SHARP FUNDAMENTAL GAP ESTIMATE ON CONVEX DOMAINS OF SPHERE

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Abstract

In their celebrated work, B. Andrews and J. Clutterbuck proved the fundamental gap (the difference between the first two eigenvalues) conjecture for convex domains in the Euclidean space [3] and conjectured similar results hold for spaces with constant sectional curvature. We prove the conjecture for the sphere. Namely when D, the diameter of a convex domain in the unit $\mathbb{S}^n$ sphere, is $\leq \frac{\pi}{2}$, the gap is greater than the gap of the corresponding 1-dim sphere model. We also prove the gap is $\geq 3\frac{\pi^2}{D^2}$ when $n \geq 3$, giving a sharp bound. As in [3], the key is to prove a super log-concavity of the first eigenfunction.

1. Introduction

Given a bounded smooth domain Ω in a Riemannian manifold M^n, the eigenvalues of the Laplacian on Ω with respect to the Dirichlet and Neumann boundary conditions are given by

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \cdots \to \infty,$$

and

$$0 = \mu_0 < \mu_1 \leq \mu_2 \cdots \to \infty,$$

respectively. There are many works on estimating the eigenvalues, especially the first eigenvalues. Estimating the gap between the first two eigenvalues, the fundamental (or mass) gap,

$$\Gamma(\Omega) = \begin{cases} \lambda_2 - \lambda_1 > 0, & \text{Dirichlet boundary,} \\ \mu_1 > 0, & \text{Neumann boundary,} \end{cases}$$

of the Laplacian or more generally for Schrödinger operators is also very important both in mathematics and physics. For Neumann boundary condition, it is the same as estimating the first nontrivial eigenvalue. In this case, for a convex domain in a Riemannian manifold with Ricci

curvature bounded from below, a sharp lower bound for μ_1 is given by a 1-dim model [2, 4, 9, 13, 16, 33]. For Dirichlet boundary condition, a sharp upper bound for $\lambda_2 - \lambda_1$ has been obtained for domains in the space of constant sectional curvature in [7, 8, 10] in their solution of the Payne–Polya–Weinberger conjecture. The optimal bound is achieved by geodesic balls. For convex domains $\Omega \subset \mathbb{R}^n$ with diameter D and convex potential, it was independently conjectured by van den Berg, Ashbaugh and Benguria, Yau [6, 26, 30] in the 80's that the gap $\Gamma(\Omega)$ has the sharp lower bound of $\frac{3\pi^2}{D^2}$. The subject has a long history, see the excellent survey by Ashbaugh [5] for discussion of the fundamental gap and history up to 2006, and the references in [3]. We only mention that in the influential paper, Singer, Wong, Yau and Yau [25] showed that $\Gamma(\Omega) \geq \frac{\pi^2}{4D^2}$. Yu and Zhong [31] improved this to $\frac{\pi^2}{D^2}$, see also [19]. Further improvements to the gap is done by Yau [28] which depends upper bound estimate on the log-concavity of the first eigenfunction. Yau [29] also studied the case for non-convex potentials. In 2011 the conjecture was completely solved by B. Andrews and J. Clutterbuck in their celebrated work [3] by establishing a sharp log-concavity estimate for the first eigenfunction, see also [22]. For triangles in $\mathbb{R}^2$, Z. Lu and J. Rowlett [20] showed the improved gap estimate, that the gap is $\geq \frac{64\pi^2}{9D^2}$ and the minimum is realized by the equilateral triangle, where D is the diameter of the triangle. See also their work [21] for the behavior of the gap as the domain collapses to lower dimension. For convex domains on a sphere, Lee and Wang [18] showed the gap is $\geq \frac{\pi^2}{D^2}$. See [23] for an estimate on general manifolds.

In this paper we give a sharp lower bound on the gap for convex domains on a sphere. One of our main result is the following.

Theorem 1.1. *Let $\Omega \subset \mathbb{S}^n$ be a strictly convex domain with diameter D, $\lambda_i (i = 1, 2)$ be the first two eigenvalues of the Laplacian on Ω with Dirichlet boundary condition. Then*

$$(1.1) \qquad \Gamma(\Omega) = \lambda_2 - \lambda_1 \geq \bar{\lambda}_2(n, D) - \bar{\lambda}_1(n, D) \text{ if } D \leq \frac{\pi}{2},$$

where $\bar{\lambda}_i(n, D)$ $(i = 1, 2)$ are the first two eigenvalues of the operator $\frac{d^2}{ds^2} - (n-1)\tan(s)\frac{d}{ds}$ on $[-\frac{D}{2}, \frac{D}{2}]$ with Dirichlet boundary condition. Furthermore,

$$(1.2) \qquad \bar{\lambda}_2(n, D) - \bar{\lambda}_1(n, D) \geq 3\frac{\pi^2}{D^2} \quad \text{if } D < \pi, \ n \geq 3.$$

Remark 1.2. In fact, we prove some monotonicity properties for the gap of the 1-dim model (2.2) for any constant curvature, see Theorem 2.1 for more detail. In particular, when $n = 2$, the gap of the model is less than $3\frac{\pi^2}{D^2}$ in the non-Euclidean case. For sphere we expect it is still greater than $2\frac{\pi^2}{D^2}$, and for fixed D, we also expect the gap increases

when the dimension gets bigger for $n \geq 2$. See Appendix A for some numerical evidence and [14] for the asymptotics.

Corollary 1.3. *Let* $\Omega \subset \mathbb{S}^n$ *be a strictly convex domain with diameter* $D \leq \frac{\pi}{2}$, λ_i $(i = 1, 2)$ *be the first two eigenvalues of the Laplacian on* Ω *with Dirichlet boundary condition. Then*

$$(1.3) \qquad \lambda_2 - \lambda_1 \geq 3 \frac{\pi^2}{D^2} \quad \text{when } n \geq 3.$$

Remark 1.4. Same estimates are true for Schrödinger operator of the form $-\Delta + V$, where $V \geq 0$ and is convex.

Remark 1.5. In continuations of this work, the diameter $\leq \frac{\pi}{2}$ restriction in various places of this paper has been removed in [15]. The gap estimate (1.3) is also shown to hold when $n = 2$ [14]. Theorems 3.2 and 3.6 in this paper play important role in those work.

The key to proving (1.1) is the following log-concavity of the first eigenfunction.

Theorem 1.6. *Let* $\Omega \subset \mathbb{S}^n$ *be a strictly convex domain with diameter* $D \leq \frac{\pi}{2}$, $\phi_1 > 0$ *be a first eigenfunction of the Laplacian on* Ω *with Dirichlet boundary condition. Then for* $\forall x, y \in \Omega$, *with* $x \neq y$,
(1.4)

$$\langle \nabla \log \phi_1(y), \gamma'(\tfrac{d}{2}) \rangle - \langle \nabla \log \phi_1(x), \gamma'(-\tfrac{d}{2}) \rangle \leq 2 \left(\log \bar{\phi}_1 \right)' \left(\frac{d(x,y)}{2} \right),$$

where γ *is the unit normal minimizing geodesic with* $\gamma(-\frac{d}{2}) = x$, $\gamma(\frac{d}{2}) = y$, *and* $\bar{\phi}_1 > 0$ *is a first eigenfunction of the operator* $\frac{d^2}{ds^2} - (n-1) \tan(s) \frac{d}{ds}$ *on* $[-\frac{D}{2}, \frac{D}{2}]$ *with Dirichlet boundary condition with* $d = d(x,y)$. *Dividing (1.4) by* $d(x,y)$ *and letting* $d(x,y) \to 0$, *we have*

$$(1.5) \qquad \nabla^2 \log \phi_1 \leq -\bar{\lambda}_1 \, id.$$

This improves an early estimate of Lee and Wang [18] that $\nabla^2 \log \phi_1 \leq 0$.

In the proof we work on spaces with constant sectional curvature. In particular, our proof works for spheres and Euclidean spaces at the same time. Some of our results hold also for negative constant curvature (see Section 2). For the log-concavity estimate, the last step fails for negative curvature, see the proof of Theorem 3.2 for detail. In fact, we have a more general estimate, see Theorem 3.8. We also have a parabolic version, see Theorems 3.2 and 3.6. For negative constant curvature, it is not clear if the corresponding log-concavity of the first eigenfunction holds. If it were true then we also get the corresponding gap estimate, see Theorem 4.1.

The paper is organized as follows. In §2 we study properties of the eigenvalues and eigenfunctions of the 1-dimensional model space obtained by considering the rotational symmetry of constant curvature

spaces. When the curvature is not zero and dimension is not 1 or 3, the eigenvalue and eigenfunction of the 1-dimensional model cannot be solved explicitly. We obtained a gap estimate for the model by obtaining several monotonicity properties for the eigenvalues and eigenfunctions.

In §3, we prove Theorem 1.6, the key super log-concavity estimate. Following [3], the idea is to apply the maximum principle to the so called two-point functions. For $K \neq 0$, the computation is much more subtle. In $\mathbb{R}^n$, Andrews–Clutterbuck proved the preservation of modulus holds for general solutions of the heat equation. It is not clear if this is true when $K \neq 0$. We use both the heat equation and the Laplacian equation to prove several preservation of modulus. Several elliptic versions are also obtained.

Finally, in §4, with the log-concavity result we derive a gap comparison for general manifolds with lower Ricci curvature bound. Namely the gap of the Laplacian is greater or equal to the gap of the 1-dimensional model, thereby proving Theorem 1.1. The gap comparison (Theorem 4.1) can be viewed as a generalization of Neumann gap comparison, as in the Neumann case the first eigenfunction is constant and automatically satisfies the log-concave condition (4.1). We give two proofs of the gap comparison, one elliptic and one parabolic. As another application of (1.4) we also give a lower bound on the first Dirichlet eigenvalue of the Laplacian on convex domain in sphere, see Proposition 4.7.

2. The gap of 1-dimensional model spaces

Let $\mathbb{M}_K^n$ be the model space, the n-dimensional simply connected manifold with constant sectional curvature K. Denote $\operatorname{sn}_K(s)$ the coefficient of the Jacobi field starting from 0 in $\mathbb{M}_K^n$. Namely $\operatorname{sn}_K(s)$ is the solution of

$$\operatorname{sn}_K''(s) + K \operatorname{sn}_K(s) = 0, \quad \operatorname{sn}_K(0) = 0, \quad \operatorname{sn}_K'(0) = 1.$$

Let $\mathrm{cs}_K(s) = \mathrm{sn}'_K(s)$ and $\mathrm{tn}_K(s) = K\frac{\mathrm{sn}_K(s)}{\mathrm{cs}_K(s)} = -\frac{\mathrm{cs}'_K(s)}{\mathrm{cs}_K(s)}$. (This definition of tn_K has the opposite sign of the one in [1].) Explicitly we have

$$\mathrm{sn}_K(s) = \begin{cases} \frac{1}{\sqrt{K}}\sin(\sqrt{K}s), & K > 0, \\ s, & K = 0, \\ \frac{1}{\sqrt{-K}}\sinh(\sqrt{-K}s) & K < 0, \end{cases}$$

$$\mathrm{cs}_K(s) = \begin{cases} \cos(\sqrt{K}s), & K > 0, \\ 1, & K = 0, \\ \cosh(\sqrt{-K}s), & K < 0, \end{cases}$$

and

$$\mathrm{tn}_K(s) = \begin{cases} \sqrt{K}\tan(\sqrt{K}s), & K > 0, \\ 0, & K = 0, \\ -\sqrt{-K}\tanh(\sqrt{-K}s) & K < 0. \end{cases}$$

We write the metric on $\mathbb{M}_K^n$ as the following. Given a totally geodesic hypersurface $\Sigma \subset \mathbb{M}_K^n$, let s be the (signed) distance to Σ, then the metric of $\mathbb{M}_K^n$ is

$$(2.1) \qquad g = ds^2 + \mathrm{cs}_K^2(s)g_\Sigma.$$

The Laplacian operator is

$$\Delta = \frac{\partial^2}{\partial s^2} + (n-1)\frac{\mathrm{cs}'_K(s)}{\mathrm{cs}_K(s)}\frac{\partial}{\partial s} + \frac{1}{\mathrm{cs}_K^2(s)}\Delta_\Sigma.$$

The "one-dimensional" model of the equation $\Delta\phi = -\lambda\phi$ is

$$(2.2) \qquad \phi'' - (n-1)\mathrm{tn}_K(s)\phi' + \lambda\phi = 0.$$

Below we study the basic properties of the eigenvalues and eigenfunctions of this model with Dirichlet boundary condition on $[-\frac{D}{2}, \frac{D}{2}]$. (We always assume $D < \frac{\pi}{\sqrt{K}}$ if $K > 0$.) The properties are parallel to the behavior of the first two eigenvalues and eigenfunctions of balls in $\mathbb{S}^n$ established in [8], although there are some essential difference.

First, equation (2.2) is symmetric. Namely if $\phi(s)$ is a solution of (2.2) with Dirichlet boundary condition, then so is $\phi(-s)$. By taking $\phi(s) + \phi(-s)$ or $\phi(s) - \phi(-s)$, we get even or odd eigenfunction. By Courant's Theorem on nodal domains (see, e.g., [24, page 126]) the first eigenfunction does not change sign, and the second eigenfunction changes sign exactly once. Hence, we can choose the first and second eigenfunctions $\bar{\phi}_1, \bar{\phi}_2$ such that

$$(2.3) \qquad \bar{\phi}_1 > 0 \text{ is even, and } \bar{\phi}_2 \text{ is odd with } \bar{\phi}'_2(0) > 0.$$

Denote $\bar{\lambda}_1(n, D, K)$, $\bar{\lambda}_2(n, D, K)$ the corresponding eigenvalues of $\bar{\phi}_1$, $\bar{\phi}_2$.

With the change of variable $\phi(s) = \mathrm{cs}_K^{-\frac{n-1}{2}}(s)\varphi(s)$, we obtain the Schrödinger normal form of (2.2),

$$(2.4) \qquad \varphi''(s) - \frac{(n-1)K}{4}\left(\frac{n-3}{\mathrm{cs}_K^2(s)} - (n-1)\right)\varphi = -\lambda\,\varphi.$$

Since $\frac{K}{\mathrm{cs}_K^2(s)} \geq K$, this immediately gives,

$$(2.5) \qquad \bar{\lambda}_1(n, D, K) \geq \max\{\tfrac{\pi^2}{D^2} - \tfrac{n-1}{2}K, 0\} \text{ when } n \geq 3.$$

When $n = 1, 3$ or $K = 0$, (2.4) implies that we can find the eigenvalues and eigenfunctions explicitly and the gap $\bar{\lambda}_2(n, D, K) - \bar{\lambda}_1(n, D, K) = 3\frac{\pi^2}{D^2}$. Namely $D^2\left(\bar{\lambda}_2(n, D, K) - \bar{\lambda}_1(n, D, K)\right)$ is a constant. In general one cannot find the eigenvalues explicitly. But as pointed out by Chenxu He, when $n \geq 3$ and $K > 0$ (or $K < 0$, $D \in (0, a(K))$, see below), the potential term in (2.4) is convex, therefore, the gap estimate (2.6) follows directly from the solution of 1-dimensional conjecture in [17]. On the other hand, the following monotonicity property has independent interest, and some of the monotonicity of eigenfunctions obtained in the proof will be used later on, so we still keep the theorem below.

Theorem 2.1. *For $K > 0$, $D < \frac{\pi}{\sqrt{K}}$, $D^2\left(\bar{\lambda}_2(n, D, K) - \bar{\lambda}_1(n, D, K)\right)$ is increasing in D when $n > 3$, decreasing in D when $n = 2$. Therefore, when $n \geq 3$, $D \in (0, \pi)$,*

$$(2.6) \qquad\qquad \bar{\lambda}_2(n, D, 1) - \bar{\lambda}_1(n, D, 1) \geq 3\frac{\pi^2}{D^2}.$$

When $K < 0$, the same statement is true for $D \in (0, a(K)]$, where $a(K)$ is a positive constant depends on K. (see proof for its definition)

To prove this, first we derive some monotonicity properties for the eigenfunctions. For the first eigenfunction, we observe

Lemma 2.2. *$\bar{\phi}_1(s)$ is strictly decreasing on $[0, \frac{D}{2}]$.*

Proof. Multiplying (2.2) by the integrating factor $\mathrm{cs}_K^{n-1}(s)$, we have $\bar{\phi}_1(s)$ satisfies

$$\left(\mathrm{cs}_K^{n-1}(s)\bar{\phi}_1'(s)\right)' = -\bar{\lambda}_1\,\mathrm{cs}_K^{n-1}(s)\bar{\phi}_1(s) < 0.$$

Since $\bar{\phi}_1(s)$ is even, we have $\bar{\phi}_1'(0) = 0$. Integrating the above from 0 to $0 < l < \frac{D}{2}$, we have $\bar{\phi}_1'(l) < 0$.   q.e.d.

Next we show the ratio of $\bar{\phi}_1, \bar{\phi}_2$ is also monotone.

Lemma 2.3. *Let*

$$\bar{w}(s) := \frac{\bar{\phi}_2(s)}{\bar{\phi}_1(s)}.$$

Then $\bar{w}(s)$ is increasing on $[0, \frac{D}{2}]$.

Proof. Since $\bar{\phi}_2$ is odd with $\bar{\phi}_2'(0) > 0$, we have $\bar{w}(0) = 0$, $\bar{w}'(0) > 0$. By [25], $\bar{w}$ extends to $\pm\frac{D}{2}$ and $\bar{w}'(\pm\frac{D}{2}) = 0$. Direct computation shows that $\bar{w}''(s)$ satisfies

$$(2.7) \quad \bar{w}''(s) - (n-1)\operatorname{tn}_K(s)\bar{w}'(s) + 2(\log\bar{\phi}_1)'\bar{w}'(s) + (\bar{\lambda}_2 - \bar{\lambda}_1)\bar{w}(s) = 0.$$

Since on $(0, \frac{D}{2}]$, $\bar{w} > 0$, we have at points in $(0, \frac{D}{2}]$ where $\bar{w}' = 0$, (2.7) gives

$$(2.8) \qquad\qquad \bar{w}'' = -(\bar{\lambda}_2 - \bar{\lambda}_1)\bar{w} < 0.$$

In particular, $\bar{w}''(\frac{D}{2}) < 0$, which gives $\bar{w}'(\frac{D}{2} - \varepsilon) > 0$ for all small $\varepsilon > 0$. Now we prove the lemma by showing $\bar{w}' \geq 0$ on $[0, \frac{D}{2}]$. We show this by contradiction. Suppose there is some point where $\bar{w}' < 0$, since $\bar{w}'(0) > 0$, $\bar{w}'(\frac{D}{2} - \varepsilon) > 0$, there are two points a, b with $0 < a < b < \frac{D}{2}$ such that $\bar{w}'(a) = 0$ and $\bar{w}'(b) = 0$ and $\bar{w}''(a) \leq 0$ and $\bar{w}''(b) \geq 0$. This contradicts to (2.8) that $\bar{w}'' < 0$ at points where $\bar{w}' = 0$.   q.e.d.

Now we investigate the dependence of the eigenvalues $\bar{\lambda}_i(n, D, K)$ $(i = 1, 2)$ on D using perturbation theory. We define the Sturm–Liouville operator

$$L_D = -\frac{d^2}{ds^2} + (n-1)\operatorname{tn}_K(s)\frac{d}{ds},$$

with Dirichlet boundary conditions at $\pm\frac{D}{2}$. (We omit the dependence on n, K since we are interested in how the eigenvalues change when D varies.) Its Sturm–Liouville normal form is given by

$$L_D\phi = -\operatorname{cs}_K^{1-n}(s)\left(\operatorname{cs}_K^{n-1}(s)\phi'(s)\right)'.$$

Hence, L_D is a self-adjoint operator in the Hilbert space

$$L^2\left((-\frac{D}{2}, \frac{D}{2}), \ \operatorname{cs}_K^{n-1}(s)ds\right).$$

In order to work on a fixed interval $(-\frac{D}{2}, \frac{D}{2})$, we note that by making a change of variable $s = ct$, the eigenvalue problem $L_{cD}\phi(s) = \lambda\phi(s)$ on $(-\frac{cD}{2}, \frac{cD}{2})$ can be rescaled to

$$\tilde{L}_c\phi := \left(-\frac{d^2}{dt^2} + c(n-1)\operatorname{tn}_K(ct)\frac{d}{dt}\right)\phi(ct) = c^2\lambda\phi(ct),$$

for $t \in (-\frac{D}{2}, \frac{D}{2})$. And

$$(2.9) \qquad\qquad c^2\bar{\lambda}_i(n, cD, K) = \lambda_i(\tilde{L}_c).$$

$\tilde{L}_c$ is an analytic family of operator in a neighborhood of $c = 1$, and is self-adjoint in the Hilbert space $L^2\left((-\frac{D}{2}, \frac{D}{2}), \ \operatorname{cs}_K^{c(n-1)}(s)ds\right)$. We have $\tilde{L}_1 = L_D$ and

$$\tilde{L}_c - L_D = (n-1)\left[c\operatorname{tn}_K(cs) - \operatorname{tn}_K(s)\right]\frac{d}{ds}.$$

Let ϕ_c be a normalized eigenfunction associated to eigenvalue $\tilde{\lambda}_c = \lambda(\tilde{L}_c)$. Then

$$0 = \frac{d}{dc}\left((\tilde{L}_c - \tilde{\lambda}_c)\phi_c\right)\Big|_{c=1} = \left(\frac{d\tilde{L}_c}{dc} - \frac{d\tilde{\lambda}_c}{dc}\right)\phi_c\Big|_{c=1} + \left(\tilde{L}_c - \tilde{\lambda}_c\right)\frac{d\phi_c}{dc}\Big|_{c=1}.$$

Since $\tilde{L}_1 - \tilde{\lambda}_1$ is self-adjoint in the Hilbert space $L^2\left((-\frac{D}{2}, \frac{D}{2}), \mathrm{cs}_K^{n-1}(s)ds\right)$, the second term above is zero when we inner product with eigenfunction ϕ. The perturbation formula of the eigenvalue $\lambda(\tilde{L}_c)$ at $c = 1$ is given by

Lemma 2.4.

$$\frac{d\lambda(\tilde{L}_c)}{dc}\Big|_{c=1} = \left\langle \frac{d\tilde{L}_c}{dc}\phi_c, \phi_c \right\rangle\Big|_{c=1}$$

$$(2.10) \qquad\qquad = \int_{-\frac{D}{2}}^{\frac{D}{2}} \frac{d\tilde{L}_c}{dc}\Big|_{c=1} \phi(s) \cdot \phi(s)\, \mathrm{cs}_K^{n-1}(s)ds,$$

where ϕ is an eigenfunction of L_D such that $\int_{-\frac{D}{2}}^{\frac{D}{2}} \phi^2(s)\, \mathrm{cs}_K^{n-1}(s)ds = 1$.

Hence, for the first two eigenvalues of $\tilde{L}_c$, $\lambda_i(\tilde{L}_c), i = 1, 2$ we obtain

$$\frac{d\lambda_i(\tilde{L}_c)}{dc}\Big|_{c=1} = (n-1)\int_{-\frac{D}{2}}^{\frac{D}{2}} (\mathrm{tn}_K(s) + sK\,\mathrm{cs}_K^{-2}(s))\bar{\phi}_i(s)\bar{\phi}_i'(s)\, \mathrm{cs}_K^{n-1}(s)ds$$

$$(2.11) \qquad = 2(n-1)\int_0^{\frac{D}{2}} l_K(s)\, \bar{\phi}_i(s)\bar{\phi}_i'(s)\, \mathrm{cs}_K^{n-1}(s)ds,$$

where

$$(2.12) \qquad\qquad l_K(s) := \mathrm{tn}_K(s) + sK\,\mathrm{cs}_K^{-2}(s).$$

In (2.11) we used the fact that both $l_K(s)$ and $\bar{\phi}_i\bar{\phi}_i'(i = 1, 2)$ are odd.

This gives the following monotonicity formula for the first eigenvalue $\bar{\lambda}_1(n, D, K)$.

Proposition 2.5.

$$(2.13) \qquad \frac{d}{dD}\left(D^2\bar{\lambda}_1(n, D, K)\right) \begin{cases} < 0, & K > 0, \ D \in (0, \frac{\pi}{\sqrt{K}}); \\ = 0, & K = 0; \\ > 0, & K < 0. \end{cases}$$

Proof. By (2.9),

$$\frac{d\lambda_1(\tilde{L}_c)}{dc}\Big|_{c=1} = \frac{d}{dc}(c^2\bar{\lambda}_1(n, cD, K))\Big|_{c=1}$$

$$= \frac{1}{D}\frac{d}{dD}(D^2\bar{\lambda}_1(n, D, K)).$$

Since $l_K(s)$ satisfies

$$l_K(s) \begin{cases} > 0, & K > 0, \ D \in (0, \frac{\pi}{\sqrt{K}}); \\ = 0, & K = 0; \\ < 0, & K < 0. \end{cases}$$

On $(0, \frac{D}{2})$, by Lemma 2.2, $\bar{\phi}_1' < 0$, since $\bar{\phi}_1 > 0$, the result follows from (2.11).   q.e.d.

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. For convenience, we denote

$$m_K(s) := \frac{l_K'(s)}{2} = K \operatorname{cs}_K^{-2}(s)(1 + s \operatorname{tn}_K(s)).$$

Note that $l_K(s) \operatorname{tn}_K(s) = m_K(s) - K$. Using integration by parts and that $l(0) = 0$, $\bar{\phi}_i(\frac{D}{2}) = 0$, we rewrite (2.11) as

$$\frac{d\lambda_i(\tilde{L}_c)}{dc}\bigg|_{c=1} = (n-1) \int_0^{\frac{D}{2}} l_K(s)(\bar{\phi}_i^2(s))' \operatorname{cs}_K^{n-1}(s)ds$$

$$= -(n-1) \int_0^{\frac{D}{2}} \bar{\phi}_i^2(s) \left[l_K'(s) - (n-1)l_K(s) \operatorname{tn}_K(s)\right] \operatorname{cs}_K^{n-1}(s)ds$$

$$= -(n-1) \int_0^{\frac{D}{2}} \bar{\phi}_i^2(s) \left[2m_K(s) - (n-1)(m_K(s) - K)\right] \operatorname{cs}_K^{n-1}(s)ds$$

$$= (n-1) \int_0^{\frac{D}{2}} \bar{\phi}_i^2(s) \left[(n-3)m_K(s) - (n-1)K\right] \operatorname{cs}_K^{n-1}(s)ds.$$

Hence,

$$(2.14) \quad \frac{d\lambda_2(\tilde{L}_c)}{dc}\bigg|_{c=1} - \frac{d\lambda_1(\tilde{L}_c)}{dc}\bigg|_{c=1}$$

$$= (n-1)(n-3) \int_0^{\frac{D}{2}} m_K(s) \left[\bar{\phi}_2^2(s) - \bar{\phi}_1^2(s)\right] \operatorname{cs}_K^{n-1}(s)ds.$$

Note that $m_K(s)$ is increasing when $K > 0$, $D \in (0, \frac{\pi}{\sqrt{K}})$. $m_K'(s)$ has exactly one zero on $(0, \infty)$ when $K < 0$. Denote the zero point by $a(K)$. Then we will show
(2.15)

$$\int_0^{\frac{D}{2}} m_K(s) \left[\bar{\phi}_2^2(s) - \bar{\phi}_1^2(s)\right] \operatorname{cs}_K^{n-1}(s)ds \begin{cases} > 0, & K > 0, \ D \in (0, \frac{\pi}{\sqrt{K}}); \\ = 0, & K = 0; \\ > 0, & K < 0, \ D \in (0, a(K)]. \end{cases}$$

First we claim $\bar{\phi}_1(s) = \bar{\phi}_2(s)$ at exactly one point in $[0, \frac{D}{2})$. Since

$$(2.16) \qquad \int_0^{\frac{D}{2}} \bar{\phi}_1^2(s) \operatorname{cs}_K^{n-1}(s)ds = \int_0^{\frac{D}{2}} \bar{\phi}_2^2(s) \operatorname{cs}_K^{n-1}(s)ds,$$

and $\bar{\phi}_1, \bar{\phi}_2 \geq 0$ on $[0, \frac{D}{2}]$, there is at least one point in $[0, \frac{D}{2})$ such that $\bar{\phi}_1(s) = \bar{\phi}_2(s)$. By Lemma 2.3, $\frac{\bar{\phi}_2'}{\bar{\phi}_2} \geq \frac{\bar{\phi}_1'}{\bar{\phi}_1}$. Therefore,

$$\bar{\phi}_1(\bar{\phi}_1' - \bar{\phi}_2') \leq (\bar{\phi}_1 - \bar{\phi}_2)\bar{\phi}_1',$$

and $\bar{\phi}_1' - \bar{\phi}_2' \leq 0$ when $\bar{\phi}_1 = \bar{\phi}_2$. If $\bar{\phi}_1(s) = \bar{\phi}_2(s)$ at more than one point in $[0, \frac{D}{2})$, since $\bar{\phi}_1(0) - \bar{\phi}_2(0) > 0$, at the second such point we get $\bar{\phi}_1' - \bar{\phi}_2' > 0$ which is a contradiction. Let $b \in (0, \frac{D}{2})$ be the point such that $\bar{\phi}_1(b) = \bar{\phi}_2(b)$. We have

$$\bar{\phi}_2^2(s) - \bar{\phi}_1^2(s) \begin{cases} \leq 0, & s \in [0, b], \\ \geq 0, & s \in [b, \frac{D}{2}]. \end{cases}$$

When $K > 0$, $m_K(s)$ is increasing on $[0, \frac{D}{2}]$. Hence,

$$\int_0^{\frac{D}{2}} m_K(s) \left[\bar{\phi}_2^2(s) - \bar{\phi}_1^2(s)\right] \operatorname{cs}_K^{n-1}(s)ds$$

$$= \int_0^b m_K(s) \left[\bar{\phi}_2^2(s) - \bar{\phi}_1^2(s)\right] \operatorname{cs}_K^{n-1}(s)ds$$

$$+ \int_b^{\frac{D}{2}} m_K(s) \left[\bar{\phi}_2^2(s) - \bar{\phi}_1^2(s)\right] \operatorname{cs}_K^{n-1}(s)ds$$

$$\geq \int_0^b m_K(b) \left[\bar{\phi}_2^2(s) - \bar{\phi}_1^2(s)\right] \operatorname{cs}_K^{n-1}(s)ds$$

$$+ \int_b^{\frac{D}{2}} m_K(b) \left[\bar{\phi}_2^2(s) - \bar{\phi}_1^2(s)\right] \operatorname{cs}_K^{n-1}(s)ds$$

$$= m_K(b) \int_0^{\frac{D}{2}} \left[\bar{\phi}_2^2(s) - \bar{\phi}_1^2(s)\right] \operatorname{cs}_K^{n-1}(s)ds$$

$$= 0.$$

This proves (2.15) when $K > 0$. When $K < 0$, we have the same inequality since $m_K(s)$ is increasing on $[0, a(K)]$. Clearly when $K = 0$, so is m_K, thus, we have proved (2.15). Recall

$$\frac{1}{D}\frac{d}{dD}\left(D^2(\bar{\lambda}_2(n, D, K) - \bar{\lambda}_1(n, D, K))\right) = \left.\frac{d\lambda_2(\tilde{L}_c)}{dc}\right|_{c=1} - \left.\frac{d\lambda_1(\tilde{L}_c)}{dc}\right|_{c=1}.$$

Now the monotonicity part of Theorem 2.1 follows from (2.15) and (2.14).

As $D \to 0$ the gap approaches to the gap in the 1-dim Euclidean case. Namely

$$\lim_{D \to 0} D^2(\bar{\lambda}_2(n, D, K) - \bar{\lambda}_1(n, D, K)) = 3\pi^2,$$

so we have estimate (1.1) by the monotonicity. q.e.d.

Corollary 2.6. *We also have the monotonicity of the ratio. Namely for $n \geq 3$, $D \in (0, \pi/\sqrt{K})$ when $K > 0$, $D \in (0, a(K))$ when $K < 0$, we have*

$$\frac{d}{dD}\left(\frac{\bar{\lambda}_2(n, D, K)}{\bar{\lambda}_1(n, D, K)}\right) \geq 0.$$

Proof. By direct computation

$$\frac{d}{dD}\left(\frac{\bar{\lambda}_2(n, D, K)}{\bar{\lambda}_1(n, D, K)}\right) = \frac{1}{D}\left(\frac{d}{dc}\frac{\lambda_2(n, cD, K)}{\lambda_1(n, cD, K)}\right)\Bigg|_{c=1}$$

$$= \frac{1}{D\tilde{\lambda}_1}\left(\frac{d\lambda_2(\tilde{L}_c)}{dc}\Bigg|_{c=1} - \frac{d\lambda_1(\tilde{L}_c)}{dc}\Bigg|_{c=1}\right)$$

$$- \frac{(\tilde{\lambda}_2 - \tilde{\lambda}_1)}{D\tilde{\lambda}_1^2}\frac{d\lambda_1(\tilde{L}_c)}{dc}\Bigg|_{c=1},$$

where $\tilde{\lambda}_i = \lambda_i(\tilde{L}_c)$. The result now follows from (2.15), (2.14) and Proposition 2.5.   q.e.d.

Let $f = (\log \bar{\phi}_1)'$. From Lemma 2.2 we have $f < 0$ on $[0, \frac{D}{2}]$. We will also need the following equation for f.

Lemma 2.7. $f = (\log \bar{\phi}_1)'$ *satisfies*
(2.17)
$$f'' + 2ff' - \mathrm{tn}_K(s)\left[(n+1)f' + 2\bar{\lambda}_1 + 2f^2\right] - (n-1)(K - \mathrm{tn}_K^2(s))f = 0.$$

Proof. Since

$$(2.18) \qquad f' = \frac{\bar{\phi}_1''}{\bar{\phi}_1} - \left(\frac{\bar{\phi}_1'}{\bar{\phi}_1}\right)^2 = (n-1)\mathrm{tn}_K f - \bar{\lambda}_1 - f^2,$$

we have

$$f'' - (n-1)\frac{K}{\mathrm{cs}_K^2}f - (n-1)\mathrm{tn}_K f' + 2ff' = 0.$$

Using equation (2.18) we can rewrite this as

$$0 = f'' - \frac{f' + \bar{\lambda}_1 + f^2}{\mathrm{sn}_K \,\mathrm{cs}_K} - (n-1)\mathrm{tn}_K f' + 2ff'$$

$$= f'' - 2\left(\mathrm{tn}_K + \frac{\mathrm{cs}_K(2s)}{\mathrm{sn}_K(2s)}\right)(f' + \bar{\lambda}_1 + f^2) - (n-1)\mathrm{tn}_K f' + 2ff'$$

$$= f'' + 2ff' - \mathrm{tn}_K((n+1)f' + 2\bar{\lambda}_1 + 2f^2) - 2(n-1)\frac{\mathrm{cs}_K(2s)}{\mathrm{sn}_K(2s)}\mathrm{tn}_K f$$

$$= f'' + 2ff' - \mathrm{tn}_K((n+1)f' + 2\bar{\lambda}_1 + 2f^2) - (n-1)(K - \mathrm{tn}_K^2)f.$$

                                                                    q.e.d.

3. Log-concavity of the first eigenfunction

In this section, we prove Theorem 1.6. First we show the modulus of log-concavity is preserved for $u = e^{-\lambda_1 t}\phi_1$, where ϕ_1 is a positive first eigenfunction of the Laplacian with Dirichlet boundary condition with eigenvalue λ_1.

3.1. Preservation of initial modulus. Recall

Definition 3.1. Given a semi-convex function u on a domain Ω, a function $\psi : [0, +\infty) \to \mathbb{R}$ is called a modulus of concavity for u if for every $x \neq y$ in Ω

$$(3.1) \qquad \langle \nabla u(y), \gamma'(\tfrac{d}{2}) \rangle - \langle \nabla u(x), \gamma'(-\tfrac{d}{2}) \rangle \leq 2\psi\left(\frac{d(x,y)}{2} \right),$$

where γ is the unit normal minimizing geodesic with $\gamma(-\tfrac{d}{2}) = x$ and $\gamma(\tfrac{d}{2}) = y$, $d = d(x,y)$.

Theorem 3.2. *Let $\Omega \subset \mathbb{M}_K^n$ be a uniformly convex domain with diameter D, where $K \geq 0$, $D \leq D_0 < \pi/\sqrt{K}$. Let ϕ_1 be a positive first eigenfunction of the Laplacian on Ω with Dirichlet boundary condition associated to eigenvalue λ_1, and $u : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ is given by $u(x,t) = e^{-\lambda_1 t}\phi_1(x)$. Suppose $\psi_0 : [0, \tfrac{D}{2}] \to \mathbb{R}$ is a Lipschitz continuous modulus of concavity for $\log \phi_1$. If $\psi \in C^0([0, D/2] \times \mathbb{R}_+) \cap C^\infty([0, D/2] \times (0, \infty))$ is a solution of*
(3.2)
$$\begin{cases} \dfrac{\partial \psi(s,t)}{\partial t} \geq \psi''(s,t) + 2\psi(s,t)\psi'(s,t) - \mathrm{tn}_K(s)\Big[(n+1)\psi'(s,t) + 2\lambda_1 \\ \qquad\qquad\qquad + 2\psi^2(s,t)\Big] - (n-1)(K - \mathrm{tn}_K^2(s))\psi(s,t); \\ \psi(\cdot, 0) = \psi_0(\cdot); \\ \psi(0,t) = 0, \end{cases}$$

where $\psi' = \frac{\partial}{\partial s}\psi$ and $\psi'' = \frac{\partial^2}{\partial s^2}\psi$, then $\psi(\cdot, t)$ is a modulus of concavity for $\log u(\cdot, t)$ for each $t \geq 0$.

Remark 3.3. Almost all of the proof works for general $K \in \mathbb{R}$, some parts even for general manifolds, until the step in the end which requires K to be nonnegative.

Proof. We note that $u : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ satisfies the Laplacian equation

$$(3.3) \qquad\qquad \Delta u = -\lambda_1 u \ \text{ on } \ \Omega, \quad u|_{\partial\Omega} = 0,$$

and the heat equation

$$(3.4) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u, & \text{on } \Omega \times \mathbb{R}_+; \\ u = 0, & \text{on } \partial\Omega \times \mathbb{R}_+; \\ u(x,0) = \phi_1. \end{cases}$$

These are the two properties we need for u.

For every $x \neq y$ in Ω, let

$$Z(x,y,t) := \left\langle \nabla \log u(y,t), \gamma'\left(\tfrac{d}{2}\right)\right\rangle - \left\langle \nabla \log u(x,t), \gamma'\left(-\tfrac{d}{2}\right)\right\rangle$$

(3.5)
$$- 2\psi\left(\frac{d(x,y)}{2}, t\right),$$

where γ is the unit normal minimizing geodesic from x to y with $x = \gamma(-\tfrac{d}{2})$, $y = \gamma(\tfrac{d}{2})$.

We need to show $Z(x,y,t) \leq 0$ for all $x \neq y$ in Ω and $t \geq 0$. Consider

$$Z_\varepsilon(x,y,t) := Z(x,y,t) - \varepsilon e^{Ct},$$

for some suitable large C to be chosen (independent of ϵ). Then our problem reduces to showing $Z_\varepsilon(x,y,t) < 0$ on $\hat{\Omega} \times [0,T]$ for any $\varepsilon > 0$ and $T \in (0, \infty)$, where $\hat{\Omega} = \Omega \times \Omega - \{(x,x) \mid x \in \Omega\}$.

We first prove $Z_\epsilon < 0$ near the boundary of $\hat{\Omega}$. To show this we first establish the general fact that when the domain is convex, $\mathrm{Hess}\, u$ is concave at the boundary and $\mathrm{Hess} \log u$ is concave near the boundary under suitable boundary conditions. The proof is the same as the proof for the Euclidean domain in [3, Lemma 4.2].

Lemma 3.4. *Let Ω be a uniformly convex bounded domain in a Riemannian manifold M^n, and $u : \overline{\Omega} \times \mathbb{R}_+ \to \mathbb{R}$ a C^2 function such that u is positive on Ω, $u(\cdot, t) = 0$ and $\nabla u \neq 0$ on $\partial\Omega$. Given $T < \infty$, there exists $r_1 > 0$ such that $\nabla^2 \log u|_{(x,t)} < 0$ whenever $d(x, \partial\Omega) < r_1$ and $t \in [0, T]$, and $N \in \mathbb{R}$ such that $\nabla^2 \log u|_{(x,t)}(v,v) \leq N\|v\|^2$ for all $x \in \Omega$ and $t \in [0, T]$.*

Proof. Let $\alpha = \inf_{\partial\Omega \times [0,T]} \|\nabla u\|$. By assumption α is positive. Let P be such that $\|\nabla^2 u(v,v)\| \|p \leq P\|v\|^2$ at every point $p \in \overline{\Omega} \times [0,T]$ and for all $v \in T_p\Omega$. If $x_0 \in \partial\Omega$, then $\nabla u|_{x_0} = -\|\nabla u\|\nu|_{x_0}$, where ν is the outward normal vector since $\partial\Omega = \{u = 0\}$ and $\|\nabla u\| > 0$. Also, $\nabla^2 u|_{x_0}(v,v) = -\mathrm{II}(v,v)\nabla_\nu u|_{x_0}$ for $\langle v, \nu\rangle = 0$, where II is the second fundamental form of $\partial\Omega$ at x_0. This follows since

$$\nabla^2 u(v,v) = \langle \nabla_v \nabla u, v\rangle = -\langle \nabla_v(\|\nabla u\|\nu), v\rangle$$
$$= -\|\nabla u\|\langle \nabla_v \nu, v\rangle = -\mathrm{II}(v,v)\nabla_\nu u|_{x_0}.$$

Uniform convexity implies that $\mathrm{II}(v,v) \geq \kappa\|v\|^2$ for some $\kappa > 0$. The gradient direction $e = \frac{\nabla u}{\|\nabla u\|}$ is smooth near x_0 as is the projection $\pi^\perp : w \mapsto \langle w, e\rangle e$ and the orthogonal projection $\pi = \mathrm{id} - \pi^\perp$. At x_0, πw is tangent to $\partial\Omega$, we have

(3.6)
$$\nabla^2 u(\pi w, \pi w) \leq -\alpha\kappa\|\pi w\|^2.$$

Therefore, there exists $r_0 > 0$ depending on α, κ and P such that for $x \in B_{r_0}(x_0) \cap \Omega$ and $t \in [0, T]$, we have

$$\nabla^2 u|_x(\pi w, \pi w) \leq -\frac{\alpha \kappa}{2} \|\pi w\|^2 \text{ for any } w \in T_x\Omega;$$

$$\|\nabla u(x)\| \geq \tfrac{1}{2}\|\nabla u(x_0)\| \geq \frac{\alpha}{2};$$

$$0 < u(x) \leq 2\|\nabla u(x_0)\|d(x, x_0).$$

Then in such a neighborhood, we have for any w

$$\begin{aligned}
\nabla^2 u(w, w) &= \nabla^2 u(\pi w + \pi^\perp w, \pi w + \pi^\perp w) \\
&= \nabla^2 u(\pi w, \pi w) + 2\nabla^2 u(\pi w, \pi^\perp w) + \nabla^2 u(\pi^\perp w, \pi^\perp w) \\
&\leq -\frac{\alpha\kappa}{2}\|\pi w\|^2 + 2P\|\pi w\|\|\pi^\perp w\| + P\|\pi^\perp w\|^2 \\
&\leq -\frac{\alpha\kappa}{4}\|\pi w\|^2 + \left(P + \tfrac{4P^2}{\alpha\kappa}\right)\|\pi^\perp w\|^2.
\end{aligned}$$

Since

$$\langle \nabla u, w\rangle^2 = \|\nabla u\|^2\|\pi^\perp w\|^2 \geq \frac{\alpha\|\nabla u(x_0)\|}{4}\|\pi^\perp w\|^2,$$

and $u(x) \leq 2\|\nabla u(x_0)\|d(x, x_0)$, then

$$\begin{aligned}
\nabla^2 \log u|_x(w, w) &= \frac{1}{u}\left(\nabla^2 u(w, w) - \frac{(\nabla_w u)^2}{u}\right) \\
&\leq \frac{1}{u}\left(-\frac{\alpha\kappa}{4}\|\pi w\|^2 + \left(P + \frac{4P^2}{\alpha\kappa} - \frac{\alpha}{8d(x, x_0)}\right)\|\pi^\perp w\|^2\right) \\
&< 0
\end{aligned}$$

provided $d(x, x_0) < r_1 = \min\{r_0, \frac{\alpha^2\kappa}{8(P\alpha\kappa + 4P^2)}\}$.

Since $\{x \in \Omega : d(x, \partial\Omega) \geq r_1\}$ is compact, letting

$$N = \max\{0, \sup\{\nabla^2 \log u(x, t)(w, w) : |w| = 1, t \in [0, T], d(x, \partial\Omega) \geq r_1\}\}$$

finishes the proof of the lemma.   q.e.d.

Now we can show Z is almost nonpositive near the boundary of $\hat{\Omega} = \Omega \times \Omega - \{(x, x) \mid x \in \Omega\}$.

Lemma 3.5. *Let Ω and u be as in Lemma 3.4 and let ψ be continuous on $[0, D/2] \times \mathbb{R}_+$ and Lipschitz in the first argument, with $\psi(0, t) = 0$ for each t with $D = \text{diam}\,\Omega$. Then for any $T < \infty$ and $\beta > 0$, there exists an open set $U_{\beta,T} \subset M \times M$ containing $\partial\hat{\Omega}$ such that the function defined in (3.5) satisfies $Z(x, y, t) < \beta$ for all $t \in [0, T]$ and $(x, y) \in U_{\beta,T} \cap \hat{\Omega}$.*

Proof. Since ψ is Lipschitz in the first argument, there exists L such that

$$|\psi(s, t)| \leq Ls,$$

for all $s \in [0, D/2]$ and $t \in [0, T]$. We construct $U_{\beta,T}$ as a union of open balls $\bigcup_{(x_0, y_0) \in \partial\hat{\Omega}} B_r(x_0, y_0)$, where $r = r(x_0, y_0) > 0$. In order to find

$r = r(x_0, y_0) > 0$ such that $Z(x, y, t) < \beta$ for any $(x, y) \in B_r(x_0, y_0) \cap \hat{\omega}$, we consider two cases.

Case 1: $x_0 = y_0$.

Observe that the difference of the gradient and Hessian are related as follows.

$$\left\langle \nabla \log u(y, t), \gamma'\left(\tfrac{d}{2}\right) \right\rangle - \left\langle \nabla \log u(x, t), \gamma'\left(-\tfrac{d}{2}\right) \right\rangle$$

$$= \int_{-\frac{d}{2}}^{\frac{d}{2}} \frac{d}{ds} \left\langle \nabla \log u(\gamma(s), t), \gamma'(s) \right\rangle ds$$

$$= \int_{-\frac{d}{2}}^{\frac{d}{2}} \mathrm{Hess} \log u(\gamma'(s), \gamma'(s)) ds.$$

Since $\psi(0, t) = 0$, by Lemma 3.4 we have

$$Z(x, y, t) = \int_{-\frac{d}{2}}^{\frac{d}{2}} \mathrm{Hess}(\log u(\gamma(s), t))(\gamma', \gamma') ds - 2\psi\left(\frac{d(x, y)}{2}, t\right)$$

$$\leq (N + L)d.$$

Hence, $Z < \beta$ provided $(x, y) \in B_r(x_0, x_0)$ with $r < \frac{\beta}{2(N+L)}$.

Case 2: $x_0 \neq y_0$. In this case at least one of $x_0, y_0 \in \partial\Omega$. Say $x_0 \in \partial\Omega$ and $y_0 \in \Omega$ or $y_0 \in \partial\Omega$.

If $y_0 \in \Omega$, then $u(y_0) > 0$ and there is some $A > 0$ such that $\|\nabla \log u(y)\| \leq A$ for $d(y, y_0) < r_2$. Let $\alpha_0 = \|\nabla u(x_0)\| > 0$ and $\gamma_0 : [-\frac{d_0}{2}, \frac{d_0}{2}] \to \Omega$ be a normal minimal geodesic from x_0 to y_0. Then $\eta := \langle -\gamma_0'\left(-\frac{d_0}{2}\right), \nu(x_0) \rangle > 0$ by convexity. Since $\nabla u(x_0) = -\alpha_0 \nu(x_0)$, we have $\langle \nabla u(x_0), \gamma_0'\left(-\frac{d_0}{2}\right) \rangle = \eta \alpha_0$. For $x, y \in \Omega$ near (x_0, y_0), let $\gamma : [-\frac{d}{2}, \frac{d}{2}] \to \Omega$ be a normal minimal geodesic from x to y. Since $\langle \nabla u(x), \gamma' \rangle$ is smooth in x and y, $\langle \nabla u(x), \gamma' \rangle \geq \frac{1}{2} \eta \alpha_0$ and $0 < u(x) = u(x) - u(x_0) \leq 2\alpha_0 d(x, x_0)$ for $x, y \in \Omega$ with $\max\{d(y, y_0), d(x, x_0)\} < r_3$ and $0 < r_3 \leq r_2$. Then

$Z(x, y, t)$

$$= \left\langle \nabla \log u(y, t), \gamma'\left(\tfrac{d}{2}\right) \right\rangle - \left\langle \nabla \log u(x, t), \gamma'\left(-\tfrac{d}{2}\right) \right\rangle - 2\psi\left(\frac{d(x, y)}{2}, t\right)$$

$$\leq A - \frac{1}{u(x, t)} \langle \nabla u(x, t), \gamma' \rangle + L\, d(x, y)$$

$$\leq A - \frac{\eta}{4d(x, x_0)} + L\, d(x, y).$$

Therefore, $Z(x, y, t) < 0$ if $d(y, y_0) < r_3$ and $d(x, x_0) < \min\{r_3, \frac{\eta}{4(A+LD)}\}$.

If $y_0 \in \partial\Omega$, then y can also be handled in the same way as x above.

q.e.d.

Now we continue with the proof of Theorem 3.2.

Since u satisfies (3.3) and (3.4), by the Hopf boundary point lemma, $\langle \nabla u(x,t), \nu \rangle < 0$ for every $x \in \partial\Omega$ and every $t \geq 0$. Namely u satisfies the conditions in Lemma 3.5. Fix $T < \infty$ and $\epsilon > 0$. By assumption $Z_\epsilon(x, y, 0) < 0$ on $\hat{\Omega}$. By Lemma 3.5, $Z_\epsilon(x, y, t) \leq -\frac{1}{2}\epsilon$ on $(U_{\epsilon/2,T} \cap \hat{\Omega}) \times [0, T]$. Hence, if $Z_\epsilon(x, y, t) < 0$ does not hold on $\hat{\Omega} \times [0, T]$, then there exists a first time $t_0 > 0$, and point $(x_0, y_0) \in \hat{\Omega} \setminus U_{\epsilon/2,T}$, in particular, x_0, y_0 are in the interior of Ω and $x_0 \neq y_0$, such that $Z_\epsilon < 0$ on $\hat{\Omega} \times [0, t_0)$, and at (x_0, y_0, t_0),

$$(3.7) \qquad Z_\epsilon = 0, \quad \frac{\partial}{\partial t} Z_\varepsilon \geq 0, \quad \nabla_{v \oplus w} Z_\varepsilon = 0, \quad \nabla^2_{v \oplus w, v \oplus w} Z_\varepsilon \leq 0,$$

for any $v \in T_{x_0}\Omega$, $w \in T_{y_0}\Omega$. Let $\gamma(s)$ be a unit normal minimizing geodesic with $\gamma(-\frac{d_0}{2}) = x_0$ and $\gamma(\frac{d_0}{2}) = y_0$, where $d_0 = d(x_0, y_0)$. Choose a local orthonormal frame $\{e_i\}$ at x_0 such that $e_n = \gamma'(-\frac{d_0}{2})$ and parallel translate them along γ. Let $E_i = e_i \oplus e_i \in T_{(x_0, y_0)}\Omega \times \Omega$ for $1 \leq i \leq n-1$, and $E_n = e_n \oplus (-e_n)$.

For convenience, denote $\omega = \log u$ and $\frac{\partial}{\partial s}$ by $'$.

By (3.4), $(\partial_t - \Delta)\omega = \|\nabla\omega\|^2$. Hence,

$$\partial_t \nabla\omega = \nabla\partial_t\omega = \nabla\Delta\omega + \nabla\|\nabla\omega\|^2.$$

Taking the time derivative of Z_ε at (x_0, y_0, t_0):

$$\begin{aligned}
0 \leq &\frac{\partial}{\partial t} Z_\varepsilon \Big|_{(x_0, y_0, t_0)} \\
(3.8) \quad = &\langle \partial_t \nabla\omega(y_0, t_0), \gamma' \rangle - \langle \partial_t \nabla\omega(x_0, t_0), \gamma' \rangle - 2\frac{\partial\psi}{\partial t} - C\varepsilon e^{Ct_0} \\
= &\langle \nabla\Delta\omega(y_0, t_0), \gamma' \rangle + \langle \nabla\|\nabla\omega(y_0, t_0)\|^2, \gamma' \rangle - 2\frac{\partial\psi}{\partial t} - C\varepsilon e^{Ct_0} \\
&- \langle \nabla\Delta\omega(x_0, t_0), \gamma' \rangle - \langle \nabla\|\nabla\omega(x_0, t_0)\|^2, \gamma' \rangle.
\end{aligned}$$

Now take the spatial derivative of Z_ε at (x_0, y_0, t_0). We suppress t_0 at various places below when it is clear. Associated to a vector $v \oplus w \in T_{x_0}\Omega \oplus T_{y_0}\Omega$, we construct a variation $\eta(r, s)$ as follows. Let $\sigma_1(r)$ be the geodesic with $\sigma_1(0) = x_0$, $\frac{\partial}{\partial r}\sigma_1(0) = v$, $\sigma_2(r)$ be the geodesic with $\sigma_2(0) = y_0$, $\frac{\partial}{\partial r}\sigma_2(0) = w$, and $\eta(r, s)$, $s \in [-\frac{d_0}{2}, \frac{d_0}{2}]$, be the minimal geodesic connecting $\sigma_1(r)$ and $\sigma_2(r)$, with $\eta(0, s) = \gamma(s)$. Namely $\eta(r, s) = \exp_{\sigma_1(r)} sV(r)$ for some $V(r)$. Since we are in a strictly convex domain, every two points are connected by a unique minimal geodesic, the variation $\eta(r, s)$ is smooth. Denote the variation field $\frac{\partial}{\partial r}\eta(r, s)$ by $J(r, s)$. Then $J(r, s)$ is the Jacobi field along s direction satisfying $J(r, -\frac{d_0}{2}) = v$, $J(r, \frac{d_0}{2}) = w$. Denote $J(s) = J(0, s)$. Note that with this parametrization, in general, for fixed r, $\eta(r, s)$ is not unit speed when $s \neq 0$.

We will need the first and second covariant derivative of $T(r, s) = \frac{\eta'}{\|\eta'\|}$, the unit vector of $\frac{\partial}{\partial s}\eta(r, s)$, in r at $r = 0$. While one can construct

geodesic variation easily on general manifolds with initial data of Jacobi field $J(0), J'(0)$, it is not clear how to write out the geodesic variation with the data of the Jacobi field at both end points. For $\mathbb{M}_K^n$, one can write $\eta(r, s)$ explicitly though already subtle when $K \neq 0$, see Appendix B for a construction. In general, we can find the first derivative as follows,

$$\tfrac{\partial \eta}{\partial r} \tfrac{\partial \eta}{\partial s}\big|_{r=0} = \tfrac{\partial \eta}{\partial s} \tfrac{\partial \eta}{\partial r}\big|_{r=0} = \tfrac{\partial \eta}{\partial s} J(r, s)\big|_{r=0} = J'(s).$$

As $r \to 0$, write

(3.9) $$\tfrac{\partial}{\partial s}\eta(r, s) = \gamma'(s) + rJ'(s) + \frac{1}{2}r^2 \nabla_r \nabla_r \frac{\partial \eta}{\partial s}\big|_{r=0} + O(r^3).$$

Then

$$\|\eta'\| = \left[1 + 2r\langle \gamma', J'\rangle + r^2 \left(\|J'\|^2 + \langle \nabla_r \nabla_r \frac{\partial \eta}{\partial s}\big|_{r=0}, e_n\rangle\right) + O(r^3)\right]^{\frac{1}{2}}.$$

And

$$\tfrac{\partial}{\partial r}(\|\eta'\|^{-1}) = -\|\eta'\|^{-3} \left[\langle \gamma', J'\rangle + r\left(\|J'\|^2 + \langle \nabla_r \nabla_r \frac{\partial \eta}{\partial s}\big|_{r=0}, e_n\rangle\right) + O(r^2)\right],$$

$$\tfrac{\partial}{\partial r}(\|\eta'\|^{-1})\big|_{r=0} = -\langle \gamma', J'\rangle,$$

$$\tfrac{\partial^2}{\partial r^2}(\|\eta'\|^{-1})\big|_{r=0} = 3\langle \gamma', J'\rangle^2 - \|J'\|^2 - \langle \nabla_r \nabla_r \frac{\partial \eta}{\partial s}\big|_{r=0}, e_n\rangle.$$

Hence,

$$\nabla_r T(r, s)\big|_{r=0} = \tfrac{\partial}{\partial r}(\|\eta'\|^{-1})|_{r=0} T(0, s) + \nabla_r \frac{\partial \eta}{\partial s}\big|_{r=0}$$
$$= -\langle \gamma'(s), J'(s)\rangle e_n + J'(s),$$

$$\nabla_r \nabla_r T|_{r=0} = \left(3\langle \gamma', J'\rangle^2 - \|J'\|^2 - \langle \nabla_r \nabla_r \frac{\partial \eta}{\partial s}\big|_{r=0}, e_n\rangle\right) e_n$$
$$-2\langle \gamma', J'\rangle J' + \nabla_r \nabla_r \frac{\partial \eta}{\partial s}\big|_{r=0}.$$

For space with constant sectional curvature $\mathbb{M}_K^n$, $J(s)$ is a multiple of some parallel vector field. Hence, for variation in the normal direction, $\langle \gamma'(s), J'(s)\rangle = 0$ and

(3.10) $$\nabla_r T(r, s)\big|_{r=0} = J'(s).$$

If, in addition, $\nabla_r \nabla_r \frac{\partial \eta}{\partial s}\big|_{r=0}$ has only e_n component, then

(3.11) $$\nabla_r \nabla_r T|_{r=0} = -\|J'(s)\|^2 e_n.$$

For the variation in the direction $e_i \oplus e_i$, we construct the variation explicitly and verify that $\nabla_r \nabla_r \frac{\partial \eta}{\partial s}\big|_{r=0}$ has only the e_n component, see (B.5).

Now we compute the derivatives.

(i) For the first derivative in normal directions $0 \oplus e_i$, $1 \le i \le n - 1$, the Jacobi fields are $Q_i(s) = \frac{\mathrm{sn}_K(\frac{d_0}{2} + s)}{\mathrm{sn}_K(d_0)} e_i(s)$. Denote the variation by $\gamma_i(r, s)$ and $T_i(r, s) = \frac{\gamma_i'}{\|\gamma_i'\|}$. We obtain

(3.12)

$$
\begin{aligned}
0 &= \nabla_{0 \oplus e_i} Z_\varepsilon|_{(x_0, y_0, t_0)} \\
&= \frac{\partial}{\partial r} Z_\varepsilon \left(x_0, \gamma_i(r, \tfrac{d_0}{2}), t_0 \right) \Big|_{r=0} \\
&= \left(\left\langle \nabla_r \nabla \omega(\gamma_i(r, \tfrac{d_0}{2})), T_i(r, s) \right\rangle \right. \\
&\quad + \left\langle \nabla \omega(\gamma_i(r, \tfrac{d_0}{2})), \nabla_r T_i(r, s) \right\rangle \Big) \Big|_{s = \frac{d_0}{2}, r = 0} \\
&\quad - \left\langle \nabla \omega(\gamma_i(r, -\tfrac{d_0}{2})), \nabla_r T_i(r, s) \big|_{s = -\frac{d_0}{2}} \right\rangle \Big|_{r=0} \\
&\quad - \psi' \frac{\partial}{\partial r} d \left(x_0, \gamma_i(r, \tfrac{d_0}{2}) \right) \Big|_{r=0} \\
&= \langle \nabla_{e_i} \nabla \omega(y_0), e_n \rangle + \frac{\mathrm{cs}_K(d_0)}{\mathrm{sn}_K(d_0)} \langle \nabla \omega(y_0), e_i \rangle - \frac{1}{\mathrm{sn}_K(d_0)} \langle \nabla \omega(x_0), e_i \rangle.
\end{aligned}
$$

Here we applied (3.10) with $Q_i'(s) = \frac{\mathrm{cs}_K(s + \frac{d_0}{2})}{\mathrm{sn}_K(d_0)} e_i$, and the first variation of the distance is zero to get the last equality.

Similarly, for the direction $e_i \oplus 0$, choose $Q_i(s) = \frac{\mathrm{sn}_K(s - \frac{d_0}{2})}{\mathrm{sn}_K(d_0)} e_i(s)$. we obtain

(3.13)

$$
\begin{aligned}
0 &= \nabla_{e_i \oplus 0} Z_\varepsilon|_{(x_0, y_0)} = \frac{\partial}{\partial r} Z_\varepsilon \left(\gamma_i(r, -\tfrac{d_0}{2}), y_0 \right) \Big|_{r=0} \\
&= \langle \nabla_{e_i} \nabla \omega(x_0), e_n \rangle + \frac{1}{\mathrm{sn}_K(d_0)} \langle \nabla \omega(y_0), e_i \rangle - \frac{\mathrm{cs}_K(d_0)}{\mathrm{sn}_K(d_0)} \langle \nabla \omega(x_0), e_i \rangle.
\end{aligned}
$$

(ii) Taking the variation in the direction tangent to the geodesic:

(3.14)
$$
\begin{aligned}
0 &= \nabla_{e_n \oplus 0} Z_\varepsilon|_{(x_0, y_0)} = \frac{\partial}{\partial r} Z_\varepsilon \left(\gamma(-\tfrac{d_0}{2} + r), y_0 \right) \Big|_{r=0} \\
&= - \langle \nabla_{e_n} \nabla \omega(x_0), e_n \rangle + \psi' \left(\tfrac{d_0}{2} \right).
\end{aligned}
$$

Similarly,

(3.15)
$$
\begin{aligned}
0 &= \nabla_{0 \oplus (-e_n)} Z_\varepsilon|_{(x_0, y_0)} = \frac{\partial}{\partial r} Z_\varepsilon \left(x_0, \gamma'(\tfrac{d_0}{2} - r) \right) \Big|_{r=0} \\
&= - \langle \nabla_{e_n} \nabla \omega(y_0), e_n \rangle + \psi' \left(\tfrac{d_0}{2} \right).
\end{aligned}
$$

For the second derivative of normal spatial in the directions $e_i \oplus e_i$, $i = 1, \cdots, n - 1$, the Jacobi fields are $J_i(s) = \frac{\mathrm{cs}_K(s)}{\mathrm{cs}_K(\frac{d_0}{2})} e_i(s)$. Denote $\eta_i(r, s)$ its variation and $T_i(r, s) = \frac{\eta_i'}{\|\eta_i'\|}$.

We have the following formula for the first derivative.

$$\frac{\partial}{\partial r} Z_\epsilon \left(\eta_i(r, \tfrac{d_0}{2}), \eta_i(r, -\tfrac{d_0}{2}) \right)$$

$$= \left\langle \nabla_r \nabla \omega(\eta_i(r, \tfrac{d_0}{2})), T_i(r, s)|_{s=\frac{d_0}{2}} \right\rangle$$

$$+ \left\langle \nabla \omega(\eta_i(r, \tfrac{d_0}{2})), \nabla_r T_i(r, s)|_{s=\frac{d_0}{2}} \right\rangle$$

$$- \left\langle \nabla_r \nabla \omega(\eta_i(r, -\tfrac{d_0}{2})), T_i(r, s)|_{s=-\frac{d_0}{2}} \right\rangle$$

$$- \left\langle \nabla \omega(\eta_i(s, -\tfrac{d_0}{2})), \nabla_r T_i(r, s)|_{s=-\frac{d_0}{2}} \right\rangle$$

$$- \psi'\left(\tfrac{d_0}{2}\right) \frac{\partial}{\partial r} d\left(\eta_i(r, \tfrac{d_0}{2}), \eta_i(r, -\tfrac{d_0}{2}) \right).$$

Then

$$\frac{\partial^2}{\partial r^2} Z_\epsilon \left(\eta_i(r, \tfrac{d_0}{2}), \eta_i(r, -\tfrac{d_0}{2}) \right) \Big|_{r=0}$$

$$= \left\langle \nabla_r \nabla_r \nabla \omega(\eta_i(r, \tfrac{d_0}{2})), \ T_i(0, s) \right\rangle$$

$$+ 2 \left\langle \nabla_r \nabla \omega(\eta_i(r, \tfrac{d_0}{2})), \nabla_r T_i(r, s) \right\rangle \Big|_{s=\frac{d_0}{2}, r=0}$$

$$- \left\langle \nabla_r \nabla_r \nabla \omega(\eta_i(r, -\tfrac{d_0}{2}))|_{r=0}, T_i(0, s)|_{s=-\frac{d_0}{2}} \right\rangle$$

$$- 2 \left\langle \nabla_r \nabla \omega(\eta_i(r, -\tfrac{d_0}{2})), \nabla_r T_i(r, s)|_{s=-\frac{d_0}{2}} \right\rangle \Big|_{r=0}$$

$$+ \left\langle \nabla \omega(\eta_i(0, \tfrac{d_0}{2})), \nabla_r \nabla_r T_i(r, s)|_{s=\frac{d_0}{2}, r=0} \right\rangle$$

$$- \left\langle \nabla \omega(\eta_i(0, -\tfrac{d_0}{2})), \nabla_r \nabla_r T_i(r, s)|_{s=-\frac{d}{2}, r=0} \right\rangle$$

$$- \frac{1}{2} \psi''\left(\tfrac{d_0}{2}\right) \left(\frac{\partial}{\partial r} d(\eta_i(r, \tfrac{d_0}{2}), \eta_i(r, -\tfrac{d_0}{2})) \right)^2 \Big|_{r=0}$$

$$- \psi'\left(\tfrac{d_0}{2}\right) \frac{\partial^2}{\partial r^2} d\left(\eta_i(r, \tfrac{d_0}{2}), \eta_i(r, -\tfrac{d_0}{2}) \right) \Big|_{r=0}.$$

Obviously $\langle \gamma'(s), J_i'(s) \rangle = 0$. From Appendix B, $\nabla_r \nabla_r \frac{\partial \eta}{\partial s}\big|_{r=0}$ has only e_n component. So we can use (3.10) and (3.11) to get

$$\nabla_r T_i \big|_{r=0} = J_i'(s) = -K \frac{\mathrm{sn}_K(s)}{\mathrm{cs}_K(\frac{d_0}{2})} e_i,$$

and

$$\nabla_r \nabla_r T_i \big|_{r=0} = -\|J_i'(s)\|^2 e_n = -\frac{K^2 \mathrm{sn}_K^2(s)}{\mathrm{cs}_K^2(\frac{d_0}{2})} e_n.$$

Also the first and second variation of length [12, Chapter 1] are

$$\frac{\partial}{\partial r}d\left(\eta_i(r,\tfrac{d_0}{2}),\eta_i(r,-\tfrac{d_0}{2})\right)\big|_{r=0} = \frac{\partial}{\partial r}\int_{-d_0/2}^{d_0/2}\langle\frac{\partial\eta_i}{\partial s},\frac{\partial\eta_i}{\partial s}\rangle^{\frac{1}{2}}\,ds$$

$$= \langle J_i, e_n\rangle\big|_{-d_0/2}^{d_0/2} - \int_{-d_0/2}^{d_0/2}\langle J_i, \nabla_{e_n}e_n\rangle$$

$$= 0,$$

$$\frac{\partial^2}{\partial r^2}d\left(\eta_i(r,\tfrac{d_0}{2}),\eta_i(r,-\tfrac{d_0}{2})\right)\Big|_{r=0}$$

$$= \int_{-\frac{d_0}{2}}^{\frac{d_0}{2}}\left[\langle J_i', J_i'\rangle - \langle R(e_n, J_i)J_i, e_n\rangle\right]ds + \langle e_n, \nabla_r\frac{\partial\eta_i}{\partial r}\rangle\big|_{-d_0/2}^{d_0/2}$$

$$= \frac{1}{\mathrm{cs}_K^2(\tfrac{d_0}{2})}\int_{-\frac{d_0}{2}}^{\frac{d_0}{2}}\left[(\mathrm{cs}_K'(s))^2 - K\,\mathrm{cs}_K^2(s)\right]ds$$

$$= \frac{2}{\mathrm{cs}_K^2(\tfrac{d_0}{2})}\int_0^{\frac{d_0}{2}}\left[-K\,\mathrm{sn}_K(s)\,\mathrm{cs}_K'(s) - K\,\mathrm{sn}_K'(s)\,\mathrm{cs}_K(s)\right]ds$$

$$= -\frac{2K}{\mathrm{cs}_K^2(\tfrac{d_0}{2})}\int_0^{\frac{d_0}{2}}(\mathrm{sn}_K\,\mathrm{cs}_K)'\,ds = -2\,\mathrm{tn}_K(\tfrac{d_0}{2}).$$

Hence,

(3.16)

$$0 \geq \nabla_{E_i, E_i}^2 Z_\epsilon|_{(x_0, y_0)}$$

$$= \frac{\partial}{\partial r^2}Z_\epsilon\left(\eta_i(r,\tfrac{d_0}{2}),\eta_i(r,-\tfrac{d_0}{2})\right)\Big|_{r=0}$$

$$= \langle\nabla_{e_i}\nabla_{e_i}\nabla\omega(y_0), e_n\rangle - \langle\nabla_{e_i}\nabla_{e_i}\nabla\omega(x_0), e_n\rangle$$

$$- 2\,\mathrm{tn}_K\left(\tfrac{d_0}{2}\right)\left[\langle\nabla_{e_i}\nabla\omega(y_0), e_i\rangle + \langle\nabla_{e_i}\nabla\omega(x_0), e_i\rangle\right]$$

$$- \mathrm{tn}_K^2(\tfrac{d_0}{2})\left[\langle\nabla\omega(y_0), e_n\rangle - \langle\nabla\omega(x_0), e_n\rangle\right] + 2\,\mathrm{tn}_K\left(\tfrac{d_0}{2}\right)\psi'\left(\tfrac{d_0}{2}\right).$$

Next the second variation in the tangential direction is

(3.17)
$$0 \geq \nabla_{E_n, E_n}^2 Z|_{(x_0, y_0)} = \frac{\partial^2}{\partial r^2}Z_\epsilon\left(\gamma(-\tfrac{d_0}{2}+r),\gamma(\tfrac{d_0}{2}-r)\right)\Big|_{r=0}$$

$$= \langle\nabla_{e_n}\nabla_{e_n}\nabla\omega(y_0), e_n\rangle - \langle\nabla_{e_n}\nabla_{e_n}\nabla\omega(x_0), e_n\rangle - 2\psi''\left(\tfrac{d_0}{2}\right).$$

Adding up (3.16) from $i = 1, \cdots, n-1$ and (3.17) gives

(3.18)

$$0 \geq \langle\Delta\nabla\omega(y_0), e_n\rangle - \langle\Delta\nabla\omega(x_0), e_n\rangle$$

$$- 2\,\mathrm{tn}_K\left(\tfrac{d_0}{2}\right)\sum_{i=1}^{n-1}\left[\langle\nabla_{e_i}\nabla\omega(y_0), e_i\rangle + \langle\nabla_{e_i}\nabla\omega(x_0), e_i\rangle\right]$$

$$- (n-1)\operatorname{tn}_K^2(\tfrac{d_0}{2})\left[\langle\nabla\omega(y_0), e_n\rangle - \langle\nabla\omega(x_0), e_n\rangle\right] - 2\psi''\left(\tfrac{d_0}{2}\right)$$
$$+ 2(n-1)\operatorname{tn}_K(\tfrac{d_0}{2})\psi'\left(\tfrac{d_0}{2}\right).$$

Combining the inequality from the second derivative of spatial directions (3.18) with the inequality from time derivative (3.8), and use the Bochner–Weitzenböck formula for vector field (see, e.g., [11, Page 18])

$$(3.19)\qquad\qquad \Delta\nabla\omega - \operatorname{Ric}(\nabla\omega, \cdot) = \nabla\Delta\omega,$$

we have, using $\operatorname{Ric} = (n-1)Kg$,

$$\langle\nabla\|\nabla\omega(y_0, t_0)\|^2, e_n\rangle - \langle\nabla\|\nabla\omega(x_0, t_0)\|^2, e_n\rangle - 2\frac{\partial\psi}{\partial t} - C\varepsilon e^{Ct_0}$$

$$\geq -2\operatorname{tn}_K\left(\tfrac{d_0}{2}\right)\sum_{i=1}^{n-1}\left[\langle\nabla_{e_i}\nabla\omega(y_0), e_i\rangle + \langle\nabla_{e_i}\nabla\omega(x_0), e_i\rangle\right]$$

$$- 2\psi''\left(\tfrac{d_0}{2}\right) + 2(n-1)\operatorname{tn}_K\left(\tfrac{d_0}{2}\right)\psi'\left(\tfrac{d_0}{2}\right)$$

$$(3.20)\qquad + (n-1)(K - \operatorname{tn}_K^2(\tfrac{d_0}{2}))(\langle\nabla\omega(y_0), e_n\rangle - \langle\nabla\omega(x_0), e_n\rangle)$$

$$= -2\operatorname{tn}_K\left(\tfrac{d_0}{2}\right)\left[\Delta\omega(y_0) + \Delta\omega(x_0)\right]$$

$$+ 2\operatorname{tn}_K\left(\tfrac{d_0}{2}\right)\left[\langle\nabla_{e_n}\nabla\omega(y_0), e_n\rangle + \langle\nabla_{e_n}\nabla\omega(x_0), e_n\rangle\right]$$

$$- 2\psi''\left(\tfrac{d_0}{2}\right) + 2(n-1)\operatorname{tn}_K\left(\tfrac{d_0}{2}\right)\psi'\left(\tfrac{d_0}{2}\right)$$

$$+ (n-1)(K - \operatorname{tn}_K^2(\tfrac{d_0}{2}))(2\psi\left(\frac{d_0}{2}\right) + \varepsilon e^{Ct}).$$

Now $\nabla\|\nabla\omega(y_0, t_0)\|^2 = 2\nabla_{\nabla\omega(y_0)}\nabla\omega(y_0, t_0)$.

Since $\nabla\omega = \sum_{i=1}^{n}\langle\nabla\omega, e_i\rangle e_i$, we have

$$\langle\nabla\|\nabla\omega(y_0, t_0)\|^2, e_n\rangle = \sum_{i=1}^{n-1}2\langle\nabla\omega(y_0), e_i\rangle\langle\nabla_{e_i}\nabla\omega(y_0, t_0), e_n\rangle$$
$$+ 2\langle\nabla\omega(y_0), e_n\rangle\langle\nabla_{e_n}\nabla\omega(y_0, t_0), e_n\rangle.$$

Applying the first variation identity (3.12) and (3.15), we obtain

$$\langle\nabla\|\nabla\omega(y_0, t_0)\|^2, e_n\rangle$$
$$= 2\sum_{i=1}^{n-1}\langle\nabla\omega(y_0), e_i\rangle\Big(\frac{1}{\operatorname{sn}_K(d_0)}\langle\nabla\omega(x_0), e_i\rangle - \frac{\operatorname{cs}_K(d_0)}{\operatorname{sn}_K(d_0)}\langle\nabla\omega(y_0), e_i\rangle\Big)$$
$$+ 2\psi'\left(\tfrac{d_0}{2}\right)\langle\nabla\omega(y_0), e_n\rangle.$$

Similarly, applying the first variation identity (3.13) and (3.14) and combine above, we have

$$\langle\nabla\|\nabla\omega(y_0,t_0)\|^2,e_n\rangle - \langle\nabla\|\nabla\omega(x_0,t_0)\|^2,e_n\rangle$$

$$= 2\sum_{i=1}^{n-1}\langle\nabla\omega(y_0),e_i\rangle\left(\frac{1}{\mathrm{sn}_K(d_0)}\langle\nabla\omega(x_0),e_i\rangle - \frac{\mathrm{cs}_K(d_0)}{\mathrm{sn}_K(d_0)}\langle\nabla\omega(y_0),e_i\rangle\right)$$

$$+ 2\psi'\left(\tfrac{d_0}{2}\right)\langle\nabla\omega(y_0),e_n\rangle$$

$$- 2\sum_{i=1}^{n-1}\langle\nabla\omega(x_0),e_i\rangle\left(\frac{\mathrm{cs}_K(d_0)}{\mathrm{sn}_K(d_0)}\langle\nabla\omega(x_0),e_i\rangle - \frac{1}{\mathrm{sn}_K(d_0)}\langle\nabla\omega(y_0),e_i\rangle\right)$$

$$- 2\psi'\left(\tfrac{d_0}{2}\right)\langle\nabla\omega(x_0),e_n\rangle$$

$$= -\frac{2}{\mathrm{sn}_K(d_0)}\sum_{i=1}^{n-1}\left[\langle\nabla\omega(y_0),e_i\rangle - \langle\nabla\omega(x_0),e_i\rangle\right]^2$$

$$- \frac{2(\mathrm{cs}_K(d_0)-1)}{\mathrm{sn}_K(d_0)}\sum_{i=1}^{n-1}\left[\langle\nabla\omega(y_0),e_i\rangle^2 + \langle\nabla\omega(x_0),e_i\rangle^2\right]$$

$$+ 2\psi'\left(\tfrac{d_0}{2}\right)\left[\langle\nabla\omega(y_0),e_n\rangle - \langle\nabla\omega(x_0),e_n\rangle\right]$$

$$\leq -\frac{2(\mathrm{cs}_K(d_0)-1)}{\mathrm{sn}_K(d_0)}\sum_{i=1}^{n-1}\left[\langle\nabla\omega(y_0),e_i\rangle^2 + \langle\nabla\omega(x_0),e_i\rangle^2\right]$$

$$+ 2\psi'\left(\tfrac{d_0}{2}\right)\left[\langle\nabla\omega(y_0),e_n\rangle - \langle\nabla\omega(x_0),e_n\rangle\right]$$

$$= 2\,\mathrm{tn}_K(\tfrac{d_0}{2})\left\{-2\lambda_1 - \Delta\omega(y_0) - \Delta\omega(x_0)\right\}$$

$$- 2\,\mathrm{tn}_K(\tfrac{d_0}{2})\left\{\langle\nabla\omega(y_0),e_n\rangle^2 + \langle\nabla\omega(x_0),e_n\rangle^2\right\}$$

$$+ 2\psi'\left(\tfrac{d_0}{2}\right)\left[2\psi\left(\tfrac{d_0}{2}\right) + \varepsilon e^{Ct_0}\right].$$

Here in the last equality we used the identity $\mathrm{tn}_K(\tfrac{d_0}{2}) = \frac{1-\mathrm{cs}_K(d_0)}{\mathrm{sn}_K(d_0)}$, $\|\nabla\omega\|^2 = \|\nabla\log u\|^2 = \frac{\Delta u}{u} - \Delta\log u = -\lambda_1 - \Delta\omega$, and $Z_\varepsilon(x_0,y_0,t_0) = 0$.
　　Plugging this into (3.20) and using (3.14), (3.15), we obtain

$$(3.21)\quad\begin{aligned}&-4\,\mathrm{tn}_K(\tfrac{d_0}{2})\lambda_1 - 2\,\mathrm{tn}_K(\tfrac{d_0}{2})\left\{\langle\nabla\omega(y_0),e_n\rangle^2 + \langle\nabla\omega(x_0),e_n\rangle^2\right\}\\ &+ 2\psi'\left(\tfrac{d_0}{2}\right)\left[2\psi\left(\tfrac{d_0}{2}\right) + \varepsilon e^{Ct_0}\right] - 2\frac{\partial\psi}{\partial t} - C\varepsilon e^{Ct_0}\\ &\geq -2\psi''\left(\tfrac{d_0}{2}\right) + 2(n+1)\,\mathrm{tn}_K\left(\tfrac{d_0}{2}\right)\psi'\left(\tfrac{d_0}{2}\right)\\ &+ (n-1)(K - \mathrm{tn}_K^2(\tfrac{d_0}{2}))(2\psi\left(\tfrac{d_0}{2}\right) + \varepsilon e^{Ct}).\end{aligned}$$

Lastly

$$\langle \nabla \omega(y_0), e_n \rangle^2 + \langle \nabla \omega(x_0), e_n \rangle^2 \geq \frac{(\langle \nabla \omega(y_0), e_n \rangle - \langle \nabla \omega(x_0), e_n \rangle)^2}{2}$$

$$= \frac{\left(2\psi\left(\frac{d_0}{2}\right) + \varepsilon e^{Ct_0}\right)^2}{2}$$

$$\geq 2\psi^2\left(\frac{d_0}{2}\right) + 2\psi\left(\frac{d_0}{2}\right)\varepsilon e^{Ct_0}.$$

Now when $K \geq 0$,

$$- 2\operatorname{tn}_K(\tfrac{d_0}{2})\left\{\langle \nabla \omega(y_0), e_n \rangle^2 + \langle \nabla \omega(x_0), e_n \rangle^2\right\}$$

$$\leq -2\operatorname{tn}_K(\tfrac{d_0}{2})\left[2\psi^2\left(\tfrac{d_0}{2}\right) + 2\psi\left(\tfrac{d_0}{2}\right)\varepsilon e^{Ct_0}\right].$$

Choose $C > \sup_{[0, \frac{D}{2}] \times [0,T]} \{2\psi' - 4\operatorname{tn}_K \psi - (n-1)(K - \operatorname{tn}_K^2(s))\}$, (this is independent of ϵ as required) then, as $\varepsilon > 0$, (3.21) becomes

$$2\psi''\left(\frac{d_0}{2}\right) + 4\psi'\left(\frac{d_0}{2}\right)\psi\left(\frac{d_0}{2}\right)$$

$$- 2\operatorname{tn}_K\left(\frac{d_0}{2}\right)\left[(n+1)\psi'\left(\frac{d_0}{2}\right) + 2\lambda_1 + 2\psi^2\left(\frac{d_0}{2}\right)\right]$$

$$- (n-1)(K - \operatorname{tn}_K^2(\tfrac{d_0}{2}))\left(2\psi\left(\frac{d_0}{2}\right)\right) - 2\frac{\partial \psi}{\partial t} > 0,$$

which is a contradiction to our assumption. q.e.d.

With similar proof we also obtain the following preserving of log-concavity estimate.

Theorem 3.6. *Let Ω and u be as in Theorem 3.2. Suppose $\psi_0 : [0, D/2] \to \mathbb{R}$ satisfies*

$$\langle \nabla \log u(y,0), \gamma'(\tfrac{d}{2}) \rangle - \langle \nabla \log u(x,0), \gamma'(-\tfrac{d}{2}) \rangle \leq 2\psi_0|_{s=\frac{d}{2}} + (n-1)\operatorname{tn}_K(\tfrac{d}{2}).$$

Let $\psi \in C^0([0, D/2] \times \mathbb{R}_+) \cap C^\infty([0, D/2] \times (0, \infty))$ be a solution of
(3.22)
$$\begin{cases} \frac{\partial \psi}{\partial t} \geq \psi'' + 2\psi\psi' - 2\operatorname{tn}_K(s)(\psi' + \psi^2 + \lambda_1) \text{ on } [0, D/2] \times \mathbb{R}_+, \\ \psi(\cdot, 0) = \psi_0(\cdot), \\ \psi(0, t) = 0, \\ \psi(s, t) \leq 0. \end{cases}$$

Then

$$\langle \nabla \log u(y,t), \gamma'(\tfrac{d}{2}) \rangle - \langle \nabla \log u(x,t), \gamma'(-\tfrac{d}{2}) \rangle$$

$$\leq 2\psi(s,t)|_{s=\frac{d}{2}} + (n-1)\operatorname{tn}_K(\tfrac{d}{2}),$$

for all $t \geq 0$ and $D \leq \frac{\pi}{\sqrt{K}}$ if $K > 0$.

Remark 3.7. Note that the stationary solutions of ψ satisfy

$$0 = (\psi'(s) + \psi^2(s) + \lambda_1)' - 2\operatorname{tn}_K(s)(\psi' + \psi^2(s) + \lambda_1).$$

Solving the ODE $y' - 2\operatorname{tn}_K(s)y = 0$, we have $y = y(0)K\operatorname{cs}_K^{-2}(s)$. Hence, an initial condition $y(0) = 0$ would imply the trivial solution in y, which is equivalent to $\psi' + \psi^2 + \lambda_1 = 0$. The condition $y(0) = 0$ can be obtained by adding the condition $\psi'(0) = -\lambda_1$. The proof below makes no assumptions on ψ'. The difference of the stationary solutions of (3.2) and (3.22) does not have a definite sign, so the two estimates have independent interests.

Proof. Since the proof is similar, we will only specify the changes. As in the proof of Theorem 3.2, for any $\varepsilon > 0$, define the function Z_ε on $\hat{\Omega} \times \mathbb{R}_+$ by

$$Z_\varepsilon(x, y, t) := \langle \nabla \log u(y, t), \gamma'(\tfrac{d}{2}) \rangle - \langle \nabla \log u(x, t), \gamma'(-\tfrac{d}{2}) \rangle - 2\psi(\tfrac{d}{2}, t)$$
$$- (n-1)\operatorname{tn}_K(\tfrac{d}{2}) - \varepsilon e^{Ct},$$

for some $C > 0$ to be chosen later. The boundary case is handled in the same way as before. By assumption, $Z_\varepsilon(x, y, 0) < 0$ on $\hat{\Omega}$. Let t_0 be the first time that $Z_\varepsilon(x_0, y_0, t_0) = 0$. First the time derivative (3.8) and the first derivative in the normal directions (3.12) are the same. Taking the derivative tangent to the geodesic, we have

$$\langle \nabla_{e_n} \nabla \omega(x_0), e_n \rangle = \langle \nabla_{e_n} \nabla \omega(y_0), e_n \rangle = \psi'(\tfrac{d_0}{2}) + \frac{n-1}{2}K\operatorname{cs}_K^{-2}(\tfrac{d_0}{2}).$$

Taking the second variation in the normal direction as in (3.16),

$$0 \geq \langle \nabla_{e_i} \nabla_{e_i} \nabla \omega(y_0), e_n \rangle - \langle \nabla_{e_i} \nabla_{e_i} \nabla \omega(x_0), e_n \rangle$$
$$- 2\operatorname{tn}_K\left(\tfrac{d_0}{2}\right)[\langle \nabla_{e_i} \nabla \omega(y_0), e_i \rangle + \langle \nabla_{e_i} \nabla \omega(x_0), e_i \rangle]$$
$$- \operatorname{tn}_K^2(\tfrac{d_0}{2})[\langle \nabla \omega(y_0), e_n \rangle - \langle \nabla \omega(x_0), e_n \rangle]$$
$$+ 2\operatorname{tn}_K\left(\tfrac{d_0}{2}\right)\psi'\left(\tfrac{d_0}{2}\right) + (n-1)K\operatorname{tn}_K(\tfrac{d_0}{2})\operatorname{cs}_K^{-2}(\tfrac{d_0}{2}).$$

Next the second variation in the tangential direction as in (3.17),

$$0 \geq \langle \nabla_{e_n} \nabla_{e_n} \nabla \omega(y_0), e_n \rangle - \langle \nabla_{e_n} \nabla_{e_n} \nabla \omega(x_0), e_n \rangle$$
$$- 2\psi''\left(\tfrac{d_0}{2}\right) - 2(n-1)K\operatorname{tn}_K(\tfrac{d_0}{2})\operatorname{cs}_K^{-2}(\tfrac{d_0}{2}).$$

Adding up from $i = 1, \cdots, n-1$ and the second tangential variation,

$$0 \geq \langle \Delta \nabla \omega(y_0), e_n \rangle - \langle \Delta \nabla \omega(x_0), e_n \rangle$$
$$- 2\operatorname{tn}_K\left(\tfrac{d_0}{2}\right)\sum_{i=1}^{n-1}[\langle \nabla_{e_i} \nabla \omega(y_0), e_i \rangle + \langle \nabla_{e_i} \nabla \omega(x_0), e_i \rangle]$$
$$- (n-1)\operatorname{tn}_K^2(\tfrac{d_0}{2})[\langle \nabla \omega(y_0), e_n \rangle - \langle \nabla \omega(x_0), e_n \rangle] - 2\psi''\left(\tfrac{d_0}{2}\right)$$

$$+ 2(n-1)\operatorname{tn}_K(\tfrac{d_0}{2})\psi'\left(\tfrac{d_0}{2}\right) + (n-1)^2 K \operatorname{tn}_K(\tfrac{d_0}{2})\operatorname{cs}_K^{-2}(\tfrac{d_0}{2})$$
$$- 2(n-1)K\operatorname{tn}_K(\tfrac{d_0}{2})\operatorname{cs}_K^{-2}(\tfrac{d_0}{2}).$$

Combining the above with the time derivative, the first variations, and the Bochner formula as before, we have

(3.23)

$$0 \le 2\psi'' + 2\left\{\psi'\left(\tfrac{d_0}{2}\right) + \frac{(n-1)}{2}K\operatorname{cs}_K^{-2}\left(\tfrac{d_0}{2}\right)\right\}(\nabla_n\omega(y_0) - \nabla_n\omega(x_0))$$

$$- 2\frac{\partial\psi}{\partial t} - C\varepsilon e^{Ct_0} + 2\operatorname{tn}_K(\tfrac{d_0}{2})\sum_{i=1}^{n-1}(\langle\nabla_i\nabla\omega(y_0), e_i\rangle + \langle\nabla_i\nabla\omega(x_0), e_i\rangle)$$

$$- (n-1)(K - \operatorname{tn}_K^2(\tfrac{d_0}{2}))(\nabla_n\omega(y_0) - \nabla_n\omega(x_0))$$

$$+ \sum_{i=1}^{n-1}\left\{2\langle\nabla_i\nabla\omega(y_0, t)\nabla_i\omega(y_0, t), \gamma'\rangle - 2\langle\nabla_i\nabla\omega(x_0, t)\nabla_i\omega(x_0, t), \gamma'\rangle\right\}$$

$$- 2(n-1)\operatorname{tn}_K(\tfrac{d_0}{2})\psi' - (n-1)^2 K\operatorname{cs}_K^{-2}(\tfrac{d_0}{2})\operatorname{tn}_K(\tfrac{d_0}{2})$$

$$+ (n-1)2K\operatorname{cs}_K^{-2}(\tfrac{d_0}{2})\operatorname{tn}_K(\tfrac{d_0}{2}),$$

where the main difference between the setting here and Theorem 3.2 is the first, third, and last line. At the point (x_0, y_0, t_0), the middle term in the first line becomes

$$2\left\{\psi' + \frac{(n-1)}{2}K\operatorname{cs}_K^{-2}\left(\tfrac{d_0}{2}\right)\right\}(\nabla_n\omega(y_0) - \nabla_n\omega(x_0))$$

$$= 2\left\{\psi' + \frac{(n-1)}{2}K\operatorname{cs}_K^{-2}\left(\tfrac{d_0}{2}\right)\right\}\left(2\psi + (n-1)\operatorname{tn}_K(\tfrac{d_0}{2}) + \varepsilon e^{Ct_0}\right)$$

$$= 4\psi\psi' + 2(n-1)\psi'\operatorname{tn}_K(\tfrac{d_0}{2}) + 2\psi(n-1)K\operatorname{cs}_K^{-2}(\tfrac{d_0}{2})$$

$$+ (n-1)^2 K\operatorname{cs}_K^{-2}(\tfrac{d_0}{2})\operatorname{tn}_K(\tfrac{d_0}{2}) + 2\varepsilon e^{Ct_0}\left\{\psi' + \frac{(n-1)}{2}K\operatorname{cs}_K^{-2}\left(\tfrac{d_0}{2}\right)\right\}.$$

The third line becomes

$$- (n-1)(K - \operatorname{tn}_K^2(\tfrac{d_0}{2}))(\nabla_n\omega(y_0) - \nabla_n\omega(x_0))$$

$$= -(n-1)(K - \operatorname{tn}_K^2(\tfrac{d_0}{2}))(2\psi + (n-1)\operatorname{tn}_K(\tfrac{d_0}{2}) + \varepsilon e^{Ct_0}).$$

By applying the first variation identities and completing the square, the first term in the fourth line of (3.23) becomes

$$\sum_{i=1}^{n-1}\left\{2\langle\nabla_i\nabla\omega(y_0, t)\nabla_i\omega(y_0, t), \gamma'\rangle - 2\langle\nabla_i\nabla\omega(x_0, t)\nabla_i\omega(x_0, t), \gamma'\rangle\right\}$$

$$= -\frac{2}{\operatorname{sn}_K(d_0)}\sum_{i=1}^{n-1}\left\{(\nabla_i\omega(y_0) - \nabla_i\omega(x_0))^2\right.$$

$$\left. + (\operatorname{cs}_K(d_0) - 1)((\nabla_i\omega(y_0))^2 + (\nabla_i\omega(x_0))^2)\right\}$$

$$\leq -4\lambda_1 \operatorname{tn}_K(\tfrac{d_0}{2}) - 2\operatorname{tn}_K(\tfrac{d_0}{2})\sum_{i=1}^{n-1}\left(\langle\nabla_i\nabla\omega(y_0), e_i\rangle + \langle\nabla_i\nabla\omega(x_0), e_i\rangle\right)$$

$$- 4\operatorname{tn}_K(\tfrac{d_0}{2})\left(\psi'(\tfrac{d_0}{2}) + \frac{n-1}{2}K\operatorname{cs}_K^{-2}(\tfrac{d_0}{2})\right)$$

$$- 2\operatorname{tn}_K(\tfrac{d_0}{2})\left\{(\nabla_n\omega(y_0))^2 + (\nabla_n\omega(x_0))^2\right\}.$$

The last term can be bounded by

$$(\nabla_n\omega(y_0))^2 + (\nabla_n\omega(x_0))^2 \geq \frac{(\nabla_n\omega(y_0) - \nabla_n\omega(x_0))^2}{2}$$

$$= \frac{(2\psi + (n-1)\operatorname{tn}_K(\tfrac{d_0}{2}) + \varepsilon e^C t_0)^2}{2}$$

$$\geq 2\psi^2(\tfrac{d_0}{2}) + 2\psi(\tfrac{d_0}{2})(n-1)\operatorname{tn}_K(\tfrac{d_0}{2}) + \frac{(n-1)^2}{2}\operatorname{tn}_K^2(\tfrac{d_0}{2})$$

$$+ 2\varepsilon\psi(\tfrac{d_0}{2})e^{Ct_0} + \varepsilon(n-1)\operatorname{tn}_K(\tfrac{d_0}{2})e^{Ct_0},$$

so that for $K \geq 0$,

$$- 2\operatorname{tn}_K(\tfrac{d_0}{2})\left[(\nabla_n\omega(y_0))^2 + (\nabla_n\omega(x_0))^2\right]$$

$$\leq -4\operatorname{tn}_K(\tfrac{d_0}{2})\psi^2 - 4\psi(\tfrac{d_0}{2})(n-1)\operatorname{tn}_K^2(\tfrac{d_0}{2}) - (n-1)^2\operatorname{tn}_K^3(\tfrac{d_0}{2})$$

$$- 4\varepsilon\psi\operatorname{tn}_K(\tfrac{d_0}{2})e^{Ct_0} - 2\varepsilon(n-1)\operatorname{tn}_K^2(\tfrac{d_0}{2})e^{Ct_0}.$$

Inserting the above inequalities into (3.23),

$$0 \leq 2\psi''(\tfrac{d_0}{2}) + 4\psi(\tfrac{d_0}{2})\psi'(\tfrac{d_0}{2}) - 2\frac{\partial\psi}{\partial t} - 4(\psi' + \psi^2 + \lambda_1)\operatorname{tn}_K(\tfrac{d_0}{2})$$

$$+ 2\psi(n-1)K(2 - \operatorname{cs}_K^{-2}(\tfrac{d_0}{2})) - 2(n-1)(K - \operatorname{tn}_K^2(\tfrac{d_0}{2}))\psi$$

$$- (n-1)^2(K - \operatorname{tn}_K^2(\tfrac{d_0}{2}))\operatorname{tn}_K(\tfrac{d_0}{2}) - \varepsilon(n-1)(K - \operatorname{tn}_K^2(\tfrac{d_0}{2}))e^{Ct_0}$$

$$- 4\varepsilon\psi\operatorname{tn}_K(\tfrac{d_0}{2})e^{Ct_0} - 2\varepsilon(n-1)\operatorname{tn}_K^2(\tfrac{d_0}{2})e^{Ct_0} + 2\varepsilon\psi'e^{Ct_0}$$

$$+ \varepsilon(n-1)K\operatorname{cs}_K^{-2}(\tfrac{d_0}{2})e^{Ct_0} - C\varepsilon e^{Ct_0} - (n-1)^2\operatorname{tn}_K^3(\tfrac{d_0}{2})$$

$$\leq 2\psi''(\tfrac{d_0}{2}) + 4\psi(\tfrac{d_0}{2})\psi'(\tfrac{d_0}{2}) - 2\frac{\partial\psi}{\partial t} - 4(\psi' + \psi^2 + \lambda_1)\operatorname{tn}_K(\tfrac{d_0}{2})$$

$$- (n-1)^2 K\operatorname{tn}_K(\tfrac{d_0}{2}) - 4\varepsilon\psi\operatorname{tn}_K(\tfrac{d_0}{2})e^{Ct_0} - \varepsilon(n-1)\operatorname{tn}_K^2(\tfrac{d_0}{2})e^{Ct_0}$$

$$+ 2\varepsilon\psi'e^{Ct_0} + \varepsilon(n-1)K(\operatorname{cs}_K^{-2}(\tfrac{d_0}{2}) - 1)e^{Ct_0} - C\varepsilon e^{Ct_0}$$

$$\leq 2\psi''(\tfrac{d_0}{2}) + 4\psi(\tfrac{d_0}{2})\psi'(\tfrac{d_0}{2}) - 2\frac{\partial\psi}{\partial t} - 4(\psi' + \psi^2 + \lambda_1)\operatorname{tn}_K(\tfrac{d_0}{2})$$

$$- (n-1)^2 K\operatorname{tn}_K(\tfrac{d_0}{2}) - 4\varepsilon\psi\operatorname{tn}_K(\tfrac{d_0}{2})e^{Ct_0} + 2\varepsilon\psi'e^{Ct_0} - C\varepsilon e^{Ct_0}.$$

When we choose $C > \sup_{[0,\frac{D}{2}]\times[0,T]}\{2\psi' - 4\psi\operatorname{tn}_K(s)\}$ which is independent of ε, we get a contradiction.   q.e.d.

3.2. Applications. From the proof of Theorem 3.2, we get the following elliptic version.

Theorem 3.8. *Given $\Omega \subset \mathbb{M}_K^n$ a bounded strict convex domain with diameter D. Assume $K \geq 0$ and $D \leq D_0 < \pi/\sqrt{K}$ when $K > 0$. Let $\phi_1 > 0$ be a first eigenfunction of the Laplacian on Ω with Dirichlet boundary condition associated to the eigenvalue λ_1. Then for $\forall x, y \in \Omega$, with $x \neq y$,*

$$(3.24) \quad \langle \nabla \log \phi_1(y), \gamma'(\tfrac{d}{2}) \rangle - \langle \nabla \log \phi_1(x), \gamma'(-\tfrac{d}{2}) \rangle \leq 2\psi \left(\frac{d(x,y)}{2} \right),$$

where γ is the unit normal minimizing geodesic with $\gamma(-\tfrac{d}{2}) = x$ and $\gamma(\tfrac{d}{2}) = y$, and $\psi : [0, \tfrac{D}{2}] \to \mathbb{R}$ is any C^2 function with $\psi(0) = 0$, $2\psi'(s) - 4\operatorname{tn}_K(s)\psi - (n-1)(K - \operatorname{tn}_K^2(s)) \leq 0$ and

$$(3.25) \quad \begin{aligned} \psi''(s) + 2\psi(s)\psi'(s) - \operatorname{tn}_K(s) \left[(n+1)\psi'(s) + 2\psi^2(s) + 2\lambda_1 \right] \\ - (n-1)(K - \operatorname{tn}_K^2(s))\psi(s) \leq 0. \end{aligned}$$

Remark 3.9. Here we have the assumption $2\psi'(s) - 4\operatorname{tn}_K(s)\psi - (n-1)(K - \operatorname{tn}_K^2(s)) \leq 0$ which is not needed in Theorem 3.2.

In fact, we can formulate for vector fields.

Theorem 3.10. *Given $\Omega \subset \mathbb{M}_K^n$ a bounded strict convex domain with diameter D. Assume $K \geq 0$ and $D \leq D_0 < \pi/\sqrt{K}$ when $K > 0$. Let X be a vector field satisfying*

$$(3.26) \quad \begin{aligned} \Delta X &= 2\nabla_X X + \operatorname{Ric}(X, \cdot) \\ \operatorname{div} X &\leq -|X|^2 - \lambda_1, \ \text{when } K > 0, \end{aligned}$$

$\psi : [0, \tfrac{D}{2}] \to \mathbb{R}$ satisfies the same condition as in Theorem 3.8. Then

$$Z(x,y) := \langle X(y), \gamma'(\tfrac{d}{2}) \rangle - \langle X(x), \gamma'(-\tfrac{d}{2}) \rangle - 2\psi \left(\tfrac{d(x,y)}{2} \right),$$

where γ is the unit normal minimizing geodesic with $\gamma(-\tfrac{d}{2}) = x$ and $\gamma(\tfrac{d}{2}) = y$, $d = d(x,y)$ cannot attain a positive maximum in the interior of $\hat{\Omega}$.

Remark 3.11. This recovers Theorem 2.1 in [22]. Clearly $X = \nabla \log \phi_1$ satisfies (3.26).

To prove Theorem 1.6 we still need the following observation.

Lemma 3.12. *Let Ω be a convex domain in $\mathbb{M}_K^n$ with diameter D, then its first eigenvalue $\lambda_1(\Omega)$ of the Laplacian with Dirichlet boundary conditions has the following lower bound,*

$$\lambda_1 \geq \bar{\lambda}_1(n, K, D).$$

Proof. Let $\Sigma \subset \mathbb{M}_K^n$ be a totally geodesic hypersurface, and $B = [-\frac{D}{2}, \frac{D}{2}] \times \Sigma$, the "infinite strip" with the metric (2.1). Recall $\bar{\phi}_1(s)$ is the first eigenfunction of the "1-dimensional" model with Dirichlet boundary condition. Define a function on B by $v_1(s, z) = \bar{\phi}_1(s)$, then v_1 satisfies the Laplace eigenvalue equation on B with Dirichlet boundary condition. As $\bar{\phi}_1 > 0$ we see that v_1 is the first eigenfunction with eigenvalue $\bar{\lambda}_1$. Since diameter of Ω is D, we have that $\Omega \subset B$. By domain monotonicity $\lambda_1(\Omega) \geq \bar{\lambda}_1(n, K, D)$. q.e.d.

Now we are ready to prove Theorem 1.6.

Proof of Theorem 1.6. Given any $D < \frac{\pi}{2\sqrt{K}}$ when $K > 0$, let $\psi(s) = \left(\log \bar{\phi}_1\right)'$, where $\bar{\phi}_1$ is an even positive first eigenfunction satisfying (2.2) with Dirichlet boundary condition on $[-\frac{D'}{2}, \frac{D'}{2}]$, with $\frac{\pi}{2\sqrt{K}} \geq D' > D$. Then $\psi(\frac{d(x,y)}{2})$ is uniformly continuous on $\overline{\Omega} \times \overline{\Omega}$. By Lemma 3.12 $\lambda_1 \geq \bar{\lambda}_1$ and by (2.17) ψ satisfies (3.25). By (2.18),

$$2\psi' - 4\operatorname{tn}_K(s)\psi - (n-1)(K - \operatorname{tn}_K^2(s))$$
$$= -2\bar{\lambda}_1 - 2\psi^2 + 2(n-3)\operatorname{tn}_K(s)\psi - (n-1)(K - \operatorname{tn}_K^2(s)).$$

Since $s = \frac{D}{2} \leq \frac{\pi}{4\sqrt{K}}$, we have $K - \operatorname{tn}_K^2(s) \geq 0$. Also $\psi \leq 0$ by Lemma 2.2. Hence, when $n \geq 3$, we have $2\psi' - 4\operatorname{tn}_K(s)\psi - (n-1)(K - \operatorname{tn}_K^2(s)) \leq 0$.

When $n = 2$,

$$\psi'(s) - 2\operatorname{tn}_K(s)\psi(s) = -\bar{\lambda}_1 - \left(\psi + \tfrac{1}{2}\operatorname{tn}_K\right)^2 + \tfrac{1}{4}\operatorname{tn}_K^2(s)$$
$$\leq -\bar{\lambda}_1 + \tfrac{1}{4}\operatorname{tn}_K^2(\tfrac{\pi}{4\sqrt{K}}) = -\bar{\lambda}_1 + \tfrac{K}{4}.$$

For $n = 2$, since $D < \frac{\pi}{2\sqrt{K}}$, we have $s \in [0, \frac{\pi}{4\sqrt{K}}]$ and $\frac{1}{\cos^2(\sqrt{K}s)} \leq 2$ in (2.4), and $\bar{\lambda}_1 \geq 4K - \frac{3}{4}K = \frac{13}{4}K$. Hence, $\psi' - 2\operatorname{tn}_K \psi \leq 0$. All the conditions for ψ in Theorem 3.8 are satisfied. Hence, (3.24) holds. Let $D' \to D$ and $K = 1$ finish the proof. q.e.d.

4. Gap comparison

In this section, we use the log-concavity estimates (1.4) to prove the gap estimate (1.1) in Theorems 1.1. To show this we prove the following general gap comparison.

Theorem 4.1. *Let Ω be a bounded convex domain with diameter D in a Riemannian manifold M^n with $\operatorname{Ric}_M \geq (n-1)K$, ϕ_1 a positive first eigenfunction of the Laplacian on Ω with Dirichlet boundary condition. Assume ϕ_1 satisfies the log-concavity estimates*
(4.1)
$$\langle \nabla \log \phi_1(y), \gamma'(\tfrac{d}{2})\rangle - \langle \nabla \log \phi_1(x), \gamma'(-\tfrac{d}{2})\rangle \leq 2\left(\log \bar{\phi}_1\right)'\left(\frac{d(x,y)}{2}\right),$$

where γ is the unit normal minimizing geodesic with $\gamma(-\frac{d}{2}) = x$ and $\gamma(\frac{d}{2}) = y$, and $\bar{\phi}_1 > 0$ is a first eigenfunction of the operator $\frac{d^2}{ds^2} - (n-1)\operatorname{tn}_K(s)\frac{d}{ds}$ on $[-\frac{D}{2}, \frac{D}{2}]$ with Dirichlet boundary condition with $d = d(x,y)$. Then we have the gap comparison

$$(4.2) \qquad \lambda_2 - \lambda_1 \geq \bar{\lambda}_2(n, D, K) - \bar{\lambda}_1(n, D, K).$$

With (1.4), this theorem gives (1.1).

Remark 4.2. This gap comparison can be viewed as a generalization of Neumann gap comparison, as in the Neumann case the first eigenfunction is constant and automatically satisfies the log-concave condition (4.1).

We give two proofs of this theorem. First we give an elliptic proof as in [22], which is a combination of [3, 25], see also [32]. Then we give a parabolic proof as in [3]. In those two papers [3, 22], the result is proven for domains in $\mathbb{R}^n$.

We will need the following "Laplacian comparison" for two points distance function [2, Theorem 3], see also [22, Lemma 7.1]. This statement should be well known to experts as it follows from the first and second variation formulas of the distance function quickly. One finds its importance in Andrews–Clutterbuck's work [2]. As the Laplacian comparison for one point distance function is a very important tool. We present the two points version in Corollary 4.4 so it is easy to use. For completeness we give a proof.

Theorem 4.3. *Let M^n be a Riemannian manifold with $\operatorname{Ric}_M \geq (n-1)K$. Assume that $x, y \in M$ with $d(x,y) = d > 0$, and let $\gamma : [-\frac{d}{2}, \frac{d}{2}] \to M$ be a unit normal minimizing geodesic from x to y. Let $\{e_i\}$ be an orthonormal basis at x and parallel translate it along $\gamma(s)$ with $e_n = \gamma'(s)$. Denote $E_i = e_i \oplus e_i$, $i = 1, \cdots, n-1$. Then*

$$(4.3) \qquad \sum_{i=1}^{n-1} \nabla^2_{E_i, E_i} d|_{(x,y)} \leq -2(n-1)\operatorname{tn}_K\left(\tfrac{d}{2}\right)$$

in the barrier sense. Equality holds if and only if M^n has constant sectional curvature K.

Proof. To begin with assume x, y are not cut point of each other, namely $d(x,y)$ is smooth at (x,y). For each $i \in \{1, \cdots, n-1\}$, let

$$\eta_i(r, s) := \exp_{\gamma(s)}\left(rV_i(s)\right),$$

with $V_i(s) = \frac{\operatorname{cs}_K(s)}{\operatorname{cs}_K(\frac{d}{2})}e_i(s)$. Since $d\left(\eta_i(r, -\frac{d}{2}), \eta_i(r, \frac{d}{2})\right) \leq L[\eta_i(r, \cdot)]$, where $L[\eta_i(r, \cdot)]$ is the length of variation η_i, and equality holds for $r = 0$, it follows that

$$\nabla_{E_i} d|_{(x,y)} = \frac{\partial}{\partial r}L[\eta_i(r, \cdot)]\Big|_{r=0}, \quad \nabla^2_{E_i, E_i} d|_{(x,y)} \leq \frac{\partial^2}{\partial r^2}L[\eta_i(r, \cdot)]\Big|_{r=0}.$$

By the second variation of length formula [12, p. 20]

$$
\frac{\partial^2}{\partial r^2} L[\eta_i(r,s)]\Big|_{r=0}
$$

(4.4)
$$
= \int_{-\frac{d}{2}}^{\frac{d}{2}} \left[\langle V_i', V_i' \rangle - \langle R(e_n, V_i)V_i, e_n \rangle \right] ds + \langle e_n, \nabla_r \frac{\partial \eta_i}{\partial r} \rangle \Big|_{-d/2}^{d/2}
$$
$$
= \int_{-\frac{d}{2}}^{\frac{d}{2}} \left[\frac{(\mathrm{cs}_K'(s))^2}{\mathrm{cs}_K^2(\frac{d}{2})} - \frac{\mathrm{cs}_K^2(s)}{\mathrm{cs}_K^2(\frac{d}{2})} \langle R(e_n, e_i)e_i, e_n \rangle \right] ds.
$$

Summing this from $i = 1$ to $n - 1$, we have

$$
\sum_{i=1}^{n-1} \nabla_{E_i, E_i}^2 d|_{(x,y)}
$$
$$
\leq \sum_{i=1}^{n-1} \frac{\partial^2}{\partial r^2} L[\gamma_i(r, \cdot)]\Big|_{r=0}
$$
$$
= \int_{-\frac{d}{2}}^{\frac{d}{2}} \left[(n-1)\frac{(\mathrm{cs}_K'(s))^2}{\mathrm{cs}_K^2(\frac{d}{2})} - \frac{\mathrm{cs}_K^2(s)}{\mathrm{cs}_K^2(\frac{d}{2})} \mathrm{Ric}(e_n, e_n) \right] ds
$$
$$
\leq \frac{n-1}{\mathrm{cs}_K^2(\frac{d}{2})} \int_{-\frac{d}{2}}^{\frac{d}{2}} \left[(\mathrm{cs}_K'(s))^2 - K\,\mathrm{cs}_K^2(s) \right] ds
$$
$$
= \frac{2(n-1)}{\mathrm{cs}_K^2(\frac{d}{2})} \int_0^{\frac{d}{2}} \left[-K\,\mathrm{sn}_K(s)\,\mathrm{cs}_K'(s) - K\,\mathrm{sn}_K'(s)\,\mathrm{cs}_K(s) \right] ds
$$
$$
= -\frac{2(n-1)K}{\mathrm{cs}_K^2(\frac{d}{2})} \int_0^{\frac{d}{2}} (\mathrm{sn}_K\,\mathrm{cs}_K)' \, ds
$$
$$
= -2(n-1)\,\mathrm{tn}_K\left(\tfrac{d}{2}\right).
$$

If x, y are cut points of each other, $d_\epsilon\left(x, \gamma(\frac{d}{2} - \epsilon)\right)$ is a barrier for $d(x, y)$ and the estimate holds in barrier sense. See [27, Section 3] for the definition of barrier and the relation to other weak senses.   q.e.d.

We can apply this to functions which only depends on the distance.

Corollary 4.4. *Let M^n be a Riemannian manifold with $\mathrm{Ric}_M \geq (n-1)K$. Assume that $x, y \in M$ with $d(x, y) = d > 0$, and let $\gamma : [-\frac{d}{2}, \frac{d}{2}] \to M$ be a unit normal minimizing geodesic from x to y. Let $\{e_i\}$ be an orthonormal basis at x and parallel translate it along $\gamma(s)$ with $e_n = \gamma'(s)$. Denote $E_i = e_i \oplus e_i$, $i = 1, \cdots, n-1$. Then*

(4.5)
$$
\sum_{i=1}^{n-1} \nabla_{E_i, E_i}^2 \varphi(d(x,y)) \leq -2(n-1)\,\mathrm{tn}_K\left(\tfrac{d}{2}\right)\varphi' \qquad \text{if } \varphi' \geq 0,
$$

$$(4.6) \quad \sum_{i=1}^{n-1} \nabla^2_{E_i, E_i} \varphi(d(x,y)) \geq -2(n-1) \operatorname{tn}_K \left(\tfrac{d}{2}\right) \varphi' \qquad \text{if } \varphi' \leq 0$$

in the barrier sense.

Proof of Theorem 4.1 (elliptic proof). Let $w(x) = \dfrac{\phi_2(x)}{\phi_1(x)}$ and $\bar{w}(s) = \dfrac{\bar{\phi}_2(s)}{\bar{\phi}_1(s)}$, where ϕ_i are the first and second eigenfunctions of the Laplacian on Ω with Dirichlet boundary condition, and $\bar{\phi}_i$ are first and second eigenfunctions of the 1-dim model:

$$\begin{cases} \bar{\phi}_i'' - (n-1) \operatorname{tn}_K(s) \bar{\phi}_i' + \bar{\lambda}_i \bar{\phi}_i = 0, \\ \bar{\phi}_i(\pm D/2) = 0 \end{cases}$$

specified as in (2.3). Hence, $\bar{w}(0) = 0$ and $\bar{w}$ is positive on $(0, D/2)$. By direct computation,

$$\nabla w = \frac{\nabla \phi_2}{\phi_1} - w \, \nabla \log \phi_1,$$

$$(4.7) \qquad \Delta w = -(\lambda_2 - \lambda_1) w - 2 \langle \nabla \log \phi_1, \nabla w \rangle,$$

$$(4.8) \qquad \bar{w}' = \frac{\bar{\phi}_2'}{\bar{\phi}_1} - \frac{\bar{\phi}_2 \bar{\phi}_1'}{\bar{\phi}_1^2},$$

$$(4.9) \qquad \bar{w}'' - (n-1) \operatorname{tn}_K(s) \bar{w}' = -(\bar{\lambda}_2 - \bar{\lambda}_1) \bar{w} - 2(\log \bar{\phi}_1)' \bar{w}'.$$

We can extend w to a smooth function on $\overline{\Omega}$ with Neumann condition $\frac{\partial w}{\partial \nu} = 0$ on $\partial \Omega$ [25], same for $\bar{w}$.

Consider the quotient of the oscillations of w and $\bar{w}(s)$ and let

$$Q(x,y) = \frac{w(x) - w(y)}{\bar{w}\left(\frac{d(x,y)}{2}\right)}$$

on $\overline{\Omega} \times \overline{\Omega} \setminus \Delta$, where $\Delta = \{(x,x) | x \in \overline{\Omega}\}$ is the diagonal. Since

$$\lim_{y \to x} Q(x,y) = 2 \frac{\langle \nabla w(x), X \rangle}{\bar{w}'(0)},$$

where $X = \gamma'(0)$ and γ is the unique normal minimal geodesic connecting x to y, we can extend the function Q to the unit sphere bundle $U\Omega = \{(x, X) \mid x \in \bar{\Omega}, \|X\| = 1\}$ as

$$Q(x,X) = \frac{2\langle \nabla w(x), X \rangle}{\bar{w}'(0)}.$$

The maximum of Q then is achieved.

Case 1: the maximum of Q is achieved at (x_0, y_0) with $x_0 \neq y_0$. Denote $d_0 = d(x_0, y_0) > 0$, $m = Q(x_0, y_0) > 0$. At (x_0, y_0), we have $\nabla Q = 0$, $\nabla^2 Q \leq 0$. The Neumann condition $\frac{\partial w}{\partial \nu} = 0$ and strict convexity of Ω forces that both x_0 and y_0 must be in Ω. Indeed, if $x_0 \in \partial \Omega$,

then taking the derivative in the (out) normal direction at x_0, since $\nabla_\nu w|_{x_0} = 0$, we have

$$0 = \nabla_\nu Q(x, y_0)|_{x_0} = -2m \frac{\bar{w}'(\frac{d(x_0, y_0)}{2})}{\bar{w}(\frac{d(x_0, y_0)}{2})} \nabla_\nu d(x, y_0)|_{x=x_0}.$$

Now m, $\bar{w}$ are positive, by Lemma 2.3, $\bar{w}' > 0$, and since Ω is strictly convex, $\nabla_\nu d(x, y_0)|_{x=x_0} > 0$. This is a contradiction.

Let γ be the normal minimal geodesic such that $\gamma(-d_0/2) = x_0$ and $\gamma(d_0/2) = y_0$. Let $e_n := \gamma'$ and extend to an orthonormal basis $\{e_i\}$ by parallel translation along γ. Denote $E_i = e_i \oplus e_i$, $i = 1, \cdots, n$; $E_n = e_n \oplus (-e_n)$.

For $E \in T_x M \oplus T_y M$,

$$(4.10) \qquad \nabla_E Q = \frac{\nabla_E w(x) - \nabla_E w(y)}{\bar{w}} - \frac{(w(x) - w(y))}{\bar{w}^2}(\nabla_E \bar{w}),$$

and

$$\nabla^2_{E,E} Q = \frac{\nabla^2_{E,E} w(x) - \nabla^2_{E,E} w(y)}{\bar{w}} - 2 \frac{\nabla_E w(x) - \nabla_E w(y)}{\bar{w}^2} \nabla_E \bar{w}$$
$$+ 2 \frac{w(x) - w(y)}{\bar{w}^3} \nabla_E \bar{w} \nabla_E \bar{w} - \frac{w(x) - w(y)}{\bar{w}^2} \nabla^2_{E,E} \bar{w}$$

$$(4.11) \qquad = \frac{\nabla^2_{E,E} w(x) - \nabla^2_{E,E} w(y)}{\bar{w}} - \frac{2}{\bar{w}}(\nabla_E Q)(\nabla_E \bar{w}) - \frac{Q}{\bar{w}} \nabla^2_{E,E} \bar{w}.$$

Hence, at (x_0, y_0), we have

$$(4.12) \qquad 0 = \frac{\nabla_E w(x_0) - \nabla_E w(y_0)}{\bar{w}} - \frac{m}{\bar{w}}(\nabla_E \bar{w}),$$

$$(4.13) \qquad 0 \geq \frac{\nabla^2_{E,E} w(x_0) - \nabla^2_{E,E} w(y_0)}{\bar{w}} - \frac{m}{\bar{w}} \nabla^2_{E,E} \bar{w}.$$

We apply these to various directions. From $\nabla_{0 \oplus e_i} Q = \nabla_{e_i \oplus 0} Q = 0$, $i = 1, \cdots, n$ we have

$$\nabla_{e_i} \omega(y_0) = \nabla_{e_i} \omega(x_0) = 0, \ i = 1, \cdots, n-1,$$

$$\nabla_{e_n} \omega(y_0) = \nabla_{e_n} \omega(x_0) = -\frac{m}{2} \bar{w}'(d_0/2).$$

Hence,

$$(4.14) \qquad \nabla \omega(y_0) = \nabla \omega(x_0) = -\frac{m}{2} \bar{w}'(d_0/2) e_n.$$

Adding up $\nabla^2_{E_i, E_i} Q \leq 0$, $i = 1, \cdots, n$, using (4.13), we have

$$(4.15) \qquad 0 \geq \frac{\Delta w(x_0) - \Delta w(y_0)}{\bar{w}} - \frac{m}{\bar{w}} \sum_{i=1}^{n} \nabla^2_{E_i, E_i} \bar{w}.$$

By Lemma 2.3, $\bar{w}' \geq 0$, hence, by (4.5), $\sum_{i=1}^{n-1} \nabla^2_{E_i, E_i} \bar{w} \leq -(n-1) \operatorname{tn}_K \bar{w}'$. As before $\nabla^2_{E_n, E_n} \bar{w} = \bar{w}''$. As $m, \bar{w}$ are positive, plug these and (4.7) to (4.15), and using (4.14), (4.9), we have

$$0 \geq -(\lambda_2 - \lambda_1)m + 2\frac{\langle \nabla \log \phi_1, \nabla w(y_0) \rangle - \langle \nabla \log \phi_1, \nabla w(x_0) \rangle}{\bar{w}}$$
$$- m\frac{\bar{w}'' - (n-1)\operatorname{tn}_K \bar{w}'}{\bar{w}}$$
$$= -(\lambda_2 - \lambda_1)m$$
$$- m\bar{w}'\frac{(\langle \nabla \log \phi_1(y_0), e_n \rangle - \langle \nabla \log \phi_1(x_0), e_n \rangle) - 2(\log \bar{\phi}_1)'}{\bar{w}}$$
$$+ (\bar{\lambda}_2 - \bar{\lambda}_1)m.$$

Using (1.4), we have $0 \geq -(\lambda_2 - \lambda_1)m + (\bar{\lambda}_2 - \bar{\lambda}_1)m$, which is (1.1).

Case 2: the maximum of Q is attained at some $(x_0, X_0) \in U\Omega$. By Cauchy–Schwarz inequality, the corresponding maximal direction is $X_0 = \frac{\nabla w}{\|\nabla w\|}$ so that the maximum value is $m = \frac{2\|\nabla w\|}{\bar{w}'(0)}$. Furthermore, $\|\nabla w(x_0)\| \geq \|\nabla w(x)\|$ for any $x \in \bar{\Omega}$. Suppose $x_0 \in \partial\Omega$, then by (strict) convexity,

$$0 = \nabla_\nu \|\nabla w\|^2|_{x_0} = -II(\nabla w, \nabla w)|_{x_0} < 0,$$

a contradiction. Hence, $x_0 \in \Omega$. Now let $e_n := \frac{\nabla w}{\|\nabla w\|}$ and complete to an orthonormal frame $\{e_i\}$ at x_0. We further parallel translate to a neighborhood of x_0. In such a frame we have

$$\nabla_n w = \langle \nabla w, e_n \rangle = \|\nabla w\|,$$

and

$$\nabla_i w = \langle \nabla w, e_i \rangle = 0, \quad i = 1, \ldots, n-1.$$

At the maximal point x_0, we have the first derivative vanishing

$$0 = \nabla \|\nabla w\|^2 = 2\langle \nabla \nabla w, \nabla w \rangle = 2\|\nabla w\|\nabla_n \nabla w,$$

and the second derivative non-positive

$$0 \geq \nabla_k \nabla_k \|\nabla w\|^2 = 2\left(\langle \nabla_k \nabla_k \nabla w, \nabla w \rangle + \|\nabla_k \nabla w\|^2\right)$$
$$\geq 2\langle \nabla_k \nabla_k \nabla w, \nabla w \rangle = 2\|\nabla w\|\langle \nabla_k \nabla_k \nabla w, e_n \rangle.$$

In short

(4.16) $$0 \geq \langle \nabla_k \nabla_k \nabla w, e_n \rangle, \quad k = 1, \ldots, n-1.$$

Now let

$$x(s) := \exp_{x_0}(s\, e_n),$$
$$y(s) := \exp_{x_0}(-s\, e_n),$$
$$g(s) := Q(x(s), y(s)).$$

By construction, since the variations are approaching x_0 in the e_n direction, we have

$$m = Q(x_0, e_n(x_0)) = g(0) \geq g(s), \quad \text{for all } s \in (-\varepsilon, \varepsilon),$$

and so $\lim_{s \to 0} g'(s) = 0$ and $\lim_{s \to 0} g''(s) \leq 0$. By (4.10), (4.11)

$$g'(s) = \frac{\langle \nabla w, x'(s) \rangle - \langle \nabla w, y'(s) \rangle}{\bar{w}(s)} - \frac{g(s)}{\bar{w}(s)} \bar{w}',$$

and

(4.17)
$$g''(s)$$
$$= \frac{\langle \nabla_s \nabla w(x(s)), x'(s) \rangle + \langle \nabla w(x(s)), x''(s) \rangle - \langle \nabla_s \nabla w(y), y'(s) \rangle - \langle \nabla w, y''(s) \rangle}{\bar{w}}$$
$$- 2g'(s) \frac{\bar{w}'}{\bar{w}} - g(s) \left(\frac{\bar{w}''}{\bar{w}} \right).$$

From (4.9)

$$\frac{\bar{w}''}{\bar{w}} = (n-1) \operatorname{tn}_K(s) \frac{\bar{w}'}{\bar{w}} - (\bar{\lambda}_2 - \bar{\lambda}_1) - 2(\log \bar{\phi}_1)' \frac{\bar{w}'}{\bar{w}}.$$

Using the fact that $\bar{\phi}_2(0) = 0$, $\bar{\phi}_2'(0) \neq 0$ and $\bar{\phi}_1'(0) = 0, \bar{\phi}_1(0) \neq 0$, when $s \to 0$, we have

$$\lim_{s \to 0} \frac{\bar{w}''}{\bar{w}} = K(n-1) - (\bar{\lambda}_2 - \bar{\lambda}_1) + 2\bar{\lambda}_1,$$

and

$$\lim_{s \to 0} g'(s) \frac{\bar{w}'}{\bar{w}} = g''(0).$$

Since

$$x''(s) = \frac{d}{ds} x'(s) = \nabla_{x'(s)} x'(s) = 0,$$

letting $s \to 0$ in (4.17), we have

$$0 \geq 2 \frac{\langle \nabla_n \nabla_n \nabla w, e_n \rangle}{\tilde{w}'(0)} + m[(\bar{\lambda}_2 - \bar{\lambda}_1) - K(n-1) - 2\bar{\lambda}_1].$$

Combining this with (4.16), we have

$$0 \geq 2 \frac{\langle \Delta(\nabla w), e_n \rangle}{\bar{w}'(0)} + m[(\bar{\lambda}_2 - \bar{\lambda}_1) - K(n-1) - 2\bar{\lambda}_1].$$

Use the Bochner–Weitzenböck formula (3.19), we have

$$0 \geq 2 \frac{\langle \nabla(\Delta w), e_n \rangle + \operatorname{Ric}(\nabla w, e_n)}{\bar{w}'(0)} + m[(\bar{\lambda}_2 - \bar{\lambda}_1) - K(n-1) - 2\bar{\lambda}_1].$$

Inserting in (4.7), we have

$$0 \geq 2 \frac{\langle \nabla(-(\lambda_2 - \lambda_1)w - 2\langle \nabla \log \phi_1, \nabla w \rangle), e_n \rangle + \mathrm{Ric}(\nabla w, e_n)}{\bar{w}'(0)}$$
$$+ m[(\bar{\lambda}_2 - \bar{\lambda}_1) - K(n-1) - 2\bar{\lambda}_1]$$
$$= [-2(\lambda_2 - \lambda_1) - 4\langle \nabla_n \nabla \log \phi_1, e_n \rangle + 2\mathrm{Ric}(e_n, e_n)] \frac{\|\nabla w\|}{\bar{w}'(0)}$$
$$+ m[(\bar{\lambda}_2 - \bar{\lambda}_1) - K(n-1) - 2\bar{\lambda}_1].$$

By (1.5) we have $-\nabla^2 \log \phi_1 \geq \bar{\lambda}_1 \mathrm{id}$. Since $m = \frac{2\|\nabla w\|}{\bar{w}'(0)}$ and $\mathrm{Ric} \geq (n-1)K$, we get

$$\lambda_2 - \lambda_1 \geq \bar{\lambda}_2 - \bar{\lambda}_1. \qquad \text{q.e.d.}$$

For the parabolic method of proof, we first prove the following theorem which is similar to Theorem 2.1 in [3]. Recall

Definition 4.5. A function ω is a modulus of contraction for vector field X if for every $x \neq y$ in Ω

$$\langle X(y), \gamma' \rangle - \langle X(x), \gamma' \rangle \leq 2\omega \left(\frac{d(x,y)}{2} \right),$$

where γ is the unit normal minimizing geodesic with $\gamma(-\frac{d}{2}) = x$ and $\gamma(\frac{d}{2}) = y$, $d = d(x,y)$.

Theorem 4.6. *Let Ω be a strictly convex domain of diameter D with smooth boundary in a Riemannian manifold M^n with $\mathrm{Ric} \geq (n-1)K$, and X a time-dependent vector field on Ω. Suppose $v : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ is a smooth solution of the equation with Neumann boundary condition,*

$$\begin{cases} \frac{\partial v}{\partial t} = \Delta v + X \cdot \nabla v & in \ \Omega \times \mathbb{R}_+; \\ \nabla_\nu v = 0 & in \ \partial\Omega \times \mathbb{R}_+. \end{cases}$$

Suppose that

1) *$X(\cdot, t)$ has modulus of contraction $\omega(\cdot, t)$ for each $t \geq 0$, where $\omega : [0, D/2] \times \mathbb{R}_+ \to \mathbb{R}$ is smooth.*
2) *$v(\cdot, 0)$ has modulus of continuity φ_0, where $\varphi_0 : [0, D/2] \to \mathbb{R}$ is smooth with $\varphi_0(0) = 0$ and $\varphi_0'(z) > 0$ for $0 \leq z \leq D/2$.*
3) *$\varphi : [0, D/2] \times \mathbb{R}_+ \to \mathbb{R}$ satisfies*
   a) *$\varphi(z, 0) = \varphi_0(z)$ for each $z \in [0, D/2]$,*
   b) *$\frac{\partial\varphi}{\partial t} \geq \varphi'' - (n-1)\varphi' \mathrm{tn}_K + \omega\varphi'$ on $[0, D/2] \times \mathbb{R}_+$,*
   c) *$\varphi' > 0$ on $[0, D/2] \times \mathbb{R}_+$,*
   d) *$\varphi(0, t) \geq 0$ for each $t \geq 0$.*

Then $\varphi(\cdot, t)$ is a modulus of continuity for $v(\cdot, t)$ for each $t \geq 0$.

Proof. For any $\varepsilon \geq 0$, define

$$Z_\varepsilon(y, x, t) = v(y, t) - v(x, t) - 2\varphi \left(\frac{d(x,y)}{2}, t \right) - \varepsilon e^t.$$

By assumption, $Z_\varepsilon(y, x, 0) \le -\epsilon$ for every $x \ne y$ in Ω, and $Z_\varepsilon(x, x, t) \le -\varepsilon$ for every $x \in \Omega$ and $t \ge 0$. We will prove for every $\varepsilon > 0$, $Z_\varepsilon < 0$ on $\Omega \times \Omega \times \mathbb{R}_+$. If not, then there exists first time $t_0 > 0$ and $x_0 \ne y_0 \in \bar{\Omega}$ such that $Z_\varepsilon(x_0, y_0, t_0) = 0$. If $y_0 \in \partial\Omega$, then, by the Neumann boundary condition,

$$\nabla_{\nu_y} Z_\varepsilon = -\varphi' \nabla_{\nu_y} d,$$

where ν_y is the outward unit normal at y. By strict convexity, we have $\nabla_{\nu_y} d > 0$. With assumption (c), we have $\nabla_{\nu_y} Z_\varepsilon < 0$. This implies $Z(x_0, y_s, t_0) > 0$ for y_s near y_0 in the normal direction, which is a contradiction to $Z_\varepsilon \le 0$ on $\bar{\Omega} \times \bar{\Omega} \times [0, t_0]$.

Assume now that x_0, y_0 are interior points of Ω. Let $\gamma(s)$ be the unit normal minimizing geodesic such that $\gamma(-\frac{d_0}{2}) = x_0$ and $\gamma(\frac{d_0}{2}) = y_0$, where $d_0 = d(x_0, y_0)$. Choose a local orthonormal frame $\{e_i\}$ at x_0 such that $e_n = \gamma'(-\frac{d_0}{2})$ and parallel translate them along γ. Let $E_i = e_i \oplus e_i \in T_{(x_0, y_0)} \Omega \times \Omega$ for $1 \le i \le n-1$, and $E_n = e_n \oplus (-e_n)$. From the vanishing of first variation, we have

(4.18) $$\nabla v_{y_0} = \nabla v_{x_0} = \varphi' e_n.$$

Using maximum principle and (4.5), we obtain

$$0 \ge \sum_{i=1}^n \nabla^2_{E_i, E_i} Z_\varepsilon \Big|_{(x_0, y_0, t_0)}$$
$$\ge \Delta v(y_0, t_0) - \Delta v(x_0, t_0) - 2\varphi'' + 2(n-1)\varphi' \operatorname{tn}_K(d_0/2).$$

Therefore, combining these,

$$0 \le \frac{\partial}{\partial t} Z_\varepsilon$$
$$= \Delta v(y_0, t_0) - \Delta v(x_0, t_0) + \varphi'(\langle X(y_0), \gamma' \rangle$$
$$\quad - \langle X(x_0), \gamma' \rangle) - 2\frac{\partial \varphi}{\partial t} - \varepsilon e^{t_0}$$
$$\le 2\varphi'' - 2(n-1)\varphi' \operatorname{tn}_K(d_0/2) + 2\omega \varphi' - 2\frac{\partial \varphi}{\partial t} - \varepsilon e^{t_0} < 0,$$

which is a contradiction.   q.e.d.

With above theorem, we derive another proof of Theorem 4.1, the gap comparison. The proof is a minor modification from the one by Ni in [22].

Proof of Theorem 4.1 (parabolic proof). Let $v(x, t) = e^{-(\lambda_2 - \lambda_1)t} \frac{\phi_2}{\phi_1}$ with (ϕ_i, λ_i) the i-th eigenpair of the Laplacian. From (4.7), we can see that v satisfies the required heat equation with $X = \nabla \log \phi_1$. Let $\varphi(s, t) = C e^{-(\bar{\lambda}_2 - \bar{\lambda}_1)t} \frac{\bar{\phi}_2}{\bar{\phi}_1}$, with $(\bar{\phi}_i, \bar{\lambda}_i)$ the i-th eigenpair of the model, and let $\omega(s) = (\log \bar{\phi}_1)'$. Denote $\frac{d}{ds}$ by $'$. From (2.7), we see that φ satisfies the required differential inequality. By assumption, ω is a modulus

of contraction for $\nabla \log \phi_1$. By Lemma 2.3, we know that $\varphi' \geq 0$. To apply Theorem 4.6, we want φ' strictly positive, however, at the boundary $\varphi'(\frac{D}{2})$ may be zero (which can be seen explicitly when $K = 0$). We can overcome this by considering a larger domain $D_\varepsilon > D$. Let $\bar{\phi}_i^{D_\varepsilon}$ be the corresponding eigenfunctions on $[-\frac{D_\varepsilon}{2}, \frac{D_\varepsilon}{2}]$ and $\omega^{D_\varepsilon} = (\log \bar{\phi}_1^{D_\varepsilon})'$. Note that $\bar{\lambda}_1 \geq \bar{\lambda}_1^{D_\varepsilon}$. From (2.18), ω and ω^{D_ε} satisfy

$$\begin{cases} \omega' = -\omega^2 + (n-1)\operatorname{tn}_K(s)\omega - \bar{\lambda}_1, \\ \omega(0) = 0, \end{cases}$$

and

$$\begin{cases} (\omega^{D_\varepsilon})' = -(\omega^{D_\varepsilon})^2 + (n-1)\operatorname{tn}_K(s)\omega^{D_\varepsilon} - \bar{\lambda}_1^{D_\varepsilon}, \\ \omega^{D_\varepsilon}(0) = 0, \end{cases}$$

so by ODE comparison, we have $\omega \leq \omega^{D_\varepsilon}$. Hence, ω^{D_ε} is a modulus of contraction for X. Furthermore, the corresponding φ^{D_ε} will be strictly increasing on $[0, \frac{D}{2}]$. Hence, applying Theorem 4.6, we have

$$e^{-(\lambda_2 - \lambda_1)t}\left(\frac{\phi_2(y)}{\phi_1(y)} - \frac{\phi_2(x)}{\phi_1(x)}\right) \leq Ce^{-(\bar{\lambda}_2^{D_\varepsilon} - \bar{\lambda}_1^{D_\varepsilon})t}\frac{\bar{\phi}_2^{D_\varepsilon}}{\bar{\phi}_1^{D_\varepsilon}}\bigg|_{d(x,y)/2}.$$

Letting $D^\varepsilon \to D$, we have $\lambda_2 - \lambda_1 \geq \bar{\lambda}_2 - \bar{\lambda}_1$. q.e.d.

As another application of (1.4), we obtain an estimate on the first Dirichlet eigenvalue of the Laplacian on convex domains in sphere..

Proposition 4.7. *Let Ω be a strictly convex domain with diameter D in $\mathbb{M}_K^n$ with $K \geq 0$, $D < \frac{\pi}{2\sqrt{K}}$ when $K > 0$. Then the first two Dirichlet eigenvalues of the Laplacian on Ω satisfy*

$$\lambda_1 \geq n\bar{\lambda}_1 \geq \max\left\{\frac{n\pi^2}{D^2} - \frac{n(n-1)}{2}K, 0\right\} \quad \text{when } n \geq 3,$$

$$\lambda_2 \geq n\bar{\lambda}_1 + \frac{3\pi^2}{D^2} \geq \max\left\{(n+3)\frac{\pi^2}{D^2} - \frac{n(n-1)}{2}K, 0\right\} \quad \text{when } n \geq 3.$$

When $K = 0$, this recovers Corollary 7.4 in [22] in the case $q(x) = 0$. Our proof is also similar.

Proof. Let ϕ_1 be a positive eigenfunction associated to the eigenvalue λ_1. Then ϕ_1 attains the maximum at an interior point $x_0 \in \Omega$ and $\nabla\phi_1(x_0) = 0$. For r small, let $B_{x_0}(r) \subset \Omega$ be a geodesic ball centered at x_0 with radius r and $\gamma : [0, r] \to \Omega$ be a normalized geodesic from x_0 to $x \in \partial B_{x_0}(r)$. Integrating (1.4) over $\partial B_{x_0}(r)$ and applying the

divergence theorem, we have

$$2\text{vol}(\partial B_{x_0}(r))\left(\log\bar{\phi}_1(\tfrac{r}{2})\right)'$$

$$\geq \int_{\partial B_{x_0}(r)}\left[\langle\nabla\log\phi_1(x),\gamma'(r)\rangle - \langle\nabla\log\phi_1(x_0),\gamma'(0)\rangle\right]dA(x)$$

$$= \int_{\partial B_{x_0}(r)}\nabla\log\phi_1(x)\cdot\nu_x\,dA(x)$$

$$= \int_{B_{x_0}(r)}\text{div}\left(\nabla\log\phi_1(x)\right)\text{dvol}$$

$$= -\lambda_1\text{vol}(B_{x_0}(r)) - \int_{B_{x_0}(r)}|\nabla\log\phi_1|^2\text{dvol}.$$

Then

$$\lambda_1 \geq -2\frac{\text{vol}(\partial B_{x_0}(r))}{\text{vol}(B_{x_0}(r))}\left(\log\bar{\phi}_1(\tfrac{r}{2})\right)' - \frac{1}{\text{vol}(B_{x_0}(r))}\int_{B_{x_0}(r)}|\nabla\log\phi_1|^2\text{dvol}.$$

Let $r \to 0$ in the right hand side above, using $\nabla\phi_1(x_0) = 0$, the second term in the right hand of above inequality is zero. Also

$$-2\lim_{r\to 0}\frac{\text{vol}(\partial B_{x_0}(r))}{\text{vol}(B_{x_0}(r))}\left(\log\bar{\phi}_1(\tfrac{r}{2})\right)' = -2\lim_{r\to 0}\frac{\omega_{n-1}r^{n-1}}{\int_0^r\omega_{n-1}s^{n-1}ds}\left(\log\bar{\phi}_1(\tfrac{r}{2})\right)'$$

$$= -n\lim_{r\to 0}\frac{\left(\log\bar{\phi}_1(\tfrac{r}{2})\right)'}{\tfrac{r}{2}}$$

$$= n\bar{\lambda}_1,$$

where we used (2.18) and $\bar{\phi}_1'(0) = 0$. Combining this with (2.5) and Theorem 1.1 gives the result.   q.e.d.

Appendix A. Numerical computation of the gap for the sphere model

We define the normalized gap to be

$$\frac{D^2}{\pi^2}\left(\bar{\lambda}_2 - \bar{\lambda}_1\right).$$

Below we give the numerically computed values for the normalized gap for different diameter D and different dimension n. We used the finite difference method on the operator

$$\frac{d^2}{ds^2} - \frac{(n-1)}{4}\left(\frac{n-3}{\cos^2(s)} - (n-1)\right).$$

Namely, the operator in (2.4) for $K = 1$.

Here we can see that the normalized gap is decreasing for $n = 2$ and increasing for $n = 4$.

Table 1. The normalize gap when D increases

D	gap for $n = 2$	gap for $n = 4$
0.5	2.9999262845	3.0001717986
1.5	2.9940610569	3.0177628990
2.1	2.9713788083	3.0854596183
3.1	2.5564359813	3.8997988823
3.14	2.3138191920	3.9959197251
3.141	2.2836242932	3.9984557650
3.14159	2.2582889873	3.9999546561

Table 2. The normalize gap for increasing n and $D = 1.57 < \frac{\pi}{2}$

n	normalized gap
2	2.99272656
3	2.99998766
4	3.02176274
5	3.05802504
6	3.10872300
7	3.17377060
8	3.25303530
9	3.34632483

Here we see that the gap is increasing, however, not linearly. Note that when $n = 3$, we can explicitly compute the normalized gap as 3, and the difference is due to the program's rounding error.

Appendix B. Explicit variation formula

In this section, we will use notions from Section 2. First we introduce the models we will use for $\mathbb{M}_K^n$, the simply connected space with constant curvature K, and review some basic facts about geodesics in these models.

For $K > 0$, $\mathbb{M}_K^n \subset \mathbb{R}^{n+1}$ is the set given by the equation

$$x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = \tfrac{1}{K}.$$

For $K < 0$, $\mathbb{M}_K^n \subset \mathbb{R}^{n+1}$ is the set given by the equation

$$x_1^2 + x_2^2 + \cdots + x_n^2 - x_{n+1}^2 = \tfrac{1}{K},$$

with $x_{n+1} > 0$.

With these models we have the following representation for geodesics. Namely for any $x \in \mathbb{M}_K^n$ and unit vector $v \in T_x\mathbb{M}_K^n$, the geodesic start from x in the direction v is given by

(B.1) $$\exp_x rv = \operatorname{cs}_K(r)\, x + \operatorname{sn}_K(r)\, v.$$

Given any two points $x, y \in \mathbb{M}_K^n$, the geodesic connecting x, y is given by intersection $\mathbb{M}_K^n$ with the plane containing the origin and x, y, and its distance is given by the following

(B.2) $$\tfrac{1}{K} \operatorname{cs}_K(d(x, y)) = \langle x, y \rangle,$$

where one uses the Lorentz metric for the inner product when $K < 0$.

Now we are ready to construct the variation explicitly. Let $\gamma(s)$ be a unit speed geodesic on $\mathbb{M}_K^n$ such that $\gamma(-\frac{d_0}{2}) = x_0$ and $\gamma(\frac{d_0}{2}) = y_0$ (with $d_0 < \frac{\pi}{\sqrt{K}}$ when $K > 0$). Let $e_n := \gamma'(-\frac{d_0}{2})$ and $e_i \in T_{x_0}\mathbb{M}_K^n$ a unit vector which is perpendicular to e_n. Then parallel translate e_i, e_n along $\gamma(s)$ so that $e_n(s) = \gamma'(s)$. We now construct an explicit geodesic variation of $\gamma(s)$ in the e_i direction. Then construct two geodesics $\sigma_{x_0}, \sigma_{y_0}$ at the endpoints of $\gamma(s)$ such that $\sigma_{x_0}(0) = x_0$ and $\sigma_{y_0}(0) = y_0$, with initial condition $\sigma'_{x_0}(0) = \sigma'_{y_0}(0) = e_i$. By (B.1),

$$\sigma_{x_0}(r) = \operatorname{cs}_K(r)\, x_0 + \operatorname{sn}_K(r)\, e_i := p(r),$$
$$\sigma_{y_0}(r) = \operatorname{cs}_K(r)\, y_0 + \operatorname{sn}_K(r)\, e_i := q(r).$$

Let $d_r := d(p(r), q(r))$. Then

(B.3) $$\langle p, q \rangle = \tfrac{1}{K} \operatorname{cs}_K(d_r) = \tfrac{1}{K} \operatorname{cs}_K(d_0) \operatorname{cs}_K^2(r) + \operatorname{sn}_K^2(r).$$

For each r, let $f_r(s)$ denote the geodesic such that $f_r(-\frac{d_r}{2}) = p(r)$ and $f_r(\frac{d_r}{2}) = q(r)$. To obtain an explicit formula for $f_r(s)$, we use the fact that on $\mathbb{M}_K^n$, the geodesics are characterized by intersections with the plane containing the origin and the endpoints. Hence, the direction of the geodesic connecting p and q is the orthogonal projection of the vector $q - p$ onto p. Namely

$$q - p - (q - p) \cdot p \frac{p}{\|p\|^2} = q - K(\langle p, q \rangle)\, p.$$

Denote its unit vector by

$$U(r) = \frac{1}{\sqrt{K^{-1} - K(\langle p, q \rangle)^2}} \left[q - K(\langle p, q \rangle)\, p \right].$$

Then

$$f_r(s) = \exp_{\sigma_{x_0}(r)}(s + \tfrac{d_r}{2})\, U(r) = \operatorname{cs}_K(s + \tfrac{d_r}{2}) p + \operatorname{sn}_K(s + \tfrac{d_r}{2})\, U(r).$$

Reparametrize by letting $V(r) = \frac{d_r}{d_0} U(r)$. Then
(B.4)
$$\eta(r, s) = \exp_{\sigma_{x_0}(r)}(s + \tfrac{d_0}{2})\, V(r) = \operatorname{cs}_K(\tfrac{d_r}{d_0}s + \tfrac{d_r}{2}) p + \operatorname{sn}_K(\tfrac{d_r}{d_0}s + \tfrac{d_r}{2})\, U(r)$$

is the variation of geodesics. Namely for each fixed r, $\eta(r, s)$ is the geodesic such that $\eta(r, -\frac{d_0}{2}) = \sigma_{x_0}(r)$ and $\eta(r, \frac{d_0}{2}) = \sigma_{y_0}(r)$.

To compute the expansion in r up to the second order term, we compute out the expansion for the following terms:

$$d_r = d_0 - \operatorname{tn}_K(\tfrac{d_0}{2})r^2 + O(r^4),$$

$$p(r) = \left(1 - \tfrac{Kr^2}{2} + O(r^4)\right) x_0 + \left(r + O(r^3)\right) e_i,$$

$$q(r) = \left(1 - \tfrac{Kr^2}{2} + O(r^4)\right) y_0 + \left(r + O(r^3)\right) e_i,$$

$$\langle p, q \rangle = K^{-1} \operatorname{cs}_K(d_0) + (1 - \operatorname{cs}_K(d_0))r^2 + O(r^4),$$

$$\frac{1}{\sqrt{K^{-1} - K(\langle p, q \rangle)^2}} = \frac{1}{\operatorname{sn}_K(d_0)} + \frac{1}{\operatorname{tn}_K(d_0)(\operatorname{cs}_K(d_0) + 1)} K^2 r^2 + O(r^4).$$

Using this expansion, we have $V(0) = e_n$, $\nabla_r V(0) = \operatorname{tn}_K(\tfrac{d_0}{2})e_i$, and $\nabla_r \nabla_r V(0) = (-\tfrac{2}{d_0} \operatorname{tn}_K(\tfrac{d_0}{2}) - \operatorname{tn}_K^2(\tfrac{d_0}{2}))e_n$. Therefore,

$$(\text{B.5}) \qquad \nabla_r \nabla_r \partial_s \eta(0, \pm\tfrac{d_0}{2}) = \left(-\tfrac{2}{d_0} \operatorname{tn}_K(\tfrac{d_0}{2}) - \operatorname{tn}_K^2(\tfrac{d_0}{2})\right) e_n.$$

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