J. DIFFERENTIAL GEOMETRY
110 (2018) 543-572

EXISTENCE OF SOLUTIONS TO THE EVEN DUAL MINKOWSKI PROBLEM

Yiming Zhao

Abstract

Recently, Huang, Lutwak, Yang & Zhang discovered the duals of Federer's curvature measures within the dual Brunn–Minkowski theory and stated the "Minkowski problem" associated with these new measures. As they showed, this *dual Minkowski problem* has as special cases *the Aleksandrov problem* (when the index is 0) and the *logarithmic Minkowski problem* (when the index is the dimension of the ambient space) – two problems that were never imagined to be connected in any way. Huang, Lutwak, Yang & Zhang established sufficient conditions to guarantee existence of solution to the dual Minkowski problem in the even setting. In this work, existence of solution to the even dual Minkowski problem is established under new sufficiency conditions. It was recently shown by Böröczky, Henk & Pollehn that these new sufficiency conditions are also necessary.

1. Introduction

The classical Brunn–Minkowski theory sits at the core of convex geometry. The family of area measures $S_j(K,\cdot)$, introduced by Fenchel & Jessen and Aleksandrov (see Section 4.2 [**56**]), is one of the fundamental families of geometric measures in the Brunn–Minkowski theory. The Minkowski–Christoffel problem asks for necessary and sufficient conditions on a given measure so that it is precisely the j-th area measure of a convex body (compact, convex subset of $\mathbb{R}^n$ with non-empty interior). A major breakthrough regarding the Minkowski–Christoffel problem was recently achieved by Guan & Ma [**27**]. When $j = n-1$, this problem is the classical Minkowski problem, which, in the smooth case, is the problem of prescribing Gauss curvature (given as a function of the normals). Important regularity results for the Minkowski problem are due to Cheng & Yau [**13**], etc. When $j = 1$, this problem is known

Mathematics Subject Classification. 52A40, 52A38, 35J20.

Key words and phrases. The dual Minkowski problem, dual curvature measures, the dual Brunn–Minkowski theory, Monge–Ampère equation.

Received May 12, 2016.

as the Christoffel problem. Apart from area measures, another important family of measures in the Brunn–Minkowski theory consists of the curvature measures $C_j(K,\cdot)$ introduced by Federer (see page 224 [**56**]). The characterization problem for the curvature measure $C_0(K,\cdot)$, also known as Aleksandrov's integral curvature, is the famous Aleksandrov problem and is a counterpart of the classical Minkowski problem.

In addition, to the classical Brunn–Minkowski theory, the L_p Brunn–Minkowski theory came to life roughly two decades ago. The birth of the L_p Brunn–Minkowski theory can be credited to Lutwak [**42**, **43**] when he began systematically investigating the p-Minkowski combination (studied earlier by Firey, see, e.g., [**56**]) and discovered the fundamental L_p surface area measure $S^{(p)}(K,\cdot)$. Since then, the theory has quickly become a major focus of convex geometry. The characterization problem for the L_p surface area measure $S^{(p)}(K,\cdot)$ is known as the L_p Minkowski problem. When $p=1$, it is the classical Minkowski problem. When $p>1$, the L_p Minkowski problem was solved by Chou & Wang [**14**]. See also Hug, Lutwak, Yang & Zhang (Hug-LYZ) [**33**]. Two important unsolved singular cases of the L_p Minkowski problem are: the logarithmic Minkowski problem (prescribing the cone volume measure $S^{(0)}(K,\cdot)$) and the centro-affine Minkowski problem (prescribing the centro-affine surface area measure $S^{(-n)}(K,\cdot)$).

Another important theory in modern convex geometry is the dual Brunn–Minkowski theory. The dual Brunn–Minkowski theory, introduced by Lutwak in 1975, is a theory that is in a sense dual to the classical Brunn–Minkowski theory. A good discussion of the dual Brunn–Minkowski theory can be found in Section 9.3 of Schneider's classical volume [**56**]. Quoting from Gardner, Hug & Weil [**19**]: "The dual Brunn–Minkowski theory can count among its successes the solution of the Busemann–Petty problem in [**16**], [**20**], [**41**], and [**69**]. It also has connections and applications to integral geometry, Minkowski geometry, the local theory of Banach spaces, and stereology; see [**17**] and the references given there." The dual theory studies interior properties of convex bodies while the classical theory is most effective in dealing with boundary information of convex bodies. In the recent revolutionary work [**32**], Huang-LYZ discovered fundamental geometric measures, duals of Federer's curvature measures, called dual curvature measures $\widetilde{C}_j(K,\cdot)$, in the dual Brunn–Minkowski theory. This new family of measures magically connects the well-known cone volume measure ($j=n$) and Aleksandrov's integral curvature ($j=0$). These measures were never imagined to be related.

Huang-LYZ [**32**] asked for necessary and sufficient conditions on a given measure μ on S^{n-1} so that it is precisely the j-th dual curvature measure of a convex body. This problem is called *the dual Minkowski problem*. The dual Minkowski problem has the Aleksandrov problem

($j = 0$) and the logarithmic Minkowski problem ($j = n$) as special cases.

Since the unit balls of finite dimensional Banach spaces are origin-symmetric convex bodies and the dual curvature measure of an origin-symmetric convex body is even, it is of great interest to study the following *even dual Minkowski problem.*

The even dual Minkowski problem: *Given an even finite Borel measure μ on S^{n-1} and $j \in \{0, 1, \cdots, n\}$, find necessary and sufficient condition(s) on μ so that there exists an origin-symmetric convex body K such that $\mu(\cdot) = \widetilde{C}_j(K, \cdot)$.*

When the given measure μ has a density, solving the (even) dual Minkowski problem is equivalent to solving the following Monge–Ampère type equation on S^{n-1},

$$\frac{1}{n} h(v) |\nabla_{S^{n-1}} h(v) + h(v)v|^{j-n} \det(H(v) + h(v)I) = f(v), \tag{1.1}$$

where the given data f is a non-negative (even) integrable function on S^{n-1}. Here $\nabla_{S^{n-1}} h$ is the gradient of h on S^{n-1}, H is the Hessian matrix of h with respect to an orthonormal frame on S^{n-1}, and I is the identity matrix.

It is perhaps not surprising that the even dual Minkowski problem is connected to subspace concentration, since subspace concentration is key to the solution of the even logarithmic Minkowski problem, see Böröczky-LYZ [**9**]. In Huang-LYZ [**32**], subspace mass inequalities which limit the amount of concentration a measure can have within subspaces were given and they were proven to be sufficient for the existence of a solution to the dual Minkowski problem for indices other than 0. As one may see from [**32**], it turns out that the techniques and estimates needed to solve the even dual Minkowski problem for intermediate indices are significantly more delicate than those required by solving the even logarithmic Minkowski problem. In the Appendix of the current paper, when $j = 2, \cdots, n-1$, important examples of convex bodies, whose j-th dual curvature measures violate the subspace mass inequalities given in [**32**], are exhibited. A new set of subspace mass inequalities will be presented and it will be shown that they are sufficient for the existence part of the dual Minkowski problem. The key estimate is in Lemma 4.1. Very recently, Böröczky, Henk & Pollehn [**7**] showed that the new set of subspace mass inequalities are also necessary for the existence part of the even dual Minkowski problem. One should note that solving the dual Minkowski problem with measures as the given data is more difficult than solving the already complicated partial differential equation (1.1). There appears to be no approximation argument known that would reduce the general problem to solving just (1.1). In fact, when the given measure has a density, the measure trivially satisfies any subspace mass inequalities. It is precisely when the measure has a singular part that the constants appearing in the subspace mass inequalities become critical.

Note that the examples exhibited in Appendix and the new subspace mass inequalities were independently discovered by Böröczky, Henk & Pollehn [**7**] (without showing that the inequalities are sufficient for the existence of a solution to the even dual Minkowski problem).

Quermassintegrals, which include volume and surface area, are the fundamental geometric invariants in the classical Brunn–Minkowski theory. Quermassintegrals have strong geometric significance in that they are proportional to the mean of areas of orthogonal projections of the given convex body onto all lower dimensional subspaces (of a given dimension) of $\mathbb{R}^n$ and they are (up to a constant) independent of the dimension of the ambient vector space. In particular, for $j = 1, \cdots, n$, the $(n-j)$-th quermassintegral of a convex body K may be defined by

$$W_{n-j}(K) = \frac{\omega_n}{\omega_j} \int_{G(n,j)} V_j(K|\xi) d\xi, \tag{1.2}$$

where V_j is Lebesgue measure in $\mathbb{R}^j$ and the integration is with respect to the Haar measure on the Grassmannian manifold $G(n, j)$ containing all j-dimensional subspaces of $\mathbb{R}^n$. Here $K|\xi$ stands for the image of the orthogonal projection of K onto $\xi \in G(n, j)$ and ω_j is the j-dimensional volume of the unit ball in $\mathbb{R}^j$. For $j = 0, \cdots, n-1$, the area measure $S_j(K, \cdot)$ can be defined to be the unique Borel measure on S^{n-1} that makes the following variational formula for the quermassintegral valid for each convex body L in $\mathbb{R}^n$,

$$\frac{d}{dt} W_{n-j-1}([h_t])\bigg|_{t=0^+} = \int_{S^{n-1}} h_L(v) dS_j(K, v), \tag{1.3}$$

where $[h_t]$ is the family of convex bodies defined by $[h_t] = \{x \in \mathbb{R}^n : x \cdot v \le h_K(v) + t h_L(v)\}$ for all sufficiently small t, and h_K, h_L are support functions of K, L.

The Minkowski–Christoffel problem is the problem of characterizing area measures.

When $j = n-1$, the Minkowski–Christoffel problem is the classical Minkowski problem. The problem was first studied by Minkowski when the given measure is either discrete or has a continuous density and later solved by Aleksandrov and Fenchel & Jessen for arbitrary measures, see, for example, page 461 in Schneider [**56**]. Important contributions towards the regularity of the solution to the Minkowski problem include Caffarelli [**11**], Cheng & Yau [**13**], Nirenberg [**49**], etc. The solution to the Minkowski problem is a critical ingredient in the proof of Zhang's affine Sobolev inequality, an affine inequality that is stronger than the classical Sobolev inequality, see [**67**].

When $j = 1$, the Minkowski–Christoffel problem is known as the Christoffel problem, which was solved independently by Firey and Berg, see, for example, Section 8.3.2 in [**56**]. A short approach to their result can be found in Grinberg & Zhang [**22**]. A Fourier transform approach can be found in Goodey, Yaskin & Yaskina [**21**]. For polytopes, a direct treatment was given by Schneider [**54**].

For intermediate indices $1 < j < n-1$, the Minkowski–Christoffel problem has remained open for a long time. From PDE point of view, this problem and its variants were thoroughly studied in, for example, Guan & Guan [**23**], Guan & Ma [**27**], and Sheng, Trudinger & Wang [**60**].

Apart from area measures, another important family of measures in the Brunn–Minkowski theory contains the celebrated curvature measures $C_j(K,\cdot)$ introduced by Federer for sets of positive reach, see page 224 in Schneider [**56**]. A simpler and elegant introduction of curvature measures for convex bodies was given by Schneider [**55**]. The characterization problem for curvature measures may be called the general Aleksandrov problem. Progress on this problem was made by Guan, Li & Li [**24**] and Guan, Lin & Ma [**26**].

In particular, the 0-th curvature measure $C_0(K,\cdot)$ is also known as Aleksandrov's integral curvature. The characterization problem, in this case, is the Aleksandrov problem. A complete solution was given by Aleksandrov [**1**] using his mapping lemma. See also Guan & Li [**25**], Oliker [**50**], and Wang [**64**] for other works on this problem and its variant.

The L_p surface area measure $S^{(p)}(K,\cdot)$, introduced by Lutwak [**42**, **43**], is central to the rapid-developing L_p Brunn–Minkowski theory. The L_p Minkowski problem is the characterization problem for L_p surface area measures. When $p = 1$, the L_p Minkowski problem is the same as the classical Minkowski problem. The solution, when $p > 1$, was given by Chou & Wang [**14**]. See also Chen [**12**], Hug-LYZ [**33**], Lutwak [**42**], Lutwak & Oliker [**44**], LYZ [**46**], Jian, Lu & Wang [**34**], and Zhu [**73**, **74**]. The solution to the L_p Minkowski problem has led to some powerful analytic affine isoperimetric inequalities, see, for example, Haberl & Schuster [**30**], LYZ [**45**], Wang [**63**].

The L_p Minkowski problem contains two major unsolved cases.

When $p = -n$, the L_{-n} surface area measure $S^{(-n)}(K,\cdot)$ is also known as the centro-affine surface area measure whose density in the smooth case is the centro-affine Gauss curvature. The characterization problem, in this case, is the centro-affine Minkowski problem posed in Chou & Wang [**14**]. See also Jian, Lu & Zhu [**35**], Lu & Wang [**36**], Zhu [**72**], etc., on this problem.

When $p = 0$, the L_0 surface area measure $S^{(0)}(K,\cdot)$ is the cone volume measure whose total measure is the volume of K. Cone volume measure is the only one among all L_p surface area measures that is $\mathrm{SL}(n)$ invariant. It is still being intensively studied, see, for example, Barthe, Guédon, Mendelson & Naor [4], Böröczky & Henk [6], Böröczky-LYZ [8, 9, 10], Henk & Linke [31], Ludwig [38], Ludwig & Reitzner [40], Naor [47], Naor & Romik [48], Paouris & Werner [51], Stancu [61, 62], Xiong [66], Zhu [71], and Zou & Xiong [75]. The characterization problem for the cone volume measure is the logarithmic Minkowski problem. A complete solution to the existence part of the logarithmic Minkowski problem, when restricting to even measures and the class of origin-symmetric convex bodies, was recently given by Böröczky-LYZ [9]. The key condition is about measure concentration. In the general case (non-even case), different efforts have been made by Böröczky, Hegedűs & Zhu [5], Stancu [61, 62], and Zhu [71]. The logarithmic Minkowski problem has strong connections with isotropic measures (Böröczky-LYZ [10]), curvature flows (Andrews [2, 3]), and the log-Brunn–Minkowski inequality (Böröczky-LYZ [8], Colesanti, Livshyts & Marsiglietti [15], Rotem [52], Saroglou [53], Xi & Leng [65]), an inequality stronger than the classical Brunn–Minkowski inequality.

As Lutwak [41] showed, if the orthogonal projection in (1.2) is replaced by intersection, we get the fundamental geometric invariants in the dual Brunn–Minkowski theory. The $(n-j)$-th dual quermassintegral $\widetilde{W}_{n-j}(K)$ may be defined by

$$\widetilde{W}_{n-j}(K)=\frac{\omega_n}{\omega_j}\int_{G(n,j)}V_j(K\cap\xi)d\xi. \tag{1.4}$$

Compare (1.2) with (1.4). The following is a special case of the variational formula for the dual quermassintegral established in Huang-LYZ [32]. For $j=1,\cdots,n$ and a convex body K containing the origin in its interior, the following holds for each convex body L in $\mathbb{R}^n$:

$$\frac{d}{dt}\widetilde{W}_{n-j}([h_t])\bigg|_{t=0^+}=j\int_{S^{n-1}}h_L(v)d\widetilde{C}_j(K,v). \tag{1.5}$$

Here $[h_t]$ is the family of convex bodies defined by $[h_t]=\{x\in\mathbb{R}^n : x\cdot v\le h_K(v)e^{t\cdot h_L(v)}\}$ for all sufficiently small t. The Borel measure $\widetilde{C}_j(K,\cdot)$ uniquely determined by (1.5) is called the *j-th dual curvature measure*. The similarity between (1.3) and (1.5) is remarkable. There is a natural way of extending (1.5) to $j=0$ and, thus, defining $\widetilde{C}_0(K,\cdot)$, see Theorem 4.5 in [32]. Dual curvature measures are concepts belonging to the dual Brunn–Minkowski theory. Apart from the remarkable works on the Busemann–Petty problem already mentioned, see, for example, Gardner [18], Gardner, Hug & Weil [19], Zhang [68], and especially the book [17] by Gardner for a glimpse of the dual theory.

Of critical importance is the fact that dual curvature measures are valuations on the set of convex bodies containing the origin in their interiors. Valuations have been the objects of many recent works, see, for example, Haberl [**28**], Haberl & Parapatits [**29**], Ludwig [**37**, **39**], Ludwig & Reitzner [**40**], Schuster [**57**, **58**], Schuster & Wannerer [**59**] and the references therein.

The family of dual curvature measures harbors two important special cases. When $j = 0$, the dual curvature measure $\widetilde{C}_0(K, \cdot)$ is up to a constant equal to the Aleksandrov's integral curvature $C_0(K^*, \cdot)$ for the polar body. When $j = n$, the dual curvature measure $\widetilde{C}_n(K, \cdot)$ is up to a constant equal to the cone volume measure.

The characterization problem for dual curvature measures is called *the dual Minkowski problem.* When restricting attention to even measures and the class of origin-symmetric convex bodies, this problem may be called *the even dual Minkowski problem.* Since measure concentration is critical to the even logarithmic Minkowski problem, it is expected that the solution to the even dual Minkowski problem is also linked to measure concentration. This is indeed the case. The following theorem was established in [**32**]:

Theorem 1.1 ([**32**]). *Suppose μ is a non-zero even finite Borel measure on S^{n-1} and $j \in \{1, \cdots, n-1\}$. If μ satisfies*

$$\frac{\mu(S^{n-1} \cap L_i)}{\mu(S^{n-1})} < 1 - \frac{j-1}{j}\frac{n-i}{n-1}, \tag{1.6}$$

for each i-dimensional subspace $L_i \subset \mathbb{R}^n$ and each $i = 1, \cdots, n-1$, then there exists an origin-symmetric convex body K such that $\mu(\cdot) = \widetilde{C}_j(K, \cdot)$.

Note that when $j = 1$, Equation (1.6) is the same as saying the measure μ cannot be concentrated entirely in any subspaces, which is obviously necessary. However, when $j \in \{2, \cdots, n-1\}$, for each i-dimensional subspace $L_i \subset \mathbb{R}^n$, we may find a sequence of cylinders $\{T_a\}$ such that

$$\lim_{a \to 0^+} \frac{\widetilde{C}_j(T_a, L_i \cap S^{n-1})}{\widetilde{C}_j(T_a, S^{n-1})} = \begin{cases} \frac{i}{j}, & \text{if } i < j, \\ 1, & \text{if } i \geq j. \end{cases} \tag{1.7}$$

See Appendix. Note that the constant in (1.7) is always strictly larger than the constant in (1.6), except when $i = 1$ or $j = 1$ (in which case, they are equal). This implies that the constant in (1.6) can be improved.

In this paper, a new set of subspace mass inequalities is presented. Let $j = 1, \cdots, n-1$. We say the measure μ satisfies *the j-th subspace mass inequality* if

$$\frac{\mu(S^{n-1} \cap L_i)}{\mu(S^{n-1})} < \begin{cases} \frac{i}{j}, & \text{if } i < j, \\ 1, & \text{if } i \geq j, \end{cases} \tag{1.8}$$

for each i-dimensional subspace $L_i \subset \mathbb{R}^n$ and each $i = 1, \cdots, n-1$.

Our main theorem solves the existence part of the even dual Minkowski problem.

Theorem 1.2. *Suppose μ is a non-zero even finite Borel measure on S^{n-1} and $j \in \{1, \cdots, n-1\}$. If μ satisfies the j-th subspace mass inequality, then there exists an origin-symmetric convex body K such that $\mu(\cdot) = \widetilde{C}_j(K, \cdot)$.*

Recently, it was established by Böröczky, Henk & Pollehn [**7**] that the subspace mass inequalities (1.8) are also necessary for the existence part of the even dual Minkowski problem.

Theorem 1.3 ([**7**])**.** *Suppose K is an origin-symmetric convex body in $\mathbb{R}^n$ and $j \in \{1, \cdots, n-1\}$. Then the j-th dual curvature measure $\widetilde{C}_j(K, \cdot)$ of K satisfies the j-th subspace mass inequality.*

Hence, the existence part of the even dual Minkowski problem is completely settled.

Theorem 1.4. *Suppose μ is a non-zero even finite Borel measure on S^{n-1} and $j \in \{1, \cdots, n-1\}$. The measure μ satisfies the j-th subspace mass inequality if and only if there exists an origin-symmetric convex body K such that $\mu(\cdot) = \widetilde{C}_j(K, \cdot)$.*

Note that Theorem 1.4 follows directly from Theorems 1.2 and 1.3. Theorem 1.2 will be established in Section 5.

The result in this paper extends to the case $j = n$ and the proof holds with appropriate modification. The case $j = n$ will not be included because it has already been well-treated in Böröczky-LYZ [**9**]. The case $j = 0$ is the Aleksandrov problem, which also has been completely settled by Aleksandrov [**1**].

Acknowledgments. The author would like to thank the anonymous referees for their very valuable comments.

2. Preliminary

2.1. Basics regarding convex bodies. First, we agree on some standard notations used in the current work. Throughout the paper, n will be an integer such that $n \geq 2$. We will be working in $\mathbb{R}^n$ with the standard inner product between $x, y \in \mathbb{R}^n$ denoted by $x \cdot y$. The usual Euclidean norm will be written as $|x|$ for $x \in \mathbb{R}^n$. We will use S^{k-1} for the unit sphere in $\mathbb{R}^k$ for $k = 1, 2, \cdots$. The volume of the unit ball in $\mathbb{R}^k$ is written as ω_k. Recall that the area ($(k-1)$-dimensional Hausdorff measure) of S^{k-1} is $k\omega_k$. We will use $C(S^{n-1})$ for the normed vector space containing all continuous functions on the unit sphere S^{n-1} equipped with the max norm; i.e., $||f|| = \max\{|f(u)| : u \in S^{n-1}\}$ for each $f \in C(S^{n-1})$. The subspace $C^+(S^{n-1}) \subset C(S^{n-1})$ contains only positive functions while the subspace $C_e(S^{n-1}) \subset C(S^{n-1})$ contains only even functions. The subspace $C_e^+(S^{n-1}) \subset C(S^{n-1})$ contains only positive and even functions. The total measure of a given finite Borel measure μ will be written as $|\mu|$. Throughout the paper, expressions like $c(n,j)$ will be used to denote "constants" whose values might change even within the same proof. For example, the expression $c(n,j)$ is a "constant" that only depends on n and j.

A subset K of $\mathbb{R}^n$ is called a *convex body* if it is a compact convex set with non-empty interior. The boundary of K will be denoted by ∂K. The set of all convex bodies that contain the origin in the interior is denoted by $\mathcal{K}_o^n$. The set of all origin-symmetric convex bodies will be denoted by $\mathcal{K}_e^n$. Obviously $\mathcal{K}_e^n \subset \mathcal{K}_o^n$.

For general references to the theory of convex bodies, books such as [**17**] and [**56**] are recommended.

Let K be a compact convex subset in $\mathbb{R}^n$. The *support function* $h_K : S^{n-1} \to \mathbb{R}$ is defined by

$$h_K(v) = \max\{x \cdot v : x \in K\}, \text{ for each } v \in S^{n-1}.$$

When $K \in \mathcal{K}_o^n$, the radial function (with respect to the origin) $\rho_K : S^{n-1} \to \mathbb{R}$ is defined by

$$\rho_K(u) = \max\{t > 0 : tu \in K\}, \text{ for each } u \in S^{n-1}.$$

Note that if K contains the origin in its interior, both h_K and ρ_K are positive. Moreover, they are bounded away from 0 since they are continuous functions and S^{n-1} is compact. By polar coordinates, it is simple to see that the volume of K may be computed by integrating the n-th power of the radial function over the unit sphere, i.e.,

$$V(K) = \frac{1}{n}\int_{S^{n-1}} \rho_K^n(u)du. \tag{2.1}$$

Here we use $V(K)$ for the volume of K (or the usual Lebesgue measure of K).

Suppose K contains the origin in its interior. The *polar body* of K, denoted by K^*, is defined by

$$K^* = \{y \in \mathbb{R}^n : y \cdot x \le 1, \text{ for all } x \in K\}.$$

Let K_i be a sequence of convex bodies in $\mathbb{R}^n$. We say K_i converges to a compact convex set $K \subset \mathbb{R}^n$ in *Hausdorff metric* if $||h_{K_i} - h_K|| \to 0$. If K and K_i contain the origin in their interiors, then K_i converges to K in Hausdorff metric implies that

$$||\rho_{K_i} - \rho_K|| \to 0.$$

For each $f \in C^+(S^{n-1})$, define $[f] \in \mathcal{K}_o^n$ by

$$[f] = \bigcap_{v \in S^{n-1}} \{x \in \mathbb{R}^n : v \cdot x \le f(v)\}.$$

The convex body $[f]$ is called the *Wulff shape* or the *Aleksandrov body* generated by f. It is simple to see

$$h_{[f]} \leq f, \tag{2.2}$$

and if $K \in \mathcal{K}_o^n$, then

$$[h_K] = K. \tag{2.3}$$

For the rest of this section, we assume $K \in \mathcal{K}_o^n$. The *supporting hyperplane* $P(K, v)$ of K for each $v \in S^{n-1}$ is given by

$$P(K, v) = \{x \in \mathbb{R}^n : x \cdot v = h_K(v)\}.$$

At each boundary point $x \in \partial K$, a unit vector v is said to be an *outer unit normal* of K at $x \in \partial K$ if $P(K, v)$ passes through x. For each $\eta \subset S^{n-1}$, the *reverse radial Gauss image*, $\boldsymbol{\alpha}_K^*(\eta)$, of K at η, is the set of all radial directions $u \in S^{n-1}$ such that the boundary point $\rho_K(u)u$ has at least one element in η as its outer unit normal, i.e.,

$$\boldsymbol{\alpha}_K^*(\eta) = \{u \in S^{n-1} : \text{there exists } v \in \eta \text{ such that } u\rho_K(u) \in P(K, v)\}.$$

When $\eta = \{v\}$, we usually write $\boldsymbol{\alpha}_K^*(v)$ instead of $\boldsymbol{\alpha}_K^*(\{v\})$.

The fundamental geometric functionals in the dual Brunn–Minkowski theory are dual quermassintegrals. Let $j = 1, \cdots, n$. The *$(n-j)$-th dual quermassintegral* of K, denoted by $\widetilde{W}_{n-j}(K)$, is proportional to the mean of the j-dimensional volume of intersections of K with j-dimensional subspaces of $\mathbb{R}^n$; i.e.,

$$\widetilde{W}_{n-j}(K) = \frac{\omega_n}{\omega_j} \int_{G(n,j)} V_j(K \cap \xi) d\xi, \tag{2.4}$$

where the integration is with respect to the Haar measure on the Grassmannian manifold $G(n, j)$ containing all j-dimensional subspaces $\xi \subset \mathbb{R}^n$, and V_j is the Lebesgue measure in $\mathbb{R}^j$. The dual quermassintegral has the following simple integral representation (see Section 2 in Lutwak [41]):

$$\widetilde{W}_{n-j}(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K^j(u) du.$$

Clearly, the above equation suggests that the dual quermassintegral can be defined for all real j in exactly the same manner.

The *normalized dual quermassintegral* $\bar{W}_{n-j}(K)$ is given by

$$\bar{W}_{n-j}(K) = \left(\frac{1}{n\omega_n} \int_{S^{n-1}} \rho_K^j(u) du \right)^{\frac{1}{j}}, \qquad \text{for } j \neq 0, \tag{2.5}$$

and by

$$\bar{W}_n(K) = \exp\left(\frac{1}{n\omega_n} \int_{S^{n-1}} \log \rho_K(u) du \right), \qquad \text{for } j = 0.$$

To simplify our notations, we will write

$$\widetilde{V}_j(K) = \widetilde{W}_{n-j}(K) \qquad \text{and} \qquad \bar{V}_j(K) = \bar{W}_{n-j}(K).$$

The functionals $\widetilde{V}_j$ and $\bar{V}_j$ are called the *j-th dual volume* and the *normalized j-th dual volume*, respectively. From the definition of radial function, it is obvious that the j-th dual volume is homogeneous of degree j and the normalized j-th dual volume is homogeneous of degree 1. That is, for $c > 0$,

$$\widetilde{V}_j(cK) = c^j \widetilde{V}_j(K) \qquad \text{and} \qquad \bar{V}_j(cK) = c\bar{V}_j(K). \tag{2.6}$$

By Jensen's inequality, we have

$$\bar{V}_{j_1}(K) \le \bar{V}_{j_2}(K), \tag{2.7}$$

whenever $j_1 \le j_2$.

2.2. Dual curvature measures and the even dual Minkowski problem. For quick later references, we gather here some basic facts about dual curvature measures and the even dual Minkowski problem.

Using *local dual parallel bodies*, a concept dual to *local parallel sets* which give rise to area measures and curvature measures, Huang-LYZ [**32**] discovered a new family of geometric measures defined on S^{n-1} in the dual Brunn–Minkowski theory. The newly discovered measures are dual to Federer's curvature measures (see, e.g., page 224 in Schneider [**56**]) and are, thus, called *dual curvature measures.* For each $K \in \mathcal{K}_o^n$ and each $j = 0, 1, \cdots, n$, the j-th dual curvature measure $\widetilde{C}_j(K, \eta)$ of K at η has the following integral representation:

$$\widetilde{C}_j(K, \eta) = \frac{1}{n} \int_{\boldsymbol{\alpha}_K^*(\eta)} \rho_K^j(u)du, \tag{2.8}$$

for each Borel set $\eta \subset S^{n-1}$. It is easy to see that the total measure of $\widetilde{C}_j(K, \cdot)$ is equal to the $(n-j)$-th dual quermassintegral; i.e.,

$$\widetilde{C}_j(K, S^{n-1}) = \widetilde{W}_{n-j}(K).$$

Moreover, the j-th dual curvature measure is homogeneous of degree j. That is,

$$\widetilde{C}_j(cK, \cdot) = c^j \widetilde{C}_j(K, \cdot).$$

From (2.8), it is not hard to see that when $j = 0$, the measure $\widetilde{C}_0(K, \cdot)$ is up to a constant equal to the Aleksandrov's integral curvature of the polar body K^* and when $j = n$, the measure $\widetilde{C}_n(K, \cdot)$ is up to a constant equal to the cone volume measure of K.

The characterization problem for dual curvature measures is called *the dual Minkowski problem.* The dual Minkowski problem includes the Aleksandrov problem ($j = 0$) and the logarithmic Minkowski problem ($j = n$) as special cases. When the given measure is even and the solution set is restricted to the set containing all origin-symmetric convex bodies, the dual Minkowski problem may be called *the even dual Minkowski problem.*

The even dual Minkowski problem: *Given an even finite Borel measure μ on S^{n-1} and $j \in \{0, 1, \cdots, n\}$, find necessary and sufficient condition(s) on μ so that there exists an origin-symmetric convex body K such that $\mu(\cdot) = \widetilde{C}_j(K, \cdot)$.*

In this paper, a solution to the even dual Minkowski problem, except for the cases $j = 0$ and $j = n$ which are the already solved Aleksandrov problem and even logarithmic Minkowski problem, will be presented.

3. An optimization problem associated with the dual Minkowski problem

In order to solve Minkowski problems using a variational method, the first crucial step is to find an optimization problem whose Euler–Lagrange equation would imply that the given measure is exactly the geometric measure (under investigation) of an optimizer. To find such an optimization problem, it is essential that one has certain variational formula that would lead to the geometric measure being studied. In this section, both the variational formula and the optimization problem associated to the dual Minkowski problem will be stated. It is important to note that the variational formula and the optimization problem were due to Huang-LYZ [**32**] and they are included here merely for the sake of completeness.

Suppose $j \in \{1, \cdots, n-1\}$ and μ is a non-zero even finite Borel measure on S^{n-1}. Define $\Phi : C_e^+(S^{n-1}) \to \mathbb{R}$ by letting

$$\Phi(f) = -\frac{1}{|\mu|}\int_{S^{n-1}} \log f(v)d\mu(v) + \log \bar{V}_j([f]),$$

for every $f \in C_e^+(S^{n-1})$. Note that the functional Φ is homogeneous of degree 0; i.e.,

$$\Phi(cf) = \Phi(f),$$

for all $c > 0$.

The optimization problem (I):

$$\sup\{\Phi(f) : f \in C_e^+(S^{n-1})\}.$$

Since $f \in C_e^+(S^{n-1})$, it is obvious that $[f] \in \mathcal{K}_e^n$. Note by (2.2) and (2.3),

$$\Phi(f) \le \Phi(h_{[f]}).$$

Thus, we may restrict our attention in search of a maximizer to the set of all support functions of origin-symmetric convex bodies. That is, if $K_0 \in \mathcal{K}_e^n$, then h_{K_0} is a maximizer to the optimization problem (I) if and only if K_0 is a maximizer to the following optimization problem.

The optimization problem (II):

$$\sup\{\Phi_\mu(Q) : Q \in \mathcal{K}_e^n\},$$

where $\Phi_\mu : \mathcal{K}_e^n \to \mathbb{R}$ is defined by letting

$$\Phi_\mu(Q) = -\frac{1}{|\mu|}\int_{S^{n-1}} \log h_Q(v)d\mu(v) + \log \bar{V}_j(Q), \tag{3.1}$$

for each $Q \in \mathcal{K}_e^n$. Note that on $\mathcal{K}_e^n$, the functional Φ_μ is continuous with respect to the Hausdorff metric and Φ_μ is homogeneous of degree 0; i.e.,

$$\Phi_\mu(cQ) = \Phi_\mu(Q),$$

for all $c > 0$.

In order to obtain the Euler–Lagrange equation for the optimization problem (I), the following variational formula is critical (see Theorem 4.5 in [32]):

$$\frac{d}{dt}\log \bar{V}_j([h_t])\bigg|_{t=0} = \frac{1}{\widetilde{V}_j(Q)}\int_{S^{n-1}} g(v)d\widetilde{C}_j(Q,v), \tag{3.2}$$

where $h_t = h_Q e^{tg}$, the convex body $[h_t]$ is the Wulff shape generated by h_t, and g is an arbitrary even continuous function on S^{n-1}.

Suppose $K_0 \in \mathcal{K}_e^n$ is a maximizer to (II), or equivalently h_{K_0} is a maximizer to (I). Since Φ is homogeneous of degree 0, we may assume $\widetilde{V}_j(K_0) = |\mu|$. Let $g : S^{n-1} \to \mathbb{R}$ be an arbitrary even continuous function. For $\delta > 0$ small enough and $t \in (-\delta, \delta)$, define $h_t : S^{n-1} \to (0,\infty)$ by

$$h_t(v) = h_{K_0}(v)e^{tg(v)}, \qquad \text{for each } v \in S^{n-1}.$$

Obviously $h_t \in C_e^+(S^{n-1})$. Since h_{K_0} is a maximizer to (I) and $h_0 = h_{K_0}$, we have

$$\frac{d}{dt}\Phi(h_t)\bigg|_{t=0} = 0. \tag{3.3}$$

By definition of Φ and h_t, (3.2), and the fact that $\widetilde{V}_j(K_0) = |\mu|$, we have

$$\begin{aligned}
&\frac{d}{dt}\Phi(h_t)\bigg|_{t=0}\\
&= \frac{d}{dt}\left(-\frac{1}{|\mu|}\int_{S^{n-1}} \left(\log h_{K_0}(v) + tg(v)\right)d\mu(v) + \log \bar{V}_j([h_t])\right)\bigg|_{t=0}\\
&= -\frac{1}{|\mu|}\int_{S^{n-1}} g(v)d\mu(v) + \frac{1}{\widetilde{V}_j(K_0)}\int_{S^{n-1}} g(v)d\widetilde{C}_j(K_0,v)\\
&= \frac{1}{|\mu|}\left(-\int_{S^{n-1}} g(v)d\mu(v) + \int_{S^{n-1}} g(v)d\widetilde{C}_j(K_0,v)\right).
\end{aligned} \tag{3.4}$$

Equations (3.3) and (3.4) imply

$$\int_{S^{n-1}} g(v)d\mu(v) = \int_{S^{n-1}} g(v)d\widetilde{C}_j(K_0, v),$$

for each $g \in C_e(S^{n-1})$. Since both μ and $\tilde{C}_j(K_0, \cdot)$ are even measures, we have

$$\mu(\cdot) = \widetilde{C}_j(K_0, \cdot).$$

Thus, we have,

Lemma 3.1. *Suppose μ is a non-zero even finite Borel measure on S^{n-1} and $j \in \{1, \ldots, n-1\}$. Assume $K_0 \in \mathcal{K}_e^n$. If $\widetilde{V}_j(K_0) = |\mu|$ and*

$$\Phi_\mu(K_0) = \sup\{\Phi_\mu(Q) : Q \in \mathcal{K}_e^n\},$$

then

$$\mu(\cdot) = \widetilde{C}_j(K_0, \cdot).$$

Lemma 3.1 reduces the problem of finding a solution to the dual Minkowski problem to finding a maximizer to the optimization problem (II).

4. Solving the optimization problem

In this section, we show that the optimization problem (II) does have a solution. The key is to prove that any maximizing sequence (of convex bodies) to (II) will have a subsequence that converges in Hausdorff metric to an origin-symmetric convex body that has non-empty interior. Estimates of an entropy-type integral with respect to the given measure (the first term on the right side of (3.1)) and dual quermassintegral (the second term on the right side of (3.1)) must be provided.

Providing an upper bound for the dual quermassintegral of an arbitrary origin-symmetric convex body can be extremely difficult. As a result, choosing a "proper" convex body to help us with the estimation is vital in solving the optimization problem. The meaning of the word "proper" is two-fold: first, the dual quermassintegral of the body chosen must be relatively uncomplicated to estimate or compute; second, the chosen body must be "close" to the given body to ensure that the estimate is reasonably accurate. In Huang-LYZ [**32**], cross polytopes were used. It turns out that cylinders (sum of two lower dimensional ellipsoids) will give a much more accurate upper bound.

Let $e_1, \cdots, e_n$ be an orthonormal basis in $\mathbb{R}^n$. Suppose $1 \leq k \leq n-1$ is an integer and $a_1, \cdots, a_k > 0$. Define

$$T = \left\{x \in \mathbb{R}^n : \frac{|x\cdot e_1|^2}{a_1^2} + \cdots + \frac{|x \cdot e_k|^2}{a_k^2} \leq 1, \right.$$
$$\left. \text{and } |x \cdot e_{k+1}|^2 + \cdots + |x \cdot e_n|^2 \leq 1 \right\}.$$

Write $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ with $\{e_1, \cdots, e_k\} \subset \mathbb{R}^k$ and $\{e_{k+1}, \cdots, e_n\} \subset \mathbb{R}^{n-k}$. Define

$$G = \{x \in \mathbb{R}^k : \frac{|x \cdot e_1|^2}{a_1^2} + \cdots + \frac{|x \cdot e_k|^2}{a_k^2} \leq 1\} \subset \mathbb{R}^k, \tag{4.1}$$

and

$$B = \{x \in \mathbb{R}^{n-k} : |x \cdot e_{k+1}|^2 + \cdots + |x \cdot e_n|^2 \leq 1\} \subset \mathbb{R}^{n-k}.$$

Note that $T = G \times B$.

For each $u \in S^{n-1}$, consider the general spherical coordinates,

$$u = (u_1 \cos\phi, u_2 \sin\phi),$$

where $u_1 \in S^{k-1} \subset \mathbb{R}^k$, $u_2 \in S^{n-k-1} \subset \mathbb{R}^{n-k}$, and $0 \leq \phi \leq \frac{\pi}{2}$. From the definition of radial function, we have

$$\begin{aligned}
\rho_T(u) &= \max\{t > 0 : (tu_1 \cos\phi, tu_2 \sin\phi) \in T\} \\
&= \max\{t > 0 : tu_1 \cos\phi \in G, tu_2 \sin\phi \in B\} \\
&= \max\{t > 0 : t \leq \frac{\rho_G(u_1)}{\cos\phi}, t \leq \frac{1}{\sin\phi}\} \qquad (4.2) \\
&= \begin{cases} \frac{\rho_G(u_1)}{\cos\phi}, & \text{if } 0 \leq \phi \leq \arctan\frac{1}{\rho_G(u_1)}, \\ \frac{1}{\sin\phi}, & \text{if } \arctan\frac{1}{\rho_G(u_1)} < \phi \leq \frac{\pi}{2}. \end{cases}
\end{aligned}$$

By applying (2.1) in the k-dimensional subspace $\mathbb{R}^k$ and the volume of an ellipsoid, we have

$$\frac{1}{k} \int_{S^{k-1}} \rho_G^k(u_1) du_1 = \omega_k a_1 \cdots a_k. \tag{4.3}$$

The following lemma gives a critical estimate on the upper bound of the $(n-q)$-th dual quermassintegral of cylinders.

Lemma 4.1. *Suppose $1 \leq k \leq n-1$ is an integer and $k < q \leq n$. Let $e_1, \cdots, e_n$ be an orthonormal basis in $\mathbb{R}^n$ and $a_1, \cdots, a_k > 0$. Define*

$$T = \left\{x \in \mathbb{R}^n : \frac{|x \cdot e_1|^2}{a_1^2} + \cdots + \frac{|x \cdot e_k|^2}{a_k^2} \leq 1, \right.$$
$$\left. \textit{and } |x \cdot e_{k+1}|^2 + \cdots + |x \cdot e_n|^2 \leq 1 \right\}.$$

Then

$$\log \bar{V}_q(T) = \frac{1}{q} \log\left(\frac{1}{n\omega_n} \int_{S^{n-1}} \rho_T^q(u) du\right) \leq \frac{1}{q} \log(a_1 \cdots a_k) + c(n, k, q).$$

Proof. Write $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ with $\{e_1, \cdots, e_k\} \subset \mathbb{R}^k$ and $\{e_{k+1}, \cdots, e_n\} \subset \mathbb{R}^{n-k}$. For each $u \in S^{n-1}$, consider the general spherical coordinates,

$$u = (u_1 \cos\phi, u_2 \sin\phi),$$

where $u_1 \in S^{k-1} \subset \mathbb{R}^k$, $u_2 \in S^{n-k-1} \subset \mathbb{R}^{n-k}$, and $0 \le \phi \le \frac{\pi}{2}$. For spherical Lebesgue measure, we have

$$du = \cos^{k-1}\phi \sin^{n-k-1}\phi \, d\phi du_1 du_2.$$

Define G as in (4.1).

Using general spherical coordinates, by (4.2), we have

(4.4)

$$\begin{aligned}
&\int_{S^{n-1}} \rho_T^q(u)du \\
=&\int_{S^{n-k-1}}\int_{S^{k-1}}\int_0^{\arctan\frac{1}{\rho_G(u_1)}} \left(\frac{\rho_{G(u_1)}}{\cos\phi}\right)^q \cos^{k-1}\phi\sin^{n-k-1}\phi\, d\phi du_1 du_2 \\
&+\int_{S^{n-k-1}}\int_{S^{k-1}}\int_{\arctan\frac{1}{\rho_G(u_1)}}^{\frac{\pi}{2}} \left(\frac{1}{\sin\phi}\right)^q \cos^{k-1}\phi\sin^{n-k-1}\phi\, d\phi du_1 du_2 \\
=&(n-k)\omega_{n-k}\left(\int_{S^{k-1}}\int_0^{\arctan\frac{1}{\rho_G(u_1)}} \rho_G^q(u_1)\cos^{k-1-q}\phi\sin^{n-k-1}\phi\, d\phi du_1\right. \\
&\qquad\left.+\int_{S^{k-1}}\int_{\arctan\frac{1}{\rho_G(u_1)}}^{\frac{\pi}{2}} \cos^{k-1}\phi\sin^{n-k-q-1}\phi\, d\phi du_1\right) \\
=:&(n-k)\omega_{n-k}(I_1+I_2).
\end{aligned}$$

We will use the change of variable $s = \rho_G(u_1)\tan\phi$. Note that

$$\begin{aligned}
\phi &= \arctan\frac{s}{\rho_G(u_1)}, \\
\cos\phi &= \left(1+\frac{s^2}{\rho_G^2(u_1)}\right)^{-1/2}, \\
\sin\phi &= \frac{s}{\rho_G(u_1)}\left(1+\frac{s^2}{\rho_G^2(u_1)}\right)^{-1/2}, \\
d\phi &= \left(1+\frac{s^2}{\rho_G^2(u_1)}\right)^{-1}\frac{1}{\rho_G(u_1)}ds.
\end{aligned}$$

First, we compute I_1.

$$\begin{aligned}
I_1 =& \int_{S^{k-1}}\int_0^1 \rho_G^q(u_1)\left(1+\frac{s^2}{\rho_G^2(u_1)}\right)^{-(k-1-q)/2} \\
&\cdot\left(\frac{s}{\rho_G(u_1)}\left(1+\frac{s^2}{\rho_G^2(u_1)}\right)^{-1/2}\right)^{n-k-1}\left(1+\frac{s^2}{\rho_G^2(u_1)}\right)^{-1}\frac{1}{\rho_G(u_1)}dsdu_1 \\
=& \int_{S^{k-1}}\int_0^1 \rho_G^k(u_1)\left(\rho_G^2(u_1)+s^2\right)^{\frac{q-n}{2}} s^{n-k-1}dsdu_1.
\end{aligned}$$

Since $q \le n$, the function $t^{\frac{q-n}{2}}$ is non-increasing in t. This, (4.3), and the fact that $q > k$ imply

$$\begin{aligned} I_1 &\le \int_{S^{k-1}} \int_0^1 \rho_G^k(u_1) s^{q-n} s^{n-k-1} ds du_1 \\ &= \int_{S^{k-1}} \rho_G^k(u_1) du_1 \int_0^1 s^{q-k-1} ds \\ &= \frac{k}{q-k} \omega_k a_1 \cdots a_k. \end{aligned} \tag{4.5}$$

Similarly, for I_2, we have

$$\begin{aligned} I_2 = &\int_{S^{k-1}} \int_1^\infty \left(1 + \frac{s^2}{\rho_G^2(u_1)}\right)^{-(k-1)/2} \\ &\cdot \left(\frac{s}{\rho_G(u_1)}\left(1 + \frac{s^2}{\rho_G^2(u_1)}\right)^{-1/2}\right)^{n-k-q-1} \left(1 + \frac{s^2}{\rho_G^2(u_1)}\right)^{-1} \frac{1}{\rho_G(u_1)} ds du_1 \\ &= \int_{S^{k-1}} \int_1^\infty \rho_G^k(u_1) \left(\rho_G^2(u_1) + s^2\right)^{\frac{q-n}{2}} s^{n-k-q-1} ds du_1. \end{aligned}$$

Since $q \le n$, the function $t^{\frac{q-n}{2}}$ is non-increasing in t. This, together with (4.3), implies

$$\begin{aligned} I_2 &\le \int_{S^{k-1}} \int_1^\infty \rho_G^k(u_1) s^{q-n} s^{n-k-q-1} ds du_1 \\ &= \int_{S^{k-1}} \rho_G^k(u_1) du_1 \int_1^\infty s^{-k-1} ds \\ &= \omega_k a_1 \cdots a_k. \end{aligned} \tag{4.6}$$

By (4.4), (4.5), and (4.6), we have

$$\log \bar{V}_q(T) = \frac{1}{q} \log \left(\frac{1}{n\omega_n} \int_{S^{n-1}} \rho_T^q(u) du \right) \le \frac{1}{q} \log(a_1 \cdots a_k) + c(n,k,q).$$

q.e.d.

Let μ be a finite Borel measure on S^{n-1} and $j \in \{1, \cdots, n-1\}$. We say μ satisfies the *j-th subspace mass inequality* if

1) when $1 \le i < j$,

$$\frac{\mu(S^{n-1} \cap L_i)}{|\mu|} < \frac{i}{j}, \tag{4.7}$$

for each i dimensional subspace $L_i \subset \mathbb{R}^n$;

2) when $j \le i \le n-1$,

$$\frac{\mu(S^{n-1} \cap L_i)}{|\mu|} < 1, \tag{4.8}$$

for each i dimensional subspace $L_i \subset \mathbb{R}^n$.

Let $e_1, \cdots, e_n$ be an orthonormal basis of $\mathbb{R}^n$. We define the following partition of the unit sphere. For each $\delta \in (0, \frac{1}{\sqrt{n}})$, define

$$A_{i,\delta} = \{v \in S^{n-1} : |v \cdot e_i| \geq \delta, |v \cdot e_j| < \delta, \text{ for } j > i\}, \tag{4.9}$$

for each $i = 1, \cdots, n$. These sets are non-empty since $e_i \in A_{i,\delta}$. They are obviously disjoint. Furthermore, it can be seen that the union of $A_{i,\delta}$ covers S^{n-1}. Indeed, for any unit vector $v \in S^{n-1}$, by the choice of δ, there has to be at least one i such that $|v \cdot e_i| \geq \delta$. Let i_0 be the last i that makes $|v \cdot e_i| \geq \delta$. Then $v \in A_{i_0,\delta}$.

Set $\xi_i = \text{span}\{e_1, \cdots, e_i\}$ for $i = 1, \cdots, n$ and $\xi_0 = \{o\}$. Define

$$A'_{i,\delta} = \{v \in S^{n-1} : |v \cdot e_i| \geq \delta, |v \cdot e_j| = 0, \text{ for } j > i\},$$
$$A''_{i,\delta} = \{v \in S^{n-1} : |v \cdot e_i| > 0, |v \cdot e_j| < \delta, \text{ for } j > i\}.$$

Clearly, $A'_{i,\delta} \subset A_{i,\delta} \subset A''_{i,\delta}$ and as δ decreases to 0^+, the set $A'_{i,\delta}$ gets bigger while the set $A''_{i,\delta}$ gets smaller. Hence, for each finite Borel measure μ on S^{n-1},

$$\lim_{\delta\to 0^+} \mu(A'_{i,\delta}) = \mu(S^{n-1} \cap (\xi_i \setminus \xi_{i-1})),$$
$$\lim_{\delta\to 0^+} \mu(A''_{i,\delta}) = \mu(S^{n-1} \cap (\xi_i \setminus \xi_{i-1})).$$

This, together with the fact that $A'_{i,\delta} \subset A_{i,\delta} \subset A''_{i,\delta}$, implies

$$\lim_{\delta\to 0^+} \mu(A_{i,\delta}) = \mu(S^{n-1} \cap (\xi_i \setminus \xi_{i-1})),$$

and, hence,

$$\lim_{\delta\to 0^+} \sum_{\beta=1}^{i} \mu(A_{\beta,\delta}) = \mu(S^{n-1} \cap \xi_i). \tag{4.10}$$

Lemma 4.2. *Suppose μ is a non-zero finite Borel measure on S^{n-1} and $j \in \{1, \cdots, n-1\}$. Let $e_1, \cdots, e_n$ be an orthonormal basis in $\mathbb{R}^n$. If μ satisfies the j-th subspace mass inequality, then there exist $t_0 > 0$ and $0 < \delta_0 < 1/\sqrt{n}$ (depending only on $n, j, \mu, e_1, \cdots, e_n$) such that*

$$\frac{\sum_{\beta=1}^{i} \mu(A_{\beta,\delta_0})}{|\mu|} < \begin{cases} \frac{i}{j} - it_0 & \text{if } 1 \leq i < j, \\ 1 - jt_0 & \text{if } j \leq i \leq n-1. \end{cases} \tag{4.11}$$

Here $A_{i,\delta}$ is as defined in (4.9).

Proof. Let $\xi_i = \text{span}\{e_1, \cdots, e_i\}$ for each $i = 1, 2, \cdots, n$.

Since μ satisfies the j-th subspace mass inequality, by (4.10), (4.7), and (4.8),

$$\lim_{\delta\to 0^+} \frac{\sum_{\beta=1}^{i} \mu(A_{\beta,\delta})}{|\mu|} = \frac{\mu(S^{n-1} \cap \xi_i)}{|\mu|} < \begin{cases} \frac{i}{j} & \text{if } 1 \leq i < j, \\ 1 & \text{if } j \leq i \leq n-1. \end{cases}$$

Since the inequality is strict, we may choose t_0 and $0 < \delta_0 < 1/\sqrt{n}$ such that (4.11) is valid. q.e.d.

The following lemma provides an estimate on an entropy-type integral with respect to a measure satisfying the j-th subspace mass inequality.

Lemma 4.3. *Suppose μ is a non-zero finite Borel measure on S^{n-1}, $\varepsilon_0 > 0$, and $j \in \{1, \cdots, n-1\}$. Let $a_{1l} \leq \cdots \leq a_{nl}$ be n sequences of positive reals such that $a_{nl} \geq \varepsilon_0$ and $e_{1l}, \cdots, e_{nl}$ be a sequence of orthonormal bases in $\mathbb{R}^n$, that converges to an orthonormal basis $e_1, \cdots, e_n$. Define the ellipsoid E_l by*

$$E_l = \left\{ x \in \mathbb{R}^n : \frac{|x \cdot e_{1l}|^2}{a_{1l}^2} + \cdots + \frac{|x \cdot e_{nl}|^2}{a_{nl}^2} \leq 1 \right\}.$$

If μ satisfies the j-th subspace mass inequality, then there exist $t_0, L_0 > 0$ and $0 < \delta_0 < 1/\sqrt{n}$ such that for each $l > L_0$,

$$\frac{1}{|\mu|} \int_{S^{n-1}} \log h_{E_l}(v) d\mu(v) \geq \left(\frac{1}{j} - t_0 \right) \log(a_{1l} \cdots a_{jl}) + c(\delta_0, j, t_0, \varepsilon_0). \tag{4.12}$$

Proof. By Lemma 4.2, choose t_0, δ_0 (with respect to $n, j, \mu, e_1, \cdots, e_n$) so that (4.11) holds. For notational simplicity, we write $\lambda_i = \mu(A_{i,\delta_0})/|\mu|$ for $i = 1, \cdots, n$. Hence,

$$\sum_{\beta=1}^{i} \lambda_\beta < \begin{cases} \frac{i}{j} - it_0 & \text{if } 1 \leq i < j, \\ 1 - jt_0 & \text{if } j \leq i \leq n-1. \end{cases} \tag{4.13}$$

Since $e_{1l}, \cdots, e_{nl}$ converges to $e_1, \cdots, e_n$, there exists $L_0 > 0$ such that for each $l > L_0$,

$$|e_{il} - e_i| < \delta_0/2, \qquad \text{for } i = 1, \cdots, n. \tag{4.14}$$

Note that $\pm a_{il} e_{il} \in E_l$. Hence, for each $v \in A_{i,\delta_0}$, by the definition of support function, and (4.14),

$$\begin{aligned} h_{E_l}(v) &\geq |v \cdot e_{il}| a_{il} \\ &= |v \cdot e_i + v \cdot (e_{il} - e_i)| a_{il} \\ &\geq \left(|v \cdot e_i| - |v \cdot (e_{il} - e_i)| \right) a_{il} \\ &\geq \frac{\delta_0}{2} a_{il}. \end{aligned} \tag{4.15}$$

By the fact that A_{i,δ_0} for $i = 1, \cdots, n$ form a partition of S^{n-1}, (4.15), and the definition of λ_i, we have

$$\begin{aligned} \frac{1}{|\mu|} \int_{S^{n-1}} \log h_{E_l}(v) d\mu(v) &= \frac{1}{|\mu|} \sum_{i=1}^{n} \int_{A_{i,\delta_0}} \log h_{E_l}(v) d\mu(v) \\ &\geq \frac{1}{|\mu|} \sum_{i=1}^{n} \left(\log \frac{\delta_0}{2} + \log a_{il} \right) \mu(A_{i,\delta_0}) \\ &= \log \frac{\delta_0}{2} + \sum_{i=1}^{n} \lambda_i \log a_{il}. \end{aligned} \tag{4.16}$$

Let $s_i = \lambda_1 + \cdots + \lambda_i$ for $i = 1, \cdots, n$ and $s_0 = 0$. Note that $s_n = 1$. We have $\lambda_i = s_i - s_{i-1}$ for $i = 1, \cdots, n$. Thus,

$$
\begin{aligned}
\sum_{i=1}^{n} \lambda_i \log a_{il} &= \sum_{i=1}^{n} (s_i - s_{i-1}) \log a_{il} \\
&= \sum_{i=1}^{n} s_i \log a_{il} - \sum_{i=1}^{n} s_{i-1} \log a_{il} \\
&= \sum_{i=1}^{n} s_i \log a_{il} - \sum_{i=0}^{n-1} s_i \log a_{i+1,l} \\
&= \log a_{nl} + \sum_{i=1}^{n-1} s_i \left(\log a_{il} - \log a_{i+1,l}\right).
\end{aligned}
\tag{4.17}
$$

Equation (4.17), (4.13) with the definition of s_i, the fact that $a_{1l} \leq \cdots \leq a_{nl}$, and that $a_{nl} \geq \varepsilon_0$ imply

$$
\begin{aligned}
\sum_{i=1}^{n} \lambda_i \log a_{il} \geq \log a_{nl} &+ \sum_{i=1}^{j-1} \left(\frac{i}{j} - it_0\right) \left(\log a_{il} - \log a_{i+1,l}\right) \\
&+ \sum_{i=j}^{n-1} (1 - jt_0) \left(\log a_{il} - \log a_{i+1,l}\right) \\
&= \left(\frac{1}{j} - t_0\right) \log(a_{1l} \cdots a_{jl}) + jt_0 \log a_{nl} \\
&\geq \left(\frac{1}{j} - t_0\right) \log(a_{1l} \cdots a_{jl}) + jt_0 \log \varepsilon_0.
\end{aligned}
\tag{4.18}
$$

Equation (4.12) now follows from (4.18) and (4.16). q.e.d.

Recall that for each $Q \in \mathcal{K}_e^n$, the functional Φ_μ is defined by

$$
\Phi_\mu(Q) = -\frac{1}{|\mu|} \int_{S^{n-1}} \log h_Q(v) d\mu(v) + \log \bar{V}_j(Q),
$$

where $j \in \{1, \cdots, n-1\}$.

We are now ready to show that if the given measure μ satisfies the j-th subspace mass inequality, then there exists a solution to the optimization problem (II).

Lemma 4.4. *Suppose μ is a non-zero finite Borel measure on S^{n-1} and $j \in \{1, \cdots, n-1\}$. If μ satisfies the j-th subspace mass inequality, then there exists $K_0 \in \mathcal{K}_e^n$ such that*

$$
\Phi_\mu(K_0) = \sup\{\Phi_\mu(Q) : Q \in \mathcal{K}_e^n\}.
$$

Proof. Let $\{Q_l\}$ be a maximizing sequence; i.e., $Q_l \in \mathcal{K}_e^n$ and

$$
\lim_{l \to \infty} \Phi_\mu(Q_l) = \sup\{\Phi_\mu(Q) : Q \in \mathcal{K}_e^n\}.
$$

Since Φ_μ is homogeneous of degree 0, we may assume Q_l is of diameter 1. By the Blaschke selection theorem, we may assume Q_l converges in Hausdorff metric to a non-empty compact convex set $K_0 \subset \mathbb{R}^n$. Note that K_0 must be origin symmetric. By the continuity of Φ_μ on $\mathcal{K}_e^n$, if K_0 has non-empty interior, we are done. The rest of the proof will focus on showing K_0 indeed has non-empty interior. We argue by contradiction. Assume K_0 has no interior points.

Let $E_l \in \mathcal{K}_e^n$ be the John ellipsoid associated with Q_l, i.e., the ellipsoid contained in Q_l with maximal volume. Then, it is a well-known fact (see page 588 in [**56**]) that

$$E_l \subset Q_l \subset \sqrt{n} E_l. \tag{4.19}$$

Since E_l is an n-dimensional ellipsoid centered at the origin, we can find an orthonormal basis $e_{1l}, \cdots, e_{nl}$ in $\mathbb{R}^n$ and $0 < a_{1l} \leq \cdots \leq a_{nl}$ such that

$$E_l = \left\{ x \in \mathbb{R}^n : \frac{|x \cdot e_{1l}|^2}{a_{1l}^2} + \cdots + \frac{|x \cdot e_{nl}|^2}{a_{nl}^2} \leq 1 \right\}.$$

Since the diameter of Q_l is 1 and $E_l \subset Q_l$, we have $a_{1l}, \cdots, a_{nl} < 1/2$. By taking subsequences, we may assume $a_{1l}, \cdots, a_{nl}$ are convergent as $l \to \infty$ and $e_{1l}, \cdots, e_{nl}$ converges to an orthonormal basis $e_1, \cdots, e_n$ in $\mathbb{R}^n$. Since K_0 has no interior points, we can find $1 \leq k < n$ and $\varepsilon_0 > 0$ such that

$$a_{1l}, \cdots, a_{kl} \to 0^+, \tag{4.20}$$

and

$$a_{k+1,l}, \cdots, a_{nl} \geq \varepsilon_0. \tag{4.21}$$

That k cannot be n is due to the fact that the diameter of Q_l is 1 and $Q_l \subset \sqrt{n} E_l$.

We will show

$$\lim_{l \to \infty} \Phi_\mu(Q_l) = -\infty.$$

By Lemma 4.3, there exist $t_0, \delta_0, L_0 > 0$ such that for each $l > L_0$,

$$\begin{aligned} \frac{1}{|\mu|} \int_{S^{n-1}} \log h_{E_l}(v) d\mu(v) \geq \left(\frac{1}{j} - t_0 \right) \log(a_{1l} \cdots a_{jl}) \\ + c(\delta_0, j, t_0, \varepsilon_0). \end{aligned} \tag{4.22}$$

Choose $k_0 = \min\{k, j\}$ and $q_0 \in (j, n]$ such that $\frac{1}{j} - t_0 < \frac{1}{q_0} < \frac{1}{j}$. Note that

$$q_0 > k_0. \tag{4.23}$$

Define

$$T_l = \frac{1}{\sqrt{n-k_0}}\Big\{x \in \mathbb{R}^n : \frac{|x\cdot e_{1l}|^2}{k_0 a_{1l}^2} + \cdots + \frac{|x\cdot e_{k_0 l}|^2}{k_0 a_{k_0 l}^2} \le 1, \\ \text{and } \frac{|x\cdot e_{k_0+1,l}|^2}{n-k_0} + \cdots + \frac{|x\cdot e_{nl}|^2}{n-k_0} \le 1\Big\} \\ = \Big\{x \in \mathbb{R}^n : \frac{|x\cdot e_{1l}|^2}{\frac{k_0}{n-k_0} a_{1l}^2} + \cdots + \frac{|x\cdot e_{k_0 l}|^2}{\frac{k_0}{n-k_0} a_{k_0 l}^2} \le 1, \\ \text{and } |x\cdot e_{k_0+1,l}|^2 + \cdots + |x\cdot e_{nl}|^2 \le 1\Big\}.$$

Note that for each $x \in E_l$, we have $|x \cdot e_{il}| \le a_{il}$ for $i = 1, \cdots, k_0$ and $|x \cdot e_{il}| \le a_{il} < 1$ for $i = k_0 + 1, \cdots, n$. Hence, $x \in \sqrt{n-k_0}T_l$, which implies $E_l \subset \sqrt{n-k_0}T_l$. This, and (4.19) give,

$$E_l \subset Q_l \subset \sqrt{n(n-k_0)}T_l. \tag{4.24}$$

By Lemma 4.1, and (4.23),

$$\log \bar{V}_{q_0}(T_l) \le \frac{1}{q_0}\log(a_{1l}\cdots a_{k_0 l}) + c(n, k_0, q_0). \tag{4.25}$$

By (4.24), (2.6), (2.7) with the fact that $q_0 > j$, (4.22), and (4.25), we have for $l > L_0$,

$$\begin{aligned} \Phi_\mu(Q_l) &\le -\frac{1}{|\mu|}\int_{S^{n-1}} \log h_{E_l}(v)d\mu(v) + \log \bar{V}_j(\sqrt{n(n-k_0)}T_l) \\ &= -\frac{1}{|\mu|}\int_{S^{n-1}} \log h_{E_l}(v)d\mu(v) + \log \bar{V}_j(T_l) + c(n,k_0) \\ &\le -\frac{1}{|\mu|}\int_{S^{n-1}} \log h_{E_l}(v)d\mu(v) + \log \bar{V}_{q_0}(T_l) + c(n,k_0) \\ &\le -\left(\frac{1}{j} - t_0\right)\log(a_{1l}\cdots a_{jl}) + \frac{1}{q_0}\log(a_{1l}\cdots a_{k_0 l}) \\ &\qquad + c(n, k_0, q_0, \delta_0, j, t_0, \varepsilon_0). \end{aligned} \tag{4.26}$$

When $k_0 = k$, i.e., $j \ge k$, by (4.26), (4.21), the fact that $\frac{1}{j} - t_0 < \frac{1}{q_0}$, and (4.20), we have for $l > L_0$,

$$\begin{aligned} \Phi_\mu(Q_l) &\le -\left(\frac{1}{j} - t_0\right)\log(a_{1l}\cdots a_{kl}) + \frac{1}{q_0}\log(a_{1l}\cdots a_{kl}) \\ &\qquad + c(n, k_0, q_0, \delta_0, j, t_0, \varepsilon_0) \\ &= \left(\frac{1}{q_0} - \left(\frac{1}{j} - t_0\right)\right)\log(a_{1l}\cdots a_{kl}) + c(n, k_0, q_0, \delta_0, j, t_0, \varepsilon_0) \\ &\to -\infty. \end{aligned}$$

When $k_0 = j$, i.e., $j \le k$, by (4.26), the fact that $\frac{1}{j} - t_0 < \frac{1}{q_0}$, and (4.20), we have for $l > L_0$,

$$\begin{aligned}\Phi_\mu(Q_l) &\le -\left(\frac{1}{j} - t_0\right)\log(a_{1l}\cdots a_{jl}) + \frac{1}{q_0}\log(a_{1l}\cdots a_{jl}) \\ &\qquad + c(n, k_0, q_0, \delta_0, t_0, \varepsilon_0) \\ &= \left(\frac{1}{q_0} - \left(\frac{1}{j} - t_0\right)\right)\log(a_{1l}\cdots a_{jl}) + c(n, k_0, q_0, \delta_0, t_0, \varepsilon_0) \\ &\to -\infty.\end{aligned}$$

Hence,

$$\lim_{l\to\infty} \Phi_\mu(Q_l) = -\infty.$$

But this is a contradiction to $\{Q_l\}$ being a maximizing sequence. q.e.d.

5. A solution to the even dual Minkowski problem

The next theorem shows that the subspace mass inequalities are sufficient for the existence of a solution to the even dual Minkowski problem.

Theorem 5.1. *Suppose μ is a non-zero finite even Borel measure on S^{n-1} and $j \in \{1, \cdots, n-1\}$. If μ satisfies the j-th subspace mass inequality, then there exists $K \in \mathcal{K}_e^n$ such that $\mu(\cdot) = \widetilde{C}_j(K, \cdot)$.*

Proof. By Lemma 4.4, there exists $K_0 \in \mathcal{K}_e^n$ such that

$$\Phi_\mu(K_0) = \sup\{\Phi_\mu(Q) : Q \in \mathcal{K}_e^n\}.$$

Since Φ_μ is homogeneous of degree 0 and $\widetilde{V}_j$ is homogeneous of degree j, there exists $c > 0$ such that

$$\widetilde{V}_j(cK_0) = |\mu|,$$

and

$$\Phi_\mu(cK_0) = \sup\{\Phi_\mu(Q) : Q \in \mathcal{K}_e^n\}.$$

By Lemma 3.1,

$$\mu(\cdot) = \widetilde{C}_j(cK_0, \cdot).$$ q.e.d.

A remark is in order: the proofs in this paper work equally well (with some necessary changes) in the cases when $j = n$, which has been well-treated in the remarkable work [**9**] by Böröczky-LYZ, and when $n - 1 < j < n$, which will be treated in another paper. When $j < 0$, a complete solution to the dual Minkowski problem for arbitrary measures (not necessarily even), including the existence and the uniqueness part, is presented in [**70**].

As pointed out in the introduction, the necessity of the subspace mass inequalities are due to Böröczky, Henk & Pollehn [**7**] (see also Theorem 1.3). Hence, the existence part of the even dual Minkowski problem (when the index is integer $1, 2, \cdots, n$) is completely settled.

In Huang-LYZ [**32**], the existence results were established for all real numbers in the interval $(0, n]$ whereas the results here are limited to integers within the interval $(0, n]$. It would be of interest to extend the results obtained here to all real numbers.

Appendix A. An example of the subspace mass concentration for dual curvature measures

Note that the following calculation was also performed in Böröczky, Henk & Pollehn [**7**]. The example is included here since it gives critical intuition to the solution of the even dual Minkowski problem presented in the current paper.

Let $i, j \in \{1, \cdots, n-1\}$ and $e_1, \cdots, e_n$ be an orthonormal basis in $\mathbb{R}^n$. Define

$$T_a = \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_i^2 \leq a^2, \text{ and } x_{i+1}^2 + \cdots + x_n^2 \leq 1\},$$

where $x_k = x \cdot e_k$ and $a > 0$. Let $L_i = \text{span}\{e_1, \cdots, e_i\}$.

We will compute the limit of the ratio

$$\frac{\widetilde{C}_j(T_a, L_i \cap S^{n-1})}{\widetilde{C}_j(T_a, S^{n-1})},$$

as $a \to 0^+$.

Write $\mathbb{R}^n = \mathbb{R}^i \times \mathbb{R}^{n-i}$ with $\{e_1, \cdots, e_i\} \subset \mathbb{R}^i$ and $\{e_{i+1}, \cdots, e_n\} \subset \mathbb{R}^{n-i}$. For each $u \in S^{n-1}$, consider the general spherical coordinates:

$$u = (u_1 \cos\phi, u_2 \sin\phi),$$

where $u_1 \in S^{i-1} \subset \mathbb{R}^i$, $u_2 \in S^{n-i-1} \subset \mathbb{R}^{n-i}$, and $0 \leq \phi \leq \frac{\pi}{2}$. For spherical Lebesgue measure,

$$du = \cos^{i-1}\phi \sin^{n-i-1}\phi d\phi du_1 du_2.$$

From the definition of radial function, we have

$$\begin{aligned} \rho_{T_a}(u) &= \max\{t > 0 : (tu_1\cos\phi, tu_2\sin\phi) \in T_a\} \\ &= \max\{t > 0 : t^2\cos^2\phi \leq a^2, \text{ and } t^2\sin^2\phi \leq 1\} \\ &= \min\left\{\frac{a}{\cos\phi}, \frac{1}{\sin\phi}\right\}. \end{aligned} \tag{A.1}$$

We claim that

$$u \in \boldsymbol{\alpha}^*_{T_a}(L_i \cap S^{n-1}) \text{ if and only if } \rho_{T_a}(u) = \frac{a}{\cos\phi}. \tag{A.2}$$

Recall that here $\boldsymbol{\alpha}^*_{T_a}$ is the reverse radial Gauss image of T_α defined in Section 2 and the reverse radial Gauss image is well-defined even for non-smooth convex bodies.

Suppose $u \in \boldsymbol{\alpha}^*_{T_a}(L_i \cap S^{n-1})$. There exists $v \in L_i \cap S^{n-1}$ with $u \cdot v > 0$ such that

$$\rho_{T_a}(u) u \cdot v = h_{T_a}(v). \tag{A.3}$$

Since $v \in L_i \cap S^{n-1}$, by the definition of support function and the choice of T_a, it is obvious that

$$h_{T_a}(v) = a. \tag{A.4}$$

By (A.1), the fact that $u \cdot v > 0$, and that $v \in L_i$, we have

$$\rho_{T_a}(u) u \cdot v \le \frac{a}{\cos\phi}(u_1 \cos\phi, u_2 \sin\phi) \cdot v = a u_1 \cdot v. \tag{A.5}$$

Equations (A.3), (A.4), and (A.5) imply that $u_1 \cdot v \ge 1$. Since both u_1 and v are unit vectors, we have $u_1 = v$. Hence, by (A.3), and (A.4),

$$\rho_{T_a}(u) = \frac{h_{T_a}(v)}{u \cdot v} = \frac{a}{(u_1 \cos\phi, u_2 \sin\phi) \cdot v} = \frac{a}{\cos\phi}.$$

Now, let us assume $\rho_{T_a}(u) = \frac{a}{\cos\phi}$. Write $u = (u_1 \cos\phi, u_2 \sin\phi)$. Let $v = u_1 \in L_i \cap S^{n-1}$. Then

$$\rho_{T_a}(u) u \cdot v = \frac{a}{\cos\phi}(u_1 \cos\phi, u_2 \sin\phi) \cdot u_1 = a = h_{T_a}(v).$$

Hence, $u \in \boldsymbol{\alpha}^*_{T_a}(L_i \cap S^{n-1})$.

Definition of the j-th curvature measure (2.8) and (A.2) imply

$$\begin{aligned}\widetilde{C}_j(T_a, L_i \cap S^{n-1}) &= \frac{1}{n}\int_{S^{n-i-1}}\int_{S^{i-1}}\int_0^{\arctan\frac{1}{a}} \left(\frac{a}{\cos\phi}\right)^j \cos^{i-1}\phi \\ &\qquad \cdot \sin^{n-i-1}\phi \, d\phi \, du_1 \, du_2 \\ &= \frac{i\omega_i (n-i)\omega_{n-i}}{n} a^j \int_0^{\arctan\frac{1}{a}} \cos^{i-j-1}\phi \sin^{n-i-1}\phi \, d\phi.\end{aligned}$$

By using the change of variable $s = a\tan\phi$, we have

$$\widetilde{C}_j(T_a, L_i \cap S^{n-1}) = \frac{i\omega_i (n-i)\omega_{n-i}}{n} a^i \int_0^1 (a^2 + s^2)^{\frac{j-n}{2}} s^{n-i-1} ds. \tag{A.6}$$

On the other side, by the definition of the j-th curvature measure and (A.1), we have

$$\begin{aligned}\widetilde{C}_j(T_a, S^{n-1}) &= \frac{1}{n}\int_{S^{n-i-1}}\int_{S^{i-1}}\int_0^{\arctan\frac{1}{a}} \left(\frac{a}{\cos\phi}\right)^j \cos^{i-1}\phi \\ &\qquad \cdot \sin^{n-i-1}\phi \, d\phi \, du_1 \, du_2 \\ &\quad + \frac{1}{n}\int_{\arctan\frac{1}{a}}^{\frac{\pi}{2}} \sin^{-j}\phi \cos^{i-1}\phi \sin^{n-i-1}\phi \, d\phi \, du_1 \, du_2 \\ &= \frac{i\omega_i (n-i)\omega_{n-i}}{n} \left(a^j \int_0^{\arctan\frac{1}{a}} \cos^{i-j-1}\phi \sin^{n-i-1}\phi \, d\phi \right. \\ &\qquad \left. + \int_{\arctan\frac{1}{a}}^{\frac{\pi}{2}} \cos^{i-1}\phi \sin^{n-i-j-1}\phi \, d\phi \right).\end{aligned}$$

By using the change of variable $s = a\tan\phi$, we have

$$\widetilde{C}_j(T_a, S^{n-1}) = \frac{i\omega_i(n-i)\omega_{n-i}}{n} a^i \Big(\int_0^1 (a^2+s^2)^{\frac{j-n}{2}} s^{n-i-1} ds + \int_1^\infty (a^2+s^2)^{\frac{j-n}{2}} s^{n-i-j-1} ds \Big). \tag{A.7}$$

By (A.6) and (A.7), we have

$$\begin{aligned} &\frac{\widetilde{C}_j(T_a, L_i \cap S^{n-1})}{\widetilde{C}_j(T_a, S^{n-1})} \\ =&\frac{\int_0^1 (a^2+s^2)^{\frac{j-n}{2}} s^{n-i-1} ds}{\int_0^1 (a^2+s^2)^{\frac{j-n}{2}} s^{n-i-1} ds + \int_1^\infty (a^2+s^2)^{\frac{j-n}{2}} s^{n-i-j-1} ds}. \end{aligned} \tag{A.8}$$

Since $j < n$, the integrals in the above equation are increasing as a decreases to 0. Hence,

$$\lim_{a\to 0^+} \int_0^1 (a^2+s^2)^{\frac{j-n}{2}} s^{n-i-1} ds = \begin{cases} \frac{1}{j-i}, & \text{if } i < j, \\ \infty, & \text{if } i \geq j, \end{cases} \tag{A.9}$$

and

$$\lim_{a\to 0^+} \int_1^\infty (a^2+s^2)^{\frac{j-n}{2}} s^{n-i-j-1} ds = \frac{1}{i}. \tag{A.10}$$

Equations (A.8), (A.9), and (A.10) imply

$$\lim_{a\to 0^+} \frac{\widetilde{C}_j(T_a, L_i \cap S^{n-1})}{\widetilde{C}_j(T_a, S^{n-1})} = \begin{cases} \frac{i}{j}, & \text{if } i < j, \\ 1, & \text{if } i \geq j. \end{cases}$$

Proposition A.1. *Suppose $i, j \in \{1, \cdots, n-1\}$. For each i-dimensional subspace $L_i \subset \mathbb{R}^n$, there exists a family of cylinders $\{T_a\}$ such that*

$$\lim_{a\to 0^+} \frac{\widetilde{C}_j(T_a, L_i \cap S^{n-1})}{\widetilde{C}_j(T_a, S^{n-1})} = \begin{cases} \frac{i}{j}, & \textit{if } i < j, \\ 1, & \textit{if } i \geq j. \end{cases}$$

References

[1] A.D. Aleksandrov, *Existence and uniqueness of a convex surface with a given integral curvature*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **35** (1942), 131–134, MR 0007625, Zbl 0061.37604.

[2] B. Andrews, *Gauss curvature flow: the fate of the rolling stones*, Invent. Math. **138** (1999), 151–161, MR 1714339, Zbl 0936.35080.

[3] B. Andrews, *Classification of limiting shapes for isotropic curve flows*, J. Amer. Math. Soc. (JAMS) **16** (2003), 443–459 (electronic), MR 1949167, Zbl 1023.53051.

[4] F. Barthe, O. Guédon, S. Mendelson & A. Naor, *A probabilistic approach to the geometry of the l_p^n-ball*, Ann. Probab. **33** (2005), 480–513, MR 2123199, Zbl 1071.60010.

[5] K.J. Böröczky, P. Hegedűs & G. Zhu, *On the discrete logarithmic Minkowski problem*, Int. Math. Res. Not. (IMRN) (2016), 1807–1838, MR 3509941, Zbl 1345.52002.

[6] K.J. Böröczky & M. Henk, *Cone-volume measure of general centered convex bodies*, Adv. Math. **286** (2016), 703–721, MR 3415694, Zbl 1334.52003.

[7] K.J. Böröczky, M. Henk & H. Pollehn, *Subspace concentration of dual curvature measures of symmetric convex bodies*, J. Differential Geom. in press. arXiv:1604.07390.

[8] K.J. Böröczky, E. Lutwak, D. Yang & G. Zhang, *The log-Brunn–Minkowski inequality*, Adv. Math. **231** (2012), 1974–1997, MR 2964630 Zbl 1258.52005.

[9] K.J. Böröczky, E. Lutwak, D. Yang & G. Zhang, *The logarithmic Minkowski problem*, J. Amer. Math. Soc. (JAMS) **26** (2013), 831–852, MR 3037788, Zbl 1272.52012.

[10] K.J. Böröczky, E. Lutwak, D. Yang & G. Zhang, *Affine images of isotropic measures*, J. Differential Geom. **99** (2015), 407–442, MR 3316972, Zbl 1325.28004.

[11] L. Caffarelli, *Interior $W^{2,p}$ estimates for solutions of the Monge–Ampère equation*, Ann. of Math. (2) **131** (1990), 135–150, MR 1038360, Zbl 0704.35044.

[12] W. Chen, *L_p Minkowski problem with not necessarily positive data*, Adv. Math. **201** (2006), 77–89, MR 2204749, Zbl 1102.34023.

[13] S.Y. Cheng & S.T. Yau, *On the regularity of the solution of the n-dimensional Minkowski problem*, Comm. Pure Appl. Math. **29** (1976), 495–516, MR 0423267, Zbl 0363.53030.

[14] K.-S. Chou & X.-J. Wang, *The L_p-Minkowski problem and the Minkowski problem in centroaffine geometry*, Adv. Math. **205** (2006), 33–83, MR 2254308, Zbl 1245.52001.

[15] A. Colesanti, G.V. Livshyts & A. Marsiglietti, *The infinitesimal form of Brunn–Minkowski type inequalities*, arXiv:1606.06586.

[16] R.J. Gardner, *A positive answer to the Busemann–Petty problem in three dimensions*, Ann. of Math. (2) **140** (1994), 435–447, MR 1298719, Zbl 0826.52010.

[17] R.J. Gardner, *Geometric tomography*, 2nd edition, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2006, MR 2251886, Zbl 1102.52002.

[18] R.J. Gardner, *The dual Brunn–Minkowski theory for bounded Borel sets: dual affine quermassintegrals and inequalities*, Adv. Math. **216** (2007), 358–386, MR 2353261, Zbl 1126.52008.

[19] R.J. Gardner, D. Hug & W. Weil, *Operations between sets in geometry*, J. Eur. Math. Soc. (JEMS) **15** (2013), 2297–2352, MR 3120744, Zbl 1282.52006.

[20] R.J. Gardner, A. Koldobsky & T. Schlumprecht, *An analytic solution to the Busemann–Petty problem on sections of convex bodies*, Ann. of Math. (2) **149** (1999), 691–703, MR 1689343, Zbl 0937.52003.

[21] P. Goodey, V. Yaskin & M. Yaskina, *A Fourier transform approach to Christoffel's problem*, Trans. Amer. Math. Soc. **363** (2011), 6351–6384, MR 2833558, Zbl 1234.52007.

[22] E. Grinberg & G. Zhang, *Convolutions, transforms, and convex bodies*, Proc. London Math. Soc. (3) **78** (1999), 77–115, MR 1658156, Zbl 0974.52001.

[23] B. Guan & P. Guan, *Convex hypersurfaces of prescribed curvatures*, Ann. of Math. (2) **156** (2002), 655–673, MR 1933079, Zbl 1025.53028.

[24] P. Guan, J. Li & Y. Li, *Hypersurfaces of prescribed curvature measure*, Duke Math. J. **161** (2012), 1927–1942, MR 2954620, Zbl 1254.53073.

[25] P. Guan & Y. Li, *$C^{1,1}$ estimates for solutions of a problem of Alexandrov*, Comm. Pure Appl. Math. **50** (1997), 789–811, MR 1454174, Zbl 0879.53047.

[26] P. Guan, C. Lin & X.-N. Ma, *The existence of convex body with prescribed curvature measures*, Int. Math. Res. Not. (IMRN) (2009), 1947–1975, MR 2507106, Zbl 1178.53010.

[27] P. Guan, X.-N. Ma, *The Christoffel–Minkowski problem. I. Convexity of solutions of a Hessian equation*, Invent. Math. **151** (2003), 553–577, MR 1961338, Zbl 1213.35213.

[28] C. Haberl, *Minkowski valuations intertwining with the special linear group*, J. Eur. Math. Soc. (JEMS) **14** (2012), 1565–1597, MR 2966660, Zbl 1270.52018.

[29] C. Haberl & L. Parapatits, *The centro-affine Hadwiger theorem*, J. Amer. Math. Soc. (JAMS) **27** (2014), 685–705, MR 3194492, Zbl 1319.52006.

[30] C. Haberl & F.E. Schuster, *Asymmetric affine L_p Sobolev inequalities*, J. Funct. Anal. **257** (2009), 641–658, MR 2530600, Zbl 1180.46023.

[31] M. Henk & E. Linke, *Cone-volume measures of polytopes*, Adv. Math. **253** (2014), 50–62, MR 3148545, Zbl 1308.52007.

[32] Y. Huang, E. Lutwak, D. Yang & G. Zhang, *Geometric measures in the dual Brunn–Minkowski theory and their associated Minkowski problems*, Acta Math. **216** (2016), 325–388, MR 3573332.

[33] D. Hug, E. Lutwak, D. Yang & G. Zhang, *On the L_p Minkowski problem for polytopes*, Discrete Comput. Geom. **33** (2005), 699–715, MR 2132298, Zbl 1078.52008.

[34] H. Jian, J. Lu & X.-J. Wang, *Nonuniqueness of solutions to the L_p-Minkowski problem*, Adv. Math. **281** (2015), 845–856, MR 3366854, Zbl 1326.35009.

[35] H. Jian, J. Lu & G. Zhu, *Mirror symmetric solutions to the centro-affine Minkowski problem*, Calc. Var. Partial Differential Equations **55** (2016), Article ID 41, 22p, MR 3479715, Zbl 06580705.

[36] J. Lu & X.-J. Wang, *Rotationally symmetric solutions to the L_p-Minkowski problem*, J. Differential Equations **254** (2013), 983–1005, MR 2997361, Zbl 1273.52006.

[37] M. Ludwig, *Minkowski valuations*, Trans. Amer. Math. Soc. **357** (2005), 4191–4213 (electronic), MR 2159706, Zbl 1077.52005.

[38] M. Ludwig, *General affine surface areas*, Adv. Math. **224** (2010), 2346–2360, MR 2652209, Zbl 1198.52004.

[39] M. Ludwig, *Minkowski areas and valuations*, J. Differential Geom. **86** (2010), 133–161, MR 2772547, Zbl 1215.52004.

[40] M. Ludwig & M. Reitzner, *A classification of* $\mathrm{SL}(n)$ *invariant valuations*, Ann. of Math. (2) **172** (2010), 1219–1267, MR 2680490, Zbl 1223.52007.

[41] E. Lutwak, *Intersection bodies and dual mixed volumes*, Adv. in Math. **71** (1988), 232–261, MR 963487, Zbl 0657.52002.

[42] E. Lutwak, *The Brunn–Minkowski–Firey theory. I. Mixed volumes and the Minkowski problem*, J. Differential Geom. **38** (1993), 131–150, MR 1231704, Zbl 0788.52007.

[43] E. Lutwak, *The Brunn–Minkowski–Firey theory. II. Affine and geominimal surface areas*, Adv. Math. **118** (1996), 244–294, MR 1378681, Zbl 0853.52005.

[44] E. Lutwak & V. Oliker, *On the regularity of solutions to a generalization of the Minkowski problem*, J. Differential Geom. **41** (1995), 227–246, MR 1316557, Zbl 0867.52003.

[45] E. Lutwak, D. Yang & G. Zhang, *Sharp affine L_p Sobolev inequalities*, J. Differential Geom. **62** (2002), 17–38, MR 1987375, Zbl 1073.46027.

[46] E. Lutwak, D. Yang & G. Zhang, *On the L_p-Minkowski problem*, Trans. Amer. Math. Soc. **356** (2004), 4359–4370, MR 2067123, Zbl 1069.52010.

[47] A. Naor, *The surface measure and cone measure on the sphere of l_p^n*, Trans. Amer. Math. Soc. **359** (2007), 1045–1079 (electronic), MR 2262841, Zbl 1109.60006.

[48] A. Naor & D. Romik, *Projecting the surface measure of the sphere of l_p^n*, Ann. Inst. H. Poincaré Probab. Statist. **39** (2003), 241–261, MR 1962135, Zbl 1012.60025.

[49] L. Nirenberg, *The Weyl and Minkowski problems in differential geometry in the large*, Comm. Pure Appl. Math. **6** (1953), 337–394, MR 0058265, Zbl 0051.12402.

[50] V. Oliker, *Embedding $\mathbf{S}^n$ into $\mathbf{R}^{n+1}$ with given integral Gauss curvature and optimal mass transport on $\mathbf{S}^n$*, Adv. Math. **213** (2007), 600–620, MR 2332603, Zbl 1233.49024.

[51] G. Paouris & E. Werner, *Relative entropy of cone measures and L_p centroid bodies*, Proc. Lond. Math. Soc. (3) **104** (2012), 253–286, MR 2880241, Zbl 1246.52008.

[52] L. Rotem, *A letter: The log-Brunn–Minkowski inequality for complex bodies*, arXiv:1412.5321.

[53] C. Saroglou, *More on logarithmic sums of convex bodies*, Mathematika **62** (2016), 818–841, MR 3521355, Zbl 06642373.

[54] R. Schneider, *Das Christoffel-Problem für Polytope*, Geometriae Dedicata **6** (1977), 81–85, MR 0470865, Zbl 0363.52007.

[55] R. Schneider, *Curvature measures of convex bodies*, Ann. Mat. Pura Appl. (4) **116** (1978), 101–134, MR 506976, Zbl 0389.52006.

[56] R. Schneider, *Convex bodies: the Brunn–Minkowski theory*, 2nd edition, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2014, MR 3155183, Zbl 1287.52001.

[57] F.E. Schuster, *Valuations and Busemann–Petty type problems*, Adv. Math. **219** (2008), 344–368, MR 2435426, Zbl 1146.52003.

[58] F.E. Schuster, *Crofton measures and Minkowski valuations*, Duke Math. J. **154** (2010), 1–30, MR 2668553, Zbl 1205.52004.

[59] F.E. Schuster & T. Wannerer, $\mathrm{GL}(n)$ *contravariant Minkowski valuations*, Trans. Amer. Math. Soc. **364** (2012), 815–826, MR 2846354, Zbl 1246.52009.

[60] W. Sheng, N. Trudinger & X.-J. Wang, *Convex hypersurfaces of prescribed Weingarten curvatures*, Comm. Anal. Geom. **12** (2004), 213–232, MR 2074877, Zbl 1065.53008.

[61] A. Stancu, *The discrete planar L_0-Minkowski problem*, Adv. Math. **167** (2002), 160–174, MR 1901250, Zbl 1005.52002.

[62] A. Stancu, *On the number of solutions to the discrete two-dimensional L_0-Minkowski problem*, Adv. Math. **180** (2003), 290–323, MR 2019226, Zbl 1054.52001.

[63] T. Wang, *The affine Sobolev–Zhang inequality on* $BV(\mathbb{R}^n)$, Adv. Math. **230** (2012), 2457–2473, MR 2927377, Zbl 1257.46016.

[64] X.-J. Wang, *Existence of convex hypersurfaces with prescribed Gauss–Kronecker curvature*, Trans. Amer. Math. Soc. **348** (1996), 4501–4524, MR 1357407, Zbl 0926.53024.

[65] D. Xi & G. Leng, *Dar's conjecture and the log-Brunn–Minkowski inequality*, J. Differential Geom. **103** (2016), 145–189, MR 3488132, Zbl 1348.52006.

[66] G. Xiong, *Extremum problems for the cone volume functional of convex polytopes*, Adv. Math. **225** (2010), 3214–3228, MR 2729006, Zbl 1213.52011.

[67] G. Zhang, *The affine Sobolev inequality*, J. Differential Geom. **53** (1999), 183–202, MR 1776095, Zbl 1040.53089.

[68] G. Zhang, *Dual kinematic formulas*, Trans. Amer. Math. Soc. **351** (1999), 985–995, MR 1443203, Zbl 0961.52001.

[69] G. Zhang, *A positive solution to the Busemann–Petty problem in* $\mathbf{R}^4$, Ann. of Math. (2) **149** (1999), 535–543, MR 1689339, Zbl 0937.52004.

[70] Y. Zhao, *The dual Minkowski problem for negative indices*, Calc. Var. Partial Differential Equations (2) **56** (2017), Art. 18, 16pp, MR 3605843, Zbl 06715617.

[71] G. Zhu, *The logarithmic Minkowski problem for polytopes*, Adv. Math. **262** (2014), 909–931, MR 3228445, Zbl 1321.52015.

[72] G. Zhu, *The centro-affine Minkowski problem for polytopes*, J. Differential Geom. **101** (2015), 159–174, MR 3356071, Zbl 1331.53016.

[73] G. Zhu, *The* L_p *Minkowski problem for polytopes for* $0 < p < 1$, J. Funct. Anal. **269** (2015) 1070–1094, MR 3352764, Zbl 1335.52023.

[74] G. Zhu, *The* L_p *Minkowski problem for polytopes for* $p < 0$, Indiana Univ. Math. J. (4) **66** (2017), 1333–1350, MR 3689334, Zbl 06790082.

[75] D. Zou & G. Xiong, *Orlicz–John ellipsoids*, Adv. Math. **265** (2014), 132–168, MR 3255458, Zbl 1301.52015.

Department of Mathematics
Tandon School of Engineering
New York University
New York, NY
USA

E-mail address: yiming.zhao@nyu.edu