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MAXIMIZING STEKLOV EIGENVALUES ON SURFACES

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Abstract

We study the Steklov eigenvalue functionals $\sigma_k(\Sigma, g) L_g(\partial\Sigma)$ on smooth surfaces with non-empty boundary. We prove that, under some natural gap assumptions, these functionals do admit maximal metrics which come with an associated minimal surface with free boundary from Σ into some Euclidean ball, generalizing previous results by Fraser and Schoen in [10].

Let Σ be a smooth compact connected surface with a smooth boundary $\partial\Sigma \neq \emptyset$. We denote by γ its genus and by m the number of connected components of its boundary, which, together with orientability, characterize topologically the surface. Given a Riemannian metric g on Σ, the Dirichlet-to-Neumann operator, $L : C^\infty(\partial\Sigma) \mapsto C^\infty(\partial\Sigma)$, is defined as follows: for any $u \in C^\infty(\partial\Sigma)$, consider the harmonic extension $\hat{u}$ of u in Σ, which is unique, then $Lu = \partial_\nu \hat{u}$ where ν is the outward unit conormal along $\partial\Sigma$. This operator is self-adjoint and has a discrete spectrum

$$0 = \sigma_0 < \sigma_1(\Sigma, g) \leq \sigma_2(\Sigma, g) \leq \cdots \leq \sigma_k(\Sigma, g) \leq \cdots \to +\infty$$

of so-called Steklov eigenvalues counted with multiplicity. These are the σ's for which there exists a non-trivial solution $u \in C^\infty(\Sigma)$, smooth up to the boundary, of

$$\begin{cases} \Delta_g u = 0 & \text{in } \Sigma, \\ \partial_\nu u = \sigma u & \text{on } \partial\Sigma, \end{cases}$$

where $\Delta_g = -div_g(\nabla)$ is the Laplace–Beltrami operator. These eigenvalues are also characterized by the following variational problem:

$$\sigma_k(\Sigma, g) = \inf_{E_{k+1}} \sup_{\phi \in E_{k+1} \setminus \{0\}} \frac{\int_\Sigma |\nabla\phi|_g^2 \, dv_g}{\int_{\partial\Sigma} \phi^2 d\sigma_g},$$

where the infimum is taken over the vector space of smooth functions E_{k+1} of dimension $k+1$.

These eigenvalues may be seen as functionals depending on the metric g. For obvious scaling reasons, it is more interesting to consider the functionals $\sigma_k(\Sigma, g) L_g(\partial\Sigma)$. There has been a recent interest in studying these Steklov eigenvalue functionals because of the connection

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between critical metrics for these functionals and minimal immersions of Σ with free boundary into some Euclidean ball. A smooth immersion $\Phi : \Sigma \mapsto \mathbb{B}^{n+1}$ is a minimal surface with free boundary if $\Phi(\Sigma)$ is a minimal surface with $\Phi(\partial\Sigma) \subset \mathbb{S}^n$ which hits the boundary orthogonally (that is, $\partial_\nu \Phi$ is parallel to Φ on $\partial\Sigma$). These free boundary minimal surfaces arise as critical points of the area when the surface is constrained to lie in the ball but is free to vary on the boundary of the ball. This link between this purely geometric problem and the Steklov eigenvalues was first discovered by Fraser–Schoen [**8**]. In particular, it is proved in Fraser–Schoen [**9**], proposition 2.4, that a metric g_0 on Σ such that $\sigma_k(\Sigma, g_0) L_{g_0}(\partial\Sigma)$ is maximal among smooth metrics on Σ comes with a conformal minimal immersion with free boundary $\Phi : \Sigma \mapsto \mathbb{B}^{n+1}$ for some n such that Φ is an isometry on $\partial\Sigma$, up to scaling. Note that, conversely, see again Fraser–Schoen [**9**], the coordinates of any conformal minimal immersion with free boundary are Steklov eigenfunctions corresponding to some σ_k. This link has led Fraser and Schoen to start an intensive study of the first Steklov eigenvalue (see [**8**], [**10**], [**9**]).

Thus, it is geometrically interesting to look for maximal metrics for Steklov eigenvalues in order to get conformal minimal immersions with free boundary. That's a good reason to introduce the topological invariant

$$\sigma_k(\gamma, m) = \sup_g \sigma_k(\Sigma, g) L_g(\partial\Sigma),$$

where Σ is an oriented surface of genus γ with m boundary components. Girouard and Polterovich [**13**] proved that

$$\sigma_k(\gamma, m) \leq 2\pi k(\gamma + m),$$

generalizing for $k \geq 2$ an estimate due to Fraser and Schoen [**8**] in the case $k = 1$. Very few exact values of $\sigma_k(\gamma, m)$ are known. Weinstock [**26**] proved in 1954 that

$$\sigma_k(0,1) = 2\pi k, \tag{0.1}$$

and that for $k = 1$, the case of equality holds for the Euclidean disk. The exact value of $\sigma_1(0,2)$ was found by Fraser–Schoen [**10**] and the maximizing metric was characterized as coming from the critical catenoid. In this same paper, an asymptotic of $\sigma_1(0,m)$ as $m \to +\infty$ was obtained.

It can also be shown by standard gluing procedures (even if a bit technical, see [**7**]) that the following inequalities between these topological invariants hold:

$$\sigma_k(\gamma, m) \geq \max_{\substack{i_1+\cdots+i_s=k \\ \forall q, i_q \geq 1 \\ \gamma_1+\cdots+\gamma_s \leq \gamma \\ \gamma_1+\cdots+\gamma_s+m_1+\cdots+m_s \leq \gamma+m \\ \gamma_1<\gamma \text{ or } \gamma_1+m_1<\gamma+m \text{ if } s=1}} \sum_{q=1}^{s} \sigma_{i_q}(\gamma_q, m_q). \tag{0.2}$$

We prove the following existence result:

Theorem 1. *Let Σ be a compact orientable surface of genus γ, with a smooth boundary with $m \geq 1$ connected components. Let $k \geq 1$. If the inequality (0.2) is strict, then there exists a smooth metric g on Σ such that $\sigma_k(\gamma, m) = \sigma_k(\Sigma, g) L_g(\partial\Sigma)$. Moreover, up to scaling, this maximizing metric is the pull-back of the Euclidean metric by some conformal minimal immersion with free boundary in the unit Euclidean ball $\mathbb{B}^{n+1}$ for some n.*

This theorem was proved for the first eigenvalue $k = 1$, with $\gamma = 0$ and any m in Fraser–Schoen [**10**]. In this case, the condition that (0.2) is strict reads as $\sigma_1(0, m) > \sigma_1(0, m-1)$. They also proved that this condition holds true for any m so that $\sigma_1(0, m)$ is achieved by a smooth maximal metric for all $m \geq 1$. Their proof easily extends to higher genus, still for $k = 1$, except that we do not know if the gap condition holds for $\gamma \geq 1$.

Note that our theorem gives suitable conditions for the existence of conformal minimal immersion with free boundary with specified genus and number of boundary components given by k-th Steklov eigenfunctions for any $k \geq 1$. Note also that the gap assumption, i.e., the fact that (0.2) is strict, is necessary to get an existence result. Indeed, it was proved by Girouard–Polterovich [**12**] that $\sigma_2(0,1)$ is not achieved by a maximizing metric. Note that, in this case, we have $\sigma_2(0,1) = 2\sigma_1(0,1)$ by (0.1) so that (0.2) is not strict.

Even in the case $k = 1$, our proof differs a little bit from that of Fraser–Schoen [**10**]. And for higher eigenvalues, compared to the first, we have to deal with possible bubbling phenomena and, thus, to analyze them precisely in order to rule them out thanks to the gap assumption. The starting point of our proof is the following simple remark: it is somewhat more convenient (even if not easy) to maximize the Steklov eigenvalue among metrics in a given conformal class since everything depends then from a single function. Then we pick up a special maximizing sequence for $\sigma_k(\gamma, m)$ consisting in maximizers in their own conformal class. These maximizers come, as we shall see, with a corresponding harmonic map with free boundary from Σ into some Euclidean ball and the proof of Theorem 1 relies on a careful asymptotic analysis of these harmonic maps when the conformal class degenerates. Quantification results for such sequences of harmonic maps with free boundary were recently obtained in Laurain–Petrides [**19**].

In order to carry out this program, we introduce the conformal invariant

$$\sigma_k(\Sigma, [g]) = \sup_{\tilde{g} \in [g]} \sigma_k(\tilde{g}) L_{\tilde{g}}(\partial\Sigma),$$

for any smooth compact Riemannian surface (Σ, g) with a non-empty boundary. Here $[g]$ denotes the conformal class of g, that is all the metrics on Σ which are a multiple of g by a smooth positive function.

Then, if Σ is orientable of genus γ and with m boundary components, we have of course that

$$\sigma_k(\gamma, m) = \sup_{[g]} \sigma_k(\Sigma, [g]) .$$

Once again, one can prove by standard gluing techniques (see [7]) that

$$\sigma_k(\Sigma,[g]) \geq \max_{\substack{1\leq j\leq k \\ i_1+\dots+i_s=j}} \left(\sigma_{k-j}(\Sigma,[g]) + \sum_{m=1}^{s} \sigma_{i_m}(\mathbb{D},[\xi]) \right). \tag{0.3}$$

Note that thanks to (0.1), this inequality reads completely as

$$\sigma_k(\Sigma,[g]) \geq \max_{\substack{1\leq j\leq k \\ i_1+\dots+i_s=j}} \left(\sigma_{k-j}(\Sigma,[g]) + 2\pi j \right),$$

but, for a reason which will become clear in the proofs, we prefer to state it in the form of (0.3). Then we have the following existence result:

Theorem 2. *Let (Σ, g) be a compact Riemannian surface with a non-empty smooth boundary. Let $k \geq 1$. Then, if (0.3) is strict, there exists a smooth maximal metric $\tilde{g} \in [g]$, such that $\sigma_k(\Sigma,[g]) = \sigma_k(\Sigma,\tilde{g})L_{\tilde{g}}(\Sigma)$.*

Note that by (0.1) and (0.3), the gap condition of our theorem would be a consequence of

$$\sigma_k(\Sigma,[g]) > \sigma_{k-1}(\Sigma,[g]) + 2\pi.$$

If a maximal metric $\tilde{g}$ for $\sigma_k(\Sigma,[g])$ exists, the conformal factor related to g of a maximal metric $\tilde{g}$, is $\Phi.\partial_\nu\Phi$ on $\partial\Sigma$, where Φ is some harmonic map from Σ into $\mathbb{B}^{n+1}$ with free boundary whose coordinates are eigenfunctions for the k-th Steklov eigenvalue. Such a map takes values in the Euclidean ball, is harmonic inside Σ, satisfies that $|\Phi| = 1$ and $\partial_\nu\Phi$ is orthogonal to $T_\Phi\mathbb{S}^n$ on the boundary of Σ. These harmonic maps with free boundary have been studied in particular in Da Lio [5], Da Lio–Rivière [6], Laurain–Petrides [19] and Scheven [23].

The strategy of proof of Theorem 2 is the following. We do not prove either that any maximizing sequence does converge, up to a subsequence, to a maximizer nor that maximizers in a possible "weaker sense" are regular. Instead, as was initiated by Fraser–Schoen [10], we carefully select a maximizing sequence by a regularization process which does converge to a smooth maximizer. This special maximizing sequence is the solution of an approached variational problem and comes with a sequence of "almost" harmonic maps with free boundary in some Euclidean ball. The core of the proof is to carefully analyze the asymptotic behavior of these maps to prove that they do converge to a real smooth harmonic map with free boundary, leading to a maximal metric for the Steklov eigenvalue under consideration. The main difficulty is that, contrary to the case $k = 1$, one cannot a priori avoid phenomenon of concentration, with multiple bubbles appearing. We, thus, have to perform a bubble tree decomposition for this sequence, to understand precisely the behavior of these maps at a concentration point, to prove a no-neck energy result, in order to get a quantification result, and enough test-functions to use the variational characterization of the k-th Steklov eigenvalue in order to violate the gap assumption of the theorem.

The proof of Theorem 1 starts from the existence of maximal metrics in their own conformal class: this gives once again a special maximizing sequence. We then understand the behavior of this sequence if the conformal class degenerates in order to prove that it cannot happen under the gap assumption of the theorem. Then we rely on a compactness result by Laurain–Petrides [**19**] to finally prove that our maximizing sequence does converge to a smooth maximizer once degeneracy of the conformal class has been ruled out.

Analogous questions can be considered concerning the maximization of Laplace eigenvalues on closed surfaces. Inequalities (0.2) and (0.3) were proved in this situation by Colbois–El Soufi [**4**]. Maximizing metrics for Laplace eigenvalues come with minimal immersion of the surface into some sphere. If one adds the conformal class constraint, they come with smooth harmonic maps into the sphere. The analog of Theorem 1 for Laplace eigenvalues was proved in Petrides [**22**]. The analog of Theorem 2 was recently announced with a very brief sketch of proof in Nadirashvili–Sire [**20**] and proved in Petrides [**22**]. The proofs in the Steklov case are somewhat more difficult since one has to deal with "almost" harmonic maps with free boundary in some Euclidean ball instead of "almost" harmonic maps in some sphere. The analysis of such maps is more tricky: regularity and quantification results are, for instance, more recent (see Scheven [**23**], Da Lio–Rivière [**6**], Da Lio [**5**], Laurain–Petrides [**19**] compared to Hélein [**14**], Parker [**21**]) and the description of the bubbling phenomenon in the case of the present paper was explicitly asked for by Fraser–Schoen [**9**].

The paper is organized as follows:

In Section 1, we introduce some notations and recall some more or less classical tools that we shall use during the proof. Section 2 is devoted to the set up of the proof of Theorem 2, proof carried out in Sections 3 to 5. We refer to the end of Section 2 for a detailed sketch of the proof of Theorem 2.

We prove Theorem 1 in Section 6, dealing with a maximizing sequence of metrics for $\sigma_k(\gamma, m)$ whose k-th eigenvalue is maximal in its conformal class. We then study the asymptotics of the harmonic maps on Σ with free boundary into some $\mathbb{B}^{n+1}$ they define, and thanks to the gap assumption of the theorem, we remove all the problems of convergence which could occur for this maximizing sequence.

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1. Preliminaries

1.1. Notations. Let (M,g) be a smooth Riemannian surface with a boundary of length $L_g(\partial M)=1$. Let $x\in M$ and $r>0$. We denote by $B_g(x,r)$ the open ball of radius r centered at x. If $x\in\partial M$, we let $I_g(x,r)=\partial M\cap B_g(x,r)$. In the Euclidean upper half-plane $\mathbb{R}^2_+=\{(s,t)\in\mathbb{R}^2; t\geq 0\}$, we let for $x\in\mathbb{R}\times\{0\}$, $\mathbb{D}_r^+(x)=\mathbb{D}_r(x)\cap\mathbb{R}^2_+$ and $I_r(x)=(-r,r)\times\{0\}$.

We denote by $\mathcal{M}(\partial M)$ the set of positive Radon measures equipped with the weak* topology on ∂M and by $\mathcal{M}_1(\partial M)$ the subset of probability measures.

As already said, we denote by $\sigma_k(M,g)$ the k-th eigenvalue of the Dirichlet-to-Neumann operator on M. It satisfies the classical min-max variational characterization:

$$\sigma_k(M,g)=\inf_{E_{k+1}}\sup_{\phi\in E_{k+1}\setminus\{0\}}\frac{\int_M|\nabla\phi|_g^2\,dv_g}{\int_{\partial M}\phi^2 d\sigma_g},\tag{1.1}$$

where the infimum is taken over the spaces of smooth functions E_{k+1} of dimension $k+1$.

For an open set $\Omega\subset M$ such that $\partial\Omega=\Gamma\cup\tilde{\Gamma}$ where $\Gamma=\partial\Omega\cap\partial M$ and $\tilde{\Gamma}=\partial\Omega\setminus\partial M$ are non-empty piecewise smooth curves, and a smooth density e^u on Γ we denote by $\sigma_\star(\Omega,g,\Gamma,e^u)$ the first eigenvalue for the following problem

$$\begin{cases}\Delta_g\phi=0 & \text{in } \Omega,\\ \partial_\nu\phi=\sigma_\star(\Omega,g,\Gamma,e^u)\,e^u\phi & \text{on } \Gamma,\\ \phi=0 & \text{on } \tilde{\Gamma},\end{cases}$$

that is

$$\sigma_\star(\Omega,g,\Gamma,e^u)=\inf_{\phi\in H}\frac{\int_\Omega|\nabla\phi|_g^2\,dv_g}{\int_\Gamma\phi^2e^u d\sigma_g},$$

where

$$H=\{\phi\in W^{1,2}(\Omega),\phi=0\text{ on }\tilde{\Gamma}\},$$

the value of ϕ on $\partial\Omega$ being understood taken in the sense of the Sobolev trace.

For all the paper, we fix $\delta>0$, a constant $C_0>1$ and a family $(x_l)_{l=1,\dots,L}$ of points in ∂M and smooth functions $v_l: M\mapsto\mathbb{R}$ such that

- for any $l\in\{1,\dots,L\}$, $g_l=e^{-2v_l}g$ is a flat metric in $\Omega_l=B_{g_l}(x_l,2\delta)$, and $\Gamma_l=I_{g_l}(x_l,2\delta)$ is a geodesic line for g_l so that the exponential map $\exp_{g_l,x_l}$ defines an isometry between $\mathbb{D}^+_{2\delta}(0)$ and $(B_{g_l}(x_l,2\delta),g_l)$.

- $\partial M = \bigcup_{l=1}^{L} \gamma_l$ where $\gamma_l = I_{g_l}(x_l, \delta)$.
- For any $1 \leq l \leq L$, $C_0^{-2} \leq e^{2v_l} \leq C_0^2$.
- For any $x \in \omega_l$ and $0 < r < \delta$, $B_g(x, C_0^{-1} r) \subset B_{g_i}(x, r) \subset B_g(x, C_0 r)$.

For $1 \leq l \leq L$ and a point $z \in \mathbb{D}_{2\delta}^+(0)$, we let

$$e^{2\tilde{v}_l(z)} = e^{2v_l(\exp_{g_l,x_l}(z))} \text{ and } \bar{z}^l = \exp_{g_l,x_l}(z),$$

and for $x \in \Omega_l$ and a set $\Omega \subset \Omega_l$,

$$\tilde{x}^l = \exp_{g_l,x_l}^{-1}(x) \text{ and } \widetilde{\Omega}^l = \exp_{g_l,x_l}^{-1}(\Omega).$$

For a smooth density e^u on ∂M we let

$$e^{\tilde{u}^l(z)} = e^{\tilde{v}_l(z)} e^{u(\exp_{g_l,x_l}(z))},$$

so that for $\Gamma \subset \Gamma_l$,

$$\int_\Gamma e^u d\sigma_g = \int_{\tilde{\Gamma}^l} e^{\tilde{u}^l} ds.$$

For other functions $\phi \in L^1(M)$ or measures $\nu \in \mathcal{M}(\partial M)$, we let

$$\tilde{\phi}^l(z) = \phi(\exp_{g_l,x_l}(z)) \text{ and } \tilde{\nu}^l = \exp_{g_l,x_l}^\star(\nu).$$

Let $p_\epsilon(x, y)$ be the heat kernel of ∂M at time $\epsilon > 0$ for the induced measure $d\sigma_g$. Then, for $y, z \in \Gamma_l$, we let

$$\tilde{p}_\epsilon^l(z, y) = e^{\tilde{v}_l(z)} p_\epsilon(\exp_{g_l,x_l}(z), \exp_{g_l,x_l}(y)),$$

so that for a density $e^{u(x)} = \int_\Gamma p_\epsilon(x, y) d\nu(y)$ for $\Gamma \subset \Gamma_l$ and some measure ν, we have

$$e^{\tilde{u}^l(z)} = \int_{\tilde{\Gamma}^l} \tilde{p}_\epsilon^l(z, y) d\tilde{\nu}(y),$$

and for $\phi \in L^1(\partial M)$,

$$\int_{\tilde{\Gamma}^l} \tilde{\phi}^l(s, 0) \tilde{p}_\epsilon^l((s, 0), \tilde{y}^l) ds = \int_\Gamma \phi(x) p_\epsilon(x, y) d\sigma_g(x).$$

When the context is clear, we drop the exponent l in all the notations.

Now, for parameters $a \in \mathbb{R} \times \{0\}$ and $\alpha > 0$, we define the following rescaled objects

$$\hat{x} = \frac{\tilde{x} - a}{\alpha}, \hat{\Omega} = \frac{\widetilde{\Omega} - a}{\alpha}, \hat{\Gamma} = \frac{\widetilde{\Gamma} - a}{\alpha},$$

$$e^{2\hat{u}(z)} = \alpha^2 e^{2\tilde{u}(\alpha z + a)}, \hat{\phi}(z) = \tilde{\phi}(\alpha z + a),$$

$$\hat{\nu} = H_{a,\alpha}^\star(\tilde{\nu}), \hat{p}_\epsilon(z, y) = \alpha \tilde{p}_\epsilon^l(\alpha z + a, \alpha y + a),$$

where $H_{a,\alpha}(x) = \alpha x + a$, so that if $e^{u(x)} = \int_\Gamma p_\epsilon(x, y) d\nu(y)$, we have

$$e^{\hat{u}(z)} = \int_{\hat{\Gamma}} \hat{p}_\epsilon(z, y) d\hat{\nu}(y),$$

and

$$\int_{\hat{\Gamma}} \phi((s,0))\hat{p}_\epsilon((s,0),\hat{y})ds = \int_{\Gamma} \phi(x)p_\epsilon(x,y)d\sigma_g(y).$$

We also denote for $z \in \mathbb{R}^2$,

$$\breve{z} = \exp_{g_l,x_l}(\alpha z + a),$$

so that $\hat{\breve{z}} = z$ and

$$\breve{\Omega} = \exp_{g_l,x_l}(\alpha\Omega + a).$$

1.2. Estimates on the heat kernel. The heat kernel $p_\epsilon(x,y)$ of a the union of circles ∂M at time $\epsilon > 0$ with respect to the measure $d\sigma_g$ satisfies the following uniform estimates as $\epsilon \to 0$

(1.2)

$$p_\epsilon(x,y) =_{\epsilon\to 0} \frac{e^{-\frac{d_g(x,y)^2}{4\epsilon}}}{\sqrt{4\pi\epsilon}} \left(a_0(x,y) + \epsilon a_1(x,y) + \epsilon^2 a_2(x,y) + o(\epsilon^2)\right),$$

with $a_0, a_1, a_2 \in \mathcal{C}^\infty(\partial M \times \partial M)$ are Riemannian invariants such that $a_0(x,x) = 1$ as proved, for instance, in [2]. We have also a uniform bound: there exists $A_0 > 0$ such that for any $\epsilon > 0$,

$$\forall x,y \in \partial M, \frac{1}{A_0\sqrt{4\pi\epsilon}} e^{-\frac{d_g(x,y)^2}{4\epsilon}} \leq p_\epsilon(x,y) \leq \frac{A_0}{\sqrt{4\pi\epsilon}} e^{-\frac{d_g(x,y)^2}{4\epsilon}}. \tag{1.3}$$

We deduce the same uniform properties for the rescaled heat kernel $\hat{p}_\epsilon(x,y)$ by some parameters $a_\epsilon \in \mathbb{R} \times \{0\}$ and $\alpha_\epsilon > 0$ such that $a_\epsilon \to a \in \mathbb{R} \times \{0\}$ and $\alpha_\epsilon \to 0$ as $\epsilon \to 0$. We have for any $R > 0$,

$$\hat{p}_\epsilon(z,y) = \frac{e^{-\frac{|y-z|^2}{4\theta_\epsilon}(1+o(1))}}{\sqrt{4\pi\theta_\epsilon}}(1+o(1)) \text{ uniformly on } \mathbb{D}_R \times \mathbb{D}_R, \tag{1.4}$$

where $\theta_\epsilon = \frac{\epsilon}{e^{2\tilde{v}_l(a)}\alpha_\epsilon^2}$ and we have the following bound for any fixed $0 < \rho < 1$

$$\frac{e^{-\frac{|y-z|^2}{4\theta_\epsilon}(1+\rho)}}{\sqrt{4\pi\theta_\epsilon}}(1-\rho) \leq \hat{p}_\epsilon(z,y) \leq \frac{e^{-\frac{|y-z|^2}{4\theta_\epsilon}(1-\rho)}}{\sqrt{4\pi\theta_\epsilon}}(1+\rho), \tag{1.5}$$

for all $\epsilon > 0$ small enough.

Let's prove (1.4). We fix $R > 0$ and we have uniformly for $(x,y) \in I_R \times I_R$ as $\epsilon \to 0$

$$\begin{aligned}\hat{p}_\epsilon(x,y) &= \frac{\alpha_\epsilon e^{v_l(\breve{x})}}{\sqrt{4\pi\epsilon}} e^{-\frac{d_g(\breve{x},\breve{y})^2}{4\epsilon}} \left(a_0(\breve{x},\breve{y}) + o(1)\right) \\ &= \frac{\alpha_\epsilon e^{\tilde{v}_l(a)}}{\sqrt{4\pi\epsilon}}(1+o(1)) e^{-\frac{d_g(\breve{x},\breve{y})^2}{4\epsilon}}\end{aligned}$$

by (1.2). It remains to notice that

$$d_g(\breve{x},\breve{y}) = e^{\tilde{v}_l(a)} |x-y| \alpha_\epsilon(1+o(1))$$

uniformly for $(x,y) \in \mathbb{D}_R^+ \times \mathbb{D}_R^+$ and we get the desired approximation (1.4).

For a sequence of measures $\nu_\epsilon \in \mathcal{M}(\partial M)$, we also have uniform bounds for $R > r > 0$ and $\theta_\epsilon \to 0$ as $\epsilon \to 0$

$$\sup_{x \in I_{R-r}} \int_{\partial M \setminus \breve{I}_R} \alpha_\epsilon p_\epsilon(\breve{x}, y) d\nu_\epsilon(y) = O\left(\frac{e^{-\frac{(R-r)^2}{8\theta_\epsilon}}}{\sqrt{\theta_\epsilon}} \right). \tag{1.6}$$

We prove it thanks to (1.3) and (1.5). Let $x \in I_{R-r}$.

$$\begin{aligned}
\alpha_\epsilon \int_{\partial M \setminus \breve{I}_R} p_\epsilon(\breve{x}, y) d\nu_\epsilon(y) &= e^{-v_l(\breve{x})} \int_{I_{C_0^2 R} \setminus I_R} \hat{p}_\epsilon(x,z) d\hat{\nu}_\epsilon(z) \\
&\quad + \int_{\partial M \setminus \breve{I}_{C_0^2 R}} \alpha_\epsilon p_\epsilon(\breve{x}, y) d\nu_\epsilon(y) \\
&\leq C_0 \int_{I_{C_0^2 R} \setminus I_R} \frac{e^{-\frac{|x-z|^2}{8\theta_\epsilon}}}{\sqrt{\pi\theta_\epsilon}} d\hat{\nu}_\epsilon(z) \\
&\quad + \int_{\partial M \setminus I_g(\bar{a}_\epsilon, \frac{\alpha_\epsilon C_0^2 R}{C_0})} \frac{\alpha_\epsilon A_0}{\sqrt{4\pi\epsilon}} e^{-\frac{d_g(\breve{x},y)^2}{4\epsilon}} d\nu_\epsilon(y) \\
&\leq O\left(\frac{e^{-\frac{(R-r)^2}{8\theta_\epsilon}}}{\sqrt{\theta_\epsilon}} \right) + \frac{A_0 \alpha_\epsilon}{\sqrt{4\pi\epsilon}} e^{-\frac{\alpha_\epsilon^2 (R-r)^2}{4\epsilon}},
\end{aligned}$$

where $\breve{I}_r \subset I_g(\bar{a}_\epsilon, \alpha_\epsilon C_0 r) \subset I_g(\bar{a}_\epsilon, \alpha_\epsilon C_0 R)$. This proves (1.6). We also have

$$\sup_{x \in \partial M \setminus \breve{I}_R} \int_{\breve{I}_r} p_\epsilon(x,y) d\sigma_g(y) = O\left(\frac{e^{-\frac{(R-r)^2}{8\theta_\epsilon}}}{\sqrt{\theta_\epsilon}} \right). \tag{1.7}$$

Let $x \in \partial M \setminus \breve{I}_R$. We assume that $x \in I_{C_0^2 R} \setminus I_R$. Then,

$$\begin{aligned}
\int_{\breve{I}_r} p_\epsilon(x,y) d\sigma_g(y) &= \int_{I_r} \hat{p}_\epsilon(z, \breve{x}) dz \\
&\leq \frac{1}{\sqrt{\pi\theta_\epsilon}} \int_{I_r} e^{-\frac{|x-z|^2}{8\theta_\epsilon}} dz \\
&\leq \frac{2r}{\sqrt{\pi\theta_\epsilon}} e^{-\frac{(R-r)^2}{8\theta_\epsilon}}.
\end{aligned}$$

Now, if ϵ is small enough and if $x \in \partial M \setminus \breve{I}_{C_0^2 R} \subset \partial M \setminus I_g(\bar{a}_\epsilon, \alpha_\epsilon R C_0)$, we have

$$\int_{\breve{I}_r} p_\epsilon(x,y) d\sigma_g(y) \leq \int_{I_g(\bar{a}_\epsilon, \alpha_\epsilon C_0 r)} p_\epsilon(x,y) d\sigma_g(y)$$

$$\leq \frac{A_0}{\sqrt{4\pi\epsilon}} \int_{I_g(\bar{a}_\epsilon, \alpha_\epsilon C_0 r)} e^{-\frac{d_g(x,y)^2}{4\epsilon}} d\sigma_g(y)$$

$$\leq O\left(\frac{e^{-\frac{\alpha_\epsilon^2 (R-r)^2}{4\epsilon}}}{\sqrt{\theta_\epsilon}}\right).$$

We proved (1.7). Now let's prove that

$$\lim_{R\to+\infty} \lim_{\epsilon\to 0} \sup_{x\in I_r} \left| \int_{I_R} \hat{p}_\epsilon(z,x)dz - 1 \right| = 0. \tag{1.8}$$

We fix $0 < \rho < \frac{1}{2}$ and $R > 0$. Then, for ϵ small enough, we have by (1.5) that

$$\int_{I_R} \hat{p}_\epsilon(z,x)dz \leq \int_{\mathbb{R}\times\{0\}} \frac{e^{-\frac{|x-z|^2(1-\rho)}{4\theta_\epsilon}}}{\sqrt{4\pi\theta_\epsilon}}(1+\rho)dz = \frac{1+\rho}{\sqrt{1-\rho}}$$

for any $x \in I_r$ and

$$\begin{aligned} \int_{I_R} \hat{p}_\epsilon(z,x)dz &\geq \int_{I_R} \frac{e^{-\frac{|x-z|^2(1+\rho)}{4\theta_\epsilon}}}{\sqrt{4\pi\theta_\epsilon}}(1-\rho)dz \\ &\geq \int_{\mathbb{R}\times\{0\}} \frac{e^{-\frac{|x-z|^2(1+\rho)}{4\theta_\epsilon}}}{\sqrt{4\pi\theta_\epsilon}}(1-\rho)dz - \int_{\mathbb{R}\times\{0\}\setminus I_R} \frac{e^{\frac{-|x-z|^2}{8\theta_\epsilon}}}{\sqrt{\pi\epsilon}}dz \\ &\geq \frac{1-\rho}{\sqrt{1+\rho}} + o(1) \text{ as } \epsilon\to 0 \end{aligned}$$

uniformly on I_r. Letting $\epsilon \to 0$, then $R \to +\infty$ and then $\rho \to 0$ gives (1.8).

1.3. Capacity and Poincaré inequalities. We first notice the following consequence of the classical computation of the capacity of annuli in $\mathbb{R}^2$.

Claim 1. *Let (M,g) be a compact Riemannian surface. Then, there is $C > 0$ and $r_0 > 0$ such that for all $x \in M$ and all $0 < r_2 < r_1 < r_0$, there exists a smooth function $\eta_{g,x,r_1,r_2} : M \to \mathbb{R}$ with*

- $0 \leq \eta_{g,x,r_1,r_2} \leq 1$,
- $\eta_{g,x,r_1,r_2} = 1$ *on* $B_g(x,r_2)$,
- $\eta_{g,x,r_1,r_2} \in \mathcal{C}_c^\infty(B_g(x,r_1))$,
- $\int_M |\nabla \eta_{g,x,r_1,r_2}|_g^2 dv_g \leq \frac{C}{\ln\left(\frac{r_1}{r_2}\right)}$.

We now recall two theorems giving Poincaré inequalities on surfaces.

Theorem 3 ([1], Lemma 8.3.1)**.** *Let (M,g) be a Riemannian manifold. Then, there exists a constant $B > 0$ such that for any $L \in W^{-1,2}(M)$ with $L(1) = 1$, we have the following Poincaré inéquality*

$$\forall f \in W^{1,2}(M), \int_M (f - L(f))^2 dv_g \leq B \|L\|_{W^{-1,2}(M)}^2 \int_M |\nabla f|_g^2 dv_g.$$

We denote by

$$C_{1,2}(K) = \inf\left\{\int_{\mathbb{R}^2}\phi^2 dv_g + \int_{\mathbb{R}^2}|\nabla\phi|_g^2\, dv_g; \phi\in\mathcal{C}_c^\infty(\mathbb{R}^2), \phi\geq 1 \text{ on } K\right\},$$

the capacity of a compact set $K\subset\mathbb{R}^2$ and

$$Cap_2(K,\Omega) = \inf\left\{\int_{\Omega}|\nabla\phi|_g^2\, dv_g; \phi\in\mathcal{C}_c^\infty(\Omega), \phi\geq 1 \text{ on } K\right\},$$

the relative capacity of $K\subset\subset\Omega$.

Theorem 4 ([1], Corollary 8.2.2)**.** *Let $\Omega\subset\mathbb{R}^2$ be a bounded extension domain. Then, there exists a constant C_Ω such that for any compact $K\subset\Omega$ with $C_{1,2}(K)>0$ and for any function $f\in\mathcal{C}^\infty(\Omega)$ such that $f=0$ on K,*

$$\|f\|_{L^2(\Omega)} \leq \frac{C_\Omega}{C_{1,2}(K)}\|\nabla f\|_{L^2(\Omega)}.$$

Ω is a bounded extension domain means that the extension by 0 on $\mathbb{R}^2$ of every function in $W_0^{1,2}(\Omega)$ is $W^{1,2}$ in $\mathbb{R}^2$. This is true for the family of sets we consider during the proof:

$$\Omega = \mathbb{D}^+_{\frac{1}{\rho}}\setminus\bigcup_{i=1}^s\mathbb{D}_\rho(x_i),$$

where $\rho>0$, $x_i\in\mathbb{D}_{\frac{1}{\rho}}$ such that if $i\neq j$, then $x_i\neq x_j$ and

$$10\rho < \min\left(\min_i d(x_i,\partial\mathbb{D}_{\frac{1}{10\rho}}); \min_{i\neq j}\frac{|x_i-x_j|}{2}\right).$$

We now set

$$\Omega_K = \mathbb{D}^+_{\frac{1}{K\rho}}\setminus\bigcup_{i=1}^s\mathbb{D}_{K\rho}$$

for some fixed number $1<K<10$ chosen independent of the problem we consider. We obtain the corollary:

Corollary. *Let $r>0$ fixed. Then, we have a constant $C_r>0$ such that for every $f\in\mathcal{C}^\infty(\Omega)$ which vanishes on a smooth piecewise curve $\Gamma\subset\subset\Omega_K$ which connects two points of distance $r>0$,*

$$\|f\|_{L^2(\Omega)}\leq C_r\|\nabla f\|_{L^2(\Omega)}.$$

Indeed, it is proved in ([15], pages 95–97) that

$$Cap_2(\Gamma,\Omega)\geq\frac{K_0}{\ln(\frac{1}{r})},$$

and that

$$C_{1,2}(\Gamma)\geq K_1 Cap_2(\Gamma,\Omega)$$

for constants $K_0>0$ and $K_1>0$ which only depend on Ω and K.

2. Selection of a maximizing sequence

We fix $k \geq 1$. In this section, we build a specific maximizing sequence for $\sigma_k(M,[g])$ thanks to the heat equation on ∂M. Let $\epsilon > 0$. We denote by K_ϵ the heat operator on ∂M so that for a positive Radon measure $\nu \in \mathcal{M}(M)$, $K_\epsilon[\nu]d\sigma_g$ is the solution at time $\epsilon > 0$ of the heat equation on the curves $(\partial M, d\sigma_g)$ which converges to ν as $\epsilon \to 0$ in $\mathcal{M}(\partial M)$. Given $x, y \in \mathcal{M}(\partial M)$, we denote by $p_\epsilon(x,y)$ the heat kernel of $(\partial M, g)$ so that for $\nu \in \mathcal{M}(\partial M)$,

$$K_\epsilon[\nu](x) = \int_{\partial M} p_\epsilon(x,y)d\nu(y).$$

For $f \in L^1(\partial M)$, we set $K_\epsilon[f] := K_\epsilon[f d\sigma_g]$ so that

$$\int_{\partial M} K_\epsilon[f]d\nu = \int_{\partial M} f K_\epsilon[\nu]d\sigma_g.$$

For $\epsilon > 0$, we set

$$\sigma_\epsilon = \sup_{\nu \in \mathcal{M}(\partial M)} \sigma_k(M, g, \partial M, K_\epsilon[\nu]). \tag{2.1}$$

By continuity of $\nu \in \mathcal{M}_1(\partial M) \mapsto \sigma_k(M, g, \partial M, K_\epsilon[\nu])$, a maximizing sequence for the variational problem (2.1) converges in $\mathcal{M}_1(\partial M)$, up to the extraction of a subsequence, to a measure $\nu_\epsilon \in \mathcal{M}_1(\partial M)$ such that

$$\sigma_\epsilon = \sigma_k(M, g, \partial M, K_\epsilon[\nu_\epsilon]). \tag{2.2}$$

We set

$$e^{u_\epsilon} = K_\epsilon[\nu_\epsilon], \tag{2.3}$$

a sequence of smooth positive densities satisfying

$$\sigma_\epsilon = \sigma_k(M, g, \partial M, e^{u_\epsilon}) \to \sigma_k(M,[g]) \text{ as } \epsilon \to 0. \tag{2.4}$$

Indeed, $\sigma_\epsilon \leq \sigma_k(M,[g])$ for all $\epsilon > 0$ and for $\eta > 0$, there exists some density e^u such that $\int_{\partial M} e^u d\sigma_g = 1$ and $\sigma_k(M, g, \partial M, e^u) \geq \sigma_k(M,[g]) - \frac{\eta}{2}$. By uniform estimates on the heat operator, $K_\epsilon[e^u] \to e^u$ as $\epsilon \to 0$ in $\mathcal{C}^0(\partial M)$. Then, there exists $\epsilon_0 > 0$ such that

$$\sigma_\epsilon \geq \sigma_k(M, g, \partial M, K_\epsilon[e^u]) \geq \sigma_k(M, g, \partial M, e^u) - \frac{\eta}{2} \geq \sigma_k(M,[g]) - \eta$$

for $\epsilon < \epsilon_0$. We get (2.4). Now, thanks to the choice of the maximizing sequence (2.3) the variational problem (2.1) gives

Proposition 1. *Fix $\epsilon > 0$. Then, there exists a family $\Phi_\epsilon = (\phi_\epsilon^0, \cdots, \phi_\epsilon^{n(\epsilon)})$ of smooth independent functions in $L^2(\partial M, e^{u_\epsilon} d\sigma_g)$ such that*

(i) *For $i \in \{0, \cdots, n(\epsilon)\}$, $\phi_\epsilon^i \in E_k(M, g, \partial M, e^{u_\epsilon})$, that is to say*

$$\begin{cases} \Delta_g \phi_\epsilon^i = 0 & \text{in } M, \\ \partial_\nu \phi_\epsilon^i = \sigma_\epsilon e^{u_\epsilon} \phi_\epsilon^i & \text{in } \partial M, \end{cases}$$

(ii) $K_\epsilon[|\Phi_\epsilon|^2] \geq 1$ *on* ∂M,
(iii) $K_\epsilon[|\Phi_\epsilon|^2] = 1$ *on* $supp(\nu_\epsilon)$.

Proof. Since ϵ is fixed, we omit the indices ϵ for σ_ϵ, ν_ϵ and e^{u_ϵ} up to the end of the proof of the claim.

Let $\mu \in \mathcal{M}(\partial M)$ and $t > 0$. We set $\sigma_t = \sigma_k(M, g, \partial M, K_\epsilon[\nu + t\mu])$. Note that $\sigma = \sigma_{t=0}$ and by continuity, $\sigma_t \to \sigma$ as $t \to 0^+$. We first prove that

$$\lim_{t\to 0^+} \frac{\sigma_t - \sigma}{t} = \inf_{\phi \in E_k(M,g,\partial M, e^u)} \left(-\sigma \frac{\int_{\partial M} K_\epsilon[\phi^2] d\mu}{\int_{\partial M} \phi^2 e^u d\sigma_g} \right). \tag{2.5}$$

Let $\phi_0, \phi_1, \cdots, \phi_k$ be an orthonormal family of functions in $L^2(\partial M, e^u d\sigma_g)$ such that $\phi_i \in E_i(M, g, \partial M, e^u)$. We set $E = Vect\{\phi_0, \cdots, \phi_k\}$. Then, by the min-max variational characterization (1.1),

$$\begin{aligned} \sigma_t &\leq \sup_{\phi \in E \setminus \{0\}} \left(\frac{\int_M |\nabla \phi|_g^2 \, dv_g}{\int_{\partial M} \phi^2 K_\epsilon[\nu + t\mu] d\sigma_g} \right) \\ &= \sup_{\phi \in E \cap S^k} \left(\frac{\int_M |\nabla \phi|_g^2 \, dv_g}{\int_{\partial M} \phi^2 K_\epsilon[\nu] d\sigma_g + t \int_{\partial M} K_\epsilon[\phi^2] d\mu} \right), \end{aligned}$$

where $S^k = \{\sum_{i=0}^k \beta_i \phi_i, \beta \in \mathbb{S}^k\}$ and

$$\begin{aligned} \sigma_t &\leq \sup_{\phi = \sum_{i=0}^k \beta_i \phi_i \in S^k} \left(\sum_{i=0}^k \beta_i^2 \sigma_i \left(1 - t \int_{\partial M} K_\epsilon[\phi^2] d\mu + o(t) \right) \right) \\ &\leq \sigma \left(1 - t \frac{\int_{\partial M} K_\epsilon[\phi_k^2] d\mu}{\int_{\partial M} \phi_k^2 e^u d\sigma_g} + o(t) \right), \end{aligned}$$

uniformly as $t \to 0$, where $\sigma_i = \sigma_i(M, g, \partial M, e^u)$. Indeed, by the gap $\sigma_k(M, [g]) \geq \sigma_{k-1}(M, [g]) + 2\pi$, we have $\sigma = \sigma_k(M, g, \partial M, e^u) > \sigma_{k-1}(M, g, \partial M, e^u)$ and since we have (2.4). Then, minimizing among the $\phi_k \in E_k(M, g, \partial M, e^u)$, we get that

$$\limsup_{t \to 0^+} \frac{\sigma_t - \sigma}{t} \leq \inf_{\phi \in E_k(M,g,\partial M, e^u)} \left(-\sigma \frac{\int_{\partial M} K_\epsilon[\phi^2] d\mu}{\int_{\partial M} \phi^2 e^u d\sigma_g} \right). \tag{2.6}$$

Now, we let $\phi_t \in E_k(M, g, \partial M, K_\epsilon[\nu + t\mu])$ with $\|\phi_t\|_{L^2(\partial M, K_\epsilon[\nu] d\sigma_g)} = 1$. We have that

$$\begin{cases} \Delta_g \phi_t = 0 & \text{in } M, \\ \partial_\nu \phi_t = \sigma_t K_\epsilon[\nu + t\mu] \phi_t = \sigma_t (e^u + t K_\epsilon[\mu]) \phi_t & \text{in } \partial M. \end{cases} \tag{2.7}$$

For $t \leq \frac{\|e^u\|_{L^\infty}}{2\|K_\epsilon[\mu]\|_{L^\infty}}$, we have that

$$\frac{1}{2} e^u \leq K_\epsilon[\nu + t\mu] \leq 2e^u,$$

and that for any $\phi \in \mathcal{C}^\infty(\partial M)$,

$$\frac{1}{2}\int_{\partial M} e^u \phi^2 \leq \int_{\partial M} \phi^2 K_\epsilon[\nu + t\mu] \leq 2\int_{\partial M} \phi^2 e^u,$$

so that $L^2(K_\epsilon[\nu+t\mu]d\sigma_g)$ and $L^2(K_\epsilon[\nu]d\sigma_g) = L^2(e^u d\sigma_g)$ define the same sets with equivalent norms and constants in the equivalence independent of t. Then, $\{\phi_t\}$ is bounded in $L^2(e^u d\sigma_g)$. By elliptic regularity theory for the Dirichlet-to-Neumann operator with equation (2.7), (see [**25**], Chapter 7.11, page 37), there exists $\phi \in E_k(M, g, \partial M, e^u)$ such that up to the extraction of a subsequence, $\phi_t \to \phi$ in $\mathcal{C}^m(M)$ as $t \to 0^+$ and $\|\phi\|_{L^2(\partial M, e^u d\sigma_g)} = 1$. We denote by Π the orthogonal projection on $E_k(M, g, \partial M, e^u)$ with respect to the $L^2(\partial M, e^u d\sigma_g)$-norm. Then, we write (2.7) as

(2.8)
$$\begin{cases} \Delta_g\left(\frac{\phi_t - \Pi\phi_t}{\alpha_t}\right) = 0 & \text{in } M, \\ \partial_\nu\left(\frac{\phi_t - \Pi\phi_t}{\alpha_t}\right) - \sigma_t e^u\left(\frac{\phi_t - \Pi\phi_t}{\alpha_t}\right) = \frac{\sigma_t - \sigma}{\alpha_t} e^u \phi_t + \frac{t}{\alpha_t}\sigma_t K_\epsilon[\mu]\phi_t & \text{in } \partial M, \end{cases}$$

with

$$\alpha_t = \|\phi_t - \Pi\phi_t\|_{L^\infty} + t + (\sigma - \sigma_t). \quad (2.9)$$

Up to the extraction of a subsequence, we have that

$$t_0 = \lim_{t\to 0^+} \frac{t}{\alpha_t} \text{ and } \delta_0 = \lim_{t\to 0^+} \frac{\sigma - \sigma_t}{\alpha_t}.$$

Notice that $\delta_0 \geq 0$. By elliptic theory on the Dirichlet-to-Neumann operator (see [**25**], Chapter 7.11, page 37), since $\frac{\phi_t - \Pi\phi_t}{\alpha_t}$ is uniformly bounded as $t \to 0^+$, we get up to the extraction of a subsequence that

$$\frac{\phi_t - \Pi\phi_t}{\alpha_t} \to R_0 \text{ as } t \to 0^+ \text{ in } \mathcal{C}^m(M),$$

where $R_0 \in E_k(M, g, \partial M, e^u)^\perp$. Passing to the limit in equation (2.8), we get

$$\begin{cases} \Delta_g R_0 = 0 & \text{in } M, \\ \partial_\nu R_0 - \sigma e^u R_0 = -\delta_0 e^u \phi + t_0 \sigma K_\epsilon[\mu]\phi & \text{in } \partial M, \end{cases} \quad (2.10)$$

and by (2.9)

$$\|R_0\|_\infty + t_0 + \delta_0 = 1. \quad (2.11)$$

Testing (2.10) against ϕ, and using the fact that $R_0 \in E_k(M, g, \partial M, e^u)^\perp$, we have that

$$\delta_0 = \delta_0 \int_{\partial M} e^u \phi^2 d\sigma_g = t_0 \sigma \int_{\partial M} K_\epsilon[\mu]\phi^2 d\sigma_g.$$

If $t_0 = 0$, then $\delta_0 = 0$ and then $R_0 = 0$ thanks to (2.10) and the fact that $R_0 \in E_k(M, g, \partial M, e^u)^\perp$. This is absurd with (2.11). Thus, $t_0 \neq 0$

and

$$\lim_{t\to 0^+}\frac{\sigma_t-\sigma}{t}=\frac{-\delta_0}{t_0}=-\sigma\frac{\int_{\partial M}K_\epsilon[\phi^2]d\mu}{\int_{\partial M}\phi^2e^u d\sigma_g}.$$

This and (2.6) gives (2.5).

Since $(1+t\int_{\partial M}d\mu)\sigma_t\le\sigma$ for all $t\ge 0$, we deduce from (2.5) that for any $\mu\in\partial\mathcal{M}$, there exists $\phi\in E_k(M,g,\partial M,e^u)$ such that

$$\int_{\partial M}\phi^2e^u d\sigma_g=1 \text{ and } \int_{\partial M}(1-K_\epsilon[\phi^2])d\mu\le 0. \tag{2.12}$$

We define the following subsets of $\mathcal{C}^0(\partial M)$

$$K=\left\{\psi=\sum_{i=0}^n K_\epsilon[\phi_i^2]-1\in\mathcal{C}^0(M);\phi_0,\cdots,\phi_n\in E_k,\int_{\partial M}\psi d\nu=0\right\},$$

where $E_k=E_k(M,g,\partial M,e^u)$ and

$$F=\{f\in\mathcal{C}^0(\partial M),f\ge 0\}.$$

F is closed and convex. The set K is convex since it is a translation of the convex hull of

$$C=\{K_\epsilon[\phi^2];\phi\in E_k(M,g,\partial M,e^u),\|\phi\|_{L^2(M,g,\partial M,e^u)}=1\}.$$

Since $E_k(M,g,\partial M,e^u)$ is finite dimensional, the vector space spanned by C is finite dimensional and C is compact. Caratheodory's theorem gives that K is compact.

If $F\cap K=\emptyset$, Hahn–Banach theorem gives the existence of some $\mu\in\mathcal{M}(\partial M)$ such that

$$\forall f\in F,\int_{\partial M}fd\mu\ge 0, \tag{2.13}$$

and

$$\forall\psi\in K,\int_{\partial M}\psi d\mu<0. \tag{2.14}$$

Then, μ is a non-zero, by (2.13), positive, by (2.14), measure and μ contradicts (2.12) by (2.14). Thus, $F\cap K\neq\emptyset$ and there exists $\phi^0,\cdots,\phi^n\in E_k(M,g,\partial M,e^u)$ with

$$\int_{\partial M}|\Phi|^2e^u d\sigma_g=1 \text{ and } K_\epsilon[|\Phi|^2]\ge 1, \tag{2.15}$$

where $\Phi=(\phi^0,\cdots,\phi^n)$. By Gaussian decomposition of some non-negative quadratic form, we can assume that $(\phi^0,\cdots,\phi^n)$ is a family of independent eigenfunctions in $L^2(\partial M, e^u d\sigma_g)$ and satisfies (2.15). This gives (i) and (ii). We can write that

$$1 = \int_{\partial M} |\Phi|^2 e^u d\sigma_g = \int_{\partial M} K_\epsilon[|\Phi|^2] d\nu \geq \int_{\partial M} d\nu = 1.$$

Therefore, $K_\epsilon[|\Phi|^2] = 1$ ν-a.e and since $K_\epsilon[|\Phi|^2]$ is continuous, $K_\epsilon[|\Phi|^2] = 1$ on $supp(\nu)$. This gives (iii) and ends the proof of the claim. q.e.d.

By a result of Fraser–Schoen [**9**] and Karpukhin–Kokarev–Polterovich [**18**], there exists a bound for the multiplicity of k-th Steklov eigenvalues on surfaces which only depends on k and the topology of the surface. Therefore, up to the extraction of a subsequence, we assume in the following that $n(\epsilon) = n$ is fixed.

We organize the proof of Theorem 2 as follows:

In Section 3, we give regularity estimates on the densities e^{u_ϵ} and on the associated Steklov eigenfunctions defined by Proposition 1 (see Claim 4). These estimates permit to pass to the limit on the eigenvalue equation (Proposition 1 (i)) as $\epsilon \to 0$ (see Claim 5). However, we cannot pass to the limit on the whole surface. We have to avoid some singularities for the maximizing sequence which could occur. We cannot remove a priori some concentration points of $\{e^{2u_\epsilon} dv_g\}$ even with the assumption that (0.3) is strict. Other harmless singularities are also carefully avoided (see Claim 3).

From Sections 4 to 6, we assume the existence of concentration points for the maximizing sequence and we aim at deducing the case of equality in (0.3). In Section 4, we detect all the concentration scales thanks to the construction of a bubble tree. This leads to the proof of Proposition 2, page 132.

We then give in Section 5 regularity estimates on the eigenfunctions at each scale of concentration and pass to the limit in the equation they satisfy. Notice that this work is divided into two subsections, depending on the speed of convergence to zero of the concentration scale α_ϵ as $\epsilon \to 0$.

Finally, in Section 6.1, capitalizing on the energy estimates for the limiting measures and equations given in Section 3.2 on M (see (3.28)), at the end of Section 5.1 (see (5.35)) and Section 5.2 (see (5.40)) on some disks $\mathbb{D}$, we both prove the regularity of the limiting measures at all the scales of concentration, and that no energy is lost in the necks in the bubbling process. This is given by Proposition 3, page 168. Thanks to this proposition, we prove in Section 6.2 that the presence of concentration points implies the case of equality in (0.3) by a suitable choice of test functions for the variational characterization of $\sigma_\epsilon = \sigma_k(M, g, \partial M, e^{u_\epsilon})$.

Therefore, since the specific maximizing sequence $\{e^{u_\epsilon} d\sigma_g\}$ does not concentrate with the assumption that (0.3) is strict, the end of the proof of Theorem 2 just uses the second part of Proposition 3 in Section 6.1.

3. Regularity estimates in the surface

3.1. Regularity estimates far from singularities. In this subsection, we aim at getting finer and finer regularity estimates on the eigenfunctions which appear in Proposition 1 and pass to the limit on the equation they satisfy. We denote by ν the weak* limit of ν_ϵ. Notice that ν is also the weak* limit of $\{e^{u_\epsilon} d\sigma_g\}$. Indeed, if $\zeta \in \mathcal{C}^0(\partial M)$,

$$\begin{aligned}\left|\int_{\partial M} \zeta \left(e^{u_\epsilon} d\sigma_g - d\nu_\epsilon\right)\right| &= \left|\int_{\partial M} (K_\epsilon[\zeta] - \zeta) d\nu_\epsilon\right| \\ &\leq \sup_M \left|K_\epsilon[\zeta] - \zeta\right|,\end{aligned}$$

which goes to 0 as $\epsilon \to 0$ by uniform continuity of ζ.

Hypothesis (iii) in Proposition 1 gives uniform estimates on the eigenfunctions $\{\phi_\epsilon^i\}$ on sets of points which lie at a distance to $supp(\nu_\epsilon)$ asymptotically smaller than $\sqrt{\epsilon}$.

Claim 2. *For any $R > 0$ there exists a constant $C_R > 0$ such that for any sequence (x_ϵ) of points in ∂M, with $d_g(x_\epsilon, supp(\nu_\epsilon)) \leq R\sqrt{\epsilon}$, we have*

$$\left|\phi_\epsilon^i(x_\epsilon)\right| \leq C_R \text{ for all } \epsilon > 0.$$

Proof. We refer the reader to Section 1.1 for the notations used during this proof. We can assume that $x_\epsilon \in \omega_l$ for $1 \leq l \leq L$ fixed and we set

$$\hat{\Phi}_\epsilon(x) = \widetilde{\Phi_\epsilon}^l(\sqrt{\epsilon}x + \tilde{x}_\epsilon^l)$$

for $x \in \mathbb{D}_{\delta\epsilon^{-\frac{1}{2}}} \cap \mathbb{R}^2_+$. Then,

$$\begin{cases} \Delta_\xi \hat{\Phi}_\epsilon^i = 0 & \text{in } \mathbb{D}^+_{\delta\epsilon^{-\frac{1}{2}}}, \\ \partial_t \hat{\phi}_\epsilon^i = -\sigma_\epsilon \sqrt{\epsilon} e^{\tilde{u}_\epsilon^l(\sqrt{\epsilon}x + \tilde{x}_\epsilon^l)} \hat{\phi}_\epsilon^i & \text{in } I_{\delta\epsilon^{-\frac{1}{2}}}, \end{cases}$$

for $0 \leq i \leq n$. By estimate (1.3) of Section 1.2, $\{\sqrt{\epsilon}p_\epsilon\}$ is uniformly bounded so that $\{\sqrt{\epsilon}e^{\tilde{u}_\epsilon^l(\sqrt{\epsilon}x + \tilde{x}_\epsilon^l)}\}$ is uniformly bounded. Now, we let $y_\epsilon \in supp(\nu_\epsilon)$ be such that $d_g(x_\epsilon, y_\epsilon) \leq R\sqrt{\epsilon}$. Thanks to Proposition 1, we have that $K_\epsilon[|\Phi_\epsilon|^2](y_\epsilon) = 1$. Let us write then with (1.3), Section 1.2 that for $\rho > 0$,

$$\begin{aligned} 1 = K_\epsilon\left[|\Phi_\epsilon|^2\right](y_\epsilon) &\geq \sum_{i=0}^n K_\epsilon\left[\left|\phi_\epsilon^i\right|^2\right](y_\epsilon) \\ &= \sum_{i=0}^n \int_{\partial M} p_\epsilon(y, y_\epsilon)\left(\phi_\epsilon^i(y)\right)^2 d\sigma_g(y) \\ &\geq \sum_{i=0}^n \frac{1}{A_0\sqrt{4\pi\epsilon}} e^{-\rho^2 C_0^2} \int_I \left(\phi_\epsilon^i(y)\right)^2 d\sigma_g(y) \\ &\geq \sum_{i=0}^n \frac{1}{A_0\sqrt{4\pi}C_0} e^{-\rho^2 C_0^2} \int_{I_{2\rho}(\hat{z}_\epsilon)} \left(\hat{\phi}_\epsilon^i(z)\right)^2 dz, \end{aligned}$$

where $I = I_g\left(y_\epsilon, 2\rho C_0\sqrt{\epsilon}\right)$ in the third line and we set $\hat{z}_\epsilon = \frac{1}{\sqrt{\epsilon}}(\tilde{y}^l_\epsilon - \tilde{x}^l_\epsilon)$ so that, up to the extraction of a subsequence $\hat{z}_\epsilon \to z_0 \in \partial M$ as $\epsilon \to 0$ and we deduce from the previous inequality that, for any $\rho > 0$, $\hat{\phi}^i_\epsilon$ is bounded in $L^2(I_\rho(z_0))$. Thus, by elliptic regularity of the Dirichlet-to-Neumann operator (see Taylor [**25**], Chapter 7.11, page 37), we get that $\{\hat{\phi}^i_\epsilon\}$ is uniformly bounded in I_ρ by some constant D_ρ. Setting $C_R = D_{2C_0R}$ gives the claim. q.e.d.

Now, we will restrict the estimates on the eigenfunctions ϕ^i_ϵ far from some singularities which could appear.

$\mathbf{A}_{r,\epsilon}$: We say that a point $x \in \partial M$ satisfies $\mathbf{A}_{r,\epsilon}$ for some $r > 0$ and some $\epsilon > 0$ if

$$\sigma_\star(B_g(x,r), g, I_g(x,r), e^{u_\epsilon}) \leq \frac{\sigma_k(M,[g])}{2}.$$

$\mathbf{B}_{r,\epsilon}$: We say that a point $x \in M$ satisfies $\mathbf{B}_{r,\epsilon}$ for $r > 0$ and $\epsilon > 0$ if there exists $f \in E_k(M, g, \partial M, \{e^{u_\epsilon}\})$ such that $f(x) = 0$ and the Nodal set of f which contains x does not intersect $\partial B_g(x,r) \setminus \partial M$.

Note that if $r_1 < r_2$, $\mathbf{A}_{r_1,\epsilon} \Rightarrow \mathbf{A}_{r_2,\epsilon}$ and $\mathbf{B}_{r_1,\epsilon} \Rightarrow \mathbf{B}_{r_2,\epsilon}$. We say that a point $x \in M$ satisfies $\mathbf{P}_{r,\epsilon}$ for $r > 0$ and $\epsilon > 0$ if $x \in \partial M$ and x satisfies $\mathbf{A}_{r,\epsilon}$ or if x satisfies $\mathbf{B}_{r,\epsilon}$. For a surface (M,g), a sequence of densities $\{e^{u_\epsilon}\}$ on ∂M and $r > 0$, we define the singular set

$$X_r(M, g, \partial M, \{e^{u_\epsilon}\}) = \{x \in \Omega, \exists \epsilon > 0 \text{ such that } x \text{ satisfies } \mathbf{P}_{r,\epsilon}\}.$$

Note that if $r_1 < r_2$, then $X_{r_1}(M, g, \partial M, \{e^{u_\epsilon}\}) \subset X_{r_2}(M, g, \partial M, \{e^{u_\epsilon}\})$. The following claim holds true

Claim 3. *There exists a sequence $\{e^{u_{\epsilon_m}}\}$ with $\epsilon_m \to 0$ as $m \to +\infty$ and there exist some points $p_1, \cdots, p_s \in \partial M$ with $0 \leq s \leq k$ such that*

- $\forall \rho > 0, \exists r > 0, X_r(M, g, \partial M, \{e^{u_{\epsilon_m}}\}) \subset \bigcup_{i=1}^s B_g(p_i, \rho)$,
- *For any subsequence $\{e^{u_{\epsilon_{m(j)}}}\}_{j\geq 0}$ of $\{e^{u_{\epsilon_m}}\}_{m\geq 0}$,*

(3.1)
$$\forall \rho > 0, \forall r > 0, \forall 1 \leq i \leq s, X_r(M, g, \partial M, \{e^{u_{\epsilon_{m(j)}}}\}) \cap B_g(p_i, \rho) \neq \emptyset.$$

Proof. Assume by contradiction that for any sequence $\epsilon_m \to 0$, as $m \to +\infty$, for any series of s points $p_1, \cdots, p_s \in \partial M$ with $0 \leq s \leq k$, there is $\rho > 0$ such that

$$\forall r > 0, X_r(M, g, \partial M, \{e^{u_{\epsilon_m}}\}) \setminus \bigcup_{i=1}^s B_g(p_i, \rho) \neq \emptyset. \tag{3.2}$$

Thanks to this hypothesis, we will deduce by induction the following property $\mathbf{H}_s$ for $1 \leq s \leq k+1$

$\mathbf{H}_s$: There exist sequences $\epsilon_m \to 0$, $r_m \searrow 0$ as $m \to +\infty$, some points $p_1^m, \cdots, p_s^m \in M$ and s pairwise distinct points $p_1, \cdots, p_s \in \partial M$ such that for $1 \leq i \leq s$, $p_i^m \to p_i$ as $m \to +\infty$ and p_i^m satisfies $\mathbf{P}_{r_m,\epsilon_m}$.

Let's first prove $\mathbf{H}_1$. By (3.2) applied for $s=0$ and a sequence $\{2^{-j}\}$, we have the existence of $p_1^m \in X_{2^{-m}}(M,g,\partial M,\{e^{u_{2^{-j}}}\}_{j\geq 0})$ for any fixed $m\geq 0$. For $m\geq 0$, we choose $\epsilon_m = 2^{-j(m)}$ such that p_1^m satisfies $\mathbf{P}_{2^{-m},\epsilon_m}$. It is clear that $\epsilon_m \to 0$ as $m\to+\infty$. Up to the extraction of a subsequence, there exists $p_1\in M$ such that $p_1^m \to p_1$ as $m\to+\infty$. Now, it is clear that $p_1\in\partial M$. Indeed, if $p_1 \in M\setminus\partial M$, then we choose $m_0\in\mathbb{N}$ such that for $m\geq m_0$, $B_g(p_1^m,r_m)\subset M\setminus\partial M$. Then p_1^m satisfies $\mathbf{B}_{r_m,\epsilon_m}$ and the Nodal set of some function $f_m \in E_k(M,g,\partial M,\{e^{u_{\epsilon_m}}\})$ which contains p_1^m does not intersect ∂M since it does not intersect $\partial B_g(p_1^m,r_m)$. Since f_m is harmonic, it vanishes on an open set of M by the maximum principle so that f_m vanishes on M. This contradicts the fact that f_m is a k-th eigenfunction for the Dirichlet-to-Neumann operator. Then $p_1\in\partial M$ and we get $\mathbf{H}_1$.

We assume now that $\mathbf{H}_s$ is true for some $1\leq s\leq k$. We consider the sequences $\{\epsilon_m\}$, $\{r_m\}$, $\{p_i^m\}$ and $p_1,\cdots,p_s\in\partial M$ given by $\mathbf{H}_s$. Let us prove $\mathbf{H}_{s+1}$. By (3.2), there is $\rho>0$ such that for all $r>0$,

$$X_r(M,g,\partial M,\{e^{u_{\epsilon_m}}\})\setminus\bigcup_{i=1}^{s}B_g(p_i,\rho)\neq\emptyset.$$

Let $p_{s+1}^m\in X_{r_m}(M,g,\partial M,\{e^{u_{\epsilon_j}}\}_{j\geq0})$. For $m\in\mathbb{N}$ fixed, we let $\alpha(m)$ be such that p_{s+1}^m satisfies $\mathbf{P}_{r_m,\epsilon_{\alpha(m)}}$. Since $r_m\to0$ as $m\to+\infty$, it is clear that $\alpha(m)\to+\infty$ as $m\to+\infty$. We set $\beta(m)=\min(m,\alpha(m))$. By $\mathbf{H}_s$, for $1\leq i\leq s$, $p_i^{\alpha(m)}$ satisfies $\mathbf{P}_{r_{\alpha(m)},\epsilon_{\alpha(m)}}$ and since r_m is decreasing, $p_i^{\alpha(m)}$ satisfies $\mathbf{P}_{r_{\beta(m)},\epsilon_{\alpha(m)}}$. Moreover, p_{s+1}^m satisfies $\mathbf{P}_{r_m,\epsilon_{\alpha(m)}}$ and since r_m is decreasing p_{s+1}^m satisfies $\mathbf{P}_{r_{\beta(m)},\epsilon_{\alpha(m)}}$. Up to the extraction of a subsequence, we can assume that $r_{\beta(m)}\searrow0$ as $m\to+\infty$ and we let $p_{s+1}\in M$ such that $p_{s+1}^m\to p_{s+1}$ as $m\to+\infty$. Since $p_{s+1}^m\in M\setminus\bigcup_{i=1}^s B_g(p_i,\rho)$, $p_{s+1}\notin\{p_1,\cdots,p_s\}$. By the same arguments as in the proof of $\mathbf{H}_1$, we also have that $p_{s+1}\in\partial M$. This proves $\mathbf{H}_{s+1}$.

The proof of $\mathbf{H}_{k+1}$ is complete. Now, we prove that $\mathbf{H}_{k+1}$ leads to a contradiction. We define $k+1$ test functions for the variational characterization of $\sigma_{\epsilon_m}=\sigma_k(M,g,\partial M,e^{u_{\epsilon_m}})$, η_i^m for $m\in\mathbb{N}$ and $1\leq i\leq k+1$ as follows

- If p_i^m satisfies $\mathbf{A}_{r_m,\epsilon_m}$, η_i^m is the extension by 0 in $M\setminus B_g(p_i^m,r_m)$ of an eigenfunction for $\sigma_\star(B_g(p_i^m,r_m),g,I_g(p_i^\epsilon,r_m),\{e^{u_{\epsilon_m}}\})$. In this case,

$$\frac{\int_M|\nabla\eta_i^m|_g^2\,dv_g}{\int_{\partial M}(\eta_i^m)^2\,d\sigma_g}\leq\frac{\sigma_k(M,[g])}{2}. \tag{3.3}$$

- If p_i^m does not satisfy $\mathbf{A}_{r_m,\epsilon_m}$, it satisfies $\mathbf{B}_{r_m,\epsilon_m}$ and η_i^m is some eigenfunction for $\sigma_\star(D_i^m,g,\Gamma_i^m,e^{u_{\epsilon_m}})$ extended by 0 in $M\setminus D_i^m$ where D_i^m is a nodal domain of some Steklov eigenfunction associated to σ_{ϵ_m} which is included in $B_g(p_i^m,r_m)$. Such a domain exists by assumption $\mathbf{B}_{r_m,\epsilon_m}$ and satisfies $\Gamma_i^m = \partial M \cap D_i^m \neq \emptyset$. In this case,

$$\frac{\int_M |\nabla \eta_i^m|_g^2 \, dv_g}{\int_{\partial M} (\eta_i^m)^2 \, d\sigma_g} = \sigma_\star(D_i^m, g, \Gamma_i^m, e^{u_{\epsilon_m}}) = \sigma_{\epsilon_m}. \tag{3.4}$$

For m large enough, we have

$$\min_{1 \leq i < i' \leq k+1} d_g(p_i^m, p_{i'}^m) - 3r_m \geq \frac{1}{2} \min_{1 \leq i < i' \leq k+1} d_g(p_i, p_{i'}) > 0,$$

so that the functions $\eta_1^m, \cdots, \eta_{k+1}^m$ have pairwise disjoint supports. Thanks to (3.3) and (3.4), the min-max characterization of $\sigma_{\epsilon_m} = \sigma(M, g, \partial M, e^{u_{\epsilon_m}})$ (1.1) gives that

$$\sigma_{\epsilon_m} \leq \max_{1 \leq i \leq k+1} \frac{\int_M |\nabla \eta_i^m|_g^2 \, dv_g}{\int_{\partial M} (\eta_i^m)^2 \, d\sigma_g} \leq \sigma_{\epsilon_m},$$

since for m large enough, $\sigma_{\epsilon_m} \to \sigma_k(M,[g]) > \frac{\sigma_k(M,[g])}{2}$. Then, all the inequalities are equalities and by the case of equality in the min-max characterization of the k-th eigenvalue, one of the functions η_i^m is an eigenfunction on the surface for $\sigma_{\epsilon_m} = \sigma_k(M, g, \partial M, e^{u_\epsilon})$. Since $supp(\eta_i^m) \subset B_g(p_i^m, r_m)$ and $\eta_i^m \neq 0$, we contradict the harmonicity of η_i^m.

Therefore, we have proved that there exists a subsequence $\{e^{u_{\epsilon_m}}\}$ and $p_1, \cdots, p_s \in \partial M$ for some $0 \leq s \leq k$ such that

$$\forall \rho > 0, \exists r > 0, X_r(M, g, \partial M, \{e^{u_{\epsilon_m}}\}) \subset \bigcap_{i=1}^{s} B_g(p_i, \rho),$$

which is exactly the first part of the claim.

Let's prove now the second part of the claim. If there exists a subsequence $m(j) \to +\infty$ as $j \to +\infty$ such that there exists $\rho > 0$ and $r > 0$ and $1 \leq i_0 \leq s$ with

$$X_r(M, g, \partial M, \{e^{u_{\epsilon_{m(j)}}}\}) \cap B_g(p_{i_0}, \rho) = \emptyset,$$

then, taking the subsequence $m(j)$, we can remove the index $i_0 \in \{1, \cdots, s\}$ so that

$$X_r(M, g, \partial M, \{e^{u_{\epsilon_{m(j)}}}\}) \subset \bigcup_{i \in \{1,\cdots,s\} \setminus \{i_0\}} B_g(p_i, \rho).$$

We go on with this process until we cannot find a subsequence such that (3.1) does not hold. This ends the proof of the claim. q.e.d.

Up to the extraction of a subsequence, we assume in the following that $\{e^{u_\epsilon}\}$ satisfies the conclusion of Claim 3. For $\rho > 0$, we let

$$M(\rho) = M \setminus \bigcup_{i=1}^{s} B_g(p_i, \rho),$$

and

$$I(\rho) = \partial M \setminus \bigcup_{i=1}^{s} I_g(p_i,\rho).$$

We are now able to get regularity estimates on the functions e^{u_ϵ} in $I(\rho)$ and Φ_ϵ in $M(\rho)$.

Claim 4. *We assume that* $m_0(\rho) = \lim_{\epsilon\to 0}\int_{I(\rho)} e^{u_\epsilon}dv_g > 0$ *for any* $\rho > 0$ *small enough. Then we have the following*

- *Estimates on* Φ_ϵ

$$\forall\rho>0, \exists C_1(\rho)>0, \forall\epsilon>0, \|\Phi_\epsilon\|_{W^{1,2}(M(\rho))} \leq C_1(\rho), \tag{3.5}$$

$$\forall\rho>0, \exists C_2(\rho)>0, \forall\epsilon>0, \|\Phi_\epsilon\|_{C^0(M(\rho))} \leq C_2(\rho), \tag{3.6}$$

- *Quantitative non-concentration estimates on* e^{u_ϵ} *and* $|\nabla\Phi_\epsilon|_g^2$

(3.7)

$$\forall\rho>0, \exists D_1(\rho)>0, \forall r>0, \limsup_{\epsilon\to 0}\sup_{x\in I(\rho)}\int_{I_g(x,r)} e^{u_\epsilon}dv_g \leq \frac{D_1(\rho)}{\ln(\frac{1}{r})},$$

(3.8)

$$\forall\rho>0, \exists D_2(\rho)>0, \forall r>0, \limsup_{\epsilon\to 0}\sup_{x\in I(\rho)}\int_{B_g(x,r)} |\nabla\Phi_\epsilon|_g^2 dv_g \leq \frac{D_2(\rho)}{\sqrt{\ln(\frac{1}{r})}}.$$

Proof. We first prove (3.5) by using Claim 3 and the assumption $m_0(\rho) > 0$.

For that purpose, let's prove that $\{\frac{e^{u_\epsilon}}{\int_{I(\rho)} e^{u_\epsilon}d\sigma_g}d\sigma_g\}$ is bounded in $W^{-1,2}(M(\rho))$. Let $\rho > 0$ and let $r > 0$ be such that

$$X_r\left(M, g, \partial M, \{e^{u_\epsilon}\}\right) \subset \bigcup_{i=1}^{s} B_g(p_i,\rho).$$

Then, for all $x \in I(\rho)$ and all $\epsilon > 0$, $\sigma_\star(B_g(x,r), g, I_g(x,r), e^{u_\epsilon}) > \frac{\sigma_k(M,[g])}{2}$. By the compactness of $I(\rho)$, we can find $y_1, \cdots, y_t \in I(\rho)$ such that

$$I(\rho) \subset \bigcup_{i=1}^{t} I_g(y_i, r).$$

Let $\psi_1, \cdots, \psi_t$ be a partition of unity associated to this covering, such that $\sum_{i=1}^{t}\psi_i = 1$ on $I(\rho)$ and $supp(\psi_i) \subset B_g(y_i, r)$. Let $L : W^{1,2}(M(\rho)) \to W^{1,2}(M)$ be a continuous extension operator. Then, if $\psi \in W^{1,2}(M(\rho))$, its trace on the boundary satisfies

$$\int_{I(\rho)} \psi \frac{e^{u_\epsilon}d\sigma_g}{\int_{I(\rho)} e^{u_\epsilon}d\sigma_g} = \sum_{i=1}^{t}\int_{I(\rho)\cap B_g(y_i,r)} \psi\psi_i \frac{e^{u_\epsilon}d\sigma_g}{\int_{I(\rho)} e^{u_\epsilon}d\sigma_g}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{t}\left(\int_{I(\rho)\cap B_g(y_i,r)}(\psi_i\psi)^2\frac{e^{u_\epsilon}d\sigma_g}{\int_{I(\rho)}e^{u_\epsilon}d\sigma_g}\right)^{\frac{1}{2}}\\
&\leq \sum_{i=1}^{t}\left(\int_{\partial M\cap B_g(y_i,r)}(\psi_i L(\psi))^2\frac{e^{u_\epsilon}d\sigma_g}{\int_{I(\rho)}e^{u_\epsilon}d\sigma_g}\right)^{\frac{1}{2}}\\
&\leq \sum_{i=1}^{t}\frac{\left(\int_{B_g(y_i,r)}\left|\nabla(\psi_i L(\psi))\right|_g^2 dv_g\right)^{\frac{1}{2}}}{\sigma_i^{\frac{1}{2}}\left(\int_{I(\rho)}e^{u_\epsilon}d\sigma_g\right)^{\frac{1}{2}}}\\
&\leq \frac{A_0(\rho)}{\left(\frac{\sigma_k(M,[g])}{2}\right)^{\frac{1}{2}}m_0(\rho)^{\frac{1}{2}}}\left\|L(\psi)\right\|_{W^{1,2}(M)}\\
&\leq A_1(\rho)\left\|\psi\right\|_{W^{1,2}(M(\rho))}
\end{aligned}
$$

for some constants $A_0(\rho)$ and $A_1(\rho)$ which do not depend on $\epsilon > 0$ (where $\sigma_i = \sigma_\star\left(B_g(y_i,r), g, I_g(y_i,r), e^{u_\epsilon}g\right)$).

By Theorem 3 in Section 1.3, we now get the following Poincaré inequality: there exists some constant $A_2(\rho)$ such that for any $f \in \mathcal{C}^\infty(M(\rho))$

$$
\forall \epsilon > 0, \int_{M(\rho)}\left(f - \int_{I(\rho)} f\frac{e^{u_\epsilon}d\sigma_g}{\int_{I(\rho)}e^{u_\epsilon}d\sigma_g}\right)^2 dv_g \leq A_2(\rho)\int_{M(\rho)}\left|\nabla f\right|_g^2 dv_g.
$$

We deduce from this inequality that

$$
\int_{M(\rho)} f^2 dv_g \leq 2A_2(\rho)\int_{M(\rho)}\left|\nabla f\right|_g^2 dv_g + 2V_g(M)\frac{\int_{I(\rho)} f^2 e^{u_\epsilon}d\sigma_g}{\int_{I(\rho)}e^{u_\epsilon}d\sigma_g}.
$$

Applying this inequality to the ϕ_ϵ^i's and summing for $i = 0\cdots n$, we get that

$$
\int_{M(\rho)}\left|\Phi_\epsilon\right|^2 dv_g \leq 2A_2(\rho)\sigma_\epsilon\int_{\partial M}\left|\Phi_\epsilon\right|^2 d\sigma_g + 2V_g(M)\frac{\int_{\partial M}\left|\Phi_\epsilon\right|^2 e^{u_\epsilon}d\sigma_g}{\int_{I(\rho)}e^{u_\epsilon}d\sigma_g}
$$

using the fact that

$$
\int_{M(\rho)}\left|\nabla\phi_i^\epsilon\right|_g^2 dv_g \leq \int_M\left|\nabla\phi_i^\epsilon\right|_g^2 dv_g = \sigma_\epsilon\int_{\partial M}e^{u_\epsilon}(\phi_\epsilon^i)^2 d\sigma_g,
$$

by (iii) of Proposition 1,

$$
\int_{\partial M}e^{u_\epsilon}\left|\Phi_\epsilon\right|^2 d\sigma_g = \int_{\partial M}\left|\Phi_\epsilon\right|^2 K_\epsilon[\nu_\epsilon]d\sigma_g = \int_{\partial M}K_\epsilon[\left|\Phi_\epsilon\right|^2]d\nu_\epsilon = 1.
$$

Then, we get that

$$
\int_{M(\rho)}\left|\Phi_\epsilon\right|^2 dv_g \leq 2A_2(\rho)\sigma_\epsilon + \frac{2V_g(M)}{\int_{I(\rho)}e^{u_\epsilon}d\sigma_g}.
$$

Thanks to the assumption of the claim, namely that $\int_{I(\rho)} e^{u_\epsilon} d\sigma_g \to m_0(\rho) > 0$, we get the existence of some $A_3(\rho)$ such that

$$\int_{M(\rho)} |\Phi_\epsilon|^2 \, dv_g \leq A_3(\rho).$$

Now, with what we just said, we also know that

$$\int_{M(\rho)} |\nabla \Phi_\epsilon|_g^2 \, dv_g \leq \sigma_\epsilon,$$

and (3.5) follows.

In order to get (3.6), we first prove that

$$\forall \rho > 0, \exists C_0(\rho), \forall \epsilon > 0, \|\Phi_\epsilon\|_{\mathcal{C}^0(I(\rho))} \leq C_0(\rho). \tag{3.9}$$

Let $\rho > 0$, $0 \leq i \leq n$ and up to change ϕ_ϵ^i into $-\phi_\epsilon^i$, let (x_ϵ) be a sequence of points of $I(\rho)$ such that $\phi_\epsilon^i(x_\epsilon) = \sup_{I(\rho)} \left|\phi_\epsilon^i\right|$. We set

$$\delta_\epsilon = d_g(x_\epsilon, supp(\nu_\epsilon)).$$

We divide the rest of the proof of (3.9) into three cases.

Case 1 – We assume that $\delta_\epsilon^{-1} = O(1)$. Then, $\{e^{u_\epsilon}\}$ is uniformly bounded in $I_g\left(x_\epsilon, \min\left\{\frac{\delta_\epsilon}{2}, \frac{\rho}{2}\right\}\right)$ by (1.4). By (3.5), $\{\phi_\epsilon^i\}$ is bounded in $L^2(I(\frac{\rho}{2}))$. Then, $\{\phi_\epsilon^i\}$ is bounded in $W^{1,2}(I_g(x_\epsilon, \min\{\frac{\delta_\epsilon}{2}, \frac{\rho}{2}\}))$ by elliptic theory for the Dirichlet-to-Neumann operator (see [**25**], chapter 7.11, page 37), and $\{\phi_\epsilon^i(x_\epsilon)\}$ is bounded by Sobolev embeddings.

Case 2 – We assume that $\delta_\epsilon = O(\sqrt{\epsilon})$. Then, $\{\phi_\epsilon^i(x_\epsilon)\}$ is bounded by Claim 2.

Case 3 – We assume that $\delta_\epsilon \to 0$ and $\frac{\sqrt{\epsilon}}{\delta_\epsilon} \to 0$ as $\epsilon \to 0$. We let

$$\psi_\epsilon = \tilde{\phi}_\epsilon(\delta_\epsilon x + \tilde{x}_\epsilon) \text{ for } x \in \mathbb{D}^+_{\delta\delta_\epsilon^{-1}} \text{ and } e^{w_\epsilon} = \delta_\epsilon e^{\tilde{u}_\epsilon(\delta_\epsilon x + x_\epsilon)} \text{ for } x \in I_{\delta\delta_\epsilon^{-1}},$$

so that

$$\begin{cases} \Delta \psi_\epsilon = 0 & \text{in } \mathbb{D}^+_{\delta\delta_\epsilon^{-1}}, \\ \partial_t \psi_\epsilon = -\sigma_\epsilon e^{w_\epsilon} \psi_\epsilon & \text{on } I_{\delta\delta_\epsilon^{-1}}. \end{cases} \tag{3.10}$$

Let $y_\epsilon \in supp(\nu_\epsilon)$ be such that $d_g(x_\epsilon, y_\epsilon) = \delta_\epsilon$ and set $z_\epsilon = \frac{\tilde{y}_\epsilon - \tilde{x}_\epsilon}{\delta_\epsilon}$ so that $z_\epsilon \to z_0$ as $\epsilon \to 0$ up to the extraction of a subsequence. We set $R = |z_0|$. Thanks to Claim 2, we know that $\psi_\epsilon(z_\epsilon) = \phi_\epsilon^i(y_\epsilon) = O(1)$. Thanks to estimates (1.6) on the heat kernel, there exists $D_1 > 0$ such that

$$e^{w_\epsilon} \leq D_1 \text{ on } I_{\frac{R}{2}}.$$

We first assume that ψ_ϵ does not vanish in $\mathbb{D}^+_{3R}$. Then, we can apply Harnack's inequality and get some constant $D_2 > 0$ such that

$$\psi_\epsilon \geq D_2 \psi_\epsilon(0) \text{ on } \mathbb{D}^+_{\frac{R}{4}},$$

for all $\epsilon > 0$. Since ψ_ϵ is positive on $\mathbb{D}^+_{|z_\epsilon|}(z_\epsilon) \subset \mathbb{D}^+_{3R}$, by equation (3.10), it is weakly superharmonic and we can write that

$$\psi_\epsilon(z_\epsilon) \geq \frac{1}{\pi |z_\epsilon|} \int_{\partial \mathbb{D}^+_{|z_\epsilon|}(z_\epsilon)} \psi_\epsilon d\sigma.$$

Taking only the part of the integral which lies in $\mathbb{D}^+_{\frac{R}{4}}$, we get the existence of some constant $D_3 > 0$ such that

$$\psi_\epsilon(z_\epsilon) \geq D_3 \psi_\epsilon(0),$$

and this concludes the proof of (3.6) in this case since $\phi^i_\epsilon(x_\epsilon) = \psi_\epsilon(0) = O(1)$.

We now assume that ψ_ϵ vanishes on $\mathbb{D}^+_{3R}$. Since $\delta_\epsilon \to 0$ as $\epsilon \to 0$, and $x_\epsilon \in I(\rho)$, by Claim 3, ψ_ϵ vanishes on a piecewise smooth curve in $\mathbb{D}^+_{4R}$ which connects two points of distance greater than R. By the corollary of Theorem 4 of Section 1.3 on $\Omega = \mathbb{D}^+_{5R}$, we get some constant $C_R > 0$ such that

$$\int_{\mathbb{D}^+_{4R}} \psi_\epsilon^2 \leq C_R \int_{\mathbb{D}^+_{5R}} |\nabla \psi_\epsilon|^2 dx,$$

which proves that $\{\psi_\epsilon\}$ is bounded in $W^{1,2}(\mathbb{D}^+_{4R})$ by conformal invariance of the L^2-norm of the gradient in dimension 2. By trace Sobolev properties, $\{\psi_\epsilon\}$ is bounded in $L^2(I_{4R})$ and by elliptic regularity theory for the Dirichlet-to-Neumann operator (see [**25**], Chapter 7.11, page 37), ψ_ϵ is bounded in $L^\infty(\mathbb{D}_{\frac{R}{4}})$ which gives that $\{\phi^i_\epsilon(x_\epsilon)\}$ is bounded.

The study of these three cases completes the proof of (3.9).

We now prove (3.6). Let $\rho > 0$ and $0 \leq i \leq n$. Then, since ϕ^i_ϵ is harmonic in $M\left(\frac{\rho}{2}\right)$, by elliptic regularity theory, there exists a constant $K_0(\rho) > 0$ such that

$$\left\|\phi^i_\epsilon\right\|_{\mathcal{C}^0(M(\rho))} \leq K_0(\rho) \left(\left\|\phi^i_\epsilon\right\|_{L^2\left(M\left(\frac{\rho}{2}\right)\right)} + \left\|\phi^i_\epsilon\right\|_{\mathcal{C}^0\left(I\left(\frac{\rho}{2}\right)\right)} \right),$$

so that (3.6) holds with $C_2(\rho) = K_0(\rho)\left(C_1\left(\frac{\rho}{2}\right) + C_0\left(\frac{\rho}{2}\right)\right)$.

Thanks to Claim 3, we have the existence of some $r_1(\rho) > 0$ such that for any $0 < r < r_1(\rho)$,

$$\forall \epsilon > 0, \forall x \in I(\rho), \frac{1}{\sigma_\star\left(B_g(x,r), g, I_g(x,r), e^{u_\epsilon} g\right)} \leq \frac{2}{\sigma_k(M,[g])}.$$

By isocapacity estimates,

$$\begin{aligned} \int_{I_g(x,r)} e^{u_\epsilon} d\sigma_g &\leq \frac{Cap_2(B_g(x,r), B_g(x,r_1))}{\sigma_\star(B_g(x,r), g, I_g(x,r), e^{u_\epsilon})} \\ &\leq 2\frac{Cap_2(\mathbb{D}_{\frac{r}{C_0}}, \mathbb{D}_{C_0 r_1})}{\sigma_k(M,[g])} \end{aligned}$$

$$\leq \frac{4\pi}{\sigma_k(M,[g])\ln\left(\frac{C_0^2 r_1}{r}\right)},$$

and we get (3.7).

Finally, let's prove (3.8). We set for $x \in I_\delta$ such that $\bar{x} \in M(\rho)$, where $\bar{x} = \exp_{g_l,x_l}(x)$ as defined in Section 1.1 and for $0 < r \leq \delta$

$$F_\epsilon(r) = \int_{\mathbb{D}_r^+(x)} \left|\nabla \tilde{\Phi}_\epsilon\right|^2 dx.$$

We suppose in the following that $\delta < 1$, without loss of generality. We just aim at proving that

$$F_\epsilon(r) \leq \frac{D_0(\rho)}{\sqrt{\ln\left(\frac{1}{r}\right)}}.$$

We know that $\tilde{\Phi}_\epsilon$ satisfies the equations

$$\begin{cases} \Delta\tilde{\Phi}_\epsilon = 0 & \text{in } \mathbb{D}_\delta^+, \\ \partial_t \tilde{\Phi}_\epsilon = -\sigma_\epsilon e^{\tilde{u}_\epsilon}\tilde{\Phi}_\epsilon & \text{on } I_\delta, \end{cases}$$

and we deduce that

$$F_\epsilon(r) = \sigma_\epsilon \int_{I_r(x)} e^{\tilde{u}_\epsilon}\left|\tilde{\Phi}_\epsilon\right|^2 dx + \int_{\partial\mathbb{D}_r^+(x)} \tilde{\Phi}_\epsilon.\partial_\nu\tilde{\Phi}_\epsilon d\sigma_\xi.$$

Using (3.6) and (3.7), there exist some constants $K_1(\rho)$ and $K_2(\rho)$ independent of ϵ, r and x with $\bar{x} \in I(\rho)$, such that

$$\begin{aligned} F_\epsilon(r)^2 &\leq \frac{K_1(\rho)}{\ln\left(\frac{1}{r}\right)^2} + K_2(\rho)\left(\int_{\partial\mathbb{D}_r^+(x)}\left|\nabla\tilde{\Phi}_\epsilon\right| dx\right)^2 \\ &\leq \frac{K_1(\rho)}{\ln\left(\frac{1}{r}\right)^2} + \pi r K_2(\rho)\int_{\partial\mathbb{D}_r^+(x)}\left|\nabla\tilde{\Phi}_\epsilon\right|^2 dx \\ &\leq \frac{K_1(\rho)}{\ln\left(\frac{1}{r}\right)^2} + \pi r K_2(\rho) F_\epsilon'(r) \end{aligned}$$

for any $0 < r < \delta$. We can write that

$$\begin{aligned} \left(F_\epsilon(r)\sqrt{\ln\left(\frac{1}{r}\right)}\right)'(s) &= F_\epsilon'(s)\sqrt{\ln\left(\frac{1}{s}\right)} - \frac{1}{2s\sqrt{\ln\left(\frac{1}{s}\right)}}F_\epsilon(s) \\ &\geq \frac{F_\epsilon(s)^2\sqrt{\ln\left(\frac{1}{s}\right)}}{\pi s K_2(\rho)} - \frac{K_1(\rho)}{\pi s K_2(\rho)\ln\left(\frac{1}{s}\right)^{\frac{3}{2}}} \\ &\quad -\frac{1}{2s\sqrt{\ln\left(\frac{1}{s}\right)}}F_\epsilon(s). \end{aligned}$$

Setting

$$J_\epsilon = \left\{ s \in (0,\delta); F_\epsilon(s) < \frac{\pi K_2(\rho)}{\ln\left(\frac{1}{s}\right)} \right\},$$

we have for $s \in (0,\delta) \setminus J_\epsilon$

$$\left(F_\epsilon(r)\sqrt{\ln\left(\frac{1}{r}\right)} \right)'(s) \geq -\frac{K_3(\rho)}{s \ln\left(\frac{1}{s}\right)^{\frac{3}{2}}} \tag{3.11}$$

for $K_3(\rho) = \frac{K_1(\rho)}{\pi K_2(\rho)}$. Let $r \in (0,\delta)$,

$$s_\epsilon = \inf\{s \in [r,\delta), s \in J_\epsilon\}.$$

If $s_\epsilon = r$, then

$$F_\epsilon(r)\sqrt{\ln\left(\frac{1}{r}\right)} \leq \frac{\pi K_2(\rho)}{\sqrt{\ln\left(\frac{1}{r}\right)}} \leq \frac{\pi K_2(\rho)}{\sqrt{\ln\left(\frac{1}{\delta}\right)}},$$

and if $s_\epsilon > r$, then, integrating (3.11) from r to s_ϵ leads to

$$\begin{aligned} F_\epsilon(r)\sqrt{\ln\left(\frac{1}{r}\right)} &\leq F_\epsilon(s_\epsilon)\sqrt{\ln\left(\frac{1}{s_\epsilon}\right)} + \int_r^{s_\epsilon} \frac{K_3(\rho)}{s\ln\left(\frac{1}{s}\right)^{\frac{3}{2}}} ds \\ &\leq F_\epsilon(s_\epsilon)\sqrt{\ln\left(\frac{1}{s_\epsilon}\right)} + \frac{2K_3(\rho)}{\sqrt{\ln\left(\frac{1}{s_\epsilon}\right)}}. \end{aligned}$$

If $s_\epsilon < \delta$, we deduce from this inequality and the definition of s_ϵ that

$$F_\epsilon(r)\sqrt{\ln\left(\frac{1}{r}\right)} \leq \frac{\pi K_2(\rho) + 2K_3(\rho)}{\sqrt{\ln\left(\frac{1}{\delta}\right)}},$$

and if $s_\epsilon = \delta$,

$$F_\epsilon(r)\sqrt{\ln\left(\frac{1}{r}\right)} \leq \sigma_\epsilon\sqrt{\ln\left(\frac{1}{\delta}\right)} + \frac{2K_3(\rho)}{\sqrt{\ln\left(\frac{1}{\delta}\right)}},$$

where we used conformal invariance of the L^2-norm of the gradient to get $F_\epsilon(\delta) \leq \sigma_\epsilon$.

Gathering all the cases, we get (3.8) and this ends the proof of the claim. q.e.d.

In the following claim, we aim at passing to the limit in equation (i) and the condition (ii) given by Proposition 1. The limiting functions would then satisfy (3.15) and (3.16).

Claim 5. *We assume that* $m_0(\rho) = \lim_{\epsilon\to 0}\int_{I(\rho)} e^{2u_\epsilon} dv_g > 0$ *for any* $\rho > 0$ *small enough. Then, the following assertions hold*

- *For any $\rho > 0$, there exists $\beta_\epsilon \to 0$ as $\epsilon \to 0$ such that*

$$\forall x \in I(\rho), \left|\Phi_\epsilon\right|^2(x) \geq 1 - \beta_\epsilon. \tag{3.12}$$

- *For $\rho > 0$ and $x \in I(\rho)$, we set $\Psi_\epsilon(x) = \frac{\Phi_\epsilon(x)}{|\Phi_\epsilon(x)|}$. Then for any $\rho > 0$, $\{\Psi_\epsilon\}$ is uniformly equicontinuous on $\mathcal{C}^0(I(\rho), \mathbb{S}^n)$.*
- *For any $\rho > 0$, up to the extraction of a subsequence of $\{\Phi_\epsilon\}$, there exist functions $\Phi \in W^{1,2}(M(\rho), \mathbb{R}^{n+1}) \cap L^\infty(I(\rho), \mathbb{R}^{n+1})$ and $\Psi \in W^{\frac{1}{2},2}(I(\rho), \mathbb{S}^n) \cap \mathcal{C}^0(I(\rho), \mathbb{S}^n)$ such that*

$$\Phi_\epsilon \rightharpoonup \Phi \text{ in } W^{1,2}(M(\rho), \mathbb{R}^{n+1}) \text{ as } \epsilon \to 0, \tag{3.13}$$

and

$$\Psi_\epsilon \to \Psi \text{ in } \mathcal{C}^0(I(\rho), \mathbb{S}^n) \text{ as } \epsilon \to 0, \tag{3.14}$$

with

$$|\Phi|^2 \geq_{a.e.} 1 \text{ and } \Psi = \frac{\Phi}{|\Phi|} \text{ on } I(\rho). \tag{3.15}$$

Moreover, for $0 \leq i \leq n$,

$$\begin{cases} \Delta_g \phi^i = 0 & \text{in } M(\rho), \\ \partial_\nu \phi^i = \sigma_k(M, [g]) \psi^i d\nu & \text{on } I(\rho), \end{cases} \tag{3.16}$$

in a weak sense.

Proof. Step 1 – Let $1 \leq i \leq s$. We prove that at the neighborhood of the singular points defined in Claim 3,

$$\sup_{x \in I(\rho)} \int_{I_g(p_i, \frac{\rho}{10})} \left|\Phi_\epsilon(y)\right|^2 p_\epsilon(x, y) d\sigma_g(y) = O(e^{-\frac{\rho^2}{8\epsilon}}).$$

Let $x \in I(\rho)$. Then, by estimate (1.3) of Section 1.2

$$\begin{aligned} e^{\frac{\rho^2}{8\epsilon}} \int_{I_g(p_i, \frac{\rho}{10})} \left|\Phi_\epsilon(y)\right|^2 p_\epsilon(x, y) d\sigma_g(y) &\leq \frac{A_0 e^{-\frac{31\rho^2}{400\epsilon}}}{\sqrt{4\pi\epsilon}} \frac{\int_{I_g(p_i, \frac{\rho}{10})} \left|\Phi_\epsilon\right|^2 e^{u_\epsilon} d\sigma_g}{\inf_{I_g(p_i, \frac{\rho}{10})} e^{u_\epsilon}} \\ &\leq \frac{A_0 e^{-\frac{31\rho^2}{400\epsilon}}}{\sqrt{4\pi\epsilon} \inf_{I_g(p_i, \frac{\rho}{10})}}, \end{aligned}$$

since by (iii) of Proposition 1,

$$\int_{\partial M} \left|\Phi_\epsilon\right|^2 e^{u_\epsilon} d\sigma_g = 1.$$

We assume by contradiction that

$$\inf_{I_g(p_i, \frac{\rho}{10})} e^{u_\epsilon} \leq \frac{e^{-\frac{31\rho^2}{400\epsilon}}}{\sqrt{\epsilon}}.$$

Let $y \in \overline{I_g(p_i, \frac{\rho}{10})}$ be such that $e^{u_\epsilon(y)} = \inf_{I_g(p_i,\frac{\rho}{10})} e^{u_\epsilon}$. Then, by (1.3) of Section 1.2,

$$e^{u_\epsilon(y)} = \int_{\partial M} p_\epsilon(y,x) d\nu_\epsilon(x) \geq \frac{e^{-\left(\frac{2\rho}{10}\right)^2 \frac{1}{4\epsilon}}}{A_0 \sqrt{4\pi\epsilon}} \int_{I_g(p_i,\frac{\rho}{10})} d\nu_\epsilon.$$

We deduce from this and the previous inequality that

$$\int_{I_g(p_i,\frac{\rho}{10})} d\nu_\epsilon \leq A_0 \sqrt{4\pi} e^{-\frac{27\rho^2}{400\epsilon}}.$$

Let $z \in I_g(p_i, \frac{\rho}{20})$, and let us write thanks again to (1.3) of Section 1.2 that

$$e^{u_\epsilon(z)} \leq A_0 \frac{\int_{I_g(p_i,\frac{\rho}{10})} d\nu_\epsilon + e^{-\frac{\rho^2}{4\epsilon}\frac{1}{20^2}}}{\sqrt{4\pi\epsilon}} \leq \frac{A_0^2}{\sqrt{\epsilon}} e^{-\frac{27\rho^2}{400\epsilon}} + \frac{A_0}{\sqrt{4\pi\epsilon}} e^{-\frac{\rho^2}{1600\epsilon}}.$$

Then, $\|e^{u_\epsilon}\|_{\mathcal{C}^0(I_g(p_i,\frac{\rho}{20}))} \to 0$ as $\epsilon \to 0$. This implies that

$$\sigma_\star\left(B_g(p_i, \frac{\rho}{20}), g, I_g(p_i, \frac{\rho}{20}), e^{u_\epsilon}\right) \to +\infty \text{ as } \epsilon \to 0. \tag{3.17}$$

It is clear that $\mathbf{A}_{\frac{\rho}{20},\epsilon}$ defined before Claim 3 cannot be true for p_i and ϵ small enough. By (3.1) in Claim 3, $\mathbf{B}_{\frac{\rho}{20},\epsilon}$ holds true for p_i. Then, there is an eigenfunction f associated to $\sigma_\epsilon = \sigma_k(M, g, \partial M, e^{u_\epsilon})$ such that $f_\epsilon(p_i) = 0$ and the nodal set which contains p_i does not intersect $\partial B_g(p_i, \frac{\rho}{20}) \setminus \partial M$. We obtain a nodal domain $D_\epsilon \subset B_g(p_i, \frac{\rho}{10})$ for f_ϵ such that $p_i \in D_\epsilon \cap \partial M$. By 3.17,

$$\sigma_\epsilon = \sigma_\star(D_\epsilon, g, D_\epsilon \cap \partial M, e^{u_\epsilon}) \geq \sigma_\star\left(B_g(p_i, \frac{\rho}{20}), g, I_g(p_i, \frac{\rho}{20}), e^{u_\epsilon}\right) \to +\infty$$

as $\epsilon \to 0$. Since $\sigma_\epsilon \leq \sigma_k(M, [g])$, we get a contradiction. This completes the proof of Step 1.

STEP 2 – There exists $\beta_\epsilon \to 0$ as $\epsilon \to 0$ such that

$$\forall x, y \in I(\rho), \quad d_g(x,y) \leq \frac{\sqrt{\epsilon}}{\beta_\epsilon} \Rightarrow |\Phi_\epsilon(x) - \Phi_\epsilon(y)| \leq \beta_\epsilon. \tag{3.18}$$

We set $\gamma_\epsilon = \|\sqrt{\epsilon} e^{u_\epsilon}\|^{\frac{1}{2}}_{L^\infty(I(\rho))}$. We have $\gamma_\epsilon \to 0$ as $\epsilon \to 0$. Indeed, for $r > 0$, and $x \in I(\rho)$ such that $\gamma_\epsilon^2 = \sqrt{\epsilon} e^{u_\epsilon(x)}$,

$$\begin{aligned} \sqrt{\epsilon} e^{u_\epsilon(x)} &\leq \frac{A_0}{\sqrt{4\pi}} \int_{I_g(x,r)} d\nu_\epsilon + o(1) \\ &= \frac{A_0}{\sqrt{4\pi}} \nu(I_g(x,r)) + o(1) \\ &\leq \frac{A_0 D_1(\rho)}{\sqrt{4\pi} \ln\left(\frac{1}{r}\right)} + o(1). \end{aligned}$$

By estimate (1.3), since $\nu_\epsilon \rightharpoonup_\star \nu$ as $\epsilon \to 0$ and by (3.7) of Claim 4. Letting $\epsilon \to 0$ and then $r \to 0$, we get $\gamma_\epsilon \to 0$ as $\epsilon \to 0$. We also have that $\frac{\gamma_\epsilon}{\sqrt{\epsilon}} \to +\infty$ as $\epsilon \to 0$, since $\gamma_\epsilon \geq \frac{m_0(\rho)}{2}^{\frac{1}{2}} \epsilon^{\frac{1}{4}}$ (indeed, $m_0(\rho) + o(1) = \|e^{u_\epsilon}\|_{L^1(I(\rho))} \leq \|e^{u_\epsilon}\|_{L^\infty(I(\rho))}$). Let now $x_\epsilon, y_\epsilon \in I(\rho)$ with $d_g(x_\epsilon, y_\epsilon) \leq \frac{\sqrt{\epsilon}}{\gamma_\epsilon}$. Up to the extraction of a subsequence, $x_\epsilon \in \gamma_l$ for some l fixed and we set

$$\begin{cases} \hat{\Phi}_\epsilon(x) = \tilde{\Phi}^l(\tilde{x}_\epsilon^l + \frac{\sqrt{\epsilon}}{\gamma_\epsilon} x), \\ e^{\hat{u}_\epsilon(x)} = \frac{\sqrt{\epsilon}}{\gamma_\epsilon} e^{\tilde{u}_\epsilon(\tilde{x}_\epsilon + \frac{\sqrt{\epsilon}}{\gamma_\epsilon} x)}, \end{cases}$$

which satisfy

$$\begin{cases} \Delta_\xi \hat{\Phi}_\epsilon = 0 & \text{in } \mathbb{D}^+_{3C_0} \\ \partial_t \hat{\Phi}_\epsilon = -\sigma_\epsilon e^{\hat{u}_\epsilon} \hat{\Phi}_\epsilon & \text{on } I_{3C_0} \end{cases} \tag{3.19}$$

Let α_ϵ be the mean value of $\hat{\Phi}_\epsilon$ in $\mathbb{D}^+_{3C_0}$. Then

$$\begin{aligned} \left\| \hat{\Phi}_\epsilon - \alpha_\epsilon \right\|_{L^\infty(I_{2C_0}(0))} &\leq D_0 \left\| \hat{\Phi}_\epsilon - \alpha_\epsilon \right\|_{H^1(I_{2C_0})} \\ &\leq D \left\| \partial_t \hat{\Phi}_\epsilon \right\|_{L^2(I_{3C_0})(\rho)} + D \left\| \hat{\Phi}_\epsilon - \alpha_\epsilon \right\|_{L^2(\mathbb{D}^+_{3C_0}(0))} \\ &\leq D\sigma_\epsilon \left\| \Phi_\epsilon \right\|_{L^\infty} C_0 \gamma_\epsilon + D' \left\| \nabla \hat{\Phi}_\epsilon \right\|_{L^2(\mathbb{D}^+_{3C_0}(0))} \\ &\leq D\sigma_\epsilon C_2(\rho) C_0 \gamma_\epsilon + \frac{D' \sqrt{D_2(\rho)}}{\ln\left(\frac{\gamma_\epsilon}{3C_0^2 \sqrt{\epsilon}} \right)^{\frac{1}{4}}}. \end{aligned}$$

The first inequality comes from Sobolev embeddings, the second comes from the regularity theory for the Dirichlet-to-Neumann operator (see [**25**], Chapter 7.11, page 37) looking at (3.19). The third inequality comes from the classical Poincaré inequality on $\mathbb{D}^+_{3C_0}$, and, finally, we use (3.6) and (3.8) in Claim 4. Setting

$$\beta_\epsilon = 2D\sigma_k(M, [g]) C_2(\rho) C_0 \gamma_\epsilon + \frac{2D' \sqrt{D_2(\rho)}}{\ln\left(\frac{\gamma_\epsilon}{3C_0^2 \sqrt{\epsilon}} \right)^{\frac{1}{4}}},$$

we have that $\beta_\epsilon \to 0$ as $\epsilon \to 0$ and that

$$|\Phi_\epsilon(x_\epsilon) - \Phi_\epsilon(y_\epsilon)| \leq \beta_\epsilon.$$

Up to increasing β_ϵ so that $\frac{\sqrt{\epsilon}}{\beta_\epsilon} \leq \frac{\sqrt{\epsilon}}{\gamma_\epsilon}$ we proved Step 2.

STEP 3 – For any $\rho > 0$, there exists $\beta_\epsilon \to 0$ as $\epsilon \to 0$ such that

$$\forall x \in I(\rho), \left| |\Phi_\epsilon|^2 (x) - K_\epsilon[|\Phi_\epsilon|^2](x) \right| \leq \beta_\epsilon, \tag{3.20}$$

and

$$\forall x \in I(\rho) \cap supp(\nu_\epsilon), |K_\epsilon[|\Phi_\epsilon|](x) - 1| \leq \beta_\epsilon. \tag{3.21}$$

Note that (3.20) implies (3.12) by Proposition 1. Let's prove (3.20). For $x \in I(\rho)$,

$$\begin{aligned}\left| |\Phi_\epsilon|^2 - K_\epsilon[|\Phi_\epsilon|^2] \right|(x) &\leq \int_{I_g(x,\frac{\epsilon}{\beta_\epsilon})} \left| |\Phi_\epsilon|^2(x) - |\Phi_\epsilon|^2(y) \right| p_\epsilon(x,y) d\sigma_g(y) \\ &\quad + 2C_2(\frac{\rho}{10})^2 \int_{\partial M \setminus I_g(x,\frac{\sqrt{\epsilon}}{\beta_\epsilon})} p_\epsilon(x,y) d\sigma_g \\ &\quad + \sum_{i=1}^{s} \int_{I_g(p_i,\frac{\rho}{10})} |\Phi_\epsilon|^2(y) p_\epsilon(x,y) d\sigma_g(y).\end{aligned}$$

Notice that we can assume here that $\frac{\sqrt{\epsilon}}{\beta_\epsilon} \to 0$ up to increasing β_ϵ and that we used (3.6). We can estimate the first RHS term thanks to Step 2 and (3.6), the second RHS term thanks to estimates (1.3) and the third RHS term thanks to Step 1 and we get

$$\left| |\Phi_\epsilon|^2 - K_\epsilon[|\Phi_\epsilon|^2] \right|(x) \leq 2C_2(\frac{\rho}{2})\beta_\epsilon + O(e^{-\frac{1}{4C_0^4 \beta_\epsilon^2}}) + O(e^{-\frac{\rho^2}{8\epsilon}}).$$

Up to increase β_ϵ, we get (3.20) and then (3.12).

Thanks to Point (iii) in Proposition (1), we deduce that

$$\forall x \in supp(\nu_\epsilon) \cap I(\rho), ||\Phi_\epsilon(x)| - 1| \leq \beta_\epsilon, \tag{3.22}$$

and for $x \in I(\rho)$, we have

$$\begin{aligned}||\Phi_\epsilon| - K_\epsilon[|\Phi_\epsilon|]|(x) &\leq \int_{I_g(x,\frac{\epsilon}{\beta_\epsilon})} ||\Phi_\epsilon|(x) - |\Phi_\epsilon|(y)| p_\epsilon(x,y) d\sigma_g(y) \\ &\quad + 2C_2(\frac{\rho}{10}) \int_{\partial M \setminus I_g(x,\frac{\sqrt{\epsilon}}{\beta_\epsilon})} p_\epsilon(x,y) d\sigma_g \\ &\quad + \sum_{i=1}^{s} \left(\int_{I_g(p_i,\frac{\rho}{10})} |\Phi_\epsilon|^2(y) p_\epsilon(x,y) d\sigma_g(y) \right)^{\frac{1}{2}},\end{aligned}$$

and the same arguments, together with (3.22) lead to (3.21), up to increase again β_ϵ.

Step 4 – Let $\Psi_\epsilon = \frac{\Phi_\epsilon}{|\Phi_\epsilon|}$ on $I(\rho)$. Then, for $\rho > 0$, there exists $C_3(\rho)$ such that

$$|\Psi_\epsilon(x) - \Psi_\epsilon(y)|^2 \sqrt{\ln\left(\frac{2\delta(\partial M)}{d_g(x,y)}\right)} \leq C_3(\rho)$$

for all $x, y \in I(\rho)$, where $\delta(\partial M)$ is the diameter of ∂M. In particular, Ψ_ϵ is uniformly equicontinuous on $I(\rho)$.

We first prove that there exists $D_3(\rho) > 0$ such that

$$\sup_{x \in I(\rho)} \sup_{v \in \Psi_\epsilon^\perp \cap \mathbb{S}^n} \frac{1}{Vol_g(B_g(x,r))} \int_{B_g(x,r)} (\Phi_\epsilon . v)^2 dv_g \leq \frac{D_3(\rho)}{\sqrt{\ln\left(\frac{1}{r}\right)}} \tag{3.23}$$

for all r small enough. Indeed, for $x \in I(\rho)$ and $v \in \Psi_\epsilon(x)^\perp \cap \mathbb{S}^n$, $\Phi_\epsilon.v$ vanishes at x. By Claim 3, x does not satisfy $\mathbf{B}_{r,\epsilon}$. Thus, the nodal set which contains x intersects $\partial B_g(x,r)$. By the corollary of Theorem 4 on a disk and a dilatation on this disk, we get some constant $D_4(\rho)$ such that

$$\frac{1}{Vol_g(B_g(x,r))}\int_{B_g(x,r)} (\Phi_\epsilon.v)^2\, dv_g \leq D_4(\rho)\int_{B_g(x,2r)} |\nabla(\Phi_\epsilon.v)|_g^2\, dv_g$$

for all r small enough. With (3.8) in Claim 4, we deduce that

$$\frac{1}{Vol_g(B_g(x,r))}\int_{B_g(x,r)} (\Phi_\epsilon.v)^2\, dv_g \leq \frac{D_2(\rho)D_4(\rho)}{\sqrt{\ln\left(\frac{1}{2r}\right)}}$$

for all r small enough. Thus, (3.23) is proved.

Assume now by contradiction that the conclusion of Step 4 is false: there exist $\epsilon_m \to 0$ as $m \to +\infty$, x_m and y_m some points in $I(\rho)$ such that

$$|\Psi_{\epsilon_m}(x_m) - \Psi_{\epsilon_m}(y_m)|^2\sqrt{\ln\left(\frac{1}{r_m}\right)}, \to +\infty \text{ as } m \to +\infty \tag{3.24}$$

where $r_m = d_g(x_m,y_m) \to 0$ as $m \to +\infty$. Since for a fixed m, Ψ_{ϵ_m} is not constant at the neighborhood of y_m, one can assume that for any m, $\Psi_{\epsilon_m}(y_m) \neq -\Psi_{\epsilon_m}(x_m)$ without changing (3.24). Thanks to (3.12), up to the extraction of a subsequence, there exists a fixed vector $v \in \mathbb{S}^n$ of the canonical basis of $\mathbb{R}^{n+1}$ such that

$$\frac{1}{L_g(I_g(x_m,r_m))}\int_{I_g(x_m,r_m)} (\Phi_{\epsilon_m}.v)^2\, d\sigma_g \geq \frac{1}{n+1} + o(1).$$

Since, by Sobolev trace inequalities, there exists $K > 0$ independent of m such that

$$\begin{aligned}\frac{1}{L}\int_{I_g(x_m,r_m)} (\Phi_{\epsilon_m}.v)^2\, d\sigma_g \quad &\leq \quad \frac{K}{Vol_g(B_g(x,r))}\int_{B_g(x,r)} (\Phi_{\epsilon_m}.v)^2\, dv_g \\ &\quad + K\int_{B_g(x_m,r_m)} |\nabla(\Phi_{\epsilon_m}.v)|_g^2\, dv_g,\end{aligned}$$

where $L = L_g(I_g(x_m,r_m))$. We get thanks to (3.8) of Claim 4 that

$$\begin{aligned}\frac{1}{Vol_g(B_g(x,r))}\int_{B_g(x,r)} (\Phi_{\epsilon_m}.v)^2\, dv_g \quad &\geq \quad \frac{1}{(n+1)K} - \frac{D_2(\rho)}{\sqrt{\ln\left(\frac{1}{r_m}\right)}} + o(1) \\ &= \quad \frac{1}{K(n+1)} + o(1).\end{aligned}$$

Thanks to the assumption (3.24), we now prove that there exist $X_m \in \Psi_{\epsilon_n}(x_m)^{\perp}$ and $Y_m \in \Psi_{\epsilon_m}(y_m)^{\perp}$ such that

$$v = X_m + Y_m \text{ and } |X_m|^2 + |Y_m|^2 = o\left(\sqrt{\ln \frac{1}{r_m}}\right). \tag{3.25}$$

We denote $a_m = \Psi_{\epsilon_m}(x_m) \in \mathbb{S}^{k-1}$, $b_m = \Psi_{\epsilon_m}(y_m) \in \mathbb{S}^{k-1}$ and Π_m the vector space generated by a_m and b_m. Notice that Π_m is a plane since $b_m \notin \{a_m, -a_m\}$ by assumption. Let $c_m \in \Pi_m \cap \mathbb{S}^{k-1}$ such that $\{a_m, c_m\}$ is an orthonormal basis of Π_m. We get $\theta_m \in \mathbb{R}$ such that

$$b_m = \cos\theta_m a_m + \sin\theta_m c_m,$$

and $\sin\theta_m \neq 0$. We let $v = p_m + q_m$ with $p_m \in \Pi_m$ and $q_m \in \Pi_m^{\perp}$. Notice that $|p_m| \leq 1$ and $|q_m| \leq 1$. Let $\alpha_m \in \mathbb{R}$ be such that

$$p_m = |p_m| (\cos\alpha_m a_m + \sin\alpha_m c_m).$$

We then set

$$X_m = t_m c_m + q_m \in a_m^{\perp},$$
$$Y_m = s_m(-\sin\theta_m a_m + \cos\theta_m c_m) \in b_m^{\perp},$$

with

$$s_m = -|p_m| \frac{\cos\alpha_m}{\sin\theta_m},$$
$$t_m = |p_m| \left(\sin\alpha_m + \frac{\cos\alpha_m \cos\theta_m}{\sin\theta_m}\right),$$

so that $v = X_m + Y_m$. Then,

$$|X_m|^2 + |Y_m|^2 = |q_m|^2 + t_m^2 + s_m^2 \leq 1 + f_{\theta_m}(\alpha_m),$$

where for α and $\theta \in \mathbb{R}$,

$$\begin{aligned} f_\theta(\alpha) &= \frac{\cos^2\alpha}{\sin^2\theta} + \left(\sin\alpha + \frac{\cos\alpha\cos\theta}{\sin\theta}\right)^2 \\ &= \frac{1 + \cos^2\theta\cos 2\alpha + \cos\theta\sin\theta\sin 2\alpha}{\sin^2\theta}. \end{aligned}$$

We easily prove that $f_\theta(\alpha) \leq f_\theta(\frac{\theta}{2}) = \frac{1}{1-\cos\theta}$. Then,

$$|X_m|^2 + |Y_m|^2 \leq O\left(\frac{1}{1-\cos\theta_m}\right) = O\left(\frac{1}{|a_m - b_m|^2}\right) = o\left(\sqrt{\ln\frac{1}{r_m}}\right).$$

This ends the proof of (3.25).

We now write thanks to (3.23) that

$$\begin{aligned} \frac{1}{(n+1)K} + o(1) &\leq \frac{1}{Vol_g(B_g(x,r))} \int_{B_g(x,r)} (\Phi_{\epsilon_m}.v)^2 \, dv_g \\ &\leq \frac{2}{Vol_g(B_g(x,r))} \int_{B_g(x,r)} (\Phi_{\epsilon_m}.X_m)^2 \, dv_g \end{aligned}$$

$$
\begin{aligned}
& +\frac{2}{Vol_g(B_g(x,r))}\int_{B_g(y_m,2r_m)}(\Phi_{\epsilon_m}.Y_m)^2\,dv_g \\
\leq\ & 2D_3(\rho)\,|X_m|^2\left(\ln\left(\frac{1}{r_m}\right)\right)^{-\frac{1}{2}} \\
& +8C_0^2D_3(\rho)\,|Y_m|^2\left(\ln\left(\frac{1}{2r_m}\right)\right)^{-\frac{1}{2}} \\
=\ & o(1).
\end{aligned}
$$

This clearly gives a contradiction and proves Step 4.

It is clear now that there exist some functions Φ and Ψ such that up to the extraction of a subsequence, (3.13), (3.14) and (3.15) hold. It remains to prove Step 5:

STEP 5 – We have that

$$\phi_\epsilon^i e^{u_\epsilon} d\sigma_g \rightharpoonup_\star \psi^i d\nu \text{ as } \epsilon\to 0 \text{ in } I(\rho).$$

Let $\zeta\in\mathcal{C}_c^0(I(\rho))$. Then

$$
\begin{aligned}
\int_{\partial M}\zeta\phi_\epsilon^i e^{2u_\epsilon}d\sigma_g-\int_{\partial M}\zeta\psi^i d\nu &= \int_{\partial M}\left(K_\epsilon[\zeta\phi_\epsilon^i]-\zeta K_\epsilon[\phi_\epsilon^i]\right)d\nu_\epsilon \\
&\quad+\int_{\partial M}\zeta\left(K_\epsilon[\phi_\epsilon^i]-\psi_\epsilon^i K_\epsilon[|\Phi_\epsilon|]\right)d\nu_\epsilon \\
&\quad+\int_{\partial M}\zeta\left(\psi_\epsilon^i K_\epsilon[|\Phi_\epsilon|]-\psi_\epsilon^i\right)d\nu_\epsilon \\
&\quad+\int_{\partial M}\zeta\left(\psi_\epsilon^i d\nu_\epsilon-\psi^i d\nu\right). \qquad (3.26)
\end{aligned}
$$

Let us estimate these four terms. We have for $x\in\partial M$ that

$$
\begin{aligned}
\left|K_\epsilon[\zeta\phi_\epsilon^i]-\zeta K_\epsilon[\phi_\epsilon^i]\right|(x) &= \left|\int_{\partial M}(\zeta(y)-\zeta(x))\,\phi_\epsilon^i(y)p_\epsilon(x,y)d\sigma_g(y)\right| \\
&\leq C_2\left(\frac{\rho}{10}\right)\int_{I(\frac{\rho}{10})}|\zeta(y)-\zeta(x)|\,p_\epsilon(x,y)d\sigma_g(y) \\
&\quad+|\zeta(x)|\sum_{j=1}^{s}\int_{I_g\left(p_j,\frac{\rho}{10}\right)}\left|\phi_\epsilon^i(y)\right|p_\epsilon(x,y)d\sigma_g(y),
\end{aligned}
$$

since $supp(\zeta)\subset I(\rho)$ and thanks to (3.6) of Claim 4. By Step 1 and since $supp(\zeta)\subset I(\rho)$, we deduce that this function uniformly converges to 0 in ∂M as $\epsilon\to 0$. Thus, the first RHS term in (3.26) converges to 0 as $\epsilon\to 0$. For $x\in I(\rho)$,

$$
\begin{aligned}
\left|K_\epsilon[\phi_\epsilon^i]-\psi_\epsilon^i K_\epsilon[|\Phi_\epsilon|]\right| &\leq \int_{\partial M}\left|\phi_\epsilon^i(y)-\psi_\epsilon^i(x)\,|\phi_\epsilon|\,(y)\right|p_\epsilon(x,y)d\sigma_g(y) \\
&\leq \int_{I(\frac{\rho}{10})}|\Phi_\epsilon(y)|\left|\psi_\epsilon^i(y)-\psi_\epsilon^i(x)\right|p_\epsilon(x,y)d\sigma_g(y)
\end{aligned}
$$

$$
\begin{aligned}
& +2\sum_{j=1}^{s}\int_{I_g(p_j,\frac{\rho}{10})}\left|\Phi_\epsilon(y)\right| p_\epsilon(x,y)d\sigma_g(y)\\
\leq\ & C_2\left(\frac{\rho}{10}\right)\int_{I(\frac{\rho}{10})}\left|\psi_\epsilon^i(y)-\psi_\epsilon^i(x)\right| p_\epsilon(x,y)d\sigma_g(y)\\
& +O(e^{-\frac{\rho^2}{16\epsilon}}),
\end{aligned}
$$

thanks to (3.6) of Claim 4 and Step 1. Thanks to the uniform equicontinuity of $\{\Psi_\epsilon\}$ on $I(\frac{\rho}{10})$, it uniformly converges to zero in ∂M as $\epsilon \to 0$. Thus, the second RHS term of (3.26) converges to 0 as $\epsilon \to 0$. Thanks to (3.21), we can write since $|\Psi_\epsilon| = 1$ that

$$\left|\int_{\partial M}\zeta\left(\psi_\epsilon^i K_\epsilon[|\Phi_\epsilon|]-\psi_\epsilon^i\right)d\nu_\epsilon\right| \leq \beta_\epsilon \left\|\zeta\right\|_\infty,$$

so that the third RHS term in (3.26) converges to 0 as $\epsilon \to 0$. At last, we use the convergences $\Psi_\epsilon \to \Psi$ in $\mathcal{C}^0(I(\rho))$ and $\nu_\epsilon \rightharpoonup_\star \nu$ on $I(\rho)$ to obtain that the fourth RHS term in (3.26) also converges to 0 as $\epsilon \to 0$. This clearly ends the proof of Step 5.

Finally, passing to the weak limit in $I(\rho)$ for $\rho > 0$, in the equation satisfied by ϕ_ϵ^i permits to end the proof of the claim thanks to these steps. q.e.d.

Thanks to Claim 5, with the assumption $m_0(\rho) = \lim_{\epsilon\to 0}\int_{I(\rho)} e^{u_\epsilon}dv_g > 0$, a diagonal extraction gives some functions $\Phi : M\setminus\{p_1,\cdots,p_s\} \to \mathbb{R}^{n+1}$ and $\Psi : \partial M\setminus\{p_1,\cdots,p_s\} \to \mathbb{S}^n$ such that for all $\rho > 0$ the conclusions (3.13), (3.14), (3.15) and (3.16) hold true for Φ and Ψ.

3.2. Energy estimates. Now, we give some energy estimates which will be useful later in the proof. We set a function ω on M satisfying the following equation

$$
\left\{\begin{array}{ll}\Delta_g\omega = 0 & \text{in } M,\\ \omega = |\Phi| & \text{on } \partial M,\end{array}\right. \tag{3.27}
$$

in a weak sense. Since $|\Phi| \in W^{\frac{1}{2},2}(\partial M)$, such a solution exists and satisfies $\omega \in W^{1,2}(M)$ (see [**11**], Theorem 8.3). Let's prove this energy inequality:

Claim 6.
(3.28)

$$\lim_{\rho\to 0}\lim_{\epsilon\to 0}\int_{M(\rho)}\left|\nabla\Phi_\epsilon\right|_g^2 dv_g \geq \int_M\frac{\left|\nabla\Phi\right|_g^2}{\omega}dv_g \geq \sigma_k m + \int_M\frac{\left|\Phi\right|^2\left|\nabla\omega\right|_g^2}{\omega^3}dv_g,$$

where $\sigma_k = \sigma_k\left(M,[g]\right)$, $m = \lim_{\rho\to 0} m_0(\rho) = \lim_{\rho\to 0}\int_{I(\rho)} d\nu$.

Proof. Let $\rho > 0$. By Claim 1, there exists $C > 0$ independent of ρ and a nonnegative function $\eta \in \mathcal{C}^\infty(M)$ such that $supp(\eta) \subset M(\rho)$,

$\eta = 1$ on $M(\sqrt{\rho})$, $0 \leq \eta \leq 1$, and

$$\int_M |\nabla \eta|_g^2 \, dv_g \leq \frac{C}{\ln\left(\frac{1}{\rho}\right)}.$$

By the weak maximum principle on (3.27), (see [11], Theorem 8.1),

$$\inf_M \omega \geq \inf_{\partial M} |\Phi| \geq 1,$$

and

$$\begin{aligned}
\lim_{\epsilon \to 0} \int_{M(\rho)} |\nabla \Phi_\epsilon|_g^2 \, dv_g &\geq \int_{M(\rho)} |\nabla \Phi|_g^2 \, dv_g \\
&\geq \int_M \eta \frac{|\nabla \Phi|_g^2}{\omega} dv_g \\
&= \sum_{i=0}^n \int_M \left\langle \nabla \frac{\eta \phi_i}{\omega}, \nabla \phi_i \right\rangle_g dv_g \\
&\quad - \sum_{i=0}^n \int_M \frac{\phi_i}{\omega} \left\langle \nabla \eta, \nabla \phi_i \right\rangle_g dv_g \\
&\quad - \sum_{i=0}^n \int_M \phi_i \eta \left\langle \nabla \frac{1}{\omega}, \nabla \phi_i \right\rangle_g dv_g.
\end{aligned}$$

We have that

$$\begin{aligned}
\sum_{i=0}^n \int_M \left\langle \nabla \frac{\eta \phi_i}{\omega}, \nabla \phi_i \right\rangle_g dv_g &= \sum_{i=0}^n \int_M \frac{\eta \phi_i}{\omega} \Delta_g \phi_i dv_g \\
&\quad + \sum_{i=0}^n \int_{\partial M} \frac{\eta \phi_i}{\omega} \partial_\nu \phi_i d\sigma_g \\
&= \sigma_k(M, [g]) \int_{\partial M} \eta \frac{|\Phi|}{\omega} d\nu \\
&= \sigma_k(M, [g]) \int_{\partial M} \eta d\nu,
\end{aligned}$$

thanks to (3.16) and that

$$\begin{aligned}
\sum_{i=0}^n \int_M \eta \phi_i \left\langle \nabla \frac{1}{\omega}, \nabla \phi_i \right\rangle_g dv_g &= -\int_M \left\langle \nabla \eta, \nabla \frac{1}{\omega} \right\rangle_g \frac{|\Phi|^2}{2} dv_g \\
&\quad + \int_M \eta \frac{|\Phi|^2}{2} \Delta_g \left(\frac{1}{\omega}\right) dv_g \\
&\quad + \int_{\partial M} \frac{|\Phi|^2}{2} \eta \partial_\nu \left(\frac{1}{\omega}\right) d\sigma_g \\
&= \int_M \left\langle \nabla \eta, \nabla \omega \right\rangle_g \frac{|\Phi|^2}{2\omega^2} dv_g
\end{aligned}$$

$$
\begin{aligned}
& -\int_M \eta \frac{|\Phi|^2}{\omega^3} |\nabla \omega|_g^2 dv_g \\
& -\frac{1}{2}\int_M |\Phi|^2 \eta \frac{\Delta_g \omega}{\omega^2} dv_g \\
& -\frac{1}{2}\int_{\partial M} \eta \partial_\nu \omega d\sigma_g \\
= & \int_M \langle \nabla \eta, \nabla \omega \rangle_g \frac{|\Phi|^2}{2\omega^2} dv_g \\
& -\int_M \eta \frac{|\Phi|^2}{\omega^3} |\nabla \omega|_g^2 dv_g \\
& +\frac{1}{2}\int_M \eta \Delta_g \omega dv_g \\
& -\frac{1}{2}\int_M \langle \nabla \eta, \nabla \omega \rangle_g dv_g,
\end{aligned}
$$

so that

$$
\begin{aligned}
\lim_{\epsilon \to 0} \int_{M(\rho)} |\nabla \Phi_\epsilon|_g^2 dv_g \geq & \int_{M(\rho)} |\nabla \Phi|_g^2 dv_g \\
\geq & \sigma_k(M,[g]) \int_{\partial M} \eta d\nu + \int_M \eta \frac{|\Phi|^2}{\omega^3} |\nabla \omega|_g^2 dv_g \\
& -\sum_{i=0}^n \int_M \frac{\phi_i}{\omega} \langle \nabla \eta, \nabla \phi_i \rangle_g dv_g \\
& -\int_M \langle \nabla \eta, \nabla \omega \rangle_g \frac{|\Phi|^2}{2\omega^2} dv_g \\
& +\frac{1}{2}\int_M \langle \nabla \eta, \nabla \omega \rangle_g dv_g \\
\geq & \sigma_k(M,[g]) \int_{\partial M} \eta d\nu \\
& + \int_M \eta \frac{|\Phi|^2}{\omega^3} |\nabla \omega|_g^2 dv_g - \frac{C'}{\sqrt{\ln\left(\frac{1}{\rho}\right)}},
\end{aligned}
$$

where C' is a constant independent of ρ. Indeed, $\phi_i, \omega \in W^{1,2}(M)$ and we have for $0 \leq i \leq n$ that

$$
\Delta_g (\omega - \phi_i) = 0 \text{ and } \Delta_g (\omega + \phi_i) = 0
$$

in a weak sense. By the weak maximum principle (see [**11**], Theorem 8.1),

$$
\inf_M (\omega - \phi_i) \geq \inf_{\partial M} (\omega - \phi_i) \geq 0,
$$

and

$$\inf_{M} (\omega + \phi_i) \geq \inf_{\partial M} (\omega + \phi_i) \geq 0,$$

since $|\phi_i| \leq |\Phi| \leq \omega$ on ∂M. Then,

$$\sup_{M} \frac{|\phi_i|}{\omega} \leq 1 \text{ and } \sup_{M} \frac{|\Phi|^2}{\omega^2} \leq n+1.$$

We, finally, get the claim, passing to the limit as $\rho \to 0$. q.e.d.

4. Scales of concentration for the maximizing sequence

4.1. Concentration, capacity and rescalings. In this section, we aim at describing all the concentration scales of the sequence $\{e^{u_\epsilon} d\sigma_g\}$. We denote by $Z(M, \{e^{u_\epsilon} d\sigma_g\})$ the concentration points of a sequence of measures $\{e^{u_\epsilon} d\sigma_g\}$ on the boundary ∂M of a surface (M, g) that is

$$Z(M, \{e^{u_\epsilon} d\sigma_g\}) = \{z \in M; \lim_{r \to 0} \limsup_{\epsilon \to 0} \int_{I_g(z,r)} e^{u_\epsilon} d\sigma_g > 0\}.$$

Taking the maximizing sequence $\{e^{u_\epsilon} d\sigma_g\}$ for $\sigma_k(M, [g])$ given by the previous subsection, which converges to ν in $\mathcal{M}_1(\partial M)$, we clearly have that

$$Z(M, \{e^{u_\epsilon} d\sigma_g\}) = \{z \in \partial M; \nu(\{z\}) > 0\},$$

and that

$$Z(M, \{e^{u_\epsilon} d\sigma_g\}) \subset \bigcap_{r>0} X_r(M, \{e^{u_\epsilon} \sigma_g\}) = \{p_1, \cdots, p_s\}, \tag{4.1}$$

where $p_1, \cdots, p_s$ are defined in Claim 3. This is a consequence of Claim 1 in Section 1.3: indeed, for $x \in Z(M, \{e^{u_\epsilon} d\sigma_g\})$ and for $r > 0$ small enough, let η_{g,x,r,r^2} be given by Claim 1. Then

$$\begin{aligned} \sigma_\star(B_g(x,r), g, I_g(x,r), e^{u_\epsilon}) &\leq \frac{\int_M \left|\nabla \eta_{g,x,r,r^2}\right|_g^2 dv_g}{\int_{\partial M} \left(\eta_{g,x,r,r^2}\right)^2 e^{u_\epsilon} d\sigma_g} \\ &\leq \frac{C}{\ln\left(\frac{1}{r}\right) \int_{B_g(x,r^2)} e^{u_\epsilon} d\sigma_g}, \end{aligned}$$

so that

$$\lim_{r \to 0} \limsup_{\epsilon \to 0} \sigma_\star(B_g(x,r), g, I_g(x,r), e^{u_\epsilon}) = 0.$$

Then there is a subsequence $\{\epsilon_j\}$ for which x satisfies $\mathbf{A}_{r,\epsilon_j}$ for all r small enough. Thanks to Claim 3, this gives that $x \in \{p_1, \cdots, p_s\}$.

We now define some functions which will rescale the problem at the neighborhood of the concentration points. For $a \in \mathbb{R} \times \{0\}$ and $\alpha > 0$, we let

$$H_{a,\alpha}(y) = \alpha y + a \text{ for } y \in \mathbb{R}^2.$$

For $p=(1,0)\in\mathbb{S}^1$, we define $\lambda:\mathbb{D}\setminus\{p\}\to\mathbb{R}^2_+$ the conformal diffeomorphism such that

$$F\circ\lambda\circ F^{-1}(z)=i\frac{z+1}{1-z}$$

with its inverse

$$F\circ\lambda^{-1}\circ F^{-1}(z)=\frac{z-i}{z+i},$$

where $F:\mathbb{R}^2\to\mathbb{C}$ is the canonical map $F(x,y)=x+iy$. In this section, we prove the following:

Proposition 2. *There exist some points $a_1^\epsilon,\cdots,a_N^\epsilon\in\mathbb{R}\times\{0\}$ and some scales*

$$0<\alpha_N^\epsilon<\alpha_{N-1}^\epsilon<\cdots<\alpha_1^\epsilon,$$

such that for $1\le i\le N$,

$$\alpha_i^\epsilon\to 0 \text{ as } \epsilon\to 0, \tag{4.2}$$

and letting

$$F_i=\left\{j>i;\frac{d_g(\bar a_i^\epsilon,\bar a_j^\epsilon)}{\alpha_i^\epsilon} \text{ is bounded}\right\},$$

we have for $j\neq i$ that

$$j\in F_i\Rightarrow\frac{\alpha_j^\epsilon}{\alpha_i^\epsilon}\to 0 \text{ as } \epsilon\to 0, \tag{4.3}$$

and that

$$j\notin F_i\Rightarrow\frac{d_g(\bar a_i^\epsilon,\bar a_j^\epsilon)}{\alpha_i^\epsilon}\to+\infty \text{ as } \epsilon\to 0. \tag{4.4}$$

There exist some disjoint sets $I_0^\epsilon,I_1^\epsilon,\cdots,I_N^\epsilon\subset\partial M$, some sets $\Gamma_1^\epsilon,\cdots,\Gamma_N^\epsilon\subset\mathbb{R}\times\{0\}$ and $S_1^\epsilon,\cdots,S_N^\epsilon\subset\mathbb{S}^1$ given by

$$\Gamma_i^\epsilon=H^{-1}_{a_i^\epsilon,\alpha_i^\epsilon}\left(\widetilde{I_i^\epsilon}^{l_i}\right) \text{ and } S_i^\epsilon=\left(H_{a_i^\epsilon,\alpha_i^\epsilon}\circ\lambda\right)^{-1}\left(\widetilde{I_i^\epsilon}^{l_i}\right),$$

some associated densities defined by

$$e^{\hat u_i^\epsilon}ds=\left(H_{a_i^\epsilon,\alpha_i^\epsilon}\right)^\star\left(e^{\tilde u_\epsilon^{l_i}}ds\right) \text{ and } e^{\check u_i^\epsilon}d\theta=\left(H_{a_i^\epsilon,\alpha_i^\epsilon}\circ\lambda\right)^\star\left(e^{\tilde u_\epsilon^{l_i}}ds\right),$$

some masses $m_i>0$ satisfying

$$L_{e^{u_\epsilon}d\sigma_g}(I_i^\epsilon)=L_{e^{\hat u_i^\epsilon}ds}(\Gamma_i^\epsilon)=L_{e^{\check u_i^\epsilon}d\theta}(S_i^\epsilon)\to m_i \text{ as } \epsilon\to 0 \tag{4.5}$$

for $1\le i\le N$ and some $l_i\in\{1,\cdots,L\}$, and $m_0\ge 0$ satisfying

$$L_{e^{u_\epsilon}d\sigma_g}(I_0^\epsilon)\to m_0 \text{ as } \epsilon\to 0, \tag{4.6}$$

such that

$$Z(\mathbb{S}^1,\{\mathbf{1}_{S_i^\epsilon}e^{\check u_i^\epsilon}d\theta\})=\emptyset \tag{4.7}$$

for $1\le i\le N$,

$$Z(M,\{\mathbf{1}_{I_0^\epsilon}e^{u_\epsilon}d\sigma_g\})=\emptyset, \tag{4.8}$$

and

$$\sum_{i=0}^{N} m_i = 1. \tag{4.9}$$

4.2. Proof of Proposition 2. Let us denote by $z_1, \cdots, z_{N_0}$ the atoms of ν with $N_0 \leq s \leq k$ (s is given by 4.1 or Claim 3) so that

$$e^{u_\epsilon} d\sigma_g \rightharpoonup^\star \nu_0 + \sum_{i=1}^{N_0} m_i \delta_{z_i},$$

where $\nu_0 \in \mathcal{M}(\partial M)$ has no atoms. Let $m_0 = \int_{\partial M} d\nu_0 \geq 0$. All the m_i's are positive for $1 \leq i \leq N_0$, and

$$\sum_{i=0}^{N_0} m_i = 1.$$

Let $1 \leq i \leq N_0$. We choose $l_i \in \{1, \cdots, L\}$ such that $z_i \in \gamma_{l_i}$. Up to the extraction of a subsequence, one can build a sequence $\{r_i^\epsilon\}$ such that $r_i^\epsilon > 0$ and $r_i^\epsilon \to 0$ as $\epsilon \to 0$ with

$$\int_{I_g(z_i, r_i^\epsilon)} e^{u_\epsilon} d\sigma_g \to m_i \text{ as } \epsilon \to 0.$$

We associate to sequences $a_i^\epsilon \in \mathbb{R} \times \{0\}$ and $\alpha_i^\epsilon > 0$ that we shall choose later the sets

$$\Gamma_i^\epsilon = H^{-1}_{a_i^\epsilon, \alpha_i^\epsilon}\left(\widetilde{I_g(z_i, r_i^\epsilon)}^{l_i}\right) \subset \mathbb{R} \times \{0\},$$

$$S_i^\epsilon = \lambda^{-1}(\Gamma_i^\epsilon) \subset \mathbb{S}^1,$$

$$M_i^\epsilon = B_g(z_i^\epsilon, r_i^\epsilon),$$

$$I_i^\epsilon = I_g(z_i^\epsilon, r_i^\epsilon),$$

$$M_0^\epsilon = M \setminus \bigcup_{i=1}^{N_0} M_i^\epsilon,$$

$$I_0^\epsilon = \partial M \setminus \bigcup_{i=1}^{N_0} I_i^\epsilon,$$

and the densities

$$e^{\hat{u}_i^\epsilon} = \alpha_i^\epsilon e^{(\tilde{u}_\epsilon^{l_i} + \tilde{v}_{l_i}^{l_i}) \circ H_{a_i^\epsilon, \alpha_i^\epsilon}} : \Gamma_i^\epsilon \to \mathbb{R},$$

$$e^{\check{u}_i^\epsilon} d\theta = \lambda^\star(e^{\hat{u}_i^\epsilon} ds) : S_i^\epsilon \to \mathbb{R}.$$

For the notations, we refer to Section 1.1.

Note that

$$M = M_0^\epsilon \cup \bigcup_{i=1}^{N_0} M_i^\epsilon,$$

with $L_{e^{u_\epsilon}d\sigma_g}(I_i^\epsilon) \to m_i$ as $\epsilon \to 0$ for $0 \le i \le N_0$. We assign to the subset M_i^ϵ a test function $\eta_i^\epsilon \in \mathcal{C}_c^\infty(M_i^\epsilon)$ given by Claim 1 in Section 1.3

$$\eta_i^\epsilon = \eta_{g,z_i,(r_i^\epsilon)^{\frac{1}{2}},r_i^\epsilon} \text{ for } 1 \le i \le N_0,$$

$$\eta_0^\epsilon = 1 - \sum_{i=1}^{N_0} \eta_{g,z_i,(r_i^\epsilon)^{\frac{1}{4}},(r_i^\epsilon)^{\frac{1}{2}}}.$$

Note that these test functions with pairwise disjoint supports and small Rayleigh quotient may also be used to prove that $N_0 \le k$ if $m_0 = 0$ or $N_0 \le k-1$ if $m_0 > 0$.

For $1 \le i \le N_0$, let's now adjust the parameters a_i^ϵ and α_i^ϵ in order to detect other scales of concentration of the mass at the neighborhood of z_i. By Hersch theorem (see [**16**], lemma 1.1 in the case of the circle $\mathbb{S}^1$) we can choose $a_i^\epsilon \in \mathbb{R} \times \{0\}$ and $\alpha_i^\epsilon > 0$ such that

$$\int_{\mathbb{S}^1} x e^{\check{u}_i^\epsilon} \mathbf{1}_{S_i^\epsilon} d\theta = 0. \tag{4.10}$$

Note that $\bar{a}_i^\epsilon \to z_i$ and that $\alpha_i^\epsilon \to 0$ as $\epsilon \to 0$. This normalization of the center of mass gives a dichotomy in the description of the concentration points of $\{e^{\check{u}_i^\epsilon}\mathbf{1}_{S_i^\epsilon}d\theta\}$: if $z \in Z(\mathbb{S}^1, \{e^{\check{u}_i^\epsilon}\mathbf{1}_{S_i^\epsilon}d\theta\})$, then, some mass is also concentrated in the opposite hemisphere $\{x \in \mathbb{S}^1; (x,z) \le 0\}$ and we can increase the number of test functions with small Rayleigh quotient on the manifold among $\eta_1^\epsilon, \cdots, \eta_{N_0}^\epsilon$. From this remark, we will build by induction a finite bubble tree which describes the concentrations at all the scales they appear.

A tree T is a set of finite sequences

$$\gamma = (i_1, \cdots, i_{|\gamma|}) \in \bigcup_{j \in \mathbb{N}} \mathbb{N}^j,$$

where $|\gamma|$ is the length of γ which satisfies

- $(\emptyset) \in T$ is the root of the tree.
- if $\gamma \in \bigcup_{j\in\mathbb{N}} \mathbb{N}^j$ and $i \in \mathbb{N}$, then $(\gamma, i) \in T \Rightarrow \gamma \in T$ and (γ, i) is called a son of γ.
- If $(\gamma, 0) \in T$ then $\forall i \in \mathbb{N}, (\gamma, 0, i) \notin T$. $(\gamma, 0)$ is called a leaf of T. We denote by L_T the set of leaves of T.
- If $\gamma \in T$, then $\{i \in \mathbb{N}; (\gamma, i) \in T\} = \{0, \cdots, N_\gamma\}$ with $N_\gamma \in \mathbb{N}$ and N_γ is the number of sons of γ.

Let T be a tree. We let $|T| = \sup\{|\gamma|\,; \gamma \in T\}$ be the depth of the tree. We let also $T_j = \{\gamma \in T; |\gamma| \le j\}$ be the truncated tree of depth $j \in \mathbb{N}$. We say that $\tilde{\gamma} \in T$ is a descendant of $\gamma \in T$ if there exists $\gamma' \in \bigcup_{j\in\mathbb{N}} \mathbb{N}^j$ such that $\tilde{\gamma} = (\gamma, \gamma')$.

In the following, we define by induction a tree T with

- some sets $I_\gamma^\epsilon \subset \partial M$ for $\gamma \in T$ and $\Gamma_\gamma^\epsilon \subset \mathbb{R} \times \{0\}$, $S_\gamma^\epsilon \subset \mathbb{S}^1$ for $\gamma \in T \setminus L_T$,

- some parameters $l_\gamma \in \{1, \cdots, L\}$, $r_\gamma^\epsilon > 0$, $a_\gamma^\epsilon \in \mathbb{R} \times \{0\}$ and $\alpha_\gamma^\epsilon > 0$ for $\gamma \in T \setminus L_T$,
- some points $z_\gamma \in \mathbb{S}^1$ if $\gamma \in T \setminus L_T$ and $|\gamma| \geq 2$ and $z_\gamma \in \partial M$ if $\gamma \in T \setminus L_T$ and $|\gamma| = 1$,
- some measures $\nu_0 \in \mathcal{M}(M)$ of mass $m_0 = \int_M d\nu_0 \geq 0$, $\nu_\gamma \in \mathcal{M}(\mathbb{S}^1)$ of mass $m_\gamma = \int_{\mathbb{S}^1} d\nu_\gamma \geq 0$ if $\gamma \in L_T$ and $|\gamma| \geq 2$ and some masses $m_\gamma > 0$ for $\gamma \in T \setminus L_T$,
- some functions $\hat{u}_\gamma^\epsilon : \Gamma_\gamma^\epsilon \to \mathbb{R}$ and $\check{u}_\gamma^\epsilon : S_\gamma^\epsilon \to \mathbb{R}$,
- some test functions $\eta_\gamma^\epsilon : M \to \mathbb{R}$ with $\eta_\gamma^\epsilon \in \mathcal{C}_c^\infty(M_\gamma^\epsilon)$ for $\gamma \in T$,

depending on ϵ. We describe the process of construction, by induction of this tree now and will prove in Claim 7 that it is a finite tree.

If $\gamma \in T$ and $|\gamma| = 1$, these objects are defined at the beginning of Section 4.2.

Assume now that these objects are defined for all γ of length $|\gamma| \leq j$. Let $\gamma \in T \setminus L_T$ with $|\gamma| \leq j$. Then, up to the extraction of a subsequence,

$$\mathbf{1}_{S_\gamma^\epsilon} e^{\check{u}_\gamma^\epsilon} d\theta \rightharpoonup^\star \nu_{(\gamma,0)} + \sum_{i=1}^{N_\gamma} m_{(\gamma,i)} \delta_{z_{(\gamma,i)}}, \tag{4.11}$$

where for $1 \leq i \leq N_\gamma$, $m_{(\gamma,i)} > 0$, $m_{(\gamma,0)} = \int_{\mathbb{S}^1} d\nu_{(\gamma,0)}$ and $\nu_{(\gamma,0)}$ is without atom. As we will see in the proof of Claim 7 and by the same arguments as in the previous subsection, Claim 1 provides some test functions which prove that $N_\gamma \leq k$. Notice that

$$\sum_{i=0}^{N_\gamma} m_{(\gamma,i)} = m_\gamma.$$

Let $1 \leq i \leq N_\gamma$. We define $l_{(\gamma,i)} = l_\gamma$ and up to the extraction of a subsequence, we can build $\{r_{(\gamma,i)}^\epsilon\}$ such that $r_{(\gamma,i)}^\epsilon > 0$ and $r_{(\gamma,i)}^\epsilon \to 0$ as $\epsilon \to 0$ with

$$\int_{I_\xi(z_{(\gamma,i)}, r_{(\gamma,i)}^\epsilon) \cap S_\gamma^\epsilon} e^{\check{u}_\gamma^\epsilon} d\theta \to m_{(\gamma,i)} \text{ as } \epsilon \to 0.$$

We define

$$\bar{\eta}_{(\gamma,i)}^\epsilon = \eta_{\xi, z_{(\gamma,i)}, (r_{(\gamma,i)})^{\frac{1}{2}}, r_{(\gamma,i)}} \circ \lambda^{-1} \circ H_{a_\gamma^\epsilon, \alpha_\gamma^\epsilon}^{-1} \circ \exp_{g_{l_\gamma}, x_{l_\gamma}}^{-1},$$

and

$$\bar{\eta}_{(\gamma,0)}^\epsilon = 1 - \sum_{i=1}^{N_\gamma} \eta_{\xi, z_{(\gamma,i)}, (r_{(\gamma,i)})^{\frac{1}{4}}, (r_{(\gamma,i)})^{\frac{1}{2}}} \circ \lambda^{-1} \circ H_{a_\gamma^\epsilon, \alpha_\gamma^\epsilon}^{-1} \circ \exp_{g_{l_\gamma}, x_{l_\gamma}}^{-1},$$

naturally extended by a constant on M so that $\bar{\eta}_{(\gamma,i)}^\epsilon \in \mathcal{C}^\infty(M)$. For $0 \leq i \leq N_\gamma$ the function

$$\eta_{(\gamma,i)}^\epsilon = \eta_\gamma^\epsilon \bar{\eta}_{(\gamma,i)}^\epsilon$$

satisfies (4.13) in the proof of Claim 7 and that

$$supp(\eta^\epsilon_{(\gamma,i)}) \cap supp(\eta^\epsilon_{(\gamma,j)}) = \emptyset \text{ for } i \neq j \text{ and } supp(\eta^\epsilon_{(\gamma,i)}) \subset supp(\eta^\epsilon_\gamma).$$

The use of these test functions proves that $N_\gamma \leq k$.

Let $1 \leq i \leq N_\gamma$. We define the sets

$$\Gamma^\epsilon_{(\gamma,i)} = H^{-1}_{a^\epsilon_{(\gamma,i)},\alpha^\epsilon_{(\gamma,i)}}\left(H_{a^\epsilon_\gamma,\alpha^\epsilon_\gamma}\left(\Gamma^\epsilon_\gamma \cap \lambda^{-1}\left(I_\xi(z_{(\gamma,i)}, r^\epsilon_{(\gamma,i)})\right)\right)\right),$$

$$S^\epsilon_{(\gamma,i)} = \lambda^{-1}\left(D^\epsilon_{(\gamma,i)}\right),$$

$$I^\epsilon_{(\gamma,i)} = \exp_{g_{l_\gamma},x_{l_\gamma}}\left(H_{a^\epsilon_{(\gamma,i)},\alpha^\epsilon_{(\gamma,i)}}\left(\Gamma^\epsilon_{(\gamma,i)}\right)\right) = \breve{\Gamma}^\epsilon_{(\gamma,i)},$$

$$I^\epsilon_{(\gamma,0)} = I^\epsilon_\gamma \setminus \bigcup_{i=1}^{N_\gamma} I^\epsilon_{(\gamma,i)},$$

and the densities

$$\frac{e^{\hat{u}^\epsilon_{(\gamma,i)}\left(\frac{z-a^\epsilon_{(\gamma,i)}}{\alpha^\epsilon_{(\gamma,i)}}\right)}}{\alpha^\epsilon_{(\gamma,i)}} = \frac{e^{\hat{u}^\epsilon_\gamma\left(\frac{z-a^\epsilon_\gamma}{\alpha^\epsilon_\gamma}\right)}}{\alpha^\epsilon_\gamma},$$

$$e^{\check{u}^\epsilon_{(\gamma,i)}}ds = \lambda^\star\left(e^{\hat{u}^\epsilon_{(\gamma,i)}}d\theta\right),$$

and by Hersch's normalization, we choose the parameters $a^\epsilon_{(\gamma,i)}$ and $\alpha^\epsilon_{(\gamma,i)}$ with

$$\int_{\mathbb{S}^1} x e^{\check{u}^\epsilon_{(\gamma,i)}} \mathbf{1}_{S^\epsilon_{(\gamma,i)}} d\theta = 0, \tag{4.12}$$

and

$$\int_{I^\epsilon_{(\gamma,i)}} e^{u_\epsilon} d\sigma_g = \int_{\Gamma^\epsilon_{(\gamma,i)}} e^{\hat{u}^\epsilon_{(\gamma,i)}} ds = \int_{S^\epsilon_{(\gamma,i)}} e^{\check{u}^\epsilon_{(\gamma,i)}} d\theta = m_{(\gamma,i)}.$$

Claim 7. *T is a finite tree.*

Proof. STEP 1 – We prove that if $\gamma \in T \setminus L_T$, then

$$\text{either } N_\gamma = 0 \text{ or } \sharp\{0 \leq i \leq N_\gamma; m_{(\gamma,i)} > 0\} \geq 2.$$

Since $m_{(\gamma,i)} > 0$ for $1 \leq i \leq N_\gamma$, we get Step 1 if $N_\gamma \geq 2$ or $N_\gamma = 0$. We now assume that $N_\gamma = 1$. By (4.11) and (4.12),

$$\int_{\mathbb{S}^1} (x, z_{(\gamma,1)}) d\nu_{(\gamma,0)} + m_{(\gamma,1)} = 0.$$

Since $m_{(\gamma,1)} > 0$, we get that $\nu_{(\gamma,0)} \neq 0$ and $m_{(\gamma,0)} > 0$. This proves Step 1.

STEP 2 – We prove that if $\gamma \in T \setminus L_T$, then

$$\frac{\int_M \left|\nabla \eta^\epsilon_{(\gamma,i)}\right|_g^2 dv_g}{\int_{\partial M}\left(\eta^\epsilon_{(\gamma,i)}\right)^2 e^{u_\epsilon} d\sigma_g} \to 0 \text{ as } \epsilon \to 0, \tag{4.13}$$

and that if $\gamma, \tilde{\gamma} \in T$ with $|\gamma| \leq |\tilde{\gamma}|$, then

- If $\tilde{\gamma}$ is not a descendant of γ, then $supp(\eta^\epsilon_{\tilde{\gamma}}) \cap supp(\eta^\epsilon_\gamma) = \emptyset$.
- If $\tilde{\gamma}$ is a descendant of γ, then $supp(\eta^\epsilon_{\tilde{\gamma}}) \subset supp(\eta^\epsilon_\gamma)$.

We prove (4.13) by induction on $|\gamma|$. This is clearly true for $|\gamma| = 1$. Let $j \geq 1$ and assume that (4.13)holds for all $|\gamma| \leq j$. We have

$$\frac{\int_M \left|\nabla \eta^\epsilon_{(\gamma,i)}\right|_g^2 dv_g}{\int_{\partial M} \left(\eta^\epsilon_{(\gamma,i)}\right)^2 e^{u_\epsilon} d\sigma_g} = \frac{\int_M \left|\nabla \eta^\epsilon_\gamma \bar{\eta}^\epsilon_{(\gamma,i)}\right|_g^2 dv_g}{\int_{\partial M} \left(\eta^\epsilon_\gamma \bar{\eta}^\epsilon_{(\gamma,i)}\right)^2 e^{u_\epsilon} d\sigma_g}$$

with

$$\begin{aligned} \int_M \left|\nabla \eta^\epsilon_\gamma \bar{\eta}^\epsilon_{(\gamma,i)}\right|_g^2 dv_g &\leq 2\left(\int_M \left|\nabla \eta^\epsilon_\gamma\right|_g^2 dv_g + \int_M \left|\nabla \bar{\eta}^\epsilon_{(\gamma,i)}\right|_g^2 dv_g\right) \\ &= 2\left(o\left(\int_{\partial M} \left(\eta^\epsilon_\gamma\right)^2 e^{u_\epsilon} d\sigma_g\right) + o(1)\right), \end{aligned}$$

by the induction assumption, and for $i \geq 1$,

$$\begin{aligned} \int_{\partial M} \left(\eta^\epsilon_\gamma \bar{\eta}^\epsilon_{(\gamma,i)}\right)^2 e^{u_\epsilon} d\sigma_g &\geq \int_{\mathbb{S}^1} \left(\eta_{\xi, z_{(\gamma,i)}, (r^\epsilon_{(\gamma,i)})^{\frac{1}{2}}, r^\epsilon_{(\gamma,i)}}\right)^2 e^{\check{u}^\epsilon_\gamma} \mathbf{1}_{S^\epsilon_\gamma} d\theta \\ &\geq \int_{\mathbb{S}^1} e^{\check{u}^\epsilon_\gamma} \mathbf{1}_{S^\epsilon_\gamma \cap I_\xi(z_{(\gamma,i)}, r^\epsilon_{(\gamma,i)})} d\theta \\ &= m_{(\gamma,i)}, \end{aligned}$$

and for $i = 0$, fixing $\rho > 0$,

$$\begin{aligned} \int_{\partial M} \left(\eta^\epsilon_\gamma \bar{\eta}^\epsilon_{(\gamma,0)}\right)^2 e^{u_\epsilon} d\sigma_g &\geq \int_{S^\epsilon_\gamma} \left(1 - \sum_{i=1}^{N_\gamma} \eta_{\xi, z_{(\gamma,i)}, (r^\epsilon_{(\gamma,i)})^{\frac{1}{4}}, (r^\epsilon_{(\gamma,i)})^{\frac{1}{2}}}\right)^2 e^{\check{u}^\epsilon_\gamma} \\ &\geq \int_{S^\epsilon_\gamma \setminus \bigcup_{i=1}^{N_\gamma} I_\xi(p_i,\rho)} e^{\check{u}^\epsilon_\gamma} d\theta \\ &= \int_{\mathbb{S}^1 \setminus \bigcup_{i=1}^{N_\gamma} I_\xi(p_i,\rho)} \left(d\nu_{(\gamma,0)} + \sum_{i=1}^{N_\gamma} m_{(\gamma,i)} \delta_{z_{(\gamma,i)}}\right) \\ &\quad + o(1) \\ &= \int_{\mathbb{S}^1 \setminus I_\xi(p_i,\rho)} d\nu_{(\gamma,0)} + o(1) \end{aligned}$$

as $\epsilon \to 0$. Gathering the previous inequalities, together with

$$\int_{\partial M} \left(\eta^\epsilon_\gamma\right)^2 e^{u_\epsilon} d\sigma_g \leq \int_{\partial M} e^{u_\epsilon} d\sigma_g = 1,$$

we get (4.13).

We now prove the second part of step 2, also by induction. Assume that, for some $j \geq 1$ fixed, for all $\gamma, \tilde{\gamma} \in T$ with $|\gamma| \leq |\tilde{\gamma}| \leq j$ we have that

- If $\tilde{\gamma}$ is not a descendant of γ, then $supp(\eta^\epsilon_{\tilde{\gamma}}) \cap supp(\eta^\epsilon_{\gamma}) = \emptyset$.
- If $\tilde{\gamma}$ is a descendant of γ, then $supp(\eta^\epsilon_{\tilde{\gamma}}) \subset supp(\eta^\epsilon_{\gamma})$.

Let us prove now that this is still true for any $\gamma, \tilde{\gamma} \in T$ with $|\gamma| \leq |\tilde{\gamma}| \leq j+1$. If $|\tilde{\gamma}| \leq j$, there is of course nothing to prove. Assume that $|\tilde{\gamma}| = j+1$

If $|\gamma| = j+1$, then,

$$supp(\eta^\epsilon_{\gamma}) \cap supp(\eta^\epsilon_{\tilde{\gamma}}) \subset supp(\bar{\eta}^\epsilon_{\gamma}) \cap supp(\bar{\eta}^\epsilon_{\tilde{\gamma}}),$$

which is empty if and only if $\gamma \neq \tilde{\gamma}$.

If $|\gamma| \leq j$, we denote $\tilde{\gamma} = (\hat{\gamma}, i)$ with $0 \leq i \leq N_{\hat{\gamma}}$. We can apply the induction hypothesis to $|\gamma| \leq |\hat{\gamma}| \leq j$. Then,

- if $supp(\eta_{\tilde{\gamma}}) \cap supp(\eta_{\gamma}) \neq \emptyset$, we get $supp(\eta_{\hat{\gamma}}) \cap supp(\eta_{\gamma}) \neq \emptyset$ since $supp(\eta_{\tilde{\gamma}}) \subset supp(\eta_{\hat{\gamma}})$. By the induction assumption, $\hat{\gamma}$ is a descendant of γ and $\tilde{\gamma}$ is a descendant of γ.
- If $\tilde{\gamma}$ is a descendant of γ, then, $\hat{\gamma}$ is a descendant of γ and by the induction assumption, $supp(\eta_{\tilde{\gamma}}) \subset supp(\eta_{\hat{\gamma}}) \subset supp(\eta_{\gamma})$.

The proof of Step 2 is complete.

STEP 3 – We prove the following assertion $\mathbf{H}_j$ by induction on j.

$\mathbf{H}_j$: If $T_j \neq T_{j+1}$, then, $T_{j+1} = T$ or there exist $j+1$ test functions with pairwise disjoint support in the set $\{\eta^\epsilon_\gamma, \gamma \in T_{j+1}\}$.

Notice that by (4.13) in Step 2, the assumption $T_{k+1} \neq T$ would give a contradiction. Indeed, it suffices to test the $k+1$ functions given by the assumption $\mathbf{H}_{k+1}$ in the variational characterization of $\sigma_\epsilon = \sigma_k(M, g, \partial M, e^{u_\epsilon})$, (1.1). Therefore, the increasing sequence of trees $\{T_j\}$ is stationary, and Claim 7 will follow.

Note that $\mathbf{H}_1$ is true by the existence of $\{\eta^\epsilon_1\}$.

Let $j \geq 2$ and we assume that $\mathbf{H}_{j-1}$ is true and that $T_j \neq T_{j+1}$. Then, $T_{j-1} \neq T_j$ and $\mathbf{H}_{j-1}$ gives j test functions with pairwise disjoint support in the set $\{\eta^\epsilon_\gamma; \gamma \in T_j\}$ denoted by $\eta^\epsilon_{\gamma_1} \cdots \eta^\epsilon_{\gamma_j}$. We assume that $T_{j+1} \neq T$. Then, there is $\gamma \in T_j$ such that $N_\gamma \geq 1$. By Step 1, there are two indices $i_1 \neq i_2$ such that $m_{(\gamma,i_1)} > 0$ and $m_{(\gamma,i_2)} > 0$.

If γ is not a descendant of one of $\gamma_1, \cdots, \gamma_j$, then we take the set of test functions $\{\eta^\epsilon_{\gamma_1}, \cdots, \eta^\epsilon_{\gamma_j}, \eta^\epsilon_{(\gamma,i_1)}\}$.

If γ is a descendant of one of $\gamma_1, \cdots, \gamma_j$, then, by Step 2, since the functions $\eta^\epsilon_{\gamma_1}, \cdots, \eta^\epsilon_{\gamma_j}$ have pairwise disjoint support, there is a unique $1 \leq i \leq j$ such that γ is a descendant of γ_i and we take the set of test functions with pairwise disjoint support:

$$\{\eta^\epsilon_{\gamma_1}, \cdots, \eta^\epsilon_{\gamma_{i-1}}, \eta^\epsilon_{\gamma_{i+1}}, \cdots, \eta^\epsilon_{\gamma_j}, \eta^\epsilon_{(\gamma,i_1)}, \eta^\epsilon_{(\gamma,i_2)}\}.$$

Thus, $\mathbf{H}_j$ holds. This ends the proof of Step 3 and as already said the proof of the claim. q.e.d.

Thanks to this construction, the parameters $(a_\gamma^\epsilon, \alpha_\gamma^\epsilon)$ define separated bubble or bubbles over bubbles. This reads as a formula which originates from [**3**] and [**24**] in the context of bubble tree constructions:

Claim 8. *If* $\gamma \in T \setminus L_T$, $\alpha_\gamma^\epsilon \to 0$ *as* $\epsilon \to 0$ *and if* $\gamma_1, \gamma_2 \in T \setminus L_T$ *with* $\gamma_1 \neq \gamma_2$, *then*

$$\frac{d_g(\bar{a}_{\gamma_1}^\epsilon, \bar{a}_{\gamma_2}^\epsilon)}{\alpha_{\gamma_1}^\epsilon + \alpha_{\gamma_2}^\epsilon} + \frac{\alpha_{\gamma_1}^\epsilon}{\alpha_{\gamma_2}^\epsilon} + \frac{\alpha_{\gamma_2}^\epsilon}{\alpha_{\gamma_1}^\epsilon} \to +\infty \text{ as } \epsilon \to 0.$$

Proof. We recall that there exists $C_0 > 0$ such that for all $0 < r < \delta$,

$$B_g(x, C_0^{-1}r) \subset \exp_{g_l, x_l}(\mathbb{D}_r^+(\tilde{x}^l)) \subset B_g(x, C_0 r)$$

for all $x \in \gamma_l$ with $1 \leq l \leq L$. On the disks, there also exists $C_1 > 0$ and some $\delta_1 > 0$ such that for all $0 < r < \delta_1$,

$$B_\xi(z_\gamma, C_1^{-1}r) \subset \lambda^{-1}(\mathbb{D}_r^+(\hat{z}_\gamma)) \subset B_\xi(z_\gamma, C_1 r)$$

for all $\gamma \in T \setminus L_T$ such that $|\gamma| \geq 2$ and $z_\gamma \neq p$, where $\hat{z}_\gamma = \lambda(z_\gamma)$; and

$$B_\xi(p, C_1^{-1}r) \subset \lambda^{-1}\left(\mathbb{R}_+^2 \setminus \mathbb{D}_{\frac{1}{r}}^+\right) \subset B_\xi(p, C_1 r).$$

Now, given $\gamma_1, \gamma_2 \in T \setminus L_T$, we let $\gamma \in T$ such that $\gamma_1 = (\gamma, \tilde{\gamma}_1)$, $\gamma_2 = (\gamma, \tilde{\gamma}_2)$ and $|\gamma|$ is maximal. We consider 5 cases in order to prove the claim.

Case 1 – $\gamma = (\emptyset)$. Then $\gamma_1 = (i, \hat{\gamma}_1)$ and $\gamma_2 = (j, \hat{\gamma}_1)$ with $i \neq j$. Since

$$I_{\gamma_1}^\epsilon \subset I_g(z_i, r_i^\epsilon) \subset \exp_{g_l, x_l}\left(I_{C_0 r_i^\epsilon}(\tilde{z}_i)\right),$$

we get with (4.10) that

$$|a_i^\epsilon - \tilde{z}_i| \leq C_0 r_i^\epsilon,$$

and

$$\alpha_i^\epsilon \leq C_0 r_i^\epsilon + |a_i^\epsilon - \tilde{z}_i|,$$

so that $a_i^\epsilon \to \tilde{z}_i$ as $\epsilon \to 0$ and $\alpha_i^\epsilon \to 0$ as $\epsilon \to 0$ and the same is true for j. Then, since $z_i \neq z_j$,

$$\frac{d_g(\bar{a}_i^\epsilon, \bar{a}_j^\epsilon)}{\alpha_i^\epsilon + \alpha_j^\epsilon} = \frac{d_g(z_i, z_j) + o(1)}{\alpha_i^\epsilon + \alpha_j^\epsilon} \to +\infty \text{ as } \epsilon \to 0.$$

Case 2 – $\gamma \neq (\emptyset)$, $\tilde{\gamma}_1 = (\emptyset)$, $\tilde{\gamma}_2 = (j, \hat{\gamma}_2)$ with $z_{(\gamma,j)} \neq p$. Then, we have

$$I_{\gamma_2}^\epsilon \subset I_{(\gamma,j)}^\epsilon \subset \exp_{g_l, x_l}\left(I_{C_1 r_{(\gamma,j)}^\epsilon \alpha_\gamma^\epsilon}(\alpha_\gamma^\epsilon \hat{z}_{(\gamma,j)} + a_\gamma^\epsilon)\right),$$

so that by (4.12), we have that

$$\left|\alpha_\gamma^\epsilon \hat{z}_{(\gamma,j)} + a_\gamma^\epsilon - a_{(\gamma,j)}^\epsilon\right| \leq C_1 r_{(\gamma,j)}^\epsilon \alpha_\gamma^\epsilon,$$

and

$$\alpha^\epsilon_{(\gamma,j)} \leq C_1 r^\epsilon_{(\gamma,j)}\alpha^\epsilon_\gamma + \left|\alpha^\epsilon_\gamma \hat{z}_{(\gamma,j)} + a^\epsilon_\gamma - a^\epsilon_{(\gamma,j)}\right|,$$

and $\frac{\alpha^\epsilon_\gamma}{\alpha^\epsilon_{(\gamma,j)}} \to +\infty$ as $\epsilon \to 0$.

Case 3 – $\gamma \neq (\emptyset)$, $\tilde{\gamma}_1 = (\emptyset)$, $\tilde{\gamma}_2 = (j, \hat{\gamma}_2)$ with $z_{(\gamma,j)} = p$.

We assume that $\frac{\left|a^\epsilon_{(\gamma,j)} - a^\epsilon_\gamma\right|}{\alpha^\epsilon_{(\gamma,j)} + \alpha^\epsilon_\gamma}$ is bounded and we prove by contradiction that $\frac{\alpha^\epsilon_{(\gamma,j)}}{\alpha^\epsilon_\gamma} \to +\infty$ as $\epsilon \to 0$. We assume that $\alpha^\epsilon_{(\gamma,j)} = O(\alpha^\epsilon_\gamma)$. Then, it is clear that $\frac{\left|a^\epsilon_{(\gamma,j)} - a^\epsilon_\gamma\right|}{\alpha^\epsilon_\gamma}$ is bounded and we have by (4.12) that

$$\alpha^\epsilon_{(\gamma,j)} \geq \frac{\alpha^\epsilon_\gamma}{C_1 r^\epsilon_{(\gamma,i)}} - \left|a^\epsilon_\gamma - a^\epsilon_{(\gamma,j)}\right|,$$

so that

$$\frac{\alpha^\epsilon_{(\gamma,j)}}{\alpha^\epsilon_\gamma} \geq \frac{1}{C_1 r^\epsilon_{(\gamma,i)}} - \frac{\left|a^\epsilon_\gamma - a^\epsilon_{(\gamma,j)}\right|}{\alpha^\epsilon_\gamma} \to +\infty \text{ as } \epsilon \to 0,$$

which contradicts the assumption $\alpha^\epsilon_{(\gamma,j)} = O(\alpha^\epsilon_\gamma)$. Thus, $\frac{\alpha^\epsilon_{(\gamma,j)}}{\alpha^\epsilon_\gamma} \to +\infty$ as $\epsilon \to 0$.

Case 4 – $\gamma \neq (\emptyset)$, $\tilde{\gamma}_1 = (i, \hat{\gamma}_1)$, $\tilde{\gamma}_2 = (j, \hat{\gamma}_2)$ with $i \neq j$, $z_{(\gamma,i)} \neq p$ and $z_{(\gamma,j)} \neq p$.

We have that $\left|a^\epsilon_{(\gamma,i)} - a^\epsilon_{(\gamma,j)}\right| = \alpha^\epsilon_\gamma \left(\left|\hat{z}_{(\gamma,i)} - \hat{z}_{(\gamma,j)}\right| + o(1)\right)$, $\frac{\alpha^\epsilon_{(\gamma,i)}}{\alpha^\epsilon_\gamma} = o(1)$ and $\frac{\alpha^\epsilon_{(\gamma,j)}}{\alpha^\epsilon_\gamma} = o(1)$ by Case 2 so that

$$\frac{d_g\left(\bar{a}^\epsilon_{(\gamma,i)}, \bar{a}^\epsilon_{(\gamma,j)}\right)}{\alpha^\epsilon_{(\gamma,i)} + \alpha^\epsilon_{(\gamma,j)}} \to +\infty \text{ as } \epsilon \to 0.$$

Case 5 – $\gamma \neq (\emptyset)$, $\tilde{\gamma}_1 = (i, \hat{\gamma}_1)$, $\tilde{\gamma}_2 = (j, \hat{\gamma}_2)$ with $z_{(\gamma,i)} \neq p$ and $z_{(\gamma,j)} = p$.

As in Case 3, we assume that $\frac{\left|a^\epsilon_{(\gamma,i)} - a^\epsilon_{(\gamma,j)}\right|}{\alpha^\epsilon_{(\gamma,i)} + \alpha^\epsilon_{(\gamma,j)}}$ is bounded and we will prove by contradiction that $\frac{\alpha^\epsilon_{(\gamma,j)}}{\alpha^\epsilon_{(\gamma,i)}} \to +\infty$ as $\epsilon \to 0$. Let's assume that $\alpha^\epsilon_{(\gamma,j)} = O(\alpha^\epsilon_{(\gamma,i)})$. Then,

$$\begin{aligned}
\frac{\left|a^\epsilon_{(\gamma,j)} - a^\epsilon_\gamma\right|}{\alpha^\epsilon_\gamma + \alpha^\epsilon_{(\gamma,j)}} &\leq \frac{\left|a^\epsilon_{(\gamma,j)} - a^\epsilon_{(\gamma,i)}\right|}{\alpha^\epsilon_\gamma + \alpha^\epsilon_{(\gamma,j)}} + \frac{\left|a^\epsilon_{(\gamma,i)} - a^\epsilon_\gamma\right|}{\alpha^\epsilon_\gamma + \alpha^\epsilon_{(\gamma,j)}} \\
&\leq \frac{\left|a^\epsilon_{(\gamma,j)} - a^\epsilon_{(\gamma,i)}\right|}{\alpha^\epsilon_{(\gamma,i)} + \alpha^\epsilon_{(\gamma,j)}} + \frac{\left|a^\epsilon_{(\gamma,i)} - a^\epsilon_\gamma\right|}{\alpha^\epsilon_\gamma + o(\alpha^\epsilon_\gamma)}
\end{aligned}$$

$$\leq \quad \frac{\left|a^{\epsilon}_{(\gamma,j)} - a^{\epsilon}_{(\gamma,i)}\right|}{\alpha^{\epsilon}_{(\gamma,i)} + \alpha^{\epsilon}_{(\gamma,j)}} + O(1),$$

since $\alpha^{\epsilon}_{(\gamma,i)} = o(\alpha^{\epsilon}_{\gamma})$ by Case 2, and $\left|a^{\epsilon}_{(\gamma,i)} - a^{\epsilon}_{\gamma}\right| = O(\alpha^{\epsilon}_{\gamma})$. Then, $\frac{\left|a^{\epsilon}_{(\gamma,j)} - a^{\epsilon}_{\gamma}\right|}{\alpha^{\epsilon}_{\gamma} + \alpha^{\epsilon}_{(\gamma,j)}}$ is bounded and by Case 3, $\frac{\alpha^{\epsilon}_{(\gamma,j)}}{\alpha^{\epsilon}_{\gamma}} \to +\infty$ as $\epsilon \to 0$ so that $\frac{\alpha^{\epsilon}_{(\gamma,j)}}{\alpha^{\epsilon}_{(\gamma,i)}} \to +\infty$ as $\epsilon \to 0$ which gives a contradiction. Thus, $\frac{\alpha^{\epsilon}_{(\gamma,j)}}{\alpha^{\epsilon}_{(\gamma,i)}} \to +\infty$ as $\epsilon \to 0$.

Gathering all the cases, the proof is complete. q.e.d.

Now, we are in position to prove Proposition 2. We denote by $L^+ \subset L_T$ the set of leaves $\gamma \in L_T$ such that $m_\gamma > 0$.

To simplify, we now denote the elements of L^+ by $\{1, \cdots, N\}$ and all the indices $\gamma \in L^+$ in I^{ϵ}_{γ}, $\Gamma^{\epsilon}_{\gamma}$ S^{ϵ}_{γ}, a^{ϵ}_{γ}, $\alpha^{\epsilon}_{\gamma}$, $e^{\hat{u}^{\epsilon}_{\gamma}}$, $e^{\check{u}^{\epsilon}_{\gamma}}$, ν_γ and m_γ are replaced by the corresponding index $i \in \{1, \cdots, N\}$.

Up to the extraction of a subsequence and up to reordering the α^{ϵ}_i's, we get (4.2), (4.3) and (4.4) thanks to Claim 8. By construction, we obtain the remaining facts of the proposition.

5. Regularity estimates at the concentration scales

In this section, we aim at proving some energy estimates in order to prove later Proposition 3 page 168. We fix $i \in \{1, \cdots, N\}$ given by Proposition 2 and up to the end of the section drop the index i of the parameters l_i, a^{ϵ}_i, α^{ϵ}_i the functions $\hat{u}^{\epsilon}_i$, we defined. As described in Section 1.1, we let

$$\hat{\Phi}_\epsilon(z) = \widetilde{\Phi}^l_\epsilon \circ H_{a_\epsilon,\alpha_\epsilon}(z) = \widetilde{\Phi}^l_\epsilon(\alpha_\epsilon z + a_\epsilon),$$

and

$$\hat{\nu}_\epsilon = H^{\star}_{a_\epsilon,\alpha_\epsilon}(\tilde{\nu}_\epsilon).$$

Then, for $0 \leq i \leq n$ and for $\rho > 0$ fixed, we get the equations

$$\text{(5.1)} \qquad \begin{cases} \Delta_\xi \hat{\phi}^i_\epsilon = 0 & \text{in } \mathbb{D}^+_{\frac{1}{\rho}}, \\ \partial_t \hat{\phi}^i_\epsilon = -\sigma_\epsilon e^{\hat{u}_\epsilon} \hat{\phi}^i_\epsilon & \text{on } I_{\frac{1}{\rho}}. \end{cases}$$

As we will see, the properties gathered in Proposition 1 and Claim 3 are in some sense invariant by dilatation. Indeed, this is clear in equation (5.1). We also have that if $\Omega \subset \omega_l$ and $\Gamma = \Omega \cap \partial M$,

$$\sigma_\star(\Omega, g, \Gamma, e^{u_\epsilon}) = \sigma_\star(\hat{\Omega}, \xi, \hat{\Gamma} e^{\hat{u}_\epsilon}),$$

where we set $\hat{\Omega} = H^{-1}_{a_\epsilon,\alpha_\epsilon}\left(\widetilde{\Omega}^l\right)$ and $\hat{\Gamma} = H^{-1}_{a_\epsilon,\alpha_\epsilon}\left(\widetilde{\Gamma}^l\right)$. The heat equation is also invariant by dilatation, up to some errors on the surface M we precised in Section 1.2 (see (1.2) and (1.4)), thanks to the following identity in the Euclidean case

$$\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi\epsilon}} e^{-\frac{|x-y|^2}{4\epsilon}} f(y)dy = \int_{\mathbb{R}} \frac{\alpha}{\sqrt{4\pi\epsilon}} e^{-\alpha^2 \frac{|\frac{x}{\alpha}-y|^2}{4\epsilon}} f(\alpha y)dy.$$

Therefore, we can derive regularity estimates of the eigenfunctions at all the concentration scales.

However, we have to distinguish two cases, depending on the speed of concentration α_ϵ when compared to ϵ. In Section 5.1, we treat the case when $\frac{\alpha_\epsilon^2}{\epsilon} \to +\infty$ as $\epsilon \to 0$, and in Section 5.2, we will treat the case when $\alpha_\epsilon^2 = O(\epsilon)$.

5.1. Regularity estimates when $\frac{\alpha_\epsilon^2}{\epsilon} \to +\infty$. We assume in this subsection that $\frac{\alpha_\epsilon^2}{\epsilon} \to +\infty$ as $\epsilon \to 0$. We set $\theta_\epsilon = \frac{\epsilon}{e^{2\tilde{v}_l(a)}\alpha_\epsilon^2}$, where $a_\epsilon \to a \in \mathbb{R} \times \{0\}$ as $\epsilon \to 0$, and $i_0 \in \{1, \cdots, N_0\}$ such that $\widetilde{z_{i_0}} = a$. Then

$$\theta_\epsilon \to 0 \text{ as } \epsilon \to 0. \tag{5.2}$$

We will adapt the technics of Section 3.1 in the surface $(\mathbb{D}^2, \xi, \mathbb{S}^1, e^{\check{u}_\epsilon})$. First, notice that

$$e^{\hat{u}_\epsilon} ds - d\hat{\nu}_\epsilon \rightharpoonup_\star 0 \text{ in } \mathcal{M}(\mathbb{R} \times \{0\}) \text{ as } \epsilon \to 0. \tag{5.3}$$

Indeed, for $\zeta \in \mathcal{C}_c^0(I_{R_0})$ for some $R_0 > 0$, and $R > R_0$, we can write that

$$\begin{aligned}\int_{\mathbb{R}\times\{0\}} \zeta \left(e^{2\hat{u}_\epsilon} dx - d\hat{\nu}_\epsilon\right) &= \int_{\partial M \setminus \breve{I_R}} \left(\int_{\breve{I_{R_0}}} p_\epsilon(y,x)\zeta(\hat{y})d\sigma_g(y)\right) d\nu_\epsilon(x) \\ &\quad + \int_{I_R}\int_{I_R} \left(\zeta(z) - \zeta(x)\right) \hat{p}_\epsilon(z,x) dz d\hat{\nu}_\epsilon(x) \\ &\quad + \int_{I_{R_0}} \left(\int_{I_R} \hat{p}_\epsilon(z,x)dz - 1\right) \zeta(x) d\hat{\nu}_\epsilon(x).\end{aligned}$$

By estimates (1.7) on the heat kernel, we have that

$$\begin{aligned}&\int_{\partial M \setminus \breve{I_R}} \left(\int_{\breve{I_{R_0}}} p_\epsilon(x,y) \left|\zeta(\hat{y})\right| d\sigma_g(y)\right) d\nu_\epsilon(x) \\ &\leq \|\zeta\|_\infty \sup_{x \in \partial M \setminus \breve{I_R}} \int_{\breve{I_{R_0}}} p_\epsilon(x,y) d\sigma_g(y) \\ &\leq O\left(\frac{e^{-\frac{(R-R_0)^2}{8\theta_\epsilon}}}{\sqrt{\theta_\epsilon}}\right) \to 0 \text{ as } \epsilon \to 0.\end{aligned}$$

By estimates (1.5) on the heat kernel, we have that

$$\int_{I_R} \left(\int_{I_R} \left|\zeta(z) - \zeta(x)\right| \hat{p}_\epsilon(z,x) dz\right) d\hat{\nu}_\epsilon(x)$$

$$\leq \sup_{x\in I_R}\int_{\mathbb{R}\times\{0\}} |\zeta(x)-\zeta(z)| \frac{e^{-\frac{|x-z|^2}{8\theta_\epsilon}}}{\sqrt{\pi\theta_\epsilon}}dz \to 0 \text{ as } \epsilon\to 0,$$

since ζ is uniformly continuous on $\mathbb{R}\times\{0\}$. Finally, we have by the heat kernel estimate (1.8) that

$$\lim_{R\to+\infty}\lim_{\epsilon\to 0}\sup_{x\in I_{R_0}}\left|\int_{I_R}\hat{p}_\epsilon(z,x)dz-1\right| = 0,$$

so that we get (5.3). We denote by $\hat{\nu}$ the weak star limit of both $\{e^{\hat{u}_\epsilon}dx\}$ and $\{\hat{\nu}_\epsilon\}$ in $\mathcal{M}(\mathbb{R}\times\{0\})$.

Let's tackle a generalization of Claim 3 at all the scales which appear between α_ϵ and δ_0. For a sequence $\{\gamma_\epsilon\}$, we let

$$e^{\overline{u_\epsilon}^{\gamma_\epsilon}(x)} = \gamma_\epsilon e^{\tilde{u}^l_\epsilon(\gamma_\epsilon x+a_\epsilon)} \text{ and } \overline{\Phi_\epsilon}^{\gamma_\epsilon}(x) = \widetilde{\Phi_\epsilon}^l(\gamma_\epsilon x+a_\epsilon),$$

and for a sequence of domains $\Omega_\epsilon\subset\omega_l$, with $\Gamma_\epsilon = \partial\Omega_\epsilon\cap\partial M\neq\emptyset$.

$$\overline{\Omega_\epsilon}^{\gamma_\epsilon} = H^{-1}_{a_\epsilon,\gamma_\epsilon}\left(\widetilde{\Omega}^l_\epsilon\right) \text{ and } \overline{\Gamma_\epsilon}^{\gamma_\epsilon} = H^{-1}_{a_\epsilon,\gamma_\epsilon}\left(\widetilde{\Gamma}^l_\epsilon\right),$$

so that

$$\sigma_\star\left(\Omega_\epsilon,g,\Gamma_\epsilon,e^{u_\epsilon}\right) = \sigma_\star\left(\overline{\Omega_\epsilon}^{\gamma_\epsilon},\xi,\overline{\Gamma_\epsilon}^{\gamma_\epsilon},e^{\overline{u_\epsilon}^{\gamma_\epsilon}}\right),$$

and

$$\begin{cases} \Delta_\xi\overline{\Phi_\epsilon}^{\gamma_\epsilon} = 0 & \text{in } \overline{\Omega_\epsilon}^{\gamma_\epsilon}, \\ \partial_t\overline{\Phi_\epsilon}^{\gamma_\epsilon} = -\sigma_\epsilon e^{\overline{u_\epsilon}^{\gamma_\epsilon}}\overline{\Phi_\epsilon}^{\gamma_\epsilon} & \text{on } \overline{\Gamma_\epsilon}^{\gamma_\epsilon}. \end{cases}$$

We also let $\mathbb{A}_\rho$ be the half-annulus $\mathbb{D}^+_{\frac{1}{\rho}}\setminus\mathbb{D}^+_\rho$ and $J_\rho = I_{\frac{1}{\rho}}\setminus I_\rho$.

We recall that $X_r(\Omega,\xi,\Gamma,\{e^{\overline{u_\epsilon}^{\gamma_\epsilon}}\})$ is the set of points x of $\Omega\subset\mathbb{R}^2_+$ (with $\Gamma=\Omega\cap\mathbb{R}\times\{0\}$) such that there exists $\epsilon>0$ which satisfies $\mathbf{P}_{r,\epsilon}$, that is $\mathbf{A}_{r,\epsilon}$ or $\mathbf{B}_{r,\epsilon}$, where

$\mathbf{A}_{r,\epsilon}$: $x\in\Gamma$ and $\sigma_\star(\mathbb{D}_r(x),\xi,I_r(x),e^{\overline{u_\epsilon}^{\gamma_\epsilon}}) \leq \frac{\sigma_k(M,[g])}{2}$

$\mathbf{B}_{r,\epsilon}$: There exists $f\in E_k(M,g,\partial M,e^{u_\epsilon})$ such that $\overline{f}^{\gamma_\epsilon}(x)=0$ and the Nodal set of $\overline{f}^{\gamma_\epsilon}$ which contains x does not intersect $\partial\mathbb{D}^+_r(x)$.

Note that for $\gamma_\epsilon=\alpha_\epsilon$, $e^{\overline{u_\epsilon}^{\gamma_\epsilon}} = e^{\hat{u}_\epsilon}$ and that the set of concentration points satisfies

$$Z(\Omega,\{e^{\hat{u}_\epsilon}ds\})\subset X_r(\Omega,\xi,\Gamma,\{e^{\hat{u}_\epsilon}\}) \tag{5.4}$$

for all $r>0$. We write $\omega^\epsilon_1\ll\omega^\epsilon_2$ if two sequences $\{\omega^\epsilon_1\}$ and $\{\omega^\epsilon_2\}$ satisfy $\frac{\omega^\epsilon_1}{\omega^\epsilon_2}\to 0$ as $\epsilon\to 0$.

Claim 9. *Up to the extraction of a subsequence, there exist some sequences* $\{\omega^\epsilon_i\}$ *with* $0\leq i\leq t+1$ *and* $0\leq t\leq k$ *such that*

$$\alpha^\epsilon = \omega^\epsilon_0 \ll \omega^\epsilon_1 \ll \omega^\epsilon_2 \ll \cdots \ll \omega^\epsilon_t \ll \omega^\epsilon_{t+1} = \delta_0,$$

there exist $R_0 > 0$ *and some points* $p_{i,j}$ *with* $0 \le i \le t$ *and* $1 \le j \le s_i$ *such that if* $1 \le i \le t$, $p_{i,j} \in J_{\frac{1}{R_0}}$ *and if* $i = 0$, $p_{0,j} \in I_{R_0}$, *with*

$$s - 1 + \sum_{i=0}^{t} s_i \le k,$$

and for all $0 < \rho < \frac{1}{2R_0}$, *there exists some* $r > 0$ *such that for all* $1 \le i \le t$,

$$X_r\left(\mathbb{A}_\rho, \xi, J_\rho, \{e^{\overline{u_\epsilon}^{\omega_i^\epsilon}}\}\right) \subset \bigcup_{j=1}^{s_i} \mathbb{D}_\rho^+(p_{i,j}),$$

$$X_r\left(\mathbb{D}^+_{\frac{1}{\rho}}, \xi, I_{\frac{1}{\rho}}, \{e^{\hat{u}_\epsilon}\}\right) \subset \bigcup_{j=1}^{s_0} \mathbb{D}_\rho^+(p_{0,j}),$$

for all sequence $\{\gamma_\epsilon\}$ *such that* $\frac{\omega_i^\epsilon}{\rho} < \gamma_\epsilon < \rho\omega_{i+1}^\epsilon$ *with* $0 \le i \le t$ *fixed,*

$$X_r\left(\mathbb{A}_{R_0\rho}, \xi, J_{R_0\rho}, \{e^{\overline{u_\epsilon}^{\gamma^\epsilon}}\}\right) = \emptyset,$$

and for all $0 < \rho < \frac{1}{2R_0}$, *for all* $r > 0$, *for all* $0 \le i \le t$, $1 \le j \le s_i$ *and for all subsequence* $\epsilon_m \to 0$ *as* $m \to \infty$,

$$X_r\left(\mathbb{D}^+_{\frac{1}{\rho}}, \xi, I_{\frac{1}{\rho}}, \{e^{\overline{u_{\epsilon_m}}^{\omega_i^{\epsilon_m}}}\}_{m\ge 0}\right) \cap \mathbb{D}_\rho^+(p_{i,j}) \ne \emptyset. \tag{5.5}$$

Proof. By contradiction, we assume that for all subsequence $\epsilon_m \to 0$ as $m \to +\infty$, for all $\{\omega_i^{\epsilon_m}\}_{m\ge 0}$ with $0 \le i \le t$ and

$$\alpha^\epsilon = \omega_0^\epsilon \lll \omega_1^\epsilon \lll \omega_2^\epsilon \lll \cdots \lll \omega_t^\epsilon \lll \omega_{t+1}^\epsilon = \delta_0,$$

for all families of points $p_{i,j} \in \mathbb{R}^2$ with $0 \le i \le t$ and $1 \le j \le s_i$ such that if $1 \le i \le t$, $p_{i,j} \in J_{\frac{1}{R_0}}$ and if $i = 0$, $p_{0,j} \in I_{R_0}$, with

$$s - 1 + \sum_{i=0}^{t} s_i \le k,$$

and

$$R_0 = \max\left\{\max_{1\le i\le t, 1\le j\le s_i}\left\{\max\left\{|p_{i,j}|, \frac{1}{|p_{i,j}|}\right\}\right\}, \max_{1\le j\le s_0}\{|p_{0,j}|\}, \delta_0\right\} + 1,$$

there exists $0 < \rho < \frac{1}{2R_0}$ such that for all $r > 0$, either there exists $1 \le i \le t$ such that

$$X_r\left(\mathbb{A}_\rho, \xi, J_\rho, \{e^{\overline{u_\epsilon}^{\omega_i^\epsilon}}\}\right) \setminus \bigcup_{j=1}^{s_i} \mathbb{D}_\rho^+(p_{i,j}) \ne \emptyset, \tag{5.6}$$

or

$$X_r\left(\mathbb{D}^+_{\frac{1}{\rho}}, \xi, I_{\frac{1}{\rho}}, \{e^{\hat{u}_\epsilon}\}\right) \setminus \bigcup_{j=1}^{s_0} \mathbb{D}_\rho^+(p_{0,j}) \ne \emptyset, \tag{5.7}$$

or there exists a sequence $\{\gamma_\epsilon\}$ such that $\frac{\omega_i^\epsilon}{\rho} < \gamma_\epsilon < \rho\omega_{i+1}^\epsilon$ for some $0 \leq i \leq t$, with

$$X_r\left(\mathbb{A}_{R_0\rho}, \xi, J_{R_0\rho}, \{e^{\overline{u_\epsilon}^{\gamma^\epsilon}}\}\right) \neq \emptyset. \tag{5.8}$$

With this assumption, we prove by induction the following property $\mathbf{H}_{\tilde{s}}$ for $s-1 \leq \tilde{s} \leq k+1$

$\mathbf{H}_{\tilde{s}}$: there exist sequences $\epsilon_m \to 0$ and $r_m \searrow 0$ as $m \to +\infty$, some scales

$$\alpha^\epsilon = \omega_0^\epsilon \ll \omega_1^\epsilon \ll \omega_2^\epsilon \ll \cdots \ll \omega_t^\epsilon \ll \omega_{t+1}^\epsilon = \delta_0,$$

some points $p_{i,j}^m \in \mathbb{R}_+^2 \setminus \{0\}$ and $p_{i,j} \in \mathbb{R} \times \{0\}$ if $1 \leq i \leq t$, $1 \leq j \leq s_i$; and $p_{0,j}^m \in \mathbb{R}^2$ and $p_{0,j} \in \mathbb{R} \times \{0\}$ if $1 \leq j \leq s_0$ with

$$s - 1 + \sum_{i=0}^{t} s_i = \tilde{s},$$

and $p_{i,j} \neq p_{i,j'}$ if $j \neq j'$ for $0 \leq i \leq t$, such that for all $0 \leq i \leq t$, $1 \leq j \leq s_i$, $p_{i,j}^m$ satisfies $\mathbf{P}_{r_m,\epsilon_m}$ in $\left(\mathbb{R}_+^2, \xi, \mathbb{R} \times \{0\}, \{e^{\overline{u_{\epsilon_m}}^{\omega_i^{\epsilon_m}}}\right)\}_{m \geq 0}$.

We already have $\mathbf{H}_{s-1}$, let's prove $\mathbf{H}_s$. We fix $\rho > 0$. By assumption, since we apply it with all s_i's equal to 0, either (5.8) or (5.7) happen. Let's study these two cases:

Case (5.8)$_{s-1}$: There exists a sequence $\{\gamma_{\epsilon_m}\}$ with $\frac{\alpha_{\epsilon_m}}{\rho} < \gamma_{\epsilon_m} < \rho\delta_0$ and some $x_m \in X_{2^{-m}}\left(\mathbb{A}_\rho, \xi, J_\rho, \{e^{\overline{u_\epsilon}^{\gamma^\epsilon}}\}\right)$. We choose ϵ_m such that x_m satisfies $\mathbf{P}_{2^{-m},\epsilon_m}$. It is clear that $\epsilon_m \to 0$ as $m \to \infty$.

- If $\frac{\gamma_{\epsilon_m}}{\alpha_{\epsilon_m}} \to +\infty$, we set a new scale $\omega_1^{\epsilon_m} = \gamma_{\epsilon_m}$ and $p_{1,1}^m = x_m \in \mathbb{A}_\rho$. Up to the extraction of a subsequence, $p_{1,1}^m \to p_{1,1} \in \mathbb{R}_+^2 \setminus \{0\}$ as $m \to +\infty$. It is clear by Claim 3 that $\omega_1^{\epsilon_m} \ll \delta_0$ up to reduce ρ. By the same arguments as in Claim 3, $p_{1,1} \in \mathbb{R} \times \{0\} \setminus \{0\}$ and we get $\mathbf{H}_s$ in this case.
- If $\frac{\gamma_{\epsilon_m}}{\alpha_{\epsilon_m}}$ is bounded, up to reduce ρ, one gets that (5.7) holds and we can go to Case (5.7)$_{s-1}$.

Case (5.7)$_{s-1}$: There exists $x_m \in X_{2^{-m}}\left(\mathbb{D}_{\frac{1}{\rho}}^+, \xi, I_{\frac{1}{\rho}}\{e^{\hat{u}_\epsilon}\}\right)$. We set $p_{0,1}^m = x_m$ and up to the extraction of a subsequence, $p_{0,1}^m \to p_{0,1}$ as $m \to +\infty$. By the same arguments as in Claim 3, $p_{0,1} \in \mathbb{R} \times \{0\}$ and we get $\mathbf{H}_s$ in this case.

Now, we assume that $\mathbf{H}_{\tilde{s}}$ is true for some $s \leq \tilde{s} \leq k$. Let's prove $\mathbf{H}_{\tilde{s}+1}$. We define all the parameters ϵ_m, r_m, $\omega_i^{\epsilon_m}$, $p_{i,j}^m$ and $p_{i,j}$ given by $\mathbf{H}_{\tilde{s}}$. We fix $\rho > 0$. By assumption, one of the assertions (5.6), (5.7) and (5.8) must happen. Let's study these three cases:

CASE $(5.6)_{\tilde{s}}$: Let $1 \leq i \leq t$ and $x_m \in X_{r_m}\left(\mathbb{A}_\rho, \xi, J_\rho, \{e^{\overline{u_{\epsilon_m}}^{\omega_i^{\epsilon_m}}}\}\right) \setminus \bigcup_{j=1}^{s_i} \mathbb{D}_\rho^+(p_{i,j})$. For $m \geq 0$, we set $p_{i,s_i+1}^m = x_m$ and we let $\epsilon_{\beta(m)}$ be such that p_{i,s_i+1}^m satisfies $\mathbf{P}_{r_m,\epsilon_{\beta(m)}}$. Since $r_m \searrow 0$, as $m \to +\infty$, setting $M(m) = \min\{m, \beta(m)\}$ gives that $p_{i,j}^{\beta(m)}$ satisfies $\mathbf{P}_{r_{M(m)},\epsilon_{\beta(m)}}$ for all $1 \leq j \leq s_i$ and p_{i,s_i+1}^m satisfies $\mathbf{P}_{r_{M(m)},\epsilon_{\beta(m)}}$. Up to the extraction of a subsequence, we can assume that $p_{i,s_i+1}^m \to p_{i,s_i+1}$ as $m \to +\infty$ and that $r_{M(m)} \searrow 0$ as $m \to +\infty$. Since $p_{i,s_i+1}^m \in \mathbb{A}_\rho \setminus \bigcup_{j=1}^{s_i} \mathbb{D}_\rho^+(p_{i,j})$, $p_{i,s_i+1} \in \mathbb{R}_+^2 \setminus \{0, p_{i,1} \cdots, p_{i,s_i}\}$. By the same arguments as in Claim 3, $p_{i,s_{i+1}} \in \mathbb{R} \times \{0\} \setminus \{0\}$ and the proof of $\mathbf{H}_{\tilde{s}+1}$ is complete in this case.

CASE $(5.7)_{\tilde{s}}$: The proof of $\mathbf{H}_{\tilde{s}+1}$ is the same as in $(5.6)_{\tilde{s}}$.

CASE $(5.8)_{\tilde{s}}$: Let $\{\gamma_{\epsilon_m}\}$ be a sequence such that $\frac{\omega_i^{\epsilon_m}}{\rho} < \gamma_{\epsilon_m} < \rho\omega_{i+1}^{\epsilon_m}$ and $x_m \in X_{r_m}\left(\mathbb{A}_{R_0\rho}, \xi, J_{R_0\rho}, \{e^{\overline{u_{\epsilon_q}}^{\gamma_{\epsilon_q}}}\}_{q\geq 0}\right)$.

- If $\frac{\gamma_{\epsilon_m}}{\omega_i^{\epsilon_m}} \to +\infty$ and $\frac{\gamma_{\epsilon_m}}{\omega_{i+1}^{\epsilon_m}} \to 0$, we define a new scale $\omega_{t+1}^{\epsilon_m} = \gamma_{\epsilon_m}$ and $p_{t+1,1}^m = x_m$. Up to the extraction of a subsequence, $p_{t+1,1}^m \in \mathbb{A}_\rho$ satisfies $\mathbf{P}_{r_m,\epsilon_m}$, $p_{t+1,1}^m \to p_{t+1,1} \in \mathbb{R}_+^2 \setminus \{0\}$ and $r_m \searrow 0$ as $m \to +\infty$. By the same arguments as in Claim (3), $p_{t+1,1} \in \mathbb{R}\times\{0\}\setminus\{0\}$. Up to reorder $\{\omega_i^{\epsilon_m}\}$, we get $\mathbf{H}_{\tilde{s}+1}$ in this case.
- If $i = 0$ and $\frac{\gamma_{\epsilon_m}}{\omega_0^{\epsilon_m}}$ is bounded, up to reduce ρ, we get that (5.7) holds and go back to Case $(5.7)_{\tilde{s}}$.
- The case $i = t$ and $\frac{\omega_{t+1}^{\epsilon_m}}{\gamma_{\epsilon^m}}$ is bounded leads to a contradiction by Claim 3.
- The other cases lead to the fact that (5.6) holds up to reduce ρ and we are back to Case $(5.6)_{\tilde{s}}$.

Gathering the three cases, we deduce $\mathbf{H}_{\tilde{s}+1}$. Therefore, $\mathbf{H}_{k+1}$ holds true and we now prove that this leads to a contradiction. We will define new test functions for the variational characterization of $\sigma_\epsilon = \sigma_k(M, g, \partial M, e^{u_\epsilon})$, $\eta_{i,j}^m$ for $0 \leq i \leq t$, $1 \leq j \leq s_i$.

- If $p_{i,j}^m$ satisfies $\mathbf{A}_{r_m,\epsilon_m}$, $\eta_{i,j}^m$ is defined by the extension by 0 in $M \setminus \Omega_{i,j}^m$ of an eigenfunction for $\sigma_\star\left(\Omega_{i,j}^m, g, \Gamma_{i,j}^m, e^{u_{\epsilon_m}}\right)$, where $\Omega_{i,j}^m \subset M$ and $\Gamma_{i,j}^m \subset \partial M$ are defined by $\mathbb{D}_{r_m}^+(p_{i,j}^m) = \overline{\Omega_{i,j}^m}^{\omega_i^{\epsilon_m}}$ and $I_{r_m}(p_{i,j}^m) = \overline{\Gamma_{i,j}^m}^{\omega_i^{\epsilon_m}}$.
- If $p_{i,j}^m$ does not satisfy $\mathbf{A}_{r_m,\epsilon_m}$, it satisfies $\mathbf{B}_{r_m,\epsilon_m}$ and $\eta_{i,j}^m$ is defined by an eigenfunction for $\sigma_\star\left(D_{i,j}^m, g, \Gamma_{i,j}^m, e^{u_{\epsilon_m}}\right)$ extended by 0 in $M \setminus D_{i,j}^m$, where $D_{i,j}^m \subset M$ is the Nodal domain of an eigenfunction associated to σ_{ϵ_m} such that $\overline{D_{i,j}^m}^{\omega_i^{\epsilon_m}} \subset \mathbb{D}_{r_m}^+(p_{i,j}^m)$ and $\Gamma_{i,j}^m = D_{i,j}^m \cap \partial M$.

We also use the functions η_i^m for $\{1 \leq i \leq s\}$, already defined in the proof of Claim 3. Note that these $k+1$ functions have pairwise disjoint support for m large enough. Then, by (1.1),

$$\sigma_{\epsilon_m} \leq \max_{\substack{0\leq i\leq t \\ 1\leq j\leq s_i \\ l\neq i_0}} \left\{ \frac{\int_M \left|\nabla \eta_{i,j}^m\right|_g^2 dv_g}{\int_{\partial M} \left(\eta_{i,j}^m\right)^2 e^{u_{\epsilon_m}} d\sigma_g}, \frac{\int_M \left|\nabla \eta_l^m\right|_g^2 dv_g}{\int_{\partial M} \left(\eta_l^m\right)^2 e^{u_{\epsilon_m}} d\sigma_g} \right\} \leq \sigma_{\epsilon_m}.$$

The last inequality comes from the definition of the properties **A** and **B** and we have equality if and only if one of the test functions is an eigenfunction for $\sigma_{\epsilon_m} = \sigma_k(M, g, \partial M, e^{u_{\epsilon_m}})$. This test function is a non-zero harmonic function which vanishes on an open set of the surface. This is absurd.

Therefore, we proved the first part of the claim. Up to making successive extractions of subsequences of $\{\epsilon_m\}$ and up to removing some points $p_{i,j}$, one easily proves that the last condition (5.5) also holds. q.e.d.

For $\rho > 0$, we set

$$\Omega(\rho) = \mathbb{D}^+_{\frac{1}{\rho}} \setminus \bigcup_{j=1}^{s_0} \mathbb{D}^+_\rho(p_{0,j}) \text{ and } \Gamma(\rho) = I_{\frac{1}{\rho}} \setminus \bigcup_{j=1}^{s_0} I_\rho(p_{0,j}).$$

As previously remarked, the set of concentration points of $\{e^{\hat{u}_\epsilon} ds\}$ satisfies

$$Z(\mathbb{R} \times \{0\}, \{e^{\hat{u}_\epsilon} dx\}) \subset \{p_{0,1}, \cdots, p_{0,s_0}\}, \tag{5.9}$$

and letting

$$m_i(\rho) = \lim_{\epsilon\to 0} \int_{\Gamma(\rho)} e^{\hat{u}_\epsilon} ds,$$

we have that $m_i(\rho) \geq m_i + o(1) > 0$ since we have (4.5), (4.7), (5.9) and $m_i > 0$. We aim at getting regularity estimates on $\hat{\Phi}_\epsilon$ and $e^{\hat{u}_\epsilon}$ in $\Omega(\rho)$. We follow the proof of Claim 4, thanks to the fact that $m_i(\rho) > 0$ for ρ small enough.

Claim 10. *We have the following*

- *Estimates on $\hat{\Phi}_\epsilon$:*

$$\forall \rho > 0, \exists C_1(\rho) > 0, \forall \epsilon > 0, \left\|\hat{\Phi}_\epsilon\right\|_{W^{1,2}(\Omega(\rho))} \leq C_1(\rho), \tag{5.10}$$

$$\forall \rho > 0, \exists C_2(\rho) > 0, \forall \epsilon > 0, \left\|\hat{\Phi}_\epsilon\right\|_{C^0(\Omega(\rho))} \leq C_2(\rho). \tag{5.11}$$

- *Quantitative non-concentration estimates on $e^{2\hat{u}_\epsilon}$ and $|\nabla \hat{\Phi}_\epsilon|^2$*

$$\forall \rho > 0, \exists D_1(\rho) > 0, \forall r > 0, \limsup_{\epsilon\to 0} \sup_{x\in\Gamma(\rho)} \int_{I_r(x)} e^{\hat{u}_\epsilon} \leq \frac{D_1(\rho)}{\ln(\frac{1}{r})}, \tag{5.12}$$

(5.13)
$$\forall \rho > 0, \exists D_2(\rho) > 0, \forall r > 0, \limsup_{\epsilon \to 0} \sup_{x \in \Gamma(\rho)} \int_{\mathbb{D}_r^+(x)} |\nabla \hat{\Phi}_\epsilon|^2 \leq \frac{D_2(\rho)}{\sqrt{\ln(\frac{1}{r})}}.$$

Proof. The proof of (5.10) follows exactly the proof of (3.5) in Claim 4, using the fact that $m_0^i(\rho) > 0$ for ρ small enough and Claim 9.

For the proof of (5.11), we first prove that

$$\forall \rho > 0, \exists C_0(\rho) > 0, \forall \epsilon > 0, \left\| \hat{\Phi}_\epsilon \right\|_{C^0(\Gamma(\rho))} \leq C_0(\rho). \tag{5.14}$$

We now prove (5.14). Let $0 \leq i \leq n$. Up to change $\hat{\phi}_\epsilon^i$ into $-\hat{\phi}_\epsilon^i$, there exists a subsequence $\{x_\epsilon\}$ of points in $\Gamma(\rho)$ such that $\hat{\phi}_\epsilon^i(x_\epsilon) = \sup_{\Gamma(\rho)} \left|\hat{\phi}_\epsilon^i\right|$. We set $\delta_\epsilon = d_\xi(x_\epsilon, supp(\hat{\nu}_\epsilon))$ and we let $y_\epsilon \in supp(\hat{\nu}_\epsilon)$ be such that $\delta_\epsilon = |x_\epsilon - y_\epsilon|$. We divide the proof into 3 cases:

Case 1 – $\delta_\epsilon^{-1} = O(1)$. Then, $\{e^{\hat{u}_\epsilon}\}$ is uniformly bounded in $I_{\min\left(\frac{\delta_\epsilon}{2}, \frac{\rho}{2}\right)}(x_\epsilon)$ by estimates on the heat kernel (see (1.6)). By (5.10), $\hat{\phi}_\epsilon^i$ is bounded in $L^2\left(\Gamma\left(\frac{\rho}{2}\right)\right)$. By elliptic theory for the Dirichlet-to-Neumann operator (see [**25**], Chapter 7.11, page 37), $\hat{\phi}_\epsilon^i$ is bounded in $W^{1,2}\left(\Gamma\left(\frac{\rho}{2}\right)\right)$ (see (5.1)), and $\{\hat{\phi}_\epsilon^i(x_\epsilon)\}$ is bounded by Sobolev embeddings.

Case 2 – $\delta_\epsilon = O\left(\frac{\sqrt{\epsilon}}{\alpha_\epsilon}\right)$. Using Claim 2, we get that $\{\hat{\phi}_\epsilon^i(x_\epsilon)\}$ is bounded.

Case 3 – $\delta_\epsilon \to 0$ as $\epsilon \to 0$ and $\frac{\sqrt{\epsilon}}{\alpha_\epsilon \delta_\epsilon} \to 0$ as $\epsilon \to 0$. We set

$$e^{w_\epsilon(x)} = \delta_\epsilon e^{2\hat{u}_\epsilon(x_\epsilon + \delta_\epsilon x)},$$
$$\psi_\epsilon(x) = \phi_\epsilon^i(x_\epsilon + \delta_\epsilon x),$$
$$z_\epsilon = \frac{1}{\delta_\epsilon}(x_\epsilon - y_\epsilon),$$

so that

$$\begin{cases} \Delta \psi_\epsilon = 0 & \text{in } \mathbb{D}_5^+, \\ \partial_t \psi_\epsilon = -\sigma_\epsilon e^{w_\epsilon} \psi_\epsilon & \text{on } I_5. \end{cases} \tag{5.15}$$

Up to the extraction of a subsequence, there is $z_0 \in \mathbb{R} \times \{0\}$ with $|z_0| = 1$ such that $z_\epsilon \to z_0$ as $\epsilon \to 0$. By estimates (1.6), there is $D_1 > 0$ such that

$$e^{w_\epsilon} \leq D_1 \text{ in } I_{\frac{1}{2}}.$$

By Claim 2, since $y_\epsilon \in supp(\hat{\nu}_\epsilon)$, $\psi_\epsilon(z_\epsilon) = O(1)$ as $\epsilon \to 0$.

We first assume that ψ_ϵ does not vanish in $\mathbb{D}_3^+(0)$. Since $\psi_\epsilon(0) > 0$, $\psi_\epsilon > 0$ in $\mathbb{D}_3^+(0)$. Then, by Harnack's inequality, we get $D_2 > 0$ such that

$$\forall x \in \mathbb{D}_{\frac{1}{4}}^+, \psi_\epsilon(x) \geq D_2 \psi_\epsilon(0).$$

Since ψ_ϵ is positive, ψ_ϵ is weakly superharmonic in $\mathbb{D}^+_{|z_\epsilon|}(z_\epsilon) \subset \mathbb{D}^+_3(0)$ by (5.15) so that

$$\psi_\epsilon(z_\epsilon) \geq \frac{1}{\pi |z_\epsilon|} \int_{\partial \mathbb{D}^+_{|z_\epsilon|}(z_\epsilon)} \psi_\epsilon d\sigma,$$

and keeping the part of the integral which lies in $\mathbb{D}^+_{\frac{1}{4}}$, we get a constant $D_3 > 0$ such that $\psi_\epsilon(z_\epsilon) \geq D_3 \psi_\epsilon(0)$. We conclude that $\phi^i_\epsilon(x_\epsilon) = \psi_\epsilon(0)$ is bounded.

We now assume that ψ_ϵ vanishes in $\mathbb{D}^+_3(0)$. $X_r(\Omega(\rho), \xi, \Gamma(\rho), e^{\hat{u}_\epsilon}) = \emptyset$ by Claim 9, then ψ_ϵ vanishes in $\mathbb{D}^+_4(0)$ on a piecewise smooth curve between two points of distance greater than 1. By the corollary of Theorem 4, Section 1.3, on $\Omega = \mathbb{D}^+_5(0)$ we get some constant $C_1 > 0$ such that

$$\int_{\mathbb{D}^+_4(0)} \psi_\epsilon^2 dx \leq C_1 \int_{\mathbb{D}^+_5(0)} |\nabla \psi_\epsilon|^2 dx.$$

By elliptic estimates on (5.15), $\{\psi_\epsilon\}$ is uniformly bounded on $\mathbb{D}^+_{\frac{1}{4}}(0)$ and $\phi^i_\epsilon(x_\epsilon) = \psi_\epsilon(0) = O(1)$.

We now prove (5.11). Let $\rho > 0$ and $0 \leq i \leq n$. Since $\hat{\phi}^i_\epsilon$ is harmonic in $\Omega(\frac{\rho}{2})$, by elliptic regularity theory, there exists a constant $K_0(\rho) > 0$ such that

$$\left\| \hat{\phi}^i_\epsilon \right\|_{\mathcal{C}^0(\Omega(\rho))} \leq K_0(\rho) \left(\left\| \hat{\phi}^i_\epsilon \right\|_{L^2\left(\Omega\left(\frac{\rho}{2}\right)\right)} + \left\| \hat{\phi}^i_\epsilon \right\|_{\mathcal{C}^0\left(\Gamma\left(\frac{\rho}{2}\right)\right)} \right),$$

and setting $C_2(\rho) = K_0(\rho)\left(C_1\left(\frac{\rho}{2}\right) + C_0\left(\frac{\rho}{2}\right)\right)$ gives (5.11).

As in the proof of Claim 4, Claim 9 gives some capacity estimates and we get (5.12), and (5.13) is a consequence of (5.11), (5.12) and equation (5.1). q.e.d.

We now need an estimate of $\{\Phi_\epsilon\}$ on the whole surface in order to prove later that no energy is lost in the necks.

Claim 11. *For any $\rho > 0$, there exists a constant $C_0(\rho) > 0$ such that*

$$\forall x \in M \setminus \left(\bigcup_{i \neq i_0} B_g(p_i, \rho) \cup \bigcup_{i=0}^{t} \bigcup_{j=1}^{s_i} \Omega_{i,j} \right),$$
$$|\Phi_\epsilon(x)| \leq C_0(\rho) \left(\ln\left(1 + \frac{d_g(x, \bar{a}_\epsilon)}{\alpha_\epsilon}\right) + 1 \right),$$

where

$$\widetilde{\Omega}^l_{i,j} = \omega^\epsilon_i \mathbb{D}_\rho(p_{i,j}) + a_\epsilon \text{ and } \bar{a}_\epsilon = \exp^{-1}_{g_l, x_l}(a_\epsilon).$$

Proof. Let $0 < \rho < \frac{1}{20R_0}$ and let $r > 0$ which satisfies the conclusion of Claim 9 for this ρ.

STEP 1: We prove that for $0 \le i \le t$, there exists $A_i(\rho) > 0$ such that for all $0 \le \beta \le n$, for all sequence $\{\gamma_\epsilon\}$ with $\frac{\omega_i^\epsilon}{\rho} \le \gamma_\epsilon \le \rho \omega_{i+1}^\epsilon$, either

$$\forall x \in \mathbb{A}_{12R_0\rho}, \left|\overline{\phi_\epsilon^\beta}^{\gamma_\epsilon}(x)\right| \le A_i(\rho),$$

or

$$\forall x, y \in \mathbb{A}_{12R_0\rho}, \frac{\left|\overline{\phi_\epsilon^\beta}^{\gamma_\epsilon}(y)\right|}{A_i(\rho)} \le \left|\overline{\phi_\epsilon^\beta}^{\gamma_\epsilon}(x)\right| \le A_i(\rho)\left|\overline{\phi_\epsilon^\beta}^{\gamma_\epsilon}(y)\right|.$$

We let $\widetilde{A_i}(\rho)$ be equal to

$$\max_{0\le\beta\le n} \sup_{\substack{\epsilon>0 \\ \frac{\omega_i^\epsilon}{\rho}<\gamma_\epsilon<\rho\omega_{i+1}^\epsilon}} \left\{ \min\left\{ \max_{x\in J_{10R_0\rho}} \left|\overline{\phi_\epsilon^\beta}^{\gamma_\epsilon}(x)\right|, \max_{x,y\in\mathbb{A}_{10R_0\rho}} \frac{\left|\overline{\phi_\epsilon^\beta}^{\gamma_\epsilon}(x)\right|}{\left|\overline{\phi_\epsilon^\beta}^{\gamma_\epsilon}(y)\right|} \right\} \right\},$$

where we recall that for $r > 0$ $J_r = \mathbb{A}_r \cap \mathbb{R} \times \{0\}$. We assume by contradiction that $\widetilde{A_i}(\rho) = +\infty$. Then there exist $0 \le \beta \le n$, $\frac{\omega_i^{\epsilon_m}}{\rho} < \gamma_{\epsilon_m} < \rho\omega_{i+1}^\epsilon$ such that $\epsilon_m \to 0$ as $m \to +\infty$ and

$$\min\left\{ \max_{x\in J_{10R_0\rho}} \left|\overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x)\right|, \max_{x,y\in\mathbb{A}_{10R_0\rho}} \frac{\left|\overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x)\right|}{\left|\overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(y)\right|} \right\} \to +\infty$$

as $m \to +\infty$. Let $x_m \in J_{10R_0\rho}$ be such that

$$\left|\overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x_m)\right| = \max_{x\in J_{10R_0\rho}} \left|\overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x)\right|.$$

We set $\delta_m = d(x_m, supp\,(\overline{\nu_{\epsilon_m}}^{\gamma_{\epsilon_m}}))$ and take $y_m \in supp\,(\overline{\nu_{\epsilon_m}}^{\gamma_{\epsilon_m}})$ such that $|x_m - y_m| = \delta_m$. We study 3 cases each one leading to a contradiction.

CASE 1 – $\delta_m = O\left(\frac{\sqrt{\epsilon_m}}{\gamma_{\epsilon_m}}\right)$. We apply Claim 2 for the sequence of points $\{\exp_{g_l,x_l}(\gamma_{\epsilon_m} x_m + a_{\epsilon_m})\}_m$ in ∂M and we get a contradiction.

CASE 2 – $\delta_m \to 0$ and $\frac{\delta_m\gamma_{\epsilon_m}}{\sqrt{\epsilon_m}} \to +\infty$ as $m \to +\infty$. We set

$$e^{w_m} = \delta_m e^{\overline{u_{\epsilon_m}}^{\gamma_{\epsilon_m}}}(x_m + \delta_m x),$$
$$\psi_m = \overline{\phi_{\epsilon_m}^\beta}^{\gamma_{\epsilon_m}}(x_m + \delta_m x), \text{ and}$$
$$z_m = \frac{1}{\delta_m}(y_m - x_m),$$

so that

$$\begin{cases} \Delta\psi_m = 0, \\ \partial_t\psi_m = -\sigma_{\epsilon_m} e^{w_m}\psi_m. \end{cases}$$

Up to the extraction of a subsequence, there is $z_0 \in \mathbb{R}\times\{0\}$ with $|z_0| = 1$ such that $z_m \to z_0$ as $m \to +\infty$. By (1.6), there is $D_1 > 0$ such that

$$e^{2w_m} \le D_1 \text{ on } I_{\frac{1}{2}}.$$

By Claim 2, since $y_m \in supp\,(\overline{\nu_{\epsilon_m}}^{\gamma_{\epsilon_m}})$, $\psi_m(z_m) = O(1)$ as $m \to +\infty$.

We first assume that ψ_m does not vanish in $\mathbb{D}_3^+(0)$. Up to take $-\psi_m$, we can assume that $\psi_m > 0$ on $\mathbb{D}_3^+(0)$. Then, by Harnack inequality, we get $D_2 > 0$ such that

$$\forall x \in \mathbb{D}^+_{\frac{1}{4}}, \psi_m(x) \geq D_2 \psi_m(0).$$

Since ψ_m is positive, ψ_m is weakly superharmonic in $\mathbb{D}^+_{|z_m|}(z_m) \subset \mathbb{D}_3^+(0)$. Then,

$$\psi_m(z_m) \geq \frac{1}{\pi\,|z_m|} \int_{\partial \mathbb{D}_{|z_m|}(z_m)} \psi_m d\sigma,$$

and keeping the part of the integral which lies in $\mathbb{D}^+_{\frac{1}{4}}$, we get a constant $D_3 > 0$ such that $\psi_m(z_m) \geq D_3 \psi_m(0)$. We conclude that $\overline{\phi^\beta_{\epsilon_m}}^{\gamma_{\epsilon_m}}(x_m) = \psi_m(0) = O(1)$ which is absurd.

We assume now that ψ_m vanishes in $\mathbb{D}_3^+(0)$. By Claim 9, ψ_m vanishes in $\mathbb{D}_4^+(0)$ on a piecewise smooth curve between two points of distance greater than 1. By the corollary of Theorem 4 on $\Omega = \mathbb{D}_5^+(0)$, we get a Poincaré inequality

$$\int_{\mathbb{D}_4^+(0)} \psi_m^2 dx \leq C_1 \int_{\mathbb{D}_5^+(0)} |\nabla \psi_m|^2 dx.$$

By elliptic regularity theory, ψ_m is uniformly bounded on $\mathbb{D}^+_{\frac{1}{4}}(0)$ and $\overline{\phi^\beta_{\epsilon_m}}^{\gamma_{\epsilon_m}}(x_m) = \psi_m(0) = O(1)$ which is absurd.

Case 3 – $\frac{1}{\delta_m} = O(1)$. Up to the extraction of a subsequence, we assume that $x_m \to x$ in $J_{10R_0\rho}$ as $m \to +\infty$.

We first assume that $\psi_m := \overline{\phi^\beta_{\epsilon_m}}^{\gamma_{\epsilon_m}}$ vanishes in $\mathbb{A}_{5R_0\rho}$. We get by Claim 9 and the corollary of Theorem 4 on $\Omega = \mathbb{A}_{2R_0\rho}$, a constant $C_r > 0$ such that

$$\int_{\mathbb{A}_{4R_0\rho}} \psi_m^2 dx \leq C_r \int_{\mathbb{A}_{2R_0\rho}} |\nabla \psi_m|^2 dx.$$

By (1.6), there are some constants $\tilde{r} > 0$ and $D_1 > 0$ such that

$$e^{\overline{u_{\epsilon_m}}^{\gamma_{\epsilon_m}}} \leq D_1 \text{ on } I_{\tilde{r}}(x).$$

By elliptic estimates, $\{\psi_m\}$ is uniformly bounded on $\mathbb{A}_{5R_0\rho} \cap \mathbb{D}_{\frac{\tilde{r}}{2}}(x)$ which gives a contradiction.

We assume now that $\psi_m := \overline{\phi^\beta_{\epsilon_m}}^{\gamma_{\epsilon_m}}$ does not vanish in $\mathbb{A}_{5R_0\rho}$. Up to take $-\psi_m$, we assume that $\psi_m > 0$ on $\mathbb{A}_{5R_0\rho}$.

Let's assume that $y_m \to y$ as $m \to +\infty$ with $y \in J_{7R_0\rho}$. By Claim 2, $\psi_m(y_m) = O(1)$. By (1.6), there exists a constant $D_1 > 0$ such that

$$e^{\overline{u_{\epsilon_m}}^{\gamma_{\epsilon_m}}} \leq D_1 \text{ in } I_{\delta - \tilde{\delta}}(x),$$

where $\tilde{\delta} = \min\left(\frac{\delta}{4}, \frac{R_0\rho}{4}\right)$. By Harnack's inequality, there exists $D_2 > 0$ such that

$$\forall z \in \mathbb{A}_{6R_0\rho} \cap \mathbb{D}^+_{\delta-2\tilde{\delta}}(x), \psi_m(x_m) \leq D_2\psi_m(z).$$

By weak superharmonicity on $\mathbb{D}^+_{3\tilde{\delta}}(y_m) \subset \mathbb{A}_{5R_0\rho}$,

$$\psi_m(y_m) \geq \frac{1}{\pi \times 3\tilde{\delta}} \int_{\partial\mathbb{D}^+_{3\tilde{\delta}}(y_m)} \psi_m d\sigma.$$

We keep the part of the integral which lies in $\mathbb{A}_{6R_0\rho} \cap \mathbb{D}^+_{\delta-2\tilde{\delta}}$. Since the length of $\partial\mathbb{D}^+_{3\tilde{\delta}}(y_m) \cap \mathbb{A}_{6R_0\rho} \cap \mathbb{D}^+_{\delta-2\tilde{\delta}}$ is uniformly bounded from below, we get a constant $D_3 > 0$ such that $\psi_m(y_m) \geq D_3\psi_m(x_m)$. Then, $\overline{\phi^{\beta}_{\epsilon_m}}^{\gamma_{\epsilon_m}}(x_m) = \psi_m(x_m) = O(1)$ which is absurd.

Assume now that $y_m \in \mathbb{R} \times \{0\} \setminus J_{8R_0\rho}$. By (1.6), there is a constant $D_1 > 0$ such that

$$e^{\overline{u_{\epsilon_m}}^{\gamma_{\epsilon_m}}} \leq D_1 \text{ in } \mathbb{A}_{9R_0\rho}.$$

By Harnack inequality, there exists a constant $C_1 > 0$ such that

$$\forall z, \tilde{z} \in \mathbb{A}_{10R_0\rho}, \frac{\left|\overline{\phi_{\epsilon_m}}^{\gamma_{\epsilon_m}}\right|(\tilde{z})}{C_1} \leq \left|\overline{\phi_{\epsilon_m}}^{\gamma_{\epsilon_m}}\right|(x_m) \leq C_1 \left|\overline{\phi_{\epsilon_m}}^{\gamma_{\epsilon_m}}\right|(z),$$

which also leads to a contradiction.

We get $\widetilde{A_i}(\rho) < +\infty$. We now let $A_i(\rho)$ be equal to

$$\max_{0\leq\beta\leq n} \sup_{\substack{\epsilon>0 \\ \frac{\omega_i^{\epsilon}}{\rho}<\gamma_\epsilon<\rho\omega_{i+1}^{\epsilon}}} \left\{ \min\left\{ \max_{x\in\mathbb{A}_{12R_0\rho}} \left|\overline{\phi^{\beta}_{\epsilon}}^{\gamma_\epsilon}(x)\right|, \max_{x,y\in\mathbb{A}_{12R_0\rho}} \frac{\left|\overline{\phi^{\beta}_{\epsilon}}^{\gamma_\epsilon}(x)\right|}{\left|\overline{\phi^{\beta}_{\epsilon}}^{\gamma_\epsilon}(y)\right|} \right\}\right\},$$

and we assume by contradiction that $A_i(\rho) = +\infty$. Let γ_{ϵ_m} with $\frac{\omega_i^{\epsilon_m}}{\rho} \leq \gamma_{\epsilon_m} \leq \rho\omega_{i+1}^{\epsilon_m}$ and $\epsilon_m \to 0$ as $m \to +\infty$ be such that

$$\min\left\{ \max_{x\in\mathbb{A}_{12R_0\rho}} \left|\overline{\phi^{\beta}_{\epsilon_m}}^{\gamma_{\epsilon_m}}(x)\right|, \max_{x,y\in\mathbb{A}_{12R_0\rho}} \frac{\left|\overline{\phi^{\beta}_{\epsilon_m}}^{\gamma_{\epsilon_m}}(x)\right|}{\left|\overline{\phi^{\beta}_{\epsilon_m}}^{\gamma_{\epsilon_m}}(y)\right|} \right\} \to +\infty$$

as $m \to +\infty$. Then, by elliptic estimates there is some constant $K(\rho)$ such that

$$\max_{x\in\mathbb{A}_{12R_0\rho}} \left|\overline{\phi^{\beta}_{\epsilon_m}}^{\gamma_{\epsilon_m}}(x)\right| \leq K(\rho)\left(\max_{J_{10R_0\rho}} \left|\overline{\phi^{\beta}_{\epsilon_m}}^{\gamma_{\epsilon_m}}\right| + \left\|\overline{\phi^{\beta}_{\epsilon_m}}^{\gamma_{\epsilon_m}}\right\|_{L^2(\mathbb{A}_{10R_0\rho})}\right),$$

so that since $\widetilde{A_i}(\rho) < +\infty$,

$$\left\|\overline{\phi^{\beta}_{\epsilon_m}}^{\gamma_{\epsilon_m}}\right\|_{L^2(\mathbb{A}_{10R_0\rho})} \to +\infty \text{ as } m \to +\infty.$$

By Poincaré inequalities given by the corollary of Theorem 4 on $\Omega = \mathbb{A}_{5R_0\rho}$, and by Claim 9, we clearly have that $\overline{\phi_{\epsilon_m}^{\beta}}^{\gamma_{\epsilon_m}}$ does not vanish in $\mathbb{A}_{5R_0\rho}$ and by Harnack inequalities,

$$\sup_{m\geq 0} \max_{x,y\in\mathbb{A}_{10R_0\rho}} \frac{\left|\overline{\phi_{\epsilon_m}^{\beta}}^{\gamma_{\epsilon_m}}(x)\right|}{\left|\overline{\phi_{\epsilon_m}^{\beta}}^{\gamma_{\epsilon_m}}(y)\right|} < +\infty,$$

which contradicts the fact that $A_i(\rho) = +\infty$. Then $A_i(\rho) < +\infty$ and we get Step 1.

STEP 2: We have that for $1 \leq i \leq t$, there exists $B_i(\rho) > 0$ such that for all $0 \leq \beta \leq n$, either

$$\forall x \in \mathbb{A}_\rho \setminus \bigcup_{j=1}^{s_i} \mathbb{D}_\rho^+(p_{i,j}), \left|\overline{\phi_\epsilon^{\beta}}^{\omega_i^\epsilon}(x)\right| \leq B_i(\rho),$$

or

$$\forall x,y \in \mathbb{A}_\rho \setminus \bigcup_{j=1}^{s_i} \mathbb{D}_\rho^+(p_{i,j}), \frac{\left|\overline{\phi_\epsilon^{\beta}}^{\omega_i^\epsilon}(y)\right|}{B_i(\rho)} \leq \left|\overline{\phi_\epsilon^{\beta}}^{\omega_i^\epsilon}(x)\right| \leq B_i(\rho)\left|\overline{\phi_\epsilon^{\beta}}^{\omega_i^\epsilon}(y)\right|,$$

and there exists $B_{t+1}(\rho) > 0$ such that for all $0 \leq \beta \leq n$, either

$$\forall x \in M \setminus \bigcup_{i=1}^{s} B_g(p_i,\rho), \left|\phi_\epsilon^\beta(x)\right| \leq B_{t+1}(\rho),$$

or

$$\forall x,y \in M \setminus \bigcup_{i=1}^{s} B_g(p_i,\rho), \frac{\left|\phi_\epsilon^\beta(y)\right|}{B_{t+1}(\rho)} \leq \left|\phi_\epsilon^\beta(x)\right| \leq B_{t+1}(\rho)\left|\phi_\epsilon^\beta(y)\right|.$$

The proof of Step 2 follows exactly that of Step 1. Notice that if $m_0(\rho) > 0$, the third inequality holds by Claim 4. We leave the details to the reader.

STEP 3: We prove that there exists $K_i(\rho) > 0$ such that for $0 \leq i \leq t$, and for all $x \in \mathbb{D}^+_{\tau_{i+1}^\epsilon} \setminus \mathbb{D}^+_{t_i^\epsilon}$,

$$|F_\epsilon|(x) \leq K_i(\rho)\left\{\max_{\partial\mathbb{D}^+_{t_i^\epsilon}} |F_\epsilon| + \ln\left(\frac{|x|}{t_i^\epsilon}\right)\right\}, \tag{5.16}$$

where $t_i^\epsilon = 12R_0\omega_i^\epsilon$, $\tau_{i+1}^\epsilon = \frac{\omega_{i+1}^\epsilon}{12R_0}$ and $F_\epsilon(x) = \widetilde{\Phi}_\epsilon^{\,l}(a_\epsilon + x)$.

Let $0 \leq \beta \leq n$. We set

$$N_i^\epsilon = \{t_i^\epsilon \leq t \leq \tau_i^\epsilon; \exists x \in \mathbb{R}^2, |x| = t \text{ and } F_\epsilon(x) = 0\}.$$

Then, by the Courant Nodal theorem, N_i^ϵ has a finite number of connected components, bounded by $k+1$, since each connected component adds at least one nodal domain for the eigenfunction Φ_ϵ^β. By Step 1, we clearly have that

$$\forall x \in \mathbb{R}^2; |x| \in N_i^\epsilon \Rightarrow \left| F_\epsilon^\beta(x) \right| \leq A_i(\rho). \tag{5.17}$$

We let

$$c_{i,1}^\epsilon < d_{i,1}^\epsilon < c_{i,2}^\epsilon < d_{i,2}^\epsilon < \cdots < c_{i,q_\epsilon}^\epsilon < d_{i,q_\epsilon}^\epsilon$$

be such that

$$N_i^\epsilon = [t_i^\epsilon, \tau_i^\epsilon] \setminus \bigcup_{j=1}^{q_\epsilon}]c_{i,j}^\epsilon, d_{i,j}^\epsilon[$$

with $\{q_\epsilon\}$ a bounded sequence of integers. Let $1 \leq j \leq q_\epsilon$. Then, F_ϵ^β does not vanish on $\mathbb{D}^+_{d_{i,j}^\epsilon} \setminus \mathbb{D}^+_{c_{i,j}^\epsilon}$, and we can assume that $F_\epsilon^\beta > 0$ up to take $-F_\epsilon^\beta$. By the eigenvalue equation, F_ϵ^β is then weakly superharmonic on $\mathbb{D}^+_{d_{i,j}^\epsilon} \setminus \mathbb{D}^+_{c_{i,j}^\epsilon}$. We set

$$f_\epsilon(u) = \frac{\int_{\partial \mathbb{D}_u^+} F_\epsilon^\beta(x) d\sigma(x)}{\pi u}.$$

Then,

$$\begin{aligned} f_\epsilon'(u) &= \frac{\int_{\partial \mathbb{D}_u} \partial_\nu F_\epsilon^\beta(x) d\sigma(x)}{\pi u} \\ &= \frac{-\int_{\mathbb{D}_u} \Delta F_\epsilon^\beta(x) dx + \int_{I_u} \partial_t F_\epsilon^\beta(s,0) ds}{\pi u} \\ &= \frac{\int_{I_{c_{i,j}^\epsilon}} \partial_t F_\epsilon^\beta(s,0) ds + \int_{I_u \setminus I_{c_{i,j}^\epsilon}} \partial_t F_\epsilon^\beta(s,0) ds}{\pi u}, \end{aligned}$$

so that

$$\begin{aligned} f_\epsilon(u) &= f_\epsilon(c_{i,j}^\epsilon) + \frac{\int_{I_{c_{i,j}^\epsilon}} \partial_t F_\epsilon^\beta(s,0) ds}{\pi} \ln\left(\frac{u}{c_{i,j}^\epsilon} \right) \\ &\quad + \int_{c_{i,j}^\epsilon}^u \frac{\int_{I_v \setminus I_{c_{i,j}^\epsilon}} \partial_t F_\epsilon^\beta(s,0) ds}{\pi v} dv. \end{aligned}$$

By a Hölder inequality,

$$\left| \int_{I_{c_{i,j}^\epsilon}} \partial_t F_\epsilon^\beta(s,0) ds \right| \leq \left(\int_{\partial M} \left(\phi_\epsilon^\beta \right)^2 e^{u_\epsilon} d\sigma_g \right)^{\frac{1}{2}} \left(\int_{\partial M} e^{u_\epsilon} d\sigma_g \right)^{\frac{1}{2}} \leq 1,$$

and since F_ϵ^β is positive on $I_{d_{i,j}^\epsilon} \setminus I_{c_{i,j}^\epsilon}$,

$$f_\epsilon(u) \leq f_\epsilon(c_{i,j}^\epsilon) + \frac{1}{\pi} \ln\left(\frac{u}{c_{i,j}^\epsilon} \right) \text{ for } c_{i,j}^\epsilon \leq u \leq d_{i,j}^\epsilon.$$

By the second condition of Step 1, we have for $c_{i,j}^{\epsilon} \leq u \leq d_{i,j}^{\epsilon}$ that

$$\forall x \in \partial \mathbb{D}_u^+, F_\epsilon^\beta(x) \leq A_i(\rho) f_\epsilon(u).$$

Gathering these inequalities, for $1 \leq j \leq q_\epsilon$, we get a constant $K_i(\rho) > 0$ such that,

$$\forall x \in \partial \mathbb{D}_u^+, \left| F_\epsilon^\beta(x) \right| \leq K_i(\rho) \left(\max_{\partial \mathbb{D}_{t_i^\epsilon}^+} \left| F_\epsilon^\beta \right| + \ln\left(\frac{u}{t_i^\epsilon} \right) \right), \tag{5.18}$$

which is exactly Step 3.

We are now in position to prove the claim. By Step 2, we get some constant $L_i(\rho) > 0$ such that for $1 \leq i \leq t$,

$$\sup_{\mathbb{D}_{t_i^\epsilon}^+ \setminus \mathbb{D}_{\tau_i^\epsilon}^+} |F_\epsilon| \leq L_i(\rho) \left(\inf_{\mathbb{D}_{t_i^\epsilon}^+ \setminus \mathbb{D}_{\tau_i^\epsilon}^+} |F_\epsilon| + 1 \right), \tag{5.19}$$

and we get some constant $L_{t+1}(\rho)$ such that

$$\sup_{M(\rho)} |\Phi_\epsilon| \leq L_{t+1}(\rho) \left(\max_{\partial \mathbb{D}_{\tau_{t+1}^\epsilon}^+} |F_\epsilon| + 1 \right). \tag{5.20}$$

By (5.11) in Claim 10,

$$\sup_{\mathbb{D}_{t_0^\epsilon}^+} |F_\epsilon| \leq C_2 \left(\frac{1}{12 R_0} \right). \tag{5.21}$$

Gathering (5.16), (5.19), (5.20) and (5.21), we get the claim. q.e.d.

In the following claim, we aim at passing to the limit in equation (i) and the condition (ii) given by Proposition 1 at the scale α_ϵ. The limiting function would then satisfy (5.25) and (5.26).

Claim 12. *We have that*

- *For any $\rho > 0$, there exists $\beta_\epsilon \to 0$ as $\epsilon \to 0$ such that*

$$\forall x \in \Gamma(\rho), \left| \hat{\Phi}_\epsilon \right|^2 (x) \geq 1 - \beta_\epsilon. \tag{5.22}$$

- *For $\rho > 0$ and $x \in \Gamma(\rho)$, we set $\hat{\Psi}_\epsilon = \frac{\hat{\Phi}_\epsilon}{|\hat{\Phi}_\epsilon|}$. Then for any $\rho > 0$, $\{\hat{\Psi}_\epsilon\}$ is uniformly equicontinuous on $\mathcal{C}^0(\Gamma(\rho), \mathbb{S}^n)$.*
- *For any $\rho > 0$, up to the extraction of a subsequence of $\{\hat{\Phi}_\epsilon\}$, there exist functions $\hat{\Phi} \in W^{1,2}(\Omega(\rho), \mathbb{R}^{n+1}) \cap L^\infty(\Gamma(\rho), \mathbb{R}^{n+1})$ and $\hat{\Psi} \in W^{\frac{1}{2},2}(\Gamma(\rho), \mathbb{S}^n) \cap \mathcal{C}^0(\Gamma(\rho), \mathbb{S}^n)$ such that*

$$\hat{\Phi}_\epsilon \rightharpoonup \hat{\Phi} \text{ in } W^{1,2}(\Omega(\rho), \mathbb{R}^{n+1}), \tag{5.23}$$

and

$$\hat{\Psi}_\epsilon \to \hat{\Psi} \text{ in } \mathcal{C}^0(\Gamma(\rho), \mathbb{S}^n) \text{ as } \epsilon \to 0 \tag{5.24}$$

with

$$\left|\hat{\Phi}\right|^2 \geq_{a.e.} 1 \text{ and } \hat{\Psi} = \frac{\hat{\Phi}}{\left|\hat{\Phi}\right|} \text{ on } \Gamma(\rho), \tag{5.25}$$

and for $0 \leq i \leq n$,

$$\begin{cases} \Delta \hat{\phi}^i = 0 & \text{in } \Omega(\rho), \\ \partial_t \hat{\phi}^i = -\sigma_k(M,[g])\hat{\psi}^i d\hat{\nu} & \text{on } \Gamma(\rho), \end{cases} \tag{5.26}$$

in a weak sense.

Proof. Step 1: We recall that $a_\epsilon \to a$ as $\epsilon \to 0$ with $\widetilde{z_{i_0}} = a$. For $1 \leq j \leq s_0$ and $\theta_\epsilon = \frac{\epsilon}{e^{2\tilde{v}_l(a)}\alpha_\epsilon^2}$,

$$\sup_{x\in\Gamma(\rho)} \int_{I_{\frac{\rho}{10}}(p_{0,j})} \left|\hat{\Phi}_\epsilon(z)\right|^2 \hat{p}_\epsilon(z,x)dz = O(e^{-\frac{\rho^2}{8\theta_\epsilon}}). \tag{5.27}$$

For $0 \leq i \leq t$, $1 \leq j \leq s_i$ and $\tau_i^\epsilon = \frac{\epsilon}{e^{2\tilde{v}_l(a)}\left(\omega_i^\epsilon\right)^2}$,

$$\sup_{x\in\Gamma(\rho)} \int_{I_{\frac{\rho}{10}}(p_{i,j})} \left|\overline{\Phi_\epsilon}^{\omega_i^\epsilon}(z)\right|^2 \overline{p_\epsilon}^{\omega_i^\epsilon}\left(z, \frac{\alpha_\epsilon}{\omega_i^\epsilon}x\right)dz = O(e^{-\frac{\rho^2}{8\tau_i^\epsilon}}). \tag{5.28}$$

For $1 \leq i \leq s$ and $i \neq i_0$,

$$\sup_{x\in\Gamma(\rho)} \int_{I_g(p_i,\frac{\rho}{10})} \left|\Phi_\epsilon(z)\right|^2 p_\epsilon(\breve{x},z)d\sigma_g(z) = O(e^{-\frac{\rho^2}{8\epsilon}}). \tag{5.29}$$

Note that (5.29) was already proved in Step 1 of Claim 5. Note also that the proof of (5.27) reduces to (5.28) for $i=0$. Let $0 \leq i \leq t$ and $1 \leq j \leq s_i$. Then, for $y \in \Gamma(\rho)$,

$$\begin{aligned} &e^{\frac{\rho^2}{8\tau_i^\epsilon}} \int_{I_{\frac{\rho}{10}}(p_{i,j})} \left|\overline{\Phi_\epsilon}^{\omega_i^\epsilon}(z)\right|^2 \overline{p_\epsilon}^{\omega_i^\epsilon}\left(z,\frac{\alpha_\epsilon}{\omega_i^\epsilon}y\right)dz \\ &\qquad \leq \frac{\int_{I_{\frac{\rho}{10}}(p_{i,j})} \left|\overline{\Phi_\epsilon}^{\omega_i^\epsilon}(z)\right|^2 e^{\overline{u_\epsilon}^{\omega_i^\epsilon}}}{\inf_{I_{\frac{\rho}{10}}(p_{i,j})} e^{\overline{u_\epsilon}^{\omega_i^\epsilon}}} . O\left(\frac{e^{-\frac{\rho^2}{4\tau_i^\epsilon}\left(\frac{9^2}{10^2}-\frac{1}{2}-\frac{1}{100}\right)}}{\sqrt{\tau_i^\epsilon}}\right) \\ &\qquad \leq \frac{C_0}{\inf_{I_{\frac{\rho}{10}}(p_{i,j})} e^{\overline{u_\epsilon}^{\omega_i^\epsilon}}} \frac{e^{-\frac{3\rho^2}{40\tau_i^\epsilon}}}{\sqrt{\tau_i^\epsilon}}, \end{aligned}$$

where we used the uniform bound (1.5) on $\overline{p_\epsilon}^{\omega_i^\epsilon}$ on $\mathbb{D}_{\frac{1}{\rho}} \times \mathbb{D}_{\frac{1}{\rho}}$. We assume by contradiction that

$$\inf_{I_{\frac{\rho}{10}}(p_{i,j})} e^{\overline{u_\epsilon}^{\omega_i^\epsilon}} \leq \frac{e^{-\frac{3\rho^2}{40\tau_i^\epsilon}}}{\sqrt{\tau_i^\epsilon}}.$$

Let $y \in \partial M$ be such that $\overline{y}^{\omega_i^\epsilon} \in I_{\frac{\rho}{10}}(p_{i,j})$. Then,

$$\begin{aligned} e^{\overline{u_\epsilon}^{\omega_i^\epsilon}(\overline{y}^{\omega_i^\epsilon})} &= e^{v_l(y)}\omega_i^\epsilon \int_{\partial M} p_\epsilon(x,y)d\nu_\epsilon(y) \\ &\geq \int_{I_{\frac{\rho}{10}}(p_{i,j})} \overline{p_\epsilon}^{\omega_i^\epsilon}(z,\overline{y}^{\omega_i^\epsilon})d\overline{\nu_\epsilon}^{\omega_i^\epsilon}(z) \\ &\geq \alpha_0 \frac{e^{-\frac{\rho^2}{80\tau_i^\epsilon}}}{\sqrt{\tau_i^\epsilon}} \int_{I_{\frac{\rho}{10}}(p_{i,j})} d\overline{\nu_\epsilon}^{\omega_i^\epsilon}, \end{aligned}$$

so that the assumption leads to

$$\int_{I_{\frac{\rho}{10}}(p_{i,j})} d\overline{\nu_\epsilon}^{\omega_i^\epsilon} \leq \frac{e^{-\frac{\rho^2}{16\tau_i^\epsilon}}}{\alpha_0}.$$

For $z \in I_{\frac{\rho}{20}}(p_{i,j})$,

$$\begin{aligned} e^{\overline{u_\epsilon}^{\omega_i^\epsilon}(z)} &\leq \frac{\int_{I_{\frac{\rho}{10}}(p_{i,j})} d\overline{\nu_\epsilon}^{\omega_i^\epsilon} + O\left(e^{-\frac{\rho^2}{4\tau_i^\epsilon}\left(\frac{1}{20^2}-\frac{1}{1000}\right)}\right)}{\sqrt{\tau_i^\epsilon}} \\ &\leq \frac{e^{-\frac{\rho^2}{16\tau_i^\epsilon}} + O\left(e^{-\frac{3\rho^2}{8000\tau_i^\epsilon}}\right)}{\alpha_0\sqrt{\tau_i^\epsilon}}. \end{aligned}$$

Then, $e^{\overline{u_\epsilon}^{\omega_i^\epsilon}} \to 0$ uniformly on $\mathcal{C}^0(I_{\frac{\rho}{20}}(p_{i,j}))$ as $\epsilon \to 0$ and

$$\sigma_\star(\mathbb{D}_{\frac{\rho}{20}}(p_{i,j}), \xi, I_{\frac{\rho}{20}}(p_{i,j}), e^{\overline{u_\epsilon}^{\omega_i^\epsilon}}\xi) \to +\infty \text{ as } \epsilon \to 0.$$

This contradicts (5.5) in Claim 9. The proof of Step 1 is now complete.

STEP 2: There exists a sequence $\beta_\epsilon \to 0$ as $\epsilon \to 0$ such that

$$\forall x, y \in \Gamma(\rho), |x-y| \leq \frac{\sqrt{\theta_\epsilon}}{\beta_\epsilon} \Rightarrow \left|\hat{\Phi}_\epsilon(x) - \hat{\Phi}_\epsilon(y)\right| \leq \beta_\epsilon. \tag{5.30}$$

We set $\gamma_\epsilon = \left\|\sqrt{\theta_\epsilon}e^{\hat{u}_\epsilon}\right\|_{\mathcal{C}^0(\Gamma(\rho))}^{\frac{1}{2}}$. We have that $\gamma_\epsilon \to 0$ as $\epsilon \to 0$. Indeed, for $r > 0$, and $x \in \Gamma(\rho)$,

$$\begin{aligned} \sqrt{\theta_\epsilon}e^{\hat{u}_\epsilon(z)} &\leq \left(\frac{A_0}{\sqrt{4\pi}} + o(1)\right)\int_{I_r(x)} d\hat{\nu}_\epsilon + o(1) \\ &\leq \frac{A_0\hat{\nu}(I_r(x))}{4\pi} + o(1) \\ &\leq \frac{A_0 D_1(\rho)}{\sqrt{4\pi}\ln(\frac{1}{r})} + o(1), \end{aligned}$$

since we have (5.2) and thanks successively to (1.6), (1.5) and to (5.3), (5.12). We also have $\frac{\gamma_\epsilon}{\sqrt{\theta_\epsilon}} \to +\infty$ as $\epsilon \to 0$ since $\frac{\theta_\epsilon^{\frac{1}{4}}}{\gamma_\epsilon} = \left\| e^{\hat{u}_\epsilon} \right\|_{\mathcal{C}^0(\Gamma(\rho))}^{-\frac{1}{2}} \leq m_i(\rho)^{-\frac{1}{3}}$ is bounded and we have (5.2). Let x_ϵ and $y_\epsilon \in \Gamma(\rho)$ with $|x_\epsilon - y_\epsilon| \leq \frac{\sqrt{\theta_\epsilon}}{\gamma_\epsilon}$. We set

$$F_\epsilon(z) = \hat{\Phi}_\epsilon(x_\epsilon + \frac{\sqrt{\theta_\epsilon}}{\gamma_\epsilon} z),$$

and α_ϵ the mean value of F_ϵ in $\mathbb{D}_3^+$. Then, we get constants $D_0, D, D' > 0$ such that

$$\begin{aligned}
\|F_\epsilon - \alpha_\epsilon\|_{L^\infty(I_2(0))} &\leq D_0 \|F_\epsilon - \alpha_\epsilon\|_{H^1(I_2(0))} \\
&\leq D \|\partial_\nu F_\epsilon\|_{L^\infty(I_3(0))} + D \|F_\epsilon - \alpha_\epsilon\|_{L^2(\mathbb{D}_3^+(0))} \\
&\leq D \left\| \hat{\Phi}_\epsilon \right\|_{L^\infty(\Gamma(\rho))} \sigma_\epsilon \gamma_\epsilon + D' \|\nabla F_\epsilon\|_{L^2(\mathbb{D}_3^+(0))} \\
&\leq D C_2(\rho) \sigma_\epsilon \gamma_\epsilon + D' \frac{\sqrt{D_2(\rho)}}{\ln\left(\frac{\gamma_\epsilon}{3\sqrt{\theta_\epsilon}}\right)^{\frac{1}{4}}},
\end{aligned}$$

thanks successively to (5.13) and (5.11). See also Step 2 in the Proof of Claim 5. Setting

$$\beta_\epsilon = 2 D C_2(\rho) \sigma_\epsilon \gamma_\epsilon + 2D' \frac{\sqrt{D_2(\rho)}}{\ln\left(\frac{\gamma_\epsilon}{3\sqrt{\theta_\epsilon}}\right)^{\frac{1}{4}}},$$

$\beta_\epsilon \to 0$ as $\epsilon \to 0$ and we get Step 2.

STEP 3: There exists a sequence $\beta_\epsilon \to 0$ as $\epsilon \to 0$ such that for all $x \in \partial M$,

$$\hat{x} \in \Gamma(\rho) \Rightarrow \left| \left| \hat{\Phi}_\epsilon(\hat{x}) \right|^2 - K_\epsilon[|\Phi_\epsilon|^2](x) \right| \leq \beta_\epsilon, \tag{5.31}$$

and

$$\hat{x} \in \Gamma(\rho) \cap supp(\hat{\nu}_\epsilon) \Rightarrow |K_\epsilon[|\Phi_\epsilon|](x) - 1| \leq \beta_\epsilon. \tag{5.32}$$

Note that (5.31) gives (5.22) for $x \in supp(\nu_\epsilon)$ by Proposition 1. Let $x \in \partial M$ be such that $\hat{x} \in \Gamma(\rho)$.

$$\begin{aligned}
\left| \left| \hat{\Phi}_\epsilon(\hat{x}) \right|^2 - K_\epsilon[|\Phi_\epsilon|^2](x) \right| &\leq \int_{\partial M} p_\epsilon(x,y) \left| |\Phi_\epsilon(x)|^2 - |\Phi_\epsilon(y)|^2 \right| d\sigma_g(y) \\
&\leq \int_{I_{\frac{\sqrt{\theta_\epsilon}}{\beta_\epsilon}}(\hat{x})} \hat{p}_\epsilon(z,\hat{x}) \left| \left| \hat{\Phi}_\epsilon(\hat{x}) \right|^2 - \left| \hat{\Phi}_\epsilon(z) \right|^2 \right| dz \\
&+ I_\epsilon \\
&+ \sum_{i \neq i_0} \int_{I_g(p_i, \frac{\rho}{10})} |\Phi_\epsilon|^2 p_\epsilon(x,y) d\sigma_g(y)
\end{aligned}$$

$$+\sum_{i=0}^{t}\sum_{j=1}^{s_t}\int_{I_\rho(p_{i,j})}\left|\overline{\Phi_\epsilon}^{\omega_i^\epsilon}\right|^2\overline{p_\epsilon}^{\omega_i^\epsilon}\left(z,\frac{\alpha_\epsilon}{\omega_i^\epsilon}\hat{x}\right)dz,$$

where

$$\begin{aligned}I_\epsilon &= C_2^2(\rho)\int_{\partial M\setminus \breve{I}_{\frac{\sqrt{\theta_\epsilon}}{\beta_\epsilon}}(\hat{x})} p_\epsilon(x,y)\\ &\quad + C_0^2(\rho)\int_{\partial M\setminus \breve{I}_{\frac{\sqrt{\theta_\epsilon}}{\beta_\epsilon}}(\hat{x})} p_\epsilon(x,y)\left(\ln\left(1+\frac{d_g(y,\bar{a}_\epsilon)}{\alpha_\epsilon}\right)+1\right)^2 d\sigma_g(y).\end{aligned}$$

Here, we used Claim 10 and Claim 11. By (5.27), (5.28), (5.29) and (5.30),

$$\left|\left|\hat{\Phi}_\epsilon(\hat{x})\right|^2 - K_\epsilon[|\Phi_\epsilon|^2](x)\right| \leq 2C_2(\rho)\beta_\epsilon + O(e^{-\frac{\rho^2}{8\alpha_\epsilon^2}}) + I_\epsilon,$$

and there are some constants $K_0(\rho) > 0$ and $K_1(\rho) > 0$ such that

$$\begin{aligned}I_\epsilon &\leq K_0(\rho)\ln\left(\frac{\delta(\partial M)}{\alpha_\epsilon}\right)^2\int_{\partial M\setminus\Gamma_l} p_\epsilon(x,y)d\sigma_g(y)\\ &\quad + K_1(\rho)\int_{\hat{\Gamma}_l\setminus I_{\frac{\sqrt{\theta_\epsilon}}{\beta_\epsilon}}(\hat{x})}\hat{p}_\epsilon(z,\hat{x})\left(\ln(1+|z|)^2+1\right)dz.\end{aligned}$$

Since $\frac{\alpha_\epsilon^2}{\epsilon} \to +\infty$ as $\epsilon \to 0$,

$$\begin{aligned}&\ln\left(\frac{\delta(\partial M)}{\alpha_\epsilon}\right)^2\int_{\partial M\setminus\Gamma_l} p_\epsilon(x,y)d\sigma_g(y)\\ &\qquad\leq \ln\left(\frac{\delta(\partial M)}{\alpha_\epsilon}\right)^2 .O\left(\frac{e^{-\frac{\delta(M)^2}{4\epsilon}}}{\sqrt{\epsilon}}\right) = o(1) \text{ as } \epsilon \to 0,\end{aligned}$$

and by (1.3),

$$\begin{aligned}&\int_{\hat{\Gamma}_l\setminus I_{\frac{\sqrt{\theta_\epsilon}}{\beta_\epsilon}}(\hat{x})}\hat{p}_\epsilon(z,\hat{x})\left(\ln(1+|z|)^2+1\right)dz\\ &\qquad\leq \int_{\mathbb{R}\setminus I_{\frac{\sqrt{\theta_\epsilon}}{\beta_\epsilon}}}\frac{A_0}{\sqrt{4\pi\theta_\epsilon}}e^{-\frac{|\hat{x}-z|^2}{8\theta_\epsilon}}\left(\ln(1+|z|)^2+1\right)dz\\ &\qquad\leq \int_{\mathbb{R}\setminus I_{\frac{1}{\beta_\epsilon}}(0)}\frac{A_0}{\sqrt{4\pi}}e^{-\frac{|y|^2}{8}}\left(\ln\left(1+\left|\hat{x}+\sqrt{\theta_\epsilon}y\right|\right)^2+1\right)dy\\ &\qquad= o(1) \text{ uniformly for } \hat{x}\in\Gamma(\rho).\end{aligned}$$

Up to increasing β_ϵ, we get (5.31). The same estimates can be obtained for $|\Phi_\epsilon|$ instead of $|\Phi_\epsilon|^2$, and we get up to increase β_ϵ for $x \in \partial M$ such that $\hat{x} \in \Gamma(\rho)$,

$$\left| \left| \hat{\Phi}_\epsilon(\hat{x}) \right| - K_\epsilon[|\Phi_\epsilon|](x) \right| \leq \beta_\epsilon.$$

Since, if $z \in supp(\hat{\nu}_\epsilon) \cap \Gamma(\rho)$, we have

$$\left| \left| \hat{\Phi}_\epsilon(z) \right|^2 - 1 \right| \leq \beta_\epsilon,$$

up to increase β_ϵ, we get for $x \in \partial M$ such that $\hat{x} \in supp(\hat{\nu}_\epsilon) \cap \Gamma(\rho)$,

$$|K_\epsilon[|\Phi_\epsilon|](x) - 1| \leq \beta_\epsilon.$$

We follow Step 4 in the proof of Claim 5 to prove that $\hat{\Psi}_\epsilon$ is uniformly equicontinuous on $\Gamma(\rho)$. Indeed, we can use the corollary of Theorem 4 thanks to Claim 9. Therefore, up to the extraction of a subsequence, $\hat{\Psi}_\epsilon \to \hat{\Psi}$ in $\mathcal{C}^0(\Gamma(\rho), \mathbb{S}^n)$ as $\epsilon \to 0$.

Step 4: We have that

$$\hat{\phi}_\epsilon^i e^{\hat{u}_\epsilon} ds \rightharpoonup_\star \hat{\psi}^i \hat{\nu} \text{ in } \mathcal{M}(\Gamma(\rho)) \text{ as } \epsilon \to 0.$$

Let $\zeta \in \mathcal{C}_c^0(I(\rho))$ and $R > \frac{1}{\rho}$. Then

$$\begin{aligned}
&\int_{\mathbb{R}\times\{0\}} \zeta(z) \left(\hat{\phi}_\epsilon^i(z) e^{\hat{u}_\epsilon(z)} dz - \hat{\psi}^i(z) d\hat{\nu}(z) \right) \\
&\qquad = \int_{\partial M \setminus \breve{I}_R} \left(\int_{\breve{\Gamma}(\rho)} p_\epsilon(x,y)\zeta(y)\phi_\epsilon^i(y) d\sigma_g(y) \right) d\nu_\epsilon(x) \\
&\qquad + \int_{I_R} \left(\int_{I_R} (\zeta(z) - \zeta(x)) \hat{\phi}_\epsilon^i(z) \hat{p}_\epsilon(z,x) dz \right) d\hat{\nu}_\epsilon(x) \\
&\qquad + \int_{\Gamma(\rho)} \zeta(x) \left(\int_{I_R} \left(\hat{\psi}_\epsilon^i(z) - \hat{\psi}_\epsilon^i(x) \right) \left| \hat{\Phi}_\epsilon(z) \right| \hat{p}_\epsilon(z,x) dz \right) d\hat{\nu}_\epsilon(x) \\
&\qquad + \int_{\Gamma(\rho)} \zeta(x) \hat{\psi}_\epsilon^i(x) \left(\int_{I_R} \left(\left| \hat{\Phi}_\epsilon(z) \right| - 1 \right) \hat{p}_\epsilon(z,x) dz \right) d\hat{\nu}_\epsilon(x) \\
&\qquad + \int_{\Gamma(\rho)} \zeta(x) \left(\hat{\psi}_\epsilon^i(x) \left(\int_{I_R} \hat{p}_\epsilon(z,x) dz \right) d\hat{\nu}_\epsilon(x) - \hat{\psi}^i(x) d\hat{\nu}(x) \right).
\end{aligned}$$

We have by (1.7) that

$$\begin{aligned}
&\int_{\partial M \setminus \breve{I}_R} \left(\int_{\breve{\Gamma}(\rho)} p_\epsilon(x,y)\zeta(y)\phi_\epsilon^i(y) d\sigma_g(y) \right) d\nu_\epsilon(x) \\
&\qquad \leq \|\zeta\|_\infty C_2(\rho) \sup_{y \in \partial M \setminus \breve{I}_R} \int_{\breve{I}_{\frac{1}{\rho}}} p_\epsilon(x,y) d\sigma_g(x) \\
&\qquad = o(1) \text{ as } \epsilon \to 0.
\end{aligned}$$

By Step 1, Claim 11 and (1.5),

$$\int_{I_R} \left(\int_{I_R} (\zeta(z) - \zeta(x)) \hat{\phi}_\epsilon^i(z) \hat{p}_\epsilon(z,x) dz \right) d\hat{\nu}_\epsilon(x)$$

$$
\begin{aligned}
&\leq \sup_{x\in I_R}\int_{I_R}|\zeta(z)-\zeta(x)|\left|\hat{\phi}_\epsilon^i(z)\right|\hat{p}_\epsilon(z,x)dz\\
&\leq \sum_{j=1}^{s_0}\sup_{x\in I_R}|\zeta(x)|\int_{I_{\frac{\rho}{10}}(p_{0,j})}\left|\hat{\phi}_\epsilon^i(z)\right|\hat{p}_\epsilon(z,x)dz\\
&\quad+\sup_{x\in I_R}\int_{I_R\setminus\bigcup_{j=1}^{s_0}I_{\frac{\rho}{10}}(p_{0,j})}|\zeta(z)-\zeta(x)|\left|\hat{\phi}_\epsilon^i(z)\right|\hat{p}_\epsilon(z,x)dz\\
&\leq \|\zeta\|_\infty\sum_{j=1}^{s_0}\sup_{x\in\Gamma(\rho)}\left(\int_{I_{\frac{\rho}{10}}(p_{0,j})}\left|\hat{\Phi}_\epsilon(z)\right|^2\hat{p}_\epsilon(z,x)dz\right)^{\frac{1}{2}}\\
&\quad+C_0(\rho)\left(1+\ln(1+C_0R)\right)\sup_{x\in I_R}\int_{\mathbb{R}\times\{0\}}|\zeta(z)-\zeta(x)|\frac{e^{\frac{|x-z|^2}{8\theta_\epsilon}}}{\sqrt{\pi\theta_\epsilon}}dz\\
&=o(1)\text{ as }\epsilon\to 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\Gamma(\rho)}\zeta(x)\left(\int_{I_R}\left(\hat{\psi}_\epsilon^i(z)-\hat{\psi}_\epsilon^i(x)\right)\left|\hat{\Phi}_\epsilon(z)\right|\hat{p}_\epsilon(z,x)dz\right)d\hat{\nu}_\epsilon(x)\\
&\qquad\leq 2\|\zeta\|_\infty\sup_{x\in\Gamma(\rho)}\sum_{j=1}^{s_0}\left(\int_{I_{\frac{\rho}{10}}(p_{0,j})}\left|\hat{\Phi}_\epsilon(z)\right|^2\hat{p}_\epsilon(z,x)dz\right)^{\frac{1}{2}}\\
&\qquad\quad+\|\zeta\|_\infty C_2\left(\frac{\rho}{10}\right)\sup_{x\in\Gamma(\rho)}\int_{\Gamma\left(\frac{\rho}{10}\right)}\left|\hat{\psi}_\epsilon^i(x)-\hat{\psi}_\epsilon^i(z)\right|\hat{p}_\epsilon(z,x)dz\\
&\qquad\quad+2\|\zeta\|_\infty C_0(\rho)\left(1+\ln(1+C_0R)\right)\sup_{x\in\Gamma(\rho)}\int_{I_R\setminus\Gamma\left(\frac{\rho}{10}\right)}\hat{p}_\epsilon(z,x)dz\\
&\qquad=o(1)\text{ as }\epsilon\to 0,
\end{aligned}
$$

where by (1.5),

$$
\begin{aligned}
&\sup_{x\in\Gamma(\rho)}\int_{\Gamma\left(\frac{\rho}{10}\right)}\left|\hat{\psi}_\epsilon^i(x)-\hat{\psi}_\epsilon^i(z)\right|\hat{p}_\epsilon(z,x)dz\\
&\qquad\qquad\leq \sup_{x\in\Gamma(\rho)}\int_{\Gamma\left(\frac{\rho}{10}\right)}\left|\hat{\psi}_\epsilon^i(x)-\hat{\psi}_\epsilon^i(z)\right|\frac{e^{-\frac{|x-z|^2}{8\theta_\epsilon}}}{\sqrt{\pi\theta_\epsilon}}dz\\
&\qquad\qquad=o(1)\text{ as }\epsilon\to 0.
\end{aligned}
$$

We also have that

$$
\begin{aligned}
&\int_{\Gamma(\rho)}\zeta(x)\hat{\psi}_\epsilon^i(x)\left(\int_{I_R}\left(\left|\hat{\Phi}_\epsilon(z)\right|-1\right)\hat{p}_\epsilon(z,x)dz\right)d\hat{\nu}_\epsilon(x)\\
&\qquad\qquad\leq\|\zeta\|_\infty\sup_{x\in\Gamma(\rho)\cap supp(\hat{\nu}_\epsilon)}\int_{I_R}\left(\left|\hat{\Phi}_\epsilon(z)\right|-1\right)\hat{p}_\epsilon(z,x)dz.
\end{aligned}
$$

We use (5.32) of Step 3, in order to prove that

$$\sup_{x\in\Gamma(\rho)\cap supp(\hat{\nu}_\epsilon)} \int_{I_R} \left(\left|\hat{\Phi}_\epsilon(z)\right| - 1\right)\hat{p}_\epsilon(z,x)dz \to 0 \text{ as } \epsilon \to 0. \tag{5.33}$$

Let $x \in \partial M$ be such that $\hat{x} \in \Gamma(\rho) \cap supp(\hat{\nu}_\epsilon)$,

$$\begin{aligned} K_\epsilon[|\Phi_\epsilon|](x) - 1 &= \int_{\partial M\setminus \breve{I}_R} \left(|\Phi_\epsilon(y)| - 1\right) p_\epsilon(x,y) d\sigma_g(y) \\ &\quad + \int_{I_R} \left(\left|\hat{\Phi}_\epsilon(z)\right| - 1\right)\hat{p}_\epsilon(z,\hat{x})dz, \end{aligned}$$

and

$$\begin{aligned} &\left|\int_{\partial M\setminus \breve{I}_R} \left(|\Phi_\epsilon(y)| - 1\right) p_\epsilon(x,y) d\sigma_g(y)\right| \\ &\le \int_{\partial M\setminus\Gamma_l} p_\epsilon(x,y)d\sigma_g(y) K_0(\rho) \ln\left(\frac{\delta(\partial M)}{\alpha_\epsilon}\right) \\ &\quad + K_1(\rho)\int_{\hat{\Gamma}_l\setminus I_R} \hat{p}_\epsilon(z,\hat{x})\left(1 + \ln(1+|z|)\right)dz \\ &\le O\left(\frac{e^{-\frac{\delta(\partial M)^2}{4\epsilon}}}{\sqrt{4\pi\epsilon}} \ln\left(\frac{\delta(\partial M)}{\alpha_\epsilon}\right)\right) \\ &\quad + K_1(\rho)\int_{\mathbb{R}\setminus I_R} A_0 \frac{e^{-\frac{|\hat{x}-z|^2}{8\theta_\epsilon}}}{\sqrt{4\pi\theta_\epsilon}}(1+\ln(1+|z|))dz \\ &\le O\left(\int_{\mathbb{R}\setminus I_{\frac{R}{\sqrt{\theta_\epsilon}}}} e^{-\frac{|y|^2}{8}}\left(1 + \ln(1 + \left|\hat{x} + \sqrt{\theta_\epsilon}y\right|)\right)dz\right) \\ &= o(1) \text{ as } \epsilon \to 0. \end{aligned}$$

This gives (5.33). By (1.8),

$$\lim_{R\to+\infty}\lim_{\epsilon\to 0}\sup_{x\in I_{\frac{1}{\rho}}} \left|\int_{I_R} \hat{p}_\epsilon(z,x)dz - 1\right| = 0,$$

so that

$$\lim_{\epsilon\to 0}\left(\int_{\Gamma(\rho)} \zeta(x)\left(\hat{\psi}^i_\epsilon(x)\left(\int_{I_R}\hat{p}_\epsilon(z,x)dz\right)d\hat{\nu}_\epsilon(x) - \hat{\psi}^i(x)d\hat{\nu}(x)\right)\right) \to 0$$

as $R \to +\infty$. Gathering all these computations, we get Step 4.

As a conclusion, (5.31) in Step 3 gives (5.22) for $x \in supp(\nu_\epsilon)$ by Proposition 1. In the remark before Step 4, we get (5.24). Then, (5.22), (5.23) and (5.24) give (5.25). We, finally, get (5.26) passing to the limit in the equation satisfied by $\hat{\phi}^i_\epsilon$ thanks to Step 4. This ends the proof of the Claim. q.e.d.

Thanks to Claim 12, a diagonal extraction gives some functions $\hat{\Phi} : \mathbb{R}^2_+ \setminus \{p_{0,1}, \cdots, p_{0,s_0}\} \to \mathbb{R}^{n+1}$ and $\hat{\Psi} : \mathbb{R} \setminus \{p_{0,1}, \cdots, p_{0,s_0}\} \to \mathbb{S}^n$ such that for any $\rho > 0$, the conclusions (5.23), (5.24), (5.25) and (5.26) of Claim 12 hold true for $\hat{\Phi}$ and $\hat{\Psi}$.

We now give energy estimates on these limit functions which will be useful at the end of the proof. We recall that $\lambda : \mathbb{D} \setminus \{p\} \to \mathbb{R}^2_+$ is defined page 132. We set $\check{\Phi} = \hat{\Phi} \circ \lambda : \mathbb{D} \setminus \{p, q_0, \cdots, q_{s_0}\}$ and $\check{\Psi} = \hat{\Phi} \circ \lambda : \mathbb{S}^1 \setminus \{p, q_0, \cdots, q_{s_0}\}$, where $q_j = \lambda^{-1}(p_{0,j}) \in \mathbb{S}^1$ and we set

$$D(\rho) = \mathbb{D} \setminus \left(\mathbb{D}_\rho(p) \cup \bigcup_{i=1}^{s_0} \mathbb{D}_\rho(q_i) \right) \text{ and } S(\rho) = \mathbb{S}^1 \cap D(\rho).$$

We Let $\check{\nu}$ be the measure without atom on $\mathbb{S}^1$ such that

$$e^{\hat{u}_\epsilon} d\theta \rightharpoonup_\star d\check{\nu} \text{ in } \mathcal{M}(S(\rho)) \text{ as } \epsilon \to 0$$

for any $\rho > 0$. It is equal to $\lambda^\star(\hat{\nu})$ outside $\{p, q_0, \cdots, q_{s_0}\}$.

We also set some function ω on $\mathbb{D}$ which satisfies the following equation

$$\begin{cases} \Delta \omega = 0 & \text{in } \mathbb{D}, \\ \omega = \left|\check{\Phi}\right| & \text{on } \mathbb{S}^1, \end{cases} \tag{5.34}$$

in a weak sense. Such a harmonic function exists since $\left|\check{\Phi}\right| \in W^{\frac{1}{2},2}(\mathbb{S}^1)$ and we have $\omega \in W^{1,2}(\mathbb{D})$.

Claim 13.

$$\lim_{\rho\to 0} \lim_{\epsilon \to 0} \int_{D(\rho)} \left|\nabla \check{\Phi}_\epsilon\right|^2 dx \geq \int_{\mathbb{D}} \frac{\left|\nabla \check{\Phi}\right|^2}{\omega} dx \geq \sigma_k \int_{\mathbb{S}^1} d\check{\nu} + \int_{\mathbb{D}} \frac{\left|\check{\Phi}\right|^2 \left|\nabla \omega\right|^2}{\omega^3} dx, \tag{5.35}$$

where $\sigma_k = \sigma_k\left(M, [g]\right)$ *and* $\int_{\mathbb{S}^1} d\check{\nu} \geq m_i$.

Proof. Let $\eta \in \mathcal{C}_c^\infty(D(\sqrt{\rho}))$ be given by Claim 1 with $\eta \geq 1$ on $D(\rho)$ and

$$\int_{\mathbb{D}} \left|\nabla \eta\right|^2 \leq \frac{C}{\ln\left(\frac{1}{\rho}\right)}.$$

By the weak maximum principle on (5.34),

$$\inf_{\mathbb{D}} \omega \geq \inf_{\mathbb{S}^1} \left|\check{\Phi}\right| \geq 1,$$

and by the same computations as in the proof of Claim 6,

$$\begin{aligned} \lim_{\epsilon\to 0} \int_{D(\rho)} \left|\nabla \check{\Phi}_\epsilon\right|^2 dx & \geq \int_{D(\rho)} \left|\nabla \check{\Phi}\right|^2 dx \\ & \geq \int_{\mathbb{D}} \eta \frac{\left|\nabla \check{\Phi}\right|^2}{\omega} dx \end{aligned}$$

$$\begin{aligned} &\geq \sigma_k(M,[g])\int_{\mathbb{S}^1}\eta d\check{\nu} + \int_{\mathbb{D}}\eta\frac{\left|\check{\Phi}\right|^2}{\omega^3}\left|\nabla\omega\right|^2 \\ &\quad -\sum_{i=0}^{n}\int_{\mathbb{D}}\frac{\check{\phi}_i}{\omega}\left\langle\nabla\eta,\nabla\check{\phi}_i\right\rangle - \int_{\mathbb{D}}\left\langle\nabla\eta,\nabla\omega\right\rangle\frac{\left|\check{\Phi}\right|^2}{2\omega^2} \\ &\quad +\frac{1}{2}\int_{\mathbb{D}}\left\langle\nabla\eta,\nabla\omega\right\rangle \\ &\geq \sigma_k(M,[g])\int_{\mathbb{S}^1}\eta d\check{\nu} + \int_{\mathbb{D}}\eta\frac{\left|\check{\Phi}\right|^2}{\omega^3}\left|\nabla\omega\right|^2 \\ &\quad -\frac{C'}{\sqrt{\ln\left(\frac{1}{\rho}\right)}}, \end{aligned}$$

where C' is a constant independent of ρ. Indeed, $\check{\phi}_i, \omega \in W^{1,2}(\mathbb{D})$ and we have for $0 \leq i \leq n$ that

$$\Delta\left(\omega - \check{\phi}_i\right) = 0 \text{ and } \Delta\left(\omega + \check{\phi}_i\right) = 0,$$

in a weak sense. By the weak maximum principle (see [**11**], Theorem 8.1),

$$\inf_{\mathbb{D}}\left(\omega - \check{\phi}_i\right) \geq \inf_{\mathbb{S}^1}\left(\omega - \check{\phi}_i\right) \geq 0,$$

and

$$\inf_{\mathbb{D}}\left(\omega + \check{\phi}_i\right) \geq \inf_{\mathbb{S}^1}\left(\omega + \check{\phi}_i\right) \geq 0,$$

since $\left|\check{\phi}_i\right| \leq \left|\check{\Phi}\right| \leq \omega$ on $\mathbb{S}^1$. Then,

$$\sup_{\mathbb{D}}\frac{\left|\check{\phi}_i\right|}{\omega} \leq 1 \text{ and } \sup_{\mathbb{D}}\frac{\left|\check{\Phi}\right|^2}{\omega^2} \leq n+1.$$

We, finally, get (5.35), passing to the limit as $\rho \to 0$. We have that $\int_{\mathbb{S}^1} d\check{\nu} \geq m_i$ thanks to (4.5), (4.7) and (5.9). This ends the proof of the claim. q.e.d.

5.2. Regularity estimates when $\frac{\alpha_\epsilon^2}{\epsilon} = O(1)$. We now assume that $\frac{\alpha_\epsilon^2}{\epsilon} = O(1)$, we let $\theta_0 = \lim_{\epsilon\to 0}\frac{\epsilon}{e^{2\tilde{v}_l(a)}\alpha_\epsilon}$ and we denote by $\hat{\nu}$ the weak* limit of $\hat{\nu}_\epsilon$ in $\mathcal{M}(\mathbb{R}\times\{0\})$. Let $R_0 > 0$ and $x \in I_{R_0}$. We have by (1.5) that

$$\begin{aligned} e^{\hat{u}_\epsilon}(x) &= e^{v_l(\breve{x})}\alpha_\epsilon\int_{\partial M} p_\epsilon(\breve{x},y)d\nu_\epsilon(y) \\ &\leq \frac{A_0 e^{v_l(\breve{x})}\alpha_\epsilon}{\sqrt{4\pi\epsilon}}\int_{\partial M} d\nu_\epsilon \\ &\leq \frac{A_0}{\sqrt{4\pi\epsilon}}(1+o(1)). \end{aligned}$$

Since $m_i > 0$, we get that $\theta_0 < +\infty$. Now, we let $\hat{u}$ be a smooth function on $\mathbb{R} \times \{0\}$ defined by

$$e^{\hat{u}(x)} = \int_{\mathbb{R}\times\{0\}} \frac{e^{-\frac{|x-y|^2}{4\theta_0}}}{\sqrt{4\pi\theta_0}} d\hat{\nu}(y). \tag{5.36}$$

Let $R_0 > 0$, $R > R_0$ and $x \in I_{R_0}$. We have

$$\begin{aligned}
\left|e^{\hat{u}_\epsilon(x)} - e^{\hat{u}(x)}\right| &= \left|\int_{\partial M} \alpha_\epsilon p_\epsilon(\breve{x}, y) d\nu_\epsilon(y) - e^{\hat{u}(x)}\right| \\
&\leq \int_{\partial M \setminus \breve{I}_R} \alpha_\epsilon p_\epsilon(\breve{x}, y) d\nu_\epsilon(y) \\
&\quad + \left|\int_{I_R} \hat{p}_\epsilon(x,y) d\hat{\nu}_\epsilon(y) - \int_{\mathbb{R}\times\{0\}} \frac{e^{-\frac{|x-y|^2}{4\theta_0}}}{\sqrt{4\pi\theta_0}} d\hat{\nu}(y)\right| \\
&= o(1) + \frac{A_0}{\sqrt{4\pi\theta_0}}(1+o(1)) e^{-\frac{(R-R_0)^2}{8\theta_0}} \\
&\quad + \left|\int_{I_R} \left(\hat{p}_\epsilon(x,y) - \frac{e^{-\frac{|x-y|^2}{4\theta_0}}}{\sqrt{4\pi\theta_0}}\right) d\hat{\nu}_\epsilon\right| \\
&\quad + \left|\int_{I_R} \frac{e^{-\frac{|x-y|^2}{8\theta_0}}}{\sqrt{4\pi\theta_0}} (d\hat{\nu}_\epsilon - d\hat{\nu})\right| + \int_{\mathbb{R}\times\{0\}\setminus I_R} \frac{e^{-\frac{|x-y|^2}{4\theta_0}}}{\sqrt{4\pi\theta_0}} d\hat{\nu} \\
&\to \frac{A_0}{\sqrt{4\pi\theta_0}} e^{-\frac{(R-R_0)^2}{8\theta_0}} + \int_{\mathbb{R}\times\{0\}\setminus I_R} \frac{e^{-\frac{|x-y|^2}{4\theta_0}}}{\sqrt{4\pi\theta_0}} d\hat{\nu} \text{ as } \epsilon \to 0.
\end{aligned}$$

Letting $R \to +\infty$, we get for any $R_0 > 0$ that

$$e^{\hat{u}_\epsilon} \to e^{\hat{u}} \text{ in } \mathcal{C}^0(I_{R_0}) \text{ as } \epsilon \to 0. \tag{5.37}$$

With Claim 2, $\{\hat{\phi}^i_\epsilon\}$ is bounded in $L^2(I_R)$ for any $R > 0$. With (5.37) and elliptic estimates on the Dirichlet-to-Neumann operator (see [**25**], Chapter 7.11, page 37)

$$\begin{cases} \Delta \hat{\phi}^i_\epsilon = 0 & \text{in } \mathbb{D}^+_{R_0}, \\ \partial_t \hat{\phi}^i_\epsilon = -\sigma_\epsilon e^{\hat{u}_\epsilon} \hat{\phi}^i_\epsilon & \text{on } I_{R_0}, \end{cases}$$

we get some smooth function $\hat{\Phi}$ on $\mathbb{R}^2_+$ such that for any $R_0 > 0$,

$$\hat{\phi}^i_\epsilon \to \hat{\phi}^i \text{ in } \mathcal{C}^1(\mathbb{D}^+_{R_0}) \text{ as } \epsilon \to 0, \tag{5.38}$$

and

$$\begin{cases} \Delta \hat{\phi}^i = 0 & \text{in } \mathbb{R}^2_+, \\ \partial_t \hat{\phi}^i = -\sigma_k(M,[g]) e^{\hat{u}} \hat{\phi}^i & \text{on } \mathbb{R}\times\{0\}. \end{cases} \tag{5.39}$$

We now prove the following

Claim 14. *We have the following energy inequality*

$$\int_{\mathbb{R}^2_+} \left|\nabla\hat{\Phi}(x)\right|^2 dx \geq \sigma_k(M,[g]) \int_{\mathbb{S}^1} e^{\check{u}} d\theta, \tag{5.40}$$

where $e^{\check{u}} = e^{\hat{u}} \circ \lambda$

Proof. Step 1: Up to the extraction of a subsequence, there exists some sequence $\{\omega_i^\epsilon\}$ with $0 \leq i \leq t+1$ and $0 \leq t \leq k$ and

$$\alpha_\epsilon = \omega_0^\epsilon \ll \omega_1^\epsilon \ll \cdots \ll \omega_{t+1}^\epsilon = \delta_0,$$

and for $1 \leq i \leq t$ and $1 \leq j \leq s_i$ some points $p_{i,j} \in J_{\frac{1}{R_0}}$ with $R_0 > 0$ and $s - 1 + \sum_{i=1}^t s_i \leq k$ such that for all $\rho > 0$, there exists $C_0(\rho)$ such that

$$\forall x \in M \setminus \left(\bigcup_{i \neq i_0} B_g(p_i,\rho) \cup \bigcup_{i=1}^t \bigcup_{j=1}^{s_i} \Omega_{i,j} \right),$$

$$\left|\Phi_\epsilon\right|(x) \leq C_0(\rho) \left(\ln\left(1 + \frac{d_g(\bar{a}_\epsilon, x)}{\sqrt{\epsilon}}\right) + 1 \right),$$

where $\widetilde{\Omega_{i,j}} = \omega_\epsilon^i \mathbb{D}_\rho^+(p_{i,j}) + a_\epsilon$ and $\bar{a}_\epsilon = \exp_{g_l,x_l}^{-1}(a_\epsilon)$. We also have that for all $\rho > 0$,

$$\sup_{x\in\Gamma(\rho)} \int_{I_{\frac{\rho}{10}}(p_{i,j})} \left|\overline{\Phi_\epsilon}^{\omega_i^\epsilon}(z)\right|^2 \overline{p_\epsilon}^{\omega_i^\epsilon}\left(z, \frac{\alpha_\epsilon}{\omega_i^\epsilon} x\right) dz = O(e^{-\frac{\rho^2}{8\tau_i^\epsilon}}) \tag{5.41}$$

for $1 \leq i \leq t$, $1 \leq j \leq s_i$ and $\tau_i^\epsilon = \frac{\epsilon}{e^{2\tilde{v}_l(a)}(\omega_\epsilon^i)^2}$ and

$$\sup_{x\in\Gamma(\rho)} \int_{I_g(p_i,\frac{\rho}{10})} \left|\Phi_\epsilon(z)\right|^2 p_\epsilon(\check{x},z) dz = O(e^{-\frac{\rho^2}{8\epsilon}}). \tag{5.42}$$

For $1 \leq i \leq s$ and $i \neq i_0$.

For the estimate of Φ_ϵ, we follow the proof of Claim 9 and Claim 11, using (5.37) and (5.38) instead of the estimates of Claim 10. The proof of (5.41) and (5.42) follows the proof of Step 1 in Claim 12, which is a consequence of Claim 9.

Step 2: We have that

$$\int_{\mathbb{R}\times\{0\}} \left|\hat{\Phi}(y)\right|^2 \frac{e^{-\frac{|x-y|^2}{4\theta_0}}}{\sqrt{4\pi\theta_0}} dy \geq 1. \tag{5.43}$$

In order to prove (5.43), it suffices to use Proposition 1 and prove that for $R_0 > 0$ fixed, $x \in \partial M$ such that $\hat{x} \in I_{R_0}$, we have

$$\int_{\mathbb{R}\times\{0\}} \left|\hat{\Phi}(y)\right|^2 \frac{e^{-\frac{|\hat{x}-y|^2}{4\theta_0}}}{\sqrt{4\pi\theta_0}} - K_\epsilon[|\Phi_\epsilon|^2](x) \to 0 \text{ as } \epsilon \to 0. \tag{5.44}$$

Let's prove (5.44). We fix $r > 0$ and $R > r$. Let $x \in M$ be such that $\hat{x} \in I_r$. We fix $\rho > 0$. Then,

$$\begin{aligned}\left| K_\epsilon[|\Phi_\epsilon|^2](x) - \int_{I_R} \left|\hat{\Phi}(z)\right|^2 \frac{e^{-\frac{|\hat{x}-z|^2}{4\theta_0}}}{\sqrt{4\pi\theta_0}} dz \right| &= \int_{\partial M \setminus \breve{I}_R} |\Phi_\epsilon(y)|^2 p_\epsilon(x,y) d\sigma_g(y) \\ &\quad + \int_{I_R} \hat{p}_\epsilon(z,\hat{x}) \left|\hat{\Phi}_\epsilon(z)\right|^2 dz \\ &\quad - \int_{I_R} \left|\hat{\Phi}(z)\right|^2 \frac{e^{-\frac{|\hat{x}-z|^2}{4\theta_0}}}{\sqrt{4\pi\theta_0}} dz.\end{aligned}$$

There exist some constants $K_0(\rho) > 0$ and $K_1(\rho) > 0$ such that, by Step 1,

$$\begin{aligned}&\int_{\partial M \setminus \breve{I}_R} |\Phi_\epsilon(y)|^2 p_\epsilon(x,y) d\sigma_g(y) \\ &\qquad \le K_0(\rho) \int_{\partial M \setminus \Gamma_l} \ln\left(\frac{\delta(\partial M)}{\sqrt{\epsilon}}\right)^2 p_\epsilon(x,y) d\sigma_g(y) \\ &\qquad\quad + K_1(\rho) \int_{\hat{\Gamma}_l \setminus I_R} \left(\ln(1+|z|)^2 + 1\right) \hat{p}_\epsilon(z,\hat{x}) dz \\ &\qquad\quad + \sum_{i=1}^{t} \sum_{j=1}^{s_i} \int_{I_{\frac{\rho}{10}}(p_{i,j})} \left|\overline{\Phi_\epsilon}^{\omega_i^\epsilon}(z)\right|^2 \overline{p_\epsilon}^{\omega_i^\epsilon}\left(z, \frac{\alpha_\epsilon}{\omega_i^\epsilon}\hat{x}\right) dz \\ &\qquad\quad + \sum_{i \neq i_0} \int_{I_g(p_i, \frac{\rho}{10})} |\Phi_\epsilon(y)|^2 p_\epsilon(x,y) d\sigma_g(y) \\ &\qquad \le O\left(\ln\left(\frac{\delta(\partial M)}{\sqrt{\epsilon}}\right)^2 \frac{e^{-\frac{\delta(\partial M)^2}{4\epsilon}}}{\sqrt{\epsilon}}\right) \\ &\qquad\quad + O\left(e^{-\frac{\rho^2}{8\tau_1^\epsilon}}\right) \\ &\qquad\quad + \frac{K_1(\rho) A_0}{\sqrt{\pi\theta_0}} \int_{\mathbb{R}\times\{0\} \setminus I_R} \left(\ln(1+|z|)^2 + 1\right) e^{-\frac{|\hat{x}-z|^2}{8\theta_0}} dz.\end{aligned}$$

Passing to the limit as $\epsilon \to 0$ and then as $R \to +\infty$, we get (5.44) and then (5.43). This ends the proof of Step 2.

STEP 3: We have that

$$\sigma_k(M,[g]) \int_{\mathbb{R}\times\{0\}} \left|\hat{\Phi}(y)\right|^2 e^{\hat{u}(y)} dy \le \int_{\mathbb{R}^2_+} \left|\nabla \hat{\Phi}(x)\right|^2 dx. \tag{5.45}$$

By contradiction, we assume that there is $\epsilon_0 > 0$ such that

$$\sigma_k(M,[g]) \int_{\mathbb{R}\times\{0\}} \left|\hat{\Phi}(y)\right|^2 e^{\hat{u}(y)} dy \geq \int_{\mathbb{R}^2_+} \left|\nabla\hat{\Phi}(x)\right|^2 dx + \epsilon_0.$$

We fix $R > 0$. By equation (5.39),

$$\begin{cases} \frac{1}{2}\Delta\left|\hat{\Phi}\right|^2 = -\left|\nabla\hat{\Phi}\right|^2 & \text{in } \mathbb{R}^2_+, \\ \frac{1}{2}\partial_t\left|\hat{\Phi}\right|^2 = -\sigma_k(M,[g]) e^{\hat{u}}\left|\hat{\Phi}\right|^2 & \text{on } \mathbb{R}\times\{0\}. \end{cases}$$

We integrate on $\mathbb{D}^+_R$,

$$-\frac{1}{2}\int_{\partial\mathbb{D}^+_R} \partial_\nu\left(\left|\hat{\Phi}\right|^2\right) d\sigma = \sigma_k(M,[g]) \int_{I_R} e^{\hat{u}}\left|\hat{\Phi}\right|^2 - \int_{\mathbb{D}^+_R}\left|\nabla\hat{\Phi}\right|^2 \geq \frac{\epsilon_0}{2}$$

for any $R > R_0$, for some $R_0 > 0$, since $e^{\hat{u}}\left|\hat{\Phi}\right|^2 \in L^1(\mathbb{R}\times\{0\})$ and $\left|\nabla\hat{\Phi}\right|^2 \in L^1(\mathbb{R}^2_+)$. We set

$$f(r) = \frac{\int_{\partial\mathbb{D}^+_r}\left|\hat{\Phi}\right|^2 d\sigma}{\pi r}.$$

Then, for $R > R_0$, $\pi f'(R) \leq -\frac{\epsilon_0}{R}$ so that

$$f(R) \leq -\frac{\epsilon_0}{\pi}\ln\left(\frac{R}{R_0}\right) + f(R_0) \to -\infty \text{ as } R \to +\infty,$$

which contradicts the fact that $f(R) > 0$. This ends the proof of Step 3.

We are now in position to get the claim. We integrate (5.43) against $\hat{\nu}$ and (5.36) against dx, and we obtain

$$\int_{\mathbb{R}\times\{0\}}\left|\hat{\Phi}(y)\right|^2 e^{\hat{u}(y)} dy \geq \int_{\mathbb{R}\times\{0\}} d\hat{\nu} = \int_{\mathbb{R}\times\{0\}} e^{\hat{u}(y)} dy, \tag{5.46}$$

and we get (5.40) with (5.46) and (5.45). q.e.d.

6. Proof of Theorem 2

6.1. Regularity of the limiting measures. In this subsection, we aim at proving the following no neck energy and regularity result, keeping the notations of Proposition 2.

Proposition 3. *For $i \in \{1,\cdots,N\}$, there exists $q_{i,1},\cdots,q_{i,s_i} \in \mathbb{S}^1$ and $e^{\check{u}_i} \in L^\infty(\mathbb{S}^1)$, smooth except maybe at one point, positive such that for all $\rho > 0$,*

$$e^{\check{u}_i^\epsilon} d\theta \rightharpoonup_\star e^{\check{u}_i} d\theta \text{ on } \mathcal{M}(S_i(\rho)) \text{ as } \epsilon \to 0,$$

with $S_i(\rho) = \mathbb{S}^1 \setminus \left(\mathbb{D}_\rho(p) \cup \bigcup_{i=j}^{s_i}\mathbb{D}_\rho(q_{i,j})\right)$ and $\int_{\mathbb{S}^1} e^{\check{u}_i} d\theta = m_i$.

If $m_0 > 0$, there exists $p_1, \cdots, p_s$ and a density e^{u_0} on ∂M, smooth, such that

$$e^{u_\epsilon} d\sigma_g \rightharpoonup_\star e^{u_0} d\sigma_g \text{ on } \mathcal{M}(I(\rho)) \text{ as } \epsilon \to 0$$

with $M(\rho) = M \setminus \bigcup_{i=1}^s B_g(p_i, \rho)$ and $\int_{\partial M} e^{u_0} d\sigma_g = m_0$.

Proof. Let $\tilde{N}$ be such that for $1 \leq i \leq N$,

$$1 \leq i \leq \tilde{N} \Rightarrow \frac{\alpha_\epsilon^i}{\sqrt{\epsilon}} \to +\infty \text{ as } \epsilon \to 0,$$

and

$$\tilde{N} + 1 \leq i \leq N \Rightarrow \frac{\alpha_\epsilon^i}{\sqrt{\epsilon}} \text{ is bounded.}$$

We now reintroduce the indices i we dropped in Section 5:

For $1 \leq i \leq \tilde{N}$ fixed, we recall (see just before Claim 13) that we set

$$\{q_{i,1}, \cdots, q_{i,s_i}\} = \{\lambda^{-1}(p_{0,1}), \cdots, \lambda^{-1}(p_{0,s_0})\}$$

defined by Claim 9 and we recall that (5.9), that is $q_{i,1}, \cdots, q_{i,s_i} \in \mathbb{R} \times \{0\}$ satisfy

$$Z\left(\mathbb{S}^1, \{e^{\check{u}_\epsilon^i} d\theta\}\right) \subset \{p, q_{i,1}, \cdots, q_{i,s_i}\},$$

and that the notations before Claim 13 hold:

$$D_i(\rho) = \mathbb{D} \setminus \left(\mathbb{D}_\rho(p) \cup \bigcup_{j=1}^{s_i} \mathbb{D}_\rho(q_{i,j})\right) \text{ and } S_i(\rho) = \mathbb{S}^1 \cap D_i(\rho),$$

and $\check{\nu}_i$ is the measure without atoms defined by

$$e^{\check{u}_\epsilon^i} d\theta \rightharpoonup_\star \check{\nu}_i \text{ in } \mathcal{M}(S_i(\rho)) \text{ as } \epsilon \to 0$$

for any $\rho > 0$.

For $\tilde{N} + 1 \leq i \leq N$, the notations just before Claim 14 define $e^{\check{u}_i}$ and $e^{\check{u}_i}$ as

$$e^{\hat{u}_\epsilon^i} \to e^{\hat{u}_i} \text{ in } \mathcal{C}^0(I_{\frac{1}{\rho}}) \text{ as } \epsilon \to 0, \text{ and}$$

$$e^{\check{u}_\epsilon^i} \to e^{\check{u}_i} \text{ in } \mathcal{C}^0(\mathbb{S}^1 \setminus \mathbb{D}_\rho(p)) \text{ as } \epsilon \to 0$$

for any $\rho > 0$. Notice that $e^{\check{u}_i} = e^{\hat{u}_i} \circ \lambda$.

We also have $\{p_1, \cdots, p_s\}$ such that (4.1) holds and denote

$$M(\rho) = M \setminus \bigcup_{i=1}^s B_g(p_i, \rho),$$

and

$$I(\rho) = \partial M \setminus \bigcup_{i=1}^s I_g(p_i, \rho),$$

and ν_0 the measure without atoms such that

$$e^{u_\epsilon} d\sigma_g \rightharpoonup_\star \nu_0 \text{ in } \mathcal{M}(I(\rho)) \text{ as } \epsilon \to 0.$$

Then, we have by (4.5) and (4.7) that

$$\int_{\mathbb{S}^1} d\check{\nu}_i \geq m_i \tag{6.1}$$

for $1 \leq i \leq \tilde{N}$ and

$$\int_{\mathbb{S}^1} e^{\check{u}_i} d\theta \geq m_i, \tag{6.2}$$

and by (4.6) and (4.8) that

$$\int_{\partial M} d\nu_0 \geq m_0. \tag{6.3}$$

Considering for $1 \leq i \leq N$ the set $M_i^\epsilon(\rho)$ such that

$$\left(H_{a_i^\epsilon,\alpha_i^\epsilon}\right)^{-1}\left(\widetilde{M_i^\epsilon(\rho)}^{l_i}\right) = \Omega_i(\rho),$$

(4.2), (4.7) give that

$$M(\rho) \cap M_i^\epsilon(\rho) = \emptyset, \tag{6.4}$$

and (4.4) or (4.3) and (4.8) give that

$$i \neq j \Rightarrow M_i^\epsilon(\rho) \cap M_j^\epsilon(\rho) = \emptyset \tag{6.5}$$

for ϵ small enough.

By (6.4) and (6.5), we have for $\rho > 0$ and ϵ small enough

$$\int_M |\nabla \Phi_\epsilon|_g^2 \, dv_g \geq \mathbf{1}_{m_0>0} \int_{M(\rho)} |\nabla \Phi_\epsilon|_g^2 \, dv_g + \sum_{i=1}^N \int_{\Omega_i(\rho)} \left|\nabla \hat{\Phi}_\epsilon^i\right|^2 dx, \tag{6.6}$$

Then, applying (3.28) in Claim 6 if $m_0 > 0$, (5.35) in Claim 13 for $1 \leq i \leq \tilde{N}$, (5.38) and (5.40) in Claim 14 for $\tilde{N}+1 \leq i \leq N$, (6.1), (6.3) and the conservation of the mass (4.9),

$$\sum_{i=0}^N m_i = 1,$$

we get from (6.6) that

$$\begin{aligned}
\sigma_k(M,[g]) &= \lim_{\rho\to 0}\lim_{\epsilon\to 0}\int_M |\nabla\Phi_\epsilon|_g^2 \\
&\geq \mathbf{1}_{m_0>0}\int_M \frac{|\nabla\Phi|_g^2}{\omega} dv_g + \sum_{i=1}^{\tilde{N}}\int_{\mathbb{D}} \frac{\left|\nabla\check{\Phi}_i\right|^2}{\omega_i} dx \\
&\quad + \sum_{i=\tilde{N}+1}^{N}\int_{\mathbb{R}^2_+} \left|\nabla\hat{\Phi}_i\right|^2 dx \\
&\geq \mathbf{1}_{m_0>0}\left(\sigma_k(M,[g])\int_{\partial M} d\nu_0 + \int_M \frac{|\Phi|^2 |\nabla\omega|_g^2}{\omega^3} dv_g\right)
\end{aligned}$$

$$
\begin{aligned}
& + \sum_{i=1}^{\tilde{N}} \left(\sigma_k(M,[g]) \int_{\mathbb{S}^1} d\check{\nu}_i + \int_{\mathbb{D}} \frac{\left|\check{\Phi}_i\right|^2 \left|\nabla \omega_i\right|^2}{\omega_i^3} dx \right) \\
& + \sum_{i=\tilde{N}+1}^{N} \sigma_k(M,[g]) \int_{\mathbb{S}^1} e^{\check{u}_i} d\theta \\
\geq \quad & \sigma_k(M,[g]) + \mathbf{1}_{m_0>0} \int_M \frac{\left|\Phi\right|^2 \left|\nabla \omega\right|_g^2}{\omega^3} dv_g \\
& + \sum_{i=1}^{\tilde{N}} \int_{\mathbb{D}} \frac{\left|\check{\Phi}_i\right|^2 \left|\nabla \omega_i\right|^2}{\omega_i^3} dx.
\end{aligned}
$$

Therefore, all the inequalities are equalities in Claim 6, (6.3), Claim 13, (6.1) and Claim 14. Then, we get for $1 \leq i \leq \tilde{N}$ that $\omega_i = 1$ on $\mathbb{D}$ so that

$$\left|\hat{\Phi}_i\right|^2 = 1 \text{ on } \mathbb{S}^1,$$

for $1 \leq i \leq \tilde{N}$ that

$$\int_{\mathbb{S}^1} d\check{\nu}_i = m_i,$$

and if $m_0 > 0$ that $\omega = 1$ so that

$$\left|\Phi\right|^2 = 1 \text{ on } \partial M,$$

and

$$\int_{\partial M} d\nu_0 = m_0.$$

Let $1 \leq i \leq \tilde{N}$. Then, $\hat{\Psi}_i = \hat{\Phi}_i$ on $\mathbb{R} \times \{0\}$ and equation (5.26) gives that

$$\begin{cases} \Delta \hat{\Phi}_i = 0, \\ (-\partial_t)\hat{\Phi}_i = \sigma_k(M,[g]) \hat{\Phi}_i d\nu_i, \end{cases}$$

in a weak sense on $\mathbb{R} \times \{0\} \setminus \{q_{i,1}, \cdots, q_{i,s_i}\}$. Then, $d\hat{\nu}_i = \frac{\hat{\Phi}_i.(-\partial_t)\hat{\Phi}_i}{\sigma_k(M,[g])} ds$ which means that $\hat{\nu}_i$ is absolutely continuous with respect to ds and

$$\begin{cases} \left|\hat{\Phi}_i\right|^2 = 1 & \text{in } \mathbb{R} \times \{0\}, \\ (-\partial_t)\hat{\Phi}_i \wedge \hat{\Phi}_i = 0 & \text{in } \mathbb{R} \times \{0\}. \end{cases}$$

This means that $\hat{\Phi}_i$ is weakly $\frac{1}{2}$-harmonic on $\mathbb{R}^2_+ \setminus \{q_{i,1}, \cdots, q_{i,s_i}\}$. Then, by Da Lio (see [**5**], Proposition 2.2), since $\int_{\mathbb{R}^2_+} \left|\nabla \hat{\Phi}_i\right|^2 dx < +\infty$, we can extend $\hat{\Phi}_i$ as a $\frac{1}{2}$-harmonic map on $\mathbb{R}^2_+$. By the regularity theory for weakly $\frac{1}{2}$-harmonic maps of Da Lio–Rivière, see [**6**], $\hat{\Phi}_i$ is smooth and $\frac{1}{2}$-harmonic on $\mathbb{R}^2_+$. Setting $e^{\hat{u}_i} = \frac{\hat{\Phi}_i.(-\partial_t)\hat{\Phi}_i}{\sigma_k(M,[g])}$, and coming back to the disk, we get the first part of the claim for $1 \leq i \leq \tilde{N}$.

For $\tilde{N}+1 \leq i \leq N$, the convergence (5.37) ends the proof of the first part of the proposition.

If $m_0 > 0$, then, $\Psi = \Phi$ and equation (3.16) gives that

$$\begin{cases} \Delta_g \Phi = 0, \\ \partial_\nu \Phi = \sigma_k(M,[g]) \Phi d\nu, \end{cases}$$

in a weak sense on $M \setminus \{p_1, \cdots, p_s\}$. Then, $d\nu = \frac{\Phi.\partial_\nu \Phi}{\sigma_k(M,[g])} d\sigma_g$ which means that ν is absolutely continuous with respect to $d\sigma_g$ and

$$\begin{cases} |\Phi|^2 = 1 & \text{in } \partial M, \\ \partial_\nu \Phi \wedge \Phi = 0 & \text{in } \partial M. \end{cases}$$

This means that Φ is weakly harmonic on $M \setminus \{p_1, \cdots, p_s\}$ with free boundary. Then, since $\int_M |\nabla \Phi|^2 dv_g < +\infty$, by Laurain–Petrides (see [**19**], Claim 4), we can extend Φ as a harmonic map on M with free boundary and Φ is smooth on M. The smoothness of weakly harmonic maps with free boundary was proved in [**23**] and [**19**]. Setting $e^u = \frac{\Phi.\partial_\nu \Phi}{\sigma_k(M,[g])}$, we get the second part of the proposition. q.e.d.

6.2. Gaps and no concentration. We prove now by contradiction that $N = 0$, so that the maximizing sequence $\{e^{u_\epsilon} d\sigma_g\}$ does not have any concentration points. Therefore, by Proposition 3 with $m_0 = 1$, the proof of Theorem 2 will follow.

We now assume that $N \geq 1$ and we use Proposition 3 and the gap assumption that (0.3) is strict in order to get a contradiction.

For $1 \leq i \leq N$, let θ_i be the maximal integer such that

$$\frac{\sigma_{\theta_i}(\mathbb{D})}{m_i} < \sigma_k(M,[g]), \tag{6.7}$$

and let θ_0 be the maximal integer such that

$$\frac{\sigma_{\theta_0}(M,[g])}{m_0} < \sigma_k(M,[g]), \tag{6.8}$$

if $m_0 > 0$. We set $\theta_0 = -1$ if $m_0 = 0$. We get that for $i \in \{1, \cdots, N\}$,

$$\sigma_{\theta_i+1}(\mathbb{D}) \geq m_i \sigma_k(M,[g]), \tag{6.9}$$

and

$$\sigma_{\theta_0+1}(M,[g]) \geq m_0 \sigma_k(M,[g]). \tag{6.10}$$

Then, by the spectral gap assumption that (0.3) is strict, we have that

$$\sum_{i=0}^{N} (\theta_i + 1) \geq k+1. \tag{6.11}$$

Indeed, if $\sum_{i=0}^{N} (\theta_i + 1) \leq k$, the spectral gap gives that

$$\sum_{i=1}^{N} \sigma_{\theta_i+1}(\mathbb{D}) + \sigma_{\theta_0+1}(M,[g]) < \sigma_k(M,[g]),$$

and this contradicts (4.9) (6.9) and (6.10).

Now, we define at least $k+1$ test functions for the min-max characterization of $\sigma_\epsilon = \sigma_k(M,g,\partial M, e^{u_\epsilon})$.

Let $1 \leq i \leq N$. We denote by $(\varphi_i^0, \cdots \varphi_i^{\theta_i})$ an orthonormal family of functions in $L^2(\partial M, e^{u_0} dv_g)$ if $i=0$ and in $L^2(\mathbb{S}^1, e^{\check{u}_i} d\theta)$ if $i \neq 0$, such that if $0 \leq j \leq \theta_i$, φ_i^j is an eigenfunction for $\sigma_j(M,g,\partial M, e^{u_0})$ if $i=0$ and for $\sigma_j(\mathbb{D},\xi,\mathbb{S}^1,e^{u_i})$ if $i \neq 0$. Such functions exist by Proposition 3 and lie in $\mathcal{C}^1$.

We fix $\rho > 0$. We denote by η_i some function defined with Claim 1 by

- $\eta_0 \in \mathcal{C}_c^\infty(M(\sqrt{\rho}))$, $\eta_0 \geq 1$ on $M(\rho)$ and $\int_M |\nabla \eta_0|_g^2 dv_g \leq \frac{C}{\ln(\frac{1}{\rho})}$.
- If $i \neq 0$, $\eta_i \in \mathcal{C}_c^\infty(S_i(\sqrt{\rho}))$, $\eta_i \geq 1$ on $S_i(\rho)$ and $\int_{\mathbb{D}} |\nabla \eta_i|^2 dx \leq \frac{C}{\ln(\frac{1}{\rho})}$.

We set for $0 \leq i \leq N$ and $0 \leq j \leq \theta_i$ some test functions ξ_i^j, defined by

$$\xi_0^j = \eta_0 \varphi_0^j \text{ on } M,$$

and if $i \neq 0$, ξ_i^j depends on ϵ and satisfies for any $\epsilon > 0$

$$\left(\check{\xi_i^j}\right)_\epsilon^i = \eta_i \varphi_i^j \text{ on } \mathbb{D}$$

extended by 0 on M.

Note that all the test functions ξ_i^j lie in $\mathcal{C}^1$ and are uniformly bounded. Note also that by (6.4) and (6.5), if ϵ small enough,

$$i \neq i' \Rightarrow supp(\xi_i^j) \cap supp(\xi_{i'}^{j'}) = \emptyset$$

for $i,i' \in \{0,\cdots,N\}$, $0 \leq j \leq \theta_i$ and $0 \leq j' \leq \theta_{i'}$. For $1 \leq i \leq N$, we let E_i be the vector space spanned by $(\xi_i^0, \xi_i^1, \cdots, \xi_i^{\theta_i})$ and with (6.11), we deduce by (1.1) that

$$\sigma_\epsilon \leq \max_{0 \leq i \leq N} \sup_{\xi \in E_i \setminus \{0\}} \frac{\int_M |\nabla \xi|_g^2 dv_g}{\int_{\partial M} \xi^2 e^{u_\epsilon} d\sigma_g}. \tag{6.12}$$

Let $i \in \{1,\cdots,N\}$. For $\xi = \sum_{j=0}^{\theta_i} \mu_j \xi_i^j \in E_i$, with $\mu_j \in \mathbb{R}$ and $\sum_j \mu_j^2 = 1$, we get

$$\int_M |\nabla \xi|_g^2 dv_g = \int_{\mathbb{D}} \left| \nabla \left(\eta_i \sum_{j=0}^{\theta_i} \mu_j \varphi_i^j \right) \right|^2 dx,$$

and denoting $\varphi = \sum_{j=0}^{\theta_i} \mu_j \varphi_i^j$, we have

$$\int_M |\nabla \xi|_g^2 dv_g = \int_{\mathbb{D}} (\eta_i)^2 |\nabla \varphi|^2 dx + 2 \int_{\mathbb{D}} \eta_i \varphi \langle \nabla \eta_i, \nabla \varphi \rangle dx$$

$$\begin{aligned}
& + \int_{\mathbb{D}} \varphi^2 \left|\nabla \eta_i\right|^2 dx \\
\leq \; & \int_{\mathbb{D}} \left|\nabla \varphi\right|^2 dx \\
& +2 \left\|\eta_i \varphi\right\|_\infty \left(\int_{\mathbb{D}} \left|\nabla \varphi\right|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{D}} \left|\nabla \eta_i\right|^2 dx\right)^{\frac{1}{2}} \\
& \quad + \left\|\varphi\right\|_\infty^2 \int_{\mathbb{D}} \left|\nabla \eta_i\right|^2 dx \\
\leq \; & \int_{\mathbb{D}} \left|\nabla \varphi\right|^2 dx + O\left(\frac{1}{\sqrt{\ln(\frac{1}{\rho})}}\right) \text{ as } \rho \to 0.
\end{aligned}$$

We also have that

$$\int_{\partial M} \xi^2 e^{u_\epsilon} d\sigma_g = \int_{\mathbb{S}^1} \eta_i^2 \varphi^2 e^{\check{u}_\epsilon^i} d\theta.$$

By Proposition 3, we get that

$$\int_{\partial M} \xi^2 e^{u_\epsilon} d\sigma_g = \int_{\mathbb{S}^1} \eta_i^2 \varphi^2 e^{\check{u}_i} d\theta + o(1) \text{ as } \epsilon \to 0,$$

so that

$$\lim_{\epsilon \to 0} \int_{\partial M} \xi^2 e^{u_\epsilon} d\sigma_g \geq \int_{\mathbb{S}^1} \varphi^2 e^{\check{u}_i} d\theta + o(1) \text{ as } \rho \to 0.$$

The same work can be done for $\xi \in E_0$, so that passing to the limit as $\epsilon \to 0$ and then as $\rho \to 0$ in (6.12), we get

$$\sigma_k(M, [g]) \leq \max \left\{ \max_{1 \leq i \leq N} \sup_{\varphi \in F_i \setminus \{0\}} \frac{\int_{\mathbb{D}} \left|\nabla \varphi\right|^2 dx}{\int_{\mathbb{S}^1} \varphi^2 e^{\check{u}_i} d\theta}, \sup_{\varphi \in F_0 \setminus \{0\}} \frac{\int_M \left|\nabla \varphi\right|_g^2 dv_g}{\int_{\partial M} \varphi^2 e^{\check{u}_i} d\sigma_g} \right\},$$

where F_i is the space spanned by $\varphi_i^0, \cdots, \varphi_i^{\theta_i}$. Therefore,

$$\begin{aligned}
\sigma_k(M, [g]) \; & \leq \; \max \left\{ \max_{1 \leq i \leq N} \sigma_{\theta_i}(\mathbb{D}, \xi, \mathbb{S}^1, e^{\check{u}_i}), \sigma_{\theta_0}(M, g, \partial M, e^{u_0}) \right\} \\
& \leq \; \max \left\{ \max_{1 \leq i \leq N} \frac{\sigma_{\theta_i}(\mathbb{D})}{m_i}, \frac{\sigma_{\theta_0}(M, [g])}{m_0} \right\},
\end{aligned}$$

which contradicts (6.9) and (6.10). Therefore, there is no concentration of $\{e^{u_\epsilon} d\sigma_g\}$.

Therefore, $N = 0$ and by Proposition 3 with $m_0 = 1$, Theorem 2 follows.

7. Proof of Theorem 1

We prove Theorem 1 in this section. Notice that light modifications of the proof allow us to prove that if (0.3) is strict, the set of maximal metrics for $\sigma_k(M,[g])$ is compact, and if we have that (0.2) is strict, the set of maximal metrics for $\sigma_k(\gamma,m)$ is compact.

Let $\gamma \geq 0$ and $m \geq 1$ be such that $(\gamma,m) \neq (0,1)$ and $[g_\alpha]$ be a sequence of conformal classes on a compact oriented manifold of genus γ with m boundary components such that

$$\sigma_\alpha = \sigma_k(M,[g_\alpha]) \to \sigma_k(\gamma,m) \text{ as } \alpha \to +\infty, \tag{7.1}$$

where g_α denotes the unique metric in its conformal class such that

- The curvature of g_α is constant, equal to 0 if $(\gamma,m)=(0,2)$, and -1 if $(\gamma,m) \neq (0,2)$.
- The boundary ∂M of M is a union of closed geodesics with respect to g_α.

By the gap assumption that (0.2) is strict, we have, in particular, that

$$\sigma_k(M,[g_\alpha]) > \max_{\substack{1\leq j\leq k\\ i_1+\cdots+i_s=j}} \sigma_{k-j}(M,[g_\alpha]) + \sum_{m=1}^{s} \sigma_{i_m}(\mathbb{D}^2,[\xi])$$

for α large enough. By Theorem 2, this gives some smooth harmonic maps with free boundary $\phi_\alpha : (M,g_\alpha) \to \mathbb{S}^{n_\alpha}$ for some $n_\alpha > 0$, such that if $\tilde{g}_\alpha$ is a metric conformal to g_α with the induced metric on the boundary ∂M satisfying

$$d\sigma_{\tilde{g}_\alpha} = e^{u_\alpha} d\sigma_{g_\alpha},$$

where

$$e^{u_\alpha} = \frac{\Phi_\alpha.\partial_{\nu_\alpha}\Phi_\alpha}{\sigma_\alpha},$$

then $\int_{\partial M} d\sigma_{\tilde{g}_\alpha} = 1$ and $\sigma_k(M,\tilde{g}_\alpha) = \sigma_k(M,[g_\alpha])$. Since the multiplicity of σ_k is bounded by a constant which only depends on k, γ and m (see [**9**] and [**18**]), we can assume that $n = n_\alpha$ is fixed.

We have the following quantification result on sequences of harmonic maps with free boundary by Laurain–Petrides, [**19**], Theorem 1:

Proposition 4. *Let (M,g) be a smooth Riemannian surface with a smooth non-empty boundary. We refer to the notations introduced in Section 1.1 for the metric g. Let $q_1,\cdots,q_t \in M$. Let $\Phi_\alpha : (M_\alpha,g_\alpha) \to \mathbb{B}^{n+1}$ be a harmonic map with free boundary on an open set $M_\alpha \subset M$ such that*

- *For any $\rho > 0$, there exists $\alpha_\rho > 0$ such that for any $\alpha > \alpha_\rho$, $M_\alpha \supset M \setminus \bigcup_{i=1}^t B_g(q_i,\rho)$.*
- *For any $\rho > 0$, $g_\alpha \to g$ in $M \setminus \bigcup_{i=1}^t B_g(q_i,\rho)$ as $\alpha \to +\infty$.*
- *$\Phi_\alpha.\partial_{\nu_\alpha}\Phi_\alpha > 0$ on $M_\alpha \cap \partial M$ and*

$$\limsup_{\alpha\to+\infty} \int_{M_\alpha\cap\partial M} \Phi_\alpha.\partial_{\nu_\alpha}\Phi_\alpha d\sigma_{g_\alpha} < +\infty.$$

Then, up to the extraction of a subsequence, there exist

- *Some harmonic map with free boundary $\Phi : M \to \mathbb{S}^n$.*

- *Sequences of points* $p_\alpha^1, \cdots, p_\alpha^s$ *of* ∂M *converging to some points* $p^1, \cdots, p^s$ *of* ∂M *as* $\alpha \to +\infty$ *and sequences of scales* $\delta_\alpha^1, \cdots, \delta_\alpha^s$ *converging to* 0 *as* $\alpha \to +\infty$ *such that*

$$\frac{d_g(p_\alpha^i, p_\alpha^j)}{\delta_\alpha^i + \delta_\alpha^j} + \frac{\delta_\alpha^i}{\delta_\alpha^j} + \frac{\delta_\alpha^j}{\delta_\alpha^i} \to +\infty \text{ as } \alpha \to +\infty. \tag{7.2}$$

- *Some harmonic extensions of non-constant* $\frac{1}{2}$*-harmonic maps,* $\omega_1, \cdots, \omega_s : \mathbb{D} \to \mathbb{B}^{n+1}$,

such that

$$\int_M |\nabla \Phi|_g^2 \, dv_g + \sum_{i=1}^{s} \int_{\mathbb{D}} |\nabla \omega_i|^2 \, dx = \mathcal{E}, \tag{7.3}$$

where

$$\mathcal{E} = \lim_{\rho \to 0} \lim_{\alpha \to +\infty} \int_{\partial M \setminus \bigcup_{i=1}^t I_g(q_i, \rho)} \Phi_\alpha . \partial_{\nu_\alpha} \Phi_\alpha d\sigma_{g_\alpha},$$

and for all $\rho > 0$,

$$\Phi_\alpha . \partial_{\nu_\alpha} \Phi_\alpha d\sigma_{g_\alpha} \rightharpoonup_\star \Phi . \partial_\nu \Phi d\sigma_g \text{ on } I(\rho), \tag{7.4}$$

$$\hat{\Phi}_\alpha^i . \left(-\partial_t \hat{\Phi}_\alpha^i\right) ds \rightharpoonup_\star \hat{\omega}_i . \left(-\partial_t \hat{\omega}_i\right) ds \text{ on } \Gamma_i(\rho), \tag{7.5}$$

where we define the sets

$$I(\rho) = \partial M \setminus \left(\bigcup_{i=1}^{t} I_g(q_i, \rho) \cup \bigcup_{z \in Z(\partial M \setminus \bigcup_{i=1}^t I_g(q_i,\rho), \Phi_\alpha . \partial_{\nu_\alpha} \Phi_\alpha d\sigma_{g_\alpha})} I_g(z, \rho) \right),$$

$$\Gamma_i(\rho) = I_{\frac{1}{\rho}} \setminus \bigcup_{z \in Z(I_{\frac{1}{\rho}}, \hat{\Phi}_\alpha^i . (-\partial_t \hat{\Phi}_\alpha^i) ds)} I_\rho(z),$$

and the functions on $\mathbb{R}_+^2$

$$\hat{\Phi}_\alpha^i(x) = \widetilde{\Phi_\alpha}^{l_i}(\delta_\alpha^i x + \tilde{p}_\alpha^{l_i}) \text{ and } \hat{\omega}_i = \omega_i \circ \lambda^{-1},$$

where $1 \leq l_i \leq L$ *is chosen such that* $p^i \in \omega_{l_i}$ *and* λ *is defined page 132.*

Assuming that $g_\alpha \to g$ as $\alpha \to +\infty$ for some metric g with constant curvature and which defines closed geodesics boundary components, we apply Proposition 4 for $M_\alpha = M$, Φ_α, g_α and g. Notice that the use of Proposition 4 together with the gap assumption that (0.2) is strict follows exactly the same path as the use of Proposition 3 together with the gap assumption that (0.3) is strict in order to prove that the maximizing sequences do not have any concentration points. Therefore, one can easily contradict the fact that (0.2) is assumed to be strict in this case.

We assume now that the sequence of conformal classes $[g_\alpha]$ degenerates in the following sense:

- If $(\gamma, m) = (0,2)$, in the case of the annulus, this means that $R_\alpha \to +\infty$ or $R_\alpha \to 1$, where $R_\alpha > 1$ denotes the real parameter such that (M, g_α) is isometric to $\mathbb{D}_{R_\alpha} \setminus \mathbb{D}$.
- If $(\gamma, m) \neq (0,2)$, in the hyperbolic case, this means that the injectivity radius $i_{g_\alpha}(M) \to 0$ as $\alpha \to +\infty$ so that there exist closed geodesics which length goes to 0 or geodesics which cross two boundary components of (M, g_α) with length going to 0.

Let's tackle both cases in order to contradict that the gap (0.2) is strict. During all the proof, we identify $\mathbb{R}^2$ and $\mathbb{C}$ thanks to the map $F(x,y) = x + iy$.

7.1. The case of the annulus. Let $(\gamma, m) = (0,2)$. Then, (M, g_α) is isometric to $(\mathbb{D}_{R_\alpha} \setminus \mathbb{D}, \xi)$.

We first assume that $R_\alpha \to +\infty$ as $\alpha \to +\infty$. We denote by $\Gamma_1 = \mathbb{S}^1$ and $\Gamma_2 = \mathbb{S}^1_{R_\alpha}$ the boundary components,

$$m_1 = \lim_{\alpha \to +\infty} \int_{\Gamma_1} e^{u_\alpha} d\sigma_\xi \text{ and } m_2 = \lim_{\alpha \to +\infty} \int_{\Gamma_2} e^{u_\alpha} d\sigma_\xi.$$

With the inversion $\iota(z) = \frac{1}{\bar{z}}$, we have $\iota(\mathbb{D}_{R_\alpha} \setminus \mathbb{D}) = \mathbb{D} \setminus \mathbb{D}_{\frac{1}{R_\alpha}}$, $\iota(\Gamma_1) = \mathbb{S}^1$ and the harmonic map with free boundary

$$\Phi^1_\alpha = \Phi_\alpha \circ \iota : \mathbb{D} \setminus \mathbb{D}_{\frac{1}{R_\alpha}} \to \mathbb{B}^{n+1}$$

satisfies the hypotheses of Proposition 4 on $(\mathbb{D}, \xi)$ since $\mathbb{D} \setminus \mathbb{D}_{\frac{1}{R_\alpha}}$ exhausts $\mathbb{D}$. We have some limits $\Phi^1, \omega^1_1, \cdots, \omega^1_{s_1}$ such that

$$\int_{\mathbb{D}} \left|\nabla \Phi^1\right|^2 dx + \sum_{i=1}^{s_1} \int_{\mathbb{D}} \left|\nabla \omega^2_i\right|^2 dx = m_1,$$

and the conclusion of Proposition 4 holds for some associated scales.

With the dilatation $H(z) = \frac{z}{R_\alpha}$, we have $H(\mathbb{D}_{R_\alpha} \setminus \mathbb{D}) = \mathbb{D} \setminus \mathbb{D}_{\frac{1}{R_\alpha}}$, $H(\Gamma_2) = \mathbb{S}^1$ and the harmonic map with free boundary

$$\Phi^2_\alpha = \Phi_\alpha \circ H^{-1} : \mathbb{D} \setminus \mathbb{D}_{\frac{1}{R_\alpha}} \to \mathbb{B}^{n+1}$$

satisfies the hypotheses of Proposition 4 on $(\mathbb{D}, \xi)$ since $\mathbb{D} \setminus \mathbb{D}_{\frac{1}{R_\alpha}}$ exhausts $\mathbb{D}$. We have some limits $\Phi^2, \omega^2_1, \cdots, \omega^2_{s_2}$ such that

$$\int_{\mathbb{D}} \left|\nabla \Phi^2\right|^2 dx + \sum_{i=1}^{s_1} \int_{\mathbb{D}} \left|\nabla \omega^2_i\right|^2 dx = m_2,$$

and the conclusion of Proposition 4 holds for some associated scales.

Following the proof of Section 6.2, we use suitable eigenfunctions associated to the previous smooth limiting maps at their respective concentration scales as test functions for σ_α. They give a contradiction for the assumption that (0.2) is strict which reads as

$$\sigma_k(0,2) > \max_{i_1+\cdots+i_s=k} \sum_{q=1}^{s} \sigma_{i_q}(0,1)$$

on the annulus, for $s = 2 + s_1 + s_2$.

We now assume that $R_\alpha \to 1$ as $\alpha \to +\infty$. Then thanks to the application

$$f(z) = \exp\left(\left(z + \frac{\pi}{4}\right)\frac{2\ln(R_\alpha)}{\pi}\right),$$

we have

$$f(T_\alpha) = \mathbb{D}_{R_\alpha} \setminus \mathbb{D}$$

with

$$T_\alpha = \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \times [0, b_\alpha] \text{ and } b_\alpha = \frac{\pi^2}{\ln(R_\alpha)} \to +\infty \text{ as } \alpha \to +\infty.$$

Notice that we identify $\{Im(z) = 0\}$ and $\{Im(z) = b_\alpha\}$ and that $\{Re(z) = -\frac{\pi}{4}\}$ and $\{Re(z) = \frac{\pi}{4}\}$ correspond to the boundary components of the annulus. We denote by

$$I_\alpha = \left(\{-\frac{\pi}{4}\} \cup \{\frac{\pi}{4}\}\right) \times [0, b_\alpha],$$

and for $0 \le r \le s \le b_\alpha$,

$$I_\alpha(r,s) = \{(x,y) \in I_\alpha; r \le y \le s\}.$$

For sequences $\{r_\alpha\}$ and $\{s_\alpha\}$, $r_\alpha \ll s_\alpha$ means $s_\alpha - r_\alpha \to +\infty$ as $\alpha \to +\infty$. Then, denoting again $\tilde{g}_\alpha$ on T_α the metric $f^\star(\tilde{g}_\alpha)$ we claim that

Claim 15. *If some sequences $\{r_\alpha^i\}$ and $\{s_\alpha^i\}$ for $1 \le i \le t$ satisfy*

$$0 = s_\alpha^0 \ll r_\alpha^1 \ll s_\alpha^1 \ll \cdots \ll r_\alpha^t \ll s_\alpha^t \ll r_\alpha^{t+1} = b_\alpha$$

and

$$m_j = \lim_{\alpha\to+\infty} L_{\tilde{g}_\alpha}(I_\alpha(r_\alpha^i, s_\alpha^i)) > 0$$

for $1 \le i \le t$, then $t \le k$.

Proof. We proceed by contradiction and assume that we have such sequences with $t \ge k+1$. Let $\theta_\alpha \to +\infty$ be such that $\theta_\alpha = o(r_\alpha^{i+1} - s_\alpha^i)$ as $\alpha \to +\infty$ for $0 \le i \le t$. We set for $1 \le i \le t$

$$\eta_\alpha^i = \begin{cases} 1 & r_\alpha^i \le y \le s_\alpha^i, \\ \dfrac{y - r_\alpha^i + \theta_\alpha}{\theta_\alpha} & r_\alpha^i - \theta_\alpha \le y \le r_\alpha^i, \\ \dfrac{s_\alpha^i + \theta_\alpha - y}{\theta_\alpha} & s_\alpha^i \le y \le s_\alpha^i + \theta_\alpha, \\ 0 & y \ge s_\alpha^i + \theta_\alpha \text{ or } y \le r_\alpha^i - \theta_\alpha. \end{cases}$$

Then,

$$\int_{T_\alpha} |\nabla \eta_i^\alpha|^2_{\tilde g_\alpha} dv_{\tilde g_\alpha} = \int_{T_\alpha} |\nabla \eta_i^\alpha|^2 dx = \frac{2}{\theta_\alpha} = o(1) \text{ as } \alpha \to +\infty,$$
$$\int_{I_\alpha} (\eta_i^\alpha)^2 d\sigma_{\tilde g_\alpha} \geq m_j + o(1) \text{ as } \alpha \to +\infty.$$

Taking these at least $k+1$ functions with pairwise disjoint support for the variational characterization of $\sigma_\alpha = \sigma_k(M, \tilde g_\alpha)$ (1.1) gives that

$$\sigma_\alpha \leq \max_{1\leq i\leq k+1} \frac{\int_{T_\alpha} |\nabla \eta_i^\alpha|^2_{\tilde g_\alpha} dv_{\tilde g_\alpha}}{\int_{I_\alpha} (\eta_i^\alpha)^2 d\sigma_{\tilde g_\alpha}} = o(1) \text{ as } \alpha \to +\infty,$$

which contradicts (7.1). q.e.d.

Now, we prove that up to a rotation on M, there exist sequences $0 \ll r_\alpha \ll s_\alpha \ll b_\alpha$ such that

$$\lim_{\alpha\to+\infty} L_{\tilde g_\alpha}(I_\alpha(r_\alpha, s_\alpha)) = 1. \tag{7.6}$$

Indeed, denying (7.6) would mean that for any sequence $1 \ll u_\alpha \ll v_\alpha \ll b_\alpha$,

$$\lim_{\alpha\to+\infty} L_{\tilde g_\alpha}(I_\alpha(u_\alpha, v_\alpha)) > 0.$$

Taking for $1 \leq j \leq k+1$ $y_\alpha^j = \frac{j}{k+2} b_\alpha$ and $\theta_\alpha = \sqrt{b_\alpha}$ gives for $1 \leq j \leq k+1$

$$m_j = \lim_{\alpha\to+\infty} L_{\tilde g_\alpha}(I_\alpha(y_\alpha^j - \theta_\alpha, y_\alpha^j + \theta_\alpha)) > 0,$$

so that the $k+1$ test functions for $\sigma_\alpha = \sigma_k(M, \tilde g_\alpha)$ with pairwise disjoint support,

$$\eta_\alpha^j = \begin{cases} 1 & y_\alpha^j - \theta_\alpha \leq y \leq y_\alpha^j + \theta_\alpha, \\ \dfrac{y - y_\alpha^j + 2\theta_\alpha}{\theta_\alpha} & y_\alpha^j - 2\theta_\alpha \leq y \leq y_\alpha^j - \theta_\alpha, \\ \dfrac{y_\alpha^j + 2\theta_\alpha - y}{\theta_\alpha} & y_\alpha^j + \theta_\alpha \leq y \leq y_\alpha^j + 2\theta_\alpha, \\ 0 & y \geq y_\alpha^j + 2\theta_\alpha \text{ or } y \leq y_\alpha^j - 2\theta_\alpha, \end{cases}$$

would satisfy

$$\int_{T_\alpha} \left|\nabla \eta_j^\alpha\right|^2_{\tilde g_\alpha} dv_{\tilde g_\alpha} = \frac{2}{\theta_\alpha} = o(1) \text{ as } \alpha \to +\infty,$$
$$\int_{I_\alpha} \left(\eta_j^\alpha\right)^2 d\sigma_{\tilde g_\alpha} \geq m_j + o(1) \text{ as } \alpha \to +\infty,$$

so that $\sigma_\alpha = o(1)$ by (1.1). This contradicts again (7.1).

We take a rotation of M so that (7.6) holds. Then, by Claim 15, we can take t the maximal integer such that there exist sequences

$$0 = s_\alpha^0 \ll r_\alpha^1 \ll s_\alpha^1 \ll \cdots \ll r_\alpha^t \ll s_\alpha^t \ll r_\alpha^{t+1} = b_\alpha$$

with

$$m_j = \lim_{\alpha \to +\infty} L_{\tilde{g}_\alpha}(I_\alpha(r_\alpha^j, s_\alpha^j)) > 0,$$

and

$$\sum_{j=1}^t m_j = 1.$$

We define a sequence $r_\alpha^j < y_\alpha^j < s_\alpha^j$ such that

$$\lim_{\alpha \to +\infty} L_{\tilde{g}_\alpha}(I_\alpha(r_\alpha^j, y_\alpha^j)) = \lim_{\alpha \to +\infty} L_{\tilde{g}_\alpha}(I_\alpha(y_\alpha^j, s_\alpha^j)) = \frac{m_j}{2},$$

and

$$\Psi_\alpha^j(x+iy) = \tan(x + i(y - y_\alpha^j))$$

for $z = x+iy \in T_\alpha$. We consider the harmonic map $\check{\Phi}_\alpha^j = \Phi_\alpha \circ \left(\Psi_\alpha^j\right)^{-1}$ on $\mathbb{D}$. We let $\theta_\alpha \to +\infty$ be such that $\theta_\alpha = o(r_\alpha^{j+1} - s_\alpha^j)$ for all $0 \le j \le t$. Then,

$$D_\alpha^j = \Psi_\alpha^j(T_\alpha(r_\alpha^j - \theta_\alpha, r_\alpha^j + \theta_\alpha))$$

exhausts $\mathbb{D}$,

$$S_\alpha^j = \Psi_\alpha^j(I_\alpha(r_\alpha^j - \theta_\alpha, r_\alpha^j + \theta_\alpha))$$

exhausts $\mathbb{S}^1$, and

$$\lim_{\alpha \to +\infty} L_{\check{\tilde{g}}_\alpha}(S_\alpha^j) = m_j,$$

where $\check{\tilde{g}}_\alpha = \left(\Psi_\alpha^j\right)_\star \tilde{g}_\alpha$.

Then, we apply Proposition 4 on $(\mathbb{D}, \xi)$ to $\check{\Phi}_\alpha^j : (D_\alpha^j, S_\alpha^j) \to (\mathbb{B}^{n+1}, \mathbb{S}^n)$. In order to define suitable test functions which naturally extend to the surface, we have to prove that $\mathbf{1}_{S_\alpha^j} \check{\Phi}_\alpha^j . \partial_\nu \check{\Phi}_\alpha^j d\theta$ does not concentrate at the poles $(0,1)$ and $(0,-1)$. Let's prove it by contradiction: if, for instance, we have

$$\mathbf{1}_{S_\alpha^j} \check{\Phi}_\alpha^j . \partial_\nu \check{\Phi}_\alpha^j d\theta \rightharpoonup_\star m\delta_{(0,1)} + \nu \text{ on } \mathbb{S}^1$$

with $m > 0$, and $\nu(\{(0,1)\}) = 0$, then, $\int_{\mathbb{S}^1} d\nu > 0$ and up to the extraction of a subsequence, we can build $c_\alpha^j \ll y_\alpha^j$ such that

$$\lim_{\alpha \to +\infty} L_{\tilde{g}_\alpha}(I_\alpha(r_\alpha^j - \theta_\alpha, c_\alpha^j)) = m,$$

so that if we set $\overline{r_\alpha} = y_\alpha^j + \tau_\alpha$ and $\overline{s_\alpha} = c_\alpha^j + \tau_\alpha$ with $\tau_\alpha = \sqrt{y_\alpha^j - c_\alpha^j}$, we have

$$m_j^1 = \lim_{\alpha \to +\infty} L_{\tilde{g}_\alpha}(I_\alpha(r_\alpha^j - \theta_\alpha, \overline{s_\alpha})) > 0, \text{ and}$$

$$m_j^2 = \lim_{\alpha \to +\infty} L_{\tilde{g}_\alpha}(I_\alpha(\overline{r_\alpha}, s_\alpha^j + \theta_\alpha)) > 0$$

with $m_j^1 + m_j^2 = m_j$ and this contradicts the maximality of t.

Therefore, we use eigenfunctions associated to the densities associated to the limits of $\check{\Phi}_\alpha^j$ given by Proposition 4 and we follow the computations of Section 6.2. This defines test functions for the variational characterization (1.1) of $\sigma_\alpha = \sigma_k(M, \tilde{g}_\alpha)$. Since (0.2) is strict, as already said,

$$\sigma_k(0,2) > \max_{i_1+\cdots+i_s=k} \sum_{q=1}^{s} \sigma_{i_q}(0,1),$$

and we have at least $k+1$ test functions which would give a contradiction.

7.2. The hyperbolic case. Now, we assume that $(\gamma, m) \neq (0,2)$. We let $\gamma_\alpha^1, \cdots, \gamma_\alpha^s$ the geodesics whose length $l_\alpha^1, \cdots, l_\alpha^s$ go to 0 as $\alpha \to +\infty$, where $1 \leq s \leq 3\gamma - 3 + m$ ([**17**], IV, Lemma 4.1) satisfying one of these conditions

(i) For $1 \leq i \leq s_1$, γ_α^i is a boundary component, that is a closed geodesic such that $\gamma_\alpha^i \subset \partial M$.
(ii) For $s_1+1 \leq i \leq s_1+s_2$, γ_α^i is a closed geodesic such that $\gamma_\alpha^i \cap \partial M = \emptyset$.
(iii) For $s_1+s_2+1 \leq i \leq s_1+s_2+s_3$, γ_α^i is a geodesic which crosses two distinct boundary components at its ends.
(iv) For $s_1+s_2+s_3+1 \leq i \leq s_1+s_2+s_3+s_4 = s$, γ_α^i is a geodesic which crosses one boundary component at its ends.

The collar lemma ([**27**], Lemma 4.2) gives for $1 \leq i \leq s$ an open neighborhood P_α^i of γ_α^i isometric to the cylinder

$$\{(t,\theta), -\mu_\alpha^i < t < \mu_\alpha^i, 0 \leq \theta \leq 2\pi\},$$

if γ_α^i satisfies (ii) or (iii) and

$$\{(t,\theta), 0 \leq t < \mu_\alpha^i, 0 \leq \theta \leq 2\pi\},$$

if γ_α^i satisfies (i), endowed with the metric

$$\left(\frac{l_\alpha^i}{2\pi \cos\left(\frac{l_\alpha^i t}{2\pi}\right)} \right)^2 (dt^2 + d\theta^2)$$

with

$$\mu_\alpha^i = \frac{\pi}{l_\alpha^i}\left(\pi - 2\arctan\left(\sinh\left(\frac{l_\alpha^i}{2}\right)\right)\right).$$

Note that the geodesic γ_α^i corresponds to the line $\{t=0\}$. Note also that in the cases (i) and (ii) we identify the segments $\{\theta = 0\}$ and $\{\theta = 2\pi\}$ and that in the case (iii), the segments $\{\theta = 0\}$ and $\{\theta = 2\pi\}$ correspond to portions of the boundary components crossed by γ_α^i. In the following, we identify P_α^i with the corresponding cylinder.

We denote $M_\alpha^1, \cdots, M_\alpha^r$ the connected components of $M \setminus \bigcup_{i=1}^s P_\alpha^i$ so that

$$M = \left(\bigcup_{i=1}^s P_\alpha^i \right) \cup \left(\bigcup_{j=1}^r M_\alpha^j \right)$$

is a disjoint union. For $s_1 + s_2 + 1 \leq i \leq s$, and $-\mu_\alpha^i < a < b < \mu_\alpha^i$, we denote

$$P_\alpha^i(a,b) = \{(t,\theta); a < t < b\},$$

and for $c = \{c^{i,-}, c^{i,+}\}_{s_1+s_2+1 \leq i \leq s}$, we denote $M_\alpha^j(c)$ the connected component of

$$M \setminus \left(\bigcup_{i=1+s_1+s_2}^{s} P_\alpha^i(-\mu_\alpha^i + c^{i,-}, \mu_\alpha^i - c^{i,+}) \cup \bigcup_{i=s_1+1}^{s_2} \gamma_\alpha^i \right),$$

which contains M_α^j. We also denote

$$I_\alpha^i = M_\alpha^i \cap \partial M,$$

and for $c = \{c^{i,-}, c^{i,+}\}_{s_1+s_2+1 \leq i \leq s}$,

$$I_\alpha^i(c) = M_\alpha^i(c) \cap \partial M.$$

For all the proof, we identify $\mathbb{R}^2$ and $\mathbb{C}$ thanks to the map $F(x,y) = x + iy$.

Let $1 \leq i \leq s_1$. Then, γ_α^i satisfies the condition (i). Then, the image by the map $E : z \mapsto e^{iz}$, of P_α^i is an annulus $\mathbb{D} \setminus \mathbb{D}_{e^{-\mu_\alpha^i}}$ which exhausts $\mathbb{D}$, where $\mathbb{S}^1$ is the image of the closed geodesic. The map $\check{\Phi}_\alpha^i = \Phi_\alpha^i \circ E^{-1} : \mathbb{D} \setminus \mathbb{D}_{e^{-\mu_\alpha^i}} \to \mathbb{B}^{n+1}$ satisfies the hypotheses of Proposition 4 and we get some regular limits $\check{\Phi}^i, \omega_1^i, \cdots, \omega_{t_i}^i$ such that

$$\int_{\mathbb{D}} \left|\nabla \check{\Phi}^i\right|^2 dx + \sum_{j=1}^{t_j} \int_{\mathbb{D}} \left|\nabla \omega_1^j\right|^2 dx = \lim_{\alpha \to +\infty} \int_{\gamma_\alpha^i} e^{u_\alpha} d\sigma_{\tilde{g}_\alpha},$$

and the conclusion of the proposition holds for some associated scales and gives natural test functions.

Let $s_1 + s_2 + 1 \leq i \leq s$. Then, γ_α^i satisfies the condition (iii). We denote by

$$\Gamma_\alpha^i = \{(\theta,t) \in P_\alpha^i; \theta = 0 \text{ or } \theta = 2\pi\},$$

and for $-\mu_\alpha^i \leq a \leq b \leq \mu_\alpha^i$,

$$\Gamma_\alpha^i(a,b) = \{(\theta,t) \in \Gamma_\alpha^i; a \leq t \leq b\}.$$

We denote $a_\alpha \ll b_\alpha$ if two sequences a_α and b_α satisfy $b_\alpha - a_\alpha \to +\infty$ as $\alpha \to +\infty$. Then, we claim that

Claim 16. *If for integers $t_i \geq 0$, some sequences $a_\alpha^{i,l}$, $b_\alpha^{i,l}$ for $1 \leq l \leq t_i$, $c_\alpha = \{c_\alpha^{i,+}, c_\alpha^{i,-}\}$ and a set $J \subset \{1, \cdots, r\}$ satisfy*

$$-\mu_\alpha^i \ll -\mu_\alpha^i + c_\alpha^{i,-} = b_\alpha^{i,0} \ll a_\alpha^{i,1} \ll b_\alpha^{i,1} \ll \cdots$$
$$\ll a_\alpha^{i,t_i} \ll b_\alpha^{i,t_i} \ll a_\alpha^{i,t_{i+1}} = \mu_\alpha^i - c_\alpha^{i,+} \ll \mu_\alpha^i,$$

and for $1 \leq i \leq s$, $1 \leq l \leq t_i$, $j \in J$,

$$m_{i,l} = \lim_{\alpha \to +\infty} L_{\tilde{g}_\alpha}(\Gamma_\alpha(a_\alpha^{i,l}, b_\alpha^{i,l})) > 0,$$
$$m_j = \lim_{\alpha \to +\infty} L_{\tilde{g}_\alpha}(I_\alpha^j(c_\alpha)) > 0,$$

then, $\sum_{i=1}^s t_i + |J| \leq k$.

Proof. By contradiction, we assume that there exist such sequences with $\sum_{i=1}^s t_i + |J| \geq k+1$. Let $\theta_\alpha \to +\infty$ be such that $\theta_\alpha = o(a_\alpha^{i,l+1} - b_\alpha^{i,l})$ for $1 \leq i \leq s$ and $0 \leq l \leq t_i$. We set $\eta_\alpha^{i,l}$ be such that $supp(\eta_\alpha^{i,l}) \subset P_\alpha^i$ and

$$\eta_\alpha^{i,l} = \begin{cases} 1 & a_\alpha^{i,l} \leq t \leq b_\alpha^{i,l} + \theta_\alpha, \\ \dfrac{t - a_\alpha^{i,l} + \theta_\alpha}{\theta_\alpha} & a_\alpha^{i,l} - \theta_\alpha \leq t \leq a_\alpha^{i,l}, \\ \dfrac{b_\alpha^{i,l} + \theta_\alpha - t}{\theta_\alpha} & b_\alpha^{i,l} \leq t \leq b_\alpha^{i,l} + \theta_\alpha, \\ 0 & t \geq b_\alpha^{i,l} + \theta_\alpha \text{ or } t \leq a_\alpha^{i,l} - \theta_\alpha, \end{cases}$$

and η_α^j such that $supp(\eta_\alpha^j) \subset M_\alpha^j(c_\alpha + \theta_\alpha)$ and if $\{t = \mu_\alpha^i\}$ is on the boundary of M_α^j,

$$\eta_\alpha^j = \begin{cases} 1 & \mu_\alpha^i - c_\alpha^{i,+} \leq t \leq \mu_\alpha^i, \\ \dfrac{t - \mu_\alpha^i + c_\alpha^{i,+} + \theta_\alpha}{\theta_\alpha} & \mu_\alpha^i - c_\alpha^{i,+} - \theta_\alpha \leq t \leq \mu_\alpha^i - c_\alpha^{i,+}, \end{cases}$$

and we proceed the same way for the symmetric case $\{t = -\mu_\alpha^i\}$ with $c_\alpha^{i,-}$. Taking these at least $k+1$ test functions with pairwise disjoint support for the variational characterization (1.1) of $\sigma_\alpha = \sigma_k(M, \tilde{g}_\alpha)$, we get

$$\sigma_\alpha \leq \max\left(\max_{\substack{1 \leq i \leq s \\ 1 \leq l \leq t_i}} \frac{\int_M \left|\nabla \eta_\alpha^{i,l}\right|_{\tilde{g}_\alpha}^2 dv_{\tilde{g}_\alpha}}{\int_{\partial M} \left(\eta_\alpha^{i,l}\right)^2 d\sigma_{\tilde{g}_\alpha}}, \max_{j \in J} \frac{\int_M \left|\nabla \eta_\alpha^j\right|_{\tilde{g}_\alpha}^2 dv_{\tilde{g}_\alpha}}{\int_{\partial M} \left(\eta_\alpha^j\right)^2 d\sigma_{\tilde{g}_\alpha}} \right).$$

Then $\sigma_\alpha \leq o(1)$ which contradicts (7.1). q.e.d.

We now prove that the set of such sequences such that

$$\sum_{i=1}^s \sum_{l=1}^{t_i} m_{i,l} + \sum_{j \in J} m_j = 1$$

is not empty.

Claim 17. *We let I_0 be the set of indices $i \in \{1, \cdots, s\}$ such that there exists a sequence $0 \ll c_\alpha^i \ll \mu_\alpha^i$ such that*

$$\lim_{\alpha\to+\infty} L_{\tilde{g}_\alpha}(\Gamma_\alpha^i(-\mu_\alpha^i + c_\alpha^i, \mu_\alpha^i - c_\alpha^i)) = 0,$$

and $I_1 = \{1, \cdots, s\} \setminus I_0$. Then, there exist sequences $c_\alpha^{i,\pm} \to +\infty$ $0 \ll c_\alpha^{i,\pm} \ll \mu_\alpha^i$ for $1 \leq i \leq s$ and sequences a_α^i, b_α^i for $i \in I_1$ with

$$-\mu_\alpha^i + c_\alpha^{i,+} \ll a_\alpha^i \ll b_\alpha^i \ll \mu_\alpha^i - c_\alpha^{i,-},$$

such that

$$\lim_{\alpha\to+\infty} L_{\tilde{g}_\alpha}(\Gamma_\alpha^i(-\mu_\alpha^i + c_\alpha^{i,-}, \mu_\alpha^i - c_\alpha^{i,+})) = 0$$

for $i \in I_0$,

$$\lim_{\alpha\to+\infty} \sum_{i=1}^{s} L_{\tilde{g}_\alpha}(\Gamma_\alpha^i(a_\alpha^i, b_\alpha^i)) > 0$$

for $i \in I_1$ and

$$\lim_{\alpha\to+\infty} \sum_{i\in I_1} L_{\tilde{g}_\alpha}(\Gamma_\alpha^i(a_\alpha^i, b_\alpha^i)) + \sum_{j=1}^{r} L_{\tilde{g}_\alpha}(I_\alpha^j(c_\alpha)) = 1.$$

Proof. We proceed by contradiction, assuming the opposite to hold. Then $I_1 \neq \emptyset$ and we set for $i \in I_1$ and $1 \leq j \leq k+1$

$$\mu_\alpha^i - c_\alpha^{i,+} = \mu_\alpha^i - c_\alpha^{i,-} = t_\alpha^{i,j} + \theta_\alpha,$$

$$b_\alpha^j = -a_\alpha^j = t_\alpha^j - \theta_\alpha,$$

where $t_\alpha^j = \frac{j\mu_\alpha^i}{k+2}$ and $\theta_\alpha \to +\infty$ satisfies $\theta_\alpha = o(\mu_\alpha^i)$. Then, by assumption,

$$\sum_{i=1}^{s} \lim_{\alpha\to+\infty} L_{\tilde{g}_\alpha}\left(\Gamma_\alpha^i(-t_\alpha^{i,j} - \theta_\alpha, -t_\alpha^{i,j} + \theta_\alpha) \cup \Gamma_\alpha^i(t_\alpha^{i,j} - \theta_\alpha, t_\alpha^{i,j} + \theta_\alpha)\right) > 0$$

for any $1 \leq j \leq k+1$. We now set η_α^j some test functions for the variational characterization of $\sigma_\alpha = \sigma_k(M, \tilde{g}_\alpha)$ with pairwise disjoint support defined such that $supp(\eta_\alpha^j) \subset \bigcup_{i\in I_1} P_\alpha^i$, η_α^i is an even function on P_α^i and

$$\eta_\alpha^{i,j} = \begin{cases} 0 & 0 \leq t \leq t_\alpha^{i,j} - 2\theta_\alpha, \\ \dfrac{t - t_\alpha^{i,j} + 2\theta_\alpha}{\theta_\alpha} & t_\alpha^{i,j} - 2\theta_\alpha \leq t \leq t_\alpha^{i,j} - \theta_\alpha, \\ 1 & t_\alpha^{i,j} - \theta_\alpha \leq t \leq t_\alpha^{i,j} + \theta_\alpha, \\ \dfrac{t_\alpha^{i,j} + 2\theta_\alpha - t}{\theta_\alpha} & t_\alpha^{i,j} + \theta_\alpha \leq t \leq t_\alpha^{i,j} + 2\theta_\alpha, \\ 0 & t_\alpha^{i,j} + 2\theta_\alpha \leq t \leq \mu_\alpha^i. \end{cases}$$

With these $k+1$ test functions, we easily prove that $\sigma_\alpha \leq o(1)$ by (1.1), which contradicts (7.1). q.e.d.

Thanks to Claim 16 and Claim 17 there exist for $1 \leq i \leq s$ some integers $t_i \geq 0$, sequences $a_\alpha^{i,l}$, $b_\alpha^{i,l}$ for $1 \leq l \leq t_i$, $c_\alpha = \{c_\alpha^{i,+}, c_\alpha^{i,-}\}$ and a set $J \subset \{1, \cdots, r\}$ satisfying $c_\alpha^{i,\pm} < \mu_\alpha^i$,

$$-\mu_\alpha^i \ll -\mu_\alpha^i + c_\alpha^{i,-} = b_\alpha^{i,0} \ll a_\alpha^{i,1} \ll b_\alpha^{i,1} \ll \cdots$$
$$\ll a_\alpha^{i,t_i} \ll b_\alpha^{i,t_i} \ll a_\alpha^{i,t_{i+1}} = \mu_\alpha^i - c_\alpha^{i,+} \ll \mu_\alpha^i,$$

and for $1 \leq i \leq s$, $1 \leq l \leq t_i$, $j \in J$,

$$m_{i,l} = \lim_{\alpha \to +\infty} L_{\tilde{g}_\alpha}(\Gamma_\alpha(a_\alpha^{il}, b_\alpha^{i,l})) > 0,$$

$$m_j = \lim_{\alpha \to +\infty} L_{\tilde{g}_\alpha}(I_\alpha^j(c_\alpha)) > 0,$$

with

$$\sum_{i=1}^{s} \sum_{m=1}^{t_i} m_{i,l} + \sum_{j \in J} m_j = 1,$$

such that $\sum_{i=1}^s t_i$ is maximal.

For fixed $1 \leq i \leq s$ and $1 \leq l \leq t_i$, we focus on the asymptotic behavior of the harmonic map Φ_α on the cylinder $P_\alpha^i(a_\alpha^{i,l}, b_\alpha^{i,l})$. We define a sequence $t_\alpha^{i,l}$ such that

$$\lim_{\alpha \to +\infty} L_{\tilde{g}_\alpha}\left(\Gamma_\alpha(a_\alpha^{i,l}, t_\alpha^{i,l})\right) = \lim_{\alpha \to +\infty} L_{\tilde{g}_\alpha}\left(\Gamma_\alpha(t_\alpha^{i,l}, b_\alpha^{i,l})\right) = \frac{m_{i,l}}{2}.$$

We set

$$\Psi_\alpha^{i,l}(\theta + it) = \tan\left(\frac{\theta - \pi + i(t - t_\alpha^{i,l})}{4}\right),$$

and we consider the $\frac{1}{2}$-harmonic map $\check{\Phi}_\alpha^{i,l} = \Phi_\alpha \circ \left(\Psi_\alpha^{i,l}\right)^{-1}$ on $\mathbb{D}$. Let $\theta_\alpha \to +\infty$ be such that $\theta_\alpha = o(a_\alpha^{i,l+1} - b_\alpha^{i,l})$ for $0 \leq l \leq t_i$ and $1 \leq i \leq s$. Then,

$$D_\alpha^{i,l} = \Psi_\alpha^{i,l}\left(T_\alpha^i(a_\alpha^{i,l} - \theta_\alpha, b_\alpha^{i,l} + \theta_\alpha)\right)$$

exhausts $\mathbb{D}$,

$$S_\alpha^{i,l} = \Psi_\alpha^{i,l}\left(\Gamma_\alpha^i(a_\alpha^{i,l} - \theta_\alpha, b_\alpha^{i,l} + \theta_\alpha)\right)$$

exhausts $\mathbb{S}^1$ and

$$\lim_{\alpha \to +\infty} L_{\left(\Psi_\alpha^{i,l}\right)_\star(\tilde{g}_\alpha)}(S_\alpha^{i,l}) = m_{i,l}.$$

We can now apply Proposition 4 to $\check{\Phi}_\alpha^{i,l}(D_\alpha^{i,l}, S_\alpha^{i,l}) \to (\mathbb{B}^{n+1}, \mathbb{S}^n)$ on $(\mathbb{D}, \xi)$. In order to obtain test functions which naturally extend to the manifold, we have to prove that $\mathbf{1}_{S_\alpha^{i,l}}\check{\Phi}_\alpha^{i,l}\partial_\nu\check{\Phi}_\alpha^{i,l}d\theta$ does not concentrate at the poles $(0,1)$ and $(0,-1)$. By contradiction, if we have

$$\mathbf{1}_{S_\alpha^{i,l}}\check{\Phi}_\alpha^{i,l}\partial_\nu\check{\Phi}_\alpha^{i,l}d\theta \rightharpoonup_\star m\delta_{(0,1)} + \nu$$

with $m > 0$, $\nu(\{(0,1)\}) = 0$, then $\int_{\mathbb{S}^1} d\nu > 0$ by the hypothesis on $t_\alpha^{i,l}$ we did and up to the extraction of a subsequence, we can build $q_\alpha^{i,l} \ll t_\alpha^{i,l}$ such that

$$\lim_{\alpha\to+\infty} L_{\tilde{g}_\alpha}(\Gamma_\alpha(a_\alpha^{i,l}-\theta_\alpha, q_\alpha^{i,l})) = m.$$

Setting $\overline{b_\alpha} = q_\alpha^{i,l} + \tau_\alpha$ and $\overline{a_\alpha} = t_\alpha^{i,l} - \tau_\alpha$, with $\tau_\alpha = \sqrt{t_\alpha^{i,l} - r_\alpha^{i,l}}$, we have

$$m_{i,l}^1 = \lim_{\alpha\to+\infty} L_{\tilde{g}_\alpha}\left(\Gamma_\alpha^i(a_\alpha^{i,l}-\theta_\alpha, \overline{b_\alpha})\right) > 0,$$

$$m_{i,l}^2 = \lim_{\alpha\to+\infty} L_{\tilde{g}_\alpha}\left(\Gamma_\alpha^i(\overline{a_\alpha}, b_\alpha^{i,l}+\theta_\alpha)\right) > 0$$

with $m_{i,l}^1 + m_{i,l}^2 = m_{i,l}$ and this contradicts the maximality of $\sum_{i=1}^s t_i$.

For a fixed $j \in J$, we now focus on the asymptotic behavior of Φ_α on $M_\alpha^j(c_\alpha)$. We denote by $\widetilde{M_\alpha^j}$ the connected component of $M\setminus\left(\gamma_\alpha^1, \cdots, \gamma_\alpha^s\right)$ which contains M_α^j. There exists a diffeomorphism $\tau_\alpha : \Sigma_j \to \widetilde{M_\alpha^j}$ such that (Σ_j, h_α) is a non-compact hyperbolic surface with $h_\alpha = \tau_\alpha^\star g_\alpha$. On Σ_j, we have

$$h_\alpha \to h \text{ in } \mathcal{C}_{loc}^\infty(\Sigma_j) \text{ as } \alpha \to +\infty$$

for a hyperbolic metric h. We let $c = [h]$ and $(\hat{\Sigma}_j, \hat{c})$ the compactification of the cusps of (Σ_j, h) so that $(\hat{\Sigma}_j \setminus \{p_1, \cdots, p_t\}, \hat{c})$ is conformal to (Σ_j, c) for some punctures $p_1, \cdots, p_t$ as described in [**17**]. The sequence of sets $\Sigma_\alpha = \tau_\alpha^{-1}\left(M_\alpha^j(c_\alpha)\right)$ exhausts $\hat{\Sigma}_j$, so that the sequence of harmonic maps with free boundary $\hat{\Phi}_\alpha = \Phi_\alpha \circ \tau_\alpha : (\Sigma_\alpha, h_\alpha) \to \mathbb{B}^{n+1}$ satisfies the hypotheses of Proposition 4. In order to extend on the whole manifold the suitable test functions we define on Σ_j, we will prove that $\mathbf{1}_{\Sigma_\alpha}\hat{\Phi}_\alpha.\partial_{\nu_\alpha}\hat{\Phi}_\alpha d\sigma_{h_\alpha}$ does not concentrate at the punctures which lie in the boundary of $\hat{\Sigma}_j$ (and correspond to the degeneration of some geodesic γ_α^i which satisfies condition (iii)). By contradiction, we assume that

$$\mathbf{1}_{\Sigma_\alpha}\hat{\Phi}_\alpha.\partial_{\nu_\alpha}\hat{\Phi}_\alpha d\sigma_{h_\alpha} \rightharpoonup_\star m\delta_{p_l} + \nu \text{ on } \hat{\Sigma}_j$$

for some puncture $p_l \in \{p_1, \cdots, p_t\} \cap \partial\hat{\Sigma}_j$ with $m > 0$, $\nu(\{p_l\}) = 0$. Then, up to the extraction of a subsequence, we can build $q_\alpha \to +\infty$ such that

$$\lim_{\alpha\to+\infty} L_{\tilde{g}_\alpha}\left(\Gamma_\alpha^i(-\mu_\alpha^i + q_\alpha, -\mu_\alpha^i + c_\alpha^{i,-})\right) = m$$

for $s_1+s_2+1 \leq i \leq s$ such that $\tau_\alpha^{-1}\left(\{-\mu_\alpha^i < t < 0\}\right)$ is a neighborhood of the puncture p_l of $\hat{\Sigma}_j$. We proceed the same way for the symmetric case $\{0 < t < \mu_\alpha^i\}$. Setting $d_\alpha = \sqrt{q_\alpha}$, $\overline{a_\alpha} = -\mu_\alpha^i + q_\alpha - \sqrt{q}_\alpha$ and $\overline{b_\alpha} = -\mu_\alpha^i + c_\alpha^{i,-}$, we have

$$m = \lim_{\alpha\to+\infty} L_{\tilde{g}_\alpha}\left(\Gamma_\alpha^i(\overline{a_\alpha},\overline{b_\alpha})\right) > 0,$$

$$\lim_{\alpha\to+\infty} L_{\tilde{g}_\alpha} I_\alpha^j\left(\overline{c_\alpha}\right) = m_j - m,$$

where $\overline{c_\alpha}$ comes from c_α, taking d_α instead of $c_\alpha^{i,-}$. Adding the sequences $\overline{a_\alpha} \ll \overline{b_\alpha}$ contradicts the maximality of $\sum_{i=1}^s t_i$.

As described in Proposition 4 and the computations of Section 6.2, the limit functions given by $\check{\Phi}_\alpha^i : D_\alpha^i \subset \mathbb{D} \to \mathbb{B}^{n+1}$ for $1 \leq i \leq s_1$, $\check{\Phi}_\alpha^{i,l} : D_\alpha^{i,l} \subset \mathbb{D} \to \mathbb{B}^{n+1}$ for $s_1+s_2+1 \leq i \leq s$ and $\hat{\Phi}_\alpha^j : \Sigma_\alpha \subset \hat{\Sigma}_j \to \mathbb{B}^{n+1}$ and their associated scales give at least $k+1$ well defined test functions for the variational characterization of σ_α by the gap (0.2). Indeed, denoting γ_j the genus of $\hat{\Sigma}_j$ and m_j its number of boundary components, we notice that $\sum_{j\in J}\gamma_j \leq \gamma$ and $\sum_{j\in J}\gamma_j + m_j \leq \gamma + m$ and that if $|J| = 1$, $\gamma_1 < \gamma$ or $\gamma_1 + m_1 < \gamma + m$. These at least $k+1$ test functions for the variational characterization (1.1) of σ_α give a contradiction. This ends the proof of Theorem 1.

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