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MINIMAL HYPERSURFACES AND BOUNDARY BEHAVIOR OF COMPACT MANIFOLDS WITH NONNEGATIVE SCALAR CURVATURE

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Abstract

On a compact Riemannian manifold with boundary having positive mean curvature, a fundamental result of Shi and Tam states that, if the manifold has nonnegative scalar curvature and if the boundary is isometric to a strictly convex hypersurface in the Euclidean space, then the total mean curvature of the boundary is no greater than the total mean curvature of the corresponding Euclidean hypersurface. In 3-dimension, Shi-Tam's result is known to be equivalent to the Riemannian positive mass theorem.

In this paper, we provide a supplement to Shi-Tam's result by including the boundary effect of minimal hypersurfaces. More precisely, given a compact manifold Ω with nonnegative scalar curvature, assuming its boundary consists of two parts, Σ_H and Σ_O, where Σ_H is the union of all closed minimal hypersurfaces in Ω and Σ_O is assumed to be isometric to a suitable 2-convex hypersurface Σ in a spatial Schwarzschild manifold of mass m, we establish an inequality relating m, the area of Σ_H, and two weighted total mean curvatures of Σ_O and Σ.

In 3-dimension, our inequality has implications to isometric embedding and quasi-local mass problems. In a relativistic context, the result can be interpreted as a quasi-local mass type quantity of Σ_O being greater than or equal to the Hawking mass of Σ_H. We further analyze the limit of this quantity associated with suitably chosen isometric embeddings of large spheres in an asymptotically flat 3-manifold M into a spatial Schwarzschild manifold. We show that the limit equals the ADM mass of M. It follows that our result on the compact manifold Ω is equivalent to the Riemannian Penrose inequality.

1. Introduction and statement of results

The main goal of this paper is to prove the following theorem:

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Theorem 1.1. *Let $(\Omega^{n+1}, \breve{g})$ be a compact, connected, orientable, $(n+1)$-dimensional Riemannian manifold with nonnegative scalar curvature, with boundary $\partial\Omega$. Suppose $\partial\Omega$ is the disjoint union of two pieces, $\Sigma_{_O}$ and $\Sigma_{_H}$, where*

(i) *$\Sigma_{_O}$ has positive mean curvature H; and*
(ii) *$\Sigma_{_H}$, if nonempty, is a minimal hypersurface (with one or more components) and there are no other closed minimal hypersurfaces in $(\Omega, \breve{g})$.*

Let $\mathbb{M}^{n+1}_m$ denote an $(n+1)$-dimensional spatial Schwarzschild manifold, outside the horizon, of mass $m > 0$. Suppose $\Sigma_{_O}$ is isometric to a closed, star-shaped, 2-convex hypersurface $\Sigma^n \subset \mathbb{M}^{n+1}_m$ with $\overline{\mathrm{Ric}}(\nu,\nu) \le 0$, where $\overline{\mathrm{Ric}}$ is the Ricci curvature of $\mathbb{M}^{n+1}_m$ and ν is the outward unit normal to Σ.

If $n < 7$, then

$$m + \frac{1}{n\omega_n}\int_\Sigma N H_m \, d\sigma \ge \frac{1}{2}\left(\frac{|\Sigma_{_H}|}{\omega_n}\right)^{\frac{n-1}{n}} + \frac{1}{n\omega_n}\int_{\Sigma_{_O}} N H \, d\sigma. \tag{1.1}$$

Here H_m is the mean curvature of Σ in $\mathbb{M}^{n+1}_m$, $d\sigma$ is the area element on Σ and $\Sigma_{_O}$, ω_n is the area of the standard unit sphere $\mathbb{S}^n$, N is the static potential function on $\mathbb{M}^{n+1}_m$ given by

$$N = \frac{1 - \frac{m}{2}|x|^{1-n}}{1 + \frac{m}{2}|x|^{1-n}}$$

if one writes

$$\mathbb{M}^{n+1}_m = \left(\mathbb{R}^{n+1} \setminus \left\{ |x| < \left(\frac{m}{2}\right)^{\frac{1}{n-1}} \right\}, \left(1 + \frac{m}{2}|x|^{1-n}\right)^{\frac{4}{n-1}} g_{_E} \right),$$

where $g_{_E}$ is the Euclidean metric, N is also viewed as a function on $\Sigma_{_O}$ via the isometry between Σ and $\Sigma_{_O}$, $|\Sigma_{_H}|$ denotes the area of $\Sigma_{_H}$, and $|\Sigma_{_H}|$ is taken to be 0 if $\Sigma_{_H} = \emptyset$.

Moreover, if equality in (1.1) holds, then

$$H = H_m \quad \text{and} \quad \frac{1}{2}\left(\frac{|\Sigma_{_H}|}{\omega_n}\right)^{\frac{n-1}{n}} = m.$$

In particular, $\Sigma_{_H}$ must be nonempty in this case.

Remark 1.1. Compact manifolds $(\Omega, \breve{g})$ satisfying conditions (i) and (ii) in Theorem 1.1 exist widely. For instance, given any compact, connected, orientable Riemannian manifold $(\tilde{\Omega}, \breve{g})$ with disconnected boundary $\partial\tilde{\Omega}$, if the mean curvature vector of $\partial\tilde{\Omega}$ points inward at each boundary component, then by minimizing area among all hypersurfaces that bounds a domain with a chosen boundary component, one can always construct such an $(\Omega, \breve{g})$ (under the given dimension assumption). In a relativistic context, a compact manifold $(\Omega, \breve{g})$ satisfying conditions

(i) and (ii) represents a finite body surrounding the apparent horizon of the black hole in a time-symmetric initial data set.

REMARK 1.2. A hypersurface $\Sigma \subset \mathbb{M}^{n+1}_m$ is called 2-convex if $\sigma_1 > 0$ and $\sigma_2 > 0$, where σ_1 and σ_2 are the first and second elementary symmetric functions of the principal curvatures of Σ in $\mathbb{M}^{n+1}_m$, respectively.

REMARK 1.3. Let $\Sigma^S_H = \partial \mathbb{M}^{n+1}_m$ be the minimal hypersurface boundary of $\mathbb{M}^{n+1}_m$. Using the fact $m = \frac{1}{2}\left(\frac{|\Sigma^S_H|}{\omega_n}\right)^{\frac{n-1}{n}}$, we can write (1.1) equivalently as

(1.2)
$$\frac{1}{2}\left(\frac{|\Sigma^S_H|}{\omega_n}\right)^{\frac{n-1}{n}} + \frac{1}{n\omega_n}\int_\Sigma N H_m \, d\sigma \geq \frac{1}{2}\left(\frac{|\Sigma_H|}{\omega_n}\right)^{\frac{n-1}{n}} + \frac{1}{n\omega_n}\int_{\Sigma_O} N H \, d\sigma.$$

Such an inequality has the following variational interpretation. Let g denote the induced metric on Σ from the Schwarzschild metric $\bar{g}$ on $\mathbb{M}^{n+1}_m$. Let $\mathring{\mathcal{F}}_{(\Sigma,g)}$ be the set of *fill-ins of* (Σ, g) *with outermost horizon inner boundary*, i.e. $\mathring{\mathcal{F}}_{(\Sigma,g)}$ consists of all compact, connected, orientable manifolds $(\Omega, \breve{g})$ with nonnegative scalar curvature, with boundary satisfying (i) and (ii) such that $\Sigma_O = \Sigma$ and $\breve{g}|_{\Sigma_O} = g$, where $\breve{g}|_{\Sigma_O}$ is the induced metric on Σ_O from $\breve{g}$. Let N be the function on $\Sigma_O = \Sigma$, which is the restriction of the static potential on $\mathbb{M}^{n+1}_m$ to Σ. On $\mathring{\mathcal{F}}_{(\Sigma,g)}$, consider the functional

$$(\Omega, \breve{g}) \longmapsto \frac{1}{2}\left(\frac{|\Sigma_H|}{\omega_n}\right)^{\frac{n-1}{n}} + \frac{1}{n\omega_n}\int_{\Sigma_O} N H \, d\sigma.$$

Inequality (1.2) asserts that this functional is maximized at $(\Omega^S, \bar{g})$, where Ω^S is the domain in $\mathbb{M}^{n+1}_m$ bounded by Σ and Σ^S_H. (Such an interpretation of (1.2) in terms of fill-ins relates to the work of Mantoulidis and the second author [**30**].)

Treating the assumption that Σ_O is isometric to $\Sigma \subset \mathbb{M}^{n+1}_m$ as a condition of having an isometric embedding of Σ_O into $\mathbb{M}^{n+1}_m$, we have the following result.

Theorem 1.2. *Let $(M^3, \breve{g})$ be an asymptotically flat 3-manifold. Let S_r denote the coordinate sphere of coordinate radius r in a coordinate chart defining the asymptotic flatness of $(M^3, \breve{g})$. Let g_r be the induced metric on S_r. Then, given any constant $m > 0$, there exists an isometric embedding*

$$X_r : (S_r, g_r) \longrightarrow \mathbb{M}^3_m$$

for each sufficiently large r, such that $\Sigma_r = X_r(S_r)$ is a star-shaped, convex surface in $\mathbb{M}^3_m$, with $\overline{\mathrm{Ric}}(\nu, \nu) < 0$ where ν is the outward unit *normal to Σ_r; moreover,*

$$\lim_{r\to\infty}\left(m+\frac{1}{8\pi}\int_{S_r} N(H_m-H)\,d\sigma\right)=\mathfrak{m}, \tag{1.3}$$

and

$$V(r)-V_m(r)=2\pi r^2(\mathfrak{m}-m)+o(r^2),\ \text{as } r\to\infty. \tag{1.4}$$

Here $\mathfrak{m}$ is the ADM mass of $(M^3,\breve{g})$, H is the mean curvature of S_r in $(M^3,\breve{g})$ and H_m is the mean curvature of Σ_r in $\mathbb{M}^3_m$, N is the static potential on $\mathbb{M}^3_m$, N and H_m are viewed as functions on S_r via the embedding X_r, $V(r)$ is the volume of the region enclosed by S_r in $(M^3,\breve{g})$ and $V_m(r)$ is the volume of the region enclosed by Σ_r in $\mathbb{M}^3_m$.

Now we explain the motivations to and the implications of Theorem 1.1. Our first motivation to Theorem 1.1 is the following theorem of Shi and Tam [**41**].

Theorem 1.3 ([**41**]). *Let $(\tilde{\Omega}^{n+1},\breve{g})$ be a compact, Riemannian spin manifold with nonnegative scalar curvature, with boundary $\partial\tilde{\Omega}$. Let Σ_i, $1\le i\le k$, be the connected components of $\partial\tilde{\Omega}$. Suppose each Σ_i has positive mean curvature and each Σ_i is isomeric to a strictly convex hypersurface $\hat{\Sigma}_i\subset\mathbb{R}^{n+1}$. Then*

$$\int_{\hat{\Sigma}_i} H_0\,d\sigma \ge \int_{\Sigma_i} H\,d\sigma, \tag{1.5}$$

where H_0 is the mean curvature of $\hat{\Sigma}_i$ in $\mathbb{R}^{n+1}$ and H is the mean curvature of Σ_i in $(\tilde{\Omega},\breve{g})$. Moreover, if equality holds for some i, then $k=1$ and $(\tilde{\Omega},\breve{g})$ is isometric to a domain in $\mathbb{R}^{n+1}$.

Theorem 1.3 is a fundamental result on compact manifolds with nonnegative scalar curvature with boundary, obtained via the Riemannian positive mass theorem [**42**, **46**]. For the purpose of later explaining the proof of Theorem 1.1, we outline the proof of Theorem 1.3 from [**41**] as follows. For simplicity, we assume $k=1$ and denote Σ_1 by Σ. Identifying Σ with its isometric image in $\mathbb{R}^{n+1}$ and using the assumption that Σ is convex in $\mathbb{R}^{n+1}$, one can write the Euclidean metric $g_{_E}$ on $\mathbb{E}$, the exterior of Σ, as $g_{_E}=dt^2+g_t$, where g_t is the induced metric on the hypersurface Σ_t that has a fixed Euclidean distance t to Σ. Given the mean curvature function $H>0$ on Σ, one shows that there exists a function $u>0$ on $\mathbb{E}$ such that $g_u=u^2dt^2+g_t$ has zero scalar curvature, $(\mathbb{E},g_u)$ is asymptotically flat, and the mean curvature H_u of Σ_t in $(\mathbb{E},g_u)$ satisfies $H_u=H$ at $\Sigma_0=\Sigma$. A key feature of such an $(\mathbb{E},g_u)$ is that the integral

$$\frac{1}{n\omega_n}\int_{\Sigma_t}(H_0-H_u)\,d\sigma \tag{1.6}$$

is monotone nonincreasing and it converges to $\mathfrak{m}(g_u)$, where $\mathfrak{m}(g_u)$ is the ADM mass [1] of $(\mathbb{E}, g_u)$. By gluing $(\tilde{\Omega}, \breve{g})$ and $(\mathbb{E}, g_u)$ along their common boundary Σ and applying the Riemannian positive mass theorem, which is still valid under the condition that the mean curvatures of Σ in $(\tilde{\Omega}, \breve{g})$ and $(\mathbb{E}, g_u)$ agree (see [41, 33]), one concludes that

$$\frac{1}{n\omega_n}\int_\Sigma (H_0 - H)\,d\sigma \geq \lim_{t\to\infty}\frac{1}{n\omega_n}\int_{\Sigma_t}(H_0 - H)\,d\sigma = \mathfrak{m}(g_u) \geq 0, \tag{1.7}$$

which proves (1.5).

One of the most important features of Theorem 1.3 is that, when $n = 2$, by the solution to the Weyl embedding problem ([38, 39]), Theorem 1.3 implies the positivity of the Brown-York quasi-local mass ([9, 10]) of $\partial\tilde{\Omega}$, under the assumption that $\partial\tilde{\Omega}$ is a topological 2-sphere with positive Gauss curvature.

Remark 1.4. When $n > 2$, Eichmair, Wang and the second author [17] proved that Theorem 1.3 remains valid if each component Σ_i is isometric to a star-shaped hypersurface with positive scalar curvature in $\mathbb{R}^{n+1}$. It was also noted in [17] that the spin assumption therein can be dropped when $n < 7$. Recently, Schoen and Yau [43] proved that the Riemannian positive mass theorem holds in all dimensions without a spin assumption. Therefore, by the argument in [17], results in [41, 17] also hold in all dimensions without a spin assumption.

To motivate Theorem 1.1 from Theorem 1.3, one may consider the setting $k > 1$ of Theorem 1.3. In this case, given any boundary component Σ_i, there exists a minimal hypersurface S_i, possibly disconnected, in the interior of $(\tilde{\Omega}, \breve{g})$ such that S_i and Σ_i bounds a domain Ω satisfying conditions (i) and (ii) in Theorem 1.1. Thus, besides the nonnegative scalar curvature, one wants to understand the influence of S_i on Σ_i. This is indeed related to the following Riemannian Penrose inequality, which is our second motivation to Theorem 1.1.

Theorem 1.4 ([26, 4, 6])**.** *Let M^{n+1} be an asymptotically flat manifold with nonnegative scalar curvature, with boundary ∂M, where $n < 7$. Suppose ∂M is an outer minimizing, minimal hypersurface (with one or more component), then*

$$\mathfrak{m}(M) \geq \frac{1}{2}\left(\frac{|\partial M|}{\omega_n}\right)^{\frac{n-1}{n}}, \tag{1.8}$$

where $\mathfrak{m}(M)$ is the ADM mass of M and $|\partial M|$ is the area of ∂M. Moreover, equality holds if and only if M is isometric to a spatial Schwarzschild manifold outside its horizon.

When $n = 2$, Theorem 1.4 was first proved by Huisken and Ilmanen [25, 26] for the case that ∂M is connected, and later proved by Bray [4] for the general case in which ∂M can have multiple components. For higher dimensions, Bray and Lee [6] proved inequality (1.8) for $n < 7$ and established the rigidity case assuming that M is spin. (Without the spin assumption, the rigidity case follows by combining results of Bray and Lee [6] and McFeron and Székelyhidi [31].)

To compare Theorem 1.1 and Theorem 1.4, we can write (1.1) equivalently as

$$m + \frac{1}{n\omega_n}\int_{\Sigma_O} N(H_m - H)\, d\sigma \geq \frac{1}{2}\left(\frac{|\Sigma_H|}{\omega_n}\right)^{\frac{n-1}{n}} \tag{1.9}$$

by identifying Σ_O and Σ. The quantity on the left side of (1.9) depends only on the assumption on the (outer) boundary component Σ_O of Ω, while the mass $\mathfrak{m}(M)$ in (1.8) is determined solely by the asymptotically flat end of M. In this sense, Theorem 1.1 can be viewed as a localization of Theorem 1.4 to a compact manifold with boundary satisfying conditions (i) and (ii). Indeed, by (1.3) in Theorem 1.2 and the fact that our proof of Theorem 1.1 uses (1.8), Theorem 1.1 is equivalent to the Riemannian Penrose inequality (1.8) when $n = 2$. In this case, the right side of (1.9) is the Hawking quasi-local mass [24] of Σ_H, and (1.9) describes how Σ_H, which models the apparent horizon of black hole, contributes to the quasi-local mass of a body surrounding it.

Remark 1.5. In [14], Chen, Wang, Wang and Yau introduced a notion of quasi-local energy in reference to a general static spacetime. Setting $\tau = 0$ in equation (2.10) in [14], one sees that the quasi-local energy of a 2-surface Σ defined in [14] with respect to an isometric embedding of Σ into a time-symmetric slice of Schwarzschild the Schwarzschild spacetime with mass m is given by $\frac{1}{8\pi}\int_\Sigma N(H_m - H)\, d\sigma$, which agrees with the surface integral on the left side of (1.9) with $\Sigma = \Sigma_O$.

To illustrate that Theorem 1.1 provides a supplement to Shi-Tam's result, we want to make a connection between (1.9) and an inequality that can be obtained by directly combining (1.8) and Shi-Tam's proof of Theorem 1.3. Only for the convenience of making a comparison, we list the following inequality in a theorem format:

Theorem 1.3'. *Let $(\Omega^{n+1}, \breve{g})$ be a compact Riemannian manifold with nonnegative scalar curvature, with boundary $\partial\Omega$, satisfying conditions (i) and (ii) in Theorem 1.1. Suppose $\Sigma_H \neq \emptyset$ and Σ_O is isometric to a strictly convex hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$. If $n < 7$, then*

$$\frac{1}{n\omega_n}\int_{\Sigma_O} (H_0 - H)\, d\sigma > \frac{1}{2}\left(\frac{|\Sigma_H|}{\omega_n}\right)^{\frac{n-1}{n}}, \tag{1.10}$$

where H_0 is the mean curvature of Σ in $\mathbb{R}^{n+1}$.

The proof of (1.10) is identical to Shi-Tam's proof of Theorem 1.3 outlined earlier, except that in the final inequality of (1.7), one replaces the Riemannian positive mass theorem by the Riemannian Penrose inequality to yield

$$\begin{aligned}\frac{1}{n\omega_n}\int_{\Sigma_O}(H_0-H)\,d\sigma &\geq \lim_{t\to\infty}\frac{1}{n\omega_n}\int_{\Sigma_t}(H_0-H)\,d\sigma \\ &= \mathfrak{m}(g_u) \geq \frac{1}{2}\left(\frac{|\Sigma_H|}{\omega_n}\right)^{\frac{n-1}{n}}.\end{aligned} \tag{1.11}$$

The fact that (1.8) is applicable to the manifold obtained by gluing $(\Omega,\breve{g})$ and $(\mathbb{E},g_u)$ was demonstrated in [**35**] for $n=2$ and in [**32**] for $n<7$.

REMARK 1.6. By the argument in [**17**], (1.10) holds with the assumption that $\Sigma\subset\mathbb{R}^{n+1}$ is strictly convex replaced by that Σ is star-shaped with positive scalar curvature. Such a statement is precisely the $m=0$ analogue of Theorem 1.1 for the case $\Sigma_H\neq\emptyset$.

Inequality (1.10) takes a simpler form than (1.9), however it is always a strict inequality. This is because, if the first inequality in (1.11) were equality, the function u would be identically 1 (implied by the monotonicity calculation of (1.6) in [**41**, **17**]), consequently $H_0=H$ identically, which would show $0\geq|\Sigma_H|$, contradicting the assumption $\Sigma_H\neq\emptyset$. A more intuitive reason for (1.10) to be strict is that, though Σ_H is a nonempty minimal hypersurface in Ω^{n+1}, (1.10) is obtained by comparing Σ_O to a hypersurface in $\mathbb{R}^{n+1}$ which is free of closed minimal hypersurfaces.

For the above reason, we consider an assumption Σ_O is isometric to an $\Sigma\subset\mathbb{M}^{n+1}_m$ in Theorem 1.1. In particular, (1.9) does become an equality when Ω itself is the domain in $\mathbb{M}^{n+1}_m$ bounded by Σ and the Schwarzschild horizon Σ^S_H.

The fact that (1.9) gives a refined estimate on $|\Sigma_H|$, sharper than (1.10), can be illustrated by the case in which Σ_O is isometric to a round sphere. In the following example, for simplicity, we take $n=2$.

EXAMPLE 1. Suppose Ω is a compact 3-manifold with nonnegative scalar curvature, with boundary $\partial\Omega$, satisfying conditions (i) and (ii) in Theorem 1.1. Suppose $\Sigma_H\neq\emptyset$ and Σ_O is isometric to a round sphere with area $4\pi R^2$. Then (1.10) shows

$$R-\frac{1}{8\pi}\int_{\Sigma_O}H\,d\sigma>\sqrt{\frac{|\Sigma_H|}{16\pi}}. \tag{1.12}$$

On the other hand, Theorem 1.1 applies to any $\mathbb{M}^3_m$ with $m\in\left(0,\frac{1}{2}R\right)$ since Σ_O is isometric to a rotationally symmetric sphere in such an $\mathbb{M}^3_m$. Thus, by (1.9),

$$m+\frac{1}{8\pi}\int_{\Sigma_O}N\left(N\frac{2}{R}-H\right)\,d\sigma\geq\sqrt{\frac{|\Sigma_H|}{16\pi}} \tag{1.13}$$

with $N = \sqrt{1 - \frac{2m}{R}}$. Let $\Phi(m)$ denote the quantity on the left side of (1.13). (The left side of (1.12) equals $\lim_{m\to 0+} \Phi(m)$.) By (1.13),

$$\min_{0<m<\frac{R}{2}} \Phi(m) \geq \sqrt{\frac{|\Sigma_H|}{16\pi}}. \tag{1.14}$$

Note that either (1.12) or (1.13) implies $0 < \frac{1}{8\pi R}\int_{\Sigma_O} H\, d\sigma < 1$. Therefore, via direct calculation, one has

$$\begin{aligned} R - \frac{1}{8\pi}\int_{\Sigma_O} H\, d\sigma > \min_{0<m<\frac{R}{2}} \Phi(m) &= \frac{R}{2}\left[1 - \left(\frac{1}{8\pi R}\int_{\Sigma_O} H\, d\sigma\right)^2\right] \\ &\geq \sqrt{\frac{|\Sigma_H|}{16\pi}}. \end{aligned} \tag{1.15}$$

(It is clear that, if Ω is the region bounded by a rotationally symmetric sphere and the horizon boundary in some $\mathbb{M}^3_m$, then $\min_{0<m<\frac{R}{2}} \Phi(m) = \sqrt{\frac{|\Sigma_H|}{16\pi}}$.) In (1.15), it is also intriguing to note that $\min_{0<m<\frac{R}{2}} \Phi(m)$ is achieved at $m = m_*$ where m_*, determined by $N = \frac{1}{8\pi R}\int_{\Sigma_O} H\, d\sigma$, agrees with $\min_{0<m<\frac{R}{2}} \Phi(m)$, i.e.

$$m_* = \min_{0<m<\frac{R}{2}} \Phi(m). \tag{1.16}$$

This means that an optimal background $\mathbb{M}^3_{m_*}$ that is used to be compared with Ω is indeed determined by the minimal value of $\Phi(m)$.

Remark 1.7. Calculation in relation to the example above was first carried out in [**35**] where the special case of Theorem 1.1 in which Σ_O is isometric to a round sphere was proved. The implication of (1.16) on the quasi-local mass of such round surfaces was also discussed in [**35**].

Next, we comment on the implication of Theorem 1.1 on isometric embeddings of a 2-sphere into a Schwarzschild manifold $\mathbb{M}^3_m$ with $m > 0$. It was proved by Li and Wang [**28**] that, if σ is a metric on the 2-sphere $\mathbb{S}^2$, an isometric embedding of $(\mathbb{S}^2, \sigma)$ into $\mathbb{M}^3$ may not be unique. Indeed, it was shown in [**28**] that, if σ_r is the standard round metric of area $4\pi r^2$ with $r > 2m$, then $(\mathbb{S}^2, \sigma_r)$ admits an isometric embedding into $\mathbb{M}^3_m$ that is close to but different from the standard embedding whose image is a rotationally symmetric sphere. For this reason, one knows that inequality (1.1) does depend on the choice of the isometry between Σ_O and Σ. (This contrasts with inequality (1.5) which only depends on the intrinsic metric on Σ_i.) However, in the following example, we demonstrate that (1.1) can be applied to reveal information on such different isometric embeddings into $\mathbb{M}^3_m$.

EXAMPLE 2. Let $\Sigma \subset \mathbb{M}^3_m$ be a closed, star-shaped, convex surface with $\overline{\mathrm{Ric}}(\nu,\nu) \le 0$. Let H_m denote its mean curvature. Suppose $\iota : \Sigma \to \tilde{\Sigma}$ is an isometry between Σ and another surface $\tilde{\Sigma} \subset \mathbb{M}^3_m$ with properties

(a) $\tilde{\Sigma}$ bounds a domain D with the Schwarzschild horizon $\Sigma^S_H = \partial \mathbb{M}^3_m$, and
(b) $\tilde{\Sigma}$ has positive mean curvature $\tilde{H}_m$ with respect to the outward unit normal.

Then Theorem 1.1 is applicable to the domain D to give

$$m + \frac{1}{8\pi}\int_\Sigma N H_m \, d\sigma \ge \sqrt{\frac{|\Sigma^S_H|}{16\pi}} + \frac{1}{8\pi}\int_{\tilde{\Sigma}} \tilde{N}\tilde{H}_m \, d\sigma \tag{1.17}$$

with $\tilde{N} = N \circ \iota^{-1}$. (Note that, if (a) is replaced by an assumption $\tilde{\Sigma} = \partial D$ for some D, then the term involving $|\Sigma^S_H|$ will be absent in (1.17) and the inequality is strict.) Since $m = \sqrt{\frac{|\Sigma^S_H|}{16\pi}}$, (1.17) shows

$$\int_\Sigma N H_m \, d\sigma \ge \int_{\tilde{\Sigma}} \tilde{N}\tilde{H}_m \, d\sigma \tag{1.18}$$

with equality holds only if $H_m \circ \iota^{-1} = \tilde{H}_m$. Now suppose we consider the special case in which Σ is a rotationally symmetric sphere, then N is a constant on Σ, hence $\tilde{N}$ is also a constant that equals N. In this case, (1.18) becomes

$$\int_\Sigma H_m \, d\sigma \ge \int_{\tilde{\Sigma}} \tilde{H}_m \, d\sigma. \tag{1.19}$$

(In the case of $\tilde{\Sigma} = \partial D$, one has $8\pi m N^{-1} + \int_\Sigma H_m \, d\sigma > \int_{\tilde{\Sigma}} \tilde{H}_m \, d\sigma$.) Since H_m is a constant, equality in (1.19) holds only if $\tilde{H}_m$ is a constant. By the result of Brendle [**7**], we conclude that $\tilde{\Sigma}$ must be Σ when equality holds in (1.19).

We now outline the proof of Theorem 1.1. The first step in our proof is to generalize the monotonicity of the Brown-York mass type integral (1.6) in Shi-Tam's proof of Theorem 1.3 to the monotonicity of a weighted Brown-York mass type integral

$$\int_{\Sigma_t} N(\bar{H} - H_\eta) \, d\sigma \tag{1.20}$$

in a general static background on which N is a positive static potential function. The idea of imposing a suitable weight function in (1.20) to obtain monotonicity goes back to the work of Wang and Yau [**45**] in which isometric embeddings of surfaces into hyperbolic spaces are considered. Given a static Riemannian manifold $(\mathbb{N}, \bar{g})$ (see Definition 2.1), let $\{\Sigma_t\}$ be a family of closed hypersurfaces evolving in $(\mathbb{N}, \bar{g})$ with speed $f > 0$, we show that, as long as Σ_t is 2-convex and $\frac{\partial N}{\partial \nu} > 0$, (1.20) is monotone nonincreasing along the flow. Here 2-convexity of Σ_t means that $\sigma_1 > 0$ and $\sigma_2 > 0$, where σ_1 and σ_2 are the first and second elementary symmetric functions of the principal curvatures of Σ_t in $(\mathbb{N}, \bar{g})$; ν denotes the unit normal giving the direction of the flow; and $\bar{H}$, H_η denote the mean curvature of Σ_t with respect to $\bar{g} = f^2 dt^2 + g_t$, $g_\eta = \eta^2 dt^2 + g_t$, respectively, where g_η is taken to have the same scalar curvature as $\bar{g}$. (The idea of considering such a metric g_η goes back to Bartnik [**3**].) To apply this monotonicity formula, in the next step we study a family of closed, star-shaped, hypersurfaces $\{\Sigma_t\}$ in a spatial Schwarzschild manifold $\mathbb{M}^{n+1}_m$, given by $\Sigma_t = X(t, \mathbb{S}^n)$, where $X : [0, \infty) \times \mathbb{S}^n \to \mathbb{M}^{n+1}_m$ is a smooth map evolving according to

$$\frac{\partial X}{\partial t} = \frac{n-1}{2n} \frac{\sigma_1}{\sigma_2} \nu. \tag{1.21}$$

We show that, if the initial hypersurface Σ_0 is 2-convex with $\overline{\mathrm{Ric}}(\nu, \nu) \leq 0$, then (1.21) admits a long time solution $\{\Sigma_t\}_{0 \leq t < \infty}$ so that each Σ_t is 2-convex and has positive scalar curvature. Writing the Schwarzschild background metric $\bar{g}$ on the exterior region $\mathbb{E}$ of Σ_0 as $\bar{g} = f^2 dt^2 + g_t$, we then demonstrate that there exists a positive function η on $\mathbb{E}$ such that $g_\eta = \eta^2 dt^2 + g_t$ has zero scalar curvature, the mean curvature of Σ_0 in $(\mathbb{E}, g_\eta)$ equals H which is the mean curvature of $\Sigma_{_O}$ in $(\Omega, \breve{g})$; and $(\mathbb{E}, g_\eta)$ is asymptotically flat with mass

$$\mathfrak{m}(g_\eta) = m + \lim_{t \to \infty} \frac{1}{n\omega_n} \int_{\Sigma_t} N(\bar{H} - H_\eta)\, d\sigma. \tag{1.22}$$

Finally, by gluing $(\Omega, \breve{g})$ and $(\mathbb{E}, g_\eta)$ along $\Sigma_{_O}$ (which is identified with $\Sigma = \Sigma_0$) to get an asymptotically flat manifold $(\hat{M}, \hat{h})$, we conclude

(1.23)
$$\begin{aligned} m + \frac{1}{n\omega_n} \int_{\Sigma_{_O}} N(H_m - H)\, d\sigma &\geq m + \lim_{t \to \infty} \frac{1}{n\omega_n} \int_{\Sigma_t} N(\bar{H} - H_\eta)\, d\sigma \\ &= \mathfrak{m}(g_\eta) \geq \frac{1}{2} \left(\frac{|\Sigma_{_H}|}{\omega_n} \right)^{\frac{n-1}{n}}, \end{aligned}$$

where in the last step we used the fact that the Riemannian Penrose inequality holds on such an $(\hat{M}, \hat{h})$ (see [**35**, **32**]).

It is worth of mentioning that, similar to the fact that Shi-Tam's proof of Theorem 1.3 gives an upper bound of the Bartnik mass $\mathfrak{m}_{_B}(\Sigma)$ [**2**] for a 2-surface Σ that is isometric to a convex surface in $\mathbb{R}^3$ in terms of its Brown-York mass, our proof of Theorem 1.1 yields

$$\mathfrak{m}_{_B}(\Sigma) \leq m + \frac{1}{8\pi} \int_\Sigma N(H_m - H)\, d\sigma \tag{1.24}$$

for a surface Σ that is isometric to a convex surface with $\overline{\mathrm{Ric}}(\nu, \nu) \leq 0$ in an $\mathbb{M}^3_m$ (see Theorem 5.1). Such an estimate on the Bartnik mass verifies a special case of Conjecture 4.1 in [**34**], which is formulated for a surface that admits an isometric embedding into a general static manifold.

This paper is organized as follows. In Section 2, we derive the monotonicity formula of the weighted Brown-York mass type integral (1.20) in a general static background. In Section 3, we study a family of inverse curvature flows in a spatial Schwarzschild manifold $\mathbb{M}^{n+1}_m$, which includes (1.21) as a special case. In Section 4, we prove that a warped metric of the form $g_\eta = \eta^2 dt^2 + g_t$, with zero scalar curvature, exists on the Schwarzschild exterior region $\mathbb{E}$ swept out by the solution $\{\Sigma_t\}_{0\leq t\leq\infty}$ to (1.21), and show that g_η is asymptotically flat and its mass is given by (1.22). In Section 5, we attach $(\mathbb{E}, g_\eta)$ to $(\Omega, \breve{g})$ along Σ_O and apply the Riemannian Penrose inequality to prove Theorem 1.1. We also discuss the implication of our work to the Bartnik mass. We end the paper by proving Theorem 1.2 in Section 6.

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2. Monotonicity formula in a static background

The Euclidean space $\mathbb{R}^{n+1}$ and the spatial Schwarzschild manifolds $\mathbb{M}^{n+1}_m$ both are examples of a static Riemannian manifold according to the following definition.

Definition 2.1 ([**15**])**.** A Riemannian manifold $(\mathbb{N}, \bar{g})$ is called static if there exists a nontrivial function N such that

$$(\bar{\Delta} N)\bar{g} - \bar{D}^2 N + N\bar{Ric} = 0, \tag{2.1}$$

where $\bar{Ric}$ is the Ricci curvature of $(\mathbb{N}, \bar{g})$, $\bar{D}^2 N$ is the Hessian of N and $\bar{\Delta}$ is the Laplacian of N. The function N is called a static potential.

Throughout this section, we let $(\mathbb{N}, \bar{g})$ denote a static Riemannian manifold with a static potential N. The scalar curvature $\bar{R}$ of such an $(\mathbb{N}, \bar{g})$ is necessarily a constant (see [**15**, Proposition 2.3]). Consider a smooth family of embedded hypersurfaces $\{\Sigma_t\}$ evolving in $(\mathbb{N}, \bar{g})$ according to

$$\frac{\partial X}{\partial t} = f\nu, \tag{2.2}$$

where X denotes points in Σ_t, $f > 0$ denotes the speed of the flow, and ν is a unit normal to Σ_t. Let σ_1 and σ_2 be the first and second elementary symmetric functions of the principal curvatures of Σ_t in $(\mathbb{N}, \bar{g})$, respectively. In particular, σ_1 equals the mean curvature of Σ_t.

The metric $\bar{g}$ over the region U swept by $\{\Sigma_t\}$ can be written as

$$\bar{g} = f^2 dt^2 + g_t, \tag{2.3}$$

where g_t is the induced metric of Σ_t. Now consider another metric

$$g_\eta = \eta^2 dt^2 + g_t, \tag{2.4}$$

where $\eta > 0$ is a function on U. We impose the condition that the scalar curvature $R(g_\eta)$ of g_η equals the scalar curvature of $\bar{g}$, i.e.

$$R(g_\eta) = \bar{R}. \tag{2.5}$$

Proposition 2.2. *Under the above notations and assumptions,*

$$\begin{aligned}&\frac{d}{dt}\left(\int_{\Sigma_t} N(\bar{H} - H_\eta)d\sigma\right)\\ =&-\int_{\Sigma_t}\eta^{-1}(\eta-f)^2\bar{H}\frac{\partial N}{\partial\nu}d\sigma - \int_{\Sigma_t} N\sigma_2\eta^{-1}(\eta-f)^2 d\sigma,\end{aligned}$$

where $\bar{H}$ and H_η are the mean curvature of Σ_t with respect to $\bar{g}$ and g_η, respectively.

Proof. Denote $\bar{A}$ and A_η the second fundamental form of Σ_t with respect to $\bar{g}$ and g_η, respectively. By (2.3) and (2.4),

$$H_\eta = \eta^{-1} f \bar{H}, \quad A_\eta = \eta^{-1} f \bar{A}. \tag{2.6}$$

By the second variation formula,

$$\frac{\partial}{\partial t}\bar{H} = -\Delta f - f(|\bar{A}|^2 + \bar{Ric}(\nu,\nu)) \tag{2.7}$$

and

$$\frac{\partial}{\partial t}H_\eta = -\Delta\eta - \eta(|A_\eta|^2 + Ric_{g_\eta}(\nu,\nu)), \tag{2.8}$$

where Δ is the Laplacian operator on (Σ_t, g_t) and Ric_{g_η} is the Ricci curvature of g_η.

Let R denote the scalar curvature of (Σ_t, g_t). Let $\sigma_{2\eta}$ be the second elementary symmetric functions of the principal curvatures of Σ_t in $(\mathbb{N}, g_\eta)$. By Gauss equation,

$$\sigma_2 = \frac{R - \bar{R}}{2} + \bar{Ric}(\nu,\nu), \quad \sigma_{2\eta} = \frac{R-\bar{R}}{2} + Ric_{g_\eta}(\nu,\nu). \tag{2.9}$$

Together with (2.6), we have

$$\begin{aligned}Ric_{g_\eta}(\nu,\nu) =& \bar{Ric}(\nu,\nu) + \sigma_{2\eta} - \sigma_2\\ =& \bar{Ric}(\nu,\nu) + \sigma_2(\eta^{-2}f^2 - 1).\end{aligned} \tag{2.10}$$

Putting (2.7), (2.8) and (2.10) together, we have

$$\begin{aligned}&\frac{\partial}{\partial t}(\bar{H} - H_\eta)\\ =&\Delta(\eta - f) - f(|\bar{A}|^2 + \bar{Ric}(\nu,\nu)) + \eta(|A_\eta|^2 + Ric_{g_\eta}(\nu,\nu))\\ =&\Delta(\eta-f) + \bar{Ric}(\nu,\nu)(\eta - f) + |\bar{A}|^2(\eta^{-1}f^2 - f) + \sigma_2(\eta^{-1}f^2 - \eta).\end{aligned}$$

Using the formula $\frac{\partial}{\partial t}d\sigma = fHd\sigma$, (2.6) and integrating by part, we thus have

$$\begin{aligned}
&\frac{d}{dt}\left(\int_{\Sigma_t} N(\bar{H} - H_\eta)d\sigma\right)\\
&= \int_{\Sigma_t} f\frac{\partial N}{\partial \nu}\bar{H}(1-\eta^{-1}f)d\sigma + \int_{\Sigma_t} N\bar{H}(1-\eta^{-1}f)f\bar{H}d\sigma\\
&\quad + \int_{\Sigma_t} \big(\Delta N(\eta - f) + N\bar{Ric}(\nu,\nu)(\eta-f)\big)\,d\sigma\\
&\quad + \int_{\Sigma_t} \big(N|\bar{A}|^2(\eta^{-1}f^2 - f) + N\sigma_2(\eta^{-1}f^2-\eta)\big)\,d\sigma\\
&= \int_{\Sigma_t} (\eta - f)\left(\Delta N + N\bar{Ric}(\nu,\nu) + \eta^{-1}f\bar{H}\frac{\partial N}{\partial\nu}\right)d\sigma\\
&\quad + \int_{\Sigma_t} N\sigma_2\left(2(f-\eta^{-1}f^2) + \eta^{-1}f^2 - \eta\right)d\sigma\\
&= \int_{\Sigma_t} (\eta - f)\left(\Delta N + N\bar{Ric}(\nu,\nu) + \eta^{-1}f\bar{H}\frac{\partial N}{\partial\nu}\right)d\sigma\\
&\quad - \int_{\Sigma_t} N\sigma_2\eta^{-1}(\eta-f)^2d\sigma.
\end{aligned}$$

The static equation (2.1) implies

$$\begin{aligned}
&\Delta N + N\bar{Ric}(\nu,\nu)\\
&= \bar{\Delta}N - \bar{D}^2N(\nu,\nu) - \bar{H}\frac{\partial N}{\partial\nu} + N\bar{Ric}(\nu,\nu) = -\bar{H}\frac{\partial N}{\partial \nu}.
\end{aligned}$$

Therefore, we conclude

$$\begin{aligned}
&\frac{d}{dt}\left(\int_{\Sigma_t} N(\bar{H} - H_\eta)d\sigma\right)\\
&= \int_{\Sigma_t} (\eta-f)(-1+\eta^{-1}f)\bar{H}\frac{\partial N}{\partial\nu}d\sigma - \int_{\Sigma_t} N\sigma_2\eta^{-1}(\eta-f)^2d\sigma\\
&= -\int_{\Sigma_t} \eta^{-1}(\eta-f)^2\bar{H}\frac{\partial N}{\partial\nu}d\sigma - \int_{\Sigma_t} N\sigma_2\eta^{-1}(\eta-f)^2d\sigma.
\end{aligned}$$

q.e.d.

Corollary 2.3. *Suppose* $(\mathbb{N}, \bar{g})$ *has a positive static potential* N*. Along* $\{\Sigma_t\}$*, suppose*

$$\frac{\partial N}{\partial \nu} > 0 \text{ and } \sigma_i > 0,\ i = 1, 2. \tag{2.11}$$

Then $\int_{\Sigma_t} N(\bar{H} - H_\eta)d\sigma$ *is monotone nonincreasing and it is a constant if and only if* $\eta = f$.

3. Inverse curvature flows in Schwarzschild manifolds

Corollary 2.3 suggests one consider foliations $\{\Sigma_t\}$ satisfying condition (2.11) in a static manifold with a positive static potential. In this section, we use an inverse curvature flow to construct such foliations in the Schwarzschild manifold $\mathbb{M}^{n+1}_m$.

We begin by fixing some notations. Henceforth, we will always use $\bar{g}$ to denote the metric on $\mathbb{M}^{n+1}_m$. We write

$$(\mathbb{M}^{n+1}_m, \bar{g}) = ([0,\infty) \times \mathbb{S}^n, dr^2 + \phi^2(r)\sigma), \tag{3.1}$$

where σ is the standard metric on the unit n-sphere $\mathbb{S}^n$ and $\phi = \phi(r) > 0$ satisfies $\phi(0) = (2m)^{\frac{1}{n-1}}$ and

$$\phi' = \sqrt{1 - 2m\phi^{1-n}}. \tag{3.2}$$

In terms of this coordinate r, the static potential function N in Theorem 1.1 equals ϕ'. We use $\bar{R}(\cdot,\cdot,\cdot,\cdot)$, $\overline{\mathrm{Ric}}(\cdot,\cdot)$ to denote the curvature tensor, the Ricci curvature of $\bar{g}$, respectively. The scalar curvature $\bar{R}$ of $\bar{g}$ is identically zero.

Given any integer $1 \le k \le n$, the Garding's cone $\Gamma_k \subset \mathbb{R}^n$ is defined by

$$\Gamma_k = \{(\kappa_1, \ldots, \kappa_n) \in \mathbb{R}^n \mid \sigma_j > 0,\ 1 \le j \le k\},$$

where σ_j is the j-th elementary symmetric function of $(\kappa_1, \ldots, \kappa_n)$. We also define $\sigma_0 = 1$. A hypersurface $\Sigma \subset \mathbb{M}^{n+1}_m$ is called k-convex if its principal curvature $(\kappa_1, \ldots, \kappa_n) \in \Gamma_k$.

Theorem 3.1. *Let Σ^n_0 be a star-shaped, k-convex, closed hypersurface in $\mathbb{M}^{n+1}_m$. Consider a smooth family of hypersurfaces $\{\Sigma_t\}_{t\ge 0}$ evolving according to*

$$\frac{\partial X}{\partial t} = \frac{\nu}{F}, \tag{3.3}$$

where ν is the outward unit normal and $F = n\frac{C_n^{k-1}}{C_n^k}\frac{\sigma_k}{\sigma_{k-1}} > 0$ which is evaluated at the principal curvatures of Σ_t. Then (3.3) has a smooth solution that exists for all time, each Σ_t remains star-shaped, and the second fundamental form h of Σ_t satisfies

$$|h^i_j \phi - \delta^i_j| \le Ce^{-\alpha t},$$

where ϕ is evaluated at Σ_t and C, α depends only on Σ_0, n, k.

We remark that inverse curvature flows in Euclidean spaces were first studied by Gerhardt [**19**] and Urbas [**44**]. They considered the flow equation (3.3) where F is a concave function of homogeneous degree one, evaluated at the principal curvature, and proved that the solution exists for all time and the normalized flow converges to a round sphere if the initial hypersurface is suitably star-shaped. For flows in other space forms, Gerhardt [**20**, **21**] proved the solution exists for all time and the second fundamental form converges (see also earlier work by Ding [16]). Recently, Brendle-Hung-Wang [8] and Scheuer [40] proved that the same results hold in anti-de Sitter-Schwarzschild manifold and a class of warped product manifolds for the inverse mean curvature flow, i.e. $F = \sigma_1$. However, as pointed out by Neves [37] and Hung-Wang [27], for the inverse mean curvature flow, the rescaled limiting hypersurface is not necessarily a round sphere in an anti-de Sitter-Schwarzschild manifold. The case of $F = n\frac{C_n^{k-1}}{C_n^k}\frac{\sigma_k}{\sigma_{k-1}}$ in anti-de Sitter-Schwarzschild manifolds was analyzed by Lu [29] and Chen-Mao [13] independently. They proved that the flow exists for all time and the second fundamental converges exponentially fast if the initial hypersurface is star-shaped and k-convex.

In what follows, we prove Theorem 3.1 following the steps in [29]. We divide the proof into a few subsections.

3.1. Basic formulae. We first collect some well-known formulae in Schwarzschild manifold. Given a hypersurface $\Sigma^n \subset \mathbb{M}_m^{n+1}$, we always use g to denote the induced metric on Σ. Define

$$\Phi(r) = \int_0^r \phi(\rho)d\rho, \quad u = \left\langle \phi\frac{\partial}{\partial r}, \nu \right\rangle,$$

where ν is the outer unit normal of Σ and $\langle\cdot,\cdot\rangle$ also denotes the metric product on $\mathbb{M}_m^{n+1}$. Let $i, j.. \in \{1,\ldots,n\}$ denote indices of local coordinates on Σ. Let h be the second fundamental form on Σ.

The following formula is well-known (see [22] for instance),

$$\Phi_{;ij} = \phi' g_{ij} - h_{ij}u, \tag{3.4}$$

where ";" denotes the covariant differentiation on Σ.

Let $R(\cdot,\cdot,\cdot,\cdot)$ be the curvature tensor of g on Σ. The Gauss equation and Codazzi equation are

$$R_{ijkl} = \bar{R}_{ijkl} + (h_{ik}h_{jl} - h_{il}h_{jk}) \tag{3.5}$$

$$\nabla_k h_{ij} - \nabla_j h_{ik} = \bar{R}_{\nu ijk}, \tag{3.6}$$

and the interchanging formula is

$$\begin{aligned}\nabla_i\nabla_j h_{kl} =& \nabla_k\nabla_l h_{ij} - h_l^p(h_{ip}h_{kj} - h_{ij}h_{pk}) - h_j^p(h_{pi}h_{kl} - h_{il}h_{pk}) \\ &+ h_l^p\bar{R}_{ikjp} + h_j^p\bar{R}_{iklp} + \nabla_k\bar{R}_{ijl\nu} + \nabla_i\bar{R}_{jkl\nu}.\end{aligned} \tag{3.7}$$

Here ∇ is another notation for the covariant differentiation on Σ.

The function u is known as the support function of Σ. We have (see in [29])

Lemma 3.2.

$$\nabla_i u = g^{kl}h_{ik}\nabla_l\Phi,$$
$$\nabla_i\nabla_j u = g^{kl}\nabla_k h_{ij}\nabla_l\Phi + \phi' h_{ij} - (h^2)_{ij}u + g^{kl}\nabla_l\Phi\bar{R}_{\nu jki},$$

where $(h^2)_{ij} = g^{kl}h_{ik}h_{jl}$, $\bar{R}_{\nu jki}$ *is the curvature of ambient space.*

As for the curvature, we have the following curvature estimates, for proof, we refer readers to [**8**].

Lemma 3.3. *The sectional curvature satisfies*

$$\bar{R}(\partial_i,\partial_j,\partial_k,\partial_l) = \phi^2\left(1-\phi'^2\right)(\sigma_{ik}\sigma_{jl}-\sigma_{il}\sigma_{jk})$$
$$\bar{R}(\partial_i,\partial_r,\partial_j,\partial_r) = -\phi\phi''\sigma_{ij}.$$

Together with (3.2), *this gives*

$$\bar{R}(\partial_i,\partial_j,\partial_k,\partial_l) = 2m\phi^{3-n}(\sigma_{ik}\sigma_{jl}-\sigma_{il}\sigma_{jk})$$
$$\bar{R}(\partial_i,\partial_r,\partial_j,\partial_r) = -m(n-1)\phi^{1-n}\sigma_{ij},$$

thus

$$\bar{R}_{\alpha\beta\gamma\mu} = O(r^{-n-1}), \quad \bar{\nabla}_\rho\bar{R}_{\alpha\beta\gamma\mu} = O(r^{-n-1}).$$

Here $\{\partial_i\}$ *is the coordinate frame on* $\mathbb{S}^n$, σ_{ij} *is the standard metric of* $\mathbb{S}^n$, *and* $\{e_\alpha\}$ *denotes an orthonormal frame on* $\mathbb{M}^{n+1}_m$.

We also need the following two lemmas regarding to σ_k, see in [**29**] for detailed proof.

Lemma 3.4. *let* $F = n\frac{C_n^{k-1}}{C_n^k}\frac{\sigma_k}{\sigma_{k-1}}$, *thus* F *is of homogeneous degree* 1, *and* $F(I)=n$, *then we have*

$$\sum_i F^{ii}\lambda_i^2 \geq \frac{F^2}{n}$$

Lemma 3.5. *Let* $F = n\frac{C_n^{k-1}}{C_n^k}\frac{\sigma_k}{\sigma_{k-1}}$ *and* $(\lambda_i)\in\Gamma_k$, *then*

$$n \leq \sum_i F^{ii} \leq nk$$

3.2. Parametrization on graph and C^0 estimate. Since the initial hypersurface Σ_0 is star-shaped, we can consider it as a graph on $\mathbb{S}^n$, i.e. $X=(x,r)$ where x is the coordinate on $\mathbb{S}^n$ and r is the radial function. By taking derivatives, we have

$$X_i = \partial_i + r_i\partial_r, \quad g_{ij} = r_ir_j + \phi^2\sigma_{ij} \tag{3.8}$$

and

$$\nu = \frac{1}{v}\left(-\frac{r^i}{\phi^2}\partial_i + \partial_r\right), \tag{3.9}$$

where ν is the unit normal vector, $v = (1+\frac{|\nabla r|^2}{\phi^2})^{\frac{1}{2}}$. Note that all the derivatives are on $\mathbb{S}^n$.

Thus

$$\frac{dr}{dt} = \frac{1}{Fv}, \dot{x}^i = -\frac{r^i}{\phi^2 F v}$$

we have

$$\frac{\partial r}{\partial t} = \frac{dr}{dt} - r_j \dot{x}^j = \frac{v}{F}. \tag{3.10}$$

By a direct computation, cf. (2.6) in [**16**] we have

$$h_{ij} = \frac{1}{v}(-r_{ij} + \phi\phi'\sigma_{ij} + \frac{2\phi' r_i r_j}{\phi}). \tag{3.11}$$

Now we consider a function

$$\varphi = \int_{r_0}^{r} \frac{1}{\phi} \tag{3.12}$$

thus

$$\varphi_i = \frac{r_i}{\phi}, \varphi_{ij} = \frac{r_{ij}}{\phi} - \frac{\phi' r_i r_j}{\phi^2}. \tag{3.13}$$

If we write everything in terms of φ, we have

$$\frac{\partial \varphi}{\partial t} = \frac{v}{\phi F} \tag{3.14}$$

and

(3.15)

$$v = (1 + |D\varphi|^2)^{\frac{1}{2}}, g_{ij} = \phi^2(\varphi_i\varphi_j + \sigma_{ij}), g^{ij} = \phi^{-2}\left(\sigma^{ij} - \frac{\varphi^i\varphi^j}{v^2}\right).$$

Moreover,

$$\begin{aligned} h_{ij} &= \frac{\phi}{v}\left(\phi'(\sigma_{ij} + \varphi_i\varphi_j) - \varphi_{ij}\right), \\ h^i_j &= g^{ik}h_{kj} = \frac{\phi'}{\phi v}\delta^i_j - \frac{1}{\phi v}\tilde{\sigma}^{ik}\varphi_{kj}, \end{aligned} \tag{3.16}$$

where $\tilde{\sigma}^{ij} = \sigma^{ij} - \frac{\varphi^i\varphi^j}{v^2}$.

Lemma 3.6. *Let $\bar{r}(t) = \sup_{\mathbb{S}^n} r(\cdot, t)$ and $\underline{r}(t) = \inf_{\mathbb{S}^n} r(\cdot, t)$, then we have*

$$\begin{aligned} \phi(\bar{r}(t)) &\leq e^{t/n}\phi(\bar{r}(0)) \\ \phi(\underline{r}(t)) &\geq e^{t/n}\phi(\underline{r}(0)). \end{aligned} \tag{3.17}$$

Proof. Recall that $\frac{\partial r}{\partial t} = \frac{v}{F}$, where F is a normalized operator on (h^i_j). At the point where the function $r(\cdot, t)$ attains its maximum, we have $\nabla r = 0, (r_{ij}) \leq 0$, from (3.13), we deduce that $\nabla\varphi = 0, (\varphi_{ij}) \leq 0$ at the maximum point. From (3.16), we have $(h^i_j) \geq \left(\frac{\phi'}{\phi}\delta^i_j\right)$, where we may assume (g_{ij}) and (h_{ij}) is diagonalized if necessary. Since F is homogeneous of degree 1, and $F(1,\cdots,1)=n$, we have

$$v^2 = 1 + |\nabla\varphi|^2 = 1, F(h^i_j) \geq \frac{\phi'}{\phi} F(\delta^i_j) = \frac{n\phi'}{\phi},$$

thus

$$\frac{d}{dt}\bar{r}(t) \leq \frac{\phi(\bar{r}(t))}{n\phi'(\bar{r}(t))}$$

i.e.

$$\frac{d}{dt}\log\phi(\bar{r}(t)) \leq \frac{1}{n}$$

which yields to the first inequality. Similarly, we can prove the second inequality, thus we have the lemma. q.e.d.

3.3. Evolution equations and C^1 estimate. Before we go on with the estimate, let's derive some evolution equations first. We have

$$\dot{g}_{ij} = \frac{2h_{ij}}{F}, \quad \dot{\nu} = \frac{g^{ij}F_i e_j}{F^2}, \tag{3.18}$$

and

$$\dot{h}^i_j = -\frac{1}{F}h^i_k h^k_j - \nabla^i\nabla_j\left(\frac{1}{F}\right) - \frac{1}{F}\bar{R}^i_{\nu j\nu}. \tag{3.19}$$

Together with the interchanging formula (3.7), we have

(3.20)
$$\begin{aligned}\dot{h}^i_j =& -\frac{1}{F}h^i_k h^k_j + \frac{F^{pq,rs}h_{pq}{}^i h_{rsj}}{F^2} - \frac{2F^{pq}h_{pq}{}^i F^{rs}h_{rsj}}{F^3} - \frac{1}{F}\bar{R}^i_{\nu j\nu}\\ &+ \frac{g^{ki}F^{pq}}{F^2}\big(h_{kj,pq} - h^l_q(h_{kl}h_{pj} - h_{kj}h_{lp}) - h^l_j(h_{lk}h_{pq} - h_{kq}h_{lp})\\ &+ h^l_q\bar{R}_{kpjl} + h^l_j\bar{R}_{kpql} + \nabla_p\bar{R}_{kjq\nu} + \nabla_k\bar{R}_{jpq\nu}\big),\end{aligned}$$

where $F^{ij} = \frac{\partial F}{\partial h_{pq}}$ and $F^{pq,rs} = \frac{\partial^2 F}{\partial h_{pq}\partial h_{rs}}$.

We also need the evolution equation for the support function

$$u = \left\langle \phi\frac{\partial}{\partial r}, \nu\right\rangle,$$

which is given by

$$\dot{u} = \frac{\phi'}{F} + \frac{\phi g^{ij}F_i r_j}{F^2}. \tag{3.21}$$

Now, we need to consider the curvature term. By Lemma 3.3, (3.8) and (3.9), we have

$$\bar{R}_{\nu jnk} = \frac{r_n\delta_{jk}}{v}\left(-\phi\phi'' - (1-(\phi')^2)\right) + \frac{r_k\delta_{jn}}{v}\left(\phi\phi'' + (1-\phi'^2)\right).$$

Note that $g^{pn} = \phi^{-2}\left(\sigma^{pn} - \frac{r^p r^n}{v^2\phi^2}\right)$, where $r^p = g^{pq}r_q$. Thus

$$g^{pn}\nabla_p \Phi \bar{R}_{\nu jnk} = \left(\frac{|\nabla r|^2\delta_{jk} - r_j r_k}{\phi v^3}\right)\left(-\phi\phi'' - (1-\phi'^2)\right) \leq 0. \tag{3.22}$$

Lemma 3.7. *Along the flow, $|\nabla\varphi| \leq C$, where C depends on Σ_0, n, k.*

Proof. By (3.14) and (3.16), we have

$$\frac{\partial\varphi}{\partial t} = \frac{v}{\phi F} = \frac{v^2}{\tilde{F}(\phi'\delta_{ij} - \tilde{\sigma}^{ik}\varphi_{kj})} = \frac{1}{G},$$

where $\tilde{F} = \phi v F$.

Let $G^{ij} = \frac{\partial G}{\partial \varphi_{ij}}$, $G^k = \frac{\partial G}{\partial \varphi_k}$, $G_\varphi = \frac{\partial G}{\partial\varphi}$ then

$$G^{ij} = -\frac{1}{v^2}\tilde{F}^i_l\tilde{\sigma}^{lj}, \quad G_\varphi = \frac{1}{v^2}\tilde{F}^i_i\phi\phi''.$$

Let $\omega = \frac{1}{2}|\nabla\varphi|^2$, we have

$$\begin{aligned}\frac{\partial\omega}{\partial t} &= \nabla_k\varphi\nabla_k\dot{\varphi} = -\frac{\varphi^k}{G^2}\nabla_k G = -\frac{\varphi^k}{G^2}\left(G^{ij}\varphi_{ijk} + G^l\varphi_{lk} + G_\varphi\varphi_k\right)\\ &= \frac{1}{v^2G^2}\left(\tilde{F}^i_l\tilde{\sigma}^{lj}\varphi^k\varphi_{ijk} - v^2G^l\omega_l - 2\tilde{F}^i_i\phi\phi''\omega\right).\end{aligned}$$

We want to write the term $\tilde{\sigma}^{lj}\varphi_{ijk}$ in terms of second derivative of ω. Note that

$$\begin{aligned}\omega_{ij} &= \varphi_{kij}\varphi^k + \varphi_{ki}\varphi^k_j\\ &= \varphi_{ijk}\varphi^k + (\sigma_{ij}\sigma_{kp} - \sigma_{ik}\sigma_{jp})\varphi^p\varphi^k + \varphi_{ki}\varphi^k_j\\ &= \varphi_{ijk}\varphi^k + \sigma_{ij}|\nabla\varphi|^2 - \varphi_i\varphi_j + \varphi_{ki}\varphi^k_j\end{aligned}$$

and

$$\tilde{\sigma}^{lj}\left(\sigma_{ij}|\nabla\varphi|^2 - \varphi_i\varphi_j\right) = \delta^l_i|\nabla\varphi|^2 - \varphi_i\varphi^l.$$

Thus we have

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{1}{v^2G^2}\left(\tilde{F}^i_l\tilde{\sigma}^{lj}\omega_{ij} - \tilde{F}^i_i|\nabla\varphi|^2 + \tilde{F}^i_l\varphi_i\varphi^l - v^2G^l\omega_l - 2\tilde{F}^i_i\phi\phi''\omega\right)\\ &\quad - \frac{1}{v^2G^2}\tilde{F}^i_l\tilde{\sigma}^{lj}\varphi_{ki}\varphi^k_j.\end{aligned} \tag{3.23}$$

Note that $-\tilde{F}^i_i|\nabla\varphi|^2 + \tilde{F}^i_l\varphi_i\varphi^l \leq 0$ and $-\tilde{F}^i_l\tilde{\sigma}^{lj}\varphi_{ki}\varphi^k_j \leq 0$, thus by the maximum principle, we have

$$\omega(\cdot, t) \leq \sup\omega_0.$$ q.e.d.

Lemma 3.8. *Along the flow, $u \geq ce^{\frac{t}{n}}$, where u is the support function and c depends on Σ_0, n, k.*

Proof. Recall that $u = \langle\phi\partial_r, \nu\rangle$. By Lemma 3.6, $\phi \geq Ce^{\frac{t}{n}}$; by Lemma 3.7, $\langle\partial_r, \nu\rangle \geq C$. It follows that $u \geq ce^{\frac{t}{n}}$. q.e.d.

Lemma 3.9. *Suppose* $\overline{\mathrm{Ric}}(\nu,\nu) \leq 0$ *for* Σ_0*, then* $\overline{\mathrm{Ric}}(\nu,\nu) \leq 0$ *for all* Σ_t*. If* $k \geq 2$*, this implies* $R > 0$ *for all* Σ_t*.*

Proof. By Lemma 3.3, we have

$$\overline{\mathrm{Ric}} = \left((n-1)(1-\phi'^2) - \phi\phi''\right) g_{\mathbb{S}^n} - n\frac{\phi''}{\phi} dr^2.$$

Together with (3.9), i.e. $\nu = \frac{1}{v}\left(\partial_r - \frac{r^j \partial_j}{\phi^2}\right)$, we have

$$\begin{aligned}
\overline{\mathrm{Ric}}(\nu,\nu) &= -n\frac{\phi''}{\phi v^2} + \frac{(n-1)(1-\phi'^2) - \phi\phi''}{\phi^4 v^2}|\nabla r|^2 \\
&= -n\frac{\phi''}{\phi v^2} + \frac{(n-1)(1-\phi'^2) - \phi\phi''}{\phi^2 v^2}(v^2-1) \\
&= \frac{(n-1)(1-\phi'^2) - \phi\phi''}{\phi^2} - (n-1)\frac{1-\phi'^2+\phi\phi''}{\phi^2 v^2}.
\end{aligned}$$

Since $\phi' = \sqrt{1-2m\phi^{1-n}}$, thus

$$1-\phi'^2 = 2m\phi^{1-n}, \quad \phi\phi'' = m(n-1)\phi^{1-n}.$$

Thus

$$\overline{\mathrm{Ric}}(\nu,\nu) = m(n-1)\phi^{-1-n} - m(n-1)(n+1)\phi^{-1-n}v^{-2}.$$

On the other hand, $v^2 = 1 + |\nabla\varphi|^2$ and, by Lemma 3.7, $|\nabla\varphi|$ is bounded above by the initial data. Thus it follows that, if initially $\overline{\mathrm{Ric}}(\nu,\nu) \leq 0$, i.e. $|\nabla\varphi|^2 \leq n$, then it remains true along the flow.

To prove the second assertion, it suffices to note that

$$\sigma_2 = \frac{R}{2} + \overline{\mathrm{Ric}}(\nu,\nu) > 0$$

along the flow. Thus $R > 0$ along the flow. q.e.d.

3.4. Bound for principal curvature.

Lemma 3.10. *Along the flow,* $F\phi \leq C$*, where* C *depends only on* Σ_0, n, k*.*

Proof. Consider $F\phi$, at the maximum point, we have

$$\frac{\dot\phi}{\phi} + \frac{\dot F}{F} \geq 0$$

and

$$\frac{\phi_i}{\phi} + \frac{F_i}{F} = 0, \quad \frac{\phi_{ij}}{\phi} + \frac{F_{ij}}{F} - 2\frac{F_i F_j}{F^2} \leq 0.$$

By (3.10) and (3.19), we have

$$\begin{aligned}0 &\leq \frac{\phi' v}{F\phi} + \frac{F_i^j}{F}\left(-\frac{1}{F}h_k^i h_j^k - \nabla^i\nabla_j\left(\frac{1}{F}\right) - \frac{1}{F}\bar{R}^i_{\nu j\nu}\right)\\ &= \frac{\phi' v}{F\phi} + \frac{F_i^j}{F}\left(-\frac{1}{F}h_k^i h_j^k + \frac{\nabla^i\nabla_j F}{F^2} - 2\frac{\nabla^i F\nabla_j F}{F^3} - \frac{1}{F}\bar{R}^i_{\nu j\nu}\right).\end{aligned}$$

By the critical equation above and (3.11), we have

$$\begin{aligned}0 &\leq \frac{\phi' v}{F\phi} + \frac{F_i^j}{F}\left(-\frac{1}{F}h_k^i h_j^k - \frac{\nabla^i\nabla_j\phi}{F\phi} - \frac{1}{F}\bar{R}^i_{\nu j\nu}\right)\\ &= \frac{\phi' v}{F\phi} + \frac{F_i^j}{F^2}\left(-h_k^i h_j^k - \bar{R}^i_{\nu j\nu}\right) - \frac{F^{ij}}{F^2\phi}\left(\phi'' r_i r_j + \phi' r_{ij}\right)\\ &= \frac{\phi' v}{F\phi} + \frac{F_i^j}{F^2}\left(-h_k^i h_j^k - \bar{R}^i_{\nu j\nu}\right)\\ &\quad - \frac{F^{ij}}{F^2\phi}\left(\phi'' r_i r_j + \phi'\left(\phi\phi'\sigma_{ij} + \frac{2\phi' r_i r_j}{\phi} - h_{ij}v\right)\right)\\ &= 2\frac{\phi' v}{F\phi} - \frac{F_i^j}{F^2}\left(h_k^i h_j^k + \bar{R}^i_{\nu j\nu}\right)\\ &\quad - \frac{F^{ij}}{F^2\phi}\left(\phi'' r_i r_j + \phi'\left(\phi\phi'\sigma_{ij} + \frac{2\phi' r_i r_j}{\phi}\right)\right).\end{aligned}$$

By Lemma 3.3 and property of ϕ, we have

$$\begin{aligned}0 &\leq 2\frac{\phi' v}{F\phi} - \frac{F_i^j}{F^2}h_k^i h_j^k + C\frac{F_i^i}{F^2\phi^{n+1}}\\ &\quad - \frac{F^{ij}}{F^2\phi}\left(\phi'' r_i r_j + \frac{2(\phi')^2 r_i r_j}{\phi}\right) - \frac{F^{ij}\sigma_{ij}(\phi')^2}{F^2}.\end{aligned}$$

By Lemma 3.4, we have

$$\begin{aligned}0 &\leq 2\frac{\phi' v}{F\phi} - \frac{1}{n} + C\frac{F_i^i}{F^2\phi^{n+1}}\\ &\quad - \frac{F^{ij}}{F^2\phi}\left(\phi'' r_i r_j + \frac{2(\phi')^2 r_i r_j}{\phi}\right) - \frac{F^{ij}\sigma_{ij}(\phi')^2}{F^2}.\end{aligned}$$

Using the fact that $F^{ij}r_i r_j \geq 0$, we have

$$0 \leq 2\frac{\phi' v}{F\phi} - \frac{1}{n} + C\frac{F_i^i}{F^2\phi^{n+1}} - \frac{F^{ij}\sigma_{ij}(\phi')^2}{F^2}.$$

Note that $\sigma_{ij} = \frac{g_{ij} - r_i r_j}{\phi^2}$, we have

$$0 \leq 2\frac{\phi' v}{F\phi} - \frac{1}{n} + C\frac{F_i^i}{F^2\phi^{n+1}} - \frac{F_i^i(\phi')^2}{F^2\phi^2} + \frac{F^{ij}r_i r_j(\phi')^2}{F^2\phi^2}. \tag{3.24}$$

By Lemma 3.7, we have $g^{ij} \le C\phi^{-2}$, it follows that

$$\frac{F^{ij}r_ir_j(\phi')^2}{F^2\phi^2} \le C\frac{F_i^i(\phi')^2}{F^2\phi^2}.$$

Together with Lemma 3.5 and property of ϕ, we have

$$0 \le \frac{C}{F\phi} - \frac{1}{n} + \frac{C}{F^2\phi^{n+1}} + \frac{C}{F^2\phi^2},$$

thus $F\phi$ is bounded above. q.e.d.

Lemma 3.11. *Along the flow, $|\dot\varphi| \le C$, where C depends on Σ_0, n, k.*

Proof. By (3.14) and (3.16), we have

$$\frac{\partial\varphi}{\partial t} = \frac{v}{\phi F} = \frac{v^2}{\tilde F(\phi'\delta_{ij} - \tilde\sigma^{ik}\varphi_{kj})} = \frac{1}{G},$$

where $\tilde F = \phi v F$.

Let $G^{ij} = \frac{\partial G}{\partial \varphi_{ij}}$, $G^k = \frac{\partial G}{\partial \varphi_k}$, $G_\varphi = \frac{\partial G}{\partial \varphi}$ then

$$G^{ij} = -\frac{1}{v^2}\tilde F_l^i\tilde\sigma^{lj}, \quad G_\varphi = \frac{1}{v^2}\tilde F_i^i\phi\phi'',$$

thus

$$\begin{aligned}\frac{\partial\dot\varphi}{\partial t} &= -\frac{\dot G}{G^2} = -\frac{1}{G^2}\left(G^{ij}\dot\varphi_{ij} + G^k\dot\varphi_k + G_\varphi\dot\varphi\right)\\ &= \frac{1}{v^2G^2}\left(\tilde F_l^i\tilde\sigma^{lj}\dot\varphi_{ij} - v^2G^k\dot\varphi_k - \tilde F_i^i\phi\phi''\dot\varphi\right).\end{aligned}$$

By maximum principle, we conclude that $|\dot\varphi|$ is bounded above. q.e.d.

Lemma 3.12. *Along the flow, $F\phi \ge c$, where c depends on Σ_0, n, k.*

Proof. Since $\dot\varphi = \frac{v}{\phi F}$, by Lemma 3.11, we have

$$\frac{v}{\phi F} \le C,$$

thus $F\phi \ge c$. q.e.d.

Lemma 3.13. *Along the flow, $|\kappa_i\phi| \le C$, where κ_i is the principal curvature of Σ_t, C depends on Σ_0, n, k.*

Proof. Consider $\log\eta - \log u + \frac{2t}{n}$, where

$$\eta = \sup\{h_{ij}\xi^i\xi^j : g_{ij}\xi^i\xi^j = 1\}.$$

WLOG, we suppose that the maximum point occurs at $\eta = h_1^1$, and we have

$$\frac{\dot{h_1^1}}{h_1^1} - \frac{\dot u}{u} + \frac{2}{n} \ge 0 \tag{3.25}$$

and

$$\frac{h^1_{1i}}{h^1_1} - \frac{u_i}{u} = 0, \quad \frac{h^1_{1ij}}{h^1_1} \le \frac{u_{ij}}{u}. \tag{3.26}$$

By (3.20), (3.21) and the critical equation, we have

(3.27)

$$\begin{aligned}
0 \le & \frac{1}{h^1_1}\bigg(- \frac{1}{F} h^1_k h^k_1 + \frac{F^{pq,rs} h_{pq}{}^1 h_{rs1}}{F^2} - \frac{2F^{pq} h_{pq}{}^1 F^{rs} h_{rs1}}{F^3} - \frac{1}{F}\bar{R}^1_{\nu 1\nu} \\
& + \frac{g^{k1}F^{pq}}{F^2}\big(h_{k1,pq} - h^m_q(h_{km}h_{p1} - h_{k1}h_{mp}) - h^m_1(h_{mk}h_{pq} - h_{kq}h_{mp}) \\
& + h^m_q \bar{R}_{kp1m} + h^m_1 \bar{R}_{kpqm} + \nabla_p \bar{R}_{k1q\nu} + \nabla_k \bar{R}_{1pq\nu}\big)\bigg) \\
& - \frac{\phi'}{Fu} - \frac{\phi g^{ij} F_i r_j}{F^2 u} + \frac{2}{n}.
\end{aligned}$$

Consider the term $\frac{F^{pq}}{F^2}\frac{h^1_{1,pq}}{h^1_1}$, by (3.26) and Lemma 3.2, we have

(3.28)

$$\begin{aligned}
\frac{F^{pq}}{F^2}\frac{h^1_{1,pq}}{h^1_1} \le & \frac{F^{pq}}{F^2}\frac{u_{pq}}{u} \\
= & \frac{F^{pq}}{F^2 u}\left(g^{kl}h_{pqk}\Phi_l + \phi' h_{pq} - (h^2)_{pq}u + g^{kl}\nabla_l \Phi \bar{R}_{\nu pkq}\right).
\end{aligned}$$

Inserting (3.28) into (3.27), together with the concavity of F, yields

(3.29)

$$\begin{aligned}
0 \le & \frac{g^{k1}F^{pq}}{h^1_1 F^2}\big(h^m_q \bar{R}_{kp1m} + h^m_1 \bar{R}_{kpqm} + \nabla_p \bar{R}_{k1q\nu} + \nabla_k \bar{R}_{1pq\nu} - h^m_1 h_{mk}h_{pq}\big) \\
& + \frac{1}{h^1_1}\left(- \frac{1}{F} h^1_k h^k_1 - \frac{1}{F}\bar{R}^1_{\nu 1\nu}\right) + \frac{g^{kl}F^{pq}}{F^2 u}\nabla_l \Phi \bar{R}_{\nu pkq} + \frac{2}{n}.
\end{aligned}$$

By (3.22), we have

(3.30)

$$\begin{aligned}
0 \le & \frac{g^{k1}F^{pq}}{h^1_1 F^2}\big(h^m_q \bar{R}_{kp1m} + h^m_1 \bar{R}_{kpqm} + \nabla_p \bar{R}_{k1q\nu} + \nabla_k \bar{R}_{1pq\nu} - h^m_1 h_{mk}h_{pq}\big) \\
& + \frac{1}{h^1_1}\left(- \frac{1}{F} h^1_k h^k_1 - \frac{1}{F}\bar{R}^1_{\nu 1\nu}\right) + \frac{2}{n}.
\end{aligned}$$

By Lemma 3.3, all terms involving curvature terms of the ambient space are uniformly bounded by $C\phi^{-1-n}$, i.e.

$$\begin{aligned}
& \frac{g^{k1}F^{pq}}{F^2}\big(h^m_q \bar{R}_{kp1m} + h^m_1 \bar{R}_{kpqm} + \nabla_p \bar{R}_{k1q\nu} + \nabla_k \bar{R}_{1pq\nu}\big) \\
\le & \frac{CF^i_i}{F^2 \phi^{n+1}} h^1_1 \le C h^1_1,
\end{aligned}$$

where we have used Lemma 3.5 and Lemma 3.12 in the last inequality. Plug it into (3.30), we have

$$0 \leq \frac{1}{h_1^1}\left(-\frac{2}{F}h_k^1 h_1^k - \frac{1}{F}\bar{R}_{\nu 1\nu}^1\right) + C \leq \frac{1}{h_1^1}\left(-\frac{2}{F\phi}h_k^1 h_1^k \phi + \frac{C}{F\phi^{n+1}}\right) + C$$
$$\leq -Ch_1^1\phi + \frac{C}{h_1^1\phi} + C,$$

i.e. $h_1^1\phi \leq C$.

By Lemma 3.6 and Lemma 3.8, $ce^{\frac{t}{n}} \leq u \leq \phi \leq Ce^{\frac{t}{n}}$, the lemma now follows by the definition of the test function. q.e.d.

Lemma 3.14. *Along the flow, F is uniformly elliptic.*

Proof. It follows directly from Lemma 3.10, Lemma 3.12 and Lemma 3.13. q.e.d.

3.5. Asymptotic behaviors.

Lemma 3.15. *Along the flow, $|\nabla\varphi| \leq Ce^{-\alpha t}$, where C, α depends on Σ_0, n, k.*

Proof. We proceed as in Lemma 3.7, by (3.23), we have

$$\frac{\partial w}{\partial t} = \frac{1}{v^2G^2}\left(\tilde{F}_l^i\tilde{\sigma}^{lj}\omega_{ij} - \tilde{F}_i^i|\nabla\varphi|^2 + \tilde{F}_l^i\varphi_i\varphi^l - v^2G^l\omega_l - 2\tilde{F}_i^i\phi\phi''\omega\right)$$
$$- \frac{1}{v^2G^2}\tilde{F}_l^i\tilde{\sigma}^{lj}\varphi_{ki}\varphi_j^k.$$

By Lemma 3.14, F is uniformly elliptic, i.e. $\tilde{F}^{ij}$ is uniformly elliptic, by Lemma 3.10, ϕF is bounded above, then consider $\tilde{\omega} = \omega e^{\lambda t}$, at the maximum point, we have

$$\frac{\partial\tilde{\omega}}{\partial t} \leq \frac{1}{v^2G^2}\left(-\tilde{F}_i^i|\nabla\varphi|^2 + \tilde{F}_l^i\varphi_i\varphi^l\right)e^{\lambda t} + \lambda\tilde{\omega} \leq (-\frac{c}{\phi^2F^2} + \lambda)\tilde{\omega},$$

thus $\tilde{\omega}$ is uniformly bounded, we have $|\nabla\varphi|$ decays exponentially. q.e.d.

Lemma 3.16. *Along the flow, $F\phi \leq n + Ce^{-\alpha t}$, where C, α depends only on Σ_0, n, k.*

Proof. We proceed as in Lemma 3.10, by (3.24), we have

$$0 \leq 2\frac{\phi' v}{F\phi} - \frac{1}{n} + C\frac{F_i^i}{F^2\phi^{n+1}} - \frac{F_i^i(\phi')^2}{F^2\phi^2} + \frac{F^{ij}r_ir_j(\phi')^2}{F^2\phi^2}.$$

By Lemma 3.14, F is uniformly elliptic, by Lemma 3.15, $|\nabla\varphi|$ decays exponentially, then

$$0 \leq \frac{2}{F\phi} - \frac{1}{n} - \frac{n}{F^2\phi^2} + Ce^{-\alpha t},$$

i.e.

$$F\phi \leq n + Ce^{-\alpha t}.$$

q.e.d.

Lemma 3.17. *We have*

$$|h^i_j\phi - \delta^i_j| \le Ce^{-\alpha t},$$

where C, α depends only on Σ_0, n, k. Moreover, for any $p, q \ge 0$, we have

$$\left|\left(\frac{\partial}{\partial t}\right)^p (\phi\nabla)^q \phi^2 \nabla h^i_j\right| \le Ce^{-\alpha t},$$

where ∇ is the unit gradient on Σ_t and C depends in addition on p, q.

Proof. To prove the lemma, we first notice that by (3.16) and (3.14), we have

$$h^i_j = \frac{\phi'}{\phi v}\delta^i_j - \frac{1}{\phi v}\tilde{\sigma}^{ik}\varphi_{kj}$$

and

$$\frac{\partial \varphi}{\partial t} = \frac{v}{\phi F} = \frac{v}{\tilde{F}},$$

where

$$\tilde{F} = \phi F = F(\frac{\phi'}{v}\delta^i_j - \frac{1}{v}\tilde{\sigma}^{ik}\varphi_{kj}).$$

By the Lemma 3.7 and Lemma 3.13, we know that $\nabla\varphi$ and $\nabla^2\varphi$ is uniformly bounded. By Evans-Krylov, we have $|\varphi|_{2,\alpha} \le C$. By standard interpolation inequality, we have $\nabla^2\varphi$ decays exponentially as $\nabla\varphi$ decays exponentially by Lemma 3.15. Thus from the definition of h^i_j above, we have the first inequality.

By Schauder estimate, we have $|\varphi|_l \le Ce^{-\alpha t}$ for all $l \ge 1$.

By the definition of h^i_j, we have

$$\begin{aligned}\nabla h^i_j = &\left(\frac{\phi''}{\phi v} - \frac{\phi'^2}{\phi^2 v}\right)\delta^i_j \nabla r - \frac{\phi'}{\phi v^3}\delta^i_j \varphi_k \nabla\varphi_k \\ &+ \frac{\phi'}{\phi^2 v}\tilde{\sigma}^{ik}\varphi_{kj}\nabla r + \frac{1}{\phi v^3}\tilde{\sigma}^{ik}\varphi_{kj}\varphi_l\nabla\varphi_l \\ &+ \frac{1}{\phi v}\nabla\varphi^i\varphi^k\varphi_{kj} + \frac{1}{\phi v}\nabla\varphi^k\varphi^i\varphi_{kj} - \frac{1}{\phi v}\tilde{\sigma}^{ik}\nabla\varphi_{kj}.\end{aligned}$$

Since $|\varphi|_l \le Ce^{-\alpha t}$ for all $l \ge 1$, this implies

$$|\phi^2\nabla h^i_j| \le Ce^{-\alpha t}.$$

By induction, we have

$$|\left(\frac{\partial}{\partial t}\right)^p (\phi\nabla)^q \phi^2 \nabla h^i_j| \le Ce^{-\alpha t}$$

for all $p, q \ge 0$. q.e.d.

Lemma 3.18. *Let $\tilde{g}_{ij} = \phi^{-2} g_{ij}$ be a normalized metric, then*

$$|\tilde{g}_{ij} - \sigma_{ij}| \leq Ce^{-\alpha t},$$

where σ_{ij} is the standard metric on $\mathbb{S}^n$ and C, α depends only on Σ_0, n, k. Moreover, for any $p, q \geq 0$, we have

$$\left| \left(\frac{\partial}{\partial t} \right)^p (\phi \nabla)^q \, \phi \nabla \tilde{g}_{ij} \right| \leq Ce^{-\alpha t},$$

where ∇ is the unit gradient on Σ_t and C depends in addition on p, q.

Proof. Following the step in [**19**], we consider the rescaled hypersurface as $\hat{X} = Xe^{-\frac{t}{n}}$. Then we have $\hat{r} = re^{-\frac{t}{n}}$, thus

$$\hat{g}_{ij} = \phi^2(\hat{r})\sigma_{ij} + \hat{r}_i \hat{r}_j.$$

By Lemma 3.6 and Lemma 3.15, we have $c_0 \leq \hat{r} \leq C_0$ uniformly, and $|\hat{r}_i| \leq Ce^{-\alpha t}$, thus

$$c_0 \sigma \leq \hat{g} \leq C_0 \sigma$$

for t large enough, i.e. $\hat{g}$ is well defined.

Now let's prove that $\hat{g}$ converges to $\hat{g}_\infty$. By Lemma 3.15, we have

$$\frac{\partial \hat{g}_{ij}}{\partial t} = 2\phi(\hat{r})\phi'(\hat{r}) \left(\frac{v}{F} e^{-\frac{t}{n}} - \frac{1}{n} re^{-\frac{t}{n}} \right) \sigma_{ij} + \frac{\partial}{\partial t} \left(\hat{r}_i \hat{r}_j \right) \leq Ce^{-\alpha t}.$$

Thus $\hat{g}$ converges exponentially fact to $\hat{g}_\infty$. To prove that $\hat{g}_\infty$ is a round metric, we only need to prove that $\hat{r}$ is constant. Since $\hat{r}$ is defined on $\mathbb{S}^n$, we take derivative of $\mathbb{S}^n$ on $\hat{r}$ to obtain

$$|\nabla_{\mathbb{S}^n} \hat{r}| = |\nabla_{\mathbb{S}^n} re^{-\frac{t}{n}}| \leq Ce^{-\alpha t}.$$

Thus $\hat{r}$ is constant for $t = \infty$, i.e. we have

$$r = r_0 e^{\frac{t}{n}} + O(e^{(\frac{1}{n} - \alpha)t})$$

and

$$\phi(r) = r_0 e^{\frac{t}{n}} + O(e^{(\frac{1}{n} - \alpha)t}).$$

Hence, at time t, we have

$$g_{ij} = \phi^2(r) \left(\sigma_{ij} + \varphi_i \varphi_j \right) = r_0^2 e^{\frac{2t}{n}} \sigma_{ij} + O(e^{(\frac{2}{n} - 2\alpha)t}),$$

and the normalized metric $\tilde{g}_{ij}$ satisfies

$$\tilde{g}_{ij} = \phi^{-2} g_{ij} = \sigma_{ij} + O(e^{-\alpha t}).$$

Similar to the previous lemma, high regularity decay estimates follows by Lemma 3.15 and the definition of $\tilde{g}_{ij}$. q.e.d.

REMARK 3.1. Let $k \geq 2$. Let g be a metric on $\mathbb{S}^n$ so that (S^n, g) isometrically embeds into $\mathbb{M}^{n+1}_m$ as a star-shaped, k-convex, closed hypersurface in $\mathbb{M}^{n+1}_m$ with $\mathrm{Ric}(\nu, \nu) \leq 0$. Combining results in this section and arguments in [**12**, Section 3], one knows that g can be connected to a round metric within the space of positive scalar curvature metrics on $\mathbb{S}^n$. Therefore, repeating the proof in [**12**], we know that the conclusion of [**12**, Theorem 1.2] holds for such a metric g.

4. Bartnik-Shi-Tam type asymptotically flat extensions

Let $\Sigma^n \subset \mathbb{M}^{n+1}_m$ be a closed, star-shaped, 2-convex hypersurface satisfying

$$\overline{\mathrm{Ric}}(\nu, \nu) \leq 0. \tag{4.1}$$

Here $\overline{\mathrm{Ric}}(\cdot, \cdot)$ is the Ricci curvature of the Schwarzschild manifold $\mathbb{M}^{n+1}_m$ and ν is the outward unit normal to Σ. By Theorem 3.1, there exists a smooth solution $\{\Sigma_t\}_{0\leq t\leq\infty}$, consisting of star-shaped hypersurfaces, to

$$\frac{\partial X}{\partial t} = \frac{n-1}{2n}\frac{\sigma_1}{\sigma_2}\nu \tag{4.2}$$

with initial condition $\Sigma_0 = \Sigma$. By Lemma 3.9, condition (4.1) implies that the scalar curvature R of each Σ_t is positive.

Let $\mathbb{E}$ denote the exterior of Σ in $\mathbb{M}^{n+1}_m$, which is swept by $\{\Sigma_t\}_{0\leq t\leq\infty}$. On $\mathbb{E}$, the Schwarzschild metric $\bar{g}$ can be written as

$$\bar{g} = f^2 dt^2 + g_t,$$

where g_t is the induced metric on Σ_t and

$$f = \frac{n-1}{2n}\frac{\sigma_1}{\sigma_2} > 0.$$

Prompted by Proposition 2.2, we are interested in a new metric g_η on $\mathbb{E}$, which takes the form of

$$g_\eta = \eta^2 dt^2 + g_t,$$

and has zero scalar curvature. Here $\eta > 0$ is a function on $\mathbb{E}$.

We first derive the equation for η. Adopting the notations in Section 2, by (2.6), (2.8) and Gauss equation (2.9), we have

$$\begin{aligned}
\frac{\partial}{\partial t} H_\eta =& - \Delta\eta - \eta(|A_\eta|^2 + Ric_{g_\eta}(\nu, \nu)) \\
=& - \Delta\eta - \eta\left(\eta^{-2} f^2 |\bar{A}|^2 + \eta^{-2} f^2 \sigma_2 - \frac{R}{2}\right) \\
=& - \Delta\eta - \eta^{-1} f^2 |\bar{A}|^2 - \eta^{-1} f^2 \sigma_2 + \frac{R}{2}\eta.
\end{aligned}$$

On the other hand,

$$\frac{\partial}{\partial t}H_\eta = \frac{\partial}{\partial t}(\frac{f\bar{H}}{\eta}) = -\frac{f\bar{H}}{\eta^2}\frac{\partial\eta}{\partial t} + \frac{1}{\eta}\frac{\partial}{\partial t}(f\bar{H}).$$

Thus

$$-\frac{f\bar{H}}{\eta^2}\frac{\partial\eta}{\partial t} + \frac{1}{\eta}\frac{\partial}{\partial t}(f\bar{H}) = -\Delta\eta - u^{-1}f^2|\bar{A}|^2 - \eta^{-1}f^2\sigma_2 + \frac{R}{2}\eta$$

i.e.

$$-\frac{\partial\eta}{\partial t} + \frac{\eta^2}{f\bar{H}}\Delta\eta = \frac{\eta^3 R}{2f\bar{H}} - \frac{\eta}{f\bar{H}}\left(f^2|\bar{A}|^2 + f^2\sigma_2 + \frac{\partial}{\partial t}(f\bar{H})\right). \tag{4.3}$$

Equation (4.3) is as the same as (5) in [**17**]. The following assertion on the long time existence of η on $\mathbb{E}$ follows directly from [**17**, Proposition 2] and Lemma 3.9.

Lemma 4.1. *Let Σ be a closed, star-shaped, 2-convex hypersurface in $\mathbb{M}^{n+1}_m$ with $\overline{\mathrm{Ric}}(\nu,\nu) \le 0$. Given any positive function $\psi > 0$ on Σ, the solution to (4.3) with initial condition $\eta|_{t=0} = \psi$ exists for all time and remains positive.*

In what follows, we analyze the asymptotic behavior of g_η.

4.1. C^0 estimate of η. For the convenience of estimating η, we consider

$$w = f^{-1}\eta.$$

By (4.3), (2.7) and (2.9), it is easily seen that w satisfies the equation

$$-\frac{\partial w}{\partial t} + \frac{w^2}{\bar{H}}\left(f\Delta w + 2\nabla w\nabla f\right) = \frac{1}{2\bar{H}}\left(fR - 2\Delta f\right)(w^3 - w). \tag{4.4}$$

Lemma 4.2. *w satisfies the estimate*

$$|w-1| \le C\phi^{1-n},$$

where C depends only on Σ_0 and n.

Proof. It suffices to focus on w for $t \ge t_0$ where t_0 is sufficiently large. Following the steps in [**41**], we define

$$A(t) = \min_{\Sigma_t}\frac{fR - 2\Delta f}{\bar{H}}, \quad B(t) = \max_{\Sigma_t}\frac{fR-2\Delta f}{\bar{H}}.$$

By Lemma 3.17, Lemma 3.3 and Gauss equation (2.9), we have

$$\frac{fR-2\Delta f}{\bar{H}} = \frac{n-1}{n} + Ce^{-\alpha t},$$

thus both $A(t)$ and $B(t)$ are positive for $t \ge t_0$.

We first seek an upper bound for w. Define

$$P(t) = \left(1 - C_1\exp(-\int_{t_0}^t A(s)ds)\right)^{-\frac{1}{2}}$$

with $C_1 = 1-(\max_{\Sigma_{t_0}} w+1)^{-2}$. It is clear that $P-w \geq 0$ at t_0. Taking derivative, we have

$$\begin{aligned}\frac{d}{dt}P(t) &= -\frac{1}{2}\left(1 - C_1\exp(-\int_{t_0}^{t} A(s)ds)\right)^{-\frac{3}{2}} C_1\exp(-\int_{t_0}^{t} A(s)ds)A(t)\\ &= \frac{1}{2}P^3(P^{-2}-1)A = \frac{1}{2}(P-P^3)A.\end{aligned}$$

At the minimum point of $P-w$, we have

$$\frac{d}{dt}\left(P-w\right) \leq 0, \quad \nabla w = 0, \quad \nabla^2 w \leq 0,$$

thus

$$0 \geq \frac{1}{2}(P-P^3)A + \frac{1}{2\bar{H}}\left(fR - 2\Delta f\right)(w^3-w).$$

Since $A \leq \frac{fR-2\Delta f}{\bar{H}}$, we have

$$0 \geq P - P^3 + w^3 - w,$$

i.e. $P-w \geq 0$ as $P \geq 1$. Therefore, $w \leq P$ for all time $t \geq t_0$.

Next, we seek a lower bound of w. We consider two cases.

Case 1: $\min_{\Sigma_{t_0}} w \geq 1$. Define

$$Q(t) = \left(1 + C_2\exp(-\int_{t_0}^{t} B(s)ds)\right)^{-\frac{1}{2}},$$

where $C_2 = (\min_{\Sigma_{t_0}} w)^{-2} - 1$. It's clear that $w - Q \geq 0$ at t_0. By a similar computation as above, we have

$$\frac{d}{dt}Q(t) = \frac{1}{2}(Q-Q^3)B.$$

At the minimum point of $w-Q$,

$$\frac{d}{dt}(w-Q) \leq 0, \quad \nabla w = 0, \quad \nabla^2 w \geq 0.$$

Thus

$$0 \geq -\frac{1}{2\bar{H}}\left(fR - 2\Delta f\right)(w^3-w) - \frac{1}{2}(Q-Q^3)B.$$

Since $B \geq \frac{fR-2\Delta f}{\bar{H}}$, we have

$$0 \geq w - w^3 + Q^3 - Q,$$

which implies $w \geq Q$ as $Q \geq 1$. Thus, $w \geq Q$ for all $t \geq t_0$.

Case 2: $\min_{\Sigma_{t_0}} w < 1$. Define

$$\tilde{Q}(t) = \left(1 + (C_2+\epsilon)\exp(-\int_{t_0}^{t}(A(s)-\epsilon)ds)\right)^{-\frac{1}{2}}.$$

For ϵ small enough, we have

$$\tilde{Q}(t_0) = (1 + C_2 + \epsilon)^{-\frac{1}{2}} < \min_{\Sigma_{t_0}} w.$$

Suppose now at some $t_1 > t_0$, we have $\min_{\Sigma_{t_1}} \left(w - \tilde{Q}\right) = 0$ and, for $t_0 \le t \le t_1$, we have $w - \tilde{Q} \ge 0$. Then at t_1,

$$\frac{d}{dt}(w - \tilde{Q}) \le 0, \quad \nabla w = 0, \quad \nabla^2 w \ge 0.$$

Since

$$\frac{d}{dt}\tilde{Q}(t) = \frac{1}{2}(\tilde{Q} - \tilde{Q}^3)(A - \epsilon),$$

we have

$$0 \ge -\frac{1}{2\bar{H}}\left(fR - 2\Delta f\right)(w^3 - w) - \frac{1}{2}(\tilde{Q} - \tilde{Q}^3)(A - \epsilon).$$

Since $A - \epsilon < \frac{fR - 2\Delta f}{\bar{H}}$, the above implies

$$0 \ge \tilde{Q} - \tilde{Q}^3.$$

Contradict to the fact $\tilde{Q} < 1$. Since ϵ is arbitrary, we thus have

$$w \ge \left(1 + C_2 \exp(-\int_{t_0}^{t} A(s)ds)\right)^{-\frac{1}{2}}.$$

Finally, note that $A(t) = \frac{n-1}{n} + O(e^{-\alpha t})$, we have

$$\exp(-\int_{t_0}^{t} A(t)) \le Ce^{-\frac{n-1}{n}t} \le C\phi^{1-n}.$$

Therefore, $w \ge 1 - C\phi^{1-n}$. Similarly, $w \le 1 + C\phi^{1-n}$. Thus, we conclude

$$|w - 1| \le C\phi^{1-n}.$$ q.e.d.

4.2. Asymptotic behavior of w. Following [41], we consider the rescaled metric

$$\tilde{g}_{ij} = \phi^{-2} g_{ij}$$

on each Σ_t. Here we omit writing t for the sake of convenience. Note that by Lemma 3.18, $\tilde{g}_{ij}$ converges to σ_{ij} exponentially fast.

For any function h and l,

$$< \tilde{\nabla} h, \tilde{\nabla} l >_{\tilde{g}} = \phi^2 < \nabla h, \nabla l >_g .$$

Henceforth, for convenience, we simply write the above as

$$\tilde{\nabla} h \tilde{\nabla} l = \phi^2 \nabla h \nabla l.$$

Direct calculation gives

$$\Delta = \phi^{-2}\tilde{\Delta} + (n-2)\phi^{-3}\tilde{\nabla}\phi\tilde{\nabla}.$$

In terms of $\tilde{g}_{ij}$, equation (4.4) becomes

$$
\begin{aligned}
&-\frac{\partial w}{\partial t}+\frac{w^2}{\bar{H}\phi^2}\left(f\tilde{\Delta}w+(n-2)f\phi^{-1}\tilde{\nabla}\phi\tilde{\nabla}w+2\tilde{\nabla}w\tilde{\nabla}f\right)\\
&=\frac{1}{2\bar{H}}\left(fR-2\phi^{-2}\tilde{\Delta}f-2(n-2)\phi^{-3}\tilde{\nabla}\phi\tilde{\nabla}f\right)(w^3-w),
\end{aligned}\tag{4.5}
$$

which can be re-written as

$$
\begin{aligned}
&-\frac{\partial w}{\partial t}+\tilde{\nabla}\left(\frac{fw^2}{\bar{H}\phi^2}\tilde{\nabla}w\right)-\frac{2f}{\bar{H}\phi^2}|\tilde{\nabla}w|^2\\
=\ &w^2\tilde{\nabla}\left(\frac{f}{\bar{H}\phi^2}\right)\tilde{\nabla}w-\frac{w^2}{\bar{H}\phi^2}\left((n-2)f\phi^{-1}\tilde{\nabla}\phi+2\tilde{\nabla}f\right)\tilde{\nabla}w\\
&+\frac{1}{2\bar{H}}\left(fR-2\phi^{-2}\tilde{\Delta}f-2(n-2)\phi^{-3}\tilde{\nabla}\phi\tilde{\nabla}f\right)(w^3-w).
\end{aligned}
$$

By Lemma 4.2, this is a uniformly parabolic PDE. In addition, the term $-\frac{2f}{\bar{H}\phi^2}|\tilde{\nabla}w|^2$ has a good sign and the coefficient of $\tilde{\nabla}w$ is uniformly bounded. Thus we may directly apply standard Moser iteration to conclude that $w\in C^\alpha$.

By considering the equation for $w-1$ and applying Schauder estimate and Lemma 4.2, for any $k,l\geq 0$, we have

$$
|\left(\frac{\partial}{\partial t}\right)^k\tilde{\nabla}^l(w-1)|\leq C\phi^{1-n},\tag{4.6}
$$

where C depends only on Σ_0, n and k,l. As in [**41**], we define

$$
\mathfrak{m}=\frac{1}{2}\phi^{n-1}(1-w^{-2}).\tag{4.7}
$$

Lemma 4.3. *There exists a constant m_0, such that*

$$
|\mathfrak{m}-m_0|+|\nabla_0\mathfrak{m}|+|\frac{\partial\mathfrak{m}}{\partial t}|\leq Ce^{-\alpha t},
$$

where ∇_0 is the standard gradient on $\mathbb{S}^n$ and C,α depends only on Σ_0 and n.

Proof. By (4.6) and definition of $\mathfrak{m}$, for any $k,l\geq 0$, we have

$$
|\left(\frac{\partial}{\partial t}\right)^k\tilde{\nabla}^l\mathfrak{m}|\leq C,
$$

where C depends only on Σ_0, n and k, l. By (4.5), $\mathfrak{m}$ satisfies

$$\begin{aligned}\frac{\partial \mathfrak{m}}{\partial t} &= \frac{n-1}{2}\phi^{n-2}\phi'(1-w^{-2})\frac{\partial r}{\partial t} + \phi^{n-1}w^{-3}\frac{\partial w}{\partial t}\\ &= \frac{n-1}{2}\phi^{n-2}\phi' v f(1-w^{-2})\\ &\quad + \frac{\phi^{n-3}}{\bar{H}w}\left(f\tilde{\Delta}w + (n-2)f\phi^{-1}\tilde{\nabla}\phi\tilde{\nabla}w + 2\tilde{\nabla}w\tilde{\nabla}f\right)\\ &\quad - \frac{\phi^{n-1}}{2\bar{H}w^3}\left(fR - 2\phi^{-2}\tilde{\Delta}f - 2(n-2)\phi^{-3}\tilde{\nabla}\phi\tilde{\nabla}f\right)(w^3 - w).\end{aligned}$$

Denote by p any function that satisfies

$$|\left(\frac{\partial}{\partial t}\right)^k \tilde{\nabla}^l p| \leq Ce^{-\alpha t},$$

for any $k, l \geq 0$, where C, α is uniform constants may depend on k, l. By Lemma 3.7 and Lemma 3.17, we have

$$\phi^{-1}\tilde{\nabla}\phi = p, \quad \phi^{-1}\tilde{\nabla}f = p.$$

Thus

$$\frac{\phi^{n-3}}{\bar{H}w}f\phi^{-1}\tilde{\nabla}\phi\tilde{\nabla}w = \left(\phi^{n-1}\tilde{\nabla}w\right)\left(\phi^{-1}\tilde{\nabla}\phi\right)\frac{f}{\bar{H}w\phi^2} = p.$$

Similarly,

$$\frac{\phi^{n-3}}{\bar{H}w}\tilde{\nabla}w\tilde{\nabla}f = \left(\phi^{n-1}\tilde{\nabla}w\right)\left(\phi^{-1}\tilde{\nabla}f\right)\frac{1}{\bar{H}w\phi} = p,$$

$$\frac{\phi^{n-1}}{\bar{H}w^3}\phi^{-2}\tilde{\Delta}f(w^3 - w) = \left(\phi^{n-1}(1-w^{-2})\right)\left(\phi^{-1}\tilde{\Delta}f\right)\frac{1}{\bar{H}\phi} = p,$$

and

$$\begin{aligned}&\frac{\phi^{n-1}}{\bar{H}w^3}\phi^{-3}\tilde{\nabla}\phi\tilde{\nabla}f(w^3 - w)\\ &= \left(\phi^{n-1}(1-w^{-2})\right)\left(\phi^{-1}\tilde{\nabla}f\right)\left(\phi^{-1}\tilde{\nabla}\phi\right)\frac{1}{\bar{H}\phi} = p.\end{aligned}$$

Hence,

$$\frac{\partial \mathfrak{m}}{\partial t} = \frac{n-1}{2}\phi^{n-2}\phi' v f(1-w^{-2}) + \frac{\phi^{n-3}f}{\bar{H}w}\tilde{\Delta}w - \frac{\phi^{n-1}fR}{2\bar{H}}(1-w^{-2}) + p.$$

On the other hand,

$$\begin{aligned}&\frac{n-1}{2}\phi^{n-2}\phi' v f(1-w^{-2}) - \frac{\phi^{n-1}fR}{2\bar{H}}(1-w^{-2})\\ &= \frac{\phi^{n-1}(1-w^{-2})}{2}f\left((n-1)\phi^{-1}\phi' v - \frac{R}{\bar{H}}\right) = p.\end{aligned}$$

Therefore,

$$\frac{\partial \mathfrak{m}}{\partial t} = \frac{\phi^{n-3} f}{\bar{H} w} \tilde{\Delta} w + p.$$

Note that

$$\tilde{\nabla} \mathfrak{m} = \frac{n-1}{2} \phi^{n-2} \tilde{\nabla} \phi (1 - w^{-2}) + \phi^{n-1} w^{-3} \tilde{\nabla} w,$$

and

$$\begin{aligned}\tilde{\Delta} \mathfrak{m} =& \frac{(n-1)(n-2)}{2} \phi^{n-3} |\tilde{\nabla} \phi|^2 (1 - w^{-2}) + \frac{n-1}{2} \phi^{n-2} \tilde{\Delta} \phi (1 - w^{-2}) \\ &+ 2(n-1) \phi^{n-2} w^{-3} \tilde{\nabla} \phi \tilde{\nabla} w - 3 \phi^{n-1} w^{-4} |\tilde{\nabla} w|^2 + \phi^{n-1} w^{-3} \tilde{\Delta} w.\end{aligned}$$

Thus,

$$\tilde{\Delta} \mathfrak{m} = -3 \phi^{n-1} w^{-4} |\tilde{\nabla} w|^2 + \phi^{n-1} w^{-3} \tilde{\Delta} w + p = \phi^{n-1} w^{-3} \tilde{\Delta} w + p.$$

Therefore,

$$\begin{aligned}\frac{\partial \mathfrak{m}}{\partial t} &= \frac{\phi^{n-3} f}{\bar{H} w} \left(\phi^{1-n} w^3 \tilde{\Delta} \mathfrak{m} + \phi^{1-n} w^3 p \right) + p \\ &= \frac{f w^2}{\bar{H} \phi^2} \tilde{\Delta} \mathfrak{m} + p = \frac{1}{n^2} \tilde{\Delta} \mathfrak{m} + p.\end{aligned}$$

By Lemma 3.18, we have $\tilde{g}_{ij} = \sigma_{ij} + p$, where σ_{ij} is the standard metric on $\mathbb{S}^n$. Thus $\tilde{\Delta} \mathfrak{m} = \Delta_0 \mathfrak{m} + p$, where Δ_0 is the standard Laplacian on $\mathbb{S}^n$. Now, by Lemma 2.6 in [**41**], we conclude that there exists a constant m_0, such that

$$|\mathfrak{m} - m_0| + |\nabla_0 \mathfrak{m}| + |\frac{\partial \mathfrak{m}}{\partial t}| \le C e^{-\alpha t}.$$ q.e.d.

Lemma 4.3 directly implies the following asymptotic expansion of w.

Lemma 4.4. *As $t \to \infty$, w satisfies*

$$w = 1 + m_0 \phi^{1-n} + p,$$

where $p = O(\phi^{1-n-\alpha})$ and $|\nabla_0 p| = O(\phi^{-n-\alpha})$. Here ∇_0 denotes the standard gradient on $(\mathbb{S}^n, \sigma)$.

4.3. ADM mass of g_η. We now verify that the metric g_η is asymptotically flat and we compute its ADM mass. Note that

$$g_\eta = f^2 dt^2 + g_t + (\eta^2 - f^2) dt^2 = \bar{g} + (w^2 - 1) f^2 dt^2,$$

where $\bar{g}$ is the metric on the Schwarzschild manifold $\mathbb{M}^{n+1}_m$ with mass m.

Let r be the radial coordinate in (3.1). Let $z = (z_1, \ldots, z_{n+1})$ denote the standard rectangular coordinates on the background Euclidean space

$$\mathbb{R}^{n+1} = \left([0, \infty) \times \mathbb{S}^n, dr^2 + r^2 \sigma \right).$$

Writing $\bar{g} = \bar{g}_{ij} dz_i dz_j$ and $g_\eta = g_{ij} dz_i dz_j$, we have

$$g_{ij} = \bar{g}_{ij} + b_{ij}, \tag{4.8}$$

where

$$b_{ij} = (w^2 - 1) f^2 \frac{\partial t}{\partial z_i} \frac{\partial t}{\partial z_j}.$$

We need to analyze the term $\frac{\partial t}{\partial z_i}$. As $r = |z|$,

$$\partial_{z_i} = \frac{z_i}{r} \partial_r + (\partial_{z_i})^T,$$

where $(\partial_{z_\alpha})^T$ is tangential to $\mathbb{S}^n$. By definition,

$$\frac{\partial t}{\partial z_i} = dt(\partial_{z_i}) = \langle \bar{\nabla} t, \partial_{z_i} \rangle_{\bar{g}} = f^{-1} \langle \nu, \partial_{z_i} \rangle_{\bar{g}}.$$

Plugging in $\nu = \frac{1}{v}(\partial_r - \frac{r^j \partial_j}{\phi^2})$, we have

$$\begin{aligned} \frac{\partial t}{\partial z_i} &= \frac{1}{fv} \left(\frac{z_i}{r} - \frac{r^j}{\phi^2} \langle \partial_j, (\partial_{z_i})^T \rangle_{\bar{g}} \right) \\ &= \frac{1}{fv} \left(\frac{z_i}{r} - r^j \langle \partial_j, (\partial_{z_i})^T \rangle_\sigma \right). \end{aligned} \tag{4.9}$$

Thus,

$$b_{ij} = \frac{w^2 - 1}{v^2} \left(\frac{z_i}{r} - r^k \langle \partial_k, (\partial_{z_i})^T \rangle_\sigma \right) \left(\frac{z_j}{r} - r^l \langle \partial_l, (\partial_{z_j})^T \rangle_\sigma \right).$$

By Lemma 4.4, Lemma 3.7 and the fact that $|(\partial_{z_i})^T| \le \frac{1}{r}$, we have

$$|b_{ij}| = O(|z|^{1-n}).$$

Similarly computation gives

$$|z| |\partial_z b_{ij}| + |z|^2 |\partial_z \partial_z b_{ij}| = O(|z|^{1-n}).$$

This shows that g_η is asymptotically flat.

Lemma 4.5. *The ADM mass of $g_\eta = \eta^2 dt^2 + g_t$ equals $m + m_0$.*

Proof. The ADM mass of g_η is given by

$$\frac{1}{2n\omega_n} \lim_{r \to \infty} \int_{\mathbb{S}^n} \left(\frac{\partial g_{\eta ij}}{\partial z_i} - \frac{\partial g_{\eta ii}}{\partial z_j} \right) r^{n-1} z_j d\sigma.$$

By (4.8) and the fact that the ADM mass of $\bar{g}$ is m, the above limit is equal to

$$m + \frac{1}{2n\omega_n} \lim_{r \to \infty} \int_{\mathbb{S}^n} \left(\frac{\partial b_{ij}}{\partial z_i} - \frac{\partial b_{ii}}{\partial z_j} \right) r^{n-1} z_j d\sigma. \tag{4.10}$$

Thus it suffices to calculate $\frac{\partial b_{ij}}{\partial z_i}$ and $\frac{\partial b_{ii}}{\partial z_j}$. We have

$$\begin{aligned}\frac{\partial b_{ij}}{\partial z_i} &= 2wf^2\frac{\partial w}{\partial z_i}\frac{\partial t}{\partial z_i}\frac{\partial t}{\partial z_j} + 2(w^2-1)f\frac{\partial f}{\partial z_i}\frac{\partial t}{\partial z_i}\frac{\partial t}{\partial z_j} \\ &\quad + (w^2-1)f^2\left(\frac{\partial^2 t}{\partial z_i^2}\frac{\partial t}{\partial z_j} + \frac{\partial t}{\partial z_i}\frac{\partial^2 t}{\partial z_j\partial z_i}\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial b_{ii}}{\partial z_j} &= 2wf^2\frac{\partial w}{\partial z_j}\left(\frac{\partial t}{\partial z_i}\right)^2 + 2(w^2-1)f\frac{\partial f}{\partial z_j}\left(\frac{\partial t}{\partial z_i}\right)^2 \\ &\quad + 2(w^2-1)f^2\frac{\partial t}{\partial z_i}\frac{\partial^2 t}{\partial z_j\partial z_i}.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{\partial b_{ij}}{\partial z_i} - \frac{\partial b_{ii}}{\partial z_j} &= 2wf^2\frac{\partial t}{\partial z_i}\left(\frac{\partial w}{\partial z_i}\frac{\partial t}{\partial z_j} - \frac{\partial w}{\partial z_j}\frac{\partial t}{\partial z_i}\right) \\ &\quad + 2(w^2-1)f\frac{\partial t}{\partial z_i}\left(\frac{\partial f}{\partial z_i}\frac{\partial t}{\partial z_j} - \frac{\partial f}{\partial z_j}\frac{\partial t}{\partial z_i}\right) \\ &\quad + (w^2-1)f^2\left(\frac{\partial^2 t}{\partial z_i^2}\frac{\partial t}{\partial z_j} - \frac{\partial t}{\partial z_i}\frac{\partial^2 t}{\partial z_j\partial z_i}\right).\end{aligned}$$

By Lemma 4.4, we have

$$\frac{\partial w}{\partial z_i} = (1-n)m_0\phi^{-n-1}z_i + O(\phi^{-n-\alpha}).$$

By Lemma 3.7 and (4.9), we have

$$\frac{\partial t}{\partial z_i} = \frac{1}{f}\frac{z_i}{\phi} + O(\phi^{-1-\alpha}).$$

Therefore,

$$2wf^2\frac{\partial t}{\partial z_i}\left(\frac{\partial w}{\partial z_i}\frac{\partial t}{\partial z_j} - \frac{\partial w}{\partial z_j}\frac{\partial t}{\partial z_i}\right) = O(\phi^{-n-\alpha}).$$

On other hand, by Lemma 3.17 and straightforward computation,

$$\frac{\partial f}{\partial z_i} = \frac{z_i}{n\phi} + O(\phi^{-\alpha}).$$

Thus,

$$2(w^2-1)f\frac{\partial t}{\partial z_i}\left(\frac{\partial f}{\partial z_i}\frac{\partial t}{\partial z_j} - \frac{\partial f}{\partial z_j}\frac{\partial t}{\partial z_i}\right) = O(\phi^{-n-\alpha}).$$

Again by Lemma 3.17, Lemma 3.7 and (4.9), we have

$$\frac{\partial^2 t}{\partial z_i^2} = \frac{n(n-2)}{\phi^2} + O(\phi^{-2-\alpha}),\ \ \frac{\partial^2 t}{\partial z_i\partial z_j} = -\frac{2n}{\phi^4}z_iz_j + O(\phi^{-2-\alpha}).$$

Thus,

$$(w^2-1)f^2\left(\frac{\partial^2 t}{\partial z_i^2}\frac{\partial t}{\partial z_j}-\frac{\partial t}{\partial z_i}\frac{\partial^2 t}{\partial z_j\partial z_i}\right)=2m_0\phi^{1-n}f^2\frac{n^2}{\phi^2}\frac{z_j}{f\phi}$$
$$=2nm_0\phi^{-1-n}z_j+O(\phi^{-n-\alpha}).$$

Therefore, we conclude

$$\frac{\partial b_{ij}}{\partial z_i}-\frac{\partial b_{ii}}{\partial z_j}=2nm_0\phi^{-1-n}z_j+O(\phi^{-n-\alpha}),$$

which implies that the ADM mass of g_η is $m+m_0$ by (4.10). q.e.d.

REMARK 4.1. A more geometric way to compute the ADM mass of g_η is as follows. The foliation $\{\Sigma_t\}$ is a family of nearly round hypersurfaces according to Definition 2.1 in [**36**]. Thus, if $\mathfrak{m}(g_\eta)$ is the mass of g_η, then by Theorem 1.2 in [**36**],

$$\begin{aligned}
&2n(n-1)\omega_n\mathfrak{m}(g_\eta)\\
&=\lim_{t\to\infty}\left(\frac{|\Sigma_t|}{\omega_n}\right)^{\frac{1}{n}}\int_{\Sigma_t}\left(R-\frac{n-1}{n}H_\eta^2\right)d\sigma\\
&=\lim_{t\to\infty}\left(\frac{|\Sigma_t|}{\omega_n}\right)^{\frac{1}{n}}\int_{\Sigma_t}\left(R-\frac{n-1}{n}\bar{H}^2+\frac{n-1}{n}(1-w^{-2})\bar{H}^2\right)d\sigma\\
&=2n(n-1)\omega_n(m+m_0),
\end{aligned}\tag{4.11}$$

where we have used the fact $\bar{g}$ has mass m and

$$\lim_{t\to\infty}\int_{\Sigma_t}(1-w^{-2})\bar{H}^2d\sigma=2n^2\omega_n m_0,\tag{4.12}$$

which follows from Lemma 4.4, Lemma 3.17 and Lemma 3.18.

Lemma 4.6.

$$\lim_{t\to\infty}\int_{\Sigma_t}N(\bar{H}-H_\eta)d\sigma=\lim_{t\to\infty}\int_{\Sigma_t}N\bar{H}(1-w^{-1})d\sigma=nm_0\omega_n.$$

Proof. Similar to (4.12), this is a direct consequence of Lemma 4.4, Lemma 3.17 and Lemma 3.18. q.e.d.

We summarize the results in Lemmas 4.1, 4.4, 4.5 and 4.6 in the following theorem.

Theorem 4.7. *Let $\Sigma^n\subset\mathbb{M}^{n+1}_m$ be a closed, star-shaped, 2-convex hypersurface with $\overline{\mathrm{Ric}}(\nu,\nu)\le 0$, where $\overline{\mathrm{Ric}}(\cdot,\cdot)$ is the Ricci curvature of $\mathbb{M}^{n+1}_m$ and ν is the outward unit normal to Σ. Let $\mathbb{E}$ denote the exterior of Σ^n in $\mathbb{M}^{n+1}_m$, which is swept by a family of star-shaped hypersurfaces $\{\Sigma_t\}_{0\le t\le\infty}$ that is a smooth solution to*

$$\frac{\partial X}{\partial t}=\frac{n-1}{2n}\frac{\sigma_1}{\sigma_2}\nu$$

with initial condition $\Sigma_0 = \Sigma^n$. On $\mathbb{E}$, writing the Schwarzschild metric $\bar{g}$ as

$$\bar{g} = f^2 dt^2 + g_t,$$

where g_t is the induced metric on Σ_t and $f = \frac{n-1}{2n}\frac{\sigma_1}{\sigma_2}$. Then, given any smooth function $\psi > 0$ on Σ, there exists a smooth function $\eta > 0$ on $\mathbb{E}$ such that

(i) *$\eta|_\Sigma = \psi$, the metric $g_\eta = \eta^2 dt^2 + g_t$ has zero scalar curvature, and η satisfies*

$$f^{-1}\eta = 1 + m_0 \phi^{1-n} + p,$$

where m_0 is a constant, $p = O(\phi^{1-n-\alpha})$ and $|\nabla_0 p| = O(\phi^{-n-\alpha})$;

(ii) *the Riemannian manifold $(\mathbb{E}, g_\eta)$ is asymptotically flat; and*

(iii) *the ADM mass $\mathfrak{m}(g_\eta)$ of g_η is given by*

$$\mathfrak{m}(g_\eta) = m + m_0 = m + \lim_{t\to\infty} \frac{1}{n\omega_n} \int_{\Sigma_t} N(\bar{H} - H_\eta) d\sigma.$$

Remark 4.2. Since $(\mathbb{E}, g_\eta)$ is foliated by $\{\Sigma_t\}_{0\le t\le\infty}$, which has positive mean curvature for each t, the boundary $\partial\mathbb{E} = \Sigma$ is outer minimizing in $(\mathbb{E}, g_\eta)$, meaning that Σ minimizes area among all hypersurfaces in $\mathbb{E}$ that enclose Σ.

5. Geometric applications

In this section, we give applications of results in Sections 2–4. First, we prove Theorem 1.1.

Proof of Theorem 1.1. Let $(\Omega^{n+1}, \breve{g})$ be a compact manifold given in Theorem 1.1. By assumptions (i), (ii) and the standard geometric measure theory, Σ_H minimizes area among all closed hypersurfaces in $(\Omega, \breve{g})$ that encloses Σ_H.

Let $\mathbb{E}$ denote the exterior of Σ^n in $\mathbb{M}^{n+1}_m$. Let $\eta > 0$ be the smooth function on $\mathbb{E}$ given by Theorem 4.7 with an initial condition

$$\eta|_\Sigma = f|_\Sigma H^{-1} H_m. \tag{5.1}$$

This condition implies that the mean curvature of Σ in $(\mathbb{E}, g_\eta)$ agrees with the mean curvature H of Σ_O in $(\Omega, \breve{g})$. Since Σ_O is isometric to $\Sigma = \partial\mathbb{E}$, we can attach $(\mathbb{E}, g_\eta)$ to $(\Omega, \breve{g})$ along $\Sigma = \Sigma_O$ by matching the Gaussian neighborhood of Σ in $(\mathbb{E}, g_\eta)$ to that of Σ_O in $(\Omega, \breve{g})$. Denote the resulting manifold by $\hat{M}$ and its metric by $\hat{h}$. By construction, $\hat{h}$ is Lipschitz across Σ and smooth everywhere else on $\hat{M}$; $\hat{h}$ has nonnegative scalar curvature away from Σ; and the mean curvature of Σ from both sides in $(\hat{M}, \hat{h})$ agree. Moreover, $\partial\hat{M} = \Sigma_H$ is a minimal hypersurface that is outer minimizing in $(\hat{M}, \hat{h})$. This outer minimizing property of Σ_H is guaranteed by the fact that Σ is outer minimizing in $(\mathbb{E}, g_\eta)$ and Σ_H minimizes area among closed hypersurfaces in $(\Omega, \breve{g})$ that encloses

$\Sigma_{_H}$. On such an $(\hat{M}, \hat{h})$, it is known that the Riemannian Penrose inequality, i.e. Theorem 1.4, still holds. (For a proof of this claim, see pages 279–280 in [**35**] for the case $n = 2$ and Proposition 3.1 in [**32**] for $2 \leq n \leq 6$). Therefore, we have

$$\mathfrak{m}(g_\eta) \geq \frac{1}{2} \left(\frac{|\Sigma_{_H}|}{\omega_n} \right)^{\frac{n-1}{n}} . \tag{5.2}$$

By (iii) in Theorem 4.7, this gives

$$m + \lim_{t \to \infty} \frac{1}{n\omega_n} \int_{\Sigma_t} N(\bar{H} - H_\eta) d\sigma \geq \frac{1}{2} \left(\frac{|\Sigma_{_H}|}{\omega_n} \right)^{\frac{n-1}{n}} . \tag{5.3}$$

On the other hand, since $\frac{\partial N}{\partial \nu} > 0$ as Σ_t is star-shaped and Σ_t has positive σ_1 and σ_2 in $\mathbb{M}^{n+1}_m$, Corollary 2.3 applies with $(\mathbb{N}, \bar{g})$ given by $\mathbb{M}^{n+1}_m$ to show that

$$\int_{\Sigma_t} N(\bar{H} - H_\eta) d\sigma$$

is monotone nonincreasing. At $\Sigma = \Sigma_0$, we have $H_m = \bar{H}$ and $H = H_\eta$. Therefore,

$$\begin{aligned} \int_\Sigma N(H_m - H) d\sigma &= \int_{\Sigma_0} N(\bar{H} - H_\eta) d\sigma \\ &\geq \lim_{t \to \infty} \int_{\Sigma_t} N(\bar{H} - H_\eta) d\sigma. \end{aligned} \tag{5.4}$$

Therefore, it follows from (5.3) and (5.4) that

$$m + \frac{1}{n\omega_n} \int_\Sigma N(H_m - H) d\sigma \geq \frac{1}{2} \left(\frac{|\Sigma_{_H}|}{\omega_n} \right)^{\frac{n-1}{n}} , \tag{5.5}$$

which proves (1.1). If equality in (5.5) holds, then $\int_{\Sigma_t} N(\bar{H} - H_\eta) d\sigma$ is a constant for all t. By Corollary 2.3, we have $\eta = f$ on $\mathbb{E}$, hence $H = H_m$ by (5.1). Consequently,

$$m = \frac{1}{2} \left(\frac{|\Sigma_{_H}|}{\omega_n} \right)^{\frac{n-1}{n}} .$$

This completes the proof of Theorem 1.1. q.e.d.

Remark 5.1. We conjecture that, when equality in (1.1) holds, $(\Omega, \breve{g})$ is isometric to the domain enclosed by Σ and the horizon boundary $\Sigma^S_{_H}$ in $\mathbb{M}^{n+1}_m$. It is clear from the above proof that in this case (5.2) becomes equality. Thus, if one can establish the rigidity statement for the Riemannian Penrose inequality on manifolds with corners (cf. [**33**, **41**, **31**]), then this conjecture will follow.

Next, we note an implication of Corollary 2.3 and Theorem 4.7 on the concept of Bartnik mass [**2**]. Given a pair (g, H), where g is a metric and H is a function on $\mathbb{S}^2$, the Bartnik mass of (g, H), which we denote by $\mathfrak{m}_{_B}(g, H)$, can be defined by

$$\begin{aligned}&\mathfrak{m}_{_B}(g, H)\\ =\ &\inf\left\{\mathfrak{m}(h) \,|\, (M, h) \text{ is an admissible extension of } (\mathbb{S}^2, g, H)\right\}.\end{aligned}$$

Here $\mathfrak{m}(h)$ is the ADM mass of (M, h) which is an asymptotically flat 3-manifold with boundary ∂M. (M, h) is called an *admissible extension of* $(\mathbb{S}^2, g, H)$ provided (M, h) has nonnegative scalar curvature, ∂M is isometric to $(\mathbb{S}^2, g)$, and the mean curvature of ∂M in (M, h) equals H under the identification of ∂M with $(\mathbb{S}^2, g)$ via the isometry. Moreover, it is assumed that either (M, h) contains no closed minimal surfaces or ∂M is outer minimizing in (M, h) (see [**2**, **4**, **5**, **26**]).

Theorem 5.1. *Given a pair (g, H) on $\mathbb{S}^2$, suppose $H > 0$ and $(\mathbb{S}^2, g)$ is isometric to a closed, star-shaped, convex surface Σ with $\overline{\mathrm{Ric}}(\nu, \nu) \leq 0$ in a spatial Schwarzschild manifold $\mathbb{M}^{n+1}_m$ with $m > 0$. Then*

$$\mathfrak{m}_{_B}(g, H) \leq m + \frac{1}{8\pi}\int_\Sigma N(H_m - H)\, d\sigma.$$

Moreover, equality holds if and only if $H = H_m$ and $\mathfrak{m}_{_B}(g, H) = m$.

Proof. Taking $n = 2$ in Theorem 4.7, let η be the function given on $\mathbb{E}$ with an initial condition $\eta|_\Sigma = f|_\Sigma H^{-1} H_m$. The asymptotically flat manifold $(\mathbb{E}, g_\eta)$ is an admissible extension of $(\mathbb{S}^2, g, H)$. Therefore, by (iii) in Theorem 4.7,

$$\mathfrak{m}_{_B}(g, H) \leq \mathfrak{m}(g_\eta) = m + \lim_{t\to\infty} \frac{1}{8\pi}\int_{\Sigma_t} N(\bar{H} - H_\eta)\, d\sigma. \tag{5.6}$$

By Corollary 2.3,

$$\int_\Sigma N(H_m - H)\, d\sigma \geq \lim_{t\to\infty} \int_{\Sigma_t} N(\bar{H} - H_\eta)\, d\sigma. \tag{5.7}$$

The inequality now follows from (5.6) and (5.7). If equality holds, then (5.6) and (5.7) are equalities. In this case, by Corollary 2.3, $\eta = f$. Therefore, $H = H_m$ and $\mathfrak{m}_{_B}(g, H) = m$. q.e.d.

Remark 5.2. Indeed our method shows the following is true – given a pair (g, H) on $\mathbb{S}^2$, suppose $(\mathbb{S}^2, g)$ is isometric to the boundary of a static, asymptotically flat manifold $(\mathbb{N}^3, \bar{g})$ with a positive static potential N. Suppose $(\mathbb{N}, \bar{g})$ satisfies:

(i) $\Sigma = \partial\mathbb{N}$ has positive σ_1 and σ_2;
(ii) the inverse curvature flow (1.21) in $(\mathbb{N}, \bar{g})$, with initial condition $\Sigma_0 = \Sigma$, admits a long time, smooth solution $\{\Sigma_t\}_{0\leq t<\infty}$ with $\frac{\partial N}{\partial \nu} > 0$; and

(iii) the warped metric g_η defined by (2.4), satisfying $R(g_\eta) = 0$ on $\mathbb{N}$ and $H_\eta = H$ at Σ, can be constructed on $\mathbb{N}$ such that g_η is asymptotically flat with

$$\mathfrak{m}(g_\eta) = \mathfrak{m}(\bar{g}) + \lim_{t\to\infty} \frac{1}{8\pi} \int_{\Sigma_t} N(\bar{H} - H_\eta)\, d\sigma.$$

Then, by Corollary 2.3, the Bartnik mass of (g, H) satisfies

$$\mathfrak{m}_{_B}(g, H) \le \mathfrak{m}(\bar{g}) + \frac{1}{8\pi} \int_{\Sigma} N(\bar{H} - H)\, d\sigma. \tag{5.8}$$

Here $\mathfrak{m}(\bar{g})$ is the ADM mass of $(\mathbb{N}, \bar{g})$ and $\bar{H}$ is the mean curvature of Σ in $(\mathbb{N}, \bar{g})$. Estimate (5.8) appeared as Conjecture 4.1 in [**34**].

6. Limits along isomeric embeddings of large spheres into Schwarzschild manifolds

In this section, we prove Theorem 1.2 which was inspired by the results of Fan, Shi and Tam [**18**]. We divide the proof into two parts, the existence of the embedding and the calculation of the limits.

6.1. Isometric embedding of large spheres. In [**38**], Nirenberg shows that a 2-sphere with positive Gauss curvature can be isometrically embedded in $\mathbb{R}^3$ as a strictly convex surface. By adopting the iteration scheme used in the proof of the openness part in [**38**], one can verify that a perturbation of a standard round sphere can be isometrically embedded in a 3-dimensional Schwarzschild manifold with small mass. This assertion, which is the main tool we use in this section, is indeed a special case of [**28**, Theorem 1] (see also [**11**]).

Proposition 6.1 ([**11**, **28**]). *Let σ be the standard metric on $\mathbb{S}^2$ with area 4π. There exists $\epsilon > 0$ and $\delta > 0$, such that if $\tilde{\sigma}$ is a metric on $\mathbb{S}^2$ with $\|\tilde{\sigma} - \sigma\|_{C^{2,\alpha}} < \epsilon$ and if m is a positive constant with $m < \delta$, then there exists an isometric embedding $\tilde{X}$ of $(\mathbb{S}^2, \tilde{\sigma})$ in*

$$\mathbb{M}^3_m = \left([2m, \infty) \times \mathbb{S}^2, \frac{1}{1 - \frac{2m}{\rho}} d\rho^2 + \rho^2 \sigma \right). \tag{6.1}$$

Moreover, $\tilde{X}$ can be chosen so that

$$\|\tilde{X} - X\|_{C^{2,\alpha}} \le C \|\tilde{\sigma} - \sigma\|_{C^{2,\alpha}}. \tag{6.2}$$

Here X is the isometric embedding of $(\mathbb{S}^2, \sigma)$ in $\mathbb{M}^3_m$ given by $X(\omega) = (1, \omega)$, $\forall\ \omega \in \mathbb{S}^2$.

Remark 6.1. Estimate (6.2) is not stated in the statement of theorems in [**11**, **28**], but it follows from both proofs therein.

We now consider an asymptotically flat 3-manifold $(M, \breve{g})$ given in Theorem 1.2. Precisely, this means that, outside a compact set, M

is diffeomorphic to $\mathbb{R}^3$ minus a ball and with respect to the standard coordinates on $\mathbb{R}^3$, $\breve{g}$ satisfies $\breve{g}_{ij} = \delta_{ij} + p_{ij}$ with

$$|p_{ij}| + |x||\partial p_{ij}| + |x|^2|\partial\partial p_{ij}| + |x|^3|\partial\partial\partial p_{ij}| = O(|x|^{-\tau}) \tag{6.3}$$

for some constant $\tau > \frac{1}{2}$, where ∂ denotes the partial derivative. Moreover, it is assumed that the scalar curvature of $\breve{g}$ is integrable on $(M, \breve{g})$. Under such assumptions, the ADM mass of $(M, \breve{g})$ is well defined, which we will denote by $\mathfrak{m}$.

Given a large constant $r > 0$, let $S_r = \{|x| = r\}$ denote the coordinate sphere in $(M, \breve{g})$. Let g_r be the induced metric on S_r and let $\tilde{g}_r = r^{-2} g_r$. Identifying S_r with $\mathbb{S}^2 = \{\ |y| = 1\}$ via a map $y = r^{-1}x$, one can deduce from (6.3) that

$$\|\tilde{g}_r - \sigma\|_{C^3} \le C r^{-\tau} \tag{6.4}$$

(see (2.17) in [**18**] for instance). Here σ is the standard metric on $\mathbb{S}^2$ with area 4π and $C > 0$ is a constant independent on r.

Let $m > 0$ be any fixed constant. Define $m_r = r^{-1}m$. Applying Proposition 6.1 and (6.4), we conclude, for sufficiently large r, there exists an isometric embedding

$$\tilde{X}_r : (\mathbb{S}^2, \tilde{g}_r) \longrightarrow \mathbb{M}^3_{m_r} = \left([2m_r, \infty) \times \mathbb{S}^2, \frac{1}{1 - \frac{2m_r}{\rho}} d\rho^2 + \rho^2 \sigma \right)$$

satisfying

$$\|\tilde{X}_r - X_\sigma\|_{C^{2,\alpha}} = O(r^{-\tau}), \tag{6.5}$$

where $X_\sigma(\omega) = (1, \omega)$, $\forall\ \omega \in \mathbb{S}^2$. It follows from (6.5) that $\tilde{X}_r(\mathbb{S}^2)$ is star-shaped and convex; moreover, if $\tilde{\nu}_r$ is the outward unit normal to $\tilde{X}_r(\mathbb{S}^2)$, then

$$\tilde{\nu}_r = \left(1 + O(r^{-\tau})\right) \partial_\rho + O(r^{-\tau}) \partial_{\omega_1} + O(r^{-\tau}) \partial_{\omega_2}, \tag{6.6}$$

where ω_i, $i = 1, 2$, are local coordinates on $\mathbb{S}^2$.

Let $\overline{\mathrm{Ric}}_r$ denote the Ricci curvature of $\mathbb{M}^3_{m_r}$. In the rotationally symmetric form, it is given by $\overline{\mathrm{Ric}}_r = m_r \rho^{-3} \Psi$, where

$$\Psi = -\frac{2}{1 - \frac{2m_r}{\rho}} d\rho^2 + \rho^2 \sigma.$$

By (6.5) and (6.6),

$$\begin{aligned} \overline{\mathrm{Ric}}_r(\tilde{\nu}_r, \tilde{\nu}_r) &= m_r \rho^{-3} \Psi(\tilde{\nu}_r, \tilde{\nu}_r) \\ &= -2m_r \left(1 + O(r^{-\tau})\right). \end{aligned} \tag{6.7}$$

In particular, $\overline{\mathrm{Ric}}_r(\tilde{\nu}_r, \tilde{\nu}_r) < 0$ for large r.

The map $\tilde{X}_r$ leads to an isometric embedding of (S_r, g_r) in $\mathbb{M}^3_m$ because $\mathbb{M}^3_m$ and $\mathbb{M}^3_{m_r}$ only differer by a constant scaling. More precisely, consider the map

$$F_r : \mathbb{M}^3_{m_r} \longrightarrow \mathbb{M}^3_m,$$

where $F_r(\rho,\omega)=(r\rho,\omega)$. Define $X_r = F_r \circ \tilde{X}_r$, then

$$X_r : (S_r, g_r) \longrightarrow \mathbb{M}^3_m$$

is an isometric embedding such that $X_r(S_r)$ is a star-shaped, convex surface with

$$\overline{\mathrm{Ric}}(\nu_r,\nu_r) = -2mr^{-3}\left(1+O(r^{-\tau})\right). \tag{6.8}$$

Here ν_r is the outward unit normal to $X_r(\mathbb{S}^2)$ in $\mathbb{M}^3_m$ and (6.8) follows from (6.7). Thus, we have proved the first part of Theorem 1.2 on the existence of the desired isometric embedding of (S_r, g_r) into $\mathbb{M}^3_m$.

6.2. Evaluation of the limits. To prove the remaining part of Theorem 1.2, we write $X_r = (\rho_r, \theta_r)$. By (6.5), we have

$$\|\rho_r - r\|_{C^{2,\alpha}} = O(r^{1-\tau}). \tag{6.9}$$

Similarly, if H_m denotes the mean curvature of $X_r(S_r)$ in $\mathbb{M}^3_m$, then (6.5) gives

$$H_m = 2r^{-1} + O(r^{-\tau-1}). \tag{6.10}$$

We first compute $\int_{S_r} NH d\sigma$, where $N = \left(1-\frac{2m}{\rho}\right)^{\frac{1}{2}}$ is the static potential on $\mathbb{M}^3_m$ and H is the mean curvature of S_r in $(M,\breve{g})$. Let $A(r)$ be the area of (S_r, g_r). By [18, Lemma 2.1],

$$H = 2r^{-1} + O(r^{-1-\tau}) \quad \text{and} \quad A(r) = 4\pi r^2 + O(r^{2-\tau}).$$

By [18, Lemma 2.2],

$$\int_{S_r} H d\sigma = r^{-1}A(r) + 4\pi r - 8\pi\mathfrak{m} + o(1).$$

Therefore, by (6.9),

$$\begin{aligned}\int_{S_r} NH d\sigma &= \int_{S_r}\left(1-mr^{-1}\right)H d\sigma + o(1)\\ &= r^{-1}A(r) + 4\pi r - 8\pi\mathfrak{m} - 8\pi m + o(1).\end{aligned} \tag{6.11}$$

Next, we compute $\int_{S_r} NH_m d\sigma$. Identifying S_r with its image $\Sigma_r = X_r(S_r)$, we carry out the computation in $\mathbb{M}^3_m$. Following notations in Section 3, we rewrite the Schwarzschild metric $g_m = \frac{1}{N^2}d\rho^2 + \rho^2\sigma$ as

$$g_m = ds^2 + \phi^2(s)\sigma$$

by setting $s = \int_{2m}^{\rho} \frac{1}{N(t)}dt$. Then $\phi(s)=\rho$ and $\phi'(s)=N$. Define

$$\Phi(s) = \int_0^s \phi(t)dt \quad \text{and} \quad u = \langle \phi\partial_s, \nu_r\rangle.$$

On Σ_r, (3.4) becomes

$$\Phi_{;ij} = \phi' g_{r\,ij} - h_{ij}u, \tag{6.12}$$

where h is the second fundamental form of Σ_r. Taking trace of (6.12) gives

$$0 = 2\int_{\Sigma_r} \phi' d\sigma - \int_{\Sigma_r} H_m u\, d\sigma. \tag{6.13}$$

Now we apply [23, Lemma 2.5] to get another Minkowski type identity. Precisely, let $\sigma_2^{ij} = \frac{\partial \sigma_2}{\partial h_{ij}} = \sigma_1 {g_r}^{ij} - h^{ij}$. Contracting σ_2^{ij} with Φ_{ij} shows

$$\int_{\Sigma_r} \sigma_2^{ij} \Phi_{;ij} d\sigma = \int_{S_r} H_m \phi' d\sigma - 2\int_{S_r} \sigma_2 u\, d\sigma. \tag{6.14}$$

Integrating by parts and applying the Codazzi equation, we have

$$\int_{S_r} \sigma_2^{ij} \Phi_{;ij} d\sigma = -\int_{S_r} (\sigma_2^{ij})_{;j} \Phi_{;i}\, d\sigma = \int_{S_r} \overline{\mathrm{Ric}}(\nu_r, \nabla\Phi)\, d\sigma,$$

where $\nabla\Phi$ is the gradient of Φ on Σ_r. By (6.9),

$$|\nabla\Phi|^2 = {g_r}^{ij} \Phi_{;i} \Phi_{;j} = O(r^{2-2\tau}). \tag{6.15}$$

This combined with the fact $|\overline{\mathrm{Ric}}(\nu_r, \cdot)| = O(r^{-3})$ shows

$$\int_{S_r} \sigma_2^{ij} \Phi_{;ij} d\sigma = o(1).$$

Therefore, by (6.14),

$$\int_{S_r} H_m \phi'\, d\sigma = 2\int_{S_r} \sigma_2 u\, d\sigma + o(1). \tag{6.16}$$

Note that $u^2 = |\bar\nabla\Phi|^2 - |\nabla\Phi|^2$, where $\bar\nabla$ denotes the gradient on $\mathbb{M}^3_m$. Thus, by (6.15),

$$u = r + O(r^{1-\tau}). \tag{6.17}$$

Now let K be the Gauss curvature of (S_r, g_r). By [18, Lemma 2.1], if we let $\bar K = K - r^{-2}$, then $\bar K = O(r^{-2-\tau})$. Thus, by the Gauss equation and (6.8),

$$\sigma_2 = K + \overline{\mathrm{Ric}}(\nu_r, \nu_r) = \bar K + r^{-2} - 2mr^{-3} + o(r^{-3}).$$

Following the steps in [18], we have

$$\begin{aligned}
\int_{\Sigma_r} H_m \phi' d\sigma &= 2\int_{\Sigma_r} (\bar K + r^{-2}) u\, d\sigma - 4mr^{-3} \int_{\Sigma_r} u\, d\sigma + o(1) \\
&= 2r^{-2} \int_{\Sigma_r} \langle \bar\nabla\Phi, \nu_r\rangle dvol + 2\int_{S_r} \bar K u\, d\sigma - 16\pi m + o(1) \\
&= 6r^{-2} \int_{\Omega_r} \phi' \mathrm{dvol} + 2r \int_{S_r} (K - r^{-2})\, d\sigma - 16\pi m + o(1) \\
&= 6r^{-2} \int_{\Omega_r} \phi' \mathrm{dvol} + 8\pi r - 2r^{-1} A(r) - 16\pi m + o(1),
\end{aligned} \tag{6.18}$$

where Ω_r is the bounded domain enclosed by Σ_r and the horizon boundary of $\mathbb{M}^3_m$ and $d$vol is the volume element on $\mathbb{M}^3_m$.

Next, let $\bar{H}_m = H_m - 2r^{-1}$. By (6.10), $\bar{H} = O(r^{-1-\tau})$. By (6.13),

$$\begin{aligned} 2\int_{\Sigma_r} \phi'\, d\sigma = \int_{\Sigma_r} H_m u\, d\sigma &= \int_{\Sigma_r} (2r^{-1} + \bar{H}_m) u\, d\sigma \\ &= 6r^{-1}\int_{\Omega_r} \phi' d\mathrm{vol} + \int_{\Sigma_r} \bar{H}_m u\, d\sigma. \end{aligned}$$

Since $u = r + O(r^{1-\tau})$ and $\phi' = N = 1 + O(r^{-1})$, we have $u = r\phi' + O(r^{1-\tau})$. Thus,

(6.19)
$$\begin{aligned} 2\int_{\Sigma_r} \phi'\, d\sigma &= 6r^{-1}\int_{\Omega_r} \phi' d\mathrm{vol} + \int_{\Sigma_r} \bar{H}_m(r\phi' + O(r^{1-\tau}))\, d\sigma \\ &= 6r^{-1}\int_{\Omega_r} \phi' d\mathrm{vol} + r\int_{\Sigma_r} H_m\phi'\, d\sigma - 2\int_{\Sigma_r}\phi'\, d\sigma + O(r^{2-2\tau}). \end{aligned}$$

Since $\phi' = N = 1 - mr^{-1} + O(r^{-1-\tau})$, we also have

$$\int_{\Sigma_r} \phi'\, d\sigma = A(r) - 4\pi m r + O(r^{1-\tau}). \tag{6.20}$$

Thus, it follows from (6.19) and (6.20) that

$$\int_{\Sigma_r} H_m\phi'\, d\sigma = -6r^{-2}\int_{\Omega_r}\phi' dvol + 4r^{-1}A(r) - 16\pi m + o(1). \tag{6.21}$$

Combining (6.18) and (6.21), and replacing ϕ' by N, we have

$$\int_{S_r} NH_m\, d\sigma = 4\pi r + \frac{A(r)}{r} - 16\pi m + o(1). \tag{6.22}$$

By (6.11) and (6.22), we therefore conclude

$$\int_{S_r} N(H_m - H)\, d\sigma = -8\pi m + 8\pi\mathfrak{m} + o(1),$$

or equivalently

$$\lim_{r\to\infty}\left(m + \frac{1}{8\pi}\int_{S_r} N(H_m - H)\, d\sigma\right) = \mathfrak{m},$$

which proves (1.3).

To prove (1.4), by (6.18) and (6.21), we also have

$$\int_{\Omega_r} N\, d\mathrm{vol} = \int_{\Omega_r}\phi'\, d\mathrm{vol} = \frac{1}{2}rA(r) - \frac{2}{3}\pi r^3 + o(r^2). \tag{6.23}$$

Let $V(r)$ be the volume of the region enclosed by S_r in $(M, \breve{g})$. By (2.28) in [**18**],

$$V(r) = \frac{1}{2}rA(r) - \frac{2}{3}\pi r^3 + 2\pi\mathfrak{m}r^2 + o(r^2). \tag{6.24}$$

Hence, it follows from (6.23) and (6.24) that

$$\int_{\Omega_r} N\, d\mathrm{vol} - V(r) = -2\pi\mathfrak{m}r^2 + o(r^2). \tag{6.25}$$

Next, let $V_m(r)$ denote the volume of Ω_r in $\mathbb{M}^3_m$. We claim

$$\int_{\Omega_r} N\, d\mathrm{vol} = V_m(r) - 2\pi m r^2 + o(r^2). \tag{6.26}$$

To see this, let D_ρ denote the region in $\mathbb{M}^3_m$ bounded by the rotationally symmetric sphere with area $4\pi\rho^2$ and the horizon boundary. Let $\rho_0 > 2m$ be a fixed constant such that, for any $\rho > \rho_0$,

$$\left| N - \left(1 - \frac{m}{\rho}\right)\right| \le C_1 \rho^{-2}, \tag{6.27}$$

where $C_1 > 0$ is independent on ρ. By (6.9) and (6.27), for large r, we have

$$\begin{aligned} \int_{\Omega_r} N dvol &= \int_{\Omega_r \setminus D_{\rho_0}} N\, d\mathrm{vol} + O(1) \\ &= \int_{\Omega_r \setminus D_{\rho_0}} \left(1 - \frac{m}{\rho}\right) d\mathrm{vol} + O(r) \\ &= V_m(r) - \int_{\Omega_r \setminus D_{\rho_0}} \frac{m}{\rho}\, d\mathrm{vol} + O(r). \end{aligned} \tag{6.28}$$

By (6.9), we also have

$$\int_{(D_{r - Cr^{1-\tau}}) \setminus D_{\rho_0}} \rho^{-1}\, d\mathrm{vol} \le \int_{\Omega_r \setminus D_{\rho_0}} \rho^{-1}\, d\mathrm{vol} \le \int_{(D_{r + Cr^{1-\tau}}) \setminus D_{\rho_0}} \rho^{-1}\, d\mathrm{vol},$$

which implies

$$\int_{\Omega_r \setminus D_{\rho_0}} \rho^{-1} dvol = 2\pi r^2 + o(r^2). \tag{6.29}$$

Thus, (6.26) follows from (6.28) and (6.29). By (6.25) and (6.26), we conclude that

$$V(r) - V_m(r) = 2\pi(\mathfrak{m} - m) r^2 + o(r^2),$$

which proves (1.4) of Theorem 1.2.

We end this paper with the following corollary.

Corollary 6.2. *Let $(M^3, \breve{g})$ be an asymptotically flat 3-manifold with nonnegative scalar curvature, with boundary ∂M being an outer minimizing minimal surface (with one or more components). Let S_r denote the large coordinate sphere in $(M^3, \breve{g})$ with the induced metric g_r. Let $m = \sqrt{\frac{|\partial M|}{16\pi}}$. For large r, let X_r be the isometric embedding of (S_r, g_r) into $\mathbb{M}^3_m$ given by Theorem 1.2. Let $V(r)$ and $V_m(r)$ be the volume of the region enclosed by S_r in $(M^3, \breve{g})$ and the region enclosed by $X_r(S_r)$*

in $\mathbb{M}^3_m$, *respectively. Then*

$$\lim_{r\to\infty} \frac{V(r) - V_m(r)}{2\pi r^2} \text{ exists and is } \geq 0,$$

and "=" holds if and only if $(M^3, \breve{g})$ *is isometric to* $\mathbb{M}^3_m$.

Proof. This follows directly from (1.4) and the 3-dimensional Riemannian Penrose inequality. q.e.d.

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