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CODIMENSION TWO HOLOMORPHIC FOLIATIONS

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Abstract

This paper is devoted to the study of codimension two holomorphic foliations and distributions. We prove the stability of complete intersection of codimension two distributions and foliations in the local case. Converserly we show the existence of codimension two foliations which are not contained in any codimension one foliation. We study problems related to the singular locus and we classify homogeneous foliations of small degree.

1. Introduction

There are many works devoted to the study of codimension one holomorphic foliations on complex manifolds. The local theory is well understood in small dimensions (2 and 3), with results concerning reduction of singularities ([**33**], and [**7**]) and applications to unfolding theory, topological classification ([**27**] and [**28**]), spaces of moduli ([**29**]), existence and construction of invariant hypersurfaces [**5**], first integrals ([**30**] and [**14**]), among other topics.

In the global case, there is an intensive activity concerning the description of the "irreducible components" of the space of codimension one holomorphic foliations on a compact complex manifold ([**9**], [**3**] and [**12**]). One of the most popular challenges is to know if any codimension one foliation on $\mathbb{P}^n$, $n \geq 3$, is either the meromorphic pull-back of a foliation on a complex surface, or has a "geometric" transverse structure ([**13**] and [**11**]).

In the present work, we focus our attention on singular foliations and distributions of codimension q, $q \geq 2$, with special emphasis in the case $q = 2$. Local and global results are obtained. For example, a way to construct a singular codimension two distribution is to intersect two singular codimension one distributions. In the local case we prove in theorem 2 the "stability" of such construction, under natural assumptions. As a consequence, using Malgrange's singular Frobenius theorem, we show the persistence of first integrals (theorem 1). Conversely, we

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prove the existence of codimension two foliations which are not "contained" in any codimension one foliation; this fact is proved in the local context and on rational manifolds (see proposition 3, corollary 1 and remark 2.5).

Next the following problem is studied: is there a germ at $0 \in \mathbb{C}^4$ of codimension two foliation with an isolated singularity at 0? Indeed, there are examples of holomorphic codimension two distributions on $\mathbb{C}^4$ with an isolated singularity at $0 \in \mathbb{C}^4$. An example of this type was given in [**19**] in the context of vector bundles on $\mathbb{P}^3$: it is defined by a homogeneous 2-form on $\mathbb{C}^4$, that is a 2-form with coefficients homogeneous of the same degree. In contrast with this example we prove in theorem 3 that a codimension two foliation on $\mathbb{C}^4$, defined by a homogeneous 2-form with a singularity at $0 \in \mathbb{C}^4$, has always a straight line in its singular set; in other words the singular set has dimension ≥ 1.

Finally, we describe with some details homogeneous foliations of small degree. That description is related to the classification of codimension one foliations of degree ≤ 2 on $\mathbb{P}^n$, $n \geq 3$.

2. Definitions and some results

2.1. Local definitions. A holomorphic singular distribution of codimension q (or dimension $n-q$) on a Stein open set $U \subset \mathbb{C}^n$, $0 < q < n$, can be defined by a holomorphic q-form η which is locally decomposable outside the singular set $Sing(\eta) := \{z \in U \,|\, \eta_z = 0\}$, in the sense that any $z_o \in U \setminus Sing(\eta)$ has an open neighborhood $V \subset U$ such that

$$\eta|_V = \omega_1 \wedge ... \wedge \omega_q, \tag{1}$$

where $\omega_1, ..., \omega_q \in \Omega^1(V)$. It follows that we can define in $U \setminus Sing(\eta)$ a holomorphic distribution $\mathcal{D}_\eta$ of codimension q by

$$\mathcal{D}_\eta(p) = \{v \in T_pU \,|\, i_v\eta(p) = 0\}.$$

If $p \in U \setminus Sing(\eta)$ and $\omega_1, ..., \omega_q$ are as in (1) then

$$\mathcal{D}_\eta(p) = \bigcap_{1 \leq j \leq q} Ker(\omega_j(p)).$$

A q-form η satisfying (1) is said to be locally decomposable.

A q-form η which satisfies (1) is integrable if it satisfies Frobenius' integrability condition:

$$d\omega_j \wedge \eta = 0 \ , \ \forall \ j = 1, ..., q \, , \text{ on the open set } U. \tag{2}$$

If η satisfies (1) and (2) then the distribution $\mathcal{D}_\eta$ is integrable and so η defines a holomorphic codimension q foliation on $U \setminus Sing(\eta)$. This foliation will be denoted by $\mathcal{F}_\eta$.

The leaves of $\mathcal{F}$ are the immersed codimension q submanifolds $\mathcal{L} \subset U \setminus Sing(\eta)$ for which the tangent space to $\mathcal{L}$ at $m \in U$ is $T_m\mathcal{L} := \mathcal{D}_\eta(m)$.

When $Sing(\eta) = \emptyset$ then the integrability condition (2) is equivalent to the existence of a holomorphic 1-form θ such that

$$d\eta = \eta \wedge \theta. \tag{3}$$

When $Sing(\eta) \neq \emptyset$ then (3) is only true locally in $U \setminus Sing(\eta)$, unless we allow θ to be meromorphic.

Example 1. Complete intersection. Let $\mathcal{F}_1, ..., \mathcal{F}_q$ be q foliations of codimension one defined by integrable 1-forms $\omega_1, ..., \omega_q \in \Omega^1(U)$, $1 \leq q < n$ ($\omega_k \wedge d\omega_k = 0$, $\forall k$) such that $\eta := \omega_1 \wedge ... \wedge \omega_q \not\equiv 0$. The foliation complete intersection $\mathcal{F} = \mathcal{F}_1 \cap ... \cap \mathcal{F}_q$ is associated to the q 1-forms ω_i, $1 \leq i \leq q$. The leaves of $\mathcal{F}$ are the connected components of the intersection of the leaves $\mathcal{L}_k$ of $\mathcal{F}_k$, $1 \leq k \leq q$.

Example 2. Foliations associated to a Lie algebra of vector fields or to an action of a Lie group. Let $\mathcal{L}$ be a Lie algebra of vector fields defined on an open Stein subset $U \subset \mathbb{C}^n$. Given $m \in U$ set

$$d(m) = dim_{\mathbb{C}} \langle X(m) \,|\, X \in \mathcal{L} \rangle \,.$$

Let $d = max\{d(m) \,|\, m \in U\}$ and assume that $1 \leq d < n$. Since U is connected, then the set $Z = \{m \in U \,|\, d(m) < d\}$ is a proper analytic subset of U, so that $V = U \setminus Z$ is open dense in U and connected. In particular, $\mathcal{L}$ defines a dimension d distribution on V

$$\mathcal{L}(m) = \langle X(m) \,|\, X \in \mathcal{L} \rangle \;,\; m \in V.$$

The Lie algebra $\mathcal{L}$ defines a codimension $q = n - d$ foliation $\mathcal{F}_{\mathcal{L}}$ on V. The foliation $\mathcal{F}_{\mathcal{L}}$ can be extended to U as a singular foliation with singular set Z.

When $\mathcal{L}$ is associated to a group action $G \times U \to U$ we will say that $\mathcal{F}_{\mathcal{L}}$ is associated to the action of G.

Remark 2.1. If $\eta \in \Omega^k(U)$ is integrable and $cod(Sing(\eta)) = 1$ then we can write $\eta = h.\eta'$, where $\eta' \in \Omega^k(U)$ and $cod(Sing(\Omega')) \geq 2$. We would like to observe that η' is also integrable. The foliation $\mathcal{F}_{\eta'}$ can be considered as an "extension" of $\mathcal{F}_\eta$.

Remark 2.2. If η is an integrable q-form and $d\eta \not\equiv 0$ then relation (3) implies that $d\eta$ is locally decomposable outside $Sing(\eta)$. Since $d\eta$ is closed it is integrable and defines a singular foliation of codimension $q+1$. Relation (3) implies also that any leaf of $d\eta$ is η-invariant, in the sense that, either it is contained in a leaf of η, or in $Sing(\eta)$.

Example 3. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the codimension one foliations of $\mathbb{C}^n$ defined by the 1-forms $\eta_j = x_1...x_n.\omega_j$, $j = 1, 2$, where ω_1 and ω_2 are the logarithmic closed forms

$$\omega_1 = \sum_{j=1}^{n} \lambda_j \frac{dx_j}{x_j} \;,\; \omega_2 = \sum_{j=1}^{n} \mu_j \frac{dx_j}{x_j} \;,\; \lambda_j \,,\, \mu_j \in \mathbb{C}^* \,,\, 1 \leq j \leq n.$$

We assume that ω_1 and ω_2 are not colinear, which is equivalent to the non-colinearity of the vectors $\lambda = (\lambda_1, ..., \lambda_n)$ and $\mu = (\mu_1, ..., \mu_n)$. The intersection $\mathcal{F}_1 \cap \mathcal{F}_2$ is defined outside $\Sigma := \bigcup_j \{x_j = 0\}$ by

$$\omega_1 \wedge \omega_2 = \sum_{i<j} (\lambda_i \mu_j - \lambda_j \mu_i) \frac{dx_i}{x_i} \wedge \frac{dx_j}{x_j}.$$

Note that $\omega_1 \wedge \omega_2(m) \neq 0$, for all $m \in \mathbb{C}^n \setminus \Sigma$. Moreover, the divisor of poles of $\omega_1 \wedge \omega_2$ is $x_1 ... x_n$ and the form

$$\eta := x_1 ... x_n . \omega_1 \wedge \omega_2 = \sum_{i<j} (\lambda_i \, \mu_j - \lambda_j \, \mu_i) \, x_1 ... \widehat{x_i} ... \widehat{x_j} ... x_n \, dx_i \wedge dx_j$$

is holomorphic on $\mathbb{C}^n$, so that η defines the codimension two foliation $\mathcal{F}_1 \cap \mathcal{F}_2 := \mathcal{F}$ on $\mathbb{C}^n$. By convention, $\widehat{x_i}$ means the omission of the factor x_i in the product.

Observe that the hyperplanes $(x_j = 0)$, $1 \leq j \leq n$, are $\mathcal{F}$-invariant. For instance, when $j = 1$ we have $\eta|_{(x_1=0)} = dx_1 \wedge \eta_1$, where

$$\eta_1 = \sum_{j>1} (\lambda_1 \, \mu_j - \lambda_j \, \mu_1) \, x_2 ... \widehat{x_j} ... x_n \, dx_j,$$

and $\eta_1 \not\equiv 0$, because otherwise λ and μ would be colinear. In particular, $Sing(\eta)$ has codimension ≥ 2 and is contained in Σ. In fact, if $\lambda_i \, \mu_j - \lambda_j \, \mu_i \neq 0$ for all $i < j$ then

$$Sing(\eta) = \bigcup_{i<j<k} (x_i = x_j = x_k = 0)$$

has codimension three.

Another fact, is that η is not decomposable, whereas $\frac{1}{x_1 ... x_n} \eta = \omega_1 \wedge \omega_2$.

2.2. Globalization. Let M be a complex manifold of dimension $n \geq 2$. A holomorphic (singular) foliation $\mathcal{F}$ of codimension q,$1 \leq q < n$, on M is defined by a covering $(U_j)_{j \in J}$ of M by Stein open sets, a collection of integrable q-forms $(\eta_j)_{j \in J}$, with $\eta_j \in \Omega^q(U_j)$ and $cod(Sing(\eta_j)) \geq 2$, and a collection $(g_{ij})_{U_i \cap U_j \neq \emptyset}$ with $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$, satisfying the glueing condition $\eta_i = g_{ij} \, \eta_j$ on $U_i \cap U_j \neq \emptyset$. If $U_i \cap U_j \neq \emptyset$ then the glueing condition implies that the leaves of $\mathcal{F}_{\eta_i}|_{U_i \cap U_j}$ coincides with the leaves of $\mathcal{F}_{\eta_j}|_{U_i \cap U_j}$ and that

$$Sing(\eta_i)|_{U_i \cap U_j} = Sing(\eta_j)|_{U_i \cap U_j}. \tag{4}$$

Relation (4) implies that

$$Sing(\mathcal{F}) := \bigcup_{j \in J} Sing(\eta_j)$$

is an analytic subset of M of codimension ≥ 2. Observe that this defines a non-singular foliation of codimension q on the set $M \setminus Sing(\mathcal{F})$.

2.3. Complete intersection of foliations. Let $\mathcal{F}_1, ..., \mathcal{F}_q$ be codimension one foliations on some polydisc $U \subset \mathbb{C}^n$, defined by integrable 1-forms $\omega_1, ..., \omega_q$ such that $\eta' := \omega_1 \wedge ... \wedge \omega_q \not\equiv 0$ (generically independent). Note that there exists $f \in \mathcal{O}(U)$ and $\eta \in \Omega^q(U)$ such that $\eta' = f.\, \eta$ and $cod(Sing(\eta)) \geq 2$. Clearly, η is integrable.

By definition, the foliation $\mathcal{F}_\eta$, defined by η, is the *topological intersection* of the foliations $\mathcal{F}_1, ..., \mathcal{F}_q$. When $cod(Sing(\eta')) \geq 2$ (or equivalently $f \in \mathcal{O}^*(U)$) we will say that $\mathcal{F}$ is a *complete intersection.* Clearly, these two definitions can be germified or globalized (as in §2.2).

Remark 2.3. We will see in § 2.7 that there are germs of codimension two foliations $\mathcal{F}$ that are not topological intersections of two codimension one foliations. In this case, $\mathcal{F}$ is defined by a germ of integrable 2-form, say η, which is meromorphically decomposable, $\eta = \alpha \wedge \beta$, but for any such decomposition, neither α nor β is integrable.

A direct consequence of a theorem due to Malgrange (cf. [**24**] and [**25**]) is the following:

Theorem 2.1. *Let η be a germ at $0 \in \mathbb{C}^n$ of integrable q-form holomorphically decomposable, $\eta = \alpha_1 \wedge ... \wedge \alpha_q$. If $cod(Sing(\eta)) \geq 3$ then $\mathcal{F}_\eta$, the foliation defined by η, is a complete intersection. More precisely, there are $f_1, ..., f_q \in \mathcal{O}_n$ and $h \in \mathcal{O}_n^*$ such that*

$$\eta = h.\, df_1 \wedge ... \wedge df_q.$$

On the other hand, as we have seen in example 3 of § 2.1, for generic values of λ_i and μ_i, the form $\eta = \sum_{i<j} (\lambda_i \mu_j - \lambda_j \mu_i)\, x_1 ... \widehat{x_i} ... \widehat{x_j} ... x_n\, dx_i \wedge dx_j$ is integrable and satisfies $cod(Sing(\eta)) \geq 3$, but $\mathcal{F}_\eta$ is not a complete intersection. Therefore, the hypothesis of holomorphic decomposability in Malgrange's theorem is crucial.

2.4. Special case: $q = 2$ and $n = 4$. Let η be a 2-form on a polydisc $U \subset \mathbb{C}^4$. We fix coordinates x_1, x_2, x_3, x_4 and a non-vanishing 4-form, for instance, $\nu = dx_1 \wedge ... \wedge dx_4$. The 3-form $d\eta$ can be written as

$$d\eta = i_X\, \nu,$$

where X is a holomorphic vector field on U, called the *rotational* of η: $X = rot(\eta)$. The foliation associated to the rotational is independent of the choice of ν. On the other hand, if we change η by $h\, \eta$, where $h \in \mathcal{O}^*(U)$, then $rot(h\, \eta)$ and $rot(\eta)$ are not in general colinear. Although this notion is not intrinsic, it is convenient (see [**20**]). For instance, in the two following propositions, the first valid in any dimension.

Proposition 1. *Let $\eta \in \Omega^2(U)$, a 2-form on an open set U of $\mathbb{C}^n$. Then η is meromorphically decomposable if, and only if, $\eta^2 = \eta \wedge \eta = 0$.*

The proof of proposition 1 can be found in [**16**] page 211.

Proposition 2. *Let U be a domain of $\mathbb{C}^4$ and $\eta \in \Omega^2(U)$, $\eta \not\equiv 0$. If $rot(\eta) \not\equiv 0$ then η is integrable if, and only if, $i_{rot(\eta)}\eta = 0$.*

Proof. Let us denote $Y := rot(\eta)$; $d\eta = i_Y\nu$, $\nu = dz_1 \wedge ... \wedge dz_4$. Since $\eta \in \Omega^2(U)$, we can write $\eta^2 = F.\nu$, where $F \in \mathcal{O}(U)$. If $i_Y\eta = 0$ then

$$i_Y\eta^2 = 0 \implies F.\, i_Y\nu = 0 \overset{Y\neq 0}{\implies} F = 0 \implies \eta^2 = 0.$$

Therefore, by proposition 1 we can write $\eta = \omega_1 \wedge \omega_2$, where ω_1 and ω_2 are meromorphic. We are going to prove that $d\omega_1 \wedge \eta = d\omega_2 \wedge \eta = 0$ (which implies the integrability). From $i_Y\eta = 0$ we have

$$0 = i_Y(\omega_1 \wedge \omega_2) = i_Y(\omega_1).\, \omega_2 - i_Y(\omega_2).\, \omega_1 \implies i_Y\,\omega_1 = i_Y\,\omega_2 = 0,$$

because ω_1 and ω_2 are linearly independent in an open and dense set. On the other hand, $\omega_1 \wedge \nu = 0$ and so

$$0{=}i_Y(\omega_1 \wedge \nu){=}{-}\omega_1 \wedge i_Y\nu{=}{-}\omega_1 \wedge d\eta{=}{-}\omega_1 \wedge (d\omega_1 \wedge \omega_2{-}\omega_1 \wedge d\omega_2) \implies$$
$$d\omega_1 \wedge \eta{=}d\omega_1 \wedge \omega_1 \wedge \omega_2{=}\omega_1 \wedge d\omega_1 \wedge \omega_2{=}0.$$

Similarly, $d\omega_2 \wedge \eta = 0$. The converse is left to the reader. q.e.d.

Remark 2.4. If $rot(\eta) = 0$, i.e., if η is closed, then η is integrable if, and only if, $\eta^2 = 0$. The proof can be found in [**20**].

Example 4. Codimension two distributions and the generic quadric of $\mathbb{P}^5$. A 2-form η on an open set $U \subset \mathbb{C}^4$ can be written as

$$\eta = A\,dx_2{\wedge}dx_3{+}B\,dx_3{\wedge}dx_1{+}C\,dx_1{\wedge}dx_2{+}(E\,dx_1{+}F\,dx_2{+}G\,dx_3){\wedge}dx_4.$$

As the reader can check directly, the condition $\eta^2 = 0$ is equivalent to:

$$A\,E + B\,F + C\,G = 0.$$

Therefore, η defines a singular codimension two distribution on U if, and only if, the image of the map $\Phi = (A, B, C, D, E, F)\colon U \to \mathbb{C}^6$ is contained in the quadric $Q = (z_0\,z_3 + z_1\,z_4 + z_2\,z_5 = 0)$. When the components of Φ are homogeneous polynomials of the same degree then Φ defines a rational map $\phi\colon \mathbb{P}^3 - \to \widehat{Q}$, where $\widehat{Q} \subset \mathbb{P}^5$ is the projection on $\mathbb{P}^5$ of the quadric Q. The indetermination set of ϕ is of course the projection on $\mathbb{P}^3$ of the set $\Phi^{-1}(0)$. In §3.2 we will study the homogeneous case with $\Phi^{-1}(0) = \{0\}$. This corresponds to the case in which the form η has an isolated singularity at $0 \in \mathbb{C}^4$.

2.5. Singularities and the rotational. Let η be a germ at $0 \in \mathbb{C}^4$ of integrable 2-form with a singularity at 0. We will examine two cases:

2.5.1. The Kupka–Reeb phenomenon. When $rot(\eta)(0) \neq 0$ we can find local coordinates $x = (x_1, x_2, x_3, x_4)$ around $0 \in \mathbb{C}^4$ such that $rot(\eta) = \frac{\partial}{\partial x_4}$. In these coordinates the form η does not depends on the variable x_4 and on dx_4. More precisely, we can write $\eta = i_Z\, dx_1 \wedge dx_2 \wedge dx_3$, where Z is a germ of vector field as below

$$Z = \sum_{j=1}^{3} A_j(x_1, x_2, x_3)\, \frac{\partial}{\partial x_j}. \tag{5}$$

In particular, the foliation $\mathcal{F}_\eta$ can be interpreted as the pull-back by the projection $x \mapsto (x_1, x_2, x_3)$ of the germ of foliation on $(\mathbb{C}^3, 0)$ defined by the vector field Z (cf. [**31**]).

2.5.2. The case in which $rot(\eta)$ has an isolated singularity. When $rot(\eta)(0) = 0$ and 0 is an isolated singularity of $rot(\eta)$ we can apply the division theorem of De Rham–Saito [**15**] as follows: the integrability condition $i_{rot(\eta)}\eta = 0$ implies that there exists a germ of holomorphic vector field S such that

$$\eta = i_S\, i_{rot(\eta)}\, dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4. \tag{6}$$

In other words, the tangent bundle of $\mathcal{F}$ decomposes globally outside $Sing(\mathcal{F}_\eta)$: $T\mathcal{F} = \mathbb{C}.\, S \oplus \mathbb{C}.\, rot(\eta)$. Note that (6) implies that $L_S\eta = \eta$, where L denotes the Lie derivative.

Two sub-cases were studied in [**20**]:

1^{st}. When the linear part of $rot(\eta)$ at 0 is non-degenerate. In this case, under generic conditions, it is possible to find germs of vector fields X, Y generating the $\mathcal{O}_4$-module $T\mathcal{F}$ and such that $[X, Y] = 0$: the foliation is generated by a local action of $\mathbb{C}^2$.

2^{nd}. When the linear part of $rot(\eta)$ at 0 is nilpotent. In this case, it is proved in [**20**] that the eigenvalues of $DS(0)$ are rational positive. In particular, S is holomorphically normalizable; $S = S_1 + N$, where $S_1 = DS(0)$ and N is nilpotent and $[S_1, N] = 0$. It is proved also that there exist local coordinates $x = (x_1, x_2, x_3, x_4)$ in which the rotational satisfies $[S_1, rot(\eta)] = \lambda\, rot(\eta)$, where $\lambda \in \mathbb{Q}_+$. In other words, S and $rot(\eta)$ generate a local action of the affine group of $\mathbb{C}$.

2.6. Stability of complete intersections. Case of 2-forms. This section is devoted to the following result:

Theorem 1. *Let η_0 be a germ at $0 \in \mathbb{C}^n$ of decomposable and integrable 2-form: $\eta_0 = \alpha_0 \wedge \beta_0$, where $cod(Sing(\eta_0)) \geq 3$. Let*

$$\eta_s = \eta_0 + s\,\eta_1 + ... = \sum_{j\geq 0} s^j\, \eta_j \;,\; s \in (\mathbb{C}, 0)$$

be a holomorphic family of germs of integrable 2-forms such that $\eta_0 = \eta_s|_{s=0}$. Then there exist holomorphic families of germs of functions f_s, g_s and u_s, $u_s \in \mathcal{O}_n^*$, such that $\eta_s = u_s\, df_s \wedge dg_s$. In particular, the foliation associated to η_s has two independent first integrals.

Proof. In the proof we use the following:

Theorem 2. *Let η_0 be a germ at $0 \in \mathbb{C}^n$ of holomorphic decomposable 2-form, $\eta_0 = \alpha_0 \wedge \beta_0$, where $\alpha_0, \beta_0 \in \Omega^1(\mathbb{C}^n, 0)$. Assume that $cod(Sing(\eta_0)) \geq 3$. Let η_s, $s \in (\mathbb{C}, 0)$, be a holomorphic family of germs of 2-forms such that $\eta_s^2 = 0$. Then η_s is decomposable, i.e., there exist holomorphic families of 1-forms $\alpha_s = \alpha_0 + s\,\alpha_1 + \ldots$ and $\beta_s = \beta_0 + s\,\beta_1 + \ldots$ such that $\eta_s = \alpha_s \wedge \beta_s$.*

Proof of theorem 2. It is sufficient to prove that we can write $\eta_s = \alpha_s \wedge \beta_s$, where $\alpha_s = \sum_{j\geq 0} s^j\,\alpha_j$ and $\beta_s = \sum_{j\geq 0} s^j\,\beta_j$ are formal power series. In fact, the existence of a formal solution of the equation $\eta_s - \alpha_s \wedge \beta_s = 0$ implies the existence of a convergent solution by Artin's approximation theorem (cf. [**1**]).

Given a power series $\theta_s = \sum_{s\geq 0} s^j\,\theta_j$ we set $j^k(\theta_s) = \sum_{j=0}^k s^j\,\theta_j$. We will prove by induction on $k \geq 0$ that there exist germs $\alpha_s^k := \sum_{j=0}^k s^j\,\alpha_j$ and $\beta_s^k := \sum_{j=0}^k s^j\,\beta_j$ such that

$$j^k\left(\eta_s - \alpha_s^k \wedge \beta_s^k\right) = 0. \tag{7}$$

The first step of the induction is the hypothesis $j^0(\eta_s - \alpha_0 \wedge \beta_0) = 0$. Suppose by induction that there exist $\alpha_s^{\ell-1}$ and $\beta_s^{\ell-1}$ satisfying (7) for $k = \ell - 1 \geq 0$, and let us prove that there exist α_s^ℓ and β_s^ℓ satisfying (7), $j^{\ell-1}(\alpha_s^\ell) = \alpha_s^{\ell-1}$ and $j^{\ell-1}(\beta_s^\ell) = \beta_s^{\ell-1}$. Observe first that

$$j^\ell(\eta_s - \alpha_s^{\ell-1} \wedge \beta_s^{\ell-1}) = s^\ell\,\mu,$$

where

$$\mu = \eta_\ell - \sum_{i=1}^{\ell-1} \alpha_i \wedge \beta_{\ell-i}.$$

By Saito's division theorem the induction step can be reduced to the following (cf. [**34**]):

Claim 2.1. *If μ is as above then $\alpha_0 \wedge \beta_0 \wedge \mu = 0$.*

In fact, if claim 2.1 is true then, since $cod(Sing(\alpha_0 \wedge \beta_0)) \geq 3$, by Saito's theorem there are germs α_ℓ and β_ℓ such that

$$\mu = \alpha_0 \wedge \beta_\ell + \alpha_\ell \wedge \beta_0.$$

Therefore, if we set $\alpha_s^\ell = \alpha_s^{\ell-1} + s^\ell\,\alpha_\ell$ and $\beta_s^\ell = \beta_s^{\ell-1} + s^\ell\,\beta_\ell$ then

$$j^\ell(\eta_s - \alpha_s^\ell \wedge \beta_s^\ell) = 0,$$

as the reader can verify directly.

Proof of claim 2.1. Here we use the relation $\eta_s^2 = 0$, which implies

$$0 = \eta_s^2 = \sum_r s^r \sum_{m+n=r} \eta_m \wedge \eta_n \implies \sum_{m+n=r} \eta_m \wedge \eta_n = 0\ ,\ \forall r \geq 0.$$

The induction hypothesis implies

$$\sum_{j=0}^{\ell-1} s^j\,\eta_j = j^{\ell-1}(\eta_s) = j^{\ell-1}(\alpha_s^{\ell-1}\wedge\beta_s^{\ell-1}) \implies$$

$$\eta_r = \sum_{i=0}^{r} \alpha_i\wedge\beta_{r-i}\ ,\ 0\le r\le \ell-1.$$

Therefore,

$$0 = \sum_{m+n=\ell} \eta_m\wedge\eta_n = 2\,\eta_0\wedge\eta_\ell + \sum_{\substack{i+j+r+s=\ell\\ i+j\ge 1\\ r+s\ge 1}} \alpha_i\wedge\beta_j\wedge\alpha_r\wedge\beta_s. \tag{8}$$

Since $\alpha_i\wedge\alpha_i = 0$ and $\beta_i\wedge\beta_i = 0$, $\forall i$, we can assume that the summation set in the sum of the right hand side of (8) is

$S = \{(i,j,r,s)\,|\, i+j+r+s=\ell\,,\ i+j\ge 1\,,\ r+s\ge 1\ ,\ i\ne r\ ,\ j\ne s\}.$

For simplicity of notation, given a subset $A\subset S$ we will set

$$\sum_{(i,j,r,s)\in A} \alpha_i\wedge\beta_j\wedge\alpha_r\wedge\beta_s := \sum(A).$$

The set S can be decomposed into two disjoint subsets $S = S_1\cup S_2$, where $S_1 = \{(i,j,r,s)\in S\,|\, i=s=0 \text{ or } j=r=0\}$ and $S_2 = S\setminus S_1$. Note that $S_2 = \{(i,j,r,s)\in S|$ at most one of the indexes i, j, r or s is $= 0\}$.

We assert that $\sum(S_2)=0$. This assertion follows from the fact that there exists a permutation $\phi\colon S_2\to S_2$ with the following properties:

(i). $\phi(x)\ne x$ and $\phi(\phi(x)) = x$, $\forall x\in S_2$.

(ii). If $x=(i,j,r,s)\in S_2$ and $\phi(x) = (i',j',r',s')$ then

$$\alpha_i\wedge\beta_j\wedge\alpha_r\wedge\beta_s = -\alpha_{i'}\wedge\beta_{j'}\wedge\alpha_{r'}\wedge\beta_{s'}.$$

In fact, (i) and (ii) imply that ϕ induces a partition $S_2 = S_3\cup S_4$, $\phi(S_3) = S_4$, $\phi(S_4) = S_3$, such that $\sum(S_3) = -\sum(S_4)$. Therefore, $\sum(S_2) = \sum(S_3)+\sum(S_4) = 0$.

The permutation ϕ is constructed as follows: fix $(i,j,r,s)\in S_2$. We have three possibilities:

1. $i,j,r,s\ge 1$. In this case we set $\phi(i,j,r,s) = (r,j,i,s)$.
2. $i=0$ or $r=0$. In this case we set $\phi(i,j,r,s) = (i,s,r,j)$.
3. $j=0$ or $s=0$. In this case we set $\phi(i,j,r,s) = (r,j,i,s)$.

We leave to the reader the verification that ϕ is well defined and satisfies properties (i) and (ii).

Finally, we can write $S_1 = S_5\cup S_6$ where $S_5 = \{(i,0,0,\ell-i)\,|\,1\le i\le \ell-1\}$ and $S_6 = \{(0,\ell-i,i,0)\,|\,1\le i\le\ell-1\}$. Since

$$\alpha_i\wedge\beta_0\wedge\alpha_0\wedge\beta_j = \alpha_0\wedge\beta_j\wedge\alpha_i\wedge\beta_0 = -\alpha_0\wedge\beta_0\wedge\alpha_i\wedge\beta_j,$$

we get

$$\sum(S_1) = -2\,\alpha_0 \wedge \beta_0 \wedge \sum_{i=1}^{\ell-1} \alpha_i \wedge \beta_{\ell-i} \overset{(8)}{\Longrightarrow} \alpha_0 \wedge \beta_0 \wedge \mu = 0.$$

q.e.d.

End of the proof of theorem 1. Theorem 2 implies the existence of two holomorphic families of 1-forms α_s and β_s such that $\eta_s = \alpha_s \wedge \beta_s$, $\forall s \in (\mathbb{C}, 0)$. Consider on $(\mathbb{C}^n \times \mathbb{C}, (0,0))$ the pffafian system generated by the forms α_s, β_s and ds. This system is integrable. Since $cod_{(\mathbb{C}^n,0)}(Sing(\alpha_0 \wedge \beta_0)) \geq 3$, by semi-continuity we have $cod_{(\mathbb{C}^n,0)}(Sing(\alpha_s \wedge \beta_s)) \geq 3$. Since $\alpha_s \wedge \beta_s$ does not contain terms with ds we can conclude that

$$cod_{(\mathbb{C}^n \times \mathbb{C},0)}(Sing(\alpha_s \wedge \beta_s \wedge ds)) \geq 3.$$

Therefore, by Malgrange's theorem (see theorem 2.1) there exist $F, G \in \mathcal{O}_{n+1}$ and $U \in \mathcal{O}^*_{n+1}$ such that

$$\alpha_s \wedge \beta_s \wedge ds = U\, dF \wedge dG \wedge ds.$$

Hence, we can take the families f_s, g_s and u_s as

$$f_\tau := F|_{s=\tau}\,,\; g_\tau := G|_{s=\tau} \text{ and } u_\tau := U|_{s=\tau}.$$

q.e.d.

2.7. Codimension two foliations not contained in a codimension one foliation. In the construction of the examples we will use a result due to X. Gomez-Mont and I. Luengo [**26**]:

Theorem 2.2. *There exists a polynomial vector field on $\mathbb{C}^3$ with an isolated singularity at $0 \in \mathbb{C}^3$ and without germ of analytic invariant curve through 0.*

A consequence of theorem 2.2 is the following:

Proposition 3. *Let Z be a germ at $0 \in \mathbb{C}^3$ of vector field with an isolated singularity at 0 and without germ of invariant curve through 0. Then Z cannot be tangent to a germ at $0 \in \mathbb{C}^3$ of holomorphic codimension one foliation.*

Proof. Suppose by contradiction that Z is tangent to some a germ at $0 \in \mathbb{C}^3$ of codimension one foliation $\mathcal{F}$. Let ω be a germ of integrable 1-form defining $\mathcal{F}$ and with $cod(Sing(\omega)) \geq 2$. We assert that $Sing(\omega) = \{0\}$.

Note first that the tangency condition is equivalent to $i_Z\omega = 0$, which implies that $Sing(\omega)$ is Z-invariant. Therefore, $Sing(\omega)$ cannot contain a germ of curve, for otherwise this curve would be Z-invariant. Hence, $Sing(\omega) \subset \{0\}$ and we have two possibilities, either $Sing(\omega) = \emptyset$, or $Sing(\omega) = \{0\}$. On the other hand, if $Sing(\omega)$ was empty then by integrability there exists a local chart $x = (x_1, ..., x_n)$ such that $\omega = u\, dx_1$, where $u(0) \neq 0$. This implies that $i_Z dx_1 = Z(x_1) = 0$ and this contradicts the fact that 0 is an isolated singularity of Z. Therefore, $Sing(\omega) = \{0\}$ as asserted.

Let $\eta = i_Z\, dx_1 \wedge dx_2 \wedge dx_3$ and observe that the relation $i_Z\omega = 0$ is equivalent to $\omega \wedge \eta = 0$. Since $cod(Sing(\omega)) = 3$ by De Rham's theorem [**15**] there exists a germ of 1-form θ such that $\eta = \omega \wedge \theta$. However, a decomposable 2-form $\eta = \omega \wedge \theta$ with a singularity at $0 \in \mathbb{C}^3$ vanishes necessarily on a curve through 0. This contradicts the fact that $Sing(\eta) = Sing(Z) = \{0\}$. q.e.d.

Corollary 1. *For all $n \geq 3$ there are germs at $0 \in \mathbb{C}^n$ of holomorphic codimension two foliations which are "not contained" in any holomorphic foliation of codimension one: if a germ like this is defined by an integrable 2-form η then there is no integrable 1-form ω such that $\omega \wedge \eta = 0$.*

Proof. If $n = 3$ the corollary is a direct consequence of proposition 3: take $\eta = i_Z\, dx_1 \wedge dx_2 \wedge dx_3$, where Z is like in theorem 2.2. If $n > 3$ then let $\Pi\colon \mathbb{C}^n = \mathbb{C}^3 \times \mathbb{C}^{n-3} \to \mathbb{C}^3$ be the projection $\Pi(x, y) = x$ and

$$\eta = \Pi^*(i_Z\, dx_1 \wedge dx_2 \wedge dx_3).$$

Suppose by contradiction that there exists a germ of integrable 1-form ω such that $\omega \wedge \eta = 0$. Note that in the coordinates $(x, y) \in \mathbb{C}^3 \times \mathbb{C}^{n-3}$, $y = (y_1, ..., y_{n-3})$ the definition of η implies that $i_{\frac{\partial}{\partial y_j}}\eta = 0$, $\forall\, 1 \leq j \leq n-3$. Therefore,

$$0 = i_{\frac{\partial}{\partial y_j}}(\omega \wedge \eta) = i_{\frac{\partial}{\partial y_j}}(\omega).\, \eta \implies i_{\frac{\partial}{\partial y_j}}\, \omega = 0\ ,\ \forall\, j.$$

Hence, we can write $\omega = \sum_{j=1}^{3} A_j(x, y)\, dx_j$. On the other hand, the integrability condition $\omega \wedge d\omega = 0$ implies that

$$A_i.\, \frac{\partial A_j}{\partial y_k} - A_j.\, \frac{\partial A_i}{\partial y_k} = 0\ ,\ \forall\, k = 1, ..., n-3\ ,\ \forall\, 1 \leq i < j \leq 3. \tag{9}$$

The relations in (9) imply that there exist $u \in \mathcal{O}_n^*$ and $B_1, B_2, B_3 \in \mathcal{O}_3$ such that

$$A_j(x, y) = u(x, y).\, B_j(x)\ ,\ 1 \leq j \leq 3 \implies \omega = u.\, \Pi^*(\widetilde{\omega}),$$

where $\widetilde{\omega} = \sum_{j=1}^{3} B_j(x)\, dx_j$ is integrable and $\widetilde{\omega} \wedge (i_Z\, dx_1 \wedge dx_2 \wedge dx_3) = 0$. This contradicts proposition 3. q.e.d.

Remark 2.5. In fact, corollary 1 is a local version of the main result of [**32**]. Indeed, theorem 1 of [**32**] says that on any projective manifold of dimension at least three, a "very general" foliation by curves is not contained in a foliation or web of dimension greater than one.

We would like to observe that corollary 1 also implies this result in rational manifolds: since the vector field Z is polynomial, in any rational manifold of dimension $n \geq 3$ there are examples of codimension two foliations that are not contained in a codimension one foliation.

3. Homogeneous foliations

3.1. Homogeneous foliations. In this section, we fix a coordinate system $(x_1,...,x_n)$ of $\mathbb{C}^n$. A p-form Ω is said to be homogeneous of degree m if its components are homogeneous polynomials of degree m. The radial vector field of $\mathbb{C}^n$ will be denoted by R:

$$R = \sum_{j=1}^{n} x_j \, \frac{\partial}{\partial x_j}.$$

The p-form Ω is said to be *dicritical* if $i_R\,\Omega = 0$. Otherwise we say that Ω is non-dicritical.

If Ω is a p-form of degree m we have

$$L_R\,\Omega = i_R\,d\Omega + d\,(i_R\Omega) = (m+p)\,\Omega \quad \text{(Euler's identity)}. \tag{10}$$

Next we state some useful results.

Proposition 4. *Let η be a homogeneous 2-form of degree m. Assume that η is closed and $\eta^2 = \eta\wedge\eta = 0$. Then there exists a homogeneous integrable dicritical 1-form ω such that $\eta = d\omega$. In particular, $\omega\wedge\eta = 0$.*

Remark 3.1. The relation $\omega\wedge\eta = 0$ in the statement of proposition 4 means that the leaves of the codimension two foliation $\mathcal{F}_\eta$ are contained in the leaves of the codimension one foliation $\mathcal{F}_\omega$. In this case, we will say that *the foliation $\mathcal{F}_\eta$ is contained in the foliation $\mathcal{F}_\omega$.*

Proof of proposition 4. The proof can be found in [**17**], but it is also an easy consequence of Euler's identity: set $\widetilde{\omega} = i_R\eta$. Since $d\eta = 0$, from (10) we get

$$d\widetilde{\omega} = d\,(i_R\eta) = L_R\eta = (m+2)\,\eta \implies \eta = d\omega\ ,\ \omega = (m+2)^{-1}.\,\widetilde{\omega}.$$

Finally,

$$0 = i_R(\eta\wedge\eta) = 2\,i_R(\eta)\wedge\eta \implies \omega\wedge d\omega = (m+2)^{-1}.\,i_R(\eta)\wedge\eta = 0.$$

q.e.d.

Remark 3.2. We would like to note that if the 1-form ω is integrable, but not closed, then the 2-form $d\omega$ is integrable and the foliation $\mathcal{F}_{d\omega}$ is contained in the foliation $\mathcal{F}_\omega$. For instance, if

$$\omega = f_1...,f_p\sum_j \lambda_j\,\frac{df_j}{f_j}\ ,\ \lambda_j\in\mathbb{C}^*\ ,\ \lambda_i\neq\lambda_j\,,\ \forall\ i\neq j,$$

where f_i is holomorphic $\forall i$, then

$$d\omega = d(f_1...f_p)\wedge\sum_j \lambda_j\,\frac{df_j}{f_j}.$$

In this case, the leaves of $\mathcal{F}_{d\omega}$ are the connected components of the intersection of the levels of $f_1...f_p$ with the leaves of $\mathcal{F}_\omega$.

Let us mention that the foliation $\mathcal{F}_{d\omega}$ is contained in infinitely many codimension one foliations: all members of the family of foliations $\mathcal{F}_{\omega_\lambda}$, where

$$\omega_\lambda = f_1 ..., f_p \sum_j (\lambda_j + \lambda) \frac{df_j}{f_j}\ ,\ \lambda \in \mathbb{C}.$$

An analogous result in the homogeneous non-closed and non-dicritical case is the following:

Proposition 5. *Let η be a homogeneous integrable and non-dicritical 2-form on $\mathbb{C}^n$, $n \geq 3$. Then the 1-form $\omega := i_R\,\eta$ is integrable. Moreover, the foliation $\mathcal{F}_\eta$ is contained in the foliation $\mathcal{F}_\omega$.*

Proof. We want to prove that $\omega \wedge d\omega = 0$. From $\eta^2 = 0$ we get

$$0 = i_R(\eta^2) = 2\, i_R\eta \wedge \eta = 2\,\omega \wedge \eta \implies \omega \wedge \eta = 0.$$

If v is a constant vector field such that $f := i_v\,\omega \not\equiv 0$ then

$$\omega \wedge \eta = 0 \implies i_v(\omega \wedge \eta) = f.\,\eta - \omega \wedge i_v\eta = 0 \implies \eta = \omega \wedge \widetilde{\omega},$$

where $\widetilde{\omega} = f^{-1}.\, i_v\eta$. Therefore,

$$\omega = i_R\,\eta = -i_R\,\widetilde{\omega}.\,\omega \implies i_R\,\widetilde{\omega} = -1.$$

On the other hand, the integrability of η implies that

$$\omega\wedge\widetilde{\omega}\wedge d\omega = 0 \implies 0 = i_R(\omega\wedge\widetilde{\omega}\wedge d\omega) = -(i_R\widetilde{\omega}).\,\omega\wedge d\omega + \omega\wedge\widetilde{\omega}\wedge i_R\, d\omega \implies$$
$$\omega \wedge d\omega = \widetilde{\omega} \wedge \omega \wedge i_R\, d\omega.$$

Finally, if $k = deg(\omega)$ then (10) implies

$$(k+1)\,\omega = L_R\,\omega = i_R\,d\omega \implies i_R\,d\omega \wedge \omega = 0 \implies \omega \wedge d\omega = 0.$$

q.e.d.

Let us state a result that will be important in what follows. Let ω be a homogeneous integrable 1-form on $\mathbb{C}^n$, $n \geq 4$. We assume that ω is dicritical of degree k and $cod(Sing(\omega)) \geq 2$. Euler's identity (10) implies $i_R\,d\omega = (k+1)\,\omega$. The form ω induces a codimension one foliation $\mathcal{F}$ on the space $\mathbb{P}^{n-1}$ whose singular set is the projectivisation of $Sing(\omega)$; $Sing(\mathcal{F}) = \Pi(Sing(\omega) \setminus \{0\})$, where $\Pi\colon \mathbb{C}^n \setminus \{0\} \to \mathbb{P}^{n-1}$ is the natural projection.

A *Kupka singularity* of $\mathcal{F}$ is a point $p \in Sing(\mathcal{F})$ for which there is a local generator α of the germ $\mathcal{F}_p$ with $d\alpha(p) \neq 0$. The *Kupka set* of $\mathcal{F}$ is, by definition, $K(\mathcal{F}) = \{p \in Sing(\mathcal{F})\,|\, p \text{ is a Kupka singularity of } \mathcal{F}\}$. A *Kupka component* of $\mathcal{F}$ is an irreducible component K of $Sing(\mathcal{F})$ with $K \subset K(\mathcal{F})$. It is known that a Kupka component of $\mathcal{F}$ is a smooth sub-variety of codimension two along which the foliation is locally trivial (see §2.5.1, [**18**] and [**31**]). If $\widetilde{K}$ is an irreducible component of $Sing(\omega)$ such that $d\omega(p) \neq 0$ for all $p \in \widetilde{K} \setminus \{0\}$ then $\Pi(\widetilde{K} \setminus \{0\})$ is a Kupka component of $\mathcal{F}$. The next statement resumes some results proven in [**10**], [**2**] and [**4**]:

Theorem 3.1. *Let $\mathcal{F}$ and ω be as above. If $\mathcal{F}$ has a Kupka component K then K is a complete intersection of hypersurfaces, in homogeneous coordinates $(F = G = 0)$. Moreover, if $degree(F)/degree(G) = p/q$, where $p, q \in \mathbb{N}$ and $(p, q) = 1$, then $\mathcal{F}$ is the algebraic pencil of hypersurfaces given by the rational function $\frac{F^q}{G^p}$, or equivalently by the 1-form $q\,F\,dG - p\,G\,dF$ (in homogeneous coordinates).*

Remark 3.3. In fact, theorem 3.1 says that we can choose F and G in such a way that $\omega = q\,F\,dG - p\,G\,dF$. We would like to note also that the hypothesis $n \geq 4$ is necessary; in dimension $n = 3$ the statement is false.

Corollary 2. *Let ω be an integrable homogeneous and dicritical 1-form on $\mathbb{C}^n$, $n \geq 4$. Then $0 \in \mathbb{C}^n$ cannot be an isolated singularity of the 2-form $d\omega$.*

Proof. Note first that $0 \in \mathbb{C}^n$ is not an isolated singularity of ω. This is a consequence of Malgrange's theorem: if 0 was an isolated singularity of ω then $\omega = df$, by Malgrange's theorem, where f is homogeneous. However, by Euler's identity we have

$$0 = i_R\,\omega = i_R\,df = deg(f).\,f \implies f = 0 \implies \omega = 0,$$

a contradiction.

Suppose by contradiction that $Sing(d\omega) = \{0\}$. In this case, all irreducible components of $\Pi(Sing(\omega) \setminus \{0\})$ are contained in the Kupka set $K(\mathcal{F})$. Therefore, by theorem 3.1 and remark 3.3 we can suppose that $\omega = q\,F\,dG - p\,G\,dF$, so that $d\omega = (p+q)\,dF \wedge dG$ and $d\omega$ is decomposable. However, this implies that $dim(Sing(d\omega)) \geq 1$, a contradiction. q.e.d.

3.2. Singularities of codimension two foliations. We would like to pose the following problem:

Problem 1. Is there a germ of codimension two foliation with an isolated singularity at the origin of $\mathbb{C}^4$?

First of all, in the case of dimension three there are such foliations. In fact, the codimension two foliations with an isolated singularity at the origin of $\mathbb{C}^3$ are generic.

Next, there are homogeneous codimension two distributions in $\mathbb{C}^4$ with an isolated singularity at the origin. An example, due to [**19**], is given by the decomposable but non-integrable 2-form

$$\theta = x_3^2\,dx_2 \wedge dx_3 - x_1^2\,dx_3 \wedge dx_1 + (x_1\,x_2 + x_3\,x_4)\,dx_1 \wedge dx_2 +$$
$$+ [x_4^2\,dx_1 + x_2^2\,dx_2 + (x_1\,x_2 - x_3\,x_4)\,dx_3] \wedge dx_4.$$

The form θ has an isolated singularity at $0 \in \mathbb{C}^4$. It defines a distribution of 2-planes on $\mathbb{C}^4 \setminus \{0\}$ because $\theta \wedge \theta = 0$.

In fact, we don't know the answer of problem 1 in general, but the next statement contrasts with the previous example.

Theorem 3. *Let η be a homogeneous integrable 2-form on $\mathbb{C}^4$. Then $dim(Sing(\eta)) \geq 1$.*

Proof. We denote by Z the rotational of η. We start by the case where η is closed, which means $Z \equiv 0$.

1- *η is closed.* Let $\omega = i_R\eta$. By proposition 5, ω is integrable and by Euler's identity we have

$$d\omega = (m+2)\,\eta,$$

where m is the degree of η. Since $dim(Sing(d\omega)) \geq 1$ by corollary 2, we obtain the result in this case.

2- *η is not closed, $Z \not\equiv 0$.* Let us consider first the case where $i_R\eta = 0$. In this case, since 0 is an isolated singularity of R, by De Rham's division theorem there exists a homogeneous vector field Y such that $\eta = i_R\, i_Y\, \nu$, $\nu = dx_1 \wedge ... \wedge dx_4$. Since $dim(Sing(R \wedge Y)) \geq 1$, we get $dim(Sing(\eta)) \geq 1$. In the same way, if $cod(Sing(Z)) \geq 3$, since $i_Z\, \eta = 0$ then De Rham's division theorem implies that there exists a vector field Y such that $\eta = i_Z\, i_Y\, \nu$ and again $dim(Sing(\eta)) \geq 1$.

Therefore, we can suppose that $cod(Sing(Z)) \leq 2$ and $\omega := i_R\, \eta \not\equiv 0$. In this case, by proposition 5 the form ω is integrable and induces a codimension one foliation on $\mathbb{P}^3$ of degree $\leq m$, the degree of the coefficients of η. From Euler's identity (10) we get:

$$d\omega = d\, i_R\, \eta = L_R\, \eta - i_R\, d\eta = (m+2)\, \eta - i_R\, i_Z\, \nu. \tag{11}$$

Since $i_Z\, \eta = 0$ the above equality implies that $i_Z\, d\omega = 0$. Hence,

$$L_Z\omega = i_Z\, d\omega + d\, i_Z\, \omega = d\,(i_Z\, i_R\, \eta) = 0 \implies$$
$$L_Z\, d\omega = 0. \tag{12}$$

Let us establish a technical variant of corollary 2.

Lemma 3.1. *Let ω be a homogeneous dicritical and integrable 1-form on $\mathbb{C}^n$, $n \geq 4$. Suppose that $Sing(\omega)$ contains a hypersurface $(h = 0)$, $\omega = h.\widetilde{\omega}$, where $h(0) = 0$ and $cod(Sing(\widetilde{\omega})) \geq 2$. Then $dim(Sing(d\omega)) \geq 1$.*

Proof. Suppose by contradiction that $Sing(d\omega) = \{0\}$. Since

$$d\omega = dh \wedge \widetilde{\omega} + h\, d\widetilde{\omega},$$

we obtain that

$$\widetilde{\omega}(m) = 0\ ,\ m \neq 0 \implies d\omega(m) \neq 0 \implies d\widetilde{\omega}(m) \neq 0.$$

In particular, the singularities of $\widetilde{\omega}$ in $\mathbb{C}^n \setminus \{0\}$ are of Kupka type and by theorem 3.1 we have $\widetilde{\omega} = k\, F\, dG - \ell\, G\, dF$, where F and G are homogeneous polynomials. Hence,

$$d\omega = dh \wedge (k\, F\, dG - \ell\, G\, dF) + (k+\ell)\, h\, dF \wedge dG,$$

and so $d\omega$ vanishes on the set $(h = F = G = 0)$, which in dimension $n \geq 4$ has dimension ≥ 1. q.e.d.

Let us suppose now, by contradiction, that the 2-form η has an isolated singularity at $0 \in \mathbb{C}^4$. It follows from corollary 2 and lemma 3.1 that the 2-form $d\omega$ vanishes at least on some straight line L through $0 \in \mathbb{C}^4$. Moreover, from (12) the form $d\omega$ is invariant by the local flow of Z. In particular, if we denote by S the irreducible component of $Sing(d\omega)$ that contains L, then S is invariant by the local flows of Z and of the radial vector field R. Hence, Z and R are tangent to S. By Euler's identity (11), if $m \in S \setminus \{0\}$ then $d\omega(m) = 0$ and $\eta(m) \neq 0$, imply that $R(m)$ and $Z(m)$ must be independent along S. Therefore, $dim(S) \geq 2$. Since $Z \neq 0$ along $S \setminus \{0\}$ ($S \cap Sing(Z) = \{0\}$) and we have supposed that $cod(Sing(Z)) \leq 2$, of course we must have

$$dim(S) = dim(Sing(Z)) = 2 \implies$$
$$S \setminus \{0\} \text{ is smooth and connected } \implies$$

its projectivisation in $\mathbb{P}^3$, $\Gamma = \Pi(S \setminus \{0\})$, is a smooth curve. Let $\mathcal{G}$ be the one dimensional foliation on $\mathbb{P}^3$ defined in homogeneous coordinates by the form $i_R\, i_Z\, \nu$. Since $R \wedge Z \neq 0$ along $S \setminus \{0\}$ the curve Γ is an algebraic leaf of $\mathcal{G}$ such that $Sing(\mathcal{G}) \cap \Gamma = \emptyset$. However, this is not possible by [**22**]: any algebraic curve invariant by a one dimensional foliation $\mathcal{G}$ of $\mathbb{P}^n$, $n \geq 2$, must contain at least one singularity of $\mathcal{G}$ (see proposition 2.4 in [**22**]). q.e.d.

Remark 3.4. In general we don't know the answer of problem 1. However, a case in which $dim(Sing(\mathcal{F}_\eta)) \geq 1$ is when there exists a germ of holomorphic vector field Z such that $i_Z\eta = 0$ and $cod(Sing(Z)) \geq 3$. Indeed, if this is true then by De Rham's division theorem we can write $\eta = i_Y i_Z \nu$. This implies that $dim(Sing(\mathcal{F}_\eta)) \geq 1$, as in the argument of theorem 3. In particular, when $cod(Sing(rot(\eta))) \geq 3$ then $dim(Sing(\mathcal{F}_\eta)) \geq 1$.

3.3. Homogeneous integrable 2-forms of small degree. In this section, η will be a homogeneous integrable 2-form on $\mathbb{C}^n$, $n \geq 4$. Here we will describe with some detail the foliation $\mathcal{F}_\eta$ when $0 \leq deg(\eta) \leq 2$.

3.3.1. The case $deg(\eta) = 0$. Here η has constant coefficients, and so it is closed. Since $\eta^2 = 0$, by Darboux's theorem there exists a coordinate system $x = (x_1, ..., x_n)$ such that $\eta = dx_1 \wedge dx_2$. The leaves of $\mathcal{F}_\eta$ are the level surfaces $(x_1 = c_1, x_2 = c_2)$. The space of foliations given by such forms is the projectivisation of the space of the antisymetric $n \times n$ matrices of rank two.

3.3.2. The case $deg(\eta) = 1$**.** We consider the 3-form with constant coefficients $d\eta$. Either the form η is closed or, up to linear conjugacy, $d\eta = dx_1 \wedge dx_2 \wedge dx_3$. By using these facts Medeiros shows in [**31**] that either $\eta = dL \wedge dQ$, L linear and Q quadratic, or η "depends only on three variables"; meaning that $\eta = \sigma^*(\eta_o)$, where $\sigma\colon \mathbb{C}^n \to \mathbb{C}^3$ is a linear morphism of rank three and η_o is a linear 2-form in $\mathbb{C}^3$. As a consequence we can describe the space of foliations defined by this type of form:

Proposition 6. *The space of codimension two foliations given by homogeneous 2-forms of degree one in* $\mathbb{C}^n$*,* $n \geq 4$*, has two irreducible components. Each of these components can be considered as a Zariski open and dense subset of a compact rational variety.*

3.3.3. The case $deg(\eta) = 2$**.** This case is more difficult. We again distinguish the two cases, $d\eta \equiv 0$ and $d\eta \not\equiv 0$. If $d\eta \equiv 0$ then by proposition 4 there exists a dicritical homogeneous 1-form of degree three ω such that $\eta = d\omega$. The form ω is the homogeneous expression of some codimension one foliation on $\mathbb{P}^{n-1}$ of degree two. According to [**9**] the space of such foliations has six irreducible components:

1^{st}: $\overline{R(2,2)}$. Here the generic member has a rational first integral of the form $\frac{P}{Q}$, where P and Q are quadrics. In this case, $\omega = \frac{1}{2}(P\,dQ - Q\,dP)$ and $\eta = dP \wedge dQ$. The foliation $\mathcal{F}_\eta$ has the first integral $(P, Q)\colon \mathbb{C}^n \to \mathbb{C}^2$.

2^{nd}: $\overline{R(1,3)}$. Here the generic member has a rational first integral of the form $\frac{C}{L^3}$, where C is a cubic and L linear. In this case, $\omega = \frac{1}{4}(L\,dC - 3\,C\,dL)$ and $\eta = dL \wedge dC$. The foliation $\mathcal{F}_\eta$ has the first integral $(L, C)\colon \mathbb{C}^n \to \mathbb{C}^2$.

3^{rd}: $\overline{L(1,1,1,1)}$. Here the generic member can be expressed in homogeneous coordinates by a the 1-form

$$\omega = L_1.\, L_2.\, L_3.\, L_4 \left(\lambda_1 \frac{dL_1}{L_1} + \lambda_2 \frac{dL_2}{L_2} + \lambda_3 \frac{dL_3}{L_3} + \lambda_4 \frac{dL_4}{L_4} \right),$$

where the L_j is linear, $\lambda_j \in \mathbb{C}^*$, $1 \leq j \leq 4$, and $\sum_j \lambda_j = 0$. In this case, we have

$$\frac{1}{L_1.\, L_2.\, L_3.\, L_4}\, \eta = \sum_i \frac{dL_i}{L_i} \wedge \sum_j \lambda_j \frac{dL_j}{L_j},$$

and $\mathcal{F}_\eta$ is the intersection of the two foliations $\mathcal{F}_{d(L_1 L_2 L_3 L_4)}$ and $\mathcal{F}_\omega$.

4^{th}: $\overline{L(1,1,2)}$. Here the generic member can be expressed as

$$\omega = L_1.\, L_2.\, Q \left(\lambda_1 \frac{dL_1}{L_1} + \lambda_2 \frac{dL2}{L_2} + \lambda \frac{dQ}{Q} \right),$$

where L_1 and L_2 are linear, Q a quadric and $\lambda_1+\lambda_2+2\lambda=0$. Again the foliation $\mathcal{F}_\eta$ is the intersection of two others: the foliations $\mathcal{F}_{d(L_1L_2\,Q)}$ and $\mathcal{F}_\omega$.

5^{th}: $\overline{E(n-1)}$. Here the generic member has a first integral of the form $F=\frac{C^2}{Q^3}$ where C is a cubic and Q a quadric. The form $C.\,Q\,\frac{dF}{F}=2\,Q\,dC-3\,C\,dQ$ has a linear factor. It is proved in [9] that in some homogeneous coordinate system $x=(x_1,...,x_4,...,x_n)$ we can write

$$C=x_3\,x_4^2-x_1\,x_2\,x_4+\frac{x_1^3}{3}\ \text{ and }\ Q=x_2\,x_4-\frac{x_1^2}{2},$$

and the linear factor is x_4. In these coordinates we have $\omega=\frac{1}{x_4}(2\,Q\,dC-3\,C\,dQ)$ and

$$\eta=d\left(\frac{C\,Q}{x_4}\right)\wedge\left(2\frac{dC}{C}-3\frac{dQ}{Q}\right)\ \Longrightarrow$$

the foliation $\mathcal{F}_\eta$ is the intersection of $\mathcal{F}_{d(CQ/x_4)}$ and $\mathcal{F}_\omega$.

6^{th}: $S(2,n)$. Here the foliation induced by ω in $\mathbb{P}^{n-1}$ is a linear pull-back of a degree two foliation on $\mathbb{P}^2$. This means that there exist homogeneous coordinates $x=(x_1,x_2,x_3,...,x_n)$ on $\mathbb{C}^n$ and homogeneous polynomials of degree three P, Q and R, depending only on x_1,x_2,x_3, such that $x_1\,P+x_2\,Q+x_3\,R\equiv 0$ and

$$\omega=P(x_1,x_2,x_3)\,dx_1+Q(x_1,x_2,x_3)\,dx_2+R(x_1,x_2,x_3)\,dx_3\quad\Longrightarrow$$

$$\eta=dP\wedge dx_1+dQ\wedge dx_2+dR\wedge dx_3.$$

In other words, $\mathcal{F}_\eta$ is the pull-back by a projection $x\in\mathbb{C}^n\mapsto(x_1,x_2,x_3)\in\mathbb{C}^3$, of a homogeneous foliation of degree two and codimension two in $\mathbb{C}^3$.

Let us mention that in the above case, all leaves of $\mathcal{F}_\eta$ are ruled: they contain the fibers of the projection $(x_1,...,x_n)\mapsto(x_1,x_2,x_3)$. On the other hand, in general a foliation of degree two on $\mathbb{P}^2$ has no algebraic leaves. Therefore, in general the leaves of the foliations $\mathcal{F}_\omega$ and $\mathcal{F}_\eta$ are Zariski dense. We obtain the following result:

Theorem 4. *The space of foliations of codimension two in $\mathbb{C}^n$, $n\geq 4$, defined by closed 2-forms homogeneous of degree two, has six irreducible components corresponding to the six components of the space of foliations of codimension one and degree two on $\mathbb{P}^{n-1}$.*

In the case $d\eta\not\equiv 0$ we need some definitions. If $F\colon\mathbb{C}^n\to\mathbb{C}^3$ is a polynomial map of degree two, and α is a polynomial 2-form, also of degree two, in general the 2-form $F^*\alpha$ is of degree five. However, there are special configurations of pairing (F,α) with the property that $deg(F^*\alpha)<5$. The next four families of examples will be important to us. The first three examples will be of the form $F_j^*(\alpha)$ where

$\alpha = du \wedge [A(z_1, z_2)\, dz_1 + B(z_1, z_2)\, dz_2] + [a\, u + q(z_1, z_2)]\, dz_1 \wedge dz_2$ and $F_j(z_1, z_2, ..., z_n) = (z_1, z_2, u_j(z)) = (z_1, z_2, u)$, $1 \le j \le 3$. In the form α the polynomials A and B are linear and q is quadratic. In the map F_j the polynomial $u_j(z)$ is a quadratic form on $\mathbb{C}^n$, $1 \le j \le 3$. The following possibilities will appear in our considerations:

(1). $\eta_1 := F_1^*(\alpha_1)$, where $u_1(z) = z_3\, z_4$ and $\alpha_1 = \alpha$.

(2). $\eta_2 := F_2^*(\alpha_2)$, where $u_2(z) = z_3^2 - 2\, z_1\, z_4$ and $\alpha_2 = \alpha$.

(3). $\eta_3 := F_3^*(\alpha_3)$, where $u_3(z) = z_2\, z_3 - z_1\, z_4$ and $\alpha_3 = \alpha$.

(4). $\eta_4 := F_4^*(\alpha_4)$, where $\alpha_4 = \lambda\, u\, dz_1 \wedge dz_2 + \lambda_1\, z_1\, dz_2 \wedge du + \lambda_2\, z_2\, dz_1 \wedge du$ and $F_4(z) = (z_1, z_2, u_4(z)) = (z_1, z_2, u)$, with $u_4(z)$ a quadratic form on $\mathbb{C}^n$.

In each case the 2-form η_i is integrable and homogeneous of degree two. Another remark is that the foliations defined in $\mathbb{C}^3$ by the 2-forms α and α_4 are Liouville integrable (in other words, the forms α and α_4 can be written as $i_Z\, dz_1 \wedge dz_2 \wedge du$ and the flow associated to the vector fields Z can be explicitly integrated).

We will say that the configuration (F_j, α_j) is *nice* of type (j), $1 \le j \le 4$.

The next result describes all possibilities for the foliations defined by non-closed 2-forms homogeneous of degree two:

Theorem 5. *Let η be a non-closed homogeneous integrable 2-form of degree two in $\mathbb{C}^n$, $n \ge 4$, and $\mathcal{F}_\eta$ be the codimension two foliation associated to it. Then $\mathcal{F}_\eta$ is of one of the following types:*

(a). *Associated to a two dimensional lie algebra $\mathcal{L}$ of linear vector fields, either abelian or affine.*

(b). *A linear pull-back of a one dimensional foliation on $\mathbb{C}^3$ defined by a homogeneous vector field of degree two.*

(c). *Up to linear conjugacy, associated to a nice pair (F_j, α_j), $1 \le j \le 4$.*

In the proof of the theorem, normal forms will appear naturally. These normal forms will be useful for the reader interested in a precise description of the irreducible components of the space of such foliations, in particular, to understand the possible degenerated cases.

Proof of theorem 5. The proof will be divided in several cases and subcases. First of all we consider the case $n = 4$.

Case 1. $n = 4$ and $cod(Sing(d\eta)) \ge 3$. This is equivalent to $cod(Sing(X)) \ge 3$, where $X = rot(\eta)$. As we will see below this case corresponds to case (a) of theorem 5.

Lemma 3.2. *In case 1 we have*

$$\eta = i_Y\, i_X \nu\ ,\ \nu = dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4,$$

where Y is a linear vector field satisfying $[Y,X] = \lambda X$, $\lambda = 1 - tr(Y)$. Moreover,

(a). *If X is not nilpotent then $\lambda = 0$ and $tr(Y) = 1$. In particular, X and Y commute.*

(b). *If X is nilpotent and $\lambda \neq 0$ then after a linear change of variables we have*

$$X = z_1 \frac{\partial}{\partial_{z_2}} + z_2 \frac{\partial}{\partial_{z_3}} + z_3 \frac{\partial}{\partial_{z_4}},$$

and

$$Y = \rho z_1 \frac{\partial}{\partial z_1} + (\rho - \lambda) z_2 \frac{\partial}{\partial z_2} + (\rho - 2\lambda) z_3 \frac{\partial}{\partial z_3} + (\rho - 3\lambda) z_4 \frac{\partial}{\partial z_4},$$

where $4\rho - 5\lambda = 1$.

Proof. Recall that $d\eta = i_X \nu$, $\nu = dx_1 \wedge ... \wedge dx_4$. Since η is homogeneous of degree two, X is a linear vector field with $tr(X) = 0$. Since $cod(sing(X)) \geq 3$, by the division theorem, there exists another linear vector field Y such that

$$\eta = i_Y \, i_X \, \nu \implies L_Y \, \eta = i_Y \, d\eta = \eta \implies L_Y \, d\eta = d\eta.$$

The last relation implies

$$d\eta = i_X \, \nu = L_Y (i_X \, \nu) = i_{[Y,X]} \, \nu + i_X \, L_Y \, \nu = i_{[Y,X]} \, \nu + tr(Y) \, i_X \, \nu \implies$$
$$[Y,X] = (1 - tr(Y)) \, X := \lambda \, X \; , \; \lambda = 1 - tr(Y).$$

Consider X as a derivation $X = \sum_{j=1}^{4} X_j \, \frac{\partial}{\partial x_j}$ on $(\mathbb{C}^4)^*$. Since X is linear the k^{th} power operator X^k, $k \geq 2$, is also a derivation $X^k = \sum_{j=1}^{4} X_j^k \, \frac{\partial}{\partial x_j}$. Moreover, if the eigenvalues of X are $\lambda_1, ..., \lambda_4$ then the eigenvalues of X^k are $\lambda_1^k, ..., \lambda_4^k$.

As a derivation, the relation $[Y,X] = \lambda \, X$ can be written as $Y. \, X - X. \, Y = \lambda \, X$. It implies that $Y. \, X^k - X^k. \, Y = k \, \lambda \, X^k$, for all $k \geq 1$. The proof is by induction on $k \geq 1$. Let us assume, by induction, that $Y. \, X^k - X^k. \, Y = k \, \lambda \, X^k$ for some $k \geq 1$. Then

$$\left.\begin{matrix} Y. \, X^k - X^k. \, Y = k \, \lambda \, X^k \\ Y. \, X - X. \, Y = \lambda \, X \end{matrix}\right\} \implies \left.\begin{matrix} Y. \, X^{k+1} - X^k. \, Y. \, X = k \, \lambda \, X^{k+1} \\ X^k. \, Y. \, X - X^{k+1}. \, Y = \lambda \, X^{k+1} \end{matrix}\right\} \implies$$

$$Y. \, X^{k+1} - X^{k+1}. \, Y = (k+1) \, \lambda \, X^{k+1}.$$

Therefore, $Y. \, X^k - X^k. \, Y = k \, \lambda \, X^k$ for all $k \geq 1$. If $\lambda \neq 0$ then $tr(X^k) = k^{-1}. \, \lambda^{-1}. \, tr(Y. \, X^k - X^k. \, Y) = 0$ for all $k \geq 1$. However, this implies that $\lambda_1 = ... = \lambda_4 = 0$ and that X is nilpotent. Therefore, if X is not nilpotent then $\lambda = 0$, which proves (a).

Assume that X is nilpotent and $\lambda \neq 0$. After a linear change of variables we can assume that $X = z_1 \frac{\partial}{\partial z_2} + z_2 \frac{\partial}{\partial z_3} + z_3 \frac{\partial}{\partial z_4}$. As a derivation we have $[Y,X] = Y. \, X - X. \, Y = \lambda \, X$. If we apply both members in z_1 then we get $X(Y(z_1)) = 0$ which implies that $Y(z_1)$ is an eigenvector of

X: $Y(z_1) = \rho\, z_1$, $\rho \in \mathbb{C}$. When we apply in z_2 then $(Y.X - X.Y)(z_2) = Y(z_1) - X(Y(z_2)) = \lambda\, z_1 \implies$

$$X(Y(z_2)) = (\rho - \lambda)\, z_1 \implies Y(z_2) = (\rho - \lambda)\, z_2 + a\, z_1\ ,\ a \in \mathbb{C}.$$

By a similar argument we obtain

$$\begin{aligned} Y(z_3) &= (\rho - 2\,\lambda) z_3 + a\, z_2 + b\, z_1 \text{ and} \\ Y(z_4) &= (\rho - 3\,\lambda) z_4 + a\, z_3 + b\, z_2 + c\, z_1\ ,\ b\,,\, c \in \mathbb{C}. \end{aligned}$$

In particular, the eigenvalues of Y are ρ, $\rho - \lambda$, $\rho - 2\,\lambda$ and $\rho - 3\,\lambda$. Hence, $tr(Y) = 4\,\rho - 6\,\lambda$ and since $\lambda = 1 - tr(Y)$ we get the relation $4\rho - 5\lambda = 1$. The eigenvalues of Y are two by two distinct and so it is diagonalizable. Therefore, Y has an eigenvector $w = z_4 + \alpha\, z_3 + \beta\, z_2 + \gamma\, z_1$, where $Y(w) = (\rho - 3\lambda) w$. Set $z := X(w) = z_3 + \alpha\, z_2 + \beta\, z_1$, $y := X(z) = z_2 + \alpha\, z_1$ and $x := X(y) = z_1$. Finally,

$$Y(X(w)) - X(Y(w)) = \lambda\, X(w) \implies Y(z) = (\rho - 2\lambda)\, z.$$

Similarly, $Y(y) = (\rho - \lambda)\, y$ and $Y(x) = \rho\, x$. This finishes the proof. q.e.d.

In case (a) of lemma 3.2, where $[X, Y] = 0$, the vector fields X and Y generate an action of $\mathbb{C}^2$ on $\mathbb{C}^4$. We will assume the generic case, in which the X and Y are diagonalizable in the same basis of $\mathbb{C}^4$. This means that after a linear change of variables we can assume that $X = \sum_{j=1}^{4} \lambda_j\, z_j \frac{\partial}{\partial z_j}$ and $Y = \sum_{j=1}^{4} \mu_j\, z_j \frac{\partial}{\partial z_j}$, where $\sum_j \lambda_j = 0$ and $\sum_j \mu_j = 1$. We will assume also that $\lambda_i\, \mu_j - \lambda_j\, \mu_i \neq 0$ if $i \neq j$. In this case, we have

$$\eta = i_Y i_X \nu = z_1\, z_2\, z_3\, z_4 \sum_{i<j} \rho_{ij} \frac{dz_i \wedge dz_j}{z_i\, z_j}, \tag{13}$$

where $\nu = dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4$, $\rho_{k\ell} = \pm(\lambda_k \mu_\ell - \lambda_\ell \mu_k)$ and $\{k, \ell\} = \{1, 2, 3, 4\} \setminus \{i, j\}$. In particular, $f := z_1 z_2 z_3 z_4$ is an integrating factor of η: $d\left(\frac{\eta}{f}\right) = 0$.

As we will see next, in case (b) of lemma 3.2 the form η also has an integrating factor. Indeed, in case (b) we can write $Y = \rho\, R - \lambda\, S$, where $R = \sum_{j=1}^{4} z_j \frac{\partial}{\partial z_j}$ is the radial vector field and $S = \sum_{j=2}^{4} (j-1) z_j\, \frac{\partial}{\partial z_j}$. In particular, S, R, X generate a Lie algebra of linear vector fields with the relations $[S, R] = [R, X] = 0$ and $[S, X] = -X$. Set $\alpha := i_S\, i_X \nu$ and $\beta := i_R\, i_X \nu$, so that $\eta = \rho.\, \beta - \lambda.\, \alpha$. Let $f := g.h/z_1$, where

$$g = z_2^3 - 3\, z_1\, z_2\, z_3 + 3\, z_1^2\, z_4 \text{ and } h = z_2^2 - 2\, z_1\, z_3. \tag{14}$$

The reader can check directly that $df \wedge \alpha = f.\, d\alpha$ and $df \wedge \beta = f.\, d\beta$, which is equivalent to $d\left(\frac{1}{f}\, \alpha\right) = 0$ and $d\left(\frac{1}{f}\, \beta\right) = 0$ and this implies $d\left(\frac{1}{f}\, \eta\right) = 0$. It follows also that $\frac{1}{f}\, \alpha$ and $\frac{1}{f}\, \beta$ are logarithmic 2-forms with pole divisor $z_1.\,g.\,h$; they belong to the vector space generated by $\frac{dg\wedge dh}{gh}$, $\frac{dh\wedge dz_1}{hz_1}$ and $\frac{dz_1\wedge dg}{z_1 g}$. In fact, the reader can check directly that

$$\frac{z_1\alpha}{gh} = \left(\frac{1}{3}\frac{dg}{g} - \frac{1}{2}\frac{dh}{h}\right)\wedge\frac{dz_1}{z_1} \text{ and } \frac{z_1\beta}{gh} = \frac{1}{6}\frac{dh\wedge dg}{hg} + \frac{z_1\alpha}{gh},$$

so that

$$\frac{z_1\eta}{gh} = A\,\frac{dh\wedge dg}{hg} + B\,\frac{dg\wedge dz_1}{gz_1} + C\,\frac{dz_1\wedge dh}{z_1 h}, \tag{15}$$

where $A = \rho/6$, $B = (\rho-\lambda)/3$ and $C = (\rho-\lambda)/2$.

Remark 3.5. As we have seen in proposition 5 the form $\omega := i_R\eta$ is integrable; $\omega\wedge d\omega = 0$. Since $i_R d\omega = 4\,\omega$, if $\omega \not\equiv 0$ then $d\omega \not\equiv 0$. We would like to observe that for every $s\in\mathbb{C}$ the form $\eta_s := \eta + s\,d\omega$ is integrable. Let us prove this fact.

First of all Euler's identity implies that

$$d\omega = d\,i_R\eta = 4\,\eta - i_R\,d\eta = 4\,\eta - i_R i_X\nu \implies \tag{16}$$

$$d\omega\wedge\eta = -i_R i_X\nu\wedge i_Y i_X\nu = 0 \implies$$

$$\eta_s^2 = (\eta + s\,d\omega)^2 = 0,$$

because $\eta^2 = d\omega^2 = 0$. On the other hand, $rot(\eta_s) = rot(\eta) = X$ and from (16) we get $i_X\eta_s = 0$.

Case 2. $n = 4$ and $cod(Sing(d\eta)) \le 2$. This case will be divided in two sub-cases: η is dicritical or non-dicritical.

Case 2.1. $i_R\eta = 0$. We assert that in this case we have: $\eta = \frac{1}{4}\,i_R\,i_X\,\nu$, $\nu = dz_1\wedge\ldots\wedge dz_4$. In particular, $\mathcal{F}_\eta$ is defined by a commutative action. In fact, $i_R\eta = 0$ implies that $\eta = i_R\mu$, where μ is a 3-form homogeneous of degree one. In particular, there exits a linear vector field Y such that $\mu = i_Y\nu$, so that $\eta = i_R i_Y\nu$. On the other hand, Euler's identity implies

$$i_X\,\nu = d\eta = d\,i_R\mu = L_R\mu - i_R\,d\mu = 4\,\mu - i_R\,d\mu = 4\,i_Y\,\nu - i_R\,d\mu.$$

Since $d\mu$ is homogeneous of degree zero, then $d\mu = \rho\,\nu$, where $\rho\in\mathbb{C}$. Therefore, the above relation implies that

$$X = 4\,Y - \rho\,R \implies \eta = \frac{1}{4}\,i_R\,i_X\,\nu,$$

as asserted.

Case 2.2. $i_R\eta \ne 0$. We will divide in two sub-cases: $Sing(X)$ has codimension one or two.

Case 2.2.1. $cod(Sing(X)) = 1$. We will see that this case corresponds to case (b) in the statement of theorem 5.

We can write $X = H.Y$, where H is linear and Y a constant vector field. After a linear change of variables we can assume that $Y = \frac{\partial}{\partial z_4}$, and so

$$d\eta = H\, dz_1 \wedge dz_2 \wedge dz_3 \implies \frac{\partial H}{\partial z_4} = 0 \implies H = H(z_1, z_2, z_3).$$

Since $i_X \eta = 0$ we can write

$$\eta = A_1\, dz_2 \wedge dz_3 + A_2\, dz_3 \wedge dz_1 + A_3\, dz_1 \wedge dz_2 = i_Z\, dz_1 \wedge dz_2 \wedge dz_3,$$

where $Z = \sum_{j=1}^{3} A_j \frac{\partial}{\partial z_j}$. From $d\eta = H\, dz_1 \wedge dz_2 \wedge dz_3$, we get $\frac{\partial A_j}{\partial z_4} = 0$, $1 \le j \le 4$, and $\sum_{j=1}^{3} \frac{\partial A_j}{\partial z_j} = H$. Therefore, $\mathcal{F}_\eta$ is a linear pull-back of a degree two homogeneous one dimensional foliation on $\mathbb{C}^3$: the foliation defined by Z on $\mathbb{C}^3$.

Remark 3.6. In general η has no rational integrating factor in this case. Indeed, if it had a rational integrating factor, say $f = f_1/f_2$, $d\frac{1}{f}\eta = 0$, then $df \wedge \eta = f.\, d\eta$, which implies that $\frac{\partial f}{\partial z_4} = 0$ and $Z(f_j) = g_j.\, f_j$, $j = 1, 2$. In other words, the foliation defined by Z has at least one invariant homogeneous hypersurface. However, this is not true in general. In fact, consider the 1-form

$$\omega = i_R i_Z\, dz_1 \wedge dz_2 \wedge dz_3.$$

The form ω can be considered as the homogeneous expression of a degree two foliation on $\mathbb{P}^2$, say $\mathcal{G}$. Also, ω has the same invariant homogeneous hypersurfaces as η. A homogeneous invariant hypersurface for η gives origin to an algebraic invariant curve for $\mathcal{G}$. However, it is known that a generic foliation of degree two on $\mathbb{P}^2$ has no invariant algebraic curve (cf. [**17**] and [**21**]).

Case 2.2.2. $cod(Sing(X)) = 2$. Since X is linear and $tr(X) = 0$, it corresponds to a rank two 4×4 matrix with vanishing trace. There are three possible Jordan canonical forms: $z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2}$, $z_1 \frac{\partial}{\partial z_2} + z_2 \frac{\partial}{\partial z_3}$ and $z_1 \frac{\partial}{\partial z_3} + z_2 \frac{\partial}{\partial z_4}$.

Case 2.2.2.1. $X = z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2}$, or $d\eta = d(z_1 z_2) \wedge dz_3 \wedge dz_4$.

We will see that this case corresponds to the nice pair (F_1, α_1). In fact, From $i_X \eta = 0$ we get $z_1.\, i_{\frac{\partial}{\partial z_1}} \eta = z_2.\, i_{\frac{\partial}{\partial z_2}} \eta \implies$ there exists a 1-form β, homogeneous of degree one, such that $i_{\frac{\partial}{\partial z_1}} \eta = z_2.\, \beta$ and $i_{\frac{\partial}{\partial z_2}} \eta = z_1.\, \beta$. Note that $i_{\frac{\partial}{\partial z_1}} \beta = i_{\frac{\partial}{\partial z_2}} \beta = 0 \implies \beta = A\, dz_3 + B\, dz_4$, where A and B are linear. If we set $\eta = \sum_{1 \le i < j \le 4} P_{ij}\, dz_i \wedge dz_j$ then

$$\begin{cases} z_2.\, (A\, dz_3 + B\, dz_4) = i_{\frac{\partial}{\partial z_1}} \eta = \sum_{j>1} P_{1j}\, dz_j \\ z_1.\, (A\, dz_3 + B\, dz_4) = i_{\frac{\partial}{\partial z_2}} \eta = \sum_{j \ne 2} P_{2j}\, dz_j \end{cases} \implies$$

$$P_{12} = 0\,,\ P_{13} = z_2.\, A\,,\ P_{14} = z_2.\, B\,,\ P_{23} = z_1.\, A \text{ and } P_{24} = z_1.\, B.$$

It follows that

$$\eta = d(z_1 z_2) \wedge (A\,dz_3 + B\,dz_4) + C\,dz_3 \wedge dz_4,$$

where $C = P_{34}$ is quadratic and A, B are linear. Now, recall that $L_X\eta = 0$ and so

$$0 = L_X\left[d(z_1 z_2) \wedge (A\,dz_3 + B\,dz_4) + C\,dz_3 \wedge dz_4\right] =$$
$$= d(z_1 z_2) \wedge (X(A)\,dz_3 + X(B)\,dz_4) + X(C)\,dz_3 \wedge dz_4 \implies$$
$$X(A) = X(B) = X(C) = 0 \implies$$

$A = A(z_3, z_4)$, $B = B(z_3, z_4)$ and $C = \widetilde{C}(z_1.z_2, z_3, z_4) = a.\,z_1.z_2 + q(z_3, z_4)$, q quadratic. Therefore,

$$\eta = d(z_1\,z_2) \wedge (A(z_3, z_4)\,dz_3 + B(z_3, z_4)\,dz_4) + (a.z_1.z_2 + q(z_3, z_4))\,dz_3 \wedge dz_4.$$

In particular, η is a pull-back of a 2-form on $\mathbb{C}^3$, $\eta = \Phi^*(\widetilde{\eta})$, where $\Phi(z_1, z_2, z_3, z_4) = (z_1.z_2, z_3, z_4) = (u, z_3, z_4)$ and

$$\widetilde{\eta} = du \wedge (A(z_3, z_4)\,dz_3 + B(z_3, z_4)\,dz_4) + (a\,u + q(z_3, z_4))\,dz_3 \wedge dz_4.$$

Note also that $\mathcal{F}_\eta$ and $\mathcal{F}_{\widetilde{\eta}}$ are contained in $\mathcal{F}_\beta$, $\beta = A\,dz_3 + B\,dz_4$, because $\eta \wedge \beta = \widetilde{\eta} \wedge \beta = 0$.

Case 2.2.2.2. $X = z_1 \frac{\partial}{\partial z_2} + z_2 \frac{\partial}{\partial z_3}$, or $d\eta = -z_1\,dz_1 \wedge dz_3 \wedge dz_4 + z_2\,dz_1 \wedge dz_2 \wedge dz_4$.

We will see that this case corresponds to the nice pair (F_2, α_2). In fact, the integrability relation $i_X\eta = 0$ implies that $z_1.\,i_{\frac{\partial}{\partial z_2}}\,\eta = -z_2.\,i_{\frac{\partial}{\partial z_3}}\,\eta$ $\implies i_{\frac{\partial}{\partial z_2}}\,\eta = z_2.\beta$ and $i_{\frac{\partial}{\partial z_3}}\,\eta = -z_1.\beta$, where $\beta = A\,dz_1 + B\,dz_4$, A and B linear. With an argument similar to the preceding case, we get

$$\eta = (z_2\,dz_2 - z_1\,dz_3) \wedge (A\,dz_1 + B\,dz_4) + C\,dz_1 \wedge dz_4,$$

where C is homogeneous of degree two. The reader can check that the condition $L_X\eta = 0$ is equivalent to $X(A) = X(B) = 0$ and $X(C) + z_2\,B = 0$. Since the first integrals of X are generated by z_1, z_4 and $z_2^2 - 2z_1\,z_3$, we get

$$A = A(z_1, z_4)\,,\ B = B(z_1, z_4) \text{ and}$$
$$C = -z_3\,B(z_1, z_4) + a(z_2^2 - 2z_1\,z_3) + q(z_1, z_4),$$

where $a \in \mathbb{C}$ and q is homogeneous of degree two. In particular, we get $\eta = \Phi^*(\widetilde{\eta})$, where

$$\Phi(z_1, z_2, z_3, z_4) = (z_1, z_4, z_2^2 - 2z_1\,z_3) = (z_1, z_4, u).$$

and

$$\widetilde{\eta} = \frac{1}{2}\,du \wedge [A\,dz_1 + B\,dz_4] + [a.\,u + q]\;dz_1 \wedge dz_4.$$

Case 2.2.2.3. $X = z_1 \frac{\partial}{\partial z_3} + z_2 \frac{\partial}{\partial z_4}$.

We will see that this case corresponds to the nice pair (F_3, α_3). In fact, with an argument similar to the preceding cases we get

$$\eta = (z_2\, dz_3 - z_1\, dz_4) \wedge (A\, dz_1 + B\, dz_2) + C\, dz_1 \wedge dz_2,$$

where A, B are linear and C homogeneous of degree two. From the condition $L_X \eta = 0$ we get

$$X(A) = X(B) = 0 \text{ and } X(C) + z_1\, A + z_2\, B = 0.$$

Since the first integrals of X are generated by z_1, z_2 and $z_2\, z_3 - z_1\, z_4$ we get $A = A(z_1, z_2)$ $B = B(z_1, z_2)$ and

$C = -z_3\, A(z_1, z_2) - z_4\, B(z_1, z_2) + a(z_2\, z_3 - z_1\, z_4) + q(z_1, z_2)$, where $a \in \mathbb{C}$ and q is homogeneous of degree two. Here we obtain $\eta = \Phi^*(\widetilde{\eta})$ where $\Phi(z_1, z_2, z_3, z_4) = (z_1, z_2, z_2\, z_3 - z_1\, z_4) = (z_1, z_2, u)$ and

$$\widetilde{\eta} = du \wedge [A(z_1, z_2)\, dz_1 + B(z_1, z_2)\, dz_2] + [a\, u + q(z_1, z_2)]\; dz_1 \wedge dz_2.$$

Next we will extend the result to $n \geq 5$. Let us consider first the case where η is dicritical: $i_R \eta = 0$. From Euler's identity we get: $4\,\eta = L_R \eta = i_R\, d\eta$. The form $d\eta$ is integrable and has degree one. Here we use a result due to Medeiros [**31**]: we have two possibilities:

- either there exists a projection $\Pi\colon \mathbb{C}^n \to \mathbb{C}^4$ and a linear 3-form θ on $\mathbb{C}^4$ such that $d\eta = \Pi^*(\theta)$,
- or $d\eta = 4\, dL_1 \wedge dL_2 \wedge dQ$, where Q is quadratic and L_1, L_2 are linear.

In the first possibility we can assume that $\Pi(z_1, ..., z_n) = (z_1, ..., z_4)$ and θ can be written as $\theta = 4\, i_Z\, dz_1 \wedge ... \wedge dz_4$, where Z is a linear vector field in $\mathbb{C}^4$. We have

$$\eta = i_{R_4}\, i_Z\, dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 = \pm i_{R_4}\, i_Z\, i_{\frac{\partial}{\partial z_5}} ... i_{\frac{\partial}{\partial z_n}}\, dz_1 \wedge ... \wedge dz_n.$$

We are in case (a) of the statement of theorem 5.

In the second possibility we have

$$\eta = \frac{1}{4}\, i_R\, d\eta = 2\, Q\, dL_1 \wedge dL_2 + L_1\, dL_2 \wedge dQ - L_2\, dL_1 \wedge dQ.$$

This case corresponds to a nice pair (F_4, α_4).

Let us assume that $\omega = i_R \eta \neq 0$. We can write $\omega = h.\, \omega_1$ where:

1. $cod(Sing(\omega)) = 2$ and $\omega_1 = \omega$. In this case ω defines a codimension one foliation $\mathcal{G}$ of degree two on $\mathbb{P}^{n-1}$.
2. $deg(h) = 1$ and ω_1 defines a foliation $\mathcal{G}$ of degree one on $\mathbb{P}^{n-1}$.
3. $deg(h) = 2$ and ω_1 defines a foliation $\mathcal{G}$ of degree zero on $\mathbb{P}^{n-1}$.

In the proof we will use the classification of the components of the space of codimension one foliations of degree ≤ 3.

In some components of these spaces the form ω_1 can be expressed with k variables, where $k \in \{2, 3, 4\}$. In other words, ω_1 can be written as

$$\omega_1 = \sum_{1 \le i < k} A_i(z_1, ..., z_k)\, dz_i.$$

This happens in the following cases:

1.a. $\mathcal{G} \in L(1,1,1,1) \cup E(n-1) \cup S(2, n-1)$, if the degree of $\mathcal{G}$ is two.
2.a. $\mathcal{G} \in L(1,1,1)$, if the degree of $\mathcal{G}$ is one.
3. If the degree of $\mathcal{G}$ is zero.

We will see that, in cases 1.a and 2.a the 2-form η can be written with four variables. This reduces the study of these cases to the case $n = 4$. In case 3 this is not the case, as we will see.

Claim 3.1. *If* $\omega_1 = \sum_{i=1}^{k} B_i(z_1, ..., z_k)\, dz_i \wedge dz_j$ *then*

$$\eta = \eta_o + \alpha \wedge \omega_1,$$

where

- $\eta_o = \sum_{1 \le i < j \le k} A_{ij}(z)\, dz_i \wedge dz_j$, *and*
- $\alpha = 0$ *if* $\omega = \omega_1$ *and* $\alpha = \sum_{j > k} C_j(z)\, dz_j$ *if* $\omega = h.\omega_1$, $deg(h) > 0$.

Proof. If $\eta = \sum_{1 \le i < j \le n} A_{ij}(z)\, dz_i \wedge dz_j$ then set $\eta_o = \sum_{1 \le i < j \le k} A_{ij}(z)\, dz_i \wedge dz_j$. Note that

$$2\,\omega \wedge \eta = i_R(\eta^2) = 0 \implies \omega_1 \wedge \eta = 0.$$

If we assume $B_k \neq 0$ then

$$0 = dz_1 \wedge ... \wedge dz_{k-1} \wedge \omega_1 \wedge \eta = B_k\, dz_1 \wedge ... \wedge dz_k \wedge \eta \implies$$
$$dz_1 \wedge ... \wedge dz_k \wedge \eta = 0 \implies$$
$$\eta = \eta_0 + \sum_{j > k} \alpha_j \wedge dz_j \text{ , where } \alpha_j = \sum_{i=1}^{k} A_{ij}\, dz_i.$$

From $\omega_1 \wedge \eta = 0$ we get

$$\omega_1 \wedge \eta_o + \sum_{j > k} \omega_1 \wedge \alpha_j \wedge dz_j = 0 \implies \omega_1 \wedge \alpha_j = 0\,, \forall\, j > k.$$

From the division theorem [15] we get $\alpha_j = h_j.\omega_1$, where $h_j = 0$ if $\omega_1 = \omega$ and $deg(h_j) = 2 - deg(\omega_1)$ if $deg(\omega_1) < 3$. When $deg(\omega_1) < 3$ we get $\eta = \eta_o + \alpha \wedge \omega_1$, where $\alpha = -\sum_{j > k} h_j\, dz_j$. q.e.d.

Claim 3.2. *Let* $\eta = \sum_{1 \le i < j \le k} A_{ij}(z_1, ..., z_n)\, dz_i \wedge dz_j$ *be an integrable homogeneous 2-form on* $\mathbb{C}^n$, *where* $k \in \{3, 4\}$ *and* $cod(Sing(\eta)) \ge 2$. *Then* $\frac{\partial}{\partial z_\ell} A_{ij} = 0$ *for all* $\ell > k$. *In particular,* η *can be written with* k *variables.*

Proof. We first note that $i_{\frac{\partial}{\partial z_j}} \eta = 0$ for all $j > k$. In particular, the singular distribution defined by $ker(\eta)$ contains $\left\langle \frac{\partial}{\partial z_j} \,|\, j > k \right\rangle_{\mathcal{O}}$. In fact, in the case $k = 3$, we have

$$ker(\eta) = \left\langle Y, \frac{\partial}{\partial z_4}, ..., \frac{\partial}{\partial z_n} \right\rangle_{\mathcal{O}} := \mathcal{D},$$

where $Y = A_{23} \frac{\partial}{\partial z_1} - A13 \frac{\partial}{\partial z_2} + A_{12} \frac{\partial}{\partial z_3}$. Note also that in the case $k = 4$ we have $ker(\eta) \supset \mathcal{D}$, where

$$\mathcal{D} = \left\langle Y_1, Y_2, Y_3, Y_4, \frac{\partial}{\partial z_5}, ..., \frac{\partial}{\partial z_n} \right\rangle_{\mathcal{O}},$$

and Y_j is defined by

$$i_{Y_j}\, i_{\frac{\partial}{\partial z_j}}\, dz_1 \wedge ... \wedge dz_4 = \sum_{\substack{1 \le r < s \le 4 \\ r,s \ne j}} A_{rs}(z)\, dz_r \wedge dz_s.$$

Since η is integrable, in both cases the distribution $\mathcal{D}$ is involutive and we can use a result of [**6**]: the coefficients of Y, or of Y_1, ...,Y_4, do not depend on the variables z_{k+1},..., z_n. This proves the claim. q.e.d.

Let us return to the cases in which we can reduce the variables.

Case 1.a. When $\mathcal{G} \in L(1,1,1,1) \cup E(n-1)$ then ω can be written with four variables: after a linear change of variables we can write $\omega = i_{R_4} i_Y i_X\, dz_1 \wedge ... \wedge dz_4$, where R_4 is the radial in $\mathbb{C}^4$, $Y = \sum_{j=1}^4 \mu_j\, z_j \frac{\partial}{\partial z_j}$ and, either $X = \sum_{j=1}^4 \lambda_j\, z_j \frac{\partial}{\partial z_j}$ if $\mathcal{G} \in L(1,1,1,1)$, or $X = z_1 \frac{\partial}{\partial z_2} + z_2 \frac{\partial}{\partial z_3} + z_3 \frac{\partial}{\partial z_4}$ if $\mathcal{G} \in E(n-1)$ (see lemma 3.2). In this case, we can apply directly claims 3.1 and 3.2.

In the case $\mathcal{G} \in S(2, n-1)$, in which $k = 3$ in claim 3.1, we have $\omega = i_{R_3}\, i_X\, dz_1 \wedge dz_2 \wedge dz_3$, $R_3 = \sum_{1 \le i \le 3} z_i \frac{\partial}{\partial z_i}$ and X is a quadratic vector field on $\mathbb{C}^3$. In this case, applying claim 3.2 we get $\eta = i_{\widetilde{X}}\, dz_1 \wedge dz_2 \wedge dz_3$, where $\widetilde{X} = X + g.\, R_3$ and g is linear. Therefore, η is like in (b) of the statement of theorem 5.

Case 2.a. In this case, $\omega_1 = i_{R_3} i_X dz_1 \wedge dz_2 \wedge dz_3$, where X is a linear vector field on $\mathbb{C}^3$, and $\eta = \eta_o + \alpha \wedge \omega_1$ where α is a 1-form with constant coefficients. Hence, $\alpha = df$, where f is linear. If $df \wedge dz_1 \wedge dz_2 \wedge dz_3 = 0$ then $\eta = \sum_{1 \le i < j \le 3} A_{ij}(z_1, z_2, z_3)\, dz_i \wedge dz_j$ by claim 3.2. If $df \wedge dz_1 \wedge dz_2 \wedge dz_3 \ne 0$ then we can assume that $df = dz_4$ and $\eta = \sum_{1 \le i < j \le 4} A_{ij}(z_1, z_2, z_3, z_4)\, dz_i \wedge dz_j$ by claim 3.2.

Case 3. If the degree of $\mathcal{G}$ is zero then we can assume $\omega_1 = z_1\, dz_2 - z_2\, dz_1$. By claim 3.1 we can write $\eta = A(z)\, dz_1 \wedge dz_2 + \alpha \wedge \omega_1$, where A is a homogeneous polynomial of degree two and $\alpha = \sum_{\ell > 2} C_\ell(z)\, dz_\ell$, with C_ℓ linear, $3 \le \ell \le n$. Let us consider the blowing-up $\Pi(t, z_1, z_3, ..., z_n) =$

$(z_1, t\,z_1, z_3, ..., z_n)$. Then

$$\begin{aligned}\Pi^*(\eta) &= A(z_1, t\,z_1, z_3, ..., z_n)\,\Pi^*(dz_1 \wedge dz_2)\\ &\quad + \sum_{j>2} C_j(z_1, t\,z_1, z_3, ..., z_n)\,dz_j \wedge \Pi^*(\omega_1)\\ &= z_1\,\widetilde{A}(t, z_1, z_3, ..., z_n)\,dz_1 \wedge dt\\ &\quad\quad + z_1^2 \sum_{j>2} \widetilde{C}_j(t, z_1, z_3, ..., z_n)\,dz_j \wedge dt = z_1.\,\beta \wedge dt,\end{aligned}$$

where $\beta = \widetilde{A}(t, z_1, z_3, ..., z_n)\,dz_1 + z_1\,\sum_{j>2} \widetilde{C}_j(t, z_1, z_3, ..., z_n)\,dz_j$. The strict transform of $\Pi^*(\eta)$ is $\widetilde{\eta} = \beta \wedge dt$. Since β does not depends on dt, it can be considered as a 1-parameter family of 1-forms in $\mathbb{C}^{n-1}$. Given $c \in \mathbb{C}$, set $\beta_c := \beta|_{t=c}$. The integrability condition for $\widetilde{\eta}$ is $d\beta \wedge \beta \wedge dt = 0$, which implies $\beta_t \wedge d\beta_t = 0$, so that β_t is integrable and homogeneous of degree two for each fixed $t \in \mathbb{C}$. We assert that $z_1\,h_t$ is an integrating factor for β_t, where $h_t(z_1, z_3, ..., z_n) = h(z_1, t\,z_1, z_3, ..., z_n)$ (recall that $i_R\eta = h.\,\omega_1$).

In fact, let $\widetilde{R}$ be the radial vector field on $\mathbb{C}^{n-1}$; $\widetilde{R} = z_1\frac{\partial}{\partial z_1} + \sum_{3\le j\le n} z_j\,\frac{\partial}{\partial z_j}$. In [**14**] it is proven that if $f_t := i_{\widetilde{R}}\beta_t \neq 0$ then it is an integrating factor of β_t: $d\left(\frac{1}{f_t}\,\beta_t\right) = 0$. Since $\widetilde{R} = \Pi^*(R)$, we get

$$\begin{aligned}&z_1^2\,h(z_1, t\,z_1, z_3, ..., z_n)\,dt\\ &\quad = \Pi^*(h.\,\omega_1) = \Pi^*(i_R\eta) = i_{\widetilde{R}}(z_1\,\beta \wedge dt) = z_1\,(i_{\widetilde{R}}\beta)\,dt \implies\\ &f_t = i_{\widetilde{R}}\beta = z_1\,h(z_1, t\,z_1, z_3, ..., z_n) := z_1.h_t,\end{aligned}$$

which proves the assertion.

We have two possibilities, either h_t is irreducible for generic t, or h_t is reducible for all t. For simplicity, in the second possibility we will assume that $h_t = f_t.\,g_t$, where f_t and g_t are linear and $dz_1 \wedge df_t \wedge dg_t \neq 0$ for generic t. In both cases $\frac{1}{z_1\,h_t}\beta_t$ is a logarithmic 1-form. According to [**14**] we can write:

3.a. If h_t is irreducible then

$$\frac{1}{z_1\,h_t}\beta_t = a(t)\,\frac{dz_1}{z_1} + b(t)\,\frac{dh_t}{h_t}\,, \text{ where } a(t), b(t) \in \mathbb{C}.$$

3.b. If $h_t = f_t.\,g_t$, where f_t and g_t are linear and $dz_1 \wedge df_t \wedge dg_t \neq$ for generic t then

$$\begin{aligned}\frac{1}{z_1\,h_t}\beta_t &= \frac{1}{z_1\,f_t\,g_t}\beta_t\\ &= a(t)\,\frac{dz_1}{z_1} + b(t)\,\frac{df_t}{f_t} + c(t)\,\frac{dg_t}{g_t}\,, \text{ where } a(t), b(t), c(t) \in \mathbb{C}.\end{aligned}$$

Now, from $\Pi^*(\eta) = z_1\,\beta \wedge dt$ we get

$$\frac{1}{z_1\,h_t}\beta_t \wedge dt = \frac{z_1\,\beta_t \wedge dt}{z_1^2\,h_t} = \Pi^*\left(\frac{\eta}{z_1^2\,h}\right) \implies \eta = z_1^2\,h\,\Pi_*\left(\frac{1}{z_1\,h_t}\beta_t \wedge dt\right).$$

Therefore, in case 3.a we get

$$\begin{aligned}\eta &= z_1^2\,h\,\Pi_*\left(\left[a(t)\,\frac{dz_1}{z_1} + b(t)\,\frac{dh_t}{h_t}\right] \wedge dt\right)\\ &= a(z_2/z_1)\,h\,dz_1 \wedge dz_2 + b(z_2/z_1)\,dh \wedge \omega_1.\end{aligned}$$

Since η is homogeneous of degree two, we obtain that a and b are constant. This case corresponds to the nice pair (F_4, α_4).

In case 3.b, by the same type of computation we get

$$\eta = a\,f\,g\,dz_1 \wedge dz_2 + (b\,g\,df + c\,f\,dg) \wedge \omega_1,$$

where $a, b, c \in \mathbb{C}^*$ and $h = f.g$, with f and g linear. This case corresponds to (a) in theorem 5

In case 2, where $\omega = \omega_1$ and $cod(Sing(\omega)) = 2$ there are three cases more: $\mathcal{G} \in R(1,3)$, $\mathcal{G} \in R(2,2)$ and $\mathcal{G} \in L(2,1,1)$.

Let us consider the case $\mathcal{G} \in L(2,1,1)$. In this case we have

$$\omega = f_1\,f_2\,f_3\,\left(\lambda_1\,\frac{df_1}{f_1} + \lambda_2\,\frac{df_2}{f_2} + \lambda_3\,\frac{df_3}{f_3}\right),$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$, f_3 is quadratic and f_j linear, $j = 1, 2$. We will assume that $\lambda_j \in \mathbb{C}^*$, $\forall j$, that $cod(Sing(df_3)) \geq 5$ and $df_1 \wedge df_2 \neq 0$. We assert that in this case we have

$$\eta = f_1\,f_2\,f_3\,\left(\mu_1\,\frac{df_2}{f_2} \wedge \frac{df_3}{f_3} + \mu_2\,\frac{df_3}{f_3} \wedge \frac{df_1}{f_1} + \mu_3\,\frac{df_1}{f_1} \wedge \frac{df_2}{f_2}\right), \tag{17}$$

where $\mu, \mu_1, \mu_2 \in \mathbb{C}$. Since $df_1 \wedge df_2 \neq 0$ we can assume that $f_1 = z_1$ and $f_2 = z_2$.

From $\omega \wedge \eta = 0$ we obtain $\omega \wedge d\eta = 0 \implies \lambda_3\,z_1\,z_2\,df_3 \wedge d\eta = f_3\,\alpha$, where $\alpha = -(\lambda_1\,z_2\,dz_1 + \lambda_2\,z_1\,dz_2) \wedge d\eta$. This implies that f_3 divides $df_3 \wedge d\eta$, because f_3 does not divide $z_1.\,z_2$. In particular, we can write $df_3 \wedge d\eta = f_3\,\widetilde{\alpha}$, where $\widetilde{\alpha}$ is a 4-form with constant coefficients. This implies that $df_3 \wedge \widetilde{\alpha} = 0$. Since $cod(Sing(df_3)) \geq 5$ the division theorem [**15**] and [**34**] and the fact that $\widetilde{\alpha}$ has constant coefficients imply that $\widetilde{\alpha} = 0$. Hence, $df_3 \wedge d\eta = 0$ and again by the division theorem $d\eta = df_3 \wedge \beta$, where β is a 2-form with constant coefficients. Therefore,

$$0 = \omega \wedge d\eta = \omega \wedge df_3 \wedge \beta = f_3\,(\lambda_1\,z_2\,dz_1 + \lambda_2\,z_1\,dz_2) \wedge \beta \implies \alpha_1 \wedge \beta = 0$$

where $\alpha_1 = \lambda_1\,z_2\,dz_1 + \lambda_2\,z_1\,dz_2$. However, since β has constant coefficients, $\omega_1 \wedge \beta = 0$ implies that $\beta = \rho\,dz_1 \wedge dz_2$ and $d\eta = \rho\,df_3 \wedge dz_1 \wedge dz_2$, where we can assume $\rho \in \mathbb{C}^*$. Now, from Euler's identity we have $4\,\eta = d\omega + i_R\,d\eta$ and the reader can check directly that $d\omega$, $i_R\,d\eta \in$

$\langle f_3\, dz_1 \wedge dz_2,\, z_1\, dz_2 \wedge df_3,\, z_2\, dz_1 \wedge df_3 \rangle_{\mathbb{C}}$ which implies (17). This case corresponds to the nice pair (F_4, α_4).

Let us assume now that $\mathcal{G} \in R(p,q)$, where $(p,q) \in \{(1,3),(2,2)\}$. In this case,

$$\omega = f.\, g.\, \left(p\, \frac{dg}{g} - q\, \frac{df}{f} \right) = p\, f\, dg - q\, g\, df.$$

We will assume that $cod(Sing(df))$, $cod(Sing(dg)) \geq 5$, which is the generic case. We assert that η is closed and we are in the situation of theorem 4.

Let us consider the case $q = 3$; $deg(g) = 3$. By an argument similar to the case of $L(2,1,1)$, we have $dg \wedge d\eta = 0$. Since $deg(d\eta) = 1$ and $deg(dg) = 2$, the division theorem implies in this case that $d\eta = 0$. In the case, $p = q = 2$ we get similarly that $df \wedge d\eta = dg \wedge d\eta = 0$. Hence, $d\eta = df \wedge \alpha_1 = dg \wedge \alpha_2$, by the division theorem, and this implies again that $d\eta = 0$. q.e.d.

Remark 3.7. In the proof of theorem 5 we have assumed in some of the sub-cases a generic situation for the foliation $\mathcal{G}$ induced by $\omega = i_R \eta$. For instance, when we consider the case $\mathcal{G} \in X$, where $X = L(2,1,1)$ or $X = L(1,1,1,1)$, we have supposed that ω has a reduced integrating factor. In the complement $\overline{X} \setminus X$ there are foliations of codimension one and degree two that are represented in $\mathbb{C}^n$ by a form ω with non-reduced integrating factors. In the case of $\mathcal{G} \in \overline{L(2,1,1)} \setminus L(2,1,1)$ we have non-reduced integrating factors of the form $L_1^2.\, L_2^2$ and $L_1^2.\, Q$, where L_1, L_2 are linear and Q quadratic, whereas in the case of $\mathcal{G} \in \overline{L(1,1,1,1)} \setminus L(1,1,1,1)$ we have non-reduced integrating factors of the form $L_1^2.\, L_2.\, L_3$, $L_1^2.\, L_2^2$ and $L_1^3\, L_2$, where L_1, L_2 and L_3 are linear. We would like to mention that all these cases can be treated as in the previous proof. At the end we find degenerations of the cases we have found, but the assertions of statement of the theorem are still valid in all these cases.

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