives the quantum multiplication by $\mathbf{1}_{b_3}$.

5.4.4. The case where $\mathfrak{X} = \mathbb{P}^2$ and $G \neq \varnothing$. We take $\mathbf{N} = \mathbb{Z}^2$, $b_1 = (1,0)$, $b_2 = (0,1)$, $b_3 = (-1,-1)$ and $G = \{b_4\}$ with $b_4 = (1,1) = b_1 + b_2$. Let Q be the variable corresponding to $(1,1,1) \in \mathbf{\Lambda} = \mathbb{L} = \mathbb{Z}(1,1,1)$ and let y be the variable corresponding to $b_4 \in G$. We have

$$\chi_1 = z\vartheta_1 - z\vartheta_3, \quad \chi_2 = z\vartheta_2 - z\vartheta_3.$$

In the reduced fan D-module $\mathcal{M}_{\mathrm{red}}(\mathbf{\Sigma}_{\mathbb{P}^2}, G)$, we have the relation

$$((z\vartheta_2 - zy\partial_y)(z\vartheta_1 - zy\partial_y) - z\partial_y)\,\mathbb{1}_0 = 0,$$
$$(z\vartheta_3(z\vartheta_2 - zy\partial_y)(z\vartheta_1 - zy\partial_y) - Q)\,\mathbb{1}_0 = 0.$$

Let us consider the non-equivariant limit where $\chi_1 = \chi_2 = 0$. Then we have $z\vartheta := z\vartheta_1 = z\vartheta_2 = z\vartheta_3 = zQ(\partial/\partial Q)$. We can see that the D-

module $\mathcal{M}_{\mathrm{red}}(\mathbf{\Sigma}_{\mathbb{P}^2}, G)/(\chi_1, \chi_2)$ is of rank 4 at $y \neq 0$. On the other hand, the y-adic completion of this D-module has a basis $\{\mathbb{1}_0, z\vartheta\mathbb{1}_0, (z\vartheta)^2\mathbb{1}_0\}$ over $\mathbb{C}[z, Q][\![y]\!]$ and is isomorphic to the quantum D-module of $\mathbb{P}^2$. For example, $z\partial_y\mathbb{1}_0$ can be expressed as a linear combination of these basis elements, by using the above two equations recursively.

6. The higher residue pairing and the Poincaré pairing match

We now construct a version of K. Saito's *higher residue pairing* [**71**] on our (equivariant) Gauss–Manin system, and show that it matches the (equivariant) Poincaré pairing on quantum cohomology under the mirror isomorphism in Theorem 4.28. As in §4.2, we fix a finite subset $G \subset (\mathbf{N} \cap |\Sigma|) \setminus \{b_1, \dots, b_m\}$ and consider the unfolding $F_\chi(x; y)$ associated with G. We again set $S := \{b_1, \dots, b_m\} \cup G$.

6.1. Critical points. We start with a description of critical points of F_χ. The (logarithmic) critical scheme of F_χ is defined to be the formal spectrum of the Jacobian ring

$$\mathrm{Jac}(F_\chi) = \mathbb{C}\{\mathbb{O}_+\}[\![y]\!][\chi] \Big/ \left\langle x_i \frac{\partial F_\chi}{\partial x_i} : 1 \leq i \leq n \right\rangle,$$

where $x_i \frac{\partial F_\chi}{\partial x_i} = \sum_{\mathbf{k} \in S} y_{\mathbf{k}} k_i w_{\mathbf{k}} - \chi_i$, with k_i being the ith component of $\overline{\mathbf{k}} \in \mathbf{N}_{\mathbb{R}} \cong \mathbb{R}^n$. The mirror isomorphism Θ in Theorem 4.28 induces, at $z = 0$, an isomorphism between $\mathrm{Jac}(F_\chi)$ and $H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \otimes_{R_{\mathbb{T}}} R_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![y]\!]$ as $R_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![y]\!]$-modules. In particular, $\mathrm{Jac}(F_\chi)$ is a free module over $R_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![y]\!]$ of rank $\dim H^*_{\mathrm{CR}}(\mathfrak{X})$. Set $N := \dim H^*_{\mathrm{CR}}(\mathfrak{X})$. Let $\overline{S}_{\mathbb{T}}$ be the algebraic closure of the fraction field $S_{\mathbb{T}}$ of $R_{\mathbb{T}}$. We show that, after base change to $\overline{S}_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![y]\!]$, the critical scheme consists of N distinct points over $\overline{S}_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![y]\!]$, each of which is characterized by its limit at $Q = 0, y = y^*$ – see (4.2) for y^*. We write $\mathrm{Jac}(F_\chi)_{\overline{S}_{\mathbb{T}}} := \mathrm{Jac}(F_\chi) \otimes_{R_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![y]\!]} \overline{S}_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![y]\!]$ for the base change.

Notation 6.1. Recall that maximal cones $\sigma \in \Sigma$ are in one-to-one correspondence with $\mathbb{T}$-fixed points z_σ of $\mathfrak{X}$. We write $u_i(\sigma) \in H^2_{\mathbb{T}}(\mathrm{pt}) = \mathbf{M}_{\mathbb{Q}}$ for the restriction of the toric divisor class u_i to the fixed point z_σ. We set $\mathbf{N}(\sigma) := \mathbf{N}/\sum_{i \in \sigma} \mathbb{Z} b_i$; this gives the orbifold isotropy group at the fixed point z_σ. We also write $\Sigma(n)$ for the set of n-dimensional (i.e. maximal) cones in Σ.

Lemma 6.2. (1) *The ring* $\mathrm{Jac}(F_\chi)_{\overline{S}_{\mathbb{T}}}$ *is isomorphic to* $\overline{S}_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![y]\!]^{\oplus N}$ *as an* $\overline{S}_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![y]\!]$*-algebra, with* $N = \dim H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})$. *In other words, the critical scheme* $\mathrm{Spf}(\mathrm{Jac}(F_\chi)_{\overline{S}_{\mathbb{T}}})$ *consists of* N *distinct points over* $\overline{S}_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![y]\!]$.

(2) *For a maximal cone σ, define* $\mathrm{Crit}(\sigma)$ *to be the set:*

$$\mathrm{Crit}(\sigma) := \left\{ c = (c_{\mathbf{k}}) \in \overline{S}_{\mathbb{T}}^{\,\mathbf{N}\cap\sigma} : c_{\mathbf{k}} c_{\boldsymbol{\ell}} = c_{\mathbf{k}+\boldsymbol{\ell}},\ c_{b_i} = u_i(\sigma)\ \forall i \in \sigma \right\},$$

where we set $\mathbf{N} \cap \sigma = \{\mathbf{k} \in \mathbf{N} : \overline{\mathbf{k}} \in \sigma\}$. Then $\mathrm{Crit}(\sigma)$ *is a torsor over the character group* $\widehat{\mathbf{N}}(\sigma) = \mathrm{Hom}(\mathbf{N}(\sigma), \mathbb{C}^\times)$ *of* $\mathbf{N}(\sigma)$*.*

(3) *Note that a critical point p over $\overline{S}_{\mathbb{T}}[\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$ is by definition an $\overline{S}_{\mathbb{T}}[\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$-algebra homomorphism* $\mathrm{Jac}(F_\chi)_{\overline{S}_{\mathbb{T}}} \to \overline{S}_{\mathbb{T}}[\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$*, $[w_{\mathbf{k}}] \mapsto w_{\mathbf{k}}(p)$. For each maximal cone $\sigma \in \Sigma$ and an element $c \in \mathrm{Crit}(\sigma)$, there exists a unique critical point p such that*

$$w_{\mathbf{k}}(p)\Big|_{Q=0,y=y^*} = \begin{cases} c_{\mathbf{k}} & \mathbf{k} \in \mathbf{N} \cap \sigma, \\ 0 & \textit{otherwise.} \end{cases}$$

This gives a bijection between critical points over $\overline{S}_{\mathbb{T}}[\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$ and the set $\bigcup_{\sigma\in\Sigma(n)} \mathrm{Crit}(\sigma)$.

Proof. Part (2) is obvious. Note that $\mathrm{Jac}(F_\chi)_{\overline{S}_{\mathbb{T}}}$ is a free $\overline{S}_{\mathbb{T}}[\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$-module of rank N. Therefore $\mathrm{Jac}(F_\chi)_{\overline{S}_{\mathbb{T}}}$ is a direct sum of copies of $\overline{S}_{\mathbb{T}}[\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$ as a ring if and only if the restriction to $Q = 0, y = y^*$

$$\mathrm{Jac}(F_\chi)_{\overline{S}_{\mathbb{T}},0} := \mathrm{Jac}(F_\chi)_{\overline{S}_{\mathbb{T}}} \otimes_{\overline{S}_{\mathbb{T}}[\![\boldsymbol{\Lambda}_+]\!][\![y]\!]} \overline{S}_{\mathbb{T}}$$

is a direct sum of copies of $\overline{S}_{\mathbb{T}}$ as a ring. To establish (1) and (3), therefore, it suffices to prove that $\mathrm{Spec}(\mathrm{Jac}(F_\chi)_{\overline{S}_{\mathbb{T}},0})$ is a finite set of reduced points and equals $\bigcup_\sigma \mathrm{Crit}(\sigma)$. Note that we have $F_\chi|_{Q=0,y=y^*} = w_1 + \cdots + w_m - \sum_{i=1}^n \chi_i \log x_i$ and

$$\mathrm{Jac}(F_\chi)_{\overline{S}_{\mathbb{T}},0} \cong \frac{\bigoplus_{\mathbf{k}\in\mathbf{N}\cap|\Sigma|} \overline{S}_{\mathbb{T}} w_{\mathbf{k}}}{\langle \chi_i - \sum_{j=1}^m b_{j,i} w_j : 1 \le i \le n \rangle},$$

where the product structure on $\bigoplus_{\mathbf{k}\in\mathbf{N}\cap|\Sigma|} \overline{S}_{\mathbb{T}} w_{\mathbf{k}}$ is defined as in (2.4) (the Stanley–Reisner presentation of $H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})$), but replacing $\phi_{\mathbf{k}}$ with $w_{\mathbf{k}}$, and $b_{j,i}$ denotes the ith component of $\overline{b}_j \in \mathbf{N}_{\mathbb{R}} \cong \mathbb{R}^n$. Therefore an $\overline{S}_{\mathbb{T}}$-algebra homomorphism $\mathrm{Jac}(F_\chi)_{\overline{S}_{\mathbb{T}},0} \to \overline{S}_{\mathbb{T}}$ is specified by a tuple $(c_{\mathbf{k}}) \in \overline{S}_{\mathbb{T}}^{\,\mathbf{N}\cap|\Sigma|}$ satisfying the conditions:

$$c_{\mathbf{k}} c_{\boldsymbol{\ell}} = \begin{cases} c_{\mathbf{k}+\boldsymbol{\ell}} & \text{if } \mathbf{k}, \boldsymbol{\ell} \text{ are in the same cone of } \Sigma; \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \chi_i = \sum_{j=1}^m b_{i,j} c_{b_j}.$$

The first condition implies that the support $\{\mathbf{k} : c_{\mathbf{k}} \neq 0\}$ has to be contained in some maximal cone σ of Σ. The second condition then determines $c_{b_1}, \ldots, c_{b_m}$ uniquely. Notice that $u_j(\sigma) = 0$ for $j \notin \sigma$ since the toric divisor $\{Z_j = 0\}$ does not pass through the fixed point z_σ, and that $\sum_{j=1}^m b_{j,i} u_j = \chi_i$ by the description of the $R_{\mathbb{T}}$-module structure of $H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})$ in §2.3. It follows that $c_{b_j} = 0$ for $j \notin \sigma$ and $c_{b_j} = u_j(\sigma)$ for $j \in \sigma$. Thus closed points of $\mathrm{Spec}(\mathrm{Jac}(F_\chi)_{\overline{S}_{\mathbb{T}},0})$ correspond bijectively with elements of $\bigcup_\sigma \operatorname{Crit}(\sigma)$. Since we have $\dim_{\overline{S}_{\mathbb{T}}} \operatorname{Jac}(F_\chi)_{\overline{S}_{\mathbb{T}},0} = N = \#\bigcup_\sigma \operatorname{Crit}(\sigma)$, $\operatorname{Spec}(\operatorname{Jac}(F_\chi)_{\overline{S}_{\mathbb{T}},0})$ consists only of reduced points and is identified with $\bigcup_\sigma \operatorname{Crit}(\sigma)$. q.e.d.

Remark 6.3. In the above proof, we have shown that

$$\chi_i = \sum_{j=1}^m b_{j,i} u_j(\sigma) = \sum_{j\in\sigma} b_{j,i} u_j(\sigma) \tag{6.1}$$

for any maximal cone σ.

We study the $Q \to 0$, $y \to y^*$ asymptotics of critical values of F_χ. The problem here is that the critical value $F_\chi(p)$ of F_χ does not lie in the ring $\overline{S}_{\mathbb{T}}[\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$ because of the $\log x_i$ term. Let p be a critical point corresponding to $c = (c_{\mathbf{k}})_{\mathbf{k}\in\mathbf{N}\cap\sigma} \in \operatorname{Crit}(\sigma)$, where σ is a maximal cone of Σ, as in Lemma 6.2. We extend the function $\mathbf{k} \mapsto c_{\mathbf{k}} \in \overline{S}_{\mathbb{T}}$ for arbitrary $\mathbf{k} \in \mathbf{N}$ by requiring that $c_{\mathbf{k}} c_{\boldsymbol{\ell}} = c_{\mathbf{k}+\boldsymbol{\ell}}$ for all $\mathbf{k}, \boldsymbol{\ell} \in \mathbf{N}$. This is possible since $c_{\mathbf{k}} \neq 0$ for all $\mathbf{k} \in \mathbf{N} \cap \sigma$. Recall that we chose a decomposition $\mathbf{N} \cong \mathbb{Z}^n \times \mathbf{N}_{\mathrm{tor}}$ in §4.1. We write $\mathbf{e}_i$, $1 \le i \le n$, for the element of $\mathbf{N}$ corresponding to the ith basis vector of $\mathbb{Z}^n$. Let $\overline{\varsigma}_\sigma \colon \mathbf{N}_{\mathbb{Q}} \to \mathbb{Q}^m$ denote the splitting of the fan sequence defined by $\overline{\varsigma}_\sigma(b_i) = e_i$ for $i \in \sigma$. Then we have

$$x_i = w^{(\overline{\varsigma}(\mathbf{e}_i),\mathbf{e}_i)} = Q^{(\overline{\varsigma}-\overline{\varsigma}_\sigma)(\mathbf{e}_i)} w^{(\overline{\varsigma}_\sigma(\mathbf{e}_i),\mathbf{e}_i)} \tag{6.2}$$

and

$$w^{(\overline{\varsigma}_\sigma(\mathbf{e}_i),\mathbf{e}_i)}(p) = c_{\mathbf{e}_i} + \mathfrak{m}^G \overline{S}_{\mathbb{T}}[\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$$

by the extension of the definition of $c_{\mathbf{k}}$ above. (Recall that $\mathfrak{m}^G$ is the ideal corresponding to $Q = 0$, $y = y^*$ introduced before Proposition 4.24.) Therefore we can decompose the critical value $F_\chi(p)$ as

$$F_\chi(p) = F_\chi^{\mathrm{cl}}(p) + F_\chi^{\mathrm{qu}}(p),$$

where the classical part $F_\chi^{\mathrm{cl}}(p)$ given by

$$F_\chi^{\mathrm{cl}}(p) = \sum_{j\in\sigma} c_{b_j} - \sum_{i=1}^n \chi_i \left(\log Q^{(\overline{\varsigma}-\overline{\varsigma}_\sigma)(\mathbf{e}_i)} + \log c_{\mathbf{e}_i}\right)$$

and the quantum part $F_\chi^{\mathrm{qu}}(p)$ belongs to $\mathfrak{m}^G \overline{S}_{\mathbb{T}}[\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$:

$$F_\chi^{\mathrm{qu}}(p) = \sum_{j\in S} y_{\mathbf{k}} w_{\mathbf{k}}(p) - \sum_{j\in\sigma} c_{b_j} + \sum_{i=1}^n \chi_i \log(c_{\mathbf{e}_i}^{-1} w^{(\varsigma_\sigma(\mathbf{e}_i),\mathbf{e}_i)}(p)). \tag{6.3}$$

Lemma 6.4. *Let p be a critical point corresponding to $c \in \operatorname{Crit}(\sigma)$, where σ is a maximal cone of Σ. Write $b_j = \sum_{i=1}^n b_{j,i}\mathbf{e}_i + \zeta_j$ with $\zeta_j \in \mathbf{N}_{\mathrm{tor}}$. Then we have*

$$F_\chi^{\mathrm{cl}}(p) = \sum_{j\in\sigma} \left(u_j(\sigma) - u_j(\sigma) \log \frac{u_j(\sigma)}{c_{\zeta_j}}\right) + \log Q^{\mathbb{X}(\sigma)},$$

where $\mathbb{X} \in H^2_{\mathbb{T}}(X,\mathbb{Q}) \otimes \mathbb{L} = \mathrm{Hom}(\mathbb{L}^\star_{\mathbb{Q}}, H^2_{\mathbb{T}}(X,\mathbb{Q}))$ denotes the element corresponding to the splitting $\mathbb{L}^\star_{\mathbb{Q}} \to (\mathbb{Q}^m)^\star = H^2_{\mathbb{T}}(X,\mathbb{Q})$ dual to ς (in the sense explained before Proposition 4.22), and $\mathbb{X}(\sigma) \in \mathbf{M}_{\mathbb{Q}} \otimes \mathbb{L}$ denotes the restriction of $\mathbb{X}$ to the $\mathbb{T}$-fixed point $z_\sigma \in \mathfrak{X}$.

Proof. Recall that $c_{b_j} = u_j(\sigma)$ for $j \in \sigma$. Using (6.1), we find that

$$\sum_{i=1}^n \chi_i \log c_{\mathbf{e}_i} = \sum_{i=1}^n \sum_{j\in\sigma} u_j(\sigma) b_{j,i} \log c_{\mathbf{e}_i} = \sum_{j\in\sigma} u_j(\sigma) \log(c_{\zeta_j}^{-1} u_j(\sigma)),$$

where we used $u_j(\sigma) = c_{b_j} = c_{\zeta_j} \prod_{i=1}^n c_{\mathbf{e}_i}^{b_{j,i}}$. It remains to show that

$$\mathbb{X}(\sigma) = \sum_{i=1}^n \chi_i(\overline{\varsigma}_\sigma(\mathbf{e}_i) - \overline{\varsigma}(\mathbf{e}_i)). \tag{6.4}$$

Evaluating the right-hand side at $\xi \in \mathbb{L}^\star_{\mathbb{Q}}$, we obtain

$$\begin{aligned}\sum_{i=1}^n \chi_i \left(\xi \cdot (\overline{\varsigma}_\sigma - \overline{\varsigma})(\mathbf{e}_i)\right) &= \sum_{i=1}^n \chi_i \left(\hat{\xi} \cdot \overline{\varsigma}_\sigma(\mathbf{e}_i) - \hat{\xi} \cdot \overline{\varsigma}(\mathbf{e}_i)\right) \\ &= \sum_{j\in\sigma} u_j(\sigma)(\hat{\xi} \cdot e_j) = \hat{\xi}(\sigma),\end{aligned}$$

where $\hat{\xi} \in (\mathbb{Q}^m)^\star \cong H^2_{\mathbb{T}}(X,\mathbb{Q})$ denotes the lift of ξ with respect to the splitting $\mathbb{L}^\star_{\mathbb{Q}} \to (\mathbb{Q}^m)^\star$ and $\hat{\xi}(\sigma) \in \mathbf{M}_{\mathbb{Q}}$ denotes the restriction of $\hat{\xi}$ to z_σ. We also used (6.1) and the fact that $\hat{\xi}$ vanishes on the image of $\overline{\varsigma}$. Evaluation of the left-hand side at ξ gives the same answer, and the conclusion follows. q.e.d.

We also study the limit of the Landau–Ginzburg potential at the central fiber $Q = 0$, $y = y^*$. The fiber at $Q = 0$ of the total space $\operatorname{Spec} \mathbb{C}[\mathbb{O}_+]$ of the mirror is reducible and decomposed as follows:

$$\operatorname{Spec} \mathbb{C}[\mathbb{O}_+]\Big|_{Q=0} = \bigcup_{\sigma\in\Sigma(n)} \bigcup_{\theta\in\mathrm{Hom}(\mathbf{N}_{\mathrm{tor}},\mathbb{C}^\times)} A_{\sigma,\theta},$$

where $A_{\sigma,\theta} = \operatorname{Spec} \mathbb{C}[(\mathbf{N}/\mathbf{N}_{\mathrm{tor}}) \cap \sigma]$ is an affine toric variety. The restriction of F_χ to the central fiber is ill-defined because of the logarithmic term, but we show that for each critical point p, the difference $F_\chi(x;y) - F_\chi^{\mathrm{cl}}(p)$ has a well-defined limit as y approaches y^* and (x,Q) approaches the component $A_{\sigma,\theta}$ on which the critical point p lies.

Lemma 6.5. *Let p be a critical point of F_χ corresponding to $c \in \mathrm{Crit}(\sigma)$, where σ is a maximal cone of Σ. Let $A_{\sigma,\theta}$ be the component of the central fiber determined by the character θ defined as the restriction* *of* $\mathbf{k} \mapsto c_{\mathbf{k}}$ *to* $\mathbf{k} \in \mathbf{N}_{\mathrm{tor}}$*. Then we have*

$$F_\chi - F_\chi^{\mathrm{cl}}(p)\Big|_{Q=0, y=y^*, A_{\sigma,\theta}} = \sum_{j\in\sigma}\left(w_j - u_j(\sigma) - u_j(\sigma)\log\frac{w_j}{u_j(\sigma)}\right).$$

Proof. Let p, c, σ, θ be as given. We have

$$\begin{aligned} F_\chi &= \sum_{\mathbf{k}\in S} y_{\mathbf{k}} w_{\mathbf{k}} - \sum_{i=1}^n \chi_i \log(Q^{(\overline{\varsigma}-\overline{\varsigma}_\sigma)(\mathbf{e}_i)} w^{(\overline{\varsigma}_\sigma(\mathbf{e}_i), \mathbf{e}_i)}) \\ &= \sum_{\mathbf{k}\in S} y_{\mathbf{k}} w_{\mathbf{k}} + \log Q^{\mathbb{X}(\sigma)} - \sum_{j\in\sigma} u_j(\sigma)\log\frac{w_i}{w^{(0,\zeta_j)}}, \end{aligned}$$

where we used (6.2), (6.4) and (6.1). (The quantity $\zeta_j \in \mathbf{N}_{\mathrm{tor}}$ here is given in Lemma 6.4.) Note that $w_{\mathbf{k}}|_{A_{\sigma,\theta}}$ equals $u_j(\sigma)$ if $\mathbf{k} = b_j \in \sigma$ and is zero otherwise; also $w^{(0,\zeta_j)}|_{A_{\sigma,\theta}} = c_{\zeta_j}$. The conclusion follows easily from this and Lemma 6.4. q.e.d.

6.2. Higher residue pairing via asymptotic expansion. We define the higher residue pairing in our setting in terms of (formal) asymptotic expansion, following the method of Pham [**66**, 2ème Partie, 4]. We first recall asymptotic expansion in analytic setting. Let $f(t) = f(t^1, \dots, t^n)$ be a holomorphic function on $\mathbb{C}^n$ with a non-degenerate critical point p. Let $\Gamma(p)$ denote a stable manifold for the Morse function $\Re(f(t))$ associated to p and consider the oscillatory integral:

$$\int_{\Gamma(p)} e^{f(t)/z} g(t) dt^1 \cdots dt^n$$

with $z < 0$ and $g(t)$ a holomorphic function. As $z \nearrow 0$, the integral is dominated by the contribution near the critical point $t = p$, and we obtain its asymptotic expansion by expanding the integrand in Taylor series at p (under appropriate assumptions on f and g). A concrete method is as follows: we expand the functions $f(t)$, $g(t)$ in Taylor series at p as

$$\begin{aligned} f(t) &= \sum_{k\ge 0}\frac{1}{k!}\sum_{i_1,\dots,i_k} f_{i_1,\dots,i_k}(p) s^{i_1}\cdots s^{i_k}, \\ g(t) &= \sum_{k\ge 0}\frac{1}{k!}\sum_{i_1,\dots,i_k} g_{i_1,\dots,i_k}(p) s^{i_1}\cdots s^{i_k} \end{aligned}$$

with $s^i = t^i - p^i$, and make a linear change of variables $s^i = \sqrt{-z}\sum_{j=1}^n c^i_j v^j$ such that

$$\frac{1}{2z}\sum_{i,j} f_{ij}(p) s^i s^j = -\frac{1}{2}\sum_i (v^i)^2.$$

Then the above integral can be expanded as:

$$
\begin{aligned}
(6.5)\quad & e^{f(p)/z}\frac{(-z)^{n/2}}{\sqrt{\det(f_{ij}(p))}} \\
&\times \int_{\mathbb{R}^n} e^{-\frac{1}{2}\sum_i (v^i)^2}\left(\sum_{\substack{k,l\ge 0,\\ k\equiv l \bmod 2}} \frac{1}{k!}a^{(l)}_{i_1,\dots,i_k} v^{i_1}\cdots v^{i_k} z^{l/2}\right) dv^1\cdots dv^n,
\end{aligned}
$$

where $\sum_{k,l\ge 0}\frac{1}{k!}a^{(l)}_{i_1,\dots,i_k}v^{i_1}\cdots v^{i_k}z^{l/2}$ is the expansion of

$$
\begin{aligned}
&\exp\left(-\sum_{k\ge 3}\sum_{i_1,\dots,i_k}\sum_{j_1,\dots,j_k}\frac{(-z)^{k/2-1}}{k!} f_{i_1,\dots,i_k}(p) c^{i_1}_{j_1}\cdots c^{i_k}_{j_k} v^{j_1}\cdots v^{j_k}\right) \\
&\qquad \times \sum_{k\ge 0}\sum_{i_1,\dots,i_k}\sum_{j_1,\dots,j_k}\frac{(-z)^{k/2}}{k!} g_{i_1,\dots,i_k}(p) c^{i_1}_{j_1}\cdots c^{i_k}_{j_k} v^{j_1}\cdots v^{j_k}.
\end{aligned}
$$

By performing the above Gaussian integral termwise, we obtain a *formal asymptotic expansion* of the original oscillatory integral (note that half-integer powers of z vanish automatically). We denote this expansion as

$$
e^{f(p)/z}(-2\pi z)^{n/2}\,\mathrm{Asym}_p\left(e^{f(t)/z}g(t)dt\right)
$$

so that $\mathrm{Asym}_p(e^{f(t)/z}g(t)dt)$ is of the form:

$$
\frac{1}{\sqrt{\det(f_{ij}(p))}}\left(g(p) + a_1 z + a_2 z^2 + \cdots\right).
$$

We remark that the formal asymptotic expansion vanishes if $e^{f(t)/z}g(t)dt$ is an exact form (regardless of the formal asymptotic expansion being the actual asymptotic expansion, in which case the remark is obvious), since the integrand of the corresponding Gaussian integral (6.5) becomes also exact, and it is straightforward to check that the termwise Gaussian integral of a formal exact form is zero.

Note that the above procedure only involves the Taylor expansions of the functions $f(t)$, $g(t)$ at p and an orthogonalization of the Hessian form $(f_{ij}(p))$. Therefore we can generalize the above procedure to our setting where $f(t)$ is given by $F_\chi(x;y)$ and $g(t)dt$ is an element of $\mathbb{C}[z]\{\mathbb{O}_+\}[\![y]\!][\chi]\omega$, by using $(\log x_1,\dots,\log x_n)$ as co-ordinates $(t^1,\dots,t^n)$ here. To be more precise, we work over the ring $\overline{S}_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![y]\!]$ and consider the asymptotic expansion at one of N non-degenerate critical points over $\overline{S}_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![y]\!]$ from Lemma 6.2. Let p be a critical point of F_χ over $\overline{S}_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![y]\!]$. Then the Taylor expansion of a function in $\mathbb{C}[z]\{\mathbb{O}_+\}[\![y]\!][\chi]$ at p (with respect to the co-ordinates $\log x_1,\dots,\log x_n$)

is well-defined as formal power series with coefficients in $\overline{S}_{\mathbb{T}}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]$. When the critical point p corresponds to an element of $\mathrm{Crit}(\sigma)$ (under the bijection in Lemma 6.2),

$$\frac{\partial^2 F_\chi}{\partial \log x_i \partial \log x_j}(p)\bigg|_{Q=0,y=y^*} = \sum_{k\in\sigma} b_{k,i} b_{k,j} u_k(\sigma)$$

is diagonalizable by the matrix $(b_{k,i}\sqrt{u_k(\sigma)})_{k\in\sigma,1\le i\le n}$, and thus the Hessian form is diagonalizable over $\overline{S}_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![y]\!]$. Thus we obtain a well-defined map:

$$\mathrm{Asym}_p\colon e^{F_\chi/z}\mathbb{C}[z]\{\mathbb{O}_+\}[\![y]\!][\chi]\omega \to \overline{S}_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![y]\!][\![z]\!]$$

for each critical point p over $\overline{S}_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![y]\!]$. By the remark above, $\mathrm{Asym}_p(e^{F_\chi/z}\Omega)$ vanishes if $e^{F_\chi/z}\Omega$ is exact, and thus Asym_p descends to cohomology:

$$\mathrm{Asym}_p\colon e^{F_\chi/z}\,\mathrm{GM}(F_\chi) \to \overline{S}_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![y]\!][\![z]\!].$$

Definition 6.6. We define the higher residue pairing $P\colon \mathrm{GM}(F_\chi)\times \mathrm{GM}(F_\chi) \to \overline{S}_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![y]\!][\![z]\!]$ by

$$P(\Omega_1,\Omega_2) = \sum_p \overline{\mathrm{Asym}_p(e^{F_\chi/z}\Omega_1)}\,\mathrm{Asym}_p(e^{F_\chi/z}\Omega_2)$$

for $\Omega_1,\Omega_2\in \mathrm{GM}(F_\chi)$, where $\overline{\mathrm{Asym}_p(e^{F_\chi/z}\Omega_1)} = \mathrm{Asym}_p(e^{F_\chi/z}\Omega_1)|_{z\to -z}$, and the sum is over critical points p of F_χ over $\overline{S}_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![y]\!]$ in Lemma 6.2. Note that the higher residue pairing is invariant under the Galois group $\mathrm{Gal}(\overline{S}_{\mathbb{T}}/S_{\mathbb{T}})$, which permutes critical points, and thus P takes values in $S_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![y]\!][\![z]\!]$. (In fact it takes values in $S_{\mathbb{T}}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]$ by Theorem 6.11.)

We establish standard properties of the higher residue pairing. We first observe that Asym_p gives another solution to the equivariant Gauss–Manin system (cf. the localization map Loc in §4.5). We remark that, despite the fact that $e^{F_\chi^{\mathrm{qu}}(q)/z}$ is a formal power series in z^{-1} and that $\mathrm{Asym}_p(\Omega)$ is a formal power series in z, the product $e^{F_\chi^{\mathrm{qu}}(p)/z}\,\mathrm{Asym}_p(\Omega)$ is well-defined as an element of $\overline{S}_{\mathbb{T}}(\!(z)\!)[\![\mathbf{\Lambda}_+]\!][\![y]\!]$ because $F_\chi^{\mathrm{qu}}(p) \in \mathfrak{m}^G\overline{S}_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![y]\!]$.

Lemma 6.7. *Let p be a critical point of F_χ over $\overline{S}_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![y]\!]$. The map Asym_p is linear over $R_{\mathbb{T}}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]$ and satisfies the following dif-*

ferential equations:

$$
\begin{aligned}
&e^{F_\chi^{\mathrm{qu}}(p)/z}\operatorname{Asym}_p(e^{F_\chi/z}\nabla_{\xi Q\frac{\partial}{\partial Q}}\Omega)\\
&\qquad\qquad = \left(\xi Q\frac{\partial}{\partial Q}+\frac{1}{z}\hat{\xi}(\sigma)\right)\left(e^{F_\chi^{\mathrm{qu}}(p)/z}\operatorname{Asym}_p(e^{F_\chi/z}\Omega)\right),\\
&e^{F_\chi^{\mathrm{qu}}(p)/z}\operatorname{Asym}_p(e^{F_\chi/z}\nabla_{\frac{\partial}{\partial y_{\mathbf{k}}}}\Omega)\\
&\qquad\qquad = \frac{\partial}{\partial y_{\mathbf{k}}}\left(e^{F_\chi^{\mathrm{qu}}(p)/z}\operatorname{Asym}_p(e^{F_\chi/z}\Omega)\right),\\
&\operatorname{Asym}_p(e^{F_\chi/z}\operatorname{Gr}^{\mathrm{B}}\Omega) = \left(z\frac{\partial}{\partial z}+\mathcal{E}^{\mathrm{B}}+\frac{n}{2}\right)\operatorname{Asym}_p(e^{F_\chi/z}\Omega),
\end{aligned}
$$

where $\xi\in\mathbb{L}^\star_{\mathbb{C}}$, $\hat{\xi}\in(\mathbb{C}^m)^\star=H^2_{\mathbb{T}}(X,\mathbb{C})$ *is the lift of* ξ *introduced before Proposition 4.22, and* $\hat{\xi}(\sigma)\in\mathbf{M}_{\mathbb{C}}$ *denotes the restriction of* $\hat{\xi}$ *to the fixed point* $z_\sigma\in\mathfrak{X}$*. Moreover, the quantum part* $F_\chi^{\mathrm{qu}}(p)$ *of the critical value is homogeneous of degree one, i.e.* $\mathcal{E}^{\mathrm{B}}F_\chi^{\mathrm{qu}}(p)=F_\chi^{\mathrm{qu}}(p)$.

Proof. It is clear from the definition that Asym_p is linear over $\mathbb{C}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!][\chi]$, and since Asym_p is continuous with respect to the $\mathfrak{m}^G$-adic topology (see the discussion before Proposition 4.24 for $\mathfrak{m}^G$), it is linear over $R_{\mathbb{T}}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]$. The first two differential equations follow from:

- the fact that $e^{F_\chi/z}\Omega\mapsto e^{F_\chi(p)/z}\operatorname{Asym}_p(e^{F_\chi/z}\Omega)$ commutes with differentiation in the parameters (y,Q);
- $\partial_{\vec{v}}(e^{F_\chi/z}\Omega)=e^{F_\chi/z}\nabla_{\vec{v}}\Omega$; and
- $\xi Q\frac{\partial}{\partial Q}e^{F_\chi^{\mathrm{cl}}(p)/z}=\frac{1}{z}\hat{\xi}(\sigma)e^{F_\chi^{\mathrm{cl}}(p)/z}$, which follows from Lemma 6.4.

Let us establish the third equation. Recall that the grading operator on $\mathbb{C}[z]\{\mathbb{O}_+\}[\![y]\!]\omega$ is induced by $e_1^\star+\cdots+e_m^\star\in(\mathbb{Q}^m)^\star=\operatorname{Hom}(\mathbb{O},\mathbb{Q})$, $\deg z=1$, $\deg y_{\mathbf{k}}=1-|\mathbf{k}|$, $\deg\omega=0$. The potential function $F_\chi(x;y)$ is not homogeneous because of the $\log x_i$ term, but the logarithmic derivative $x_i\frac{\partial}{\partial x_i}F_\chi(x;y)$ is homogeneous of degree 1. Thus the critical point p is homogeneous in the sense that $\mathcal{E}^{\mathrm{B}}w_{\mathbf{k}}(p)=|\mathbf{k}|w_{\mathbf{k}}(p)$ for all $\mathbf{k}\in\mathbf{N}\cap|\Sigma|$, and so the quantum part $F_\chi^{\mathrm{qu}}(p)$ (6.3) of the critical value is homogeneous of degree 1. We can also see that the variables v_i appearing in the Gaussian integral (6.5) is of degree zero. Therefore, if Ω is of degree k, $(-2\pi z)^{n/2}e^{F_\chi^{\mathrm{qu}}(p)/z}\operatorname{Asym}_p(e^{F_\chi/z}\Omega)$ is of degree k, and $\operatorname{Asym}_p(e^{F_\chi/z}\Omega)$ is of degree $k-\frac{n}{2}$. The conclusion follows. q.e.d.

In view of Definition 6.6, Lemma 6.7 implies the following.

Proposition 6.8. *The higher residue pairing satisfies the following properties:*

1) $P(c(-z)\Omega_1,\Omega_2)=P(\Omega_1,c(z)\Omega_2)=c(z)P(\Omega_1,\Omega_2)$ *for* $c(z)\in R_{\mathbb{T}}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]$*;*

2) $\xi Q \frac{\partial}{\partial Q} P(\Omega_1, \Omega_2) = P(\nabla_{\xi Q \frac{\partial}{\partial Q}} \Omega_1, \Omega_2) + P(\Omega_1, \nabla_{\xi Q \frac{\partial}{\partial Q}} \Omega_2)$ *for* $\xi \in \mathbb{L}^\star_{\mathbb{C}}$*;*

3) $\frac{\partial}{\partial y_{\mathbf{k}}} P(\Omega_1, \Omega_2) = P(\nabla_{\frac{\partial}{\partial y_{\mathbf{k}}}} \Omega_1, \Omega_2) + P(\Omega_1, \nabla_{\frac{\partial}{\partial y_{\mathbf{k}}}} \Omega_2)$ *for* $\mathbf{k} \in S$*;*

4) $(z\frac{\partial}{\partial z} + \mathcal{E}^{\mathrm{B}} + n) P(\Omega_1, \Omega_2) = P(\mathrm{Gr}^{\mathrm{B}} \Omega_1, \Omega_2) + P(\Omega_1, \mathrm{Gr}^{\mathrm{B}} \Omega_2)$.

6.3. The pairings match.

Proposition 6.9. *Let p be the critical point of F_χ that corresponds, via Lemma 6.2, to $c \in \mathrm{Crit}(\sigma)$ where $\sigma \in \Sigma$ is a maximal cone. We have:*

$$e^{F_\chi^{\mathrm{qu}}(p)/z} \operatorname{Asym}_p \left(e^{F_\chi/z}\Omega\right) = \sum_{v \in \mathrm{Box}(\sigma)} \mathrm{Loc}(\Omega)\big|_{(\sigma,v)} c_v \, \Delta_{(\sigma,v)}(z) \tag{6.6}$$

for $\Omega \in \mathrm{GM}(F_\chi)$, where $(\cdots)|_{(\sigma,v)}$ denotes the restriction to the fixed point $z_\sigma \in \mathfrak{X}_v$ on the sector $\mathfrak{X}_v \subset I\mathfrak{X}$, and

$$\Delta_{(\sigma,v)}(z) = \frac{1}{|\mathbf{N}(\sigma)|} \prod_{i \in \sigma} \frac{1}{\sqrt{u_i(\sigma)}} \exp\left(-\sum_{k=2}^{\infty} \frac{B_k(v_i)}{k(k-1)} \left(\frac{z}{u_i(\sigma)}\right)^{k-1}\right)$$

with $B_k(h)$ the Bernoulli polynomial defined by $\sum_{k=0}^{\infty} B_k(h) \frac{t^k}{k!} = \frac{te^{ht}}{e^t - 1}$ and $v_i = \Psi_i(v)$.

Remark 6.10. Equality (6.6) here should be interpreted with care. Each coefficient of $Q^\lambda (y - y^*)^I$ of $\mathrm{Loc}(\Omega)|_{(\sigma,v)}$ is a rational function in z (Remark 4.21). We expand these rational functions as Laurent series at $z = 0$ and multiply by $\Delta_\sigma(z)$, obtaining an element of $\overline{S}_{\mathbb{T}}(\!(z)\!)[\![\mathbf{\Lambda}_+]\!][\![y]\!]$, and equate with the left-hand side. As discussed just before Lemma 6.7, the left-hand side is also well-defined as an element of $\overline{S}_{\mathbb{T}}(\!(z)\!)[\![\mathbf{\Lambda}_+]\!][\![y]\!]$.

Proof. Recall from Proposition 4.22 and Lemma 6.7 that Loc and $e^{F_\chi^{\mathrm{qu}}(p)/z} \operatorname{Asym}_p$ satisfy similar differential equations. From this we deduce that $e^{F_\chi^{\mathrm{qu}}(p)/z} \operatorname{Asym}_p(e^{F_\chi/z}\Omega)$ can be written as a linear combination of $\mathrm{Loc}(\Omega)|_{(\sigma',v')}$, with coefficients independent of Q and $y_{\boldsymbol{\ell}}$. We regard Loc as a map taking values in $H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \otimes_{R_{\mathbb{T}}} S_{\mathbb{T}}(\!(z)\!)[\![\mathbf{\Lambda}_+]\!][\![y]\!]$ as discussed in Remark 6.10. Extending the ground ring, we obtain an isomorphism

$$\begin{aligned} \mathrm{Loc}\colon \mathrm{GM}(F_\chi) \otimes_{R_{\mathbb{T}}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]} & S_{\mathbb{T}}(\!(z)\!)[\![\mathbf{\Lambda}_+]\!][\![y]\!] \\ & \cong H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \otimes_{R_{\mathbb{T}}} S_{\mathbb{T}}(\!(z)\!)[\![\mathbf{\Lambda}_+]\!][\![y]\!]. \end{aligned}$$

We set $C_p(\alpha) := e^{F_\chi^{\mathrm{qu}}(p)/z} \operatorname{Asym}_p(e^{F_\chi/z} \mathrm{Loc}^{-1} \alpha)$ for $\alpha \in H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})$. The differential equations in Proposition 4.22 and Lemma 6.7 show that

$$\left(\xi Q \frac{\partial}{\partial Q} + \frac{1}{z}\hat{\xi}(\sigma)\right) C_p(\alpha) = \frac{1}{z} C_p(\hat{\xi}\alpha), \qquad \frac{\partial}{\partial y_{\boldsymbol{\ell}}} C_p(\alpha) = 0.$$

Note that C_p belongs to $\mathrm{Hom}(H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}), R_{\mathbb{T}}) \otimes_{R_{\mathbb{T}}} \overline{S}_{\mathbb{T}}(\!(z)\!)[\![\mathbf{\Lambda}_+]\!][\![y]\!]$. The second equation shows that C_p is independent of $y_{\boldsymbol{\ell}}$. Expanding $C_p = \sum_{\lambda\in\mathbf{\Lambda}_+}\sum_{k\in\mathbb{Z}} C_{p;\lambda,k}Q^\lambda z^k$, we find from the first equation that

$$(\xi\cdot\lambda)C_{p;\lambda,k} = \hat{\xi}(\sigma)C_{p;\lambda,k+1} - C_{p;\lambda,k+1}\circ\hat{\xi}.$$

For a fixed λ, $C_{p;\lambda,k}$ vanishes for sufficiently negative $k\in\mathbb{Z}$. Therefore, if $\xi\cdot\lambda\neq 0$, repeated applications of the operation $C\mapsto \hat{\xi}(\sigma)C - C\circ\hat{\xi}$ on $\mathrm{Hom}(H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}), R_{\mathbb{T}}) \otimes_{R_{\mathbb{T}}} \overline{S}_{\mathbb{T}}$ to the coefficients $C_{p;\lambda,k}$ yield zero. On the other hand, it follows easily from the Atiyah-Bott localization theorem that this operation is a semisimple endomorphism, and thus we must have $\hat{\xi}(\sigma)C_{p;\lambda,k+1} - C_{p;\lambda,k+1}\circ\xi = 0$. This implies that $C_{p;\lambda,k}=0$ for non-zero λ and that C_p is independent of Q. Therefore we have shown that

$$e^{F_\chi^{\mathrm{qu}}(p)/z}\,\mathrm{Asym}_p(e^{F_\chi/z}\Omega) = \sum_{(\sigma',v')} C_{p,(\sigma',v')}\,\mathrm{Loc}(\Omega)\big|_{(\sigma',v')}$$

for some $C_{p,(\sigma',v')}\in\overline{S}_{\mathbb{T}}(\!(z)\!)$ independent of Q and $y_{\boldsymbol{\ell}}$.

Let us take $\Omega = w_{\mathbf{k}}\omega$ and evaluate the coefficients $C_{p,(\sigma',v')}$ by taking the limit as $Q\to 0$, $y\to y^*$. In that limit, $\mathrm{Loc}(w_{\mathbf{k}}\omega)$ becomes

$$\mathrm{Loc}^{(0)}\left(w_{\mathbf{k}}\omega\right)\Big|_{(\sigma',v')} = \begin{cases} \displaystyle\prod_{i\in\sigma'}\prod_{\substack{0\le c<\Psi_i(\mathbf{k})\\ \langle c\rangle = v_i}} (u_i(\sigma')-cz) & \text{if } v=v'\in\mathrm{Box}(\sigma'),\\ 0 & \text{otherwise,}\end{cases}$$

where $v := \mathbf{k}-\sum_{i=1}^m \lfloor\Psi_i(\mathbf{k})\rfloor b_i$.

It remains to compute the limit as $Q\to 0$, $y\to y^*$ of $e^{F_\chi^{\mathrm{qu}}(p)/z}\,\mathrm{Asym}_p\left(e^{F_\chi/z}w_{\mathbf{k}}\omega\right)$. The limit of $e^{F_\chi^{\mathrm{qu}}(p)/z}$ is 1. We have seen that $\mathrm{Asym}_p\left(e^{F_\chi/z}w_{\mathbf{k}}\omega\right)$ has a well-defined limit, because $\mathrm{Asym}_p\left(e^{F_\chi/z}w_{\mathbf{k}}\omega\right)$ lies in $\overline{S}_{\mathbb{T}}(\!(z)\!)[\![\mathbf{\Lambda}_+]\!][\![y]\!]$. Also

$$\mathrm{Asym}_p\left(e^{F_\chi/z}w_{\mathbf{k}}\omega\right) = \mathrm{Asym}_p\left(e^{(F_\chi-F_\chi^{\mathrm{cl}}(p))/z}w_{\mathbf{k}}\omega\right),$$

and we saw in the discussion before Lemma 6.5 that $F_\chi(x;y)-F_\chi^{\mathrm{cl}}(p)$ has a well-defined limit as y approaches y^* and (x,Q) approaches the component $A_{\sigma,\theta}$ of the central fiber on which p lies. When $v\notin\mathrm{Box}(\sigma)$, the restriction of $w_{\mathbf{k}}$ to $A_{\sigma,\theta}$ is zero, and thus $\mathrm{Asym}_p(e^{F_\chi/z}w_{\mathbf{k}}\omega)|_{Q=0,y=y^*}=0$. We can therefore assume that $v\in\mathrm{Box}(\sigma)$. On the component $A_{\sigma,\theta}$ we have

$$\int_{\Gamma(p)} e^{(F_\chi-F_\chi^{\mathrm{cl}}(p))/z}w_{\mathbf{k}}\omega$$
$$= \int_{\Gamma(p)}\exp\left(\sum_{i\in\sigma}\frac{w_i-u_i(\sigma)-u_i(\sigma)\log\left(\frac{w_i}{u_i(\sigma)}\right)}{z}\right) w_{\mathbf{k}}\frac{\det(b^{\alpha,\beta})}{|\mathbf{N}_{\mathrm{tor}}|}\bigwedge_{i\in\sigma}\frac{dw_i}{w_i}.$$

Here $\Gamma(p)$ is an appropriate cycle in $A_{\sigma,\theta}$ through p, the precise choice of which is irrelevant as we calculate the formal asymptotic expansion of this integral at p, and $(b^{\alpha,\beta})$ are the entries of the matrix inverse to $(b_{\alpha,\beta})$. Proceeding as in §6.2, we set $w_i = u_i(\sigma)e^{T_i}$, so that $w_{\mathbf{k}} = c_{\mathbf{k}} \exp\left(\sum_{i\in\sigma} \Psi_i(k)T_i\right)$. Thus the integral becomes

$$\frac{c_{\mathbf{k}}}{|\mathbf{N}(\sigma)|}\int_{\mathbb{R}^n} \exp\left(\sum_{i\in\sigma} \tfrac{u_i(\sigma)}{z}\left(e^{T_i}-1-T_i\right)\right)\exp\left(\sum_{i\in\sigma}\Psi_i(k)T_i\right)\bigwedge_{i\in\sigma} dT_i,$$

where we used $\det(b_{\alpha,\beta}) = |\mathbf{N}(\sigma)|/|\mathbf{N}_{\mathrm{tor}}|$. This is essentially a Γ-function. To see this, assume that $z<0$ and $u_i(\sigma)>0$, and make the change of variables $T_i \mapsto T_i + \log\left(\frac{-z}{u_i(\sigma)}\right)$. The integral becomes

$$\frac{c_{\mathbf{k}}}{|\mathbf{N}(\sigma)|}\prod_{i\in\sigma} e^{\frac{u_i(\sigma)}{-z}}\left(\tfrac{u_i(\sigma)}{-z}\right)^{\frac{u_i(\sigma)}{z}-\Psi_i(\mathbf{k})}\Gamma\Big(\Psi_i(\mathbf{k})-\tfrac{u_i(\sigma)}{z}\Big).$$

Using the functional equation for the Γ function (which is also satisfied by its asymptotic expansion) and the fact that $c_{\mathbf{k}} = c_v \prod_{i\in\sigma} u_i(\sigma)^{\lfloor \Psi_i(\mathbf{k})\rfloor}$, this is

$$\frac{c_v}{|\mathbf{N}(\sigma)|}\prod_{i\in\sigma} e^{\frac{u_i(\sigma)}{-z}}\left(\tfrac{u_i(\sigma)}{-z}\right)^{\frac{u_i(\sigma)}{z}-v_i}\left(\prod_{\substack{0\le c<\Psi_i(\mathbf{k})\\ \langle c\rangle = v_i}} u_i(\sigma)-cz\right)\Gamma\big(v_i-\tfrac{u_i(\sigma)}{z}\big).$$

Replacing the Γ function by its asymptotic expansion, using [**24**, 5.11.8]

$$\log\Gamma(z+h)\sim (z+h-\tfrac{1}{2})\log z - z + \tfrac{1}{2}\log(2\pi) + \sum_{k=2}^{\infty}\frac{(-1)^k B_k(h)}{k(k-1)z^{k-1}},$$

where $B_k(\cdot)$ are the Bernoulli polynomials, yields:

$$(-2\pi z)^{n/2}\frac{c_v}{|\mathbf{N}(\sigma)|}\prod_{i\in\sigma}\left(\prod_{\substack{0\le c<\Psi_i(\mathbf{k})\\ \langle c\rangle = v_i}} u_i(\sigma)-cz\right) \times\prod_{i\in\sigma}\frac{1}{\sqrt{u_i(\sigma)}}\exp\left(-\sum_{k=2}^{\infty}\frac{B_k(v_i)}{k(k-1)}\left(\frac{z}{u_i(\sigma)}\right)^{k-1}\right).$$

Thus the limit as $Q\to 0$, $y\to y^*$ of $\mathrm{Asym}_p\left(e^{F_X/z}w_{\mathbf{k}}\omega\right)$ is

$$\frac{c_v}{|\mathbf{N}(\sigma)|}\prod_{i\in\sigma}\left(\prod_{\substack{0\le c<\Psi_i(\mathbf{k})\\ \langle c\rangle = v_i}} u_i(\sigma)-cz\right) \times\prod_{i\in\sigma}\frac{1}{\sqrt{u_i(\sigma)}}\exp\left(-\sum_{k=2}^{\infty}\frac{B_k(v_i)}{k(k-1)}\left(\frac{z}{u_i(\sigma)}\right)^{k-1}\right)$$

if $v\in\mathrm{Box}(\sigma)$ and zero otherwise. The result follows. q.e.d.

Theorem 6.11. (1) *The mirror isomorphism* Θ *from Theorem 4.28 intertwines the higher residue pairing on* $\mathrm{GM}(F_\chi)$ *with the Poincaré pairing on* $H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X}) \otimes_{R_\mathbb{T}} R_\mathbb{T}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]$*:*

$$P(\Omega_1, \Omega_2) = \Big(\Theta(\Omega_1)\big|_{z\mapsto -z}, \Theta(\Omega_2)\Big).$$

(2) *The localization map intertwines the higher residue pairing on* $\mathrm{GM}(F_\chi)$ *with the Poincaré pairing on the Givental space:*

$$P(\Omega_1, \Omega_2) = \Big(\operatorname{Loc}(\Omega_1)\big|_{z\mapsto -z}, \operatorname{Loc}(\Omega_2)\Big).$$

Proof. First we prove (2). Using Definition 6.6 and Proposition 6.9, we find that $P(\Omega_1, \Omega_2)$ equals

$$\begin{aligned}\sum_{\sigma\in\Sigma(n)} \sum_{c\in\operatorname{Crit}(\sigma)} \sum_{v,w\in\operatorname{Box}(\sigma)} & c_v c_w \Delta_{(\sigma,v)}(-z)\Delta_{(\sigma,w)}(z) \\ & \times\overline{\operatorname{Loc}(\Omega_1)}\big|_{(\sigma,v)} \operatorname{Loc}(\Omega_2)\big|_{(\sigma,w)},\end{aligned} \tag{6.7}$$

where $\overline{\operatorname{Loc}(\Omega_1)} = \operatorname{Loc}(\Omega_1)|_{z\to -z}$. Recall that $\operatorname{Crit}(\sigma)$ is a torsor over $\widehat{\mathbf{N}}(\sigma)$. Choose a base point $c^* \in \operatorname{Crit}(\sigma)$ and write a general element $c \in \operatorname{Crit}(\sigma)$ as $c = \theta \cdot c^*$ for $\theta \in \widehat{\mathbf{N}}(\sigma)$. Orthogonality of characters implies that

$$\sum_{c\in\operatorname{Crit}(\sigma)} c_v c_w = \sum_{\theta\in\widehat{\mathbf{N}}(\sigma)} c_v^* c_w^* \theta(v)\theta(w) = c^*_{v+w}|\mathbf{N}(\sigma)|\delta_{v,-w},$$

where $\delta_{v,-w}$ equals 1 if $v \equiv -w$ in $\mathbf{N}(\sigma)$ and zero otherwise. Let $\sigma(v) \subset \sigma$ be the minimal cone of Σ containing v. We have

$$c^*_{v+w} = \prod_{j\in\sigma(v)} u_j(\sigma),$$

whenever $v \equiv -w$ in $\mathbf{N}(\sigma)$. Using the fact [**24**, 24.4.3] that $B_k(1-h) = (-1)^k B_k(h)$, we find that the quantity (6.7) equals

$$\sum_{\sigma\in\Sigma(n)} \sum_{v\in\operatorname{Box}(\sigma)} \frac{1}{|\mathbf{N}(\sigma)|} \frac{1}{\prod_{j\in\sigma\setminus\sigma(v)} u_j(\sigma)} \overline{\operatorname{Loc}(\Omega_1)}\big|_{(\sigma,v)} \operatorname{Loc}(\Omega_2)\big|_{(\sigma,-v)}.$$

The Atiyah–Bott localization formula now yields part (2) of the Theorem. In view of Remark 4.30, and the fact that $M(\tau, z)$ in Remark 4.30 is pairing-preserving (see Proposition 3.1), we have that (2) implies (1). q.e.d.

7. Convergence

In this section, we discuss convergence of the mirror isomorphism and the mirror map from Theorem 4.28. Recall that Q is a Novikov variable, $y = \{y_\mathbf{k} : \mathbf{k} \in S\}$ is a deformation parameter, and χ is an equivariant parameter. The main result in this section says that the mirror map is analytic in all the parameters (Q, y, χ) and that the mirror isomorphism is a formal power series in z with coefficients in analytic functions in (Q, y, χ). This implies the convergence of the big equivariant quantum product, and thus generalizes the convergence result [**48**] for compact toric varieties to arbitrary semi-projective toric Deligne–Mumford stacks in the *big* and *equivariant* setting.

7.1. Result. In order to state the convergence result, we introduce a co-ordinate system. We choose a $\mathbb{Z}$-basis of $\mathbf{\Lambda}$ such that $\mathbf{\Lambda}_+$ is contained in the cone spanned by the basis. This basis defines a co-ordinatization $Q = (Q_1, \dots, Q_r)$ of the variable Q (where $r = \operatorname{rank} \mathbf{\Lambda}$). Note that any power series in Q whose exponents are supported in $\mathbf{\Lambda}_+$ can be expressed as a nonnegative power series in $Q_1, \dots, Q_r$. Choosing a basis of $\mathbf{M}$, we have co-ordinates $\chi = (\chi_1, \dots, \chi_n)$ on $\operatorname{Lie} \mathbb{T} = \mathbf{N}_{\mathbb{C}}$ as before (so that $R_{\mathbb{T}} = \mathbb{C}[\chi_1, \dots, \chi_n]$). We write

$$q := (q_1, \dots, q_s) := (Q_1, \dots, Q_r, \log y_1, \dots, \log y_m, \{y_{\mathbf{k}} : \mathbf{k} \in G\})$$

with $s = r + m + |G|$. Note that $q = 0$ corresponds to $Q = 0$ and $y = y^*$ – see (4.2) for y^*. We also choose a homogeneous basis $\{T_i\}_{i=1}^N$ of $H_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})$ over $R_{\mathbb{T}}$ and homogeneous algebraic differential forms $\Omega_i \in \bigoplus_{\mathbf{k} \in \mathbf{N} \cap |\Sigma|} \mathbb{C}[z] w_{\mathbf{k}} \omega$ on the Landau–Ginzburg model (see §4.4) such that

$$\operatorname{Loc}^{(0)}(\Omega_i) = T_i, \qquad 1 \le i \le N.$$

Here $\operatorname{Loc}^{(0)}$ is the restriction of the localization map (see §4.5) to the origin $Q = 0, y = y^*$; such Ω_i exist by Proposition 4.24. By Theorem 4.26 (and its proof), $\{\Omega_i\}_{i=1}^N$ form a basis of the equivariant Gauss–Manin system $\mathrm{GM}(F_\chi)$ over $R_{\mathbb{T}}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]$. Let $\mathbb{N} = \{0, 1, 2, \dots\}$ denote the set of nonnegative integers and write $q^d = q_1^{d_1} \cdots q_s^{d_s}$ for $d = (d_1, \dots, d_s) \in \mathbb{N}^s$.

Definition 7.1. We define $\mathscr{O}^z$ to be the space of (possibly divergent) formal power series in $q = (q_1, \dots, q_s)$ and z of the form

$$\sum_{d \in \mathbb{N}^s} \sum_{k=0}^{\infty} a_{d,k}(\chi) q^d z^k,$$

where the coefficients $a_{d,k}(\chi)$ are holomorphic functions of $\chi = (\chi_1, \dots, \chi_n)$ defined in a uniform neighbourhood of $\chi = 0$ and satisfying the following estimate: there exist positive constants C_1, C_2, $\epsilon > 0$ such that we have

$$|a_{d,k}(\chi)| \le C_1 C_2^{|d|+k} |d|^k \quad \text{for} \quad |\chi| \le \epsilon, \tag{7.1}$$

where $|\chi| = (\sum_{i=1}^n |\chi_i|^2)^{1/2}$ and $|d| = \sum_{i=1}^s |d_i|$. We adopt the convention that $|d|^k = 1$ if $|d| = k = 0$. The constants C_1, C_2, ϵ here are allowed to depend on the element of $\mathscr{O}^z$. Note that the condition (7.1) implies

$a_{0,k}(\chi) = 0$ if $k > 0$. Note also that the subseries $\sum_{d\in\mathbb{N}^s} a_{d,k}(\chi)q^d$ coverges on $|q_a| < 1/C_2$, $|\chi| \le \epsilon$ for each $k \in \mathbb{N}$.

It is easy to check that $\mathcal{O}^z$ is a ring; moreover, it is a *local ring* – see Lemma 7.6.

Theorem 7.2. *Expand the mirror map $\tau(y)$ and the mirror isomorphism Θ in Theorem 4.28 with respect to the bases $\{T_i\}$, $\{\Omega_i\}$ as follows:*

$$\tau(y) = \sum_{i=1}^{N} \tau^i(Q,y,\chi)T_i, \qquad \Theta(\Omega_i) = \sum_{j=1}^{N} \Theta_i^j(Q,y,\chi,z)T_j.$$

Then:

1. *the coefficients $\tau^i(Q,y,\chi) \in R_{\mathbb{T}}[\![\mathbf{\Lambda}_+]\!][\![y]\!]$ are convergent and analytic in a neighbourhood of $Q = 0$, $y = y^*$ and $\chi = 0$;*
2. *the coefficients $\Theta_j^i(Q,y,\chi,z) \in R_{\mathbb{T}}[z][\![\mathbf{\Lambda}_+]\!][\![y]\!]$ lie in the ring $\mathcal{O}^z$; in particular they are formal power series in z with coefficients in analytic functions of (Q,y,χ) defined in a uniform neighbourhood of $Q = 0$, $y = y^*$ and $\chi = 0$.*

Corollary 7.3. *The structure constants $(\alpha \star \beta, \gamma)$ of the big and equivariant quantum product (3.1) of a semi-projective toric Deligne–Mumford stack are convergent power series in τ, Q and χ.*

Remark 7.4. To motivate the definition of $\mathcal{O}^z$, we give the following example. Suppose for simplicity that q is one variable. If $\sum_{k=0}^{\infty} a_k t^k$ is a convergent power series, the differential operator $\sum_{k=0}^{\infty} a_k (zq\frac{\partial}{\partial q})^k$ applied to $\sum_{d=0}^{\infty} q^d = 1/(1-q)$ yields a power series $\sum_{d=0}^{\infty}\sum_{k=0}^{\infty} a_k d^k q^d z^k$ that belongs to $\mathcal{O}^z$.

Remark 7.5. The coefficients Θ_j^i of the mirror isomorphism are in general divergent power series in z. If we restrict ourselves to the "extended weak Fano" situation [**50**, §3.1.4], that is, if $\mathfrak{X}$ is weak Fano and the extension G is contained in $\mathrm{Box}^{\le 1} := \mathrm{Box} \cap \{v \in \mathbf{N} \cap |\Sigma| : |v| \le 1\}$, then the mirror isomorphism Θ becomes fully convergent [**50**, Proposition 4.8]. On the other hand, if $\mathfrak{X}$ is not weak Fano or G is not contained in $\mathrm{Box}^{\le 1}$, the I-function and the mirror isomorphism are typically divergent (see [**48**, Proposition 5.13]). The convergence issue is also related to "good asymptotics" of the I-function, see [**14**, §2.6].

Lemma 7.6. *The space $\mathcal{O}^z$ is a local ring.*

Proof. We claim that if $x = \sum_{d\in\mathbb{N}^s}\sum_{k\ge 0} a_{d,k}(\chi)q^d z^k \in \mathcal{O}^z$ satisfies $a_{0,0}(0) \ne 0$, then it is invertible in $\mathcal{O}^z$. Without loss of generality we may assume that $a_{0,0}(\chi) = 1$. There exist constants C_1, C_2, $\epsilon > 0$ such that $|a_{d,k}(\chi)| \le C_1 C_2^{|d|+k} |d|^k$ for $|\chi| \le \epsilon$. We have

$$x^{-1} = 1 + \sum_{l=1}^{\infty} (-1)^l \sum_{\substack{d(1)\neq 0,\dots,d(l)\neq 0 \\ k(1),\dots,k(l)\geq 0}} a_{d(1),k(1)} \cdots a_{d(l),k(l)} \\ \times q^{d(1)+\cdots+d(l)} z^{k(1)+\cdots+k(l)}.$$

The coefficient of $q^d z^k$ can be estimated as:

$$\begin{aligned} &\left| \sum_{l=1}^{|d|} \sum\nolimits^{(l)} a_{d(1),k(1)} \cdots a_{d(l),k(l)} \right| \\ &\qquad \le \sum_{l=1}^{|d|} \sum\nolimits^{(l)} |d(1)|^{k(1)} \cdots |d(l)|^{k(l)} C_1 C_2^{|d|+k} \\ &\qquad \le |d|^k C_1 C_2^{|d|+k} \sum_{l=1}^{|d|} \sum_{\substack{d(1)+\cdots+d(l)=d \\ d(1)\neq 0,\dots,d(l)\neq 0}} 1, \end{aligned}$$

where $\sum^{(l)}$ means the sum over all $d(1), \dots, d(l) \in \mathbb{N}^s$ and $k(1), \dots, k(l) \in \mathbb{N}$ such that $d(1) \neq 0, \dots, d(l) \neq 0$, $d(1) + \cdots + d(l) = d$ and $k(1) + \cdots + k(l) = k$. It is easy to check that the sum in the second line is of exponential growth in $|d|$. q.e.d.

The rest of this section (§7) is devoted to the proof of Theorem 7.2 and Corollary 7.3. The proof of Theorem 7.2 consists of two steps: first we give an estimate for the connection matrices of the Gauss–Manin connection in the basis $\{\Omega_i\}$, and then we use a theorem on gauge fixing to show that a gauge transformation that transforms the Gauss–Manin connection into the quantum connection is defined over $\mathcal{O}^z$.

7.2. Estimates for the Gauss–Manin connection. Let $A_a = (A_a{}^j{}_i(q, \chi, z))$ denote the connection matrix of the equivariant Gauss–Manin system with respect to the basis $\{\Omega_i\}_{i=1}^N$ defined by

$$z \nabla_{q_a \frac{\partial}{\partial q_a}} \Omega_i = \sum_{j=1}^{N} A_a{}^j{}_i(q, \chi, z) \Omega_j,$$

where $A_a{}^j{}_i(q, \chi, z) \in R_{\mathbb{T}}[z][\![\boldsymbol{\Lambda}_+]\!][\![y]\!] \subset \mathbb{C}[z, \chi][\![q_1, \dots, q_s]\!]$. In this section we prove the following proposition.

Proposition 7.7. *There exist open neighbourhoods $U \subset \mathbb{C}^s$, $V \subset \mathbb{C}^n$ of the origin such that the following hold. Entries of the connection matrix A_a can be expanded in power series $\sum_{k=0}^{\infty} a_k(q, \chi) z^k$ that satisfy*

(1) *for every k, $a_k(q, \chi)$ is convergent and holomorphic for $(q, \chi) \in U \times V$;*

(2) *there exist contants* $C_1, C_2 > 0$ *such that*

$$|a_k(q,\chi)| \le C_1 C_2^k k! \qquad \text{for all } (q,\chi) \in U \times V. \tag{7.2}$$

Remark 7.8. Power series $\sum_{k=0}^\infty a_k z^k$ satisfying the estimate (7.2) are called *Gevrey series of order* 1. The same estimate also appears as a convergence condition for microdifferential operators (see [**56**, §2]). We write

$$\mathbb{C}\{\{z\}\} = \left\{ \sum_{k=0}^\infty a_k z^k : a_k \in \mathbb{C},\ \exists C_1,\ C_2 > 0 \text{ s.t. } |a_k| \le C_1 C_2^k k! \right\}.$$

It is well-known that $\mathbb{C}\{\{z\}\}$ is a local ring.

We start by noting that the Landau–Ginzburg potential F_χ is a globally defined (multi-valued) analytic function in the arguments x, y, Q, χ. Therefore the (finitely many) critical points of F_χ over $\overline{S}_{\mathbb{T}}[\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$ described in Lemma 6.2 depend analytically on the parameters (Q, y, χ). Recall from Lemma 6.2(2–3) that the co-ordinates $w_{\mathbf{k}}(p)$ of a critical point at $Q = 0$, $y = y^*$ are analytic functions of χ defined on $(\mathbb{C}^n \setminus \mathcal{D})^\sim$, where $^\sim$ means the universal cover and $\mathcal{D} = \bigcup_{i=1}^m \bigcup_{\sigma \in \Sigma(n)} \{u_i(\sigma) = 0\}$ (see Notation 6.1). Thus co-ordinates of each critical point are power series in $q = (Q, y)$ with coefficients in analytic functions on $(\mathbb{C}^n \setminus \mathcal{D})^\sim$. We fix an arbitrary compact subset $K \subset (\mathbb{C}^n \setminus \mathcal{D})^\sim$, and then choose a sufficiently small neighbourhood $U \subset \mathbb{C}^s$ of the origin such that co-ordinates of these critical points are convergent on $(\chi, q) \in K \times U$.

Let p be an analytic branch of the critical scheme of F_χ defined over $K \times U$ as above. Recall from §6.2 that $\operatorname{Asym}_p(e^{F_\chi/z}\Omega_i) \in \overline{S}_{\mathbb{T}}[z][\![\boldsymbol{\Lambda}_+]\!][\![y]\!]$ is defined as a formal asymptotic expansion of an oscillatory integral at the critical branch p:

$$\int_{\Gamma(p)} e^{F_\chi/z}\Omega_i \sim e^{F_\chi(p)/z}(-2\pi z)^{n/2} \operatorname{Asym}_p(e^{F_\chi/z}\Omega_i).$$

By the definition of the formal asymptotic expansion in §6.2 and the analyticity of F_χ, it follows that $\operatorname{Asym}_p(e^{F_\chi/z}\Omega_i)$ is a formal power series in z with coefficients in analytic functions in $(\chi, q) \in K \times U$ (where we also used the fact that Ω_i is an algebraic differential form). Moreover, these coefficients satisfy the following:

Proposition 7.9. *After shrinking U if necessary, we can find constants C_1, $C_2 > 0$ such that the power series expansion $\sum_{k=0}^\infty a_k(q,\chi) z^k$ of $\operatorname{Asym}_p(e^{F_\chi/z}\Omega_i)$ satisfies the estimate $|a_k(q,\chi)| \le C_1 C_2^k k!$ for all $k \ge 0$ and $(q,\chi) \in U \times K$.*

This proposition follows immediately from:

Lemma 7.10. *Let f, g be holomorphic functions on $\mathbb{C}^n$ and let $p \in \mathbb{C}^n$ be a non-degenerate critical point of f. (It suffices that f, g are* *defined near p.) The formal asymptotic expansion at p, as defined in* §*6.2:*

$$\int e^{f(t)/z} g(t) dt^1 \cdots dt^n \sim e^{f(p)/z}(-2\pi z)^{n/2}(a_0 + a_1 z + a_2 z^2 + \cdots)$$

satisfies the estimate $|a_k| \le C_1 C_2^k k!$ *for some constants* C_1, $C_2 > 0$. *If* B_1, $B_2 > 0$ *are constants such that the Taylor expansions* $f(p+t) = \sum_I c_I t^I$, $g(p+t) = \sum_I d_I t^I$ *at* p *satisfy* $\max(|c_I|, |d_I|) \le B_1 B_2^{|I|}$, *then the constants* C_1, C_2 *here depend only on* B_1 *and* B_2. *Here* $I \in \mathbb{N}^n$ *denotes a multi-index and we write* $t^I = (t^1)^{i_1} \cdots (t^n)^{i_n}$ *and* $|I| = i_1 + \cdots + i_n$ *for* $I = (i_1, \ldots, i_n)$.

Proof. By the holomorphic Morse lemma, we can find local co-ordinates $v = (v^1, \ldots, v^n)$ centered at p such that $f(t) = f(p) + \frac{1}{2}((v^1)^2 + \cdots + (v^n)^2)$. Changing co-ordinates, we get

$$\int e^{f(t)/z} g(t) dt^1 \cdots dt^n = e^{f(p)/z} \int e^{\sum_{i=1}^n (v^i)^2/(2z)} h(v) \frac{\partial(t^i)}{\partial(v^j)} dv^1 \cdots dv^n,$$

where $h(v)$ is a holomorphic function near $v = 0$ and $\partial(t^i)/\partial(v^j)$ denotes the Jacobian of the co-ordinate change. Let $\sum_I e_I v^I$ denote the Taylor expansion of $h(v)(\partial(t^i)/\partial(v^j))$; then we have an estimate $|e_I| \le C_1 C_2^{|I|}$ with constants C_1, $C_2 > 0$ depending only on B_1 and B_2. The asymptotic expansion of the above integral gives

$$\sum_{I=(i_1,\ldots,i_n)\in(2\mathbb{N})^n} e_I (-z)^{|I|/2} \prod_{a=1}^n (i_a - 1)!! \tag{7.3}$$

multiplied by $e^{f(p)/z}(-2\pi z)^{n/2}$, where $(2k-1)!! = (2k-1)(2k-3)\cdots 3\cdot 1$ (we set $(-1)!! = 1$). The coefficient in front of z^k in (7.3) has the estimate

$$\begin{aligned}\left| \sum_{I\in\mathbb{N}^n:|I|=k} e_{2I}(-1)^{|I|} \prod_{a=1}^n (2i_a - 1)!! \right| &\le C_1 C_2^{2k} 2^k \sum_{I\in\mathbb{N}^n:|I|=k} i_a! \\ &\le C_1 C_2^{2k} 2^k \binom{k+n-1}{k} k!\end{aligned}$$

which gives the desired estimate since $\binom{a}{b} \le 2^a$. q.e.d.

Proof of Proposition 7.7. We use the fact from Lemma 6.7 that Asym_p is a solution to the Gauss–Manin system. Let $\mathbf{R}$ be the square matrix with entries $\mathbf{R}_{p,i} = \mathrm{Asym}_p(e^{F_\chi/z}\Omega_i)$ where p ranges over all critical points in Lemma 6.2 and $1 \le i \le N$. Let $\mathbf{U}$ be the diagonal matrix with entries $F_\chi^{\mathrm{qu}}(p)$ with p ranging over the same set. The differential equation in Lemma 6.7 shows that

$$\left(z q_a \frac{\partial}{\partial q_a} + \xi_a\right)\left(e^{\mathbf{U}/z}\mathbf{R}\right) = e^{\mathbf{U}/z}\mathbf{R} A_a$$

for some $\xi_a \in H^2_{\mathbb{T}}(\mathrm{pt}, \mathbb{C}) = \bigoplus_{i=1}^n \mathbb{C}\chi_i$. This is equivalent to

$$A_a = \xi_a I + \mathbf{R}^{-1} z q_a \frac{\partial \mathbf{R}}{\partial q_a} + \mathbf{R}^{-1} q_a \frac{\partial \mathbf{U}}{\partial q_a} \mathbf{R},$$

where I denotes the identity matrix. Proposition 6.9 together with the fact that $\mathrm{Loc}^{(0)}(\Omega_i) = T_i$ shows that $\mathbf{R}|_{q=0}$ is invertible for $\chi \in (\mathbb{C}^n \setminus \mathcal{D})^\sim$. Recall from Proposition 7.9 that $\mathbf{R}_{p,i}$ is a Gevrey series of order 1 as a power series in z (see Remark 7.8). Since $\mathbb{C}\{\{z\}\}$ is a local ring, we conclude that the entries of $\mathbf{R}^{-1}$ satisfy estimates similar to Proposition 7.9, after shrinking U if necessary. Also, the entries of $z q_a \frac{\partial \mathbf{R}}{\partial q_a}$ satisfy estimates similar to Proposition 7.9 after shrinking U if necessary; this follows from the Cauchy integral formula for derivatives. The matrix $\mathbf{U}$ is convergent and analytic on $(q, \chi) \in U \times K$. Therefore each entry of A_a can be expanded in a series $\sum_{k=0}^\infty a_k(q,\chi) z^k$ that satisfies the estimate $|a_k(q,\chi)| \le C_1 C_2^k k!$ for $(q, \chi) \in U \times K$. On the other hand, we know that $a_k(q,\chi)$ here lies in $\mathbb{C}[\chi][\![q_1, \dots, q_s]\!]$, i.e. coefficients of the Taylor expansion of $a_k(q,\chi)$ with respect to q are polynomials in χ. Expand $a_k(q,\chi) = \sum_{d \in \mathbb{N}^s} a_{k,d}(\chi) q^d$. Then we have the estimate

$$|a_{k,d}(\chi)| \le C_1 C_2^{k+|d|} k! \qquad \forall \chi \in K$$

for possibly bigger constants C_1, $C_2 > 0$. Recall that K can be taken to be an arbitrary compact subset in $(\mathbb{C}^n \setminus \mathcal{D})^\sim$ (the constants C_1, C_2 depend on the choice of K); we can choose K so that the holomorphically convex hull (or polynomially convex hull) of the image of K in $\mathbb{C}^n \setminus \mathcal{D}$ contains the origin in its interior. Then the above estimate holds in a neighbourhood of the origin $\chi = 0$. This shows that each entry of A_a satisfies the estimate in Proposition 7.7. q.e.d.

Remark 7.11. The consideration of critical points in this section together with the study of the Jacobi ring in the non-equivariant limit in [**50**, Proposition 3.10(ii)] shows that *the non-equivariant quantum cohomology of a compact toric stack is generically semisimple.* See also [**48**, §5.4], [**50**, Corollary 4.9].

7.3. Gauge fixing. We begin with the following lemma.

Lemma 7.12. *Matrix entries of A_a belong to $\mathcal{O}^z$.*

Proof. Since the basis $\{\Omega_i\}$ is homogeneous with respect to Gr^{B} – see (4.10) – it follows from Proposition 4.11 that the connection matrices A_a have homogeneous entries with respect to the grading $\deg Q^d = c_1(\mathfrak{X})\cdot d$, $\deg y_{\mathbf{k}} = 1-|\mathbf{k}|$, $\deg \chi_i = \deg z = 1$. Since there are finitely many matrix entries of A_a's, we have a uniform constant $C > 0$ such that every entry $\sum_{k=0}^\infty a_k(q,\chi) z^k$ of $A_1, \dots, A_s$ satisfies $\deg a_k(q,\chi) + k \le C$ for all $k \in \mathbb{N}$. Expanding $a_k(q,\chi) = \sum_{d \in \mathbb{N}^s} a_{d,k}(\chi) q^d$, we obtain $\deg q^d + k \le C$ since $a_{d,k}(\chi) \in \mathbb{C}[\chi]$ has non-negative degree. This implies $k \le C + C'|d|$

for a uniform constant $C' > 0$. On the other hand, the estimate in Proposition 7.7 gives $|a_{d,k}(\chi)| \le C_1 C_2^{k+|d|} k!$ for $\chi \in V$. Combining the two inequalities we obtain, whenever $d \neq 0$,

$$(7.4)\quad |a_{d,k}(\chi)| \le C_1 C_2^{k+|d|}(C + C'|d|)^k \le C_1 C_3^{k+|d|}|d|^k \qquad \text{with } \chi \in V$$

for $C_3 = 2\max(C, C', 1)C_2$. We claim that $a_{0,k}(\chi) = 0$ for $k > 0$. Since $\mathrm{Loc}^{(0)}(\Omega_i) = T_i$, Proposition 4.22 implies that $A_a|_{q=0}$ is conjugate to the multiplication by $\hat{\xi}_a$ on $H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})$ for some $\hat{\xi}_a \in H^2_{\mathbb{T}}(\mathfrak{X})$ ($\hat{\xi}_a = 0$ if q_a corresponds to $\log y_1, \dots, \log y_m$ or to $y_{\mathbf{k}}$ with $\mathbf{k} \in G$). Therefore $A_a|_{q=0}$ is independent of z, and the claim follows. The claim implies that the estimate (7.4) also holds for $d = 0$. The lemma is proved. q.e.d.

We now give a result on gauge fixing, which says that a logarithmic flat connection defined over $\mathcal{O}^z$ with nilpotent residue at $q = \chi = 0$ can be made z-independent by a unique gauge transformation defined over $\mathcal{O}^z$. This result is a refinement of [**48**, Proposition 4.8] which proved a similar result in the absence of the parameter χ. The proof is almost parallel to [**48**, Proposition 4.8], but is different from it in a subtle way. We repeat the argument for the convenience of the reader.

Theorem 7.13 (gauge fixing)**.** *Let ∇ be a logarithmic flat connection of the form*

$$\nabla = d + \frac{1}{z}\sum_{i=1}^{s} A_i \frac{dq_i}{q_i},$$

where $A_i = A_i(q,\chi,z) \in \mathrm{Mat}_N(\mathcal{O}^z)$ is a square matrix with entries in $\mathcal{O}^z$ (see Definition 7.1). Note from the definition of $\mathcal{O}^z$ that $A_i(0,\chi,z)$ is independent of z. Suppose that the residue matrices $A_i(0,0,z)$ are nilpotent. Then there exists a unique gauge transformation $G = G(q,\chi,z) \in \mathrm{GL}_N(\mathcal{O}^z)$ with entries in $\mathcal{O}^z$ such that $G(0,\chi,z) = I$ and $\widehat{A}_i := G^{-1} z q_i \frac{\partial G}{\partial q_i} + G^{-1} A_i G$ is independent of z for all $1 \le i \le s$. In particular $\widehat{A}_i$ is convergent and analytic near $q = \chi = 0$.

Proof. It is easy to see that such a gauge transformation G exists uniquely over the formal power series ring $\mathbb{C}[\![\chi, z, q]\!]$; G is given as a positive Birkhoff factor of a fundamental solution[7] of the connection ∇. (See [**17**], [**45**, Proposition 3.2] and [**49**, Theorem 4.6] for the discussion in the context of quantum cohomology.) Note that the following argument also gives a recursive construction of G.

We expand $A_i = \sum_{d\in\mathbb{N}^s}\sum_{k=0}^{\infty} A_{i;d,k}(\chi) q^d z^k$, $\widehat{A}_i = \sum_{d\in\mathbb{N}^s} \widehat{A}_{i;d}(\chi) q^d$ and $G = \sum_{d\in\mathbb{N}^s}\sum_{k=0}^{\infty} G_{d,k}(\chi) q^d z^k$. Note that we have $G_{0,k} = \delta_{k,0} I$,

[7]We use the nilpotence of $A_i|_{q=\chi=0}$ to construct a fundamental solution in the formal setting.

$A_{i;0,k} = \delta_{k,0} A_{i;0,0}$ and $\widehat{A}_{i;0} = \widehat{A}_i|_{q=0} = A_{i;0,0}$. Expanding the relation $zq_i \frac{\partial G}{\partial q_i} + A_i G = G\widehat{A}_i$, we obtain

$$d_i G_{d,k-1} + \sum_{d'+d''=d} \sum_{k'+k''=k} A_{i;d',k'} G_{d'',k''} = \sum_{d'+d''=d} G_{d',k} \widehat{A}_{i;d''}.$$

This can be rewritten as follows:

$$d_i G_{d,k-1} + \operatorname{ad}(N_i) G_{d,k} = H_{i;d,k} \qquad \text{for } k \geq 1, \tag{7.5a}$$

$$\widehat{A}_{i;d} = \operatorname{ad}(N_i) G_{d,0} - H_{i;d,0}, \tag{7.5b}$$

where we set $N_i = N_i(\chi) := A_{i;0,0}(\chi)$, $\operatorname{ad}(X)Y = XY - YX$, and

$$H_{i;d,k} := \sum_{\substack{d'+d''=d \\ |d'|>0, |d''|>0}} G_{d',k} \widehat{A}_{i;d''} - \sum_{\substack{d'+d''=d \\ |d'|>0}} \sum_{k'+k''=k} A_{i;d',k'} G_{d'',k''}. \tag{7.6}$$

Suppose that we know $G_{d,k}$ and $\widehat{A}_{i;d}$ for all (i,d,k) with $|d| < e$ (for some e). This information determines $H_{i;d,k}$ for all (i,d,k) with $|d| = e$. Then we can determine $G_{d,k}$ and $\widehat{A}_{i;d}$ recursively as follows:

(1) we can solve for $G_{d,k}$ with $|d| = e$ for all k using (7.5a) – see (7.8) below;
(2) next we can solve for $\widehat{A}_{i;d}$ with $|d| = e$ using (7.5b).

We need to give estimates for $G_{d,k}$, $\widehat{A}_{i,d}$. We set, for $(e,k) \in \mathbb{N}^2$,

$$a_{e,k}(\chi) := \max_{1 \leq i \leq s} \sum_{|d|=e} \|A_{i;d,k}(\chi)\|, \quad g_{e,k}(\chi) := \sum_{|d|=e} \|G_{d,k}(\chi)\|,$$

$$h_{e,k}(\chi) := \max_{1 \leq i \leq s} \sum_{|d|=e} \|H_{i;d,k}(\chi)\|, \quad \hat{a}_e(\chi) := \max_{1 \leq i \leq s} \sum_{|d|=e} \|\widehat{A}_{i;d}(\chi)\|,$$

where $\|\cdot\|$ denotes the operator norm. By assumption there exist constants C_1, C_2, $\epsilon > 0$ such that $a_{e,k}(\chi) \leq e^k C_1 C_2^{e+k}$ for $|\chi| \leq \epsilon$ (we set $e^k = 1$ for $e = k = 0$ as before). Let $C > 0$ be a constant such that $\|N_i(\chi)\| = \|A_{i;0,0}(\chi)\| = \|\widehat{A}_{i;0}(\chi)\| \leq C$ for all $1 \leq i \leq s$ and $|\chi| \leq \epsilon$. Suppose by induction that

$$g_{e,k}(\chi) \leq \frac{e^k}{(e+1)^M} B_1^e B_2^k \text{ and } \hat{a}_e(\chi) \leq \frac{1}{(e+1)^M} C B_1^e \text{ whenever } |\chi| \leq \delta$$

for all (e,k) with $e < e_0$. Here we set $B_2 := 2C_2$ and the other positive constants B_1, M, $\delta > 0$ are specified later; they depend only on C, C_1, C_2 and $N_i(\chi)$. We will choose δ so that $0 < \delta \leq \epsilon$. Note that this induction hypothesis holds for $e_0 = 1$ since $g_{0,k} = \|G_{0,k}(\chi)\| = \delta_{k,0}$ and $\hat{a}_0 = \max_{1 \leq i \leq s} \|N_i(\chi)\| \leq C$. Under the induction hypothesis, we have from (7.6) that, whenever $|\chi| \le \delta$,

$$
\begin{aligned}
h_{e_0,k} &\le \sum_{0<e<e_0} g_{e,k}\hat{a}_{e_0-e} + \sum_{0<e\le e_0}\sum_{l=0}^{k} a_{e,l} g_{e_0-e,k-l} \\
&\le \sum_{0<e<e_0} e^k \frac{B_1^e B_2^k}{(1+e)^M}\frac{C B_1^{e_0-e}}{(1+e_0-e)^M} \\
&\qquad + \sum_{0<e\le e_0}\sum_{l=0}^{k} e^l C_1 C_2^{e+l}\frac{(e_0-e)^{k-l}}{(1+e_0-e)^M} B_1^{e_0-e}B_2^{k-l}, \\
&\le \left(C\epsilon_1(M) + 2C_1\epsilon_2(B_1,M)\right)\frac{e_0^k}{(1+e_0)^M} B_1^{e_0}B_2^k,
\end{aligned}
\tag{7.7}
$$

where we omit χ from the notation and set

$$
\begin{aligned}
\epsilon_1(M) &:= \sup_{e_0\ge 1}\sum_{0<e<e_0}\frac{(1+e_0)^M}{(1+e)^M(1+e_0-e)^M}, \\
\epsilon_2(B_1,M) &:= \sup_{e_0\ge 1}\sum_{0<e\le e_0}\frac{(1+e_0)^M}{(1+e_0-e)^M}\left(\frac{C_2}{B_1}\right)^e.
\end{aligned}
$$

Next we estimate $g_{e_0,k}$. For $d \in \mathbb{N}^s$ with $|d| = e_0$, we choose $1 \le i \le s$ such that $d_i = \max\{d_1,\dots,d_s\}$. Then we have from (7.5a) that

$$
\begin{aligned}
G_{d,k} &= \frac{1}{d_i}H_{i;d,k} - \frac{\operatorname{ad}(N_i)}{d_i}G_{d,k+1} \\
&= \frac{1}{d_i}H_{i;d,k} - \frac{\operatorname{ad}(N_i)}{d_i^2}H_{i;d,k+1} + \cdots + \frac{(-\operatorname{ad}(N_i))^l}{d_i^{l+1}}H_{i;d,k+l} + \cdots .
\end{aligned}
\tag{7.8}
$$

Note that this infinite sum converges in the χ-adic topology since $\operatorname{ad}(N_i|_{\chi=0})$ is nilpotent, and hence $G_{d,k}$'s are uniquely determined by (7.5a). We will see that this sum is convergent in the classical topology. Using $d_i \ge e_0/s$, we have

$$
g_{e_0,k} \le \sum_{l=0}^{\infty}\left(\max_{1\le i\le s}\|\operatorname{ad}(N_i)^l\|\right)\left(\frac{s}{e_0}\right)^{l+1} h_{e_0,k+l}.
$$

Since $N_i(0) = N_i|_{\chi=0}$ is nilpotent, we have $N_i(0)^{l_0} = 0$ for some $l_0 > 0$; then we have $\operatorname{ad}(N_i(0))^{2l_0} = 0$. This implies the following estimate:

$$
\max_{1\le i\le s}\|\operatorname{ad}(N_i(\chi))^l\| \le C_3(C_4|\chi|)^{\lfloor l/(2l_0)\rfloor} \quad \text{if } |\chi| \le \epsilon
$$

for some C_3, $C_4 > 0$. Thus we get

$$
g_{e_0,k} \le \sum_{l=0}^{2l_0-1}\sum_{j=0}^{\infty} C_3 C_4^j |\chi|^j \left(\frac{s}{e_0}\right)^{2l_0 j+l+1} h_{e_0,k+2l_0 j+l}.
$$

Using the estimate (7.7) for $h_{e_0,k}$ and after some computations, we find that the infinite sum converges if $s^{2l_0}B_2^{2l_0}C_4|\chi| < 1$ and that

$$g_{e_0,k} \leq \epsilon_3(B_1, M, \chi)\frac{e_0^k}{(1+e_0)^M}B_1^{e_0}B_2^k \tag{7.9}$$

with

$$\epsilon_3(B_1, M, \chi) := (C\epsilon_1(M) + 2C_1\epsilon_2(B_1, M))\,\frac{C_5}{1 - s^{2l_0}B_2^{2l_0}C_4|\chi|}$$

and $C_5 := sC_3\sum_{l=0}^{2l_0-1}(sB_2)^l$. Finally we estimate $\hat{a}_{e_0}$ by (7.5b). We have for $|\chi| \leq \delta$,

(7.10)

$$\hat{a}_{e_0} \leq 2Cg_{e_0,0} + h_{e_0,0} \leq \frac{2C\epsilon_3(B_1, M, \chi) + C\epsilon_1(M) + 2C_1\epsilon_2(B_1, M)}{(1+e_0)^M}B_1^{e_0},$$

where we used (7.7) and (7.9) in the second inequality.

We now specify the constants B_1, M, $\delta > 0$ to complete the induction step. We use the following fact [**48**, Lemma 4.9]:

$$\lim_{M\to\infty}\epsilon_1(M) = 0, \qquad \lim_{B_1\to\infty}\epsilon_2(B_1, M) = 0.$$

We can choose $B_1, M, \delta > 0$ in the following way.

(1) Choose $\delta > 0$ so that $\delta < \epsilon$ and $s^{2l_0}B_2^{2l_0}C_4\delta < 1/2$.
(2) Choose $M > 0$ so that $\epsilon_1(M) \leq 1/3$ and $2C_5C\epsilon_1(M) \leq 1/12$.
(3) Choose $B_1 > 0$ so that $2C_1\epsilon_2(B_1, M) \leq C/3$ and $4C_1C_5\epsilon_2(B_1, M) \leq 1/12$.

Then we have, when $|\chi| \leq \delta$,

$$\epsilon_3(B_1, M, \chi) \leq 2C_5C\epsilon_1(M) + 4C_5C_1\epsilon_2(B_1, M) \leq \frac{1}{6},$$

$$2C\epsilon_3(B_1, M, \chi) + C\epsilon_1(M) + 2C_1\epsilon_2(B_1, M) \leq \frac{1}{3}C + \frac{1}{3}C + \frac{1}{3}C \leq C.$$

These inequalities together with the estimates (7.9), (7.10) complete the induction step. The theorem is proved. q.e.d.

Remark 7.14. Theorem 7.13 could be viewed as an analogue of Malgrange's theorem [**59**, Theorem 4.3]. There a similar analytification of a flat connection is discussed when the connection has no singularities at $q = 0$.

7.4. Proof of Theorem 7.2 and Corollary 7.3. First we prove Theorem 7.2. Recall that the connection matrices A_a of the Gauss–Manin connection are defined over $\mathcal{O}^z$ (Lemma 7.12) and the connection matrices of the quantum connection with respect to the basis $\{T_i\}$ are independent of z. The matrix $(\Theta_i^j(Q, y, \chi, z))$ gives a gauge transformation which transforms the Gauss–Manin connection into the quantum connection; it also satisfies $\Theta_i^j|_{q=0} = \delta_i^j$ since $\Theta(\Omega_i)|_{q=0} = \mathrm{Loc}^{(0)}(\Omega_i) = T_i$.

As remarked in the proof of Theorem 7.13, the uniqueness of such a gauge transformation holds over the formal power series ring $\mathbb{C}[\![z,\chi,q]\!]$, and thus Theorem 7.13 shows that $\Theta_i^j(Q,y,\chi,z)$ belongs to $\mathcal{O}^z$. Theorem 7.13 also shows that the pull-back $\tau^*\nabla$ of the quantum connection via τ is analytic in (Q,y,χ).

Since Θ intertwines the Gauss–Manin connection with the quantum connection, we have

$$(\Theta\circ\nabla_{zq_a\frac{\partial}{\partial q_a}}\circ\Theta^{-1})(1)=\nabla_{zq_a\frac{\partial\tau}{\partial q_a}}1=q_a\frac{\partial\tau}{\partial q_a},\tag{7.11}$$

where 1 is the identity class in $H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})$. Since $\mathcal{O}^z$ is a local ring (Lemma 7.6), the inverse matrix of (Θ_i^j) has coefficients in $\mathcal{O}^z$. Thus the left-hand side can be written as an $\mathcal{O}^z$-linear combination of $\{T_i\}$. Since the right-hand side is independent of z, this shows that $q_a\frac{\partial\tau^i}{\partial q_a}$ is analytic in (q,χ). Therefore the mirror map $\tau^i(Q,y,\chi)$ is analytic in all the arguments.

Next we prove Corollary 7.3. We claim that the mirror map τ is submersive when the extension G is sufficiently large. Indeed, using the formula (7.11), we have for $\mathbf{k}\in S$,

$$\begin{aligned}\left.\frac{\partial\tau}{\partial y_{\mathbf{k}}}\right|_{Q=0,y=y^*}&=\left.(\Theta\circ\nabla_{z\frac{\partial}{\partial y_{\mathbf{k}}}}\circ\Theta^{-1})(1)\right|_{q=0}\\&=\left.\Theta\left(\nabla_{z\frac{\partial}{\partial y_{\mathbf{k}}}}(\omega+O(q))\right)\right|_{q=0}\\&=\Theta(w_{\mathbf{k}}\omega+O(z)+O(q))|_{q=0}\\&=\phi_{\mathbf{k}}+O(z),\end{aligned}$$

where we used $\Theta|_{q=0}=\mathrm{Loc}^{(0)}$ and the computation in Proposition 4.24. Since the left-hand side is independent of z, it is equal to $\phi_{\mathbf{k}}$. Since finitely many $\phi_{\mathbf{k}}$'s span $H^*_{\mathrm{CR},\mathbb{T}}(\mathfrak{X})$ over $R_{\mathbb{T}}$, the claim follows. We already showed that the pull-back $\tau^*\nabla$ of the quantum connection and the mirror map τ are analytic in a neighbourhood of $Q=0$, $y=y^*$, $\chi=0$. This immediately implies the analyticity of the big equivariant quantum product.

References

[1] Dan Abramovich, Tom Graber, and Angelo Vistoli, *Gromov-Witten theory of Deligne-Mumford stacks*, Amer. J. Math. **130** (2008), no. 5, 1337–1398, MR2450211, Zbl 1193.14070.

[2] Pedro Acosta and Mark Shoemaker, *Gromov-Witten theory of toric birational transformations* (2016), arXiv:1604.03491 [math.AG].

[3] Serguei Barannikov, *Semi-infinite Hodge structure and mirror symmetry for projective spaces* (2001), arXiv:math.AG/0010157.

[4] Victor V. Batyrev, *Quantum cohomology rings of toric manifolds*, Astérisque (1993), no. 218, 9–34, Journées de Géométrie Algébrique d'Orsay (Orsay, 1992), MR1265307, Zbl 0806.14041.

[5] Lev A. Borisov, Linda Chen, and Gregory G. Smith, *The orbifold Chow ring of toric Deligne-Mumford stacks*, J. Amer. Math. Soc. **18** (2005), no. 1, 193–215 (electronic), MR2114820, Zbl 1178.14057.

[6] Lev A. Borisov and R. Paul Horja, *On the better behaved version of the GKZ hypergeometric system*, Math. Ann. **357** (2013), no. 2, 585–603, MR3096518, Zbl 1284.33012.

[7] Alexander Braverman, Davesh Maulik, and Andrei Okounkov, *Quantum cohomology of the Springer resolution*, Adv. Math. **227** (2011), no. 1, 421–458, MR2782198, Zbl 1226.14069.

[8] Jeff Brown, *Gromov-Witten invariants of toric fibrations*, Int. Math. Res. Not. IMRN (2014), no. 19, 5437–5482, arXiv:0901.1290 [math.AG], MR3267376, Zbl 1307.14077.

[9] Kwokwai Chan, Siu-Cheong Lau, Naichung-Conan Leung, and Hsian-Hua Tseng, *Open Gromov-Witten invariants, mirror maps, and Seidel representations for toric manifolds*, Duke Math. J. **166** (2017), no. 8, 1405–1462, arXiv:1209.6119, MR3659939.

[10] Weimin Chen and Yongbin Ruan, *Orbifold Gromov-Witten theory*, Orbifolds in mathematics and physics (Madison, WI, 2001), Contemp. Math., vol. 310, Amer. Math. Soc., Providence, RI, 2002, pp. 25–85, MR1950941, Zbl 1091.53058.

[11] Weimin Chen and Yongbin Ruan, *A new cohomology theory of orbifold*, Comm. Math. Phys. **248** (2004), no. 1, 1–31, MR2104605, Zbl 1063.53091.

[12] Daewoong Cheong, Ionuţ Ciocan-Fontanine, and Bumsig Kim, *Orbifold quasimap theory*, Math. Ann. **363** (2015), no. 3–4, 777–816, arXiv:1405.7160 [math.AG], MR3412343, Zbl 1337.14012.

[13] Tom Coates, Alessio Corti, Hiroshi Iritani, and Hsian-Hua Tseng, *Computing genus-zero twisted Gromov-Witten invariants*, Duke Math. J. **147** (2009), no. 3, 377–438, MR2510741, Zbl 1176.14009.

[14] Tom Coates, Alessio Corti, Hiroshi Iritani, and Hsian-Hua Tseng, *Some applications of the mirror theorem for toric stacks* (2014), arXiv:1401.2611 [math.AG].

[15] Tom Coates, Alessio Corti, Hiroshi Iritani, and Hsian-Hua Tseng, *A mirror theorem for toric stacks*, Compos. Math. **151** (2015), no. 10, 1878–1912, arXiv:1310.4163 [math.AG], MR3414388, Zbl 1330.14093.

[16] Tom Coates, Alessio Corti, Yuan-Pin Lee, and Hsian-Hua Tseng, *The quantum orbifold cohomology of weighted projective spaces*, Acta Math. **202** (2009), no. 2, 139–193, MR2506749, Zbl 1213.53106.

[17] Tom Coates and Alexander Givental, *Quantum Riemann-Roch, Lefschetz and Serre*, Ann. of Math. (2) **165** (2007), no. 1, 15–53, MR2276766, Zbl 1189.14063.

[18] Tom Coates and Hiroshi Iritani, *A Fock sheaf for Givental quantization*, Kyoto J. Math. **58** (2018), no. 4, 695–864, arXiv:1411.7039 [math.AG], MR3880240, Zbl 07000589.

[19] Tom Coates, Hiroshi Iritani, and Yunfeng Jiang, *The crepant transformation conjecture for toric complete intersections*, Adv. Math. **329** (2018), 1002–1087, arXiv:1410.0024 [math.AG], MR3783433, Zbl 1394.14036.

[20] Tom Coates, Hiroshi Iritani, and Hsian-Hua Tseng, *Wall-crossings in toric Gromov-Witten theory. I. Crepant examples*, Geom. Topol. **13** (2009), no. 5, 2675–2744, MR2529944, Zbl 1184.53086.

[21] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011, MR2810322, Zbl 1223.14001.

[22] Ignacio de Gregorio and Étienne Mann, *Mirror fibrations and root stacks of weighted projective spaces*, Manuscripta Math. **127** (2008), no. 1, 69–80, MR2429914, Zbl 1221.14047.

[23] Colin Diemer, Ludmil Katzarkov, and Gabriel Kerr, *Symplectomorphism group relations and degenerations of Landau-Ginzburg models*, J. Eur. Math. Soc. (JEMS) **18** (2016), no. 10, 2167–2271, arXiv:1204.2233 [math.AG], MR3551190, Zbl 1354.53084.

[24] *NIST Digital Library of Mathematical Functions*, `http://dlmf.nist.gov/`, Release 1.0.10 of 2015-08-07, Online companion to [**64**].

[25] Antoine Douai and Etienne Mann, *The small quantum cohomology of a weighted projective space, a mirror D-module and their classical limits*, Geom. Dedicata **164** (2013), 187–226, MR3054624, Zbl 1273.14112.

[26] Antoine Douai and Claude Sabbah, *Gauss-Manin systems, Brieskorn lattices and Frobenius structures. I*, Proceedings of the International Conference in Honor of Frédéric Pham (Nice, 2002), vol. 53, 2003, pp. 1055–1116, MR2033510, Zbl 1079.32016.

[27] Antoine Douai and Claude Sabbah, *Gauss-Manin systems, Brieskorn lattices and Frobenius structures. II*, Frobenius manifolds, Aspects Math., E36, Vieweg, Wiesbaden, 2004, pp. 1–18, MR2115764, Zbl 1079.32017.

[28] Boris Dubrovin, *Geometry of 2D topological field theories*, Integrable systems and quantum groups (Montecatini Terme, 1993), Lecture Notes in Math., vol. 1620, Springer, Berlin, 1996, pp. 120–348, MR1397274, Zbl 0841.58065.

[29] Barbara Fantechi, Etienne Mann, and Fabio Nironi, *Smooth toric Deligne-Mumford stacks*, J. Reine Angew. Math. **648** (2010), 201–244, MR2774310, Zbl 1211.14009.

[30] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono, *Lagrangian Floer theory on compact toric manifolds. I*, Duke Math. J. **151** (2010), no. 1, 23–174, MR2573826, Zbl 1190.53078.

[31] I. M. Gel'fand, A. V. Zelevinskiĭ, and M. M. Kapranov, *Hypergeometric functions and toric varieties*, Funktsional. Anal. i Prilozhen. **23** (1989), no. 2, 12–26, MR1011353, Zbl 0721.33006.

[32] Alexander Givental, *Homological geometry and mirror symmetry*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), Birkhäuser, Basel, 1995, pp. 472–480, MR1403947, Zbl 0863.14021.

[33] Alexander Givental, *Homological geometry. I. Projective hypersurfaces*, Selecta Math. (N.S.) **1** (1995), no. 2, 325–345, MR1354600, Zbl 0920.14028.

[34] Alexander Givental, *A symplectic fixed point theorem for toric manifolds*, The Floer memorial volume, Progr. Math., vol. 133, Birkhäuser, Basel, 1995, pp. 445–481, MR1362837, Zbl 0835.55001.

[35] Alexander Givental, *Equivariant Gromov-Witten invariants*, Internat. Math. Res. Notices (1996), no. 13, 613–663, MR1408320, Zbl 0881.55006.

[36] Alexander Givental, *Elliptic Gromov-Witten invariants and the generalized mirror conjecture*, Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), World Sci. Publ., River Edge, NJ, 1998, pp. 107–155, MR1672116, Zbl 0961.14036.

[37] Alexander Givental, *A mirror theorem for toric complete intersections*, Topological field theory, primitive forms and related topics (Kyoto, 1996), Progr. Math., vol. 160, Birkhäuser Boston, Boston, MA, 1998, pp. 141–175, MR1653024, Zbl 0936.14031.

[38] Alexander Givental, *Gromov-Witten invariants and quantization of quadratic Hamiltonians*, Mosc. Math. J. **1** (2001), no. 4, 551–568, 645, Dedicated to the memory of I. G. Petrovskii on the occasion of his 100th anniversary, MR1901075, Zbl 1008.53072.

[39] Alexander Givental, *Symplectic geometry of Frobenius structures*, Frobenius manifolds, Aspects Math., E36, Friedr. Vieweg, Wiesbaden, 2004, 91–112, MR2115767, Zbl 1075.53091.

[40] Eduardo González and Hiroshi Iritani, *Seidel elements and mirror transformations*, Selecta Math. (N.S.) **18** (2012), no. 3, 557–590, MR2960027, Zbl 1263.14054.

[41] Eduardo González and Chris Woodward, *Quantum cohomology and toric minimal model programs* (2012), arXiv:1010.2118.

[42] Tom Graber and Rahul Pandharipande, *Localization of virtual classes*, Invent. Math. **135** (1999), no. 2, 487–518, MR1666787, Zbl 0953.14035.

[43] Mark Gross, *Mirror symmetry for $\mathbb{P}^2$ and tropical geometry*, Adv. Math. **224** (2010), no. 1, 169–245, MR2600995, Zbl 1190.14038.

[44] Mark Gross, *Tropical geometry and mirror symmetry*, CBMS Regional Conference Series in Mathematics, vol. 114, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2011, MR2722115, Zbl 1215.14061.

[45] Martin A. Guest, *Quantum cohomology via D-modules*, Topology **44** (2005), no. 2, 263–281, MR2114708, Zbl 1081.53077.

[46] Kentaro Hori and Cumrum Vafa, *Mirror symmetry* (2000), arXiv:hep-th/0002222.

[47] Hiroshi Iritani, *Quantum D-modules and equivariant Floer theory for free loop spaces*, Math. Z. **252** (2006), no. 3, 577–622, MR2207760, Zbl 1121.53062.

[48] Hiroshi Iritani, *Convergence of quantum cohomology by quantum Lefschetz*, J. Reine Angew. Math. **610** (2007), 29–69, MR2359850, Zbl 1160.14044.

[49] Hiroshi Iritani, *Quantum D-modules and generalized mirror transformations*, Topology **47** (2008), no. 4, 225–276, MR2416770, Zbl 1170.53071.

[50] Hiroshi Iritani, *An integral structure in quantum cohomology and mirror symmetry for toric orbifolds*, Adv. Math. **222** (2009), no. 3, 1016–1079, MR2553377, Zbl 1190.14054.

[51] Hiroshi Iritani, *Quantum cohomology and periods*, Ann. Inst. Fourier (Grenoble) **61** (2011), no. 7, 2909–2958, MR3112512, Zbl 1300.14055.

[52] Hiroshi Iritani, *A mirror construction for the big equivariant quantum cohomology of toric manifolds*, Math. Ann. **368** (2017), no. 1–2, 279–316, `arXiv:1503.02919 [math.AG]`, MR3651574, Zbl 06726653.

[53] Isamu Iwanari, *The category of toric stacks*, Compos. Math. **145** (2009), no. 3, 718–746, MR2507746, Zbl 1177.14024.

[54] Isamu Iwanari, *Logarithmic geometry, minimal free resolutions and toric algebraic stacks*, Publ. Res. Inst. Math. Sci. **45** (2009), no. 4, 1095–1140, MR2597130, Zbl 1203.14058.

[55] Yunfeng Jiang and Hsian-Hua Tseng, *Note on orbifold Chow ring of semi-projective toric Deligne-Mumford stacks*, Comm. Anal. Geom. **16** (2008), no. 1, 231–250, MR2411474, Zbl 1165.14008.

[56] Masaki Kashiwara, *Microlocal analysis*, Proceedings of the International Congress of Mathematicians (Helsinki, 1978), Acad. Sci. Fennica, Helsinki, 1980, pp. 139–150, MR562603.

[57] Ludmil Katzarkov, Maxim Kontsevich, and Tony Pantev, *Hodge theoretic aspects of mirror symmetry*, From Hodge theory to integrability and TQFT tt*-geometry, Proc. Sympos. Pure Math., vol. 78, Amer. Math. Soc., Providence, RI, 2008, 87–174, MR2483750, Zbl 1206.14009.

[58] Chiu-Chu Melissa Liu, *Localization in Gromov-Witten theory and orbifold Gromov-Witten theory*, Handbook of moduli. Vol. II, Adv. Lect. Math. (ALM), vol. 25, Int. Press, Somerville, MA, 2013, 353–425, MR3184181, Zbl 1322.14010.

[59] B. Malgrange, *Déformations de systèmes différentiels et microdifférentiels*, Mathematics and physics (Paris, 1979/1982), Progr. Math., vol. 37, Birkhäuser Boston, Boston, MA, 1983, 353–379, MR728429, Zbl 0528.32016.

[60] Etienne Mann, *Orbifold quantum cohomology of weighted projective spaces*, J. Algebraic Geom. **17** (2008), no. 1, 137–166, MR2357682, Zbl 1146.14029.

[61] Etienne Mann and Thomas Reichelt, *Logarithmic degenerations of Landau-Ginzburg models for toric orbifolds and global tt* geometry* (2016), arXiv:1605.08937 [math.AG].

[62] Dusa McDuff and Susan Tolman, *Topological properties of Hamiltonian circle actions*, IMRP Int. Math. Res. Pap. (2006), 72826, 1–77, MR2210662, Zbl 1123.53044, Zbl 1115.53062.

[63] Takuro Mochizuki, *Twistor property of GKZ-hypergeometric systems* (2015), arXiv:1501.04146.

[64] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, NY, 2010. Print companion to [**24**].

[65] Rahul Pandharipande, *Rational curves on hypersurfaces (after A. Givental)*, Astérisque (1998), no. 252, Exp. No. 848, 5, 307–340, Séminaire Bourbaki. Vol. 1997/98, MR1685628, Zbl 0932.14029.

[66] Frédéric Pham, *La descente des cols par les onglets de Lefschetz, avec vues sur Gauss-Manin*, Astérisque (1985), no. 130, 11–47, Differential systems and singularities (Luminy, 1983), MR804048, Zbl 0597.32012.

[67] Andrew Pressley and Graeme Segal, *Loop groups*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1986, Oxford Science Publications, MR900587, Zbl 0618.22011.

[68] Thomas Reichelt and Christian Sevenheck, *Logarithmic Frobenius manifolds, hypergeometric systems and quantum $\mathcal{D}$-modules*, J. Algebraic Geom. **24** (2015), no. 2, 201–281, arXiv:1010.2118, MR3311584, Zbl 1349.14139.

[69] Thomas Reichelt and Christian Sevenheck, *Non-affine Landau-Ginzburg models and intersection cohomology*, Ann. Sci. Éc. Norm. Supér. (4) **50** (2017), no. 3, 665–753. arXiv:1210.6527 [math.AG], MR3665553, Zbl 1395.14033.

[70] Claude Sabbah, *Hypergeometric period for a tame polynomial*, C. R. Acad. Sci. Paris Sér. I Math. **328** (1999), no. 7, 603–608, A longer version published in: Port. Math. (N.S.) **63** (2006), no.2, 173–226, MR1679978, Zbl 0967.32028.

[71] Kyoji Saito, *The higher residue pairings $K_F^{(k)}$ for a family of hypersurface singular points*, **40** (1983), 441–463, MR713270, Zbl 0565.32005.

[72] Kyoji Saito, *Period mapping associated to a primitive form*, Publ. Res. Inst. Math. Sci. **19** (1983), no. 3, 1231–1264, MR723468, Zbl 0539.58003.

[73] Morihiko Saito, *On the structure of Brieskorn lattice*, Ann. Inst. Fourier (Grenoble) **39** (1989), no. 1, 27–72, MR1011977, Zbl 0644.32005.

[74] Paul Seidel, π_1 *of symplectic automorphism groups and invertibles in quantum homology rings*, Geom. Funct. Anal. **7** (1997), no. 6, 1046–1095, arXiv:dg-ga/9511011, MR1487754, Zbl 0928.53042.

[75] Hsian-Hua Tseng, *Orbifold quantum Riemann-Roch, Lefschetz and Serre*, Geom. Topol. **14** (2010), no. 1, 1–81, MR2578300, Zbl 1178.14058.

[76] Oscar Zariski and Pierre Samuel, *Commutative algebra. Vol. II*, Springer-Verlag, New York-Heidelberg, 1975, Reprint of the 1960 edition, Graduate Texts in Mathematics, Vol. 29, MR0389876, Zbl 0322.13001.

Department of Mathematics
Imperial College London
180 Queen's Gate
London SW7 2AZ
United Kingdom

E-mail address: t.coates@imperial.ac.uk

Department of Mathematics
Imperial College London
180 Queen's Gate
London SW7 2AZ
United Kingdom

E-mail address: a.corti@imperial.ac.uk

Department of Mathematics
Graduate School of Science
Kyoto University
Kitashirakawa-Oiwake-cho
Sakyo-ku
Kyoto 606-8502
Japan

E-mail address: iritani@math.kyoto-u.ac.jp

Department of Mathematics
Ohio State University
100 Math Tower, 231 West 18th Ave.
Columbus
OH 43210
USA

E-mail address: hhtseng@math.ohio-state.edu