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LORENTZIAN EINSTEIN METRICS WITH PRESCRIBED CONFORMAL INFINITY

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Abstract

We prove a local well-posedness theorem for the $(n+1)$-dimensional Einstein equations in Lorentzian signature, with initial data $(\widetilde{g}, K)$ whose asymptotic geometry at infinity is similar to that anti-de Sitter (AdS) space, and compatible boundary data $\widehat{g}$ prescribed at the time-like conformal boundary of space-time. More precisely, we consider an n-dimensional asymptotically hyperbolic Riemannian manifold $(M, \widetilde{g})$ such that the conformally rescaled metric $x^2\widetilde{g}$ (with x a boundary defining function) extends to the closure $\overline{M}$ of M as a metric of class $C^{n-1}(\overline{M})$ which is also polyhomogeneous of class $C^p_{\mathrm{polyhom}}(\overline{M})$. Likewise we assume that the conformally rescaled symmetric $(0,2)$-tensor x^2K extends to $\overline{M}$ as a tensor field of class $C^{n-1}(\overline{M})$ which is polyhomogeneous of class $C^{p-1}_{\mathrm{polyhom}}(\overline{M})$. We assume that the initial data $(\widetilde{g}, K)$ satisfy the Einstein constraint equations and also that the boundary datum is of class C^p on $\partial M \times (-T_0, T_0)$ and satisfies a set of natural compatibility conditions with the initial data. We then prove that there exists an integer r_n, depending only on the dimension n, such that if $p \geqslant 2q + r_n$, with q a positive integer, then there is $T > 0$, depending only on the norms of the initial and boundary data, such that the Einstein equations (1.1) has a unique (up to a diffeomorphism) solution g on $(-T, T) \times M$ with the above initial and boundary data, which is such that $x^2 g \in C^{n-1}((-T,T) \times \overline{M}) \cap C^q_{\mathrm{polyhom}}((-T,T) \times \overline{M})$. Furthermore, if $x^2\widetilde{g}, x^2K$ are polyhomogeneous of class C^∞ and $\widehat{g}$ is in $C^\infty((-T_0, T_0) \times \partial M)$, then $x^2 g$ is in $C^\infty_{\mathrm{polyhom}}((-T,T) \times \overline{M})$.

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1. Introduction

Our goal in this paper is to prove a local well-posedness theorem for the $(n+1)$-dimensional Einstein equations

$$\mathrm{Ric}(g) = -ng, \tag{1.1}$$

in Lorentzian signature, with initial data $(\widetilde{g}, K)$ corresponding to the asymptotic geometry of anti-de Sitter (AdS) space, and compatible boundary data $\widehat{g}$ prescribed at the time-like conformal boundary of space-time. More precisely, we consider an n-dimensional asymptotically hyperbolic Riemannian manifold $(M, \widetilde{g})$, such that the conformally rescaled metric $x^2\widetilde{g}$ extends to $\overline{M}$, the union of M with its boundary ∂M (given by $x = 0$), as a metric of class $C^{n-1}(\overline{M})$ which is polyhomogeneous of class $C^p_{\mathrm{polyhom}}(\overline{M})$. Here and in what follows, x is a boundary defining function, that is a non-negative function on $\overline{M}$, smooth up to the boundary ∂M of M, with $\partial M = \{x = 0\}$ and such that the differential of x is nonzero on ∂M. We refer to Section 4 for the definition of polyhomogeneity.

Likewise we assume that the conformally rescaled symmetric $(0,2)$-tensor $x^2 K$ extends to $\overline{M}$ as a tensor field of class $C^{n-1}(\overline{M})$ which is polyhomogeneous of class $C^{p-1}_{\mathrm{polyhom}}(\overline{M})$. We assume that the initial data $(\widetilde{g}, K)$ satisfy the Einstein constraint equations and also give boundary data of class C^p on $\partial M \times (-T_0, T_0)$ satisfying a set of natural compatibility conditions with the initial data (we refer to Appendix A for a discussion of the constraint equations and compatibility conditions). The main result of our paper, which asserts that these initial and boundary data determine an Einstein metric, can be stated as follows:

Theorem 1.1. *Suppose that we are given initial and boundary conditions* $(\widetilde{g}, K, \widehat{g})$ *with* $x^2\widetilde{g} \in C^{n-1}(\overline{M}) \cap C^p_{\mathrm{polyhom}}(\overline{M})$, $x^2 K \in C^{n-1}(\overline{M}) \cap$

$C^{p-1}_{\mathrm{polyhom}}(\overline{M})$ and $\widehat{g} \in C^p((-T_0, T_0) \times \partial M)$ satisfying the constraint equations and the compatibility conditions to order p. There exists an integer r_n, depending only on the dimension n, such that if $p \geqslant 2q + r_n$, then there is $T > 0$, depending only on the norms of the initial and boundary data, such that the Einstein equations (1.1) has a unique (up to a diffeomorphism) solution g on $(-T, T) \times M$ with the above initial and boundary data, which is such that $x^2 g \in C^{n-1}((-T, T) \times \overline{M}) \cap C^q_{\mathrm{polyhom}}((-T, T) \times \overline{M})$. Furthermore, if $x^2\widetilde{g}, x^2 K \in C^\infty_{\mathrm{polyhom}}(\overline{M})$, $\widehat{g} \in C^\infty((-T_0, T_0) \times \partial M)$ and the compatibility conditions are satisfied to all orders, then $x^2 g \in C^\infty_{\mathrm{polyhom}}((-T, T) \times \overline{M})$.

Hence, the main result of our paper gives an extension to higher dimensions of the fundamental pioneering work of Friedrich [**20**], in which a general existence theorem is proved for anti-de Sitter type space-times in dimension $n + 1 = 4$. The approach of [**20**] is based on a reduction of the problem with boundary at infinity to a finite maximally dissipative initial-boundary value problem, achieved through an ingenious conformal representation of the Einstein equations in dimension four. This leads to a general existence result for solutions of the Einstein equations with negative cosmological constant admitting a smooth conformal extension at space-like infinity. It is should be noted that even though the results of [**20**] are proved the assumption of smooth initial and boundary data, the method used in [**20**] is flexible enough to allow for results on metrics of C^k regularity with large but finite k.

The reason for which the method in [**20**] does not extend to the Einstein equations in odd space-time dimensions is that the metrics obtained through this approach are smooth up to the boundary, while the Fefferman–Graham expansion [**19**] implies that in odd dimension $n + 1 > 3$, the corresponding Einstein metric cannot have this type of boundary regularity due to the appearance of log terms, which are present since the obstruction tensor does not vanish for a generic boundary datum in odd space-time dimensions. In the case of even (e.g., four) space-time dimensions, this technical point has another subtle but significant effect: while the results of [**20**] are finer than ours in the sense that initial data that are smooth up the boundary are shown to yield Einstein metrics that are also smooth up to the boundary (which is a stronger boundary regularity result than the one we obtain), our result has the advantage that it also applies to initial data that are only assumed to be polyhomogeneous, yielding polyhomogeneous Einstein metrics. This is relevant because, even in four dimensions, the solutions to the constraint equations constructed in [**4**] are generically polyhomogeneous (in fact, in $C^{n-1}(\overline{M}) \cap C^\infty_{\mathrm{polyhom}}(\overline{M})$) but not smooth up to the boundary. (Notice, however, that, despite this generic lack of smoothness up to the boundary, [**4**] does yield many nontrivial solutions to the constraint equations that are smooth up to the boundary and which give rise to many nontrivial Einstein metrics in four dimensions directly using the breakthrough result of [20].) Finally, it is worth mentioning that we obtain explicit values for the constant r_n appearing in the statement of Theorem 1.1 (e.g., in four dimensions one can take $r_3 = 17$) but that they are by no means sharp.

We shall see below that our purely PDE approach to the formulation of the Einstein equations is of a different nature from that of [20], and that it uses instead as its starting point some of the key similarities in the algebraic structure of the Einstein equations between the cases of Lorentzian and Euclidean signature. The existence of Einstein metrics in latter case is well understood thanks to the work of Graham–Lee [23], Anderson [1, 3], Biquard [8] and others on the global existence and regularity of Riemannian Einstein metrics with prescribed conformal infinity that are close, in a suitable sense, to the hyperbolic metric. The situation in Lorentzian signature is fundamentally different since it corresponds to a hyperbolic evolution problem. Both the available analytical techniques and the expected results are, thus, vastly different. In particular, the metric g is only guaranteed to exist locally in time (that is, for $|t| < T$), even for small data, a reflection of the fact that the anti-de Sitter space is not expected to enjoy the good stability properties of Minkowski space [14] (we refer to [9] and [22] for important recent work on the stability problem for anti-de Sitter space).

We would also like to mention that besides the case of the Einstein equations considered in [20], the study of wave equations on asymptotically anti-de Sitter spaces has attracted much attention in the last few years. To the best of our knowledge, the wave equation on AdS_4 was first considered by Breitenlohner and Freedman in [10] using the strong symmetry of the problem to separate variables. Again for AdS_4, Choquet-Bruhat [11, 12] proved global existence for the Yang–Mills equation under a radiation condition, and Ishibashi and Wald [29] gave a proof of the well-posedness of the Cauchy problem for the Klein–Gordon equation in AdS_{n+1} using spectral theory. More refined results for the Klein–Gordon equation in an AdS space were developed by Bachelot [5, 6, 7], who used energy methods and dispersive estimates to study the decay of the solutions and prove some results on the propagation of singularities. In [35], Vasy established fine results on the propagation of singularities are proved for the Klein–Gordon equation on asymptotically AdS spaces using microlocal analysis. Holzegel and Warnick, both independently and in joint work [26, 36, 27], used energy methods to prove the well-posedness of the Cauchy problem for this equation in asymptotically AdS_4 space-times and discussed the boundedness of solutions to the Klein–Gordon equation in stationary AdS black hole geometries. The local well-posedness for semilinear Klein–Gordon equations in asymptotically anti-de Sitter spaces with nontrivial boundary conditions at infinity was established in [**18**]. Spherically symmetric Einstein–Klein–Gordon systems have been considered in [**28**].

Finally, we mention that besides its interest as a question in geometric analysis and mathematical General Relativity, an important motivation for the problem of constructing Lorentzian Einstein manifolds with prescribed conformal infinity arises in the context of the AdS/CFT correspondence in string theory [**31**, **37**] (see [**2**, **17**] for further details on this point). The AdS/CFT correspondence is a conjectural relation which posits that a gravitational field on a Lorentzian $(n+1)$-manifold endowed with an asymptotically anti-de Sitter metric can be recovered from a conformal gauge field defined on the conformal boundary of the manifold. The gravitational field is typically modeled as a Lorentzian metric g satisfying the Einstein equation and the conformal gauge field corresponds to the conformal infinity $[\widehat{g}]$ of the metric. In this setting, the *holographic principle* asserts that the boundary data (which in the context of the Einstein equation would be the boundary metric $\widehat{g}$), defined on the n-dimensional boundary, propagates through a suitable $(n+1)$-manifold (referred to as the *bulk* in the physics literature) to determine the field (here the metric g) via a locally well-posed problem.

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2. Strategy of the proof

In this section, we will present the overall strategy of the proof of Theorem 1.1. We will also point out where the main points of the argument can be found in the article, so this section also serves as a guide to the paper.

Our first step is replace the Einstein equations (1.1) with modified Einstein equations taking the form of a quasilinear hyperbolic system, using what is often called DeTurck's trick [**15**, **16**]. In the Riemannian case, this is amounts to writing the Einstein equations as an equivalent elliptic quasilinear system.

The specific features of the quasilinear hyperbolic system corresponding to Theorem 1.1 give rise to difficulties that make its proof rather involved, both technically and conceptually. A first difficulty lies in the fact that asymptotically anti-de Sitter metrics are not globally hyperbolic, so the classical local well-posedness result of Choquet-Bruhat [**33**] does not apply. This is also reflected in the fact that the modified Einstein equation when expressed in terms of the conformally rescaled metric $\bar{g} = x^2 g$ contains terms that are strongly singular at the boundary $x = 0$, so that the usual hyperbolic estimates are not enough to control the behavior of the solutions of this equation. This requires the introduction of a functional framework adapted to the geometry of these spaces. For this we rely on a scale of twisted, weighted Sobolev spaces that are closely related to the spaces used in the edge differential calculus [**32**] but which we find more convenient for our purposes.

A second difficulty is that in contrast to the globally hyperbolic case, where the modified Einstein equations correspond to a quasi-diagonal system (meaning that the leading part of the hyperbolic system is given by a scalar second-order differential operator, in our case the wave operator $g^{\mu\nu}\partial_\mu\partial_\nu$), the leading part of the equations in the asymptotically anti-de Sitter setting is no longer given by a quasi-diagonal system. This is because the leading terms of the equation (meaning the ones that cannot be absorbed into constants in the estimates) are not only given by the second-order derivatives, but also by additional terms that are singular at leading order when $x = 0$, and reflects the fact that, in the adapted coordinates, the singularity at $x = 0$ is critical from the point of view of scalings. When these additional terms are taken into account, the equation is no longer quasi-diagonal, so one must construct approximate diagonalizations of the operators and take into account the fact that the estimates that we obtain in different "eigenspaces" are not equivalent. It is remarkable, though, that the various powers of x that appear in scattered through the equations work together to allow us to prove Theorem 1.1.

A third difficulty is that, in general, it is notoriously hard to impose boundary conditions in the Einstein equations (see, e.g., [**21**] and references therein). The way that we circumvent this problem is by constructing the solution metric g as a sum of two terms, one that is "large" at infinity and which we construct using essentially algebraic methods, and one which is "small" at infinity, whose existence must be proved using analytic techniques, so that for all practical purposes one does not need to consider the boundary conditions here.

Hence, we are led to considering the following strategy in order to tackle the problem:

Step 1: The modified Einstein equation. In Section 3, we discuss how one can replace the Einstein equations (1.1) by a quasilinear hyperbolic system $Q(g) = 0$ using DeTurck's trick. Although from a conceptual point of view the argument goes along familiar lines, the lack of global hyperbolicity makes technically nontrivial some arguments needed to prove that both equations are equivalent. This is established in Section 10 using ideas developed in the paper (Theorem 10.1).

Step 2: Peeling off the metric. In Section 4, we construct asymptotically anti-de Sitter metrics γ_l that are "approximate solutions" to the modified Einstein equation $Q(g) = 0$ and satisfy the desired boundary conditions (Theorem 4.5). These metrics have the property that $Q(\gamma_l)$ is suitably small and are obtained from the boundary datum $\widehat{g}$ in an essentially algebraic way that can be understood as peeling off the leading "layers" of the solution at $x = 0$, step by step. The parameter l corresponds to the number of steps that one considers and is related to the norms in which γ_l is an approximate solution of the modified Einstein equations. One should notice that, in general, the rescaled metrics $\bar{\gamma}_l := x^2\gamma_l$ are not smooth up to the boundary, but in some polyhomogeneous space $C^{n-1} \cap C^{p_l}_{\text{polyhom}}$.

Step 3: Setting an iteration within a suitable functional framework. To construct the metric g that solves the modified Einstein equation, we write it as

$$g = \gamma + x^{\frac{n}{2}} u \,,$$

where we have set $\gamma := \gamma_l$ for a large enough l. There γ is going to be the "large" part at $x = 0$ and the other terms is going to be "small" at the boundary.

To construct u, we set up an iteration in Section 5. The convergence of this iteration will not be proved until Section 9, however. Before that, we need to define suitable Sobolev spaces adapted to the geometry of the anti-de Sitter space in which we can derive suitable estimates for u. In Sections 6 and 7, we consider two related scales of Sobolev spaces, $\mathbf{H}^{m,r}_\alpha$ and $\mathcal{H}^{m,r}$, and derive several key estimates for them. It should be noticed that not only the are proofs of these estimates different from those of the usual Sobolev spaces $H^k(\mathbb{R}^n)$, but so is also the case for the range of parameters for which, e.g., we have pointwise estimates (Corollary 6.3) or can obtain estimates for the product of two functions (Theorem 7.1).

Step 4: Linear estimates and convergence of the iteration. Using the above adapted Sobolev spaces, in Section 8, we obtain estimates for the linear operators that appear in the iteration under certain assumptions about the structure of the metric. Here the way that the various powers of x appear is crucial to deriving the estimates that are analogous (although the spaces and range of parameters are different) to the usual ones obtained for globally hyperbolic quasilinear wave equations. It should be emphasized though that the combination of the equation being effectively not quasi-diagonal with the fall-off of the nonlinearities at the boundary make the analysis of the linear equations and the treatment of the functional spaces much subtler than in our previous paper [18], which was only concerned with scalar equations.

With these estimates in hand and equipped with the results about the adapted Sobolev spaces established in the previous step, the proof of the convergence of the iteration goes along the lines of the classical result for globally hyperbolic spaces. The details are presented in Section 9 although, as we have already mentioned, one has to wait until Theorem 10.1 to show that these metrics are, in fact, Einstein.

The paper concludes with two appendices. In Appendix A we recall the constraint and compatibility conditions that must be imposed on the initial and boundary data and the Andersson–Chrusciel result on the existence of solutions to the constraint equations. In Appendix B we record some results about the integral operators A_α and A_α^*, defined in (6.7), that we established in [18]. These operators play an important role in Sections 6 and 7. For the benefit of the reader, we also include a sketch of the proof.

3. The modified Einstein equation

When dealing with the Einstein equation, a first difficulty, well understood by now, is that the gauge invariance of the Einstein equation under changes of coordinates makes it a very degenerate system. A standard way of solving this difficulty is using a technique that is often called "DeTurck's trick" [15, 16], which employs a reference metric to get rid of this gauge freedom. In the setting that we are considering, it is important to choose a reference metric, which we will denote by γ_0, which a certain asymptotic behavior at infinity. To avoid unnecessary repetitions, let us then begin by introducing the following definition, where $I := (-T_0, T_0)$ denotes a small interval of the real line containing 0.

Definition 3.1. A metric g on $I \times M$ is called *weakly asymptotically AdS* if the following conditions hold:

(i) The rescaled reference metric $\bar{g} := x^2 g$ is of class C^2 up to the boundary.
(ii) The differential of the function x satisfies $\bar{g}^{\mu\nu}(\partial_\mu x)(\partial_\mu x) = 1$ on $I \times \partial M$.

This definition is motivated by the formal calculations of Graham and Lee in [23], many of which carry over verbatim to the case of Lorentzian signature. The definition should be compared with that of an asymptotically AdS metric, cf. [25].

We will choose the reference metric γ_0 to be a weakly asymptotically AdS metric on $I \times M$ such that the pullback of

$$\bar{\gamma}_0 := x^2 \gamma_0$$

to $I \times \overline{M}$ is $\widehat{g}$. A convenient way of doing this in terms of the initial metric $g_0 := g|_{t=0}$, which we write in terms of the initial data as described in Appendix A, is the following (we recall that the pullback of $\bar{g}_0 := x^2 g_0$ to the boundary is precisely $\widehat{g}$). Identifying $TI = I \times \mathbb{R}$, for any $(t, z) \in I \times \partial M$ let us consider the tensor on $T_{(t,z)}(I \times \partial M) = \mathbb{R} \times T_z \partial M$ given by

$$G' := \widehat{g}|_{(t,z)} - \widehat{g}|_{(0,z)} \,.$$

Now let G be the only tensor on $T_{(t,z)}(I \times M) = \mathbb{R} \times T_z M$ which satisfies

$$(j_{(-T,T)\times \partial M})^* G = G' \,, \qquad (\bar{g}_0|_z + G)^{-1} dx = \bar{g}_0^{-1}|_z dx$$

at (t, z). Notice that, by continuity, the inverse appearing in the second equation is well defined provided that the interval I is small enough. This defines a tensor field on $I \times \partial M$.

We can now extend G to a tensor field $E(G)$ defined on a small neighborhood of $I \times \partial M$, for instance, by parallel transport with respect to the metric $\bar{g}_0$ along integral curves of the gradient of x. A suitable reference metric can then be constructed as $\gamma_0 := x^{-2} \bar{\gamma}_0$ with

$$\bar{\gamma}_0 := \bar{g}_0 + \chi\, E(G) \,, \tag{3.1}$$

with χ a suitable cutoff function that is equal to 1 in a neighborhood of the boundary. Notice that the reference metric depends on the boundary and initial data and that it is a (non-degenerate) Lorentzian metric because $E(G)$ is small if the interval is small.

Let us now denote by $\Gamma^{\nu}_{\lambda\rho}$ and $\widetilde{\Gamma}^{\nu}_{\lambda\rho}$ the Christoffel symbols of the metrics g and γ_0, respectively. DeTurck's trick consists in looking for solutions to the modified Einstein equation

$$Q(g) = 0 \,, \tag{3.2}$$

where the components of the tensor $Q(g)$ are given in terms of those of the Ricci tensor, $R_{\mu\nu}$, by

$$Q_{\mu\nu} := R_{\mu\nu} + n g_{\mu\nu} + \frac{1}{2}(\nabla_\mu W_\nu + \nabla_\nu W_\mu) \,. \tag{3.3}$$

Here the covariant derivatives and the Ricci tensor are those of the metric g and the 1-form W is

$$W_\mu := g_{\mu\nu}\, g^{\lambda\rho}\, (\Gamma^{\nu}_{\lambda\rho} - \widetilde{\Gamma}^{\nu}_{\lambda\rho}) \,. \tag{3.4}$$

We will discuss the relationship between the solutions of the Einstein equations (1.1) and those of the modified equation (3.2) in Section 10, as the lack of global hyperbolicity introduces some peculiarities. It is worth mentioning that $Q(g)$ also depends on the initial and boundary conditions through the reference metric γ_0.

It is well-known that the advantage of Equation (3.2) over the Einstein equations is that the nondegeneracy has been taken care of; indeed, (3.2) is a quasilinear wave equation because

$$Q_{\mu\nu} = -\frac{1}{2} g^{\lambda\rho} \partial_\lambda \partial_\rho g_{\mu\nu} + B_{\mu\nu}(g, \partial g)\,, \tag{3.5}$$

with the second term quadratic in ∂g. Our goal now is to solve the modified Einstein equation (3.2) together with the compatible initial and boundary conditions

$$g|_{t=0} = g_0\,, \qquad \partial_t g|_{t=0} = g_1\,, \qquad (j_{(-T,T)\times\partial M})^* \bar{g} = \widehat{g}\,.$$

For the class of metrics that we are considering, the coefficients are strongly singular at $x = 0$. Indeed, it essentially follows from a computation by Graham and Lee [**23**, Equation (2.19)] that for a weakly asymptotically AdS metric g one can express (3.5) in terms of $\bar{g}$ as

$$Q_{\mu\nu} = \frac{1}{x^2}\left(n(1 - \bar{g}^{\lambda\rho} x_\lambda x_\rho)\, \bar{g}_{\mu\nu} - \frac{1}{2}(B_\mu x_\nu + B_\nu x_\mu) \right) + \frac{1}{x}\mathcal{P}^1(\bar{g}) + \mathcal{P}^2(\bar{g})\,, \tag{3.6}$$

where $x_\mu := \partial_\mu x$,

$$B_\mu := \bar{g}^{\lambda\rho} (\bar{\gamma}_0)_{\lambda\rho}\, \bar{g}_{\mu\nu} (\bar{\gamma}_0)^{\nu\lambda} x_\lambda - (n+1) x_\mu\,,$$

$\bar{\gamma}_0 := x^2 \gamma_0$ and $\mathcal{P}^1(\bar{g})$ (respectively $\mathcal{P}^2(\bar{g})$) stands for terms that depend smoothly on x, $\bar{g}$, $\bar{\gamma}_0$ and $\partial\bar{\gamma}_0$ and are linear in $\partial\bar{g}$ (respectively linear in $\partial^2\bar{g}$ and quadratic in $\partial\bar{g}$, depending also on $\partial^2\bar{\gamma}_0$). Here all the indices are raised and lowered using the metric $\bar{g}_{\mu\nu}$ but $(\bar{\gamma}_0)^{\mu\nu}$, which is the inverse of $\bar{\gamma}_0$.

In view of Equation (3.6), we can immediately make the following important observation:

Proposition 3.2. *Suppose that g is a weakly asymptotically AdS metric. Then $Q(g) = \mathcal{O}(x^{-1})$ if and only if the following relations hold true on $(-T,T) \times \partial M$:*

$$\bar{g}^{\mu\nu} (\bar{\gamma}_0)_{\mu\nu} = n + 1 \qquad \textit{and} \qquad \bar{g}^{\mu\nu} x_\nu = (\bar{\gamma}_0)^{\mu\nu} x_\nu\,.$$

4. Peeling off the metric

Throughout the defining function x will be a C^∞ positive function on M that vanishes to first order at the boundary, which ensures that one can take it as a coordinate in a certain neighborhood of the boundary ∂M in M, which we will denote by $\mathcal{A}$. To parametrize $\mathcal{A}$ we will always take coordinates (x, θ), where $\theta = (\theta^1, \dots, \theta^{n-1})$ are local coordinates on ∂M. Since the analysis of the equation $Q(g) = 0$ is only problematic in a neighborhood of the boundary, these are the most convenient coordinates to carry out the key estimates that are needed in this paper.

Let us start with some preliminary results that we will need to prove the main result of this section. Here we denote by $\mathcal{S}^2$ the space of symmetric covariant 2-tensors on $I \times \overline{M}$. In the following proposition we provide a convenient decomposition of this space at any point close to, or lying on, the boundary $I \times \partial M$. *Throughout the section, we will assume that g is a weakly asymptotically AdS metric.*

Proposition 4.1. *In $I \times \overline{\mathcal{A}}$, the space of symmetric tensors can be decomposed as*

$$\mathcal{S}^2 = \mathcal{V}_0^g \oplus \mathcal{V}_1^g \oplus \mathcal{V}_2^g \oplus \mathcal{V}_3^g\,,$$

where

$$\begin{aligned}
\mathcal{V}_0^g &:= \big\{H \in \mathcal{S}^2 : H_{\mu\nu} = \varphi \bar g_{\mu\nu} \text{ with } \varphi \text{ scalar}\big\}\,,\\
\mathcal{V}_1^g &:= \big\{H \in \mathcal{S}^2 : H_{\mu\nu}\bar g^{\nu\lambda}x_\lambda = 0 \text{ and } H_{\mu\nu}\bar g^{\mu\nu} = 0\big\}\,,\\
\mathcal{V}_2^g &:= \big\{H \in \mathcal{S}^2 : H_{\mu\nu} = \varphi\,[(n+1)x_\mu x_\nu - \bar g_{\mu\nu}] \text{ with } \varphi \text{ scalar}\big\}\,,\\
\mathcal{V}_3^g &:= \big\{H \in \mathcal{S}^2 : H_{\mu\nu} = a_\mu x_\nu + a_\nu x_\mu \text{ with } \bar g^{\lambda\rho}a_\lambda x_\rho = 0\big\}\,.
\end{aligned}$$

Proof. Since the 1-form dx does not vanish in $I \times \overline{\mathcal{A}}$, it is easy to check that $\mathcal{V}_i^g \cap \mathcal{V}_j^g = \{0\}$ if $i \neq j$ and that the dimensions of the spaces $\mathcal{V}_j^g$ at each point of $I \times \overline{\mathcal{A}}$ are

$$1\,, \quad \frac{n(n+1)}{2} - 1\,, \quad 1 \quad \text{and} \quad n\,,$$

respectively. The sum of these numbers gives

$$\frac{(n+1)(n+2)}{2}\,,$$

that is, the dimension of $\mathcal{S}^2$ at any point. The proposition then follows. q.e.d.

In what follows we will need more information about the structure of the modified Einstein operator $Q(g)$ in a neighborhood of the boundary. To analyze $Q(g)$, we will restrict our attention to the set $I \times \mathcal{A}$ and use coordinates (t, x, θ), where θ are local coordinates on ∂M. It was computed by Graham and Lee [**23**, Proposition 2.10] that the action of the differential of the map (3.3) on a symmetric tensor $h = h_0 + h'$, with $h_0 \in \mathcal{V}_0^g$ and $h' \in \mathcal{V}_1^g \oplus \mathcal{V}_2^g \oplus \mathcal{V}_3^g$, is of the form

$$(DQ)_g(h) = -\frac{1}{2}\big((\Box_g - 2n)h_0 + (\Box_g + 2)h'\big) + x\,\mathcal{L}^1 h\,, \tag{4.1}$$

where $\Box_g h_{\mu\nu} := g^{\lambda\rho}\nabla_\lambda\nabla_\rho h_{\mu\nu}$ is the wave operator on tensor fields and we, henceforth, use the notation $\mathcal{L}^m$ for a matrix m^{th} order linear differential operator in the conormal derivatives $(x\partial_x, \partial_\theta, \partial_t)$ whose coefficients are smooth functions of $(x, \bar{g}, \partial\bar{g}, \bar{\gamma}_0, \partial\bar{\gamma}_0, \partial^2\bar{\gamma}_0)$ up to $x = 0$. In the case $m = 1$, the operator will not depend on $\partial^2\bar{\gamma}_0$.

In particular, the part with second-order derivatives of the linearized operator $(DQ)_g$ is the same as that of the wave operator $-\frac{1}{2}\Box_g$. Regarding the terms that are most singular at $x = 0$, it was shown in [**23**, Proposition 2.7] that, in terms of the coordinates (t, x, θ), the Laplacian on a symmetric tensor h can be expanded in x as

$$\begin{aligned}\Box_g h_{\mu\nu} = \big(x^2\partial_x^2 + (1-n)x\partial_x\big)h_{\mu\nu} + 2h_{\lambda\rho}\bar{g}^{\lambda\lambda}\bar{g}^{\rho\rho}x_\lambda x_\rho \bar{g}_{\mu\nu} \\ - (n+1)\big(h_{\mu\lambda}\bar{g}^{\lambda\rho}x_\rho x_\nu + h_{\nu\lambda}\bar{g}^{\lambda\rho}x_\rho x_\mu\big) + 2\bar{g}^{\lambda\rho}h_{\lambda\rho}x_\mu x_\nu \\ + x\mathcal{L}^1(h)_{\mu\nu} + x^2\mathcal{L}^2(h)_{\mu\nu}\,.\end{aligned}$$

To further simplify this expression, let us define the quadratic polynomials

$$p_j(s) := -\frac{1}{2}\left(s - \frac{n}{2} + \alpha_j\right)\left(s - \frac{n}{2} - \alpha_j\right),$$

where $0 \leqslant j \leqslant 3$ and α_j are the constants

$$\alpha_0 := \frac{\sqrt{n(n+8)}}{2}\,, \quad \alpha_1 := \frac{n}{2}\,, \quad \alpha_2 := \alpha_0\,, \quad \alpha_3 := \frac{\sqrt{n(n+4)}}{2}\,. \tag{4.2}$$

In the following lemma, which we borrow from [**23**, Lemma 2.9] with a minor change the notation, we use the subspaces $\mathcal{V}_j^g$ to effectively diagonalize $(DQ)_g$ up to terms that are smaller at $x = 0$. Here $p_j(x\partial_x)$ has the obvious meaning.

Lemma 4.2 ([**23**]). *If $h \in \mathcal{V}_j^g$, we have that*

$$(DQ)_g(h)_{\mu\nu} = x^{-2}\,p_j(x\partial_x)\bar{h}_{\mu\nu} + x^{-1}(\mathcal{L}^1\bar{h})_{\mu\nu} + (\mathcal{L}^2\bar{h})_{\mu\nu}\,.$$

We will also need some information on the second derivative $(D^2Q)_g$, understood as a quadratic form. For our purposes, it will be enough to have the following symbolic description of $(D^2Q)_g(h)$, where we are not displaying indices for the ease of notation:

Lemma 4.3. *The second derivative of Q is of the form*

$$(D^2Q)_g(h) = \mathcal{O}(1)\bar{h}\,\partial^2\bar{h} + \mathcal{O}(1)\,\partial\bar{h}\,\partial\bar{h} + \mathcal{O}(x^{-1})\,\bar{h}\,\partial\bar{h} + \mathcal{O}(x^{-2})\,\bar{h}\,\bar{h}\,.$$

Here we are using the notation $\bar{h} := x^2 h$ and each term $\mathcal{O}(x^{-s})$ above stands for a smooth function of x, $\bar{g}$, $\partial\bar{g}$ and the derivatives of $\bar{\gamma}_0$ up to order s.

Proof. Ignoring the indices, we can use Eqs. (3.5) and (3.6) to symbolically write the structure of $Q(g)$ as

$$Q(g)=\bar g^{-1}\partial^2\bar g+a_0(\bar g)\,\partial\bar g\,\partial\bar g+\frac{a_1(\bar g)}{x}\,\partial\bar g+\frac{a_2(\bar g)}{x^2}\,,$$

where $a_j(\bar g)$ stands for a smooth function of x, $\bar g$ and the derivatives of $\bar\gamma_0$ up to order j. Since

$$Q(g+\varepsilon h)=Q(g)+\varepsilon\,(DQ)_g(h)+\frac{1}{2}\varepsilon^2(D^2Q)_g(h)+\mathcal{O}(\varepsilon^3)\,,$$

an elementary computation using that

$$(\bar g+\varepsilon\bar h)^{-1}=\bar g^{-1}-\varepsilon\bar g^{-1}\bar h\bar g^{-1}+\varepsilon^2\bar g^{-1}\bar h\bar g^{-1}\bar h\bar g^{-1}+\mathcal{O}(\varepsilon^3),$$

readily yields the desired expression for $(D^2Q)_g$. q.e.d.

We will also need the following elementary fact:

Lemma 4.4. *For any integers $\sigma\geqslant 0$ and s there is a polynomial f of degree σ or $\sigma+1$ such that*

$$p_j(x\partial_x)\big(x^s f(\log x)\big)=x^s(\log x)^\sigma\,.$$

Furthermore, f has degree $\sigma+1$ if and only if $p_j(s)=0$.

Proof. Since $p_j(0)\neq 0$, it is clear that

$$p_j(x\partial_x)\bigg(\frac{1}{p_j(0)}\bigg)=1\,.$$

We now proceed by induction on s and σ. Indeed, assume that the statement holds true for all $s\leqslant s_0$ and $\sigma\leqslant\sigma_0$. The key observation is that

(4.3)
$$\begin{aligned}p_j(x\partial_x)\big(x^s(\log x)^\sigma\big)=p_j(s)x^s(\log x)^\sigma-\frac{\sigma(2s+4-n)}{2}x^s(\log x)^{\sigma-1}\\-\frac{\sigma(\sigma-1)}{2}x^s(\log x)^{\sigma-2}\,.\end{aligned}$$

If $p_j(s_0+1)\neq 0$, by the induction hypothesis there is a polynomial F of degree at most σ_0 such that

$$p_j(x\partial_x)\varphi=x^{s_0+1}(\log x)^{\sigma_0},$$

with

$$\varphi:=x^{s_0+1}\bigg(\frac{(\log x)^{\sigma_0}}{p_j(s_0+1)}+F(\log x)\bigg)\,.$$

On the other hand, if $p_j(s_0+1)=0$ we have that s_0 is $\frac{n}{2}\pm\alpha_j$, and in this case $2s_0+2-n$ is always nonzero. Hence, the induction hypothesis and the identity (4.3) ensure that we can then take a polynomial F of degree at most σ_0 such that

$$p_j(x\partial_x)\varphi=x^{s_0+1}(\log x)^{\sigma_0},$$

with

$$\varphi := x^{s_0+1}\bigg(-\frac{2(\log x)^{\sigma_0+1}}{(\sigma_0+1)(2s_0+2-n)} + F(\log x)\bigg) .$$

The same argument yields analogous functions ψ with

$$p_j(x\partial_x)\psi = x^{s_0}(\log x)^{\sigma_0+1},$$

and deals with the case of negative s, thereby completing the induction argument. q.e.d.

Armed with these auxiliary results, we are now ready for the analysis of the equation $Q(g) = 0$ that we will carry out in this section. For this we need to impose more stringent regularity assumptions on the metric $\bar\gamma_0$ than those in Section 3. Specifically, hereafter we make the following regularity assumption:

Assumption (Regularity of the reference metric)**.** The metric $\bar\gamma_0$ is of class $C^{n-1}\cap C^p_{\text{polyhom}}$ on $I\times\overline{M}$.

Here $p \geqslant n-1$ a given integer and we recall that a function h is in $C^p_{\text{polyhom}}(I\cap\overline{M})$ (polyhomogeneous of class C^p) if it is of class C^p away from the boundary (say, on $I\times(M\backslash\mathcal{A})$) and a C^p function of $(t,x,\theta,\log x)$ on a neighborhood of the boundary, say $I\times\overline{\mathcal{A}}$. The last condition means that in a small neighborhood of each point of $I\times\overline{\mathcal{A}}$ there is a C^p function h' of $n+2$ arguments such that

$$h = h'(t,x,\theta,\log x) .$$

Since the pullback of the reference metric $\bar\gamma_0$ to the boundary is $\widehat{g}$, this regularity assumption implies that the boundary metric $\widehat{g}$ must be of class $C^p(I\times\partial M)$.

To state the following theorem, we will introduce the space $C^m_r(I\times M)$ of functions with $m+r$ continuous derivatives, with the peculiarity that the last r derivatives with respect to x are regularized by multiplying by x. This way, for instance, for all $k,l,m\geqslant 1$ we have that

$$x^m\,(\log x)^l \tag{4.4}$$

is in C^{m-1}_k but not in C^{m+k-1}. To define the space $C^m_r(I\times M)$, we will also use a smooth nonnegative function $\chi_{\mathcal{A}}$ of x that vanishes outside $I\times\mathcal{A}$ and is equal to 1 in a neighborhood of $I\times\partial M$. With these objects at our disposal, we can now define $C^m_r(I\times M)$ as the space of functions φ such that

$$\begin{aligned}\|\varphi\|_{C^m_r(I\times M)} &:= \|(1-\chi_{\mathcal{A}})\varphi\|_{C^{m+r}(I\times M)}\\ &\qquad + \sum_{|\beta|+j+k\leqslant r} \|(x\partial_x)^j\partial_t^k\partial_\theta^\beta(\chi_{\mathcal{A}}\varphi)\|_{C^m(I\times M)}\end{aligned} \tag{4.5}$$

is finite. The space $C^p_r(M)$ is defined analogously. (Of course, the notation ∂_θ^β is somewhat heuristic as ∂M is not covered by a global chart. To define it rigorously, it is standard that one can resort to either covering ∂M with a fixed finite collection of charts and use a subordinate partition of unity, or to taking vector fields $X_1, \dots, X_M$ on ∂M that span the whole tangent space $T_p\partial M$ at each point $p \in \partial M$ and replace $\partial_\theta^\beta \varphi$ by

$$X_1^{\beta_1} \cdots X_M^{\beta_M} \varphi\,,$$

with $|\beta| = \beta_1 + \cdots + \beta_M$. For notational simplicity, we will stick to the notation ∂_θ^β, which must be interpreted in the aforementioned sense.)

We shall next present the main result of this section, which is a procedure to obtain asymptotically anti-de Sitter metrics γ that satisfy the boundary condition $(j_{I\times\partial M})^*\bar\gamma = \widehat{g}$ and for which $Q(\gamma)$ is suitably small. To state the theorem, we need to introduce some notation. Given nonnegative integers s and σ, we will say that a symmetric tensor field q, of class C^p in the interior of $I \times M$, is in $\mathcal{O}_j(x^s \log^{\leqslant\sigma} x)$ if it can be written in $\mathcal{A}$ as

$$q = x^s \sum_{\sigma'=0}^{\sigma} (\log x)^{\sigma'} B^{\sigma'}\,,$$

where $B^{\sigma'}$ is a smooth symmetric tensor field in $I \times \overline{M}$ satisfying the bounds

$$\|B^{\sigma'}\|_{C^k(I\times M)} \leqslant F_k\big(\|\bar\gamma_0\|_{C^{k'}_{r'}(I\times M)}\big),$$

for each $k \leqslant p-j$, where $k' := \min\{k+j, n-2\}$, $r' := \max\{0, k+j-n+2\}$ and F_k is a polynomial with $F_k(0) = 0$. Although we will not say it explicitly hereafter, it is important that in all the terms of the form $\mathcal{O}_j(x^s \log^{\leqslant\sigma} x)$ that will appear in this section, the coefficients of the corresponding polynomials F_k will be uniformly bounded in terms of the $C^{n-1} \cap C^p_{p-n+1}$ norm of $\bar\gamma_0$.

Theorem 4.5. *Let us take a nonnegative integer $n-1 \leqslant l \leqslant p$ and a small real $\delta > 0$. Then there is a weakly asymptotically AdS metric γ_l on $I \times M$ of the form*

$$\gamma_l = \sum_{k=0}^{l} \mathcal{O}_k(x^{k-2} \log^{\leqslant\sigma_k} x)\,,$$

where each nonnegative integer σ_k is zero for $k \leqslant n-1$, such that:

(i) *The pullback to the boundary of $\bar\gamma_l := x^2\gamma_l$ is*

$$(j_{I\times\partial M})^*\bar\gamma_l = \widehat{g}\,.$$

(ii) *The metric γ_l is uniformly close to $\bar\gamma_0$ in the sense that*

$$\|\bar\gamma_l - \bar\gamma_0\|_{L^\infty} < \delta,$$

and, furthermore,

$$\|\bar{\gamma}_l\|_{C^{n-1}_{p-n+1}(I\times M)} < C,$$

with a constant that depends only on $\|\bar{\gamma}_0\|_{C^{n-1}_{p-n+1}}$ *and* δ.

(iii) *The metric* γ_l *is a solution of the modified Einstein equation almost to order* $l-1$ *in the sense that*

$$Q(\gamma_l) = \mathcal{O}_{l+1}(x^{l-1}\log^{\leqslant\sigma_l'} x) + \mathcal{O}_{l+2}(x^l\log^{\leqslant\sigma_l'} x)\,,$$

where σ_k' *is a nonnegative integer that is equal to zero for all* $k\leqslant n-1$.

Proof. Proposition 3.2 trivially proves the result for $l=0$. To see how things work for $l=1$, let us write the $\mathcal{O}_1(x^{-1})$ terms that appear in

$$Q(\gamma_0) = \mathcal{O}_1(x^{-1}) + \mathcal{O}_2(1),$$

as

$$\mathcal{O}_1(x^{-1}) = \frac{H_1}{x} + \mathcal{O}_1(1)\,,$$

where the tensor field H_1 is defined in terms of this quantity as

$$H_1 := E\big(x\,\mathcal{O}_1(x^{-1})|_{x=0}\big), \tag{4.6}$$

and is $\mathcal{O}_1(1)$. Here E denotes the extension operator that we introduced in Equation (3.1), and for the time being we will restrict our attention to small values of x.

Let us now use the direct sum decomposition of $\mathcal{S}^2$ proved in Proposition 4.1 to write in a unique way

$$H_1 = \sum_{j=0}^{3} H_{1j}\,,$$

with $H_{1j}\in\mathcal{V}_j^{\gamma_0}$. We will take now

$$\gamma_1 := \gamma_0 - \sum_{j=0}^{3} f_{1j}(x) H_{1j},$$

with suitably chosen functions $f_{1j}(x)$. By Lemma 4.2 and Taylor's formula,

$$\begin{aligned} Q(\gamma_1) &= Q(\gamma_0) + (DQ)_\gamma(\gamma_1-\gamma_0) + I_1\\ &= x^{-2}\sum_{j=0}^{3}\Big(x - p_j(x\partial_x) f_{1j}\Big) H_{1j} + \mathcal{O}_2(1)\\ &\qquad\qquad + (x\mathcal{L}^1 + x^2\mathcal{L}^2)(\gamma_1-\gamma_0) + I_1\,,\end{aligned}$$

where the error term is

$$I_1 := \int_0^1 (D^2Q)_{(1-s)\gamma_0+s\gamma_1}(\gamma_1-\gamma_0)\, ds\,.$$

Since $p_j(-1)\neq 0$, Lemma 4.4 ensures that we can take functions $f_{1j} = \mathcal{O}(x^{-1})$ (indeed, $f_{1j}(x) = x^{-1}/p_j(-1)$) such that

$$p_j(x\partial_x)f_{1j} = x\,.$$

Since H_1, in principle, is only defined in a neighborhood of the boundary, we should include in f_{1j} a suitable cut-off function, which we, henceforth, omit for the ease of notation. In any case, with this choice of f_{1j} and Lemma 4.3, we obtain that the error term is controlled by

$$I_1 = \mathcal{O}_2(1) + \mathcal{O}_3(x)\,,$$

which immediately implies that

$$Q(\gamma_1) = \mathcal{O}_2(1) + \mathcal{O}_3(x)\,.$$

The general case follows by an induction argument that also relies on Taylor's formula and Lemmas 4.2–4.4. To sketch the proof, let us assume that the claim holds for all integers up to $l-1$, with

$$\bar\gamma_{l-1} = \sum_{k=0}^{l-1}\mathcal{O}_k(x^k\log^{\leqslant\sigma_k}x),$$

and $\sigma_k = 0$ for all $k\leqslant n-2$. To prove it for l, we argue as above to write

$$Q(\gamma_{l-1}) = x^{l-2}\sum_{j=0}^{3}\sum_{k=0}^{\sigma'_{l-1}}(\log x)^k H_{lkj} + \mathcal{O}_{l+1}(x^{l-1}\log^{\leqslant\sigma'_{l-1}}x)\,, \tag{4.7}$$

with $H_{lkj} = \mathcal{O}_l(1)$ a tensor field in $\mathcal{V}_j^{\gamma_{l-1}}$ and σ'_{l-1} an integer, related to σ_{l-1} and to the regularity of $\bar\gamma_0$ up to the boundary, which will be discussed later. Notice that H_{lkj} can be assumed to be related to the extension via the operator E of a suitable tensor field defined on the boundary, in an analogous fashion to (4.6).

Lemma 4.4 allows us to take polynomials f_{lkj}, of degree k if $p_j(l)\neq 0$ and $k+1$ otherwise, so that

$$p_j(x\partial_x)\big(x^l f_{lkj}(\log x)\big) = x^l(\log x)^k\,.$$

If we now set

$$\bar\gamma_l := \bar\gamma_{l-1} - x^l\sum_{j=0}^{3}\sum_{k=0}^{\sigma_{l-1}} f_{lkj}(\log x)H_{lkj}\,,$$

a computation analogous to the one for γ_1 then shows that

$$Q(\gamma_l) = \mathcal{O}_{l+1}(x^{l-1}\log^{\leqslant\sigma_l}x) + \mathcal{O}_{l+2}(x^l\log^{\leqslant\sigma_l}x),$$

for some integer σ_l.

Let us now complete our analysis of the log terms that appear in this computation by discussing the values that σ'_{l-1} can take. We have seen that $\sigma_l = 0$ as long as $\sigma'_k = 0$ for all $k \leqslant l-1$ and $p_j(l) \neq 0$. That is, log terms appear in γ_l either through log terms the right hand side of Equation (4.7) (where they can come from log terms in $\bar{\gamma}_{l-1}$ or from the reference metric $\bar{\gamma}_0$, which is in $C^{n-1} \cap C^p_{\text{polyhom}}$ and, therefore, such that its first non-smooth term is of the form $x^n \log x$) or due to the existence of integer roots of a polynomial $p_j(s)$, as shown in Lemma 4.4. It follows from Equation (4.2) that the first integer root of a polynomial $p_j(s)$ is $p_1(n) = 0$, so log terms can only appear at order $x^n \log x$ in $\bar{\gamma}_l$ and we, therefore, get that $\bar{\gamma}_l$ is of class $C^{n-1} \cap C^p_{\text{polyhom}}$.

Since $\bar{\gamma}_l - \bar{\gamma}_0$ vanishes at $x = 0$, it suffices to take the support of the aforementioned cut-off functions to be small enough to ensure that $\|\bar{\gamma}_l - \bar{\gamma}_0\|_{L^\infty}$ is as small as one wishes. Besides, it is apparent from the construction that the tensor fields H_{lkj} that appear at the l^{th} step of the induction that the coefficients are bounded in terms of $\bar{\gamma}_0$ and its l^{th} order derivatives which yields the estimate

$$\|\bar{\gamma}_l\|_{C^{n-1}_{p-n+1}} < C\,,$$

with C a constant that depends on $\|\bar{\gamma}_0\|_{C^{n-1}_{p-n+1}}$. Of course, the reason for which in general we get this estimate in C^{n-1}_{p-n+1} but not in C^p is the presence of log terms in the expression for $\bar{\gamma}_l$ starting with $x^n \log x$.
q.e.d.

5. Setting the iteration

Our goal in this section is to set up an iterative procedure that will eventually lead to a solution of the equation $Q(g) = 0$ with the desired initial and boundary conditions. To this end, let us write the solution as

$$g =: \gamma + h\,,$$

where

$$\gamma := \gamma_l$$

is the metric constructed in Theorem 4.5 with some large enough value of the parameter l that we will specify later. We will also assume that the number p appearing the regularity assumption of Section 4 is large enough. Intuitively, the weakly asymptotically AdS metric γ is the part of the metric that is "large" at the boundary and h is "smaller".

Let us recall from Equation (3.5) that one can write $Q(g)$ in local coordinates as

$$Q(g) = \widetilde{P}_g g + B(g)\,,$$

where we define the g-dependent linear differential operator $\widetilde{P}_g$ as

$$(\widetilde{P}_g g')_{\mu\nu} := -\frac{1}{2} g^{\lambda\rho}\partial_\lambda\partial_\rho g'_{\mu\nu},$$

and $B(g)$ depends on g and quadratically on ∂g. Taylor's formula ensures that

$$B(g) = B(\gamma) + (DB)_\gamma h - \widetilde{\mathcal{E}}(h)\,, \tag{5.1}$$

where the error term is

$$\widetilde{\mathcal{E}}(h) := -\int_0^1 (D^2B)_{\gamma+sh}(h)\,ds, \tag{5.2}$$

and the second order differential of B is understood as a quadratic form. The equation $Q(g) = 0$ can then be written as

$$\widetilde{P}_g h + (DB)_g h + (\widetilde{P}_\gamma\gamma - \widetilde{P}_g\gamma) + Q(\gamma) - \widetilde{\mathcal{E}}(h) = 0\,. \tag{5.3}$$

Let us now define a linear operator, depending on g, as

$$T_g h := -3h(\nabla^{(\gamma)}x, \nabla^{(\gamma)}x)\,\bar{g}\,,$$

where $\nabla^{(\gamma)}$ stands for the connection associated with the metric γ. As easy computation shows that T_g is the differential of the function $g \mapsto \widetilde{P}_\gamma\gamma - \widetilde{P}_g\gamma$ at $g = \gamma$. Hence, we will set

$$\widetilde{\mathcal{F}}(h) := T_g h + \widetilde{P}_g\gamma - \widetilde{P}_\gamma\gamma\,, \tag{5.4}$$

which, in view of (5.3), allows us to write the equation $Q(g) = 0$ as

$$\widetilde{P}_g h + (DB)_g h + T_g h = -Q(\gamma) + \widetilde{\mathcal{F}}(h) + \widetilde{\mathcal{E}}(h)\,.$$

Let us now define another g-dependent linear differential operator P_g by setting

$$\widetilde{P}_g h + (DB)_g h + T_g h =: x^{\frac{n}{2}+2} P_g u\,,$$

where we have introduced the new unknown u as

$$h =: x^{\frac{n}{2}} u\,.$$

Full details about the structure of the differential operator will be given in Section 8. In terms of u, the equation $Q(g) = 0$ can be finally written as

$$P_g u = \mathcal{F}_0 + \mathcal{G}(u)\,, \tag{5.5}$$

where

$$\mathcal{G}(u) := \mathcal{F}(u) + \mathcal{E}(u),$$

and

$$\mathcal{F}_0 := -x^{-\frac{n}{2}-2}Q(\gamma), \qquad \mathcal{F}(u) := x^{-\frac{n}{2}-2}\widetilde{\mathcal{F}}(h), \qquad \mathcal{E}(u) := x^{-\frac{n}{2}-2}\widetilde{\mathcal{E}}(h).$$

In the forthcoming sections our objective will be to solve this equation using an iterative procedure that will produce u as the limit of a sequence u^m, with $u^1 := 0$ and

$$P_{g^m} u^{m+1} = \mathcal{F}_0 + \mathcal{G}(u^m)\,.$$

Of course, here $g^m := \gamma + x^{\frac{n}{2}} u^m$ and the initial conditions that we need to impose are

$$u^{m+1}|_{t=0} = u_0\,, \qquad \partial_t u^{m+1}|_{t=0} = u_1\,,$$

where we have set

$$u_j := x^{-\frac{n}{2}}(g_j - \partial_t^j \gamma|_{t=0}), \tag{5.6}$$

for each nonnegative integer j, with $g_j := \partial_t^j g|_{t=0}$. As we will see, the compatibility conditions of the initial and boundary data boil down to assumption that a certain number of the functions u_j fall off fast enough at $x = 0$ to be in a suitable space of square-integrable functions over M. Since g_j is just a time derivative of the metric at $t = 0$, and, therefore, determined by the initial datum of the problem (that is, a Riemannian metric on M and a second fundamental form satisfying the constraint equations), and γ was determined by algebraically solving the Einstein equations to a certain order near the boundary, this just means that the formal series expansions for the solution that we get from the initial and boundary data must be compatible to a certain order. In the terminology of [**20**], this is means imposing corner conditions to a finite order.

6. Adapted Sobolev spaces

In this section, we will introduce some twisted Sobolev spaces that are adapted to the AdS geometry near the conformal boundary. They will be key in our derivation of the estimates that will allow us to prove the convergence of the iteration presented in the previous section. Specifically, we will consider two kinds of adapted Sobolev spaces, $\mathbf{H}^m_\alpha$ and $\mathcal{H}^m$, as well as certain modifications of them, $\mathbf{H}^{m,r}_\alpha$ and $\mathcal{H}^{m,r}$, that play a role somewhat similar to that of the spaces C^m_r introduced in (4.5). The first kind of adapted spaces depends on a parameter α that in our applications will ultimately be one of the quantities α_j defined in (4.2), so *we will assume throughout that* $\alpha > 1$ without further mention. The properties of these spaces for $\alpha < 1$ are quite different, as discussed in [**18**].

To define the spaces $\mathbf{H}^m_\alpha$, let us begin by introducing the twisted derivative with parameter α as

$$\mathbf{D}_{x,\alpha}\varphi := \partial_x \varphi + \frac{\alpha}{x}\varphi\,.$$

Its formal adjoint in the Hilbert space

$$\mathbf{L}^2_x := L^2((0,\infty), x\,dx) \tag{6.1}$$

is

$$\mathbf{D}^*_{x,\alpha}\varphi := -\partial_x\varphi + \frac{\alpha-1}{x}\varphi\,,$$

and we will set

$$\mathbf{D}^{(k)}_{x,\alpha}\varphi := \begin{cases} (\mathbf{D}^*_{x,\alpha}\mathbf{D}_{x,\alpha})^{\frac{k}{2}}\varphi & \text{if } k \text{ is even,} \\ \mathbf{D}_{x,\alpha}(\mathbf{D}^*_{x,\alpha}\mathbf{D}_{x,\alpha})^{\frac{k-1}{2}}\varphi & \text{if } k \text{ is odd,} \end{cases} \tag{6.2}$$

with the proviso that $\mathbf{D}^{(0)}_{x,\alpha}\varphi := \varphi$.

The twisted Sobolev space $\mathbf{H}^m_\alpha \equiv \mathbf{H}^m_\alpha(M)$ is defined as follows. Let us suppose that the function u is supported in a small neighborhood of the boundary ∂M, which we will take as

$$\mathcal{A} := \big\{(x,\theta) \in (0,a) \times \partial M\big\}\,.$$

We can then define its $\mathbf{H}^m_\alpha$ norm as

$$\|u\|^2_{\mathbf{H}^m_\alpha(\mathcal{A})} := \sum_{j+|\beta|\leqslant m} \int_{\partial M}\int_0^a |\mathbf{D}^{(j)}_{x,\alpha}\partial^\beta_\theta u|^2\, x\,dx\,d\theta,$$

where $d\theta$ is the canonical measure on the sphere and the twisted derivative acts on u in the obvious way. Using a suitable cutoff function that is equal to 1 in a neighborhood of the boundary and vanishes outside $\mathcal{A}$, for a function u defined on the ball we can then set

$$\|u\|_{\mathbf{H}^m_\alpha} := \|\chi u\|_{\mathbf{H}^m_\alpha(\mathcal{A})} + \|(1-\chi)u\|_{H^m(M)}\,, \tag{6.3}$$

where H^m is the usual Sobolev space. The space $\mathbf{H}^m_\alpha$ can then be defined as the closure in this norm of the space of smooth functions on M of compact support, the definition being also applicable to tensor-valued functions using standard arguments. For $m=0$ the norm, which does not depend on α, will be simply denoted by $\|u\|_{\mathbf{L}^2}$ or occasionally by $\|u\|$.

For real $s>0$, we can use interpolation to define the space $\mathbf{H}^s_\alpha \equiv \mathbf{H}^s_\alpha(M)$. Equivalently, since $\mathbf{D}^*_{x,\alpha}\mathbf{D}_{x,\alpha}$ is an essentially self-adjoint operator in $L^2(\mathbb{R}^+, x\,dx)$ with the domain $C^\infty_0(\mathbb{R}^+)$, we can write

$$\|u\|_{\mathbf{H}^s_\alpha} := \big\|\Lambda^s_\alpha(\chi u)\big\|_{\mathbf{L}^2} + \|(1-\chi)u\|_{H^s(M)}\,, \tag{6.4}$$

where

$$\Lambda^s_\alpha := (1-\Delta_{\partial M} + \mathbf{D}^*_{x,\alpha}\mathbf{D}_{x,\alpha})^{s/2} \tag{6.5}$$

is defined using the spectral theorem. As we did in (4.5), we can also consider the space with m derivatives as above and r "regularized" derivatives. For this we use the norm that is defined as

$$\|u\|_{\mathbf{H}^{m,r}_\alpha} := \sum_{j+|\beta|\leqslant r} \|(x\,\partial_x)^j\partial^\beta_\theta(\chi u)\|_{\mathbf{H}^m_\alpha} + \|(1-\chi)u\|_{H^{m+r}(M)}\,.$$

Closely related scales of Sobolev spaces are $\mathcal{H}^m \equiv \mathcal{H}^m(M)$ and $\mathcal{H}^{m,r} \equiv \mathcal{H}^{m,r}(M)$, which do not depend on any parameters and are weighted variations of the spaces typically considered in the theory of differential edge operators (see, e.g., [**32**]). They are respectively defined as the closure of $C_0^\infty(M)$ in the norm

$$\|u\|_{\mathcal{H}^m} := \sum_{j+|\beta|\leqslant m} \|x^{j-m}\partial_x^j\partial_\theta^\beta(\chi u)\|_{\mathbf{L}^2} + \|(1-\chi)u\|_{H^{m+r}(M)}\,,$$

$$\|u\|_{\mathcal{H}^{m,r}} := \sum_{j=0}^{r} \|x^j u\|_{\mathcal{H}^{m+j}}\,,$$

in each case. Notice that these norms are constructed by including in each derivative a singular weight that depends on the number of x-derivatives that one is taking. These spaces can also be defined for non-integer values using interpolation or, denoting by $\partial_x^* := -\partial_x - 1/x$ the formal adjoint of ∂_x with respect to the $\mathbf{L}_x^2$ product, directly through the formula

$$\|u\|_{\mathcal{H}^s} := \left\|\left(\frac{1-\Delta_{\partial M}}{x^2} + \partial_x^*\partial_x\right)^{s/2}(\chi u)\right\|_{\mathbf{L}^2} + \|(1-\chi)u\|_{H^s(M)}\,. \tag{6.6}$$

In particular, this ensures that the usual interpolation formulas are valid for these scales of Sobolev spaces.

We shall need estimates relating the various adapted Sobolev spaces that we have introduced. A simple observation is the following, which show how multiplication by powers of x can help us redistribute the "standard" and "regularized" derivatives in the spaces $\mathbf{H}_\alpha^{m,r}$ and $\mathcal{H}^{m,r}$:

Proposition 6.1. *Given nonnegative integers m, r and an integer $l \in [-m, r]$, we have the inequality*

$$\|x^l u\|_{\mathcal{H}^{m,r}} \leqslant C\|u\|_{\mathcal{H}^{m+l,r-l}}\,.$$

Proof. It is enough to expand the various terms appearing in the definitions of the norm and use some elementary algebra. q.e.d.

To explore the properties of these spaces we will make use of the integral operators

$$A_\alpha\varphi(x) := x^{-\alpha}\int_0^x y^\alpha\varphi(y)\,dy\,, \tag{6.7a}$$

$$A_\alpha^*\varphi(x) := x^{\alpha-1}\int_x^1 y^{1-\alpha}\varphi(y)\,dy\,, \tag{6.7b}$$

which act on functions of one variable and will play an essential role in the rest of this section. Notice that these operators are right inverses of $\mathbf{D}_{x,\alpha}$ and $\mathbf{D}_{x,\alpha}^*$ in the sense that

$$\mathbf{D}_{x,\alpha}(A_\alpha\varphi) = \mathbf{D}_{x,\alpha}^*(A_\alpha^*\varphi) = \varphi\,;$$

in particular, A_α^* is the adjoint of A_α in $\mathbf{L}^2_x$. Obviously A_α, A_α^* also act on functions defined on $\mathcal{A}$. In Appendix B we record some important properties of these operators, extracted from [**18**].

A simple but important estimate is the following, which gives an L^∞ bound for functions belonging to an adapted Sobolev space. Notice that, contrary to what happens in the usual Sobolev embedding theorem, we are not asking for the square-integrability of $\frac{n}{2}+\varepsilon$ derivatives but actually of $\frac{n+1}{2}+\varepsilon$:

Theorem 6.2. *Let $u \in \mathbf{H}^{1,r}_\alpha$ with $r > \frac{n-1}{2}$. Then we have the pointwise estimate in the ball*

$$\|u\|_{L^\infty} \leqslant C\, \|u\|_{\mathbf{H}^{1,r}_\alpha}\,.$$

Proof. By the definition of the norm and the Sobolev embedding, it is obviously enough to prove the result for u supported in $\overline{\mathcal{A}}$. But for a.e. (x,θ) in $\mathcal{A}$ we then have

$$\begin{aligned} |u(x,\theta)| &= \left|A_\alpha^*(\mathbf{D}_{x,\alpha}u)(x,\theta)\right| \\ &\leqslant C\|\mathbf{D}_{x,\alpha}u(\cdot,\theta)\|_{\mathbf{L}^2_x} \\ &\leqslant C\|\mathbf{D}_{x,\alpha}u\|_{\mathbf{L}^2_x H^r_\theta} \\ &\leqslant C\|u\|_{\mathbf{H}^{1,r}_\alpha}\,, \end{aligned}$$

where $H^r_\theta \equiv H^r(\partial M)$ is the Sobolev space of functions on ∂M with r square-integrable derivatives and to pass to the first, second and third lines we have respectively used the properties (i) and (iii) in Theorem B.1 and the Sobolev embedding. The theorem then follows. q.e.d.

Corollary 6.3. *For any $\rho > \frac{n-1}{2}$, $\|u\|_{C^m_r(M)} \leqslant C\|u\|_{\mathcal{H}^{m+1,r+\rho}}$. Furthermore, we have the bound*

$$\|x^{-m}(x\,\partial_x)^j\partial_\theta^\beta u\|_{L^\infty(\mathcal{A})} \leqslant C\|u\|_{\mathcal{H}^{m+1,r+\rho}},$$

for all indices with $j+|\beta| \leqslant m+r$.

Proof. It stems Theorem 6.2 and the fact that $x^{-m}(x\,\partial_x)^j\partial_\theta^\beta u \in \mathcal{H}^{1,\rho}$ for the above range of indices whenever $u \in \mathcal{H}^{m+1,r+\rho}$. q.e.d.

The connection between the spaces $\mathbf{H}^{m,r}_\alpha$ and $\mathcal{H}^{m,r}$ is subtler. Of course, the estimate

$$\|u\|_{\mathbf{H}^{m,r}_\alpha} \leqslant C\|u\|_{\mathcal{H}^{m,r}} \tag{6.8}$$

follows from an elementary computation. That for some range of the parameters there is a converse to this inequality, so that the norms $\mathbf{H}^{m,r}_\alpha$ and $\mathcal{H}^{m,r}$ are equivalent, is more sophisticated. The following theorem is the partial converse to the inequality (6.8) that we need:

Theorem 6.4. *For any $k \leqslant m$, if $\alpha > k-1$,*

$$\|u\|_{\mathcal{H}^{k,r+m-k}} \leqslant C\|u\|_{\mathbf{H}^{m,r}_\alpha}\,.$$

In particular, both norms are equivalent if $\alpha > k-1$.

Proof. Since $u \in \mathbf{H}^{m,r}_\alpha$ if and only if $(x\,\partial_x)^j \partial_\theta^\beta u \in \mathbf{H}^m_\alpha$ for all $j+|\beta| \leqslant r$, it is clearly enough to prove that

$$\|u\|_{\mathcal{H}^{k,m-k}} \leqslant C\|u\|_{\mathbf{H}^m_\alpha},$$

whenever $\alpha > k-1$. There is no loss of generality in proving the result for functions supported in $\mathcal{A}$, since away from the boundary both norms are equivalent.

With $m=1$, it suffices to see that one can write

$$u = A_\alpha(\mathbf{D}_{x,\alpha} u),$$

as a consequence of Theorem B.1 and that, due to this theorem,

$$\left\|\frac{u}{x}\right\|_{\mathbf{L}^2} \leqslant C\|\mathbf{D}_{x,\alpha}u\|_{\mathbf{L}^2} \leqslant C\|u\|_{\mathbf{H}^1_\alpha}\,.$$

Hence,

$$\|\partial_x u\|_{\mathbf{L}^2} = \left\|\mathbf{D}_{x,\alpha}u - \alpha\frac{u}{x}\right\|_{\mathbf{L}^2} \leqslant \|\mathbf{D}_{x,\alpha}u\|_{\mathbf{L}^2} + \alpha\left\|\frac{u}{x}\right\|_{\mathbf{L}^2} \leqslant C\|u\|_{\mathbf{H}^1_\alpha}\,,$$

as we wanted to prove.

Let us now consider the case $m=2$. A moment's thought reveals that it is enough to keep track of derivatives with respect to x in the argument, which is what we will do here, because we have that $\partial_\theta^\beta u \in \mathbf{H}^{m-|\beta|}_\alpha$. Hence, let us start by using Theorem B.1 to write

$$\mathbf{D}_{x,\alpha}u = A^*_\alpha(\mathbf{D}^{(2)}_{x,\alpha}u) + x^{\alpha-1}f_1(\theta)\,,$$

where $f_1(\theta)$ is a function on the sphere satisfying $\|f_1\|_{L^2_\theta} \leqslant C\|u\|_{\mathbf{H}^2_\alpha}$. Here we are using the notation $L^2_\theta \equiv L^2(\partial M)$. Again by Theorem B.1, this implies

$$u = A^{(2)}_\alpha(\mathbf{D}^{(2)}_{x,\alpha}u) + cx^\alpha f_1(\theta)\,,$$

where c is a constant and we are using the notation

$$A^{(l)}_\alpha\varphi := \begin{cases} (A^*_\alpha A_\alpha)^{\frac{l}{2}}\varphi & \text{if } l \text{ is even,}\\ A_\alpha(A^*_\alpha A_\alpha)^{\frac{l-1}{2}}\varphi & \text{if } l \text{ is odd.}\end{cases}$$

The desired estimates follow from this formula and the properties of the operators A_α and A^*_α listed in Theorem B.1. In order to see this, we start by noticing that

$$\begin{aligned}\left\|\frac{A_\alpha^{(2)}\varphi}{x^2}\right\|_{\mathbf{L}^2} &= \left\|\frac{1}{x^{\alpha+2}}\int_0^x y^\alpha A_\alpha^*\varphi(y)\,dy\right\|_{\mathbf{L}^2}\\ &= \left\|\frac{1}{x}A_{\alpha+1}\left(\frac{A_\alpha^*\varphi}{x}\right)\right\|_{\mathbf{L}^2}\\ &\leqslant C\left\|\frac{A_\alpha^*\varphi}{x}\right\|_{\mathbf{L}^2}\\ &\leqslant C\|\varphi\|_{\mathbf{L}^2}\,, \end{aligned}\tag{6.9}$$

which readily yields

$$\begin{aligned}\left\|\frac{u}{x^2}\right\|_{\mathbf{L}^2} &\leqslant \left\|\frac{A_\alpha^{(2)}(\mathbf{D}_{x,\alpha}^{(2)}u)}{x^2}\right\|_{\mathbf{L}^2} + |c|\|x^{\alpha-2}f_1(\theta)\|_{\mathbf{L}^2}\\ &\leqslant C\|\mathbf{D}_{x,\alpha}^{(2)}u\|_{\mathbf{L}^2} + |c|\|x^{\alpha-2}\|_{\mathbf{L}^2_x}\|f_1\|_{L^2_\theta}\,,\\ &\leqslant C\|u\|_{\mathbf{H}^2_\alpha}\end{aligned}$$

provided $\alpha > 1$, which is the condition for $x^{\alpha-2}$ to be in $\mathbf{L}^2_x$. If $\alpha\in(0,1]$, one can easily fix the argument by multiplying by a factor of x, which yields the estimate

$$\|u\|_{\mathcal{H}^{1,1}} \leqslant C\|u\|_{\mathbf{H}^2_\alpha},$$

for α in this range. A similar argument shows that

$$\begin{aligned}\left\|\frac{\partial_x u}{x}\right\|_{\mathbf{L}^2} &\leqslant \left\|\frac{A_\alpha^*(\mathbf{D}_{x,\alpha}^{(2)}u)}{x}\right\|_{\mathbf{L}^2} + C\left\|\frac{u}{x^2}\right\|_{\mathbf{L}^2} \leqslant C\|u\|_{\mathbf{H}^2_\alpha}\,,\\ \|\partial_x^2 u\|_{\mathbf{L}^2} &\leqslant \|\mathbf{D}_{x,\alpha}^{(2)}u\| + C\left\|\frac{\partial_x u}{x}\right\|_{\mathbf{L}^2} + C\left\|\frac{u}{x^2}\right\|_{\mathbf{L}^2} \leqslant C\|u\|_{\mathbf{H}^2_\alpha}\end{aligned}$$

provided $\alpha > 1$. This proves the claim for $m = 2$.

The general case follows by induction using the same argument using that if $u\in\mathbf{H}^m_\alpha$, one can write it as

$$u = A_\alpha^{(m)}(\mathbf{D}_{x,\alpha}^{(m)}u) + \sum_{0<l\leqslant m/2} x^{\alpha+2(j-1)}f_j(\theta)\,,$$

with $\|f_j\|_{L^2_\theta} \leqslant C\|u\|_{\mathbf{H}^m_\alpha}$. As before, the constraint on α appears from the fact that, for u to be in $\mathcal{H}^{k,j}$, $x^{\alpha-k}$ must be in $\mathbf{L}^2_x$, which forces $\alpha > k-1$. The only aspect that is slightly different than above is that the way in which the powers of x must the distributed when we have an expression of the form $x^{-l}A_\alpha^{(l)}$ is by recursively using the formulas

$$\|x^{-l}A_\alpha^*\varphi\|_{\mathbf{L}^2} \leqslant C\|x^{1-l}\varphi\|_{\mathbf{L}^2}\,, \qquad x^{-l}A_\alpha\varphi = \frac{1}{x}A_{\alpha+l-1}(x^{1-l}\varphi)\,.$$

q.e.d.

Combining Theorem 6.4 with Proposition 6.1 we arrive at the following useful:

Corollary 6.5. *If $\alpha > m - l - 1$,*

$$\|u\|_{\mathcal{H}^{m,r}} \leqslant C\|x^l u\|_{\mathbf{H}_\alpha^{m,r}}\,.$$

Proof. It is enough to consider $l \leqslant m$. We then have

$$\|x^l u\|_{\mathcal{H}^{m,r}} \leqslant C\|u\|_{\mathcal{H}^{m-l,r+l}} \leqslant C\|u\|_{\mathbf{H}_\alpha^{m-l,r+l}} \leqslant C\|u\|_{\mathbf{H}_\alpha^{m,r}}\,,$$

where we have used Theorem 6.4 to pass to the second inequality. q.e.d.

7. Nonlinear estimates for adapted Sobolev spaces

We shall next provide estimates that help us deal with nonlinear functions of elements of an adapted Sobolev space. To obtain estimates for products of functions in adapted Sobolev spaces, a basic result will be the following. To state it, we will use the notation

$$\mathcal{D}_{k,\beta} := (x\partial_x)^k \partial_\theta^\beta\,. \tag{7.1}$$

Theorem 7.1. *Given $r > \frac{n-1}{2}$, consider functions $w_1, \dots, w_{m-1} \in \mathcal{H}^{1,r}$ and $u \in \mathcal{H}^{0,r}$, which we can assume to be supported in $\mathcal{A}$. Then, given multiindices with*

$$\sum_{i=1}^m (k_i + |\beta_i|) \leqslant r\,,$$

we have that

$$\begin{aligned} &\|(\mathcal{D}_{k_1,\beta_1} w_1) \cdots (\mathcal{D}_{k_{m-1},\beta_{m-1}} w_{m-1})\,(\mathcal{D}_{k_m,\beta_m} u)\|_{\mathbf{L}^2} \\ &\qquad \leqslant C\|u\|_{\mathcal{H}^{0,r}} \prod_{i=1}^{m-1} \|w_i\|_{\mathcal{H}^{1,r}}\,. \end{aligned}$$

Proof. Notice that for any $\alpha > 1$ we have

$$\begin{aligned} &\left\| \bigg(\prod_{j=1}^{m-1} \mathcal{D}_{k_j,\beta_j} w_j \bigg) \mathcal{D}_{k_m,\beta_m} u \right\|_{\mathbf{L}^2}^2 \\ &\quad = \int \bigg(\prod_{j=1}^{m-1} (\mathcal{D}_{k_j,\beta_j} w_j)^2 \bigg)\, (\mathcal{D}_{k_m,\beta_m} u)^2\, x\, dx\, d\theta \\ &\quad \leqslant \int \bigg(\prod_{j=1}^{m-1} \sup_{x'} |\mathcal{D}_{k_j,\beta_j} w_j(x',\theta)|^2 \bigg)\, (\mathcal{D}_{k_m,\beta_m} u)^2\, x\, dx\, d\theta \\ &\quad \leqslant \int \bigg(\prod_{j=1}^{m-1} \|\mathbf{D}_{x,\alpha} \mathcal{D}_{k_j,\beta_j} w_j(\cdot,\theta)\|_{\mathbf{L}^2_x} \bigg)^2\, (\mathcal{D}_{k_m,\beta_m} u)^2\, x\, dx\, d\theta \\ &\quad \leqslant \int_{\partial M} \prod_{j=1}^m V_j^2\, d\theta\,, \end{aligned} \tag{7.2}$$

where we have defined

$$V_m := \|\mathcal{D}_{k_m,\beta_m} u\|_{\mathbf{L}^2_x} \qquad \text{and} \qquad V_j := \|\mathbf{D}_{x,\alpha}\mathcal{D}_{k_j,\beta_j} w_j(\cdot,\theta)\|_{\mathbf{L}^2_x},$$

for $1 \leqslant j \leqslant m-1$ and in order to pass to the third line we have used that, by Theorem B.1, for any one-variable function $\varphi(x) \in \mathbf{H}^1_\alpha$ with $\alpha > 1$ we have the inequality:

$$\|\varphi\|_{L^\infty_x} = \|A_\alpha(\mathbf{D}_{x,\alpha}\varphi)\|_{L^\infty_x} \leqslant C\|\mathbf{D}_{x,\alpha}\varphi\|_{\mathbf{L}^2_x}\,.$$

By definition and the Sobolev embedding, when $r - k_j - |\beta_j| < \frac{n-1}{2}$ we have

$$V_j \in H^{r-k_j-|\beta_j|}_\theta \subset L^{p_j}_\theta\,, \qquad p_j := \frac{2n-2}{n-1-2r-2k_j-2|\beta_j|}\,,$$

while for $r - k_j - |\beta_j| > \frac{n-1}{2}$ the function V_j is in L^∞_θ. For convenience, we will also relabel the functions V_j so that $r - k_j - |\beta_j| > \frac{n-1}{2}$ if and only if $j > m'$, so that $V_j \in L^{p_j}_\theta$ with $p_j = \infty$ for $j > m'$. We will also relabel the functions so that $r-k_j-|\beta_j| = \frac{n-1}{2}$ exactly for $m'' < j \leqslant m'$, and for this range of j's we will take p_j to be any finite but very large number. Of course, these last two sets can obviously be empty. Since ∂M is compact, the generalized Schwartz inequality ensures that the integral (7.2) can be estimated as

$$\begin{aligned}\int_{\partial M} \prod_{j=1}^m V_j^2\, d\theta &\leqslant \prod_{j=1}^m \|V_j\|^2_{L^{p_j}_\theta}\\ &\leqslant C\prod_{j=1}^m \|V_j\|^2_{H^{r-k_j-|\beta_j|}_\theta} \leqslant C\|u\|^2_{\mathcal{H}^{0,r}} \prod_{j=1}^{m-1} \|w_j\|^2_{\mathcal{H}^{1,r}}\end{aligned}$$

provided that

$$\sum_{j=1}^m \frac{2}{p_j} \leqslant 1\,. \tag{7.3}$$

Let us show that the condition (7.3) holds, which completes the proof of the theorem. For this, let us write

$$r = (1+\rho)\frac{n-1}{2}\,,$$

where $\rho > 0$ by hypothesis. Since $p_j = \infty$ for $m > m'$ and p_j is arbitrarily large for $m'' <\leqslant j \leqslant m'$, we can then take an arbitrarily small constant δ such that

$$\begin{aligned}\sum_{j=1}^m \frac{2}{p_j} &\leqslant \sum_{j=1}^{m''} \frac{2}{p_j} + \delta\\ &= \frac{1}{n-1}\sum_{j=1}^{m''}(n-1-2r+2k_j+2|\beta_j|) + \delta\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n-1}\Big(m''(n-1-2r) + 2\sum_{j=1}^{m''}(k_j + |\beta_j|)\Big) + \delta \\
&\leqslant m'' - \frac{2r(m''-1)}{n-1} + \delta \\
&= 1 - (m''-1)\rho + \delta\,. \qquad (7.4)
\end{aligned}
$$

Therefore, the claim follows for $m'' \geqslant 2$ by taking δ smaller than $(m''-1)\rho$. To conclude the proof, let us discuss the remaining cases. When $m''=0$, the claim is immediate. For $m''=1$ one can go over the proof of (7.4) and observe that the only problematic case is when $k_1+|\beta_1|=r$. But in this case $k_j+|\beta_j|=0$ for all $j>1$, which implies that there are not any j's for which $r-k_j-|\beta_j|=\frac{n-1}{2}$ and, thus, one can take $\delta=0$. The theorem then follows. q.e.d.

Theorem 7.1 will be key in the rest of the paper. It should be noticed that this theorem provides a wide range of estimates for nonlinear functions of elements of an adapted Sobolev space. In particular, we have the following result, where, although we do not emphasize it notationally, here the function $F(w_1,\dots,w_N)$ can also depend on the space variables:

Corollary 7.2. *Let $u\in\mathcal{H}^{0,r}$ and $w_1,\dots,w_m\in\mathcal{H}^{1,r}$ with $r>\frac{n-1}{2}$. Then, if F is a C^r function of w_j and a C^0_r function of the space variables (whose dependence will not be made explicit), we have*

$$
\|F(w_1,\dots,w_m)u\|_{\mathcal{H}^{0,r}} \leqslant C\|u\|_{\mathcal{H}^{0,r}}\,, \qquad (7.5)
$$

where C depends on $\|w_1\|_{\mathcal{H}^{1,r}}+\cdots+\|w_m\|_{\mathcal{H}^{1,r}}$.

Proof. The result follows by applying Theorem 7.1 to the various terms that appear after using the Leibniz rule on

$$
\mathcal{D}_{k,\beta}\big[F(w_1,\dots,w_m)u\big],
$$

with $k+|\beta|\leqslant r$. q.e.d.

8. Estimates for the linearized equation

For future convenience, we will assume that the metric g possesses the following properties, which will be needed in the following section to prove the convergence of the iteration set in Section 5. While some parameters could have been chosen in a different range for the purposes of this section, this way the application of these results in the following section will be transparent.

Assumption. Throughout this section we will assume that the metric g satisfies the following hypotheses:

(i) The metric g is weakly asymptotically AdS and can be written as

$$\bar g = \bar\gamma + xw\,,$$

with $\gamma \equiv \gamma_l$ is the metric constructed in Theorem 4.5 with $l \geqslant \frac{n}{2}+s+2$, for some integer s satisfying

$$2 \leqslant s < \frac{n}{2}+2\,.$$

We also assume that $\|\bar g^{\mu\nu}\|_{L^\infty} < \Lambda$.

(ii) The tensor field w is bounded as

$$\sum_{k=0}^{s-1} \|\partial_t^k w\|_{L_t^\infty \mathcal{H}^{2,r+s-k-2}} + \|\partial_t^s w\|_{L_t^\infty \mathcal{H}^{1,r-1}} < \Lambda\,, \tag{8.1}$$

for some integer $r > \frac{n-1}{2}$ and some constant Λ.

(iii) The metric $\bar\gamma$ satisfies

$$\|\bar\gamma\|_{C^{n-1}_{p-n+1}} < \Lambda,$$

with $p \geqslant l+r+s+1$, which is equivalent to demanding that the initial and boundary data $(\widetilde g, K, \widehat g)$ satisfy

$$\|x^2\widetilde g\|_{C^{n-1}_{p-n+1}} + \|x^2 K\|_{C^{n-1}_{p-n}} + \|\widehat g\|_{C^p(I\times\partial M)} < \Lambda'\,.$$

Using the formula (4.1), which ensures that the principal part of P_g is $\bar g^{\mu\nu}\partial_\mu\partial_\nu$, together with the small-x behavior described in Lemma 4.2 and the fact that g is weakly asymptotically AdS, is easy to derive a manageable expression for $P_g u$. Specifically, if we take $u \in \mathcal{V}_j^g$, a direct calculation shows that $P_g u$ can be written in $\mathcal{A}$ using local coordinates as

$$\begin{aligned}(P_g u)_{\mu\nu} = -\frac{1}{2}\bar g^{tt}\Big(&\partial_t^2 + \partial_{\theta^i}^* G^{ij}\partial_{\theta^j} + \mathbf{D}^*_{x,\alpha_j} b^1 \mathbf{D}_{x,\alpha_j} + x\,\partial^*_{\theta^i}(b^2)^i\partial_x \\ &+ x\partial_x b^3\partial_t + \partial_{\theta^i}(b^4)^i\partial_t\Big)u_{\mu\nu} \\ &+ \Big(b^5\partial_x u + b^6\partial_t u + b^7\partial_\theta u + \frac{b^8}{x}u\Big)_{\mu\nu}\,,\end{aligned} \tag{8.2}$$

where as usual the local coordinates $\theta = (\theta^1,\dots,\theta^{n-1})$ parametrize the boundary ∂M, the star denotes the formal adjoint of a differential operator computed with respect to the scalar product of $\mathbf{L}^2$, and the quantities b^l are scalar functions or tensor fields that depend smoothly on $\widehat g$, $\partial\widehat g$ (through γ and $\partial\gamma$), w and ∂w. Observe that the principal part of P_g is scalar. Although we do not make explicit the tensorial structure of the tensor fields b^l appearing in the non-principal part of the operator, their action must be understood in the obvious fashion, e.g.,

$$(b^6\partial_x u)_{\mu\nu} \equiv (b^6)_{\mu\nu}^{\lambda\rho}\,\partial_x u_{\lambda\rho}.$$

Notice that, in particular,

$$b^1 = -\frac{\bar{g}^{xx}}{\bar{g}^{tt}}\,, \quad G^{ij} = -\frac{\bar{g}^{\theta^i\theta^j}}{\bar{g}^{tt}}\,, \quad x(b^2)^i = -\frac{2\bar{g}^{x\theta^i}}{\bar{g}^{tt}}\,,$$
$$xb^3 = -\frac{2\bar{g}^{xt}}{\bar{g}^{tt}}\,, \quad (b^4)^i = -\frac{2\bar{g}^{t\theta^i}}{\bar{g}^{tt}}\,. \tag{8.3}$$

Since the metric is weakly asymptotically AdS, all the quantities b^j are of order $\mathcal{O}(1)$, with $b^1 > 0$ and G^{ij} a positive definite matrix.

We shall next derive estimates for a function satisfying the scalar equation

$$L_{g,\alpha} u = F\,, \qquad u|_{t=0} = u_0\,, \qquad \partial_t u|_{t=0} = u_1\,, \tag{8.4}$$

where

$$L_{g,\alpha} u := \Big(\partial_t^2 + \partial^*_{\theta^i} G^{ij} \partial_{\theta^j} + \mathbf{D}^*_{x,\alpha} b^1 \mathbf{D}_{x,\alpha} + x\,\partial^*_{\theta^i} (b^2)^i \partial_x + x\partial_x b^3 \partial_t + \partial_{\theta^i} (b^4)^i \partial_t\Big) u\,.$$

Taking $\alpha = \alpha_j$, $L_{g,\alpha}$ would be the part of $P_g u$ containing both the highest order derivatives and the more singular terms at $x = 0$, which is a scalar differential operator for $u \in \mathcal{V}^g_j$. The metric g is assumed to satisfy the above hypotheses, and we will also assume that $\alpha \geqslant n/2$. The reason for which we introduce this auxiliary equation is to postpone the treatment of the tensorial nature of the equation until the end of this section, but we have chosen to keep the notation u for the unknown as we will eventually replace u by a tensor field satisfying $P_g u = F$.

In the following theorem we provide a priori estimates for the problem (8.4). To state the theorem in a notationally concise way, let us denote by

$$u_k := \partial_t^k u|_{t=0}\,, \qquad 2 \leqslant k \leqslant s, \tag{8.5}$$

the value of the k^{th} time derivative of u at $t = 0$. Notice that, by isolating the term with the highest number of time derivatives in (8.4) and differentiating $k-2$ times with respect to t, one can write u_k in terms of derivatives of the initial data and source term (u_0, u_1, F). The functions u_k will often appear in arguments via the quantity

$$\mathcal{C}_{s,r} := \sum_{k=0}^{s-1} \|u_k\|_{\mathcal{H}^{1,r+s-k-1}} + \|u_s\|_{\mathcal{H}^{0,r}}\,. \tag{8.6}$$

For the tensor-valued equation $P_g u = F$, this quantity will correspond to the quantity that appears in the statement of Theorem 1.1.

To state the results, we will make use of the following norms (here the prime does not refer to any sort of duality):

$$\|u\|_{s,r} := \sum_{k=0}^{s-1} \|\partial_t^k u\|_{L^\infty_t \mathcal{H}^{1,r+s-k-1}} + \|\partial_t^s u\|_{L^\infty_t \mathcal{H}^{0,r}}\,, \tag{8.7a}$$

$$\|F\|'_{s,r} := \sum_{k=0}^{s-1} \|\partial_t^k F\|_{L^\infty_t \mathcal{H}^{0,r+s-k-1}}\,. \tag{8.7b}$$

Throughout, we will use the notation C_0 for constants depending only on δ and Λ.

Theorem 8.1. *For any $F \in L^\infty_t \mathbf{L}^2$ there is a unique solution $u \in L^\infty_t \mathcal{H}^1 \cap W^{1,\infty}_t \mathbf{L}^2$ to the Cauchy problem (8.4), which satisfies the following estimate in $(-T,T)\times M$:*

$$\|u\|_{s,r} \leqslant \mathrm{e}^{C_0 T} \mathcal{C}_{s,r} + C_0 T \|F\|'_{s,r}\,.$$

For small T, the constant depends only on Λ.

Proof. It is standard that it suffices to prove the a priori estimate. For this, there is no loss of generality in assuming that u is supported in $\mathcal{A}$, since the estimate is known to hold for u supported away from the boundary. Let us then define the energy functional

$$E_1[v] := \frac{1}{2}\int_M \big((\partial_t v)^2 + G^{ij}\partial_i v\,\partial_j v + b^1(\mathbf{D}_{x,\alpha} v)^2 + x(b^2)^i \partial_x v\,\partial_i v\big)\, x\,dx\,d\theta\,, \tag{8.8}$$

where in the rest of this section we will write $\partial_i \equiv \partial_{\theta^i}$. It is apparent that at any time $E_1[v]^{\frac12}$ is equivalent to the norm $\|v\|_{\mathbf{H}^1_\alpha} + \|\partial_t v\|_{\mathbf{L}^2}$ (which is in turn equivalent to $\|v\|_{\mathcal{H}^1} + \|\partial_t v\|_{\mathbf{L}^2}$ by Theorem 6.4) in the sense that

$$\frac{1}{C} E_1[v]^{\frac12} \leqslant \|v\|_{\mathcal{H}^1} + \|\partial_t v\|_{\mathbf{L}^2} \leqslant C E_1[v]^{\frac12}\,, \tag{8.9}$$

where the constant C only depends on

$$\|\bar g\|_{C^1_1} + \|\partial_t \bar g\|_{C^0_1} + \|\partial_t^2 \bar g\|_{C^0}\,.$$

In particular, by Corollary 6.3, $C \equiv C_0$ only depends on Λ.

Now let us use the energy functional (8.8) to define

$$E_{1,r'}[v] := \sum_{k+|\beta|\leqslant r'} E_1[\mathcal{D}_{k,\beta} v]\,,$$

where again we are using the shorthand notation $\mathcal{D}_{k,\beta} := (x\partial_x)^k \partial_\theta^\beta$. In view of the norm equivalence (8.9), it is clear that $E_{1,r'}[v]$ is equivalent to the norm

$$\|v\|_{\mathcal{H}^{1,r'}} + \|\partial_t v\|_{\mathcal{H}^{0,r'}},$$

with a constant that only depends on Λ. We can now define a higher analog of the energy E_1 by setting

$$E_{s,r}[v] := \sum_{k=0}^{s-1} E_{1,r+s-k-1}[\partial_t^k v]\,. \tag{8.10}$$

In view of the norm equivalence (8.9), it is clear that $E_{s,r}[v]^{1/2}$ is equivalent to the norm

$$\sum_{k=0}^{s-1} \|\partial_t^k v\|_{\mathcal{H}^{1,r+s-k-1}} + \|\partial_t^s v\|_{\mathcal{H}^{0,r}}, \tag{8.11}$$

in the same sense as above, which implies that

$$\sup_{|t|<T} E_{s,r}[v]^{\frac{1}{2}}$$

is equivalent to $\|v\|_{s,r}$.

Our goal now is to show that, if u is a solution of (8.4), the energy $E_{s,r}[u]$ satisfies the differential inequality

$$\partial_t E_{s,r}[u] \leqslant C_0 E_{s,r}[u] + C_0 E_{s,r}[u]^{\frac{1}{2}} \sum_{k=0}^{s-1} \|\partial_t^k F\|_{\mathcal{H}^{0,r+s-k-1}}\,. \tag{8.12}$$

Indeed, by Grönwall's inequality it is standard that this implies

$$E_{s,r}[u](t)^{\frac{1}{2}} \leqslant \mathrm{e}^{C_0'|t|}\left(E_{s,r}[u](0)^{\frac{1}{2}} + C_0' \sum_{k=0}^{s-1} \int_{-|t|}^{|t|} \|\partial_t^k F\|_{\mathcal{H}^{0,r+s-k-1}} \right).$$

Since $E_{s,r}[u]^{\frac{1}{2}}$ is equivalent to $\|u\|_{s,r}$, the a priori estimate of the theorem then follows from the above inequality.

Armed with Theorems 6.2 and 7.1, the proof of (8.12) is now standard. Let us begin by computing the evolution of $E_{1,r+s-1}[u]$. One readily finds that it is given by

$$\begin{aligned}\partial_t E_{1,r+s-1}[u] = \sum_{k+|\beta|\leqslant r+s-1} \bigg[\int \partial_t(\mathcal{D}_{k,\beta}u)\, L_{g,\alpha}(\mathcal{D}_{k,\beta}u) \\ - \int x\partial_t \mathcal{D}_{k,\beta}u\, \partial_x(b^3 \partial_t \mathcal{D}_{k,\beta}u) - \int \partial_t \mathcal{D}_{k,\beta}u\, \partial_i((b^4)^i \partial_t \mathcal{D}_{k,\beta}u) \\ + \int \mathcal{O}(1) \partial_t \mathcal{D}_{k,\beta}u\, \partial \mathcal{D}_{k,\beta}u + \int \frac{\mathcal{O}(1)}{x} \mathcal{D}_{k,\beta}u\, \partial_t \mathcal{D}_{k,\beta}u \\ + \int \frac{\mathcal{O}(1)}{x} (\mathcal{D}_{k,\beta}u)^2 + \int \mathcal{O}(1)\, (\partial \mathcal{D}_{k,\beta}u)^2 \bigg],\end{aligned} \tag{8.13}$$

where all the integrals hereafter correspond to integration over the ball with respect to the natural measure $x\,dx\,d\theta$ and we are denoting by

$\mathcal{O}(1)$ well-behaved functions of $\bar{\gamma}$, w and ∂w. We claim that this can be estimated as

(8.14)
$$\begin{aligned}&\partial_t E_{1,r+s-1}[u]\\ &\quad\leqslant C_0 E_{1,r+s-1}[u] + C_0 E_{1,r+s-1}[u]^{\frac{1}{2}} \sum_{k+|\beta|\leqslant r+s-1} \|L_{g,\alpha}(\mathcal{D}_{k,\beta}u)\|\,,\end{aligned}$$

where $\|\cdot\|$ stands for the $\mathbf{L}^2$ norm. Indeed, for $k+|\beta|\leqslant r+s-1$ the first term in (8.13) is bounded as

$$\int |\partial_t \mathcal{D}_{k,\beta}u\, L_{g,\alpha}\mathcal{D}_{k,\beta}u| \leqslant C_0 E_{1,r+s-1}[u]^{\frac{1}{2}}\, \|L_{g,\alpha}\mathcal{D}_{k,\beta}u\|,$$

and the last for summands can be easily upper bounded by

$$C_0 E_{1,r+s-1}[u]$$

using Theorems 6.2 and 7.1. Let us now consider the first of the two remaining terms. We have that

$$\begin{aligned}&\left|\int x\partial_t\mathcal{D}_{k,\beta}u\,\partial_x(b^3\partial_t\mathcal{D}_{k,\beta}u)\right|\\ &\qquad=\left|\int(\partial_t\mathcal{D}_{k,\beta}u)^2x\partial_x b^3+\frac{1}{2}\int b^3x\partial_x[(\partial_t\mathcal{D}_{k,\beta}u)^2]\right|\\ &\qquad\leqslant\int\left|\frac{1}{2}x\partial_x b^3-b^3\right|(\partial_t\mathcal{D}_{k,\beta}u)^2\\ &\qquad\leqslant C_0E_{1,r+s-1}[u],\end{aligned}$$

and an analogous argument shows that

$$\left|\int \partial_t\mathcal{D}_{k,\beta}u\,\partial_i((b^4)^i\partial_t\mathcal{D}_{k,\beta}u)\right|\leqslant C_0E_{1,r+s-1}[u]\,.$$

Putting everything together, this yields (8.14). To conclude, we can now estimate the commutator using Theorems 6.2 and 7.1 to infer that

$$\begin{aligned}\|L_{g,\alpha}(\mathcal{D}_{k,\beta}u)\|&\leqslant\|\mathcal{D}_{k,\beta}(L_{g,\alpha}u)\|+\|[L_{g,\alpha},\mathcal{D}_{k,\beta}]u\|\\ &\leqslant\|\mathcal{D}_{k,\beta}F\|+\|[L_{g,\alpha},\mathcal{D}_{k,\beta}]u\|\\ &\leqslant\|F\|_{\mathcal{H}^{0,r+s-1}}+C_0E_{s,r}[u]^{\frac{1}{2}}\,,\end{aligned}$$

which shows that

$$\partial_t E_{1,r+s-1}[u]\leqslant C_0E_{s,r}[u]+C_0E_{s,r}[u]^{\frac{1}{2}}\|F\|'_{s,r}\,.$$

The computation of the time evolution of the other quantities $E_{1,r+s-k-1}[\partial_t^k u]$ appearing in the definition of $E_{s,r}[u]$ (cf. Equation (8.10)) is similar, the only difference being that one needs to control the commutator

$$\begin{aligned}\|L_{g,\alpha}(\mathcal{D}_{j,\beta}\partial_t^k u)\| &\leqslant \|\mathcal{D}_{j,\beta}\partial_t^k F\| + \|[L_{g,\alpha}, \mathcal{D}_{j,\beta}\partial_t^k]u\| \\ &\leqslant \|\partial_t^k F\|_{\mathcal{H}^{0,r+s-k-1}} + C_0 E_{s,r}[u]^{\frac{1}{2}}\,.\end{aligned}$$

Summing over k, this readily yields the differential inequality (8.12). q.e.d.

REMARK 8.2. Notice that we are not imposing that $u(t) \in \mathbf{H}^2_\alpha$ for a.e. t, so Equation (8.4) has to be understood using the energy formulation, as it is customary.

Promoting the estimates proved in Theorem 8.1 to estimates for the tensor-valued equation

$$P_g u = F\,, \qquad u|_{t=0} = u_0\,, \qquad \partial_t u|_{t=0} = u_1 \tag{8.15}$$

is now immediate as the norms (8.7) can be trivially extended to tensor-valued functions. As before, we will state the theorem in terms of the quantity $\mathcal{C}_{s,r}$, which we can still define in terms of the initial data and source term at $t=0$ as in Equation (8.6).

Theorem 8.3. *For all times $T < T_0$, if u solves the problem (8.15) one has the estimates*

$$\|u\|_{s,r} \leqslant e^{C_0 T}\mathcal{C}_{s,r} + C_0 T \|F\|'_{s,r}\,,$$

where the constant C_0 depends only on Λ.

A final simple result that will come in handy in the following section is the following, which controls the difference between the solution to two Cauchy problems of the form (8.15) with different metrics and source terms. For concreteness we will control the difference in the $\|\cdot\|_{1,0}$ norm and assume that we have the same initial conditions (u_0, u_1), but we could have used any norm $\|\cdot\|_{s',r}$ with $s' \leqslant s-1$ and allowed for distinct initial conditions. It is worth emphasizing that estimating the difference is not completely trivial a priori because the leading part of the equation, as represented by the operator P_g, is not scalar: we have seen that the parameter $\alpha = \alpha_j$ takes a different value depending on the subspace $\mathcal{V}^g_j$ that u is assumed to belong to. However, the structure of the metrics under consideration allows to prove the result quite easily.

Proposition 8.4. *Let*

$$\bar{g} := \bar{\gamma} + xw \qquad \textit{and} \qquad \bar{g}' := \bar{\gamma} + xw'$$

be metrics satisfying the assumptions (i)–(iii) above. Suppose that $u, u' \in L^\infty\mathcal{H}^1 \cap H^1_t\mathbf{L}^2$ satisfy the equations

$$P_g u = F \qquad \textit{and} \qquad P_{g'}u' = F',$$

with the same initial conditions (u_0, u_1). *Then the difference is bounded by*

$$\|u-u'\|_{1,0} \leqslant C\mathrm{e}^{CT}T(\|F-F'\|'_{1,0}+\|w-w'\|_{1,0})\,,$$

where the constant C *only depends on* Λ, $\|F\|'_{s,r}$ *and* $\mathcal{C}_{s,r}$.

Proof. A short computation using the expression for P_g shows that the differential operator P_g, whose leading part at $x=0$ is *not* scalar, can be symbolically written in a neighborhood of $x=0$ as

$$P_g u = A_2(\bar g)\,\partial^2 u + \left(\frac{A_1(\bar g)}{x} + A_1'(\bar g, \partial\bar g)\right)\partial u + \left(\frac{A_0(\bar g)}{x^2} + \frac{A_0'(\bar g,\partial\bar g)}{x}\right)u\,, \tag{8.16}$$

where A_j, A_j' are tensor-valued functions. Furthermore, we know that the term with second-order derivatives is scalar, and given by (8.2).

With $\bar g = \bar\gamma + xw$, it then follows that P_g agrees with P_γ modulo terms that are subdominant at $x=0$. More precisely, Theorem 7.1 yields

$$\begin{aligned}
\|(P_g - P_{g'})u\| &\leqslant \sum_{k=0}^{2}\left\|\frac{A_k(\bar g)-A_k(\bar g')}{x^{2-k}}\partial^k u\right\| \\
&\qquad + \sum_{0}^{1}\left\|\frac{A_k'(\bar g,\partial\bar g)-A_k'(\bar g',\partial\bar g')}{x^{1-k}}\partial^k u\right\| \\
&\leqslant C\|w-w'\|_{1,0}\,, \qquad (8.17)
\end{aligned}$$

with $\|\cdot\|$ denoting the $\mathbf{L}^2$ norm and the constant C depending only on the quantities discussed at the statement as a consequence of the estimates for u proved in Theorem 8.3.

To see why this is true, let us consider a term that does not depend on $\partial\bar g$, such as $A_2(\bar g)\,\partial^2 u$. Observe that, as the L^∞ norm of w and ∂w is bounded by a constant that depends on Λ by Theorem 6.2, it is standard that we have

$$|A(\bar g,\partial\bar g) - A(\bar g',\partial\bar g')| \leqslant C_0(|w-w'| + x|\partial w - \partial w'|)\,.$$

Therefore,

$$\begin{aligned}
\|(A_2(\bar g)-A_2(\bar g'))\,\partial^2 u\| &\leqslant \|(xw-xw')H(xw,xw')\partial^2 u\| \\
&\leqslant C\|(w-w')x\partial^2 u\| \\
&\leqslant C\|w-w'\|_{L^\infty_x L^2_\theta}\|x\partial^2 u\|_{\mathbf{L}^2_x L^\infty_\theta} \\
&\leqslant C\|w-w'\|_{\mathcal{H}^1}\|u\|_{\mathcal{H}^{1,r'+1}} \\
&\leqslant C\|w-w'\|_{1,0}\,.
\end{aligned}$$

Here H is a smooth tensor-valued function, r' is any number larger in $(\frac{n-1}{2}, r]$ and the constant C is as above. When derivatives of $\bar g$ are involved, the argument is similar. For instance,

$$\begin{aligned}\|(A_1'(\bar g,\partial\bar g)-A_1'(\bar g',\partial\bar g'))\,\partial u\| &\leqslant \|(w-w')H_1\partial u\|+\|x(\partial w-\partial w')H_2\partial u\|\\ &\leqslant C\|(w-w')\partial u\|+C\|(\partial w-\partial w')x\partial u\|\\ &\leqslant C\|w-w'\|_{L^\infty_x L^2_\theta}\|\partial u\|_{\mathbf{L}^2_x L^\infty_\theta}\\ &\qquad + C\|\partial w-\partial w'\|\|x\partial u\|_{L^\infty}\\ &\leqslant C\|w-w'\|_{\mathcal{H}^1}\|u\|_{\mathcal{H}^{1,r'+1}}\\ &\leqslant C\|w-w'\|_{1,0}\,.\end{aligned}$$

To conclude the proof of the proposition, let us notice that

$$P_{g'}(u-u')=F'-F+(P_g-P_{g'})u\,.$$

Since

$$\|(P_g-P_{g'})u\|'_{1,0}=\|(P_g-P_{g'})u\|_{L^\infty_t\mathbf{L}^2}\leqslant C\|w-w'\|_{1,0},$$

by (8.17), Theorem 8.3 then provides the desired control for the difference $u-u'$. q.e.d.

9. Convergence of the iteration

We are now ready to prove the existence of solutions to the equation $Q(g)=0$ with the desired initial and boundary conditions. With the technical tools that we have already developed, the argument is now standard.

To present the result, let us introduce a new norm that is stronger than $\|u\|_{s,r}$ in the sense that it also includes additional (adapted) derivatives with respect to the variable x. To define it, we can assume that the tensor field u is supported in $\mathcal{A}$ and consider its decomposition

$$u=u^0+u^1+u^2+u^3\,,$$

where $u^j\in\mathcal{V}^\gamma_j$. The norm is then defined using the metric γ as

$$|\!|\!|u|\!|\!|_{s,r}:=\|u\|_{s,r}+\sum_{j=0}^3\sum_{i+k+m\leqslant s-2}\|\mathbf{D}^{(2+i)}_{x,\alpha_j}\partial_t^k u^j\|_{\mathcal{H}^{0,r+m}}\,.$$

For $s=1$ we simply take $|\!|\!|u|\!|\!|_{1,r}:=\|u\|_{1,r}$. By Theorem 6.4 and the fact that $\alpha_j\geqslant\frac n2$, for $s<\frac n2+1$ this is equivalent to

$$|\!|\!|u|\!|\!|_{s,r}:=\|u\|_{s,r}+\sum_{i+k+m\leqslant s-2}\|\partial_t^k u\|_{\mathcal{H}^{2+i,r+m}}\,,$$

so, in particular, it does not depend on γ. Likewise, for $s\in[\frac n2+1,\frac n2+2)$ one can write

$$|\!|\!|u|\!|\!|_{s,r}:=\|u\|_{s,r}+\sum_{j=0}^3\|\mathbf{D}^{(s)}_{x,\alpha_j}u^j\|_{\mathcal{H}^{0,r}}+\sum_{i+k+m\leqslant s-2\text{ and }i\leqslant s-3}\|\partial_t^k u\|_{\mathcal{H}^{2+i,r+m}}.$$

Of course, when u is not supported in $\mathcal{A}$ one defines its triple norm using a compactly supported function χ, e.g., as in Equation (6.3). It should be noticed that we will not only estimate u, but also $x^\rho u$, as in the bound (9.2) below. The reason for this is that this not only amounts to redistributing standard and regularized derivatives as in Proposition 6.1, but, in fact, allows us to control ρ additional time derivatives of u. This will be useful to prove Theorem 1.1.

Theorem 9.1. *Let us choose numbers s, r, l and p and take $\gamma \equiv \gamma_l$ as in the assumptions (i)–(iii) of Section 8. For any compatible initial and boundary data $(\widetilde{g}, K, \widehat{g})$, there is some time $T > 0$ and a function u such that the weakly asymptotically AdS metric*

$$g := \gamma + x^{\frac{n}{2}} u$$

solves the modified Einstein equation $Q(g) = 0$ in $(-T, T) \times M$ with the specified initial and boundary conditions and is bounded as

$$\left|\!\left|\!\left| u \right|\!\right|\!\right|_{s,r} < C, \tag{9.1}$$

with a constant depending only on

$$\|x^2 \widetilde{g}\|_{C^{n-1}_{p-n+1}} + \|x^2 K\|_{C^{n-1}_{p-n}} + \|\widehat{g}\|_{C^p(I \times \partial M)} \,.$$

Furthermore, if $r > \frac{n-1}{2} + \rho$ with ρ a positive integer, we also have

$$\left|\!\left|\!\left| x^\rho u \right|\!\right|\!\right|_{s+\rho, r-\rho} < C\,, \tag{9.2}$$

and $\bar{g} \in C^\infty_{\mathrm{polyhom}}$ if $\widetilde{g} \in C^\infty$ and $x^2 \widetilde{g}, x^2 K \in C^\infty_{\mathrm{polyhom}}$.

Proof. For simplicity we will divide the proof in four steps. As usual, it is enough to prove the estimates in a small neighborhood $\mathcal{A}$ of the boundary. As before, we will write the metric as $\bar{g} = \bar{\gamma} + x^{\frac{n}{2}+2} u$ and write the equation $Q(g) = 0$ in the convenient form (5.5).

Estimates for the source terms. Let us begin by deriving some estimates for the functions $\mathcal{F}(u)$ and $\mathcal{E}(u)$ under the assumptions that

$$\|\bar{\gamma}\|_{C^{n-1}_{p-n+1}} < \Lambda, \quad \|u\|_{s,r} < \Lambda, \quad \|\bar{g}^{\mu\nu}\|_{L^\infty} < \Lambda\,; \tag{9.3}$$

cf. Section 8. Just as in that section, we will write the metric as $\bar{g} = \bar{\gamma} + xw$ with $w := x^{\frac{n}{2}+1} u$ bounded in the norm (8.1). Throughout, we will denote by C_0 a constant that only depends on Λ and δ and we will use without further mention the properties of the adapted Sobolev spaces that we established in Sections 6 and 7.

A close look at Equation (5.4) reveals that the function $\mathcal{F}(u)$ can be written as

$$\mathcal{F}(u) = \frac{F(\bar{g}) u}{x}\,,$$

where $F(\bar{g})$ is a smooth function of $\bar{g} := \bar{\gamma} + x^{\frac{n}{2}+2}u$ (in particular, $\mathcal{F}(u)$ does not involve any derivatives of u). Hence, at any fixed time we have

$$\|\mathcal{F}(u) - \mathcal{F}(u')\| \leqslant C_0 \left\| \frac{u-u'}{x} \right\| \leqslant C_0 \|u-u'\|_{\mathcal{H}^1} ,$$

where $\|\cdot\|$ again stands for the $\mathbf{L}^2$ norm, which implies

$$\|\mathcal{F}(u) - \mathcal{F}(u')\|'_{1,0} \leqslant C_0 \|u-u'\|_{1,0} .$$

Furthermore, by the elementary inequality $\|v/x\|_{\mathcal{H}^{k,s}} \leqslant C\|v\|_{\mathcal{H}^{k+1,s}}$,

$$\begin{aligned} \|\mathcal{F}(u)\|'_{s,r} &= \sup_{|t|<T} \sum_{k=0}^{s-1} \|\partial_t^k \mathcal{F}(u)\|_{\mathcal{H}^{0,r+s-k-1}} \\ &\leqslant C_0 \sup_{|t|<T} \sum_{k=1}^{s-1} \|\partial_t^k u\|_{\mathcal{H}^{1,r+s-k-1}} \\ &\leqslant C_0 \|u\|_{s,r} . \end{aligned}$$

Using the formula for $\mathcal{E}(u)$ given in Equation (5.2) and computing the second derivative of B as in Lemma 4.3, we infer that $\mathcal{E}(u)$ can be symbolically written as

$$\mathcal{E}(u) = \int_0^1 x^{\frac{n}{2}} B(u, x\,\partial u)\, d\sigma ,$$

where B is a quadratic form whose coefficients are smooth functions of $\bar{\gamma} + \sigma x^{\frac{n}{2}+2}u$ and the integral is with respect to the parameter σ. Using this formula and arguing essentially as in the case of $\mathcal{F}(u)$ one can prove the analogous estimates

$$\begin{aligned} \|\mathcal{E}(u) - \mathcal{E}(u')\|'_{1,0} &\leqslant C_0 \|u-u'\|_{1,0} , \\ \|\mathcal{E}(u)\|'_{s,r} &\leqslant C_0 \|u\|_{s,r} . \end{aligned}$$

Hence, it stems that the function $\mathcal{G}(u) := \mathcal{F}(u) + \mathcal{E}(u)$ that appears in Equation (5.5) satisfies the same bounds, that is,

$$\|\mathcal{G}(u) - \mathcal{G}(u')\|'_{1,0} \leqslant C_0 \|u-u'\|_{1,0} , \tag{9.4}$$

$$\|\mathcal{G}(u)\|'_{s,r} \leqslant C_0 \|u\|_{s,r} . \tag{9.5}$$

Convergence in the low norm. Our objective will be to solve the equation using the iteration

$$P_{g^m} u^{m+1} = \mathcal{F}_0 + \mathcal{G}(u^m) , \tag{9.6a}$$

where $g^m := \gamma + x^{\frac{n}{2}} u^m$ and the initial conditions that we impose are

$$u^{m+1}|_{t=0} = u_0 , \qquad \partial_t u^{m+1}|_{t=0} = u_1 , \tag{9.6b}$$

where of course $u_j := x^{-\frac{n}{2}}(g_j - \partial_t^j \gamma|_{t=0})$. We can start the iteration with $u^1 := 0$ and the desired solution to the equation $Q(u) = 0$ will arise as the limit of u^m as $m \to \infty$. Notice that we are using superscripts both for the sequence of iterates and for the components of u in the space $\mathcal{V}_j^\gamma$, but this should not cause any confusion because only the former will appear in the study of the convergence of the sequence.

Let us assume that the condition (9.3) is satisfied, where Λ is chosen so that

$$\|\bar\gamma\|_{C^{n-1}_{p-n+1}} + \|\bar g^{\mu\nu}|_{t=0}\|_{L^\infty} + \mathcal{C}_{s,r} + \|\mathcal{F}_0\|'_{s,r} < \frac{\Lambda}{2}\,. \tag{9.7}$$

Recall that, by Theorem 4.5,

$$\|\mathcal{F}_0\|'_{s,r} \leqslant C\|\mathcal{F}_0\|_{C^1_{s+r-1}(I\times M)} \leqslant C\|\widehat{g}\|_{C^p(I\times\partial M)}\,,$$

where we have used that $l \geqslant s + \frac{n}{2} + 2$ and $p \geqslant l+s+r+1$, so this just means that we choose Λ in terms of the sizes of the initial and boundary data.

To prove the convergence of the sequence in the norm $\|\cdot\|_{1,0}$, then we can use Proposition 8.4 and the estimate (9.4) to write, for $T<T_0$,

$$\begin{aligned}\|u^{m+1}-u^m\|_{1,0} &\leqslant C_0T\|\mathcal{G}(u^m)-\mathcal{G}(u^{m-1})\|'_{1,0} + C_0T\|u^m-u^{m-1}\|_{1,0}\\ &\leqslant C_0T\|u^m-u^{m-1}\|_{1,0}\,.\end{aligned} \tag{9.8}$$

It then follows that the sequence $(u^m)_{m=1}^\infty$ converges in the norm $\|\cdot\|_{1,0}$ to some $u\in L^\infty_t\mathcal{H}^1\cap W^{1,\infty}_t\mathbf{L}^2$, provided that T is smaller than some constant depending only on Λ (i.e., $T<1/(2C_0)$).

Boundedness in the high norm. Let us assume that the bound (9.3) is satisfied up to the m^{th} step of the iteration with Λ chosen so that (9.7) holds. Writing $g^m=\gamma+xw^m$ with $w^m := x^{\frac{n}{2}+1}u^m$, we then infer that the assumptions on the metric of Section 8 are satisfied too. Hence, applying Theorem 8.3 to Equation (9.6) immediately yields, for $T<T_0$,

$$\|u^{m+1}\|_{s,r} \leqslant e^{C_0T}\mathcal{C}_{s,r} + C_0T\big(\|\mathcal{F}_0\|'_{s,r} + \|\mathcal{G}(u^m)\|'_{s,r}\big)\,. \tag{9.9}$$

If we employ that $\|\mathcal{F}_0\|'_{s,r}<\Lambda/2$ in the inequality (9.9) and use the estimate (9.5), we arrive at

$$\begin{aligned}\|u^{m+1}\|_{s,r} &\leqslant e^{C_0T}\mathcal{C}_{s,r} + C_0T\|u^m\|_{s,r}\\ &\leqslant (e^{C_0T}+C_0T)\frac{\Lambda}{2}\\ &< \Lambda\end{aligned} \tag{9.10}$$

provided that T is small enough.

Since the sequence (u^m) is bounded in $\|\cdot\|_{s,r}$ by (9.10) and converges to u in $\|\cdot\|_{1,0}$ by (9.8), together with the fact that these spaces possess good interpolation properties (essentially as a consequence of the formula (6.6)), we immediately obtain that $u^m\to u$ in $\|\cdot\|_{s',r}$ for any real $s'<s$ and that u also satisfies the bound $\|u\|_{s,r}\leqslant\Lambda$. The usual argument then shows (cf., e.g., [**33**, Chapter 9]) that u is, indeed, a

solution of the equation $Q(g) = 0$ in $(-T, T) \times M$, with T small enough, and that u is bounded by

$$\|u\|_{s,r} < \Lambda, \tag{9.11}$$

as a consequence of (9.10).

Higher spatial regularity. Our goal now is to show that, if u satisfies the equation $Q(g) = 0$, up to s adapted derivatives of u can then be controlled in terms of the energy $E_{s,r}[u]$. More precisely, need to prove that

$$\begin{aligned}\sum_{j=0}^{3} \sum_{i+k+m \leqslant s-2} \|\mathbf{D}^{(2+i)}_{x,\alpha_j} \partial_t^k u^j\|_{L^\infty_t \mathcal{H}^{0,r+m}} \\ \leqslant C_0 \|u\|_{s,r} + C_0 \sum_{i+k+m \leqslant s-2} \|\partial_t^k \mathcal{F}_0\|_{\mathcal{H}^{i,r+m}} .\end{aligned} \tag{9.12}$$

Since $l \geqslant s + \frac{n}{2} + 2$ and $p \geqslant l + s + r + 1$, Theorem 4.5 then asserts that

$$\begin{aligned}\sum_{i+k+m \leqslant s-2} \|\partial_t^k \mathcal{F}_0\|_{\mathcal{H}^{i,r+m}} &\leqslant C \|x^{2-s} \mathcal{F}_0\|_{C^0_{s+r-2}(I \times M)} \\ &\leqslant C\Lambda .\end{aligned}$$

Hence, the desired bound (9.1) follows from the inequality (9.12) and the estimate (9.11).

The estimates (9.12) are proved by isolating the term $\mathbf{D}^{(2)}_{x,\alpha} u$ in the equation $Q(g) = 0$, which we write as

$$P_g u = \mathcal{F}_0 + \mathcal{G}(u),$$

with $g = \gamma + x^{\frac{n}{2}} u$. Once the term $\mathbf{D}^{(2)}_{x,\alpha} u$ has been isolated, we can take the necessary number of adapted x-derivatives for which we need a priori estimates. For concreteness, let us spell out the details for the first quantity, namely the norm $\|\mathbf{D}^{(2)}_{x,\alpha} u\|_{L^\infty_t \mathcal{H}^{0,r+s-2}}$.

From Equation (8.4) we can write

$$\begin{aligned}\mathbf{D}^{(2)}_{x,\alpha_j} u^j = \frac{1}{b^1} \Big(&\mathcal{F}_0^j + \mathcal{G}(u)^j + \partial_t^2 u^j - (\partial_x b^1) \mathbf{D}_{x,\alpha_j} u^j - \partial_i^* (G^{ik} \partial_k u^j) \\ &- x \partial_i^* [(b^2)^i \partial_x u^j] - x \partial_x (b^3 \partial_t u^j) - \partial_i [(b^4)^i \partial_t u^j] + \text{l.o.t.} \Big) ,\end{aligned} \tag{9.13}$$

where the superscript j indicates the component in $\mathcal{V}^\gamma_j$ and we have employed the identity (8.16) to write

$$P_g u = P_\gamma u + \text{l.o.t.}$$

using the same ideas as in the proof of Proposition 8.17. Besides, we have used that, as thanks to our choice of the number s, r we have the uniform bound

$$\|\bar{g} - \bar{g}|_{t=0}\|_{L^\infty} \leqslant CT\,.$$

Equation (8.3) guarantees that we can, indeed, divide by b^1 to solve the equation for $\mathbf{D}^{(2)}_{x,\alpha_j} u$. To compute the norm $\|\mathbf{D}^{(2)}_{x,\alpha_j} u^j\|_{\mathcal{H}^{0,r+s-2}}$ we must now consider the action of the differential operator $\mathcal{D}_{k,\beta}$ on this equation, with $l+|\beta| \leqslant r+s-2$ and $\mathcal{D}_{k,\beta}$ defined as in (7.1). Given the dependence on u of the various terms that appear in the equation, a straightforward computation shows that, in fact, the terms that appear can, indeed, be controlled using the norm $\|u\|_{s,r}$ and Theorems 6.2 and 7.1 as

$$\|\mathbf{D}^{(2)}_{x,\alpha_j} u^j\|_{L^\infty_t \mathcal{H}^{0,r+s-2}} \leqslant C_0\|u\|_{s,r} + C_0\|\mathcal{F}_0\|_{L^\infty_t \mathcal{H}^{0,r+s-2}}\,. \tag{9.14}$$

Although we will not write down the tedious but straightforward minutiae, it is clear from (9.13), e.g., that the most dangerous terms that can appear when one estimates $\|\mathbf{D}^{(2)}_{x,\alpha_j} u^j\|_{\mathcal{H}^{0,r+s-2}}$ are of the symbolic form

$$\|F(u)\partial_t^2 u\|_{\mathcal{H}^{0,r+s-2}} + \|F(u)x\partial_x\partial_\theta u\|_{\mathcal{H}^{0,r+s-2}} + \|F(u)x\partial_x\partial_t u\|_{0,r+s-2}\,,$$

and these are clearly controlled by $\|u\|_{s,r}$.

Now that we have estimated $\|\mathbf{D}^{(2)}_{x,\alpha_j} u^j\|_{\mathcal{H}^{0,r+s-2}}$, which gives control over $\|u\|_{L^\infty_t \mathcal{H}^{2,r+s-2}}$, we can easily obtain bounds for $\|\mathbf{D}^{(2)}_{x,\alpha_j}\partial_t^k u\|_{L^\infty_t \mathcal{H}^{0,r+s-2-l}}$ by taking time derivatives in Equation (9.13) and repeating the argument. Estimates for the other terms $\|\mathbf{D}^{(2+i)}_{x,\alpha_j}\partial_t^k u^j\|_{L^\infty_t \mathcal{H}^{0,r+k}}$ are then obtained by successively acting with $\mathbf{D}^{(i)}_{x,\alpha}$ on Equation (9.13), with $i = 1, 2\dots, s-2$. The only difference is that one has to use that, by the choice of the range of parameters made in the assumptions (i)–(iii), the norms $\|\cdot\|_{\mathbf{H}^{s',r'}_\alpha}$ and $\|\cdot\|_{\mathcal{H}^{s',r'}}$ are equivalent by Theorem 6.4 for all $s' < \frac{n}{2}+1$.

Additional time derivatives and C^∞ estimates. The proof of the a priori estimate (9.2) is, in a way, analogous to that of (9.12). If we now isolate $\partial_t^2 u$ in Equation (9.13), we find that the component $u^j \in \mathcal{V}^\gamma_j$ satisfies the equation

$$\begin{aligned}\partial_t^2 u^j = \mathcal{G}(u)^j - \mathbf{D}^{(2)}_{x,\alpha_j} u^j - (\partial_x b^1)\mathbf{D}_{x,\alpha_j} u^j - \partial_i^*(G^{ik}\partial_k u^j) - x\partial_i^*[(b^2)^i\partial_x u^j] \\ - x\partial_x(b^3\partial_t u^j) - \partial_i[(b^4)^i\partial_t u^j] + \text{l.o.t.}\end{aligned} \tag{9.15}$$

Multiplying by x^ρ, taking $s-1$ derivatives with respect to t and using the bound $|\!|\!|u|\!|\!|_{s,r} < C\delta$, we immediately find that $x^\rho\partial_t^{s+1}u$ satisfies

$$\|x^\rho\partial_t^{s+1}u\|_{L^\infty_t \mathcal{H}^{0,r-1}} < C\,.$$

Likewise, by successively taking $s-2+i$ time derivatives in (9.15) and repeating the argument, we readily obtain the bound

$$\|x^\rho \partial_t^{s+i} u\|_{L_t^\infty \mathcal{H}^{0,r-i}} < C,$$

for $2 \leqslant i \leqslant \rho$.

The fact that the solution is smooth in the polyhomogeneous sense if the initial and boundary data are is a straightforward consequence of Theorem 6.3 and the persistence of regularity principle (see, e.g., [**34**]), which just means that the time of existence T does not depend on the choice of the integer s, r as long as they are large enough, so that in this case $\|\!|u|\!\|_{s,r}$ is finite (although not uniformly bounded) for all s, r (of course, $\bar{\gamma}$ is smoothly polyhomogeneous by construction). This completes the proof of the theorem. q.e.d.

The statement about the existence of C^q metrics that appears in the statement of Theorem 1.1 is an immediate consequence of Theorem 9.1 due to Corollary 6.3 provided that the initial and boundary data are smooth enough. Specifically, by keeping track of the various choices of exponents that we have made in the preceding sections we arrive at the following:

Corollary 9.2. *Given any $q \geqslant n-1$, let us choose an integer $p > 2q + \frac{5}{2}n + 7$. If $\widehat{g} \in C^p$, $x^2\widetilde{g}, x^2 K \in C^{n-1} \cap C^p_{\mathrm{polyhom}}$ and they satisfy the constraint equations and the compatibility conditions to order q, then there exists a $T > 0$ and a unique solution to the equation $Q(g) = 0$ on $(-T, T) \times M$ with the above initial and boundary data, which is of class $\bar{g} \in C^{n-1} \cap C^q_{\mathrm{polyhom}}$.*

10. DeTurck's trick revisited

Corollary 9.2 provides a weakly asymptotically AdS metric g that solves the equation $Q(g) = 0$ in $(-T, T) \times M$, satisfies the desired initial and boundary conditions. Our objective in this section is to show that g is also a solution of the Einstein equation $\mathrm{Ric}(g) = -ng$, which completes the proof of Theorem 1.1. The standard way of proving this is via the so-called DeTurck's trick. A textbook presentation of this method can be found in [**33**, Chapter 14] (see also [**24**]), so we will only sketch the main ideas and refer to this book for further details. It should be noticed, however, that the lack of global hyperbolicity and the fact that the equations that appear are singular at the conformal boundary ensure that an additional effort is necessary to show that DeTurck's method actually works in the situation that we are considering. Fortunately, the estimates that we have derived in the previous sections of this paper are well suited for this task.

The key idea in DeTurck's method is that, if $Q(g) = 0$, the 1-form W introduced in (3.4) to break the gauge invariance of the Einstein equation must satisfy the linear hyperbolic equation

$$\Box_g W_\mu + R^\nu_\mu W_\nu = 0\,, \tag{10.1}$$

where $R^\nu_\mu := g^{\nu\lambda}R_{\mu\lambda}$ is the tensor obtained by raising an index of the Ricci tensor of the metric g. When the metric g is globally hyperbolic, it is immediate that if $W_\mu = 0$ and $\partial_t W_\mu = 0$ at $t = 0$, then $W \equiv 0$ for all time, which readily implies that the metric satisfies the Einstein equation $\mathrm{Ric}(g) = -ng$ because of the structure of the operator Q.

The difficulty here is that Equation (10.1) is not globally hyperbolic. In fact, since g is weakly asymptotically AdS (which ensures that $g = x^{-2}\bar g$ for some $\bar g$ smooth enough up to the boundary and such that $\bar g^{\mu\nu}x_\mu x_\nu = 1$ on $(-T,T)\times\partial M$), a tedious computation shows that, in $\mathcal{A}$, Equation (10.1) reads as

$$g^{\lambda\nu}\partial_\lambda\partial_\nu W_\mu + \frac{(3-n)\,\partial_x W_\mu}{x} - \frac{nW_\mu + (n-1)\bar g^{\lambda\nu}x_\lambda W_\nu\, x_\mu}{x^2} + \text{l.o.t.}\,, \tag{10.2}$$

where l.o.t. stand for terms with at most one derivative of W that are smaller at $x=0$ (i.e., they are of the form $\mathcal{O}(1)\,\partial W + \mathcal{O}(x^{-1})W$).

Let us now write $W =: W^0 + W^3$, with

$$(W^0)_\mu := \frac{\bar g^{\lambda\nu}x_\lambda W_\nu}{|dx|^2_{\bar g}}x_\mu\,.$$

This decomposition diagonalizes (10.2) in the sense that the leading terms of the equation (both in terms of derivatives and singular behavior at the boundary) are now controlled by scalar operators:

$$\begin{aligned}
\mathcal{L}_0 W_0 &:= \left(g^{\lambda\nu}\partial_\lambda\partial_\nu + \frac{3-n}{x}\partial_x - \frac{3n-1}{x^2}\right)W_0 + \text{l.o.t.}\,,\\
\mathcal{L}_3 W_3 &:= \left(g^{\lambda\nu}\partial_\lambda\partial_\nu + \frac{3-n}{x}\partial_x - \frac{2n}{x^2}\right)W_3 + \text{l.o.t.}\,,
\end{aligned}$$

where again l.o.t. stands for lower-order terms that are smaller at $x=0$. Setting $W_j =: x^{\frac{n}{2}-1}V_j$ for $j = 0,3$, we can now write

$$\mathcal{L}_j W_j =: x^{\frac{n}{2}-1}\mathcal{P}_j V_j\,,$$

where in $\mathcal{A}$ the linear operator $\mathcal{P}_j$ reads as

$$\begin{aligned}
\mathcal{P}_j V_j = \bar g^{00}\Big(\partial_t^2 + \partial^*_{\theta^i}G^{ik}\partial_{\theta^k} + \mathbf{D}^*_{x,\alpha_j}b^1\mathbf{D}_{x,\alpha_j} + x\,\partial^*_\theta\tilde b^2\partial_x\\
+ x\partial^*_x\tilde b^3\partial_\theta + x\partial_x\tilde b^4\partial_t + x\partial_\theta\tilde b^5\partial_t\Big)V_j\\
+ \Big(\tilde b^6 x\partial_x V_j + x\tilde b^7\partial_t V_j + x\tilde b^8\partial_\theta V_j + \tilde b^9 V_j\Big),
\end{aligned}$$

with α_0 and α_3 defined in Equation (4.2)

Since this has the same structure as the operator $\mathcal{P}_g$ considered in (8.2), a minor variation of Theorem 8.3 proves, in particular, that any solution $V := V_0 + V_3$ must vanish identically in $(-T,T) \times M$ if it has zero boundary and initial conditions. The compatibility conditions for the initial and boundary conditions guarantee that this is, indeed, the case (cf. Appendix A), so we have proved the following:

Theorem 10.1. *The metric g constructed in Theorem 9.1 (or Corollary 9.2) solves the Einstein equation* $\mathrm{Ric}(g) = -ng$ *in* $(-T,T) \times M$*.*

The main result of the paper (Theorem 1.1) then follows.

Appendix A. Constraint equations and compatibility conditions

In this appendix we recall the constraints that must be satisfied by the initial and boundary data of the Einstein equations $\mathrm{Ric}(g)+ng = 0$. We refer to [4] for details.

The initial and boundary conditions are a Riemannian metric $\widetilde{g}_{ij}$ on the n-dimensional manifold M, a second-order tensor K_{ij} on M and a Lorentzian metric $\widehat{g}_{\alpha\beta}$ on $\mathbb{R} \times \partial M$. We also need a function x on $\overline{M}$, which we assume to be C^∞ up to the boundary. The connection of these objects with the Lorentzian Einstein metric g on $(-T,T) \times M$ is that $\widehat{g}_{\alpha\beta}$ is the pullback of $\bar{g}_{\mu\nu} := x^2 g_{\mu\nu}$ to $(-T,T) \times \partial M$, $\widetilde{g}_{ij}$ is the pullback of $g_{\mu\nu}$ to the Cauchy surface $\{0\} \times M$ and K_{ij} is the second fundamental form of the Cauchy surface in $(-T,T) \times M$ with respect to the metric $g_{\mu\nu}$. In terms of regularity, we assume that $\widehat{g}_{\alpha\beta}$ is of class $C^p((-T_0,T_0) \times \partial M)$, that $x^2\widetilde{g}_{ij}$ is in $C^{n-1}(\overline{M}) \cap C^p_{\mathrm{polyhom}}(\overline{M})$ and that K_{ij} can be written as

$$K_{ij} = \frac{1}{x} L_{ij} + \frac{1}{n}\mathcal{K}\widetilde{g}_{ij}\,,$$

where L_{ij} is traceless (that is, $\widetilde{g}^{ij} L_{ij} = 0$, so $\mathcal{K} = \widetilde{g}^{ij} K_{ij}$) and $L_{ij}, \mathcal{K} \in C^{n-1}(\overline{M}) \cap C^{q-1}_{\mathrm{polyhom}}(\overline{M})$. With some abuse of notation, throughout this paper we use the shorthand notation

$$\|x^2 K\|_{C^{p-1}_{n-1}} := \|L_{ij}\|_{C^{p-1}_{n-1}} + \|\mathcal{K}\|_{C^{p-1}_{n-1}}\,,$$

and when we say that $x^2 K_{ij}$ is in $C^{n-1} \cap C^{p-1}_{\mathrm{polyhom}}$ we mean that $L_{ij}, \mathcal{K} \in C^{n-1} \cap C^{p-1}_{\mathrm{polyhom}}$. We recall that the estimates in [4] control precisely these quantities (in addition, to $x^2\widetilde{g}_{ij}$).

The way to compute $\partial_t^k g_{\mu\nu}|_{t=0}$ from the initial data $(\widetilde{g}, K)$ is well know, the only difference being that one must take care of the powers of x that characterize the behavior at infinity of the metric. An economic way of doing this (see, e.g., [13, Section 7.5] for details) is by embedding M in the product $(-T_0,T_0) \times M$ and choosing $t \in (-T_0,T_0)$ as a time coordinate. We can then identity $\{t=0\}$ with M and set, for any local coordinates on M,

$$g_{ij}|_{t=0} = \widetilde{g}_{ij}\,, \qquad g_{ti}|_{t=0} = 0\,,\,, \qquad g_{tt} = -x^{-2}\,.$$

The condition that K be the second fundamental form of the spatial hypersurface $\{t=0\}$ translates into

$$\partial_t g_{ij}|_{t=0} = \frac{2}{x} K_{ij},$$

while the time derivatives of the coefficients $g_{t\mu}$ at 0 are chosen so as to ensure that the 1-form W (cf. Equation (3.4)) vanishes at $t=0$. Higher order time derivatives of the metric a time 0 can then be computed from the equation $Q(g)=0$. Because of Proposition 3.2 we assume that

$$1 = \bar{g}^{\mu\nu} x_\mu x_\nu|_{\{0\}\times\partial M} = \overline{\overline{g}}^{ij}\,\partial_i x\,\partial_j x|_{\partial M}\,,$$

where $\overline{\widetilde{g}}^{ij}$ is the inverse of $\overline{\widetilde{g}}_{ij} := x^2 g_{ij}$.

The initial data $(\widetilde{g}_{ij}, K_{ij})$ cannot be chosen freely, as the following *constraint equations* must be satisfied:

(A.1a) $$\widetilde{R} - K_{ij}K^{ij} + \mathcal{K}^2 = -n(n-1)\,,$$

(A.1b) $$\widetilde{\nabla}^j K_{ji} - \widetilde{\nabla}_i \mathcal{K} = 0\,.$$

Here the quantities with tildes are computed using the Riemannian metric $\widetilde{g}$, $\widetilde{R}$ stands for the scalar curvature of $\widetilde{g}$ and indices are raised and lowered using this metric. The proof goes exactly as in [**33**].

This kind of initial data, with the assumption that the objects should be C^∞ up to the boundary, were considered by Friedrich in his breakthrough paper [**20**] to construct space-times with AdS-type behavior at space-like infinity in dimension 4. This has been discussed in more generality in Kánnár [**30**]. Andersson and Chrusciel [**4**] have established the existence of many solutions with the right behavior at infinity to the constraint equations under the additional assumption that $\mathcal{K}$ is constant, that it,

$$\widetilde{\nabla}_i \mathcal{K} = 0\,.$$

This extra hypothesis is used to decouple the scalar and vector constraint equations. These solutions are "labeled" by a symmetric traceless tensor A^{ij} that is sufficiently smooth up to the boundary (say, in $C^\infty(\overline{M})$). It is worth mentioning that, generically, the resulting solutions $(\overline{\widetilde{g}}_{ij}, L_{ij}, \mathcal{K})$ are not arbitrarily smooth up to the boundary due to the appearance of log terms: they are generically in $C^{n-1} \cap C^\infty_{\text{polyhom}}$, although there are also "many" nontrivial solutions that are smooth up to the boundary, in which the log terms are absent.

Additionally, one must consider compatibility conditions between the initial conditions $(\widetilde{g}, K)$ and the boundary datum $\widehat{g}$. As is well-known, solving the Einstein equation in a bounded domain with nontrivial boundary conditions on the boundary is usually problematic (see, e.g., [**21**] and references therein). Fortunately, in this setting we can exploit the fact that the metric we want to construct is asymptotically anti-de Sitter to obtain a manageable set of compatibility conditions: one have fixed the integers s, r (with $s + r \leqslant p$), we only need to impose that the functions u_k, defined in (5.6) and (8.5), belong to $\mathcal{H}^{1,r}$ for $0 \leqslant k \leqslant s-1$ and to $\mathbf{L}^2$ for $k = s$. This integrability condition at infinity is enough to ensure that the arguments in the paper make sense, essentially because we can integrate by parts in the proof of Theorem 8.3. A more intuitive way of understanding this condition is that it is tantamount to saying that the formal solutions that we calculate at $t = 0$ using $(\widetilde{g}, K)$ (that is, $\partial_t^k \bar{g}|_{t=0}$ as computed above) and at $x = 0$ using the boundary data (the metrics $\bar{\gamma}_l$ of Theorem 4.5 with $l \geqslant q$) must agree to order q.

Appendix B. Some estimates for the operators A_α and A_α^*

The integral operators A_α and A_α^*, defined in (6.7), play a key role in some arguments presented in Sections 6 and 7. Therefore, we will record here some estimates the we proved in [**18**, Theorem 3.1 and Proposition 3.3], where as usual we assume that $\alpha > 1$. For the benefit of the reader, we also include a sketch of the proof.

Theorem B.1 ([**18**]). *The following statements hold:*

(i) *Acting on one-variable functions, the operators A_α and A_α^* define continuous maps*

$$\mathbf{L}^2_x \to L^\infty_x\,.$$

(ii) *The operators $\frac{1}{x}A_\alpha$ and $\frac{1}{x}A_\alpha^*$ are continuous maps*

$$\mathbf{L}^2_x \to \mathbf{L}^2_x \qquad \text{and} \qquad \mathbf{L}^2 \to \mathbf{L}^2\,.$$

(iii) *If u is a function in $\mathbf{L}^2(\mathcal{A})$ with $\mathbf{D}_{x,\alpha}u$ in $\mathbf{L}^2(\mathcal{A})$, then*

$$u(x,\theta) = (A_\alpha \mathbf{D}_{x,\alpha} u)(x,\theta)\,.$$

(iv) *If u is a function in $\mathbf{L}^2(\mathcal{A})$ with $\mathbf{D}^*_{x,\alpha}u$ in $\mathbf{L}^2(\mathcal{A})$, then*

$$u(x,\theta) = (A^*_\alpha \mathbf{D}^*_{x,\alpha} u)(x,\theta) + f(\theta)\, x^{\alpha-1}\,,$$

the function $f(\theta)$ being bounded in $L^2_\theta \equiv L^2(\partial M)$ by

$$\|f\|_{L^2_\theta} \leqslant C(\|u\|_{\mathbf{L}^2} + \|\mathbf{D}_{x,\alpha}u\|_{\mathbf{L}^2})\,.$$

Proof. We can assume that u is smooth and supported in the region $0 < x < 1$. Let us begin analyzing the mapping properties of A^*_α.

In view of the expression for A_α^*, we will use the Hardy inequality

$$\int_0^1 x^{2\alpha-2r-1}\left(\int_x^1 y^{1-\alpha}\varphi(y)\,dy\right)^2 dx \leqslant C\int_0^1 x^{3-2r}\varphi(x)^2\,dx\,, \tag{B.1}$$

with $r=0,1$. To prove this, let us set

$$\psi(x) := \int_x^1 y^{1-\alpha}\varphi(y)\,dy\,.$$

Then integrating by parts and using the Cauchy–Schwarz inequality we find

$$\begin{aligned}\int_0^1 x^{2\alpha-2r-1}\psi^2\,dx &= \frac{1}{\alpha-1}\int_0^1 \varphi\,\psi\,x^{\alpha-2r-1}\,dx\\ &= \frac{1}{\alpha-r}\int_0^1 (x^{\alpha-r-\frac12}\psi)\,(x^{\frac32-r}\varphi)\,dx\\ &\leqslant \frac{1}{\alpha-r}\left(\int_0^1 x^{2\alpha-2r-1}\psi^2\,dx\right)^{\frac12}\left(\int_0^1 x^{3-2r}\varphi^2\,dx\right)^{\frac12}.\end{aligned}$$

This proves (B.1). This implies that, with $r=0,1$, $\frac{1}{x}A_\alpha^*$ is a bounded map

$$L^2((0,1),x^{3-2r}dx)\to L^2((0,1),x^{1-2r}\,dx)\,,$$

and with $r=1$ this implies that $\frac{1}{x}A_\alpha^* : \mathbf{L}_x^2 \to \mathbf{L}_x^2$. Since the star denotes the adjoint with respect to the $\mathbf{L}_x^2$ product, a standard duality argument then ensures that A_α is a bounded map

$$L^2((0,1),x^{1+2r}dx)\to L^2((0,1),x^{2r-1}dx)\,,$$

which with $r=0$ implies that $\frac{1}{x}A_\alpha : \mathbf{L}_x^2 \to \mathbf{L}_x^2$. The fact that this also corresponds to $\mathbf{L}^2\to\mathbf{L}^2$ bounds is immediate.

Let us now pass to the pointwise bounds. To prove (i) for A_α^* we utilize the Cauchy–Schwarz inequality to write

$$\begin{aligned}\left|A_\alpha^*\varphi(x)\right| &= x^{\alpha-1}\left|\int_x^1 y^{1-\alpha}\varphi(y)\,dy\right|\\ &\leqslant x^{\alpha-1}\left(\int_x^1 y^{1-2\alpha}\,dy\right)^{\frac12}\left(\int_x^1 y\,\varphi(y)^2\,dy\right)^{\frac12}\\ &\leqslant \|\varphi\|_{\mathbf{L}_x^2}\left(\frac{1-x^{\alpha-1}}{2-2\alpha}\right)^{1/2}\\ &\leqslant (2-2\alpha)^{-\frac12}\|\varphi\|_{\mathbf{L}_x^2}\,.\end{aligned}$$

The L_x^∞ estimate for A_α is similar.

To prove (iv), notice that if $u_1 := \mathbf{D}_{x,\alpha}^* u \in \mathbf{L}^2$, we can solve the ODE

$$\mathbf{D}_{x,\alpha}^* u = u_1,$$

to write

$$u = A^*_\alpha(u_1) + f(\theta)\, x^{\alpha-1},$$

for some function $f(\theta)$. Moreover,

$$\begin{aligned}\|f\|_{L^2_\theta} = C\|f(\theta)\, x^{\alpha-1}\|_{\mathbf{L}^2} &\leqslant C(\|u\|_{\mathbf{L}^2} + \|A^*_\alpha(u_1)\|_{\mathbf{L}^2})\\ &\leqslant C(\|u\|_{\mathbf{L}^2} + \|u_1\|_{\mathbf{L}^2}),\end{aligned}$$

where we have used that $A^*_\alpha : \mathbf{L}^2 \to \mathbf{L}^2$ by (ii). To prove (iii), the reasoning is analogous: again we can solve the ODE

$$\mathbf{D}_{x,\alpha} u = u_2,$$

to write

$$u = A_\alpha(u_2) + f_2(\theta)\, x^{-\alpha},$$

but we infer that f_2 must be 0 because $x^{-\alpha}$ is not in $\mathbf{L}^2_x$. The theorem then follows. q.e.d.

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