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PROPERLY IMMERSED SURFACES IN HYPERBOLIC 3-MANIFOLDS

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Abstract

We study complete finite topology immersed surfaces Σ in complete Riemannian 3-manifolds N with sectional curvature $K_N \leq -a^2 \leq 0$, such that the absolute mean curvature function of Σ is bounded from above by a and its injectivity radius function is not bounded away from zero on each of its annular end representatives. We prove that such a surface Σ must be proper in N and its total curvature must be equal to $2\pi\chi(\Sigma)$. If N is a hyperbolic 3-manifold of finite volume and Σ is a properly immersed surface of finite topology with nonnegative constant mean curvature less than 1, then we prove that each end of Σ is asymptotic (with finite positive integer multiplicity) to a totally umbilic annulus, properly embedded in N.

1. Introduction

In the celebrated paper [**4**], Colding and Minicozzi proved that complete minimal surfaces of finite topology embedded in $\mathbb{R}^3$ are proper. Based on the proof of this result, Meeks and Rosenberg [**15**] showed that complete, connected minimal surfaces with positive injectivity radius embedded in $\mathbb{R}^3$ are proper. Meeks and Tinaglia [**16**] then extended both results by proving that complete surfaces with constant mean curvature embedded in $\mathbb{R}^3$ are proper if they have finite topology or positive injectivity radius. It is natural to ask to what extent similar properness results hold for complete surfaces of finite topology in other ambient spaces, where the surfaces are not necessarily embedded or have constant mean curvature.

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In this paper we investigate relationships between properness of a complete immersed surface of finite topology in a 3-manifold of nonpositive sectional curvature under certain restrictions on its injectivity radius function and on its mean curvature function. We first derive area estimates for certain compact surfaces with bounded absolute mean curvature in Hadamard 3-manifolds. We next apply these area estimates to obtain sufficient conditions for a complete finite topology surface immersed in a complete 3-manifold with nonpositive sectional curvature to be proper, as described in Theorem 1.2 below. In order to state this result we need the following definition.

Definition 1.1. Let e be an end of a complete Riemannian surface Σ whose injectivity radius function we denote by $I_\Sigma\colon \Sigma \to (0,\infty)$. Let $\mathcal{E}(e)$ be the collection of proper subdomains $E \subset \Sigma$, with compact boundary, that represents e. We define the (lower) *asymptotic injectivity radius of* e by

$$I_\Sigma^\infty(e) = \sup\{\inf_E I_\Sigma|_E \mid E \in \mathcal{E}(e)\} \in [0,\infty].$$

Note that if Σ has an end e which admits a one-ended representative E, then $I_\Sigma^\infty(e) = \liminf_E I_\Sigma|_E$. If Σ has finite topology, then every end of Σ has a one-ended representative which is an annulus (i.e., a surface with the topology of $\mathbb{S}^1 \times [0,\infty)$), hence, this simpler definition can be used. Moreover, Lemma 5.1 in the Appendix shows that if Σ has nonpositive Gaussian curvature and an end e has a representative E which is an annulus, then for every divergent sequence of points $\{p_n\}_{n\in\mathbb{N}}$ on E,

$$\lim_{n\to\infty} I_\Sigma(p_n) = I_\Sigma^\infty(e).$$

We define the *mean curvature function* of an immersed two-sided surface Σ with a given unit normal field in a Riemannian 3-manifold to be the pointwise average of its principal curvatures; note that if Σ does not have a unit normal field, then the absolute value $|H_\Sigma|$ of the mean curvature function of Σ still makes sense because a unit normal field locally exists on Σ and under a change of this local choice, the principal curvatures change sign.

Theorem 1.2. *Let N be a complete 3-manifold with sectional curvature $K_N \leq -a^2 \leq 0$, for some $a \geq 0$. Let $\varphi\colon \Sigma \to N$ be an isometric immersion of a complete surface Σ with finite topology, whose mean curvature function satisfies $|H_\varphi| \leq a$. Then Σ has nonpositive Gaussian curvature and the following hold:*

A. If N is simply connected, then $I_\Sigma^\infty(e) = \infty$ for every end e of Σ.
B. If N has positive injectivity radius $\mathrm{Inj}(N) = \delta > 0$, then every end e of Σ satisfies $I_\Sigma^\infty(e) \geq \delta$. In particular, Σ has positive injectivity radius.

C. If I_Σ is bounded, then Σ has finite total curvature

$$\int_\Sigma K_\Sigma = 2\pi\chi(\Sigma),$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ. Furthermore, for each annular end representative E of Σ, the induced map $\varphi_*\colon \pi_1(E) \to \pi_1(N)$ on fundamental groups is injective.

D. If $I_\Sigma^\infty(e) = 0$ for each end e of Σ, then φ is proper.

The other main theorem of the paper, Theorem 1.3 below, describes, among other things, results on the asymptotic behavior of complete, properly immersed finite topology surfaces of constant absolute mean curvature $H \in [0, 1)$ in hyperbolic 3-manifolds of finite volume; througout the paper, the term *hyperbolic 3-manifold of finite volume* will refer to noncompact examples. Our asymptotic description of these surfaces was inspired by Theorem 1.1 of Collin, Hauswirth and Rosenberg [**5**] who obtained it in the special case that $H = 0$.

For any connected, noncompact, orientable surface of finite topology S different from an annulus or a plane, there exists a hyperbolic 3-manifold N_S of finite volume that admits a properly embedded surface Σ, that is totally geodesic in N_S, homeomorphic to S and such that each end of N_S contains at most one end of Σ; then a "t-parallel" surface to Σ is a properly immersed surface of constant mean curvature $H(t) = \tanh(t)$. As t ranges from 0 to ∞, this gives examples with all the possible mean curvatures $H \in [0, 1)$. Moreover, for t sufficiently small, such parallel surfaces can be shown to be embedded, see [**1**].

Theorem 1.3. *Let N be a complete, noncompact hyperbolic 3-manifold of finite volume and $H \in [0, 1)$. Let Σ be a complete, properly immersed surface in N with $|H_\Sigma| \leq H$. Then:*

1) Σ has finite area and a finite number of connected components.

2) For any divergent sequence of points $\{p_n\}_{n\in\mathbb{N}} \subset \Sigma$, $\lim_{n\to\infty} I_\Sigma(p_n) = 0$.

3) Σ has total curvature

$$\int_\Sigma K_\Sigma = 2\pi\chi(\Sigma). \tag{1}$$

4) If Σ has infinite topology, then the norm of its second fundamental form is unbounded.

5) If $H_\Sigma = H$, then every annular end representative of an end of Σ is asymptotic (with finite multiplicity) in the C^2-norm, to a totally umbilic annulus properly embedded in N. In particular, if Σ has finite topology, then the norm of the second fundamental form of Σ is bounded.

The totally umbilic annuli described in the item 5 of last theorem are properly embedded annular ends in N whose lifts to hyperbolic 3-space are contained in equidistant surfaces to totally geodesic planes; see the discussion in Section 4.3 for a complete description.

An immediate consequence of item 2 of Theorem 1.3 and of item D of Theorem 1.2 is the following corollary.

Corollary 1.4. *Let N be a hyperbolic manifold of finite volume and let $H \in [0,1)$. Then a complete immersed surface Σ in N with finite topology and mean curvature function $|H_\Sigma| \leq H$ is proper if and only $I^\infty_\Sigma(e) = 0$ for each end e of Σ.*

We remark that the hypothesis $H < 1$ in the statement of Theorem 1.3 is a necessary one. Proposition 4.8 shows that for any $H \geq 1$, every complete hyperbolic 3-manifold N of finite volume admits a properly immersed, complete annulus Σ of constant mean curvature H with positive injectivity radius and infinite area.

The paper is organized as follows. In Section 2, we prove isoperimetric inequalities for certain compact surfaces with boundary in Hadamard 3-manifolds. These isoperimetric inequalities yield area estimates which are applied in Section 3 to prove Theorem 1.2. In Section 4, we prove Theorem 1.3.

2. An isoperimetric inequality in Hadamard manifolds

In this section, we obtain an isoperimetric inequality for certain hypersurfaces in Hadamard manifolds, see Theorem 2.1 below. By Hadamard manifold we mean a simply connected manifold with nonpositive sectional curvature.

Theorem 2.1. *Let N be a Hadamard manifold of dimension 3 with sectional curvature $K_N \leq -a^2 \leq 0$ and let Γ be a complete geodesic of N. Given $r > 0$, there exists a $C = C(a,r) > 0$ such that every smooth, immersed, compact orientable surface $\Sigma \subset N$ with mean curvature $|H_\Sigma| \leq a$ and that stays at a finite distance less than r from Γ, satisfies*

$$\text{Area}(\Sigma) \leq C \text{ Length}(\partial\Sigma). \tag{2}$$

Proof. Let $\Gamma \subset N$ be a complete geodesic, $R = d_N(\cdot, \Gamma)$ be the ambient distance function to Γ and $r > 0$ be given. We will prove the theorem using the following claim:

Claim 2.2. *Fixed $r > 0$, there exist a smooth function $f\colon [0,r] \to \mathbb{R}$ and constants $C_1, C_2 > 0$, whose construction depends uniquely on the constants r and a, such that for all $x \in [0,r]$,*

$$0 \leq f'(x) \leq C_1 \tag{3}$$

holds and that, for every smooth compact surface Σ immersed in $R^{-1}([0,r))$ with mean curvature function satisfying $|H_\Sigma| \leq a$, then

$$\Delta_\Sigma(f \circ R) \geq C_2, \quad \text{in } N. \tag{4}$$

Before we prove the claim, we apply it to obtain the constant C of Theorem 2.1. First, note that R is differentiable in $N \setminus \Gamma$, with $\|\mathrm{grad}(R)\| = 1$ in $N \setminus \Gamma$. Let f be the function provided by the claim and let Σ be a surface satisfying the hypothesis of the theorem. Denoting by ν the conormal vector field along $\partial\Sigma$, we can apply the divergence theorem to obtain that

$$\int_\Sigma \Delta_\Sigma(f \circ R) = \int_{\partial\Sigma} (f' \circ R)\langle \mathrm{grad}(R), \nu\rangle \leq C_1 \,\mathrm{Length}(\partial\Sigma), \tag{5}$$

where last inequality comes from (3). On the other hand, (4) implies

$$\int_\Sigma \Delta_\Sigma(f \circ R) \geq C_2 \,\mathrm{Area}(\Sigma). \tag{6}$$

By defining $C = C_1/C_2$, (5) and (6) show that (2) holds for Σ, thereby providing the constant C of Theorem 2.1.

Next, we prove the claim.

Proof of Claim 2.2. Let Σ be as in the claim and let $f\colon [0, r] \to \mathbb{R}$ be some smooth function, to be chosen a posteriori, such that $f' \geq 0$. Since R is not smooth in points of Γ, we will first show that (4) holds in $N \setminus \Gamma$; however, our choice of f will be of an even function, then $f \circ R$ will be smooth in N and (4) will hold everywhere by continuity.

Consider $\{E_1, E_2\}$ an orthogonal frame to Σ and let η be a normal unitary vector field orienting Σ. A straightforward calculation shows that

$$\begin{aligned}\Delta_\Sigma(f \circ R) = (f' \circ R)\sum_{i=1}^{2} \mathrm{Hess}(R)(E_i, E_i) + (f'' \circ R)\sum_{i=1}^{2}\langle \mathrm{grad}(R), E_i\rangle^2 \\ + 2H_\Sigma(f' \circ R)\langle \mathrm{grad}(R), \eta\rangle,\end{aligned} \tag{7}$$

where we denote $\mathrm{grad} = \mathrm{grad}_N$ and $\mathrm{Hess}(R)(X, Y) = \langle \nabla_X \mathrm{grad}(R), Y\rangle$, respectively, the gradient and the Hessian with respect to the ambient space.

As R is the distance function to the geodesic Γ, the Hessian of R satisfies a matrix valued Riccati type differential equation, where the independent term is the curvature tensor of N. Then, since N has sectional curvature satisfying $K_N \leq -a^2 \leq 0$, it follows from the comparison principle to the Riccati equation, a Hessian comparison principle for the distance function R, given below in (8) (see, for instance, Proposition 5.4 of [**7**] or the main result of [**8**]), which we now describe. For $\rho > 0$, let $C_\rho = R^{-1}(\{\rho\})$ be the geodesic cylinder of radius ρ around Γ and S_ρ be a geodesic sphere of radius ρ centered at a point $\Gamma(s)$ of Γ. Let ∂_θ be a unitary vector field tangent to $C_\rho \cap S_\rho$ and let ∂_s be unitary such that $\{\mathrm{grad}(R), \partial_s, \partial_\theta\}$ is an orthonormal frame in N, away from Γ; see Figure 1. Then

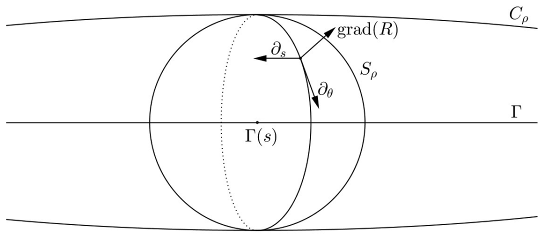

Figure 1. The frame $\{\partial_\theta,\, \partial_s,\, \mathrm{grad}(R)\}$.

$$\mathrm{Hess}(R)(\partial_\theta,\, \partial_\theta) \geq \mu_\theta(R), \quad \mathrm{Hess}(R)(\partial_s,\, \partial_s) \geq \mu_s(R), \tag{8}$$

where μ_θ, μ_s are the functions that realize equalities in (8) in the spaces of constant sectional curvature $-a^2$, and are defined by

(9)
$$\mu_\theta(x) = \begin{cases} a\coth(ax), & \text{if } a > 0 \\ 1/x, & \text{if } a = 0 \end{cases}, \quad \mu_s(x) = \begin{cases} a\tanh(ax), & \text{if } a > 0 \\ 0, & \text{if } a = 0 \end{cases}.$$

We use (8) to estimate $\mathrm{Hess}(R)(E_i,\, E_i)$. Since

$$\langle \nabla_{\partial_s}\mathrm{grad}(R),\, \partial_\theta\rangle = 0 = \langle \nabla_{\partial_\theta}\mathrm{grad}(R),\, \partial_s\rangle,$$

and also

$$E_i = \langle E_i,\, \mathrm{grad}(R)\rangle \mathrm{grad}(R) + \langle E_i,\, \partial_s\rangle\partial_s + \langle E_i,\, \partial_\theta\rangle\partial_\theta,$$

then (8) gives

$$\begin{aligned} \mathrm{Hess}(R)(E_i,\, E_i) &= \langle E_i,\, \partial_s\rangle^2 \mathrm{Hess}(R)(\partial_s,\, \partial_s) + \langle E_i,\, \partial_\theta\rangle^2 \mathrm{Hess}(R)(\partial_\theta,\, \partial_\theta) \\ &\geq \langle E_i,\, \partial_s\rangle^2 \mu_s(R) + \langle E_i,\, \partial_\theta\rangle^2 \mu_\theta(R). \end{aligned} \tag{10}$$

We sum (10) for $i = 1,\, 2$ and use that $\mu_s < \mu_\theta$ to obtain a lower estimate for the first term of (7). We suppress the variable R in the functions μ_s and μ_θ to simplify the notation.

$$\begin{aligned} \sum_{i=1}^{2} \mathrm{Hess}(R)(E_i,\, E_i) &\geq \mu_s \sum_{i=1}^{2} \langle E_i,\, \partial_s\rangle^2 + \mu_\theta \sum_{i=1}^{2} \langle E_i,\, \partial_\theta\rangle^2 \\ &= \mu_s - \mu_s\langle \eta,\, \partial_s\rangle^2 + \mu_\theta - \mu_\theta\langle \eta,\, \partial_\theta\rangle^2 \\ &\geq \mu_s + \mu_\theta - \mu_\theta\left(\langle \eta,\, \partial_s\rangle^2 + \langle \eta,\, \partial_\theta\rangle^2\right) \\ &= \mu_s + \mu_\theta\langle \eta,\, \mathrm{grad}(R)\rangle^2. \end{aligned} \tag{11}$$

Let β be the angle between $\mathrm{grad}(R)$ and η. Since $f' \geq 0$, then (11) and (7) imply that

$$\begin{aligned} \Delta_\Sigma(f \circ R) &\geq f'\left(\mu_s + \mu_\theta \cos^2(\beta)\right) + f''(1 - \cos^2(\beta)) + 2H_\Sigma f' \cos(\beta) \\ &= (f'\mu_\theta - f'')\cos^2(\beta) + 2H_\Sigma f' \cos(\beta) + f'\mu_s + f''. \end{aligned} \tag{12}$$

At this point, we are able to finish the proof in the case $a = 0$, where $\mu_s(R) = 0$, $\mu_\theta(R) = 1/R$ and $H_\Sigma = 0$. By choosing $f(x) = x^2$, (12) gives

$$\Delta_\Sigma(R^2) \geq 2,$$

and we can set $C_1 = f'(r) = 2r$ and $C_2 = 2$, which proves the claim and gives the constant $C(0, r) = r$ in the theorem.

Next, assume that $a > 0$. If we suppose that $f'\mu_\theta - f'' > 0$ (this will be shown to hold for our choice of f), algebraic manipulation in (12) gives

$$\begin{aligned}\Delta_\Sigma(f \circ R) &\geq (f'\mu_\theta - f'')\left(\cos(\beta) + H_\Sigma \frac{f'}{f'\mu_\theta - f''}\right)^2 \\ &\quad - H_\Sigma^2 \frac{(f')^2}{f'\mu_\theta - f''} + f'\mu_s + f'' \\ &\geq f'\mu_s + f'' - H_\Sigma^2 \frac{(f')^2}{f'\mu_\theta - f''}. \qquad (13)\end{aligned}$$

Using that $\mu_s\mu_\theta = a^2$ and $H_\Sigma^2 \leq a^2$, we obtain the following estimate

$$\begin{aligned}\Delta_\Sigma(f \circ R) &\geq \frac{(f')^2(a^2 - H_\Sigma^2) + f''(f'\mu_\theta - f'' - f'\mu_s)}{f'\mu_\theta - f''} \\ &\geq f''\left(1 - \frac{f'\mu_s}{f'\mu_\theta - f''}\right). \qquad (14)\end{aligned}$$

We now make our choice of f. Fix $k \in \mathbb{N}$ and let $f_k\colon \mathbb{R} \to \mathbb{R}$ be a solution to

$$f'(x)\mu_s(x) = \frac{k}{k+1}(f'(x)\mu_\theta(x) - f''(x)), \qquad (15)$$

which can be explicitly expressed by

$$f_k(x) = -\frac{k}{a(\cosh(ax))^{\frac{1}{k}}}. \qquad (16)$$

The derivatives of f_k are

$$\begin{aligned}f_k'(x) &= \frac{\tanh(ax)}{(\cosh(ax))^{\frac{1}{k}}}, \qquad (17)\\ f_k''(x) &= \frac{a}{k(\cosh(ax))^{\frac{2k+1}{k}}}[k + 1 - \cosh^2(ax)], \qquad (18)\end{aligned}$$

and, by (15) and (17) we conclude that $f_k'\mu_\theta - f_k'' > 0$; hence, (14) gives the inequality

$$\Delta_\Sigma(f \circ R) \geq a\frac{k + 1 - \cosh^2(aR)}{k(k+1)(\cosh(aR))^{\frac{2k+1}{k}}}. \qquad (19)$$

Let $g_k\colon [0, r] \to \mathbb{R}$ be defined by the expression in the right hand side of (19), i.e.,

$$g_k(x) = a\frac{k + 1 - \cosh^2(ax)}{k(k+1)(\cosh(ax))^{\frac{2k+1}{k}}}.$$

To prove the claim it suffices to find a lower positive bound for g_k, as f_k' is nonnegative and bounded by $C_1 = 1$. Note that $g_k(x) > 0$ if and only if

$$\cosh^2(ax) < k + 1.$$

Choose $n \in \mathbb{N}$ such that $\cosh^2(ar) < n + 1$. Then f_n is a function that satisfies (3) in $[0, r]$, for $C_1 = 1$. A direct calculation of the derivative of g_n shows that it is a decreasing function in $[0, r]$; thus, if we set $C_2 = g_n(r) > 0$, it follows that $g_n(x) \geq C_2 > 0$ for $x \in [0, r]$. Since n was chosen independently of Σ and C_1, C_2 depend uniquely on f_n, this completes the proof of the claim in the case when $a > 0$. q.e.d.

As explained immediately after the statement of Claim 2.2, the theorem is now proved. q.e.d.

Remark 2.3. The hypothesis $|H_\Sigma| \leq a$ in Theorem 2.1 cannot be improved. Indeed, if N is simply connected, of constant sectional curvature $K_N = -a^2 \leq 0$, then, for each $H > a$, there exists a geodesic cylinder of constant mean curvature H, containing compact subdomains with constant boundary length and with arbitrarily large area.

The next corollary is an immediate consequence of Theorem 2.1. One only needs to check that if the boundary of the compact surface Σ given below has length L, then Σ is contained in a solid geodesic cylinder of radius $L/2$, which follows from the mean curvature comparison principle.

Corollary 2.4 (Area estimate for surfaces with two boundary components)**.** *Let N be a Hadamard 3-manifold with sectional curvature $K_N \leq -a^2 \leq 0$. Then, for each $L > 0$, there is a constant $C = C(a, L)$ such that the following holds.*

Let $\Sigma \subset N$ be a compact surface immersed in N with $|H_\Sigma| \leq a$. If the boundary of Σ consists on one or two components and has total length at most L, then

$$\text{Area}(\Sigma) \leq C\, L. \tag{20}$$

3. The proof of Theorem 1.2

Throughout this section N will be a complete Riemannian 3-manifold with sectional curvature $K_N \leq -a^2 \leq 0$, and $\varphi\colon \Sigma \to N$ will be an isometric immersion of a finite topology surface Σ such that the mean curvature function H_φ of the immersion satisfies $|H_\varphi| \leq a$. A simple consequence of the Gauss equation is that the Gaussian curvature function K_Σ of Σ is nonpositive; hence, if $p \in \Sigma$ and $I_\Sigma(p)$ is finite, then there is a closed geodesic loop based at p in Σ of length $2I_\Sigma(p)$ (see, for instance, Proposition 2.12, Chapter 13 of [**6**]). Moreover, it follows from the Gauss-Bonnet formula that such loop is homotopically nontrivial in

Σ. The existence of such loops will be used in the proofs of the next two propositions.

Proposition 3.1. *Suppose N is simply connected and $E \equiv \mathbb{S}^1 \times [0, +\infty)$ is a complete, noncompact Riemannian annulus. If $\varphi\colon E \to N$ is an isometric immersion with $|H_\varphi| \le a$, then the asymptotic injectivity radius of E, which we denote by I_E^∞, is infinite. In particular, item A of Theorem 1.2 holds.*

Proof. An elementary calculation shows that there is an $\varepsilon > 0$ independent of E such that intrinsic balls $B_E(p, \varepsilon) \subset E - \partial E$ have area greater than some fixed positive constant; see Theorem 3 and Remark 4 in the appendix of [**9**]. In particular, since E is noncompact, complete and has compact boundary, then E has infinite area.

Arguing by contradiction, suppose that the asymptotic injectivity radius of E is $I_E^\infty = L \in [0, \infty)$. By the definition of I_E^∞, there is an intrinsically divergent sequence of points $\{q_n\}_n$ in E such that

$$\lim_{n\to\infty} I_E(q_n) = L.$$

After replacing by a subsequence, $I_E(q_n) < L + 1$, $\{d_E(q_n, \partial E)\}_n$ is increasing, $d_E(q_1, \partial E) \ge L+1$ and $d_E(q_n, q_{n+1}) \ge 2L+2$. Hence, there exist homotopically nontrivial geodesic loops γ_n with base points q_n and lengths equal to $2I_E(q_n) < 2L + 2$, which are pairwise disjoint by the triangle inequality.

For $n > 1$, let E_n be the compact annular region of E bounded by γ_1 and γ_n. Since the total length of ∂E_n is less than $4L + 4$, it follows from Corollary 2.4 that there is a constant C such that E_n satisfies the uniform area estimate (20):

$$\mathrm{Area}(E_n) \le C(4L + 4).$$

In particular, E has finite area, which contradicts our previous observation that the area of E was infinite, and completes the proof that $I_E^\infty = \infty$. q.e.d.

Proposition 3.2. *Suppose $\varphi\colon E \to N$ is an isometric immersion of a complete annulus $E \equiv \mathbb{S}^1 \times [0, +\infty)$ in N satisfying $|H_\varphi| \le a$. Then:*

I. If $I_E^\infty \in [0, \infty)$, then the induced homomorphism $\varphi_*\colon \pi_1(E) \to \pi_1(N)$ is injective.

II. $I_E^\infty \ge \mathrm{Inj}(N)$, where $\mathrm{Inj}(N)$ denotes the injectivity radius of N.

III. If $I_E^\infty = 0$, then φ is proper.

Proof. We first prove item I of the proposition. Arguing by contradiction, suppose I_E^∞ is finite and

$$\varphi_*\colon \pi_1(E) \to \pi_1(N)$$

is not injective. Since $\pi_1(E)$ is isomorphic to $\mathbb{Z}$, the kernel K of φ_* is a cyclic subgroup of index n, for some $n \in \mathbb{N}$. Let $\Pi\colon \widetilde{E} \to E$ be the

n-sheeted covering space of E corresponding to the subgroup K. Note that $\widetilde{E}$ is an annulus of nonpositive Gaussian curvature, $I^{\infty}_{\widetilde{E}}$ is less than or equal to nI^{∞}_{E} (since $y \in \Pi^{-1}(x) \Rightarrow I_{\widetilde{E}}(y) \leq nI_E(x)$) and the induced map from the fundamental group of $\widetilde{E}$ to N is trivial. By covering space theory, $(\varphi \circ \Pi) \colon \widetilde{E} \to N$ lifts isometrically to the universal cover $\widetilde{N}$ of N, which is a Hadamard manifold with respect to the pulled-back metric. Since $I^{\infty}_{\widetilde{E}}$ is finite, Proposition 3.1 gives a contradiction, thereby completing the proof of item I.

We next prove statement II. Suppose that $I^{\infty}_{E} < \mathrm{Inj}(N)$. Then there exists a geodesic loop γ in E with length less than $2\mathrm{Inj}(N)$. Since γ is homotopically nontrivial in E and lies in a simply connected ball in N, then for any $p \in \gamma$, the induced map $\varphi_* \colon \pi_1(E, p) \to \pi_1(N, p)$ is trivial, contradicting item I.

Finally, we prove III. If φ were not proper, there would exist an intrinsically divergent sequence of points $q_n \in E$, such that $\varphi(q_n)$ converges to a point $q \in N$. Lemma 5.1 in the Appendix implies $\lim_{n\to\infty} I_E(q_n) = 0$; hence, after replacing by a subsequence, $q_n \in B_N(q, I_N(q)/2)$ and there exist homotopically nontrivial geodesic loops γ_n based at q_n with lengths $2I_E(q_n) < I_N(q)$. By the triangle inequality, the loops $\varphi(\gamma_n)$ are contained in the simply connected geodesic ball $B_N(q, I_N(q))$, contradicting item I. Thus, φ is proper. q.e.d.

All of the assertions in Theorem 1.2, except for the first statement of item C of Theorem 1.2, follow immediately from Propositions 3.1 and 3.2. The Cohn-Vossen [**3**] inequality implies that for any complete surface M of nonpositive curvature,

$$\int_M K_M \leq 2\pi\chi(M). \tag{21}$$

In our setting where $M = \Sigma$ and each end e of Σ satisfies $I^{\infty}_{\Sigma}(e)$ is bounded, Theorem 11 of Huber [**12**] implies

$$\int_\Sigma K_\Sigma = 2\pi\chi(\Sigma),$$

which completes the proof of Theorem 1.2.

4. Proof of Theorem 1.3

In this section, we prove Theorem 1.3, which describes, among other things, the asymptotic behavior of certain immersed surfaces in hyperbolic manifolds of finite volume. Before we prove this result, we set up the notation that we use and give a brief review of the structure of the ends of orientable hyperbolic manifolds of finite volume, called *cusp ends*.

We will use the half-space model for the hyperbolic space:

$$\mathbb{H}^3 = \{(x,\, y,\, z) \in \mathbb{R}^3 \mid z > 0\},$$

endowed with the metric $ds^2 = \dfrac{dx^2 + dy^2 + dz^2}{z^2}$. In this model, the horizontal planes

$$\mathcal{H}(t) = \{(x, y, z) \in \mathbb{H}^3 \mid z = t\}$$

are horospheres, with constant mean curvature 1 with respect to its upward pointing unit normal field. Vertical planes are totally geodesic and isometric to the hyperbolic plane $\mathbb{H}^2$.

Fix two linearly independent horizontal vectors $u = (u_x,\, u_y,\, 0)$, $v = (v_x,\, v_y, 0) \in \mathbb{R}^3$ and let $G(u, v)$ be the group of parabolic translations of $\mathbb{H}^3$ generated by u, v, i.e., $G(u,\, v) = \{\tau(m, n) \mid (m, n) \in \mathbb{Z} \times \mathbb{Z}\}$, where each $\tau(m, n)$ is the parabolic isometry of $\mathbb{H}^3$ defined by $\tau(m, n)(p) = p + mu + nv$. In coordinates,

$$\tau(m,n)\colon \quad \begin{array}{ccc} \mathbb{H}^3 & \to & \mathbb{H}^3, \\ (x,y,z) & \mapsto & (x + mu_x + nv_x, y + mu_y + nv_y, z). \end{array} \tag{22}$$

If N is a complete, orientable, noncompact hyperbolic 3-manifold of finite volume, it has a finite number of ends $\mathcal{C}_i$, $i = 1,\, 2,\, \ldots, n$, called the *cusp ends* of N. For each $\mathcal{C}_i$ there exists $t_i > 0$ and linearly independent horizontal vectors u_i, v_i, such that $\mathcal{C}_i$ is represented by, and, henceforth, isometrically identified with, the quotient of

$$\mathcal{M}(t_i) = \bigcup_{t \geq t_i} \mathcal{H}(t) = \{(x, y, z) \in \mathbb{H}^3 \mid z \geq t_i\} \tag{23}$$

by the action of the group $G(u_i,\, v_i)$. Since $G(u_i, v_i)$ leaves every horosphere $\mathcal{H}(t)$ invariant and $\mathcal{M}(t_i)$ is foliated by $\{\mathcal{H}(t)\}_{t \geq t_i}$, each $\mathcal{C}_i$ admits a product foliation by the family of constant mean curvature 1 tori $\{\mathcal{T}(t) = \mathcal{H}(t)/G(u_i, v_i)\}_{t \geq t_i}$. Also, the fundamental group of each $\mathcal{C}_i$ is naturally isomorphic to $G(u_i, v_i)$, viewed as the subgroup of isometries of $\mathcal{M}(t_i)$ that commute with the covering map $\psi_i \colon \mathcal{M}(t_i) \to \mathcal{C}_i$.

4.1. Proof of items 1–4 of Theorem 1.3. With the notation concerning the structure of the cusp ends of N discussed above, we next prove the first four items of Theorem 1.3.

Let $\varphi\colon \Sigma \to N$ be the immersed surface given in the statement of the theorem. Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ be cusp end representatives of the ends of N and let $N_T = \overline{N \setminus (\mathcal{C}_1 \cup \mathcal{C}_2 \cup \ldots \cup \mathcal{C}_n)}$. Since φ is a proper map and N_T is compact, it holds that $\varphi(\Sigma) \cap N_T$ is compact. Without loss of generality, we may assume that ∂N_T is transverse to $\varphi(\Sigma)$ and consists of a finite collection of immersed closed curves.

To prove that Σ has finite area, it suffices to show that each intersection $\varphi(\Sigma) \cap \mathcal{C}_i$ has finite area. Let $\mathcal{C}$ be one of the cusp ends $\mathcal{C}_i$. Up to a reparameterization, we may assume that $\mathcal{C} = \cup_{t \geq 1} \mathcal{T}(t)$. We define

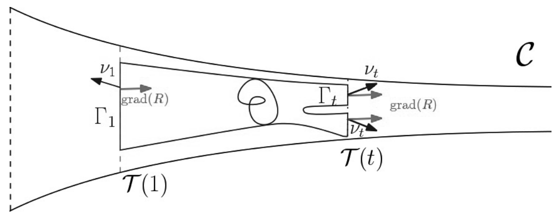

Figure 2. The immersed compact region $\varphi(E(t))$.

$E(\mathcal{C}) = \varphi^{-1}(\mathcal{C}) \subset \Sigma$ and $E(t) = \varphi^{-1}\left(\cup_{s\in[1,t]}\mathcal{T}(s)\right)$. We also use the notation $\varphi = \varphi|_{E(\mathcal{C})}$.

Let $R\colon \mathcal{C} \to [0,\infty)$ be the distance function to $\mathcal{T}(1)$. By following the arguments in the proof of Theorem 2.1, one can obtain under the assumptions of the theorem that the intrinsic Laplacian of $R\circ\varphi$ satisfies

$$\Delta_{E(\mathcal{C})}(R\circ\varphi) \le H^2 - 1 < 0. \tag{24}$$

Let $t > 1$ be a regular value for $\mathcal{R}\circ\varphi$ and let $\Gamma_t = \varphi^{-1}(\mathcal{T}(t))$. Also, we denote $\Gamma_1 = \partial E(\mathcal{C})$, then $\partial E(t) = \Gamma_1 \cup \Gamma_t$. Integrating (24) over $E(t)$, we obtain

$$\int_{E(t)} \Delta_{E(\mathcal{C})}(R\circ\varphi) \le (H^2-1)\mathrm{Area}(E(t)). \tag{25}$$

Applying the divergence theorem to the left hand side of (25), gives

$$\int_{E(t)} \Delta_{E(\mathcal{C})}(R\circ\varphi) = \int_{\Gamma_1} \langle \mathrm{grad}_{E(\mathcal{C})}(R\circ\varphi),\, \nu_1\rangle + \int_{\Gamma_t} \langle \mathrm{grad}_{E(\mathcal{C})}(R\circ\varphi),\, \nu_t\rangle, \tag{26}$$

where ν_1 and ν_t denote respectively the outward pointing conormal vectors to $E(t)$ along Γ_1 and Γ_t (see Figure 2). It follows from (25) and (26) that

$$(1-H^2)\mathrm{Area}(E(t)) \le -\int_{\Gamma_1} \langle \mathrm{grad}_{E(\mathcal{C})}(R\circ\varphi),\, \nu_1\rangle - \int_{\Gamma_t} \langle \mathrm{grad}_{E(\mathcal{C})}(R\circ\varphi),\, \nu_t\rangle. \tag{27}$$

Since φ is an isometric immersion and ν_t is tangent to $E(\mathcal{C})$, $\langle \mathrm{grad}_{E(\mathcal{C})}(R\circ\varphi),\, \nu_t\rangle = \langle \mathrm{grad}(R),\, d\varphi(\nu_t)\rangle$. Moreover, by the definition of $E(t)$, along $\varphi(\Gamma_t)$ we have

$$\langle \mathrm{grad}(R),\, d\varphi(\nu_t)\rangle > 0.$$

Hence, $\langle \mathrm{grad}_{E(\mathcal{C})}(R\circ\varphi),\, \nu_t\rangle > 0$ and (27) implies that

$$\mathrm{Area}(E(t)) < -\frac{1}{1-H^2}\int_{\Gamma_1} \langle \mathrm{grad}_{E(\mathcal{C})}(R\circ\varphi),\, \nu_1\rangle \le \frac{1}{1-H^2}\mathrm{Length}(\Gamma_1). \tag{28}$$

It follows that $\mathrm{Area}(E(\mathcal{C}))$ is bounded by $\frac{1}{1-H^2}\mathrm{Length}(\Gamma_1)$, which proves that Σ has finite area.

To finish the proof of item 1, just note that each connected component E of Σ must be such that $\varphi(E) \cap N_T \neq \emptyset$, otherwise $\varphi(E)$ would be contained in a cusp end $\mathcal{C}$ of N; since φ is proper, this implies that $R \circ \varphi|_E$ would attain a minimal value t^* on an interior point $p^* \in E$. Then, the mean curvature comparison principle applied to $\mathcal{T}(t^*)$ and to $\varphi(E)$ at $\varphi(p^*)$ gives a contradiction, since the mean curvature of $\mathcal{T}(t^*)$ is 1, $\varphi(E)$ lies in the mean convex side of $\mathcal{T}(t^*)$ and the mean curvature of $\varphi(E)$ at $\varphi(p^*)$ is strictly less than 1. Finally, since φ is proper, $\varphi^{-1}(N_T)$ must contain a finite number of connected components.

The second item of the theorem follows from item 1, as we next explain. Suppose there exist an $\varepsilon > 0$ and a divergent sequence of points $\{p_n\}_{n\in\mathbb{N}}$ in Σ such that $I_\Sigma(p_n) \geq \varepsilon$. After replacing by a subsequence, we may assume that $\{B_\Sigma(p_n, \varepsilon)\}_{n\in\mathbb{N}}$ is a collection of pairwise disjoint disks. Since Σ has nonpositive curvature, comparison theorems imply that $\mathrm{Area}(B_\Sigma(p_n, \varepsilon)) \geq \pi\varepsilon^2$. Hence, Σ has infinite area, contradicting item 1, which proves item 2.

Next, we prove item 3. Since Σ has a finite number of connected components by item 1, if Σ has infinite topology, then its Euler characteristic is $\chi(\Sigma) = -\infty$. Hence, Cohn-Vossen inequality (21) implies that Σ has infinite total curvature, which proves item 3 in this case. In the case where Σ has finite topology, item 2 implies that I_Σ is bounded; therefore, equation (1) holds by item C of Theorem 1.2.

To prove that $\varphi(\Sigma)$ has unbounded norm of the second fundamental form when it has infinite topology, just note that a uniform bound on $\|A_\varphi\|$, with the assumption that $|H_\varphi| \leq H$, would imply that K_Σ is uniformly bounded, by Gauss' equation. In particular, since Σ has finite area it would have finite total curvature, contradicting item 3, which proves item 4.

4.2. Bounds on the second fundamental form. A key step in obtaining the asymptotic description of the annular ends of Σ in the constant mean curvature setting, is the boundedness of the second fundamental form of each such end.

Proposition 4.1. *Let N be a complete, noncompact hyperbolic 3-manifold of finite volume and $H \in [0, 1)$. If E is a complete, properly immersed annulus in N with constant mean curvature H, then $\varphi(E)$ has bounded norm of its second fundamental form.*

Proof. Since φ is a proper map, then there exist a cusp end $\mathcal{C}$ of N and a subannular end $E' \subset E$ such that $\varphi(E')$ is contained in $\mathcal{C}$; since it suffices to prove that E' has bounded second fundamental form, we may assume that $\varphi(E) \subset \mathcal{C}$. Up to a reparameterization, we write $\mathcal{C} = \cup_{t\geq 1}\mathcal{T}(t)$; in particular, $\varphi(\partial E) \subset \cup_{t\in[1,\, t_0]}\mathcal{T}(t)$, for some $t_0 \geq 1$. Using this notation, we prove next claim.

Claim 4.2. *For almost every $t \geq t_0$, $\varphi(E)$ meets the torus $\mathcal{T}(t)$ transversely and for each such t there is a unique homotopically nontrivial closed curve $\alpha_t \subset E$ such that $\varphi(\alpha_t)$ is contained in $\mathcal{T}(t)$. Moreover, the annular subend E' of E determined by α_t is immersed in $\cup_{s \geq t}\mathcal{T}(s)$ and the induced map $\varphi'_*\colon \pi_1(E') \to \pi_1(\cup_{s\geq t}\mathcal{T}(s))$ is injective.*

Proof of Claim 4.2. Sard's Theorem implies that for almost all $t \geq t_0$, φ is transverse to $\mathcal{T}(t)$ and for such t, $\Gamma_t = \varphi^{-1}(\mathcal{T}(t))$ is a finite collection of pairwise disjoint simple closed curves in E. Fix a regular value $t_1 > t_0$. We next prove that exactly one of the curves in Γ_{t_1} is homotopically nontrivial in E. Since $\varphi^{-1}(\cup_{t\in[1,t_1]}\mathcal{T}(t))$ is a compact surface (possibly disconnected) containing ∂E, then the 1-cycle ∂E is $\mathbb{Z}_2$-homologous to the collection of unoriented curves in Γ_{t_1}. Since ∂E represents the nontrivial element in $H_1(E, \mathbb{Z}_2)$, then at least one of the curves in Γ_{t_1} is homotopically nontrivial.

Concerning the uniqueness part of the claim, assume that there are two homotopically nontrivial curves, α_1, α_2, in E such that $\varphi(\alpha_1)$ and $\varphi(\alpha_2)$ are both contained in $\mathcal{T}(t_1)$. Consider the subends $E_1, E_2 \subset E$ determined respectively by α_1, α_2 and assume that $E_2 \subset E_1$. Let $\widehat{E} \subset E$ be the compact annulus bounded by α_1 and α_2. Then, either $\varphi(\widehat{E})$ intersects $\cup_{t<t_1}\mathcal{T}(t)$ or it is contained in $\cup_{t\geq t_1}\mathcal{T}(t)$. If the first case occurs, there is a $t_* < t_1$ such that $\varphi(\widehat{E})$ intersects $\mathcal{T}(t_*)$ but does not intersect $\mathcal{T}(t)$ for any $t < t_*$. Because the mean curvature vector of every torus $\mathcal{T}(t)$ has length 1 and points into the cusp subend of $\mathcal{C}$ determined by $\mathcal{T}(t)$, the mean curvature comparison principle for the surfaces $\varphi(\widehat{E})$ and $\mathcal{T}(t_*)$ at a point in $\varphi(\widehat{E})\cap\mathcal{T}(t_*)$ gives a contradiction, therefore, $\varphi(\widehat{E}) \subset \cup_{t\geq t_1}\mathcal{T}(t)$. Since $\varphi(E)$ meets $\mathcal{T}(t_1)$ transversely, it follows that $\varphi(E_2)$ contains points in $\cup_{t<t_1}\mathcal{T}(t)$ near $\varphi(\alpha_2)$, and we may find $t'_* < t_1$, where t'_* is the smallest t such that $\varphi(E_2)$ intersects $\mathcal{T}(t)$, which by the previous argument gives a contradiction. This proves that φ induces an immersion $\varphi'\colon E' \to \cup_{s\geq t}\mathcal{T}(s)$.

By item 2 of Theorem 1.3, I_Σ is bounded. Thus, by part C of Theorem 1.2, $\varphi'_*\colon \pi_1(E') \to \pi_1(\cup_{s\geq t}\mathcal{T}(s))$ is injective. This completes the proof of Claim 4.2. q.e.d.

We next fix some notation. Because of Claim 4.2 we, henceforth, assume, without loss of generality, that $t_0 = 1$ and that $\varphi(E)$ intersects $\partial\mathcal{C} = \mathcal{T}(1)$ transversely in the set $\partial E = \alpha_1$. Moreover, we assume that $\mathcal{C}$ is isometric to the quotient space $\mathcal{M}(1)/G(u,v)$, for two linearly independent horizontal vectors u and v. Let $\psi\colon \mathcal{M}(1) \to \mathcal{C}$ be the covering of $\mathcal{C}$ associated to $G(u,v)$ and let $\Pi\colon \widetilde{E} \to E$ be the universal cover of E. Choose a base point $p \in \widetilde{E}$ and consider $\varphi(\Pi(p)) \in \mathcal{C}$. After choosing $\widehat{p} \in \psi^{-1}(\varphi(\Pi(p)))$, covering space theory implies that there exists a unique immersion $\phi\colon \widetilde{E} \to \mathcal{M}(1)$ such that $\phi(p) = \widehat{p}$ and $\psi\circ\phi = \varphi\circ\Pi$; in particular, it follows that ϕ is proper, since

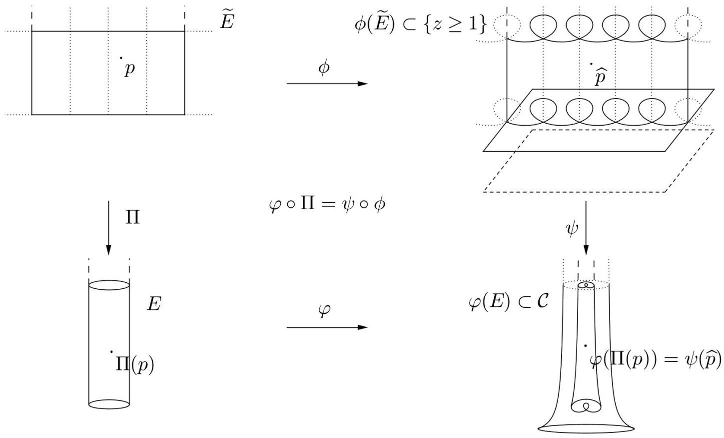

Figure 3. The immersions ϕ and φ and the covering maps $\Pi\colon \widetilde{E} \to E$ and $\psi\colon \mathcal{M}(1) \to \mathcal{C}$.

$\varphi_*\colon \pi_1(E) \to \pi_1(\cup_{s\geq t}\mathcal{T}(s))$ is injective. Moreover, $\phi(\widetilde{E})$ is an immersed half-plane in $\mathbb{H}^3$. See Figure 3.

To prove Proposition 4.1, it suffices to prove that $\phi(\widetilde{E})$ has bounded norm of the second fundamental form $\|A_\phi\|$. Arguing by contradiction, assume there exists a sequence of points $\{p_n\}_{n\in\mathbb{N}}$ in $\widetilde{E}$ such that $\|A_\phi\|(\widehat{p}_n) \geq n$, where we denote $\widehat{p}_n = \phi(p_n)$. Since $\|A_\varphi\|$ is bounded on compact sets, the sequence of the image points $\{\psi(\widehat{p}_n)\}_{n\in\mathbb{N}} \subset \varphi(E)$ is intrinsically (thus, extrinsically, since φ is proper) divergent. After choosing a subsequence, we may assume $d_{\mathcal{C}}(\psi(\widehat{p}_n), \psi(\widehat{p}_m)) > 2$, for $n \neq m$; hence, the sequence $\{\widehat{p}_n\}_{n\in\mathbb{N}}$ in $\mathbb{H}^3$ is extrinsically divergent and $d_{\mathbb{H}^3}(\widehat{p}_n, \widehat{p}_m) > 2$ for $n \neq m$. For the construction that follows, see Figure 4.

Without loss of generality, we may assume that the spheres $\{\partial\overline{B}_{\mathbb{H}^3}(\widehat{p}_n, 1)\}_n$ are transverse to ϕ. For $n \in \mathbb{N}$, let $C_n \subset \widetilde{E}$ be the connected component of $\phi^{-1}(\overline{B}_{\mathbb{H}^3}(\widehat{p}_n, 1))$ containing p_n. Consider the function

$$\begin{array}{rccl} f_n\colon & C_n & \to & \mathbb{R}, \\ & x & \mapsto & \|A_\phi\|(\phi(x))d_{\mathbb{H}^3}(\phi(x), \partial\overline{B}_{\mathbb{H}^3}(\widehat{p}_n, 1)). \end{array}$$

Let $q_n \in C_n$ be a point where f_n achieves its maximum. Let $\widehat{q}_n = \phi(q_n)$. Then,

$$\|A_\phi\|(\widehat{q}_n)d(\widehat{q}_n, \partial B_{\mathbb{H}^3}(\widehat{p}_n, 1)) = f_n(q_n) \geq f_n(p_n) = \|A_\phi\|(\widehat{p}_n) \geq n. \tag{29}$$

Let $\lambda_n = \|A_\phi\|(\widehat{q}_n)$ and $\delta_n = d_{\mathbb{H}^3}(\widehat{q}_n, \partial B_{\mathbb{H}^3}(\widehat{p}_n, 1))$. Note that, if $x \in C_n$ and $\phi(x) \in \overline{B}_{\mathbb{H}^3}(\widehat{q}_n, \delta_n/2)$, then $\|A_\phi\|(\phi(x)) \leq 2\lambda_n$, since $d_{\mathbb{H}^3}(\phi(x), \widehat{q}_n) < \delta_n/2$.

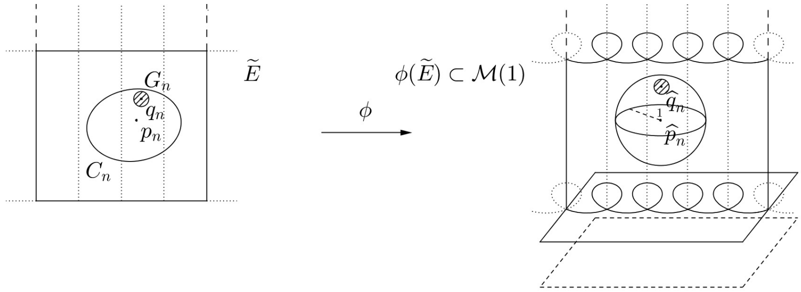

Figure 4. The domain C_n containing p_n is immersed in $B_{\mathbb{H}^3}(\widehat{p}_n, 1)$, q_n is a maximal point to f_n and the subdomain $G_n \ni q_n$ is such that $\phi(G_n)$ is an embedded graph over $T_{\widehat{q}_n}\phi(\widetilde{E})$.

In this proof we use the following notation: for $\lambda > 0$ and a Riemannian manifold $M = (M, g)$, we denote $\lambda M = (M, \lambda^2 g)$ the Riemannian manifold given by a scaling of the metric of M by λ.

Using exponential coordinates in $\mathbb{H}^3$ centered at the point $\widehat{q}_n$, consider $\lambda_n B_{\mathbb{H}^3}(\widehat{q}_n, \delta_n/2)$ to be a ball of radius $\lambda_n\delta_n/2$ in $\mathbb{R}^3$ with $\widehat{q}_n$ at the origin. From (29), we have $\lambda_n\delta_n \to \infty$, and so the sequence $\{\lambda_n B_{\mathbb{H}^3}(\widehat{q}_n, \delta_n/2)\}_{n\in\mathbb{N}}$ of Riemannian balls converges to the Euclidean space $\mathbb{R}^3$ with its flat metric. Let $\widehat{C}_n = \phi(C_n) \cap \overline{B}_{\mathbb{H}^3}(\widehat{q}_n, \delta_n/2)$ and, for $r > 0$, let $B_n(r) = B_{\lambda_n\mathbb{H}^3}(\widehat{q}_n, r) = \lambda_n B_{\mathbb{H}^3}(\widehat{q}_n, r/\lambda_n)$. The scaled surfaces $\lambda_n\widehat{C}_n$ are immersed in $B_n(\lambda_n\delta_n/2)$, with constant mean curvature $H_n = H/\lambda_n$ and satisfy

$$\|A_{\lambda_n\widehat{C}_n}\| \leq 2, \quad \|A_{\lambda_n\widehat{C}_n}\|(\widehat{q}_n) = 1. \tag{30}$$

This uniform bound on the second fundamental form of $\lambda_n\widehat{C}_n$ implies that there is a $\delta > 0$ such that for every $n \in \mathbb{N}$, there exists a connected domain G_n of C_n, containing q_n and such that:

1) $\lambda_n\phi(G_n) \subset B_n(\delta)$;
2) $\partial[\lambda_n\phi(G_n)] = \lambda_n\phi(\partial G_n) \subset \partial B_n(\delta)$;
3) $\lambda_n\phi(G_n)$ is embedded and it is a graph over its projection to $T_{\widehat{q}_n}\lambda_n\phi(G_n)$, with graphing function having uniformly bounded gradient, for all $n \in \mathbb{N}$.

A subsequence of the graphs $\{\lambda_n\phi(G_n)\}_{n\in\mathbb{N}}$ (which we still denote by $\{\lambda_n\phi(G_n)\}_{n\in\mathbb{N}}$) converges, as $n \to \infty$, to a minimal graph $G_\infty \subset \mathbb{R}^3$, embedded in $B_{\mathbb{R}^3}(\vec{0}, \delta)$ and with $\partial G_\infty \subset \partial B_{\mathbb{R}^3}(\vec{0}, \delta)$. We next use the Gauss map $g\colon G_\infty \to \mathbb{S}^2$ of G_∞ to prove that E has infinite total curvature, from which we obtain a contradiction.

$\lambda\mathbb{H}^3$ is well-known to be isometric to a Lie group together with a left invariant metric. This Lie group is the semidirect product of $\mathbb{R}^2$ with $\mathbb{R}$ having associated homomorphism $f\colon \mathbb{R} \to \mathrm{Gl}(2, \mathbb{R})$ given by

$f(t) = \exp(tI_\lambda) \in \mathrm{Gl}(2,\mathbb{R})$, where I_λ is the 2×2 diagonal matrix with $1/\lambda$ as diagonal entries; see, for instance, [**14**]. In this $\mathbb{R}^3$-coordinate model for $\lambda\mathbb{H}^3$ the horizontal planes are left and right cosets of the normal subgroup $\mathbb{R}^2$ of the Lie group, which we view as being the (x, y)-plane with the metric at the origin $(0, 0, 0)$ corresponding to the usual metric on $\mathbb{R}^3$; also left translations by elements in $\mathbb{R}^2$ correspond to parabolic isometries in the previous upper halfspace model for $\lambda\mathbb{H}^3$ given by translations by horizontal vectors.

From this point of view, for each $n \in \mathbb{N}$, the metric of $\lambda_n\mathbb{H}^3$ is left invariant and there is a left invariant Gauss map, defined for any oriented surface immersed in $\lambda_n\mathbb{H}^3$ and taking values in the unit sphere $\mathbb{S}^2$ of the tangent space to the identity element of this semidirect product. Denote by $g_n\colon \lambda_n\phi(G_n) \to \mathbb{S}^2$ the left invariant Gauss map of the oriented embedded surface $\lambda_n\phi(G_n) \subset \lambda_n\mathbb{H}^3$. Since the group structures of $\lambda_n\mathbb{H}^3$ converge to the abelian group structure of $\mathbb{R}^3$, then g_n converges to $g\colon G_\infty \to \mathbb{S}^2$, the Gauss map of G_∞. This convergence is explained in [**13**], in the last paragraph of Step 4 of the proof of Theorem 4.1.

Note that (30) implies that

$$\|A_{G_\infty}\| \leq 2, \quad \|A_{G_\infty}\|(\vec{0}) = 1.$$

Hence, g is injective near $\vec{0} \in G_\infty$, since G_∞ is a minimal surface of $\mathbb{R}^3$ with Gaussian curvature $-1/2$ at $\vec{0}$. It follows that there exists $\widetilde{\delta} \in (0, \delta)$ such that the graph $G_\infty \cap B_{\mathbb{R}^3}(\vec{0}, \widetilde{\delta})$ is a disk and $g\colon G_\infty \cap B_{\mathbb{R}^3}(\vec{0}, \widetilde{\delta}) \to \mathbb{S}^2$ is an injective diffeomorphism with its image. Note that there exists some $\varepsilon > 0$ such that for all $x \in G_\infty \cap B_{\mathbb{R}^3}(\vec{0}, \widetilde{\delta})$ and $X \in T_xG_\infty$ it holds $\|dg_x(X)\| \geq \varepsilon\|X\|$. Then, the fact that $g_n \to g$ in the $C^{1,\alpha}$-topology implies that there exists a $\delta' \in (0, \widetilde{\delta})$ such that, for n sufficiently large, $g_n\colon \lambda_n\phi(G_n) \cap B_n(\delta') \to \mathbb{S}^2$ is also injective.

The fact that G_∞ is not flat also implies that $G_\infty \cap B_{\mathbb{R}^3}(\vec{0}, \delta')$ has strictly negative total curvature; hence, there is $K_0 > 0$ such that, for n sufficiently large, $\lambda_n\phi(G_n) \cap B_n(\delta')$ has total curvature less than $-K_0^2$. Since total curvature is invariant under scalings, it follows that

$$\int_{G_n \cap \phi^{-1}(B_n(\delta'))} K_{\widetilde{E}} < -K_0^2 < 0. \tag{31}$$

The assumption $d_{\mathcal{C}}(\psi(\widehat{p}_n), \psi(\widehat{p}_m)) > 2$ if $n \neq m$ implies that, for $n \neq m$, $\Pi(G_n) \cap \Pi(G_m) = \emptyset$, therefore, $\{\Pi(G_n \cap \phi^{-1}(B_n(\delta')))\}_{n\in\mathbb{N}}$ is a collection of pairwise disjoint domains of E. Furthermore, the assumption that the restriction of g_n to $B_n(\delta')$ is injective implies that $\Pi|_{\phi^{-1}(B_n(\delta'))\cap G_n}$ is injective, then each domain $\Pi(G_n \cap \phi^{-1}(B_n(\delta')))$ has negative total curvature, uniformly bounded away from zero by (31). Hence, we conclude that the total curvature of E is infinite.

On the other hand, we next prove the total curvature of E is finite. Let $\{p_n\}_{n\in\mathbb{N}} \subset E$ be a divergent sequence of points. The proofs of items 1 and 2 apply and show that $\lim_{n\to\infty} I_E(p_n) = 0$; in particular, I_E is bounded. Since E has nonpositive Gaussian curvature, there are geodesic loops γ_n based at each p_n, and, as explained in the proof of Theorem 1.2, γ_1 and γ_n bound a compact annulus E_n such that $\{E_n\}_{n\in\mathbb{N}}$ exhaust a subend E' of E. A simple argument using Gauss-Bonnet formula shows that E' has finite absolute total curvature at most 2π, which implies that E has finite total curvature. This contradiction shows that $\|A_\phi\|$ is bounded, finishing the proof of Proposition 4.1. q.e.d.

4.3. Asymptotics of annular ends of $\varphi(\Sigma)$: proof of item 5. Next, we proceed with the proof of the theorem by proving item 5, where we analyze the asymptotic behavior of $\varphi(E)$, where E is an annular end representative of an end e of Σ. Fix a cusp end $\mathcal{C} = \mathcal{M}(1)/G(u,v)$ of N. We now describe the standard constant mean curvature annular ends in $\mathcal{C}$. Consider $(k_1,k_2) \in \mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}$ such that the greatest common divisor of k_1 and k_2 equals 1. Let $\tau(k_1,\, k_2)$ be the parabolic isometry of $\mathbb{H}^3$ given by (22) and let again $\psi\colon \mathcal{M}(1)\to\mathcal{C}$ denote the universal covering transformation related to $G(u,v)$ and $\phi(\widetilde{E}) \subset \mathcal{M}(1)$ be an immersed half-plane as in the proof of Proposition 4.1 (see Figure 3).

For each $c\in\mathbb{R}$, let $\mathcal{P}_{0,c}(k_1,k_2)$ be the vertical plane defined by

$$\mathcal{P}_{0,c}(k_1,k_2)=\{(x,y,z)\in\mathbb{H}^3 \,|\, (k_1u_y+k_2v_y)x-(k_1u_x+k_2v_x)y+c=0,\ z>0\}.$$

Then, $\tau(k_1,\, k_2)$ leaves invariant $\mathcal{P}_{0,c}(k_1,k_2)$; hence,

$$\mathcal{A}_{0,c}(k_1,k_2) = \psi(\mathcal{P}_{0,c}(k_1,k_2)\cap\mathcal{M}(1))$$

is a properly embedded totally geodesic annulus in $\mathcal{C}$. Note that the family $\{\mathcal{A}_{0,c}(k_1,k_2)\}_{c\in\mathbb{R}}$ is periodic in the sense that there exists $l>0$ such that for any $c\in\mathbb{R}$, $\mathcal{A}_{0,c}(k_1,k_2) = \mathcal{A}_{0,c+l}(k_1,k_2)$, since for each $(m,n)\in\mathbb{Z}\times\mathbb{Z}$,

$$\tau(m,n)(\mathcal{P}_{0,c}(k_1,k_2)) = \mathcal{P}_{0,c+(k_1n-k_2m)(u_xv_y-u_yv_x)}(k_1,k_2).$$

Finally, for each $c\in\mathbb{R}$, we consider the two families $\{\mathcal{P}^+_{H,c}(k_1, k_2)\}_{H\in(0,1)}$ and $\{\mathcal{P}^-_{H,c}(k_1,k_2)\}_{H\in(0,1)}$ of equidistant surfaces to $\mathcal{P}_{0,c}(k_1,k_2)$, formed by tilted planes of constant mean curvature $H\in(0,1)$ that meet $\{z=0\}$ along the line $\{(k_1u_y+k_2v_y)x-(k_1u_x+k_2v_x)y+c=0\}$, see Figure 5. Each $\mathcal{P}^\pm_{H,c}(k_1,k_2)$ is also invariant by $\tau(k_1,k_2)$, and so each of the surfaces $\mathcal{P}^\pm_{H,c}(k_1,k_2)\cap\mathcal{M}(1)$ descends to $\mathcal{C}$ as a properly embedded annulus $\mathcal{A}^\pm_{H,c}(k_1,k_2)$ of constant mean curvature H, and the family $\{\mathcal{A}^\pm_{H,c}(k_1,k_2)\}_{c\in\mathbb{R}}$ is also periodic with respect to c.

We will prove item 5 by showing that there is a pair of integers (k_1,k_2) such that $\phi(\widetilde{E})$ is asymptotic either to $\mathcal{P}^+_{H,0}(k_1,k_2)$ or to $\mathcal{P}^-_{H,0}(k_1,k_2)$, with multiplicity 1. Furthermore, we will show that $\varphi(E)=\psi(\widetilde{E})$ is asymptotic to $\mathcal{A}^+_{H,0}(k_1,k_2)$ or to $\mathcal{A}^-_{H,0}(k_1,k_2)$, with some finite multiplicity

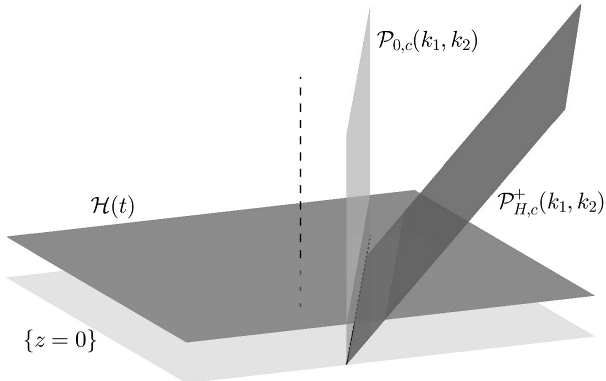

Figure 5. The tilted plane $\mathcal{P}^+_{H,c}(k_1,k_2)$ has constant mean curvature $H = \cos(\alpha) \in (0,1)$, where α is the angle to the plane $\{z=0\}$, and is equidistant to the totally geodesic vertical plane $\mathcal{P}_{0,c}(k_1,k_2)$.

k. Since $\mathcal{P}^+_{H,c}(k_1,k_2)$ is asymptotic to $\mathcal{P}^+_{H,0}(k_1,k_2)$ (and also $\mathcal{P}^-_{H,c}(k_1,k_2)$ is asymptotic to $\mathcal{P}^-_{H,0}(k_1,k_2)$) for every $c \in \mathbb{R}$, this proves the result.

Let $[\alpha_1]$ be the generator of $\pi_1(E)$. Since $\pi_1(\mathcal{C}) \equiv \mathbb{Z}\times\mathbb{Z}$, we can consider $\varphi_*([\alpha_1])$ to be an element of $\mathbb{Z}\times\mathbb{Z}$. Claim 4.2 implies that $\varphi_*([\alpha_1])$ is not the trivial element, hence, there are relatively prime integers k_1, $k_2 \in \mathbb{Z}$ and some $k \in \mathbb{N}$ such that $\varphi_*([\alpha_1]) = k(k_1, k_2)$. Thus, the map $\tau = \tau(kk_1, kk_2)$

$$\tau\colon \quad \begin{array}{ccc} \mathbb{H}^3 & \to & \mathbb{H}^3, \\ p & \mapsto & p + k(k_1 u + k_2 v) \end{array} \tag{32}$$

is such that $\tau(\phi(\widetilde{E})) = \phi(\widetilde{E})$.

Having fixed k, k_1, k_2, we simplify the notation to $\mathcal{P}^\pm_{H,c} = \mathcal{P}^\pm_{H,c}(k_1,k_2)$ and, in the $c=0$ case, to $\mathcal{P}^\pm_H = \mathcal{P}^\pm_{H,0}(k_1,k_2)$. We also let $a = kk_1u_x + kk_2v_x$ and $b = kk_1u_y + kk_2v_y$, so that the map τ of (32) is

$$\tau\colon \quad \begin{array}{ccc} \mathbb{H}^3 & \to & \mathbb{H}^3, \\ (x,y,z) & \mapsto & (x+a, y+b, z), \end{array} \tag{33}$$

the vertical planes $\mathcal{P}_{0,c}$ are

$$\mathcal{P}_{0,c} = \{(x,y,z) \in \mathbb{H}^3 \mid bx - ay + c = 0,\ z > 0\},$$

and the equidistant surfaces $\mathcal{P}^\pm_{H,c}$ are the tilted planes with boundary at $\{z=0\}$ given by the lines $\{bx - ay + c = 0\}$.

Since $\partial\phi(\widetilde{E}) \subset \mathcal{H}(1)$ is invariant under the action of τ and it is a properly immersed curve, it follows that $\partial\phi(\widetilde{E})$ stays at a finite distance to the line $\{(x,y,1) \mid bx - ay = 0\}$. Thus, there is some $c > 0$ such that

$$\partial\phi(\widetilde{E}) \subset \{(x,y,1) \in \mathbb{H}^3 \mid |bx - ay| < c\}.$$

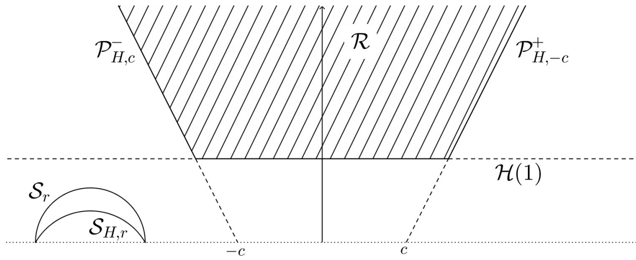

Figure 6. The region $\mathcal{R}$, limited by the three planes $\mathcal{H}(1)$, $\mathcal{P}^-_{H,c}$ and $\mathcal{P}^+_{H,-c}$, and the constant mean curvature H hyperspheres $\mathcal{S}_{H,r}$.

With this, we can prove the following convex hull type property, see Figure 6.

Claim 4.3. *$\phi(\widetilde{E})$ is contained in the region $\mathcal{R}$ of $\mathcal{M}(1)$, whose boundary contains pieces of all the three planes $\mathcal{P}^-_{H,c}$, $\mathcal{H}(1)$ and $\mathcal{P}^+_{H,-c}$.*

Proof of Claim 4.3. First, note that $\varphi(E) \subset \cup_{t\geq 1}\mathcal{T}(t)$. Hence, $\phi(\widetilde{E})$ is contained in $\mathcal{M}(1)$, and, thus, it is never below $\mathcal{H}(1)$. Next, we show that $\phi(\widetilde{E})$ is also never below $\mathcal{P}^-_{H,c}$. In the plane $\{z = 0\}$, let $\{C_r\}_{r>0}$ be a continuous family of circles of radius r, contained in the half plane $\{bx - ay + c < 0\} \cap \{z = 0\}$ and converging, when $r \to \infty$, to the line $\{bx - ay + c = 0\} \cap \{z = 0\}$. Let $\mathcal{S}_r$ be the upper half sphere of radius r, centered in the center of the circle C_r. Then, $\mathcal{S}_r$ is a totally geodesic surface of $\mathbb{H}^3$ and the family $\{\mathcal{S}_r\}_{r>0}$ converges, when $r \to \infty$, to the vertical half-plane $\mathcal{P}_{0,c}$.

Let $\mathcal{S}_{H,r}$ be an equidistant surface to $\mathcal{S}_r$ with constant mean curvature H with respect to the upwards orientation, see Figure 6. When $r \to \infty$, $\mathcal{S}_{H,r}$ converges to $\mathcal{P}^-_{H,c}$. Each $\mathcal{S}_{H,r}$ does not intersect $\partial\phi(\widetilde{E})$, by its construction. Moreover, for r sufficiently small, $\mathcal{S}_{H,r}$ does not intersect $\mathcal{M}(1)$, so it also does not intersect $\phi(\widetilde{E})$. Thus, it follows from the maximum principle that $\mathcal{S}_{H,r} \cap \phi(\widetilde{E}) = \emptyset$ for all $r > 0$, hence, there is no point of $\phi(\widetilde{E})$ below $\mathcal{P}^-_{H,c}$. The same argument proves that there is no point of $\phi(\widetilde{E})$ below $\mathcal{P}^+_{H,-c}$, proving the claim. q.e.d.

Consider the family of hyperbolic isometries of $\mathbb{H}^3$ defined, for each $t > 0$, by

$$\begin{array}{rccc} \sigma_t\colon & \mathbb{H}^3 & \to & \mathbb{H}^3, \\ & (x,y,z) & \mapsto & e^{-t}(x,y,z). \end{array} \tag{34}$$

The planes $\mathcal{P}_0$ and $\mathcal{P}^\pm_H$ are invariant under the action of σ_t, for all $t > 0$, however, the same does not hold for $c \neq 0$, since $\sigma_t(\mathcal{P}_{0,c}) = \mathcal{P}_{0,e^{-t}c}$ and $\sigma_t(\mathcal{P}^\pm_{H,c}) = \mathcal{P}^\pm_{H,e^{-t}c}$. In particular, for every c fixed, $\sigma_t(\mathcal{P}_{0,c})$ converges to

$\mathcal{P}_0$ and $\sigma_t(\mathcal{P}^{\pm}_{H,c})$ converges to $\mathcal{P}^{\pm}_H$ when $t \to \infty$. Moreover, $\sigma_t(\mathcal{H}(1)) = \mathcal{H}(e^{-t})$.

Let $\mathcal{R}_t = \sigma_t(\mathcal{R})$ and let $\widetilde{E}_t = \sigma_t(\phi(\widetilde{E}))$. Then $\mathcal{R}_t$ is the region of $\mathcal{M}(e^{-t})$ bounded by pieces of $\mathcal{P}^-_{H,e^{-t}c}$, $\mathcal{H}(e^{-t})$ and $\mathcal{P}^+_{H,-e^{-t}c}$ and $\widetilde{E}_t \subset \mathcal{R}_t$. We also let $\mathcal{R}_\infty = \lim_{t\to\infty} \mathcal{R}_t$ be the region of $\mathbb{H}^3$ in between the two planes $\mathcal{P}^-_H$ and $\mathcal{P}^+_H$. With this notation, we prove the next claim.

Claim 4.4. *Consider the limit set of the family* $\{\widetilde{E}_t\}_{t\geq 0}$, $\widetilde{E}_\infty = \{p \in \mathbb{H}^3 \mid p = \lim_{n\to\infty} p_n,\ p_n \in \widetilde{E}_{t_n},\ \lim_{n\to\infty} t_n = \infty\} \subset \mathcal{R}_\infty$. *Then, for each* $p \in \widetilde{E}_\infty$ *there exists a complete smooth surface* $\mathcal{L} \subset \widetilde{E}_\infty$, *of constant mean curvature* H, *containing* p. *Moreover,* $\mathcal{L}$ *is invariant under a* 1*-parameter group of parabolic isometries which contains* τ.

Proof of Claim 4.4. Let $p \in \widetilde{E}_\infty$. Since $\widetilde{E}_t = \sigma_t(\phi(\widetilde{E}))$ and σ_t is an ambient isometry, it follows that $\widetilde{E}_t$ is a constant mean curvature H surface with uniformly bounded norm of the second fundamental form. Hence, there is a $\delta \in (0, 1)$ such that for every $t > 0$ and every $q \in \widetilde{E}_t$ of distance at least 1 from $\partial\widetilde{E}_t$, there exists some closed disk component of q in $\widetilde{E}_t \cap \overline{B}_{\mathbb{H}^3}(q, \delta)$, with boundary contained in $\partial B_{\mathbb{H}^3}(q, \delta)$, which is a graph G^q in exponential coordinates, over a disk in $T_q\widetilde{E}_t$, with graphing function having gradient less than 1. We will also assume that δ is chosen sufficiently small so that, for $r \in (0, \delta]$, each of the spheres $\partial B_{\mathbb{H}^3}(q, r)$ intersects G^q transversely in a simple closed curve; now, define $G^q(r)$ to be the corresponding closed subdisk of $G^q = G^q(\delta)$ in the closed ball of radius $r \in (0, \delta]$ and centered at q.

In order to prove the existence of $\mathcal{L}$ as claimed, consider the sequences $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and $\{p_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} t_n = \infty$, $\lim_{n\to\infty} p_n = p$ and, for every $n \in \mathbb{N}$, $p_n \in \widetilde{E}_{t_n}$. For each $n \in \mathbb{N}$ and $r \in (0, \delta]$, let $G_n(r) = G^{p_n}(r) \subset \widetilde{E}_{t_n}$ be the graphs described above, based at p_n; without loss of generality, we may assume that every $p_n \in \widetilde{E}_{t_n}$ has distance at least 1 from ∂E_t, and so the graph $G_n := G_n(\delta)$ exists. Then $\{G_n\}_{n\in\mathbb{N}}$, up to a subsequence, converges to a constant mean curvature H graph $G_\infty \subset \mathbb{H}^3$, with $p \in G_\infty$; by its construction it follows that $G_\infty \subset \widetilde{E}_\infty$. Since δ as above was uniform, we can iterate this argument to extend G_∞ to a complete surface $\mathcal{L} \subset \widetilde{E}_\infty$, containing p and of constant mean curvature H.

For $s \in \mathbb{R}$, let $\tau_s\colon \mathbb{H}^3 \to \mathbb{H}^3$ be the parabolic isometry defined by

$$\tau_s(x, y, z) = (x + as, y + bs, z);$$

recall that $\tau_1 = \tau$ corresponds to the generating covering transformation $\widehat{\tau}$ of $\widetilde{E}$ and τ leaves invariant $\phi(\widetilde{E})$. Our next argument is to prove that for all $s \in \mathbb{R}$, $\tau_s(\mathcal{L}) \subset \mathcal{L}$, which finishes the proof of Claim 4.4.

Since $\sigma_t \circ \tau = \tau_{e^{-t}} \circ \sigma_t$ and $\tau(\phi(\widetilde{E})) = \phi(\widetilde{E})$, it follows that

$$\tau_{e^{-t_n}}(\widetilde{E}_{t_n}) = \tau_{e^{-t_n}}(\sigma_{t_n}(\phi(\widetilde{E}))) = \sigma_{t_n}(\tau(\phi(\widetilde{E}))) = \widetilde{E}_{t_n};$$

in particular, $\tau_{e^{-t_n}}$ leaves invariant $\widetilde{E}_{t_n}$.

Consider the graphs $G_n(r)$ as above. We claim that there exists an $N_0 \in \mathbb{N}$ such that for $n \geq N_0$,

$$\tau_{e^{-t_n}}(G_n(\delta/3)) \subset G_n(\delta/2). \tag{35}$$

Arguing by contradiction, suppose the above statement fails for some n arbitrarily large, and so, after replacing by a subsequence, we may assume that equation (35) fails for all n sufficiently large. Note that the isometries $\tau_{e^{-t_n}}$ converge uniformly on compact subsets of $\mathbb{H}^3$ to the identity map as n approaches infinity. In particular, for n sufficiently large, we have that $\tau_{e^{-t_n}}(\overline{B}_{\mathbb{H}^3}(p_n, \delta/3)) \subset B_{\mathbb{H}^3}(p_n, \delta/2)$. However, since $\widetilde{E}_{t_n}$ is invariant under $\tau_{e^{-t_n}}$, it follows that $\tau_{e^{-t_n}}(G_n(\delta/3)) \subset \widetilde{E}_{t_n}$; hence, there exist disks F_n, $H_n \subset \widetilde{E}$ such that

$$\sigma_{t_n}(\phi(F_n)) = G_n(\delta/2), \quad \sigma_{t_n}(\phi(H_n)) = \tau_{e^{-t_n}}(G_n(\delta/3)).$$

Since $G_n(\delta/2)$ and $\tau_{e^{-t_n}}(G_n(\delta/3))$ have their boundaries in the disjoint respective spheres $\partial B_{\mathbb{H}^3}(p_n, \delta/2)$ and $\tau_{e^{-t_n}}(\partial\overline{B}_{\mathbb{H}^3}(p_n, \delta/3))$, then elementary separation properties imply that $H_n \subset F_n$, $F_n \subset H_n$ or $H_n \cap F_n = \emptyset$. Note that $H_n \not\subset F_n$ since (35) is assumed to fail, and $F_n \not\subset H_n$ since ∂F_n is clearly disjoint from H_n; hence, $H_n \cap F_n = \emptyset$.

The property that $H_n \cap F_n = \emptyset$ gives that the generating covering transformation $\widehat{\tau}$ of the covering space $\Pi\colon \widetilde{E} \to E$ induced by τ is such that $\widehat{\tau}^{-1}(H_n) \subset F_n$ is disjoint from H_n. In particular, $\widehat{\tau}(H_n)$ is disjoint from H_n, which implies that $\Pi|_{H_n}$ is injective, for n sufficiently large. Since $\lim_{n\to\infty} t_n = \infty$, then, after replacing by a subsequence, we may assume that the projected disks $\Pi(H_n) \subset E$ are pairwise disjoint for $n \in \mathbb{N}$. Since the areas of the disks $\Pi(H_n)$ are bounded from below by some $\varepsilon > 0$ and there are an infinite number of them, we conclude that the area of E is infinite; this contradicts item 1 of Theorem 1.3, proving that equation (35) holds for n sufficiently large.

Next, we prove the claim that $\mathcal{L}$ is invariant under the 1-parameter group of isometries $\{\tau_s\}_{s\in\mathbb{R}}$, that is, $\tau_s(\mathcal{L}) = \mathcal{L}$ for every $s \in \mathbb{R}$. Let $\Gamma_n = \{\tau_s(p_n) \mid s \in \mathbb{R}\}$ denote the orbit curve of each p_n through the action of $\{\tau_s\}_{s\in\mathbb{R}}$. Also letting $\Gamma = \{\tau_s(p) \mid s \in \mathbb{R}\}$ be the orbit curve of p, then the curves Γ_n converge to Γ uniformly as $n \to \infty$.

The arguments used to derive equation (35) also show that for any $k \in \mathbb{N}$ there exists an $N_k \in \mathbb{N}$ such that for $n \geq N_k$,

$$\tau_{e^{-t_n}}(G_n(\delta/(k+1))) \subset G_n(\delta/k). \tag{36}$$

After iteration of (36), we have that for any positive integer $j \leq k$, when n is sufficiently large,

$$\tau_{e^{-jt_n}}(G_n(\delta/(k+1))) \subset G_n.$$

Hence, for any positive integer $j < k$, $\tau_{e^{-jt_n}}(p_n) \in G_n$ when n is sufficiently large. Therefore, as k is arbitrary, Γ_n intersects G_n in an arbitrarily large number of points as $n \to \infty$. Since the analytic arcs Γ_n converge smoothly to the analytic arc Γ on compact subsets of $\mathbb{H}^3$, the analytic curve Γ has infinite order contact with the disk G_∞, which implies that Γ is contained in $\mathcal{L}$. Since p was chosen arbitrarily, this proves that the orbit of any $q \in \mathcal{L}$, under the action of $\{\tau_s\}_{s\in\mathbb{R}}$ is also contained in $\mathcal{L}$, which completes the proof of Claim 4.4. q.e.d.

Claim 4.5. *For any leaf $\mathcal{L}$ as given in Claim 4.4, then either $\mathcal{L} = \mathcal{P}_H^+$ or $\mathcal{L} = \mathcal{P}_H^-$.*

Proof of Claim 4.5. Any surface invariant under a 1-parameter group $\mathbb{G}$ of parabolic isometries is called a *parabolic-invariant surface*, and the intersection of any such surface with a totally geodesic surface perpendicular to the orbit curves of $\mathbb{G}$ is called a *profile curve*. The classification of constant mean curvature parabolic-invariant surfaces is well-explained by Gomes in Chapter 3 of his doctoral thesis [**10**]. It follows from this classification that a constant mean curvature parabolic invariant surface contained in $\mathcal{R}_\infty$ is either $\mathcal{P}_H^+$, $\mathcal{P}_H^-$ or, after a homothety, it is a certain surface $\mathcal{S}$. Furthermore, such a surface $\mathcal{S}$ has a specific value of its mean curvature, is symmetric with respect to the plane $\mathcal{P}_0$ and attains a maximal height, which is isolated in any profile curve, see Figure 7. Our next argument rules out the possibility $\mathcal{L} = \mathcal{S}$, proving the claim.

Arguing by contradiction, suppose $\mathcal{L} = \mathcal{S}$. Then there exists $p \in \mathcal{L}$ where the height function of $\mathcal{L}$ attains its maximal value; in particular, $T_p\mathcal{L}$ is a horizontal plane in the sense that $T_p\mathcal{L} = T_p\mathcal{H}(h_1)$, for some $h_1 > 0$.

The uniform bound on the second fundamental form of $\phi(\widetilde{E})$ yields a $\delta > 0$ such that for all $t > 0$ and every point $x \in \widetilde{E}_t$, a neighborhood of $\widetilde{E}_t$ containing x is a graph over a disk of radius δ in $T_x\widetilde{E}_t$. We may assume such δ is sufficiently small so that a neighborhood $G \subset \mathcal{L}$ containing p is a vertical graph of small gradient over the horizontal disk $D = B(p, \delta) \subset \mathcal{H}(h_1)$. Hence, the height function f of G may be written as $f \colon D \to \mathbb{R}$, a function over D in such a way that

$$G = \{(x, y, f(x, y)) \mid (x, y, h_1) \in D\}.$$

Consider a sequence of points $p_n \in \widetilde{E}_{t_n}$ such that $\lim_{n\to\infty} p_n = p$ and $\lim_{n\to\infty} t_n = \infty$. After passing to a subsequence, we may assume that there are neighborhoods $G_n \subset \widetilde{E}_{t_n}$ containing p_n which are also vertical

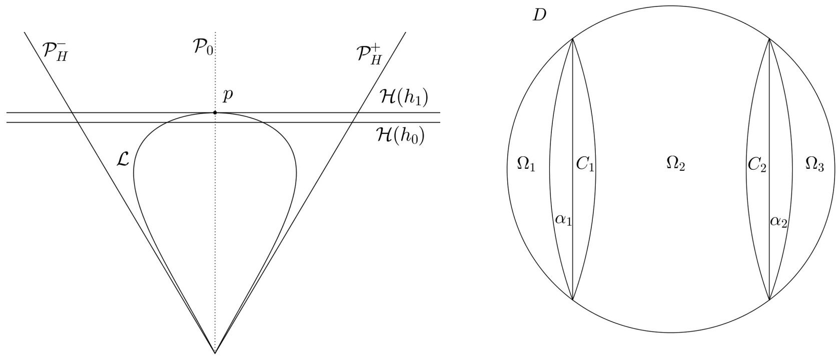

Figure 7. If $\mathcal{L} = \mathcal{S}$, a neighborhood G of $\mathcal{L}$ containing p is a horizontal graph over a domain D in the plane $\mathcal{H}(h_0)$'.

graphs over D, in the sense that there exist functions $f_n\colon D \to \mathbb{R}$ such that

$$G_n = \{(x, y, f_n(x, y)) \mid (x, y, h_1) \in D\}.$$

Since G_n converges smoothly to G, the sequence $(f_n)_{n\in\mathbb{N}}$ converges in the $C^{2,\alpha}$ norm to f.

Let Γ be the profile curve of $\mathcal{L}$ through p. Since the height function of Γ attains a strict local maximum at p, there is $h_0 < h_1$ such that $\mathcal{H}(h_0)$ intersects Γ in two points; in particular, $\mathcal{H}(h_0)$ intersects G in two disjoint horizontal segments α_1, α_2, that separate G into three disjoint regions (see Figure 7, right). Let $\varepsilon = \frac{h_1 - h_0}{2} > 0$, then there exist $n_0 \in \mathbb{N}$ and open, connected and disjoint domains Ω_1, Ω_2, $\Omega_3 \subset D$ such that for every $n \geq n_0$ it holds:

$$\begin{cases} f_n < h_0 - \varepsilon & \text{in } \Omega_1 \cup \Omega_3, \\ f_n > h_0 + \varepsilon & \text{in } \Omega_2. \end{cases}$$

In particular, $D \setminus (\Omega_1 \cup \Omega_2 \cup \Omega_3)$ is composed by two disjoint, connected domains C_1, C_2 such that $\alpha_i \subset C_i$.

For each $n \geq n_0$, consider $f_n^{-1}(\{h_0\}) \subset D$. It contains two connected curves, one contained in C_1 and the other contained in C_2, which map via f_n to two disjoint curves γ_n^1, $\gamma_n^2 \subset G_n \cap \mathcal{H}(h_0)$. Using the same arguments of the proof of Claim 4.4, we can fix points $q_n^1 \in \gamma_n^1$, $q_n^2 \in \gamma_n^2$ and obtain that the action of $\tau_{e^{-t_n}}$ is such that both $\tau_{e^{-t_n}}(q_n^1) \in \gamma_n^1$ and $\tau_{e^{-t_n}}(q_n^2) \in \gamma_n^2$.

Fix any such $n \geq n_0$ and let

$$\Gamma_n^1 = \sigma_{-t_n}(\gamma_n^1) \subset \phi(\widetilde{E}), \quad \Gamma_n^2 = \sigma_{-t_n}(\gamma_n^2) \subset \phi(\widetilde{E}),$$

$$\widehat{q}_n^1 = \sigma_{-t_n}(q_n^1) \in \Gamma_n^1, \quad \widehat{q}_n^2 = \sigma_{-t_n}(q_n^2) \in \Gamma_n^2.$$

Since $\tau_{e^{-t_n}}(q_n^1) \in \gamma_n^1$, it follows that $\tau(\widehat{q}_n^1) \in \Gamma_n^1$, hence, $\varphi^{-1}(\Pi(\Gamma_n^1))$ contains a nontrivial closed curve β_n^1 in E. Analogously, $\varphi^{-1}(\Pi(\Gamma_n^2))$ also contains a nontrivial closed curve β_n^2 in E.

We next show that $\beta_n^1 \cap \beta_n^2 = \emptyset$ for n sufficiently large; this proves the claim since $\Gamma_1, \Gamma_2 \subset \mathcal{H}(e^{t_n}h_0)$ gives $\varphi(\beta_n^1), \varphi(\beta_n^2) \subset \varphi^{-1}\mathcal{T}(e^{t_n}h_0)$, which is a contradiction with Claim 4.2 (since h_0 can be chosen generically). Note that $G(u,v)$, σ_t and τ_s preserve the left invariant Gauss map (see the discussion in the proof of Proposition 4.1) of $\varphi(E)$, $g\colon E \to \mathbb{S}^2$. Moreover, when $n \to \infty$, $g|_{\beta_n^1}$ and $g|_{\beta_n^2}$ converge respectively to the values of the Gauss map of $\mathcal{L}$ along α_1 and α_2, which are distinct. In particular, for n large enough we have $\beta_n^1 \cap \beta_n^2 = \emptyset$. As explained before, this is a contradiction that proves the claim. q.e.d.

Claim 4.6. *There exists $t_* > 1$ such that $\phi(\widetilde{E}) \cap \mathcal{M}(t_*)$ is a topological half-plane which is a horizontal graph over $\mathcal{P}_0$.*

Proof. Fix $\varepsilon > 0$. By Claim 4.5, there exist t_0 depending on ε such that the left invariant Gauss map g of $\phi(\widetilde{E}) \cap \mathcal{M}(t_0)$ lies in an ε-neighborhood of v^+, v^-, which are the values assumed by the left invariant Gauss maps of $\mathcal{P}_H^+$, $\mathcal{P}_H^-$. We denote such neighborhoods respectively by V^+, V^- and note that, if ε is sufficiently small, $V^+ \cap V^- = \emptyset$ and both V^+, V^- stay at a positive distance to the great circle in $\mathbb{S}^2$ of vectors perpendicular to the image vector of the Gauss map of $\mathcal{P}_0$.

Note that t_0 is such that $\mathcal{T}(t)$ intersects $\varphi(E)$ transversely for all $t \geq t_0$. Let $\alpha \subset E$ be the unique homotopically nontrivial closed curve in $\varphi^{-1}(\mathcal{T}(t_0))$ given by Claim 4.2 and let $E' \subset E$ be the subannular end of E determined by α, then $\varphi(E') = \varphi(E) \cap \cup_{t \geq t_0}\mathcal{T}(t)$, hence, if $\widetilde{E}' = \Pi^{-1}(E')$, then it is topologically a half plane and $\phi^{-1}(\mathcal{M}(t_0)) = \widetilde{E}'$.

To finish the proof of the claim, note that the image $g(\phi(\widetilde{E}'))$ of the Gauss map of $\phi(\widetilde{E}')$ is connected; hence, either $g(\phi(\widetilde{E}')) \subset V^+$ or $g(\phi(\widetilde{E}')) \subset V^-$. In either case, it follows that $\phi(\widetilde{E}') = \phi(\widetilde{E}) \cap \mathcal{M}(t_0)$ is a graph over $\mathcal{P}_0$. q.e.d.

Note that when $H = 0$, $\widetilde{E}_\infty = \mathcal{P}_0$ by Claim 4.4. Hence, Claim 4.6 gives that $\widetilde{E}_\infty$ contains a single leaf $\mathcal{L} = \mathcal{P}_0$, from which the asymptotic behavior follows. This special case was proved previously by Collin, Hauswirth and Rosenberg in [**5**], Theorem 1.1.

Assume now that $H \in (0,1)$ and that $\phi(\widetilde{E})$ is oriented by its mean curvature vector $\vec{H}$. To complete the proof of item 5, we use the notation in the proof of Claim 4.6: if $g(\phi(\widetilde{E}) \cap \mathcal{M}(t_0)) \subset V^+$ (resp. V^-), then $\widetilde{E}_\infty$ contains a single leaf $\mathcal{L} = \mathcal{P}_H^+$ (resp. $\mathcal{P}_H^-$). In either case, the restriction of $\phi(\widetilde{E})$ to $\mathcal{M}(t_0)$ is a graph over its limit set. Since $\|A_\phi\|$ is uniformly bounded, this graphing function converges, with multiplicity 1, smoothly to 0 along $\mathcal{H}(t)$, when $t \to \infty$.

Note that the covering transformation $\psi\colon \mathcal{M}(1) \to \mathcal{C}$ is a finite multiple k of a generator (k_1, k_2) in the fundamental group $\pi_1(\mathcal{C})$. Therefore, $\varphi(E) = \psi(\phi(\widetilde{E}))$ is asymptotic, with multiplicity k, to the embedded annulus $\mathcal{A}_H^+ = \psi(\mathcal{P}_H^+)$ or $\mathcal{A}_H^- = \psi(\mathcal{P}_H^-)$. This completes the proof of Theorem 1.3.

4.4. Some remarks on Theorem 1.3. A simple consequence of Theorem 1.3 is the following corollary, which generalizes to the bounded mean curvature case some of the corollaries of Theorem 1.1 of [**5**].

Corollary 4.7. *Let N be a complete, noncompact, hyperbolic 3-manifold of finite volume. Let Σ be a complete, properly immersed surface in N of finite topology with mean curvature function H_Σ satisfying $|H_\Sigma| \le H < 1$. Then, the area of Σ satisfies*

$$\text{Area}(\Sigma) \le \frac{2\pi}{H^2 - 1}\chi(\Sigma), \tag{37}$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ and equality in (37) holds if and only if Σ is a totally umbilic surface of constant mean curvature H. In particular, if Σ has genus zero, it has at least three ends.

Proof. The proof is straightforward and relies only on item 3 of Theorem 1.3 and on the Gauss equation, after observing that if Σ has genus g and n ends, $\chi(\Sigma) = 2 - 2g - n$. The details are left to the reader. q.e.d.

The next proposition demonstrates some differences between the $H < 1$ case treated in Theorem 1.3 and the $H \ge 1$ case.

Proposition 4.8. *For any $H \ge 1$ and any hyperbolic 3-manifold N of finite volume there exists a complete, properly immersed annulus $\mathcal{A}$ with constant mean curvature H, and $\mathcal{A}$ can be chosen to satisfy:*

1) *$\mathcal{A}$ has infinite area, positive injectivity radius and bounded norm of its second fundamental form.*
2) *Any lift of $\mathcal{A}$ to the hyperbolic 3-space is a properly embedded rotationally symmetric annulus.*

Proof. After possibly passing to the oriented two-sheeted covering of N, we may assume that N is orientable. Let $\mathcal{C} = \cup_{t\ge 1}\mathcal{T}(t)$ be a parameterized cusp end of N as described in the proof of Theorem 1.3. Let $\Pi\colon \mathbb{H}^3 \to N$ be the universal covering map of N and assume without loss of generality that $\Pi(\{z = t\}) = \mathcal{T}(t)$. Consider the fundamental group $\Gamma = \pi_1(N) \subset \text{ISO}(\mathbb{H}^3)$ of N as a subgroup of the isometry group of $\mathbb{H}^3$; each element $\varphi \in \Gamma$ satisfies $\Pi \circ \varphi = \Pi$. Since N has finite volume there exists a subgroup G of Γ isomorphic to $\mathbb{Z} \times \mathbb{Z}$ such that $\{z \ge 1\}/G \simeq \mathcal{C}$, and there exists some $\alpha \in \Gamma \setminus G$.

Let $p \in \partial_\infty \mathbb{H}^3$ be the point at infinity of the horospheres $\{z = t\}$. Note that α induces a map on $\partial_\infty \mathbb{H}^3$ such that $\alpha(p) \ne p$. Let $q = \alpha(p)$

and let γ be the complete geodesic of $\mathbb{H}^3$ whose points at infinity are p and q. Let A be an embedded rotationally symmetric annulus around γ of constant mean curvature H. If $H > 1$, A is a Delaunay surface (see [**11**]), which is periodic and, therefore, is a bounded distance from γ; in particular, $\Pi(A)$ is properly immersed in N. If $H = 1$, then A is called a catenoid cousin (see, for instance, [**2**]) and the intersections $A \cap \{z = t\}$, $A \cap \alpha(\{z = t\})$ are circles for t sufficiently large, hence, again $\Pi(A)$ is a properly immersed surface in N. In either case, the properties 1 and 2 hold. q.e.d.

5. Appendix

The proof of item D of Theorem 1.2 used the next elementary intrinsic result for annular ends of complete surfaces of nonpositive Gaussian curvature. As we did not find its statement in the literature, we present its proof in this appendix.

Lemma 5.1. *Let Σ be a complete surface of finite topology with nonpositive Gaussian curvature. Let e be an end of Σ and E be an annular end representative. Then for any divergent sequence of points $\{p_n\}_{n\in\mathbb{N}}$ in E,*

$$\lim_{n\to\infty} I_\Sigma(p_n) = I_\Sigma^\infty(e) \in [0, \infty].$$

Proof. When Σ is simply connected, I_Σ is infinite at every point of Σ; hence, the lemma holds in this case. Assume now that Σ is not simply connected, thus, $I_\Sigma(p)$ is finite for every $p \in \Sigma$. Let e and E be as in the statement of the lemma and let I_E denote the restriction of the injectivity radius function of Σ to E.

To prove the lemma it suffices to show that given any two intrinsically divergent sequences of points $\{p_n\}_{n\in\mathbb{N}}$, $\{q_n\}_{n\in\mathbb{N}}$ in E, then the two limits $\lim_{n\to\infty} I_E(p_n)$, $\lim_{n\to\infty} I_E(q_n)$ exist in $[0, \infty]$ and

$$\lim_{n\to\infty} I_E(p_n) = \lim_{n\to\infty} I_E(q_n).$$

The failure of the previous statement implies that (after possibly passing to subsequences) there exist intrinsically divergent sequences $\{p_n\}_{n\in\mathbb{N}}$, $\{q_n\}_{n\in\mathbb{N}}$ in E such that

$$\lim_{n\to\infty} I_E(p_n) = \ell \in [0, \infty), \quad \lim_{n\to\infty} I_E(q_n) = L \in (\ell, \infty].$$

There exist embedded geodesic loops γ_n, Γ_n based respectively at the points p_n, q_n with $\mathrm{Length}(\gamma_n) = 2I_E(p_n) = 2\ell_n$, $\mathrm{Length}(\Gamma_n) = 2I_E(q_n) = 2L_n$, and such that, for all $n \in \mathbb{N}$,

$$\ell_n < \ell + \varepsilon, \quad L_n > \ell + 3\varepsilon, \tag{38}$$

for some $\varepsilon > 0$. After passing to subsequences, we will assume that

1) the geodesic loops γ_n form a pairwise disjoint family;

2) for all $n, k \in \mathbb{N}$, p_{n+k} lies in the annular subend W_n of E with boundary γ_n;
3) q_n lies in the compact annulus in E bounded by γ_n and γ_{n+1}.

Since I_E is continuous and W_n is connected, there exists a point $q'_n \in W_n \setminus W_{n+1}$ with $I(q'_n) \in (\ell + 3\varepsilon, \ell + 4\varepsilon)$. Hence, replacing the points q_n by the points q'_n, we may assume that L is a finite number.

Let E_n be the compact annulus in E bounded by Γ_n and Γ_{n+1}. By the same argument as in the proof of Proposition 3.1, since ℓ and L are finite, we may assume that $\{\gamma_n, \Gamma_n\}_{n\in\mathbb{N}}$ is a collection of pairwise disjoint curves, with $\gamma_n \subset E_n$.

We claim that there exists a smooth, homotopically nontrivial, simple closed geodesic $\alpha_n \subset \mathrm{Int}(E_n)$, with length at most ℓ_n. Let Λ_n be the set of simple closed rectifiable curves in E_n homotopic to γ_n. If $\beta \in \Lambda_n$ admits a point p in $\beta \cap B_E(q_n, \varepsilon)$, then $\mathrm{Length}(\beta) > 2\ell_n$. Indeed, if $\mathrm{Length}(\beta) \leq 2\ell_n$, it would follow from the triangle inequality and from (38) that, for every $x \in \beta$ we have $d_E(q_n, x) \leq L_n$. Since $B_E(q_n, L_n)$ is simply connected, β is homotopically trivial in E, which implies that it is also homotopically trivial in E_n, contradicting $\beta \in \Lambda_n$. A similar argument shows that if β admits a point $p \in B_E(q_{n+1}, \varepsilon)$, then $\mathrm{Length}(\beta) > 2\ell_n$.

Hence, any minimizing sequence in Λ_n can be assumed to stay at least at a distance ε from the pair of points q_n, q_{n+1}, where the boundary of E_n is not smooth. Then standard minimization arguments imply that there exists a smooth closed geodesic $\alpha_n \in \Lambda_n$ which minimizes the lengths of curves in Λ_n, and, since E_n is an annulus and α_n is the generator of the fundamental group of E_n, α_n is a simple closed geodesic.

The Gauss-Bonnet formula implies that each compact annulus A_n bounded by α_1 and α_{n+1} is flat, thus, E has a subend A_∞ that is isometric to a flat cylinder with boundary being a simple closed geodesic. In fact, the flat A_∞ is easily seen to be isometric to a metric product of a circle with $[0, \infty)$, which implies that the injectivity radius function has the constant value $\mathrm{Length}(\alpha_1)/2$ on A_∞. This contradicts (38), proving the lemma. q.e.d.

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