

INVERSE PROBLEMS FOR THE CONNECTION LAPLACIAN

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Abstract

We reconstruct a Riemannian manifold and a Hermitian vector bundle with compatible connection from the hyperbolic Dirichlet-to-Neumann operator associated with the wave equation of the connection Laplacian. The boundary data is local and the reconstruction is up to the natural gauge transformations of the problem. As a corollary we derive an elliptic analogue of the main result which solves a Calderón problem for connections on a cylinder.

1. Introduction

The purpose of the present paper is to show how to reconstruct a Riemannian metric and a Hermitian vector bundle with compatible connection from partial boundary measurements associated with the wave equation of the connection Laplacian (or rough Laplacian). The recovery is possible up to the natural gauges of the problem, and the proof uses techniques from the Boundary Control method [1].

There is considerable literature on the topic, and we shall review it in due course, but the strength of our results lies in the geometric generality involved: there are no restrictions on the Riemannian manifold, Hermitian vector bundle or connection. Our methods also include a transparent and direct proof in the case of the trivial vector bundle that avoids gluing of local reconstructions. The problem is motivated by the Aharonov–Bohm effect which asserts that different gauge equivalence classes of electromagnetic potentials have different physical effects that can be detected by experiments. The solution to the inverse problem presented in this paper shows in great generality that different gauge equivalence classes of Hermitian connections (e.g., Yang–Mills potentials) will have different boundary data, and therefore, are detectable by boundary measurements.

We proceed to state our results in more detail. Let (M, g) be a smooth, compact, connected Riemannian manifold of dimension m with non-empty boundary ∂M . Let $E \rightarrow M$ be a smooth Hermitian vector

bundle of rank n , and let us denote by $\langle \cdot, \cdot \rangle_E$ the Hermitian inner product on each fiber. Let ∇ be a connection compatible with the Hermitian structure, that is, if we think of ∇ as operating on sections

$$\nabla : C^\infty(M; E) \rightarrow C^\infty(M; E \otimes T^*M),$$

then for any pair $u, v \in C^\infty(M; E)$, we have

$$d\langle u, v \rangle_E = \langle \nabla u, v \rangle_E + \langle u, \nabla v \rangle_E.$$

Note that both the sides of the above equation are differential forms, that is, sections in $C^\infty(M; T^*M)$.

We can define a natural L^2 -inner product of sections by setting

$$\langle u, v \rangle_{L^2(M; E)} = \int_M \langle u, v \rangle_E dx.$$

Here dx is the Riemannian volume measure of (M, g) , and we do not assume that M is oriented. Similarly we get a natural L^2 -inner product in $C^\infty(M; E \otimes T^*M)$. The elements in $C^\infty(M; E \otimes T^*M)$ can be thought of as 1-forms taking values in E . A pointwise product $\langle \alpha, \beta \rangle_E$ is a complex-valued 2-tensor on M which can be contracted with g to obtain a complex-valued function, and then integrated in M . In other words, if $\alpha = \alpha_i dx^i$ and $\beta = \beta_j dx^j$, then

$$\langle \alpha, \beta \rangle_{L^2(M; E \otimes T^*M)} = \int_M g^{ij} \langle \alpha_i, \beta_j \rangle_E dx.$$

We denote by ∇^* the adjoint of ∇ with respect to these L^2 -inner products, and define the *connection Laplacian* as

$$P = \nabla^* \nabla.$$

We denote by $\text{End}(E)$ the vector bundle whose fiber at $x \in M$ is the space of linear maps from the fiber E_x to itself, and say that a section $V \in C^\infty(M; \text{End}(E))$ is a *potential* if it is symmetric in the sense that for any pair of sections u, v of E ,

$$(1) \quad \langle u, Vv \rangle_E = \langle Vu, v \rangle_E.$$

Let V be a potential and consider the wave equation on sections,

$$(2) \quad \begin{aligned} (\partial_t^2 + P + V)u(t, x) &= 0, & (0, \infty) \times M, \\ u|_{(0, \infty) \times \partial M} &= f, & (0, \infty) \times \partial M, \\ u|_{t=0} = \partial_t u|_{t=0} &= 0, & \text{in } M. \end{aligned}$$

Let $T > 0$, let $\mathcal{S} \subset \partial M$ be open, and define the restricted Dirichlet-to-Neumann operator

$$\Lambda_{\mathcal{S}}^{2T} f = \nabla_\nu u|_{(0, 2T) \times \mathcal{S}}, \quad f \in C_0^\infty((0, 2T) \times \mathcal{S}; E),$$

where ν is the interior unit normal on ∂M and u is the solution of (2).

Our main result is that, for a sharp time $T > 0$, the Hermitian vector bundle $E|_{\mathcal{S}}$ and the restricted Dirichlet-to-Neumann operator

Λ_S^{2T} determine the Riemannian manifold (M, g) , the Hermitian vector bundle E , the connection ∇ and potential V . Here $E|_S$ is the pullback bundle j^*E given by the inclusion map $j : S \rightarrow M$.

Theorem 1.1. *Let $(M_i, g_i, E_i, \nabla_i, V_i)$, $i = 1, 2$, be two smooth Hermitian vector bundles that are defined on smooth, compact and connected Riemannian manifolds with boundary, and that are equipped with smooth Hermitian connections and smooth potentials. Suppose that $T > 0$ and open $S_i \subset \partial M_i$, $i = 1, 2$, satisfy*

$$T > \max_{x \in M_i} d_{g_i}(x, S_i), \quad i = 1, 2,$$

where d_{g_i} is the distance function on (M_i, g_i) . Suppose, furthermore, that there is a Hermitian vector bundle isomorphism $\phi : E_1|_{S_1} \rightarrow E_2|_{S_2}$ intertwining the Dirichlet-to-Neumann operators $\Lambda_{S_1}^{2T}$ and $\Lambda_{S_2}^{2T}$, that is, $\phi^* \Lambda_{S_2}^{2T} = \Lambda_{S_1}^{2T} \phi^*$. Then there is a Hermitian vector bundle isomorphism $\Phi : E_1 \rightarrow E_2$ that covers an isometry between (M_i, g_i) , $i = 1, 2$, and that satisfies $\Phi^* \nabla_2 = \nabla_1$, $\Phi^* V_2 = V_1$ and $\Phi|_{E_1|_{S_1}} = \phi$.

Let us denote by $\pi_i : E_i \rightarrow M_i$, $i = 1, 2$, the associated bundle projections, and recall that a vector bundle isomorphism $\Phi : E_1 \rightarrow E_2$ determines a diffeomorphism $\Psi : M_1 \rightarrow M_2$ via the equation

$$\Psi \circ \pi_1 = \pi_2 \circ \Phi.$$

The isomorphism Φ covering an isometry means that $\Psi^* g_2 = g_1$.

It is a simple exercise to check that if an isomorphism Φ as in Theorem 1.1 exists, then the restriction of Φ on $E_1|_{S_1}$ intertwines the Dirichlet-to-Neumann operators. Hence, Theorem 1.1 is optimal in terms of the gauge invariances.

We recall that a generalized Laplacian H on E is a differential operator such that its principal symbol is

$$|\xi|^2 = g^{ij}(x) \xi_i \xi_j, \quad (x, \xi) \in T^*M,$$

and we say that H is symmetric if

$$\langle u, Hv \rangle_{L^2(M;E)} = \langle Hu, v \rangle_{L^2(M;E)}, \quad u, v \in C_0^\infty(M; E).$$

A symmetric generalized Laplacian H on E can be written in the form $P + V$ for some Hermitian connection ∇ and potential V , see, e.g., [3, Proposition 2.5], and wave equations for generalized Laplacians are the most general hyperbolic equations for which unique continuation is known to hold in the whole domain of influence, see Theorem 2.3 below. Such time sharp unique continuation, that goes back to the seminal paper [33], is crucial to our proof.

Let us also point out that if the symmetry assumptions in Theorem 2.3 are weakened, then all the known uniqueness results in the scalar case require additional assumptions on the global geometry of

(M, g) , see [14, 25, 28]. We discuss the difficulties related to weaker symmetry assumptions in more detail in Remark 3.4 below.

As a corollary of Theorem 1.1, let us consider the case when (M, g) is known, E is the trivial bundle $M \times \mathbb{C}^n$ with its usual Hermitian inner product and $V = 0$. Then ∇ is of the form

$$(3) \quad d_A = d + A,$$

where $A = A_i dx^i$ and each $A_i(x)$, $x \in M$, is a skew-Hermitian $(n \times n)$ -matrix. The Dirichlet-to-Neumann operator depends on A and we write $\Lambda_{\partial M}^{2T} = \Lambda_{\partial M; A}^{2T}$.

Corollary 1.2. *Let d_A and d_B be two Hermitian connections on the trivial bundle $M \times \mathbb{C}^n$ over a fixed Riemannian manifold (M, g) , and suppose that $\Lambda_{\partial M; A}^T = \Lambda_{\partial M; B}^T$ for $T > \max_{x \in M} d_g(x, \partial M)$. Then, there exists a smooth $U : M \rightarrow U(n)$ such that $U|_{\partial M} = Id$ and*

$$(4) \quad B = U^{-1}dU + U^{-1}AU.$$

Note that if A and B satisfy (4), then $U^{-1}d_A U = d_B$ and, hence, $P_B = U^{-1}P_A U$, where $P_i = d_i^* d_i$, $i = A, B$. Thus, if u solves the wave equation for P_B , then Uu solves it for P_A . Hence, the above corollary cannot be improved, that is, if $U : M \rightarrow U(n)$ satisfies $U|_{\partial M} = Id$ and (4) holds, then $\Lambda_{\partial M; A}^T = \Lambda_{\partial M; B}^T$ for any T . In the context of the gauges in Theorem 1.1, we have that ϕ is the identity and $\Phi(x, s) = (x, U(x)s)$, where $(x, s) \in M \times \mathbb{C}^n$.

The situation of the corollary is the one that appears in the literature. For the abelian case $n = 1$, the corollary is essentially proved in [24] via the Boundary Control method. The Boundary Control method was pioneered for the isotropic wave equation on a domain in [1] and developed for manifolds in [2]. Note, however, that in [24] the boundary spectral data is used, and therefore, the result does not give the sharp time T .

In [17], the corollary is proved under the further assumptions that M is a two dimensional domain, g is the Euclidean metric tensor and the connection is small in a suitable sense. The proof uses geometric optics solutions and reduces the problem to an injectivity result about the non-abelian Radon transform, which is of independent interest; see [11] for the case of the Euclidean metric and compactly supported connections. More recently, the injectivity result for the non-abelian Radon transform was extended to any simply connected surface with strictly convex boundary and no conjugate points [32] and to higher dimensions and negative curvature [18].

There is a result due to G. Eskin [12] that implies Corollary 1.2 under the assumption that M is a domain in Euclidean space with obstacles. Our proof seems, however, simpler. Eskin also proves a

related theorem for the case of time-dependent Yang–Mills potentials in [15]. A survey on these results, including amended statements, is given in [13].

The proof of Corollary 1.2 follows directly from of our local reconstruction procedure and well-known properties of the cut locus, so the full power of Theorem 1.1 is not needed. As far as we are aware, there are no previous results for this problem when the bundle is not trivial; perhaps the closest in spirit is the result in [26] for the hyperbolic Dirac equation. However, in this reference it is assumed that the data is given on the whole boundary for an infinite time interval, whereas our main result assumes only partial data and is sharp in terms of T . One of the main contributions of the present paper is to develop a new method to glue local reconstructions. The method allows us to reconstruct an isomorphic copy of the structure (g, E, ∇, V) on the interior of M given the data Λ_S^{2T} corresponding to a sharp time T .

Let us mention that there is a recent stability result for Gel’fand’s inverse interior spectral problem [5]. There the problem is studied for compact Riemannian manifolds without boundary, and the result is closer to Corollary 1.2 than Theorem 1.1 in the sense that the global geometry needs to be considered only along the cut locus of suitable semi-geodesic coordinates. As the proof in [5] uses also techniques from the Boundary Control method, we conjecture that if d_A and d_B are as in Corollary 1.2, then $d_1(\mathcal{O}(B), \mathcal{O}(A)) \leq \omega(d_2(\Lambda_{\partial M; A}^T, \Lambda_{\partial M; B}^T))$ where ω is a modulus continuity of the same double logarithmic type as in [5], d_1 and d_2 are suitable distance functions, and

$$\mathcal{O}(A) = \{U^{-1}AU + U^{-1}dU; U : M \rightarrow U(n), U|_{\partial M} = Id\}$$

is the orbit of A under the gauge group.

As a final corollary, let us consider an elliptic analogue of Theorem 1.1. This application is very much in the spirit of [8, Theorem 1.5] where an elliptic scalar valued equation was considered.

Let (M_0, g_0) be a compact, connected Riemannian manifold with boundary, and let $C = \mathbb{R} \times M_0$ be the infinite cylinder with the product metric $g = dt^2 + g_0$. Here dt^2 is the Euclidean metric on \mathbb{R} . We consider a Hermitian vector bundle $E_0 \rightarrow M_0$ with a Hermitian connection ∇_0 , and define the operator $P_0 = \nabla_0^* \nabla_0$. Moreover, we have an induced Hermitian bundle E with connection ∇ on C , that is, $E = \pi^* E_0$ and $\nabla = \pi^* \nabla_0$, where $\pi : C \rightarrow M_0$ is the canonical projection.

Let us denote by $\lambda_1 \leq \lambda_2 \leq \dots$ the Dirichlet eigenvalues of the operator P_0 . A point $\lambda \in \mathbb{C} \setminus [\lambda_1, \infty)$ is not in the continuous spectrum of the operator $\nabla^* \nabla = -\partial_t^2 + P_0$ and, for any $f \in C_0^\infty(\partial C; E)$, the equation

$$(-\partial_t^2 + P_0 - \lambda)u = 0 \text{ in } C, \quad u|_{\partial C} = f$$

has a unique bounded solution $u \in C^\infty(C; E)$. We define the elliptic Dirichlet-to-Neumann map

$$\Lambda(\lambda)f = \nabla_\nu u|_{\partial C}, \quad \Lambda(\lambda) : C_0^\infty(\partial C; E) \rightarrow C^\infty(\partial C; E).$$

Our application is the following recovery result:

Corollary 1.3. *The Hermitian vector bundle $E|_{\partial C}$ and the elliptic Dirichlet-to-Neumann map $\Lambda(\lambda)$ for a fixed $\lambda \in \mathbb{C} \setminus [\lambda_1, \infty)$ determine the structure $(M_0, g_0, E_0, \nabla_0)$.*

Here, the structure is determined up to the natural gauge invariances as in Theorem 1.1. It is possible to prove also a version of the corollary assuming that λ is in the continuous spectrum of $-\partial_t^2 + P_0$ as long as it avoids the eigenvalues λ_i . This extension can be carried out as in [8, Theorem 1.7] but we do not include it here.

This paper is organized as follows. Section 1 is the introduction and states the main results. In Section 2, we include preliminaries, mostly having to do with the direct problem, finite speed of propagation, unique continuation and approximate controllability. The results here are standard, but some details are provided to ensure the usual techniques fit our setting. Section 3 contains the local reconstruction procedure near the boundary. We first reconstruct the metric g and the core of the section is the reconstruction of the Hermitian bundle and the connection. The main local result is Theorem 3.11 and Corollary 1.2 is immediately derived from this theorem and well-known properties of the cut locus. Section 4 contains the global reconstruction procedure, explains in detail how to build up the structure from local data and finishes the proof of Theorem 1.1. In the final Section 5 we prove Corollary 1.3.

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2. Preliminaries

2.1. Local trivializations. The connection ∇ is of the form (3) on a local trivialization of E . Let us derive local expressions for d_A^* and $P = d_A^* d_A$. To this end, we consider a section $u : M \rightarrow E$ and a E -valued 1-form $\beta = \beta_i dx^i$ supported on a local trivialization. As A is

skew-hermitian,

$$\langle Au, \beta \rangle_{L^2(M; E \otimes T^*M)} = \int_M g^{ij} \langle A_i u, \beta_j \rangle_E dx = - \int_M \langle u, g^{ij} A_i \beta_j \rangle_E dx.$$

We define $(A, \beta) = g^{ij} A_i \beta_j$ and see that $d_A^* = d^* - (A, \cdot)$. Thus,

$$Pu = d^* du + d^*(Au) - (A, du) - (A, Au).$$

We recall that for a 1-form α in local coordinates

$$d^* \alpha = -|g|^{-1/2} \frac{\partial}{\partial x^i} \left(|g|^{1/2} g^{ij} \alpha_j \right),$$

hence, $d^*(Au) = (d^*A)u - (A, du)$, and

$$(5) \quad Pu = d^* du - 2(A, du) + (d^*A)u - (A, Au).$$

This exposes the nature of P : the principal part is the usual Laplacian and the first order term given by $-2(A, du)$.

When working near the boundary ∂M , it is convenient to use boundary normal coordinates, that is, semigeodesic coordinates adapted to the boundary. Let $\Gamma \subset \partial M$ be open. Then the semigeodesic coordinates adapted to Γ are given by the map

$$(6) \quad (s, y) \mapsto \gamma(s; y, \nu), \quad y \in \Gamma, \quad s \in [0, \sigma_\Gamma(y)),$$

where the cut distance $\sigma_\Gamma : \Gamma \rightarrow (0, \infty)$ is defined by

$$(7) \quad \begin{aligned} \sigma_\Gamma(y) &= \max\{s \in (0, \tau_M(y)); d_g(\gamma(s; y, \nu), \Gamma) = s\}, \\ \tau_M(y) &= \sup\{s \in (0, \infty); \gamma(s; y, \nu) \in M^{\text{int}}\}. \end{aligned}$$

Here $\gamma(\cdot; x, \xi)$ is the geodesic with the initial data $(x, \xi) \in TM$. We recall that ν is the interior unit normal on ∂M , and define

$$(8) \quad M_\Gamma = \{\gamma(s; y, \nu); y \in \Gamma, s \in [0, \sigma_\Gamma(y))\}.$$

Then a point $x \in M_\Gamma$ is represented in the coordinates (6) by (s, y) , where s is the distance $d_g(x, \Gamma)$ and y is the unique closest point to x in Γ . Moreover, g has the form $ds^2 + h_{jk}(s, y) dy^j dy^k$ and the principal part of P is

$$(9) \quad -\partial_s^2 - h^{jk}(s, y) \partial_{y^j} \partial_{y^k}.$$

2.2. The direct problem. Let us consider the initial-boundary value problem

$$(10) \quad \begin{aligned} (\partial_t^2 + P + V)u(t, x) &= F, \quad (0, T) \times M, \\ u|_{(0, T) \times \partial M} &= f, \quad (0, T) \times \partial M, \\ u|_{t=0} &= \psi, \quad \partial_t u|_{t=0} = \phi, \quad \text{in } M, \end{aligned}$$

where $T > 0$. When $f = 0$ we have the energy estimate

$$(11) \quad \begin{aligned} \|u(t)\|_{H_0^1(M; E)} + \|\partial_t u(t)\|_{L^2(M; E)} \\ \leq C(\|\psi\|_{H_0^1(M; E)} + \|\phi\|_{L^2(M; E)} + \|F\|_{L^2((0, t) \times M; E)}), \end{aligned}$$

for all $t \in (0, T)$. For a proof in the scalar valued case, we refer to [16, Section 7.2]. The proof is analogous in the vector valued case and we omit it. We have also higher regularity results under suitable compatibility conditions. In what follows, we need only the following estimate

$$(12) \quad \|u\|_{H^m((0,T) \times M; E)} \leq C(\|\phi\|_{H^{m-1}(M; E)} + \|F\|_{H^{m-1}((0,T) \times M; E)}),$$

where $m \geq 1$, f and ψ vanish, F is compactly supported in the time interval $(0, T)$ (but not necessarily in space), and ϕ is compactly supported in M^{int} , see, e.g., [16]. We can extend $f \in C_0^\infty((0, \infty) \times \partial M; E)$ as a smooth function on the whole domain $(0, \infty) \times M$ and subtract it from u . By using (12) we see that the solution of (2) is smooth for such sources f .

We need a sharp regularity result for the Neumann trace. The result is due to Lasiecka, Lions and Triggiani in the scalar valued case [27]. The proof in the present setting is analogous but we give it for the convenience of the reader. We will use the following identity

$$(13) \quad \langle \nabla^* u, v \rangle_{L^2(M; E)} - \langle u, \nabla v \rangle_{L^2(M; E \otimes T^* M)} \\ = \langle d^* u, v \rangle_{L^2(M; E)} - \langle u, dv \rangle_{L^2(M; E \otimes T^* M)} = \int_{\partial M} \langle i_\nu u, v \rangle_E dS,$$

where $u \in C^\infty(M; E \otimes T^* M)$, $v \in C^\infty(M; E)$, and dS is the Riemannian volume of $(\partial M, g)$. This follows from [34, Prop. 2.9.1] since the principal symbol of ∇ coincides with the principal symbol of d .

Theorem 2.1. *Suppose that F, f and ψ vanish and let $\phi \in L^2(M; E)$. Then the solution u of (10) satisfies $\nabla_\nu u \in L^2((0, T) \times \partial M; E)$.*

Proof. We will first suppose that $\phi \in C_0^\infty(M; E)$. Then u is smooth by (12). We extend ν as a smooth vector field on the whole domain M , and denote this extension still by ν . We have

$$\langle Pu, \nabla_\nu u \rangle_{L^2((0,T) \times M; E)} \\ = \langle \nabla u, \nabla \nabla_\nu u \rangle_{L^2((0,T) \times M; E \otimes T^* M)} + \int_0^T \int_{\partial M} |\nabla_\nu u|_E^2 dS.$$

Here $|u|_E^2 = \langle u, u \rangle_E$. In local coordinates, the principal part of both

$$\langle \nabla_j u, \nabla_k \nabla_\nu u \rangle_E g^{jk} \quad \text{and} \quad \frac{1}{2} \nu(\langle \nabla_j u, \nabla_k u \rangle_E g^{jk})$$

is $\langle \partial_j u, \nu^p \partial_p \partial_k u \rangle_E g^{jk}$. Thus,

$$\langle \nabla u, \nabla \nabla_\nu u \rangle_{L^2((0,T) \times M; E \otimes T^* M)} = \frac{1}{2} \int_0^T \int_M \nu(\langle \nabla_j u, \nabla_k u \rangle_E g^{jk}) dx + R,$$

where the remainder term R satisfies $|R| \leq C \|u\|_{H^1((0,T) \times M; E)}^2$. Moreover,

$$\begin{aligned} \int_0^T \int_M \nu \langle \nabla_j u, \nabla_k u \rangle_E g^{jk} dx &= - \int_0^T \int_M (\operatorname{div} \nu) \langle \nabla_j u, \nabla_k u \rangle_E g^{jk} dx \\ &\quad - \int_0^T \int_{\partial M} \langle \nabla_j u, \nabla_k u \rangle_E g^{jk} dS. \end{aligned}$$

As u vanishes on the boundary, we have in the boundary normal coordinates $(s, y) \in [0, \epsilon) \times \partial M$ that

$$\langle \nabla_j u, \nabla_k u \rangle_E g^{jk} = |\partial_s u|_E^2 = |\nabla_\nu u|_E^2.$$

Hence,

$$(14) \quad \langle Pu, \nabla_\nu u \rangle_{L^2((0,T) \times M; E)} = \frac{1}{2} \|\nabla_\nu u\|_{L^2((0,T) \times \partial M; E)}^2 + R,$$

where the remainder term R satisfies $|R| \leq C \|u\|_{H^1((0,T) \times M; E)}^2$.

Analogously

$$\begin{aligned} &\langle \partial_t^2 u, \nabla_\nu u \rangle_{L^2((0,T) \times M; E)} \\ &= -\frac{1}{2} \int_0^T \int_M \nu \langle \partial_t u, \partial_t u \rangle_E dx + \left[\int_M \langle \partial_t u, \nabla_\nu u \rangle_E dx \right]_{t=0}^{t=T}, \end{aligned}$$

and

$$\begin{aligned} \int_0^T \int_M \nu \langle \partial_t u, \partial_t u \rangle_E dx &= - \int_0^T \int_M (\operatorname{div} \nu) \langle \partial_t u, \partial_t u \rangle_E dx \\ &\quad - \int_0^T \int_{\partial M} \langle \partial_t u, \partial_t u \rangle_E dx, \end{aligned}$$

where the second term on the right-hand side is zero since $u = 0$ on ∂M . Hence,

$$(15) \quad \begin{aligned} |\langle \partial_t^2 u, \nabla_\nu u \rangle_{L^2((0,T) \times M; E)}| &\leq C \|u\|_{H^1((0,T) \times M; E)}^2 \\ &\quad + C \max_{t=0, T} (\|u(t)\|_{H_0^1(M; E)}^2 + \|\partial_t u(t)\|_{L^2(M; E)}^2). \end{aligned}$$

Clearly

$$(16) \quad |\langle Vu, \nabla_\nu u \rangle_{L^2((0,T) \times M; E)}| \leq C \|u\|_{H^1((0,T) \times M; E)}^2.$$

Combining (14)–(16) with the energy estimate (11), we get

$$\|\nabla_\nu u\|_{L^2((0,T) \times \partial M; E)}^2 \leq C \|\phi\|_{L^2(M; E)}^2.$$

The claim follows since $C_0^\infty(M; E)$ is dense in $L^2(M; E)$. q.e.d.

We will next discuss how (10) can be solved for non-smooth ϕ that are supported in the interior of M . Let $K \subset M^{\text{int}}$ be compact and choose $\chi \in C_0^\infty(M)$ such that $\chi = 1$ near K . Define first the map

$$\mathcal{W} : H_0^{m-1}((0, T) \times M; E) \rightarrow H_0^m(M; E), \quad \mathcal{W}F = \chi u(T),$$

where u solves (10) with $f = 0$, $\psi = 0$ and $\phi = 0$. For $\phi \in C_0^\infty(M; E)$ satisfying $\text{supp}(\phi) \subset K$ it holds that

$$\langle \mathcal{W}F, \phi \rangle_{L^2(M; E)} = \langle F, v \rangle_{L^2((0, T) \times M; E)},$$

where v is the solution of

$$(17) \quad \begin{aligned} (\partial_t^2 + P + V)v(t, x) &= 0, & (0, T) \times M, \\ v|_{(0, T) \times \partial M} &= 0, & (0, T) \times \partial M, \\ v|_{t=T} &= 0, \quad \partial_t v|_{t=T} = -\phi, & \text{in } M. \end{aligned}$$

Then the adjoint of \mathcal{W} , restricted on the subspace

$$\dot{H}^{-m}(K; E) = \{\phi \in H^{-m}(M; E); \text{supp}(\phi) \subset K\}$$

is the unique continuous extension

$$\dot{H}^{-m}(K; E) \rightarrow H^{-m+1}((0, T) \times M; E),$$

of the map solving (17) for smooth ϕ . We may reverse time to get the solution u of (10) with $\phi \in \dot{H}^{-m}(K; E)$ and F, f and ψ vanishing.

Let us now consider the traces of such a solution u . As the principal part of $P + V$ is of the form (9) in the boundary normal coordinates $(s, y) \in [0, \epsilon) \times \partial M$, we may repeat the proof of [21, Th. B.2.9] without any changes in the present, vector valued setting. This implies that u is in $\overline{H}_{(\mu, \sigma)}^{loc}((0, T) \times (0, \epsilon) \times \Gamma; E)$ where $\mu + \sigma \leq -m + 1$ and $\Gamma \subset \partial M$ is a coordinate neighbourhood. Taking now large $\mu \in \mathbb{R}$ and small $\sigma \in \mathbb{R}$, we may apply [21, Th. B.2.7] to see that there is $m' \in \mathbb{R}$ such that that the maps $s \mapsto u(\cdot, s, \cdot)$ and $s \mapsto \nabla_\nu u(\cdot, s, \cdot)$ are continuous with values in $H^{-m'}((0, T) \times \Gamma; E)$. In particular, the traces $u|_{(0, T) \times \partial M}$ and $\nabla_\nu u|_{(0, T) \times \partial M}$ are well-defined for the solution u of the wave equation (10) with $\phi \in \dot{H}^{-m}(K; E)$ and F, f and ψ vanishing.

2.3. Finite speed of propagation, unique continuation and approximate controllability. Equation (2) has the following finite speed of propagation property:

Theorem 2.2. *Let $T > 0$, $U \subset M$ be open and define the cone*

$$\mathcal{C} = \{(t, x) \in (0, T) \times M; d_g(x, U) < T - t\}.$$

Suppose that $f \in C_0^\infty((0, T) \times \partial M; E)$ vanishes in the intersection

$$\mathcal{C} \cap ((0, T) \times \partial M).$$

Then the solution u of (2) vanishes in \mathcal{C} . In particular, if $\Gamma \subset \partial M$ is open, $r \in (0, T)$, and $\text{supp}(f) \subset (T - r, T) \times \Gamma$, then $\text{supp}(u(T))$ is contained in the domain of influence

$$M(\Gamma, r) = \{x \in M; d_g(x, \Gamma) \leq r\}.$$

We refer to [22, Lemma 4.1] for a proof in the scalar valued case. The proof in the present setting is analogous and we omit it.

The operator $P + V$ is of principally scalar form, and the local unique continuation result [9] can be applied. The local result implies the following result due to Eller and Toundykov [10] that is analogous to the semi-global Holmgren theorem.

Theorem 2.3. *Let $T > 0$ and let $\Gamma \subset \partial M$ be open. Let $s \in \mathbb{R}$, and suppose that $u \in H^s((0, 2T) \times M; E)$ satisfies $(\partial_t^2 + P + V)u = 0$ and*

$$u|_{(0,2T) \times \Gamma} = 0, \quad \nabla_\nu u|_{(0,2T) \times \Gamma} = 0.$$

Then $u(T, x) = 0$ whenever $x \in M(\Gamma, T)^{\text{int}}$.

Let us denote $Wf = u(T)$, where u is the solution of (2). The formal adjoint of W is $W^*\phi = \nabla_\nu v|_{(0,T) \times \partial M}$, where v is the solution of (17). Indeed,

$$\begin{aligned} 0 &= \langle (\partial_t^2 + P + V)u, v \rangle_{L^2((0,T) \times M; E)} - \langle u, (\partial_t^2 + P + V)v \rangle_{L^2((0,T) \times M; E)} \\ &= \left[\langle \partial_t u, v \rangle_{L^2(M; E)} - \langle u, \partial_t v \rangle_{L^2(M; E)} \right]_{t=0}^{t=T} \\ &\quad + \langle \nabla_\nu u, v \rangle_{L^2((0,T) \times \partial M; E)} - \langle u, \nabla_\nu v \rangle_{L^2((0,T) \times \partial M; E)} \\ &= \langle u(T), \phi \rangle_{L^2(M; E)} - \langle f, \nabla_\nu v \rangle_{L^2((0,T) \times \partial M; E)}. \end{aligned}$$

As discussed in the end of the previous section, for any $m \in \mathbb{R}$ and compact $K \subset M^{\text{int}}$ there is $m' \in \mathbb{R}$ such that

$$(18) \quad W^* : \dot{H}^{-m}(K; E) \rightarrow H^{-m'}((0, T) \times \partial M; E).$$

For the purposes of the present paper, apart from the case $m = 0$ described in Theorem 2.1, the optimal value of m' is irrelevant. We may consider $L^2(K; E)$ as the subspace of $L^2(M; E)$ consisting of functions supported in K (i.e., $\dot{H}^0(K; E)$).

If $\Gamma \subset \partial M$ is open and nonempty and $r > 0$, then the map

$$\phi \mapsto \nabla_\nu v|_{(0,r) \times \Gamma} : L^2(M(\Gamma, r); E) \rightarrow L^2((0, r) \times \Gamma; E)$$

is injective by Theorem 2.3. A duality argument implies that the wave equation (2) is approximately controllable in the sense of the lemma below. This is well-known in the scalar valued case, see, e.g., [23]. The proof in the present setting is analogous, however, we give it for the convenience of the reader.

Lemma 2.4. *Let $\Gamma \subset \partial M$ be open and $r > 0$. Then*

$$(19) \quad \{Wf; f \in C_0^\infty((T - r, T) \times \Gamma; E)\}$$

is dense in $L^2(M(\Gamma, r); E)$.

Proof. By the finite speed of propagation, the set (19) is a subspace of $L^2(M(\Gamma, r); E)$. It is enough to show that the orthogonal complement of this subspace contains only the origin. Suppose that $\phi \in L^2(M(\Gamma, r); E)$ satisfies

$$(20) \quad (Wf, \phi)_{L^2(M; E)} = 0, \quad f \in C_0^\infty((T - r, T) \times \Gamma; E).$$

Recall that $W^*\phi = \nabla_\nu v|_{(0, T) \times \partial M}$ where v is the solution of (17). Hence, (20) implies that $\nabla_\nu v|_{(T-r, T) \times \Gamma} = 0$. We extend v across the surface $t = T$ by using the odd reflection $v(t, x) = -v(2T - t, x)$. Then the extension satisfies the wave equation

$$\begin{aligned} (\partial_t^2 + P + V)v(t, x) &= 0, & (0, 2T) \times M, \\ v|_{(0, 2T) \times \partial M} &= 0, & (0, 2T) \times \partial M, \\ v|_{t=T} &= 0, \quad \partial_t v|_{t=T} = -\phi, & \text{in } M, \end{aligned}$$

together with the additional boundary condition $\nabla_\nu v|_{(T-r, T+r) \times \Gamma} = 0$. Theorem 2.3 implies that $\phi = 0$. Here we used also the fact that the boundary of $M(\Gamma, r)$ is of measure zero [29]. q.e.d.

As described in the scalar valued case in Section 4.4 of [28], in order to determine the cut distance σ_Γ from the restricted Dirichlet-to-Neumann map, we need to use a perturbation argument that is based on a refined version of approximate controllability and modified domains of influence. Let $\Gamma \subset \partial M$ and $h : \Gamma \rightarrow \mathbb{R}$, and define

$$M(\Gamma, h) = \{x \in M; \inf_{y \in \Gamma} (d_g(x, y) - h(y)) \leq 0\},$$

and denote for $T > 0$

$$\mathcal{B}(\Gamma, h; T) = \{(t, y) \in (0, T) \times \Gamma; T - h(y) < t\}.$$

If $r > 0$ and $h(y) = r, y \in \Gamma$, then $M(\Gamma, h)$ coincides with our earlier definition of $M(\Gamma, r)$. We denote by 1_S the indicator function of a set $S \subset M$, that is, $1_S(x) = 1$ if $x \in S$ and $1_S(x) = 0$ otherwise.

For the convenience of the reader, we give a proof of the following lemma. An analogous lemma is stated in [28] without a proof.

Lemma 2.5. *Let $T > 0$ and suppose that $\Gamma \subset \partial M$ is open. Let $L \in \mathbb{N}$, let $\Gamma_\ell \subset \Gamma$ be open and let $h_\ell \in C(\bar{\Gamma}_\ell), \ell = 1, \dots, L$. We define*

$$(21) \quad h = \sum_{\ell=1}^L h_\ell 1_{\Gamma_\ell},$$

and suppose that $h \leq T$ pointwise. Then

$$(22) \quad \{Wf; f \in C_0^\infty(\mathcal{B}(\Gamma, h; T); E)\}$$

is dense in $L^2(M(\Gamma, h); E)$.

Proof. Let $\epsilon > 0$. There is a simple function

$$h_\epsilon(y) = \sum_{j=1}^J T_j 1_{\Gamma_j}(y),$$

where $J \in \mathbb{N}$, $T_j \in (0, T)$ and $\Gamma_j \subset \Gamma$ are open and disjoint, such that $h < h_\epsilon + \epsilon$ almost everywhere on Γ and $h_\epsilon < h$ on $\bar{\Gamma}$, see, e.g., [29, Lemma 4.2].

We show by induction on J that the density holds when $h = h_\epsilon$. The base case $J = 1$ follows from Lemma 2.4. We define $\tilde{h}_\epsilon = h_\epsilon - T_J 1_{\Gamma_J}$, and use the shorthand notation $M_0 = M(\Gamma, \tilde{h}_\epsilon)$ and $M_1 = M(\Gamma_J, T_J)$. Let $\psi \in L^2(M(\Gamma, h_\epsilon); E)$. Note that $M(\Gamma, h_\epsilon) = M_0 \cup M_1$. By the induction hypothesis there is a sequence of smooth functions $(f_k^0)_{k=1}^\infty$ supported in $\mathcal{B}(\Gamma, \tilde{h}_\epsilon; T)$ such that

$$W f_k^0 \rightarrow 1_{M_0} \psi, \quad k \rightarrow \infty.$$

Moreover, by Lemma 2.4 there is a sequence of smooth functions $(f_k^1)_{k=1}^\infty$ supported in $\mathcal{B}(\Gamma_J, T_J; T)$ such that

$$W f_k^1 \rightarrow 1_{M_1}(\psi - 1_{M_0} \psi), \quad k \rightarrow \infty.$$

Thus, $W(f_k^0 + f_k^1) \rightarrow \psi$. This proves that the density holds for h_ϵ .

Suppose now that $\psi \in L^2(M(\Gamma, h); E)$. We have shown that there is a smooth function f supported in $\mathcal{B}(\Gamma, h_\epsilon; T)$ such that

$$\|1_{M(\Gamma, h_\epsilon)} \psi - W f\|_{L^2(M; E)}^2 < \epsilon.$$

Thus,

$$\|\psi - W f\|_{L^2(M; E)}^2 < \epsilon + \left(\int_{M(\Gamma, h)} |\psi|_E^2 dx - \int_{M(\Gamma, h_\epsilon)} |\psi|_E^2 dx \right).$$

The Riemannian volumes converge $|M(\Gamma, h_\epsilon)| \rightarrow |M(\Gamma, h)|$ as $\epsilon \rightarrow 0$, see [29, Lemma 4.3]. Thus, the claimed density holds. q.e.d.

3. Local reconstruction near the boundary

In this section, we show how to recover the coefficients of $P + V$, up to the gauge invariances, near the accessible part of the boundary \mathcal{S} given the map $\Lambda_{\mathcal{S}}^{2T}$. The main novelty is the recovery of the connection and potential by using such sources f that Wf localizes near a point in M . The basic idea of finding localized Wf given $\Lambda_{\mathcal{S}}^{2T}$ is described in Lemma 3.6, and the inner products appearing in this lemma are shown to be determined by $\Lambda_{\mathcal{S}}^{2T}$ in Corollary 3.2. The localization technique is refined in Lemma 3.9, and the localized solutions are then used to probe the connection and potential in the proof of Theorem 3.11.

3.1. Inner products. We begin by generalizing an integration by parts technique due to Blagovestchenskii in the 1+1 dimensional scalar valued case [4]. For a multidimensional scalar valued case this was first used by Belishev [1].

Lemma 3.1. *Let $T > 0$, let $\mathcal{S} \subset \partial M$ be open, and let f and h be functions in $C_0^\infty((0, 2T) \times \mathcal{S}; E)$. Then*

$$\begin{aligned} & \langle Wf, Wh \rangle_{L^2(M; E)} \\ &= \langle f, J\Lambda_{\mathcal{S}}^{2T} h \rangle_{L^2((0, 2T) \times \mathcal{S}; E)} - \langle f, (\Lambda_{\mathcal{S}}^{2T})^* Jh \rangle_{L^2((0, 2T) \times \mathcal{S}; E)}, \end{aligned}$$

where J is the integral operator in the time variable with the kernel $\text{sgn}(t - s)1_L(t, s)/4$. Here $L = \{(s, t) \in \mathbb{R}^2 : 0 \leq t + s \leq 2T, t, s > 0\}$.

Proof. We write $u^f = u$ for the solution of (2) and define the function $w(t, s) = \langle u^f(t), u^h(s) \rangle_{L^2(M; E)}$. We have

$$\begin{aligned} (\partial_t^2 - \partial_s^2)w(t, s) &= \langle \partial_t^2 u^f(t), u^h(s) \rangle_{L^2(M; E)} - \langle u^f(t), \partial_s^2 u^h(s) \rangle_{L^2(M; E)} \\ &= -\langle \nabla^* \nabla u^f(t), u^h(s) \rangle_{L^2(M; E)} + \langle u^f(t), \nabla^* \nabla u^h(s) \rangle_{L^2(M; E)} \\ &= -\int_{\partial M} \langle \nabla_\nu u^f(t), u^h(s) \rangle_E dS + \int_{\partial M} \langle u^f(t), \nabla_\nu u^h(s) \rangle_E dS \\ &= \int_{\partial M} \langle f(t), \Lambda_{\mathcal{S}}^{2T} h(s) \rangle_E dS - \int_{\partial M} \langle \Lambda_{\mathcal{S}}^{2T} f(t), h(s) \rangle_E dS. \end{aligned}$$

Since $w(0, s) = w(t, 0) = \partial_t w(0, s) = \partial_s w(0, s) = 0$ and w solves the above 1+1 dimensional wave equation, the result follows by considering $w(T, T)$. q.e.d.

Corollary 3.2. *Let $T > 0$, $\mathcal{S} \subset \partial M$ be open. Then $\Lambda_{\mathcal{S}}^{2T}$ determines the inner products*

$$(23) \quad \langle Wf, Wh \rangle_{L^2(M; E)}, \quad f, h \in C_0^\infty((0, 2T) \times \mathcal{S}; E).$$

Moreover, $\Lambda_{\mathcal{S}}^{2T}$ determines, for all $(f_j)_{j=1}^\infty \subset C_0^\infty((0, 2T) \times \mathcal{S}; E)$, if the sequence $(Wf_j)_{j=1}^\infty$ converges, in the strong or weak sense, in $L^2(M; E)$.

Proof. We allow the metric tensor g to be a priori unknown on \mathcal{S} . However, $\Lambda_{\mathcal{S}}^{2T}$ determines the distances $d_g(x, y)$, $x, y \in \mathcal{S}$, see, e.g., [7, Section 2.2], and these distances determine g on \mathcal{S} . Thus, we can assume without loss of generality that the Riemannian volume measure dS of (\mathcal{S}, g) is known, and Lemma 3.1 implies that $\Lambda_{\mathcal{S}}^{2T}$ determines the inner products (23).

For the second claim, we observe that the inner products (23) can be used to determine if $(Wf_j)_{j=1}^\infty$ is a Cauchy sequence in $L^2(M; E)$. This allows us to determine if $(Wf_j)_{j=1}^\infty$ converges in the strong sense. Moreover, using again (23) we can determine if $(Wf_j)_{j=1}^\infty$ is bounded in $L^2(M; E)$, and we may test the weak convergence analogously to [28, Lemma 3]. q.e.d.

3.2. Reconstruction of the metric tensor. Our reconstruction of the metric tensor is based on the proof in [28]. The following lemma is a variation of [28, Lemma 6]. We give a short proof for the convenience of the reader.

Lemma 3.3. *Let $T > 0$, $s \in (0, T]$, let $\Sigma, \Gamma \subset \partial M$ be open and let $h : \Gamma \rightarrow [0, T]$. Suppose that h is of form (21). Then the following are equivalent:*

- (i) $M(\Sigma, s) \subset M(\Gamma, h)$.
- (ii) For all $f_0 \in C_0^\infty(\mathcal{B}(\Sigma, s; T); E)$ there is a sequence $(f_j)_{j=1}^\infty$ in $C_0^\infty(\mathcal{B}(\Gamma, h; T); E)$ such that $(W(f_0 - f_j))_{j=1}^\infty$ converges to zero in $L^2(M; E)$.

Proof. The implication from (i) to (ii) follows from the density of (22) in $L^2(M(\Gamma, h); E)$. We will now show that (ii) implies (i). We denote

$$\begin{aligned} M_0 &= M(\Sigma, s), & M_1 &= M(\Gamma, h), \\ S_0 &= \mathcal{B}(\Sigma, s; T), & S_1 &= \mathcal{B}(\Gamma, h; T). \end{aligned}$$

Let us assume that (i) does not hold. There is a nonempty open set $U \subset M_0$ such that $U \cap M_1 = \emptyset$, see [28, Lemma 6]. By Lemma 2.4 there is a smooth function f_0 supported in S_0 such that $\int_U W f_0 dx \neq 0$. However, by finite speed of propagation $Wf|_U = 0$ for any f supported in S_1 . Thus,

$$\langle W(f_0 - f), 1_U \rangle_{L^2(M; E)} = \langle Wf_0, 1_U \rangle_{L^2(M; E)} \neq 0,$$

for all f supported in S_1 and (ii) does not hold. q.e.d.

By Corollary 3.2 we can determine, given the restricted Dirichlet-to-Neumann map Λ_S^{2T} , whether the condition (ii) in Lemma 3.3 holds for a function f_0 and a sequence $(f_j)_{j=1}^\infty$, assuming that $\Sigma, \Gamma \subset \mathcal{S}$.

Remark 3.4. Suppose for the moment that we weaken the symmetry assumptions by not requiring (1). Then the closest analogue of the identity in Lemma 3.1 allows us to compute the inner products $\langle \tilde{u}(T), Wh \rangle_{L^2(M; E)}$ where \tilde{u} is the solution of (2) with V replaced by its formal adjoint V^* . It seems to be difficult to use such inner products to test for convergence as in the condition (ii). In the non-symmetric scalar valued case [25], a global condition on the billiard flow of (M, g) is assumed in order for the map

$$W : L^2((0, T) \times \mathcal{S}) \rightarrow L^2(M),$$

not only to have a dense range, but to be surjective. In this case, it is easy to test for a variant of the condition (ii) where convergence in the norm is replaced by weak convergence. In [28] a similar difficulty is treated by imposing an asymptotic spectral condition of the type that was first studied in [19].

Let $\Gamma \subset \partial M$ be open and let $T > 0$. We recall that the cut distance σ_Γ is defined by (7), and define

$$(24) \quad \begin{aligned} \sigma_\Gamma^T(y) &= \min(\sigma_\Gamma(y), T), \quad y \in \Gamma, \\ M_\Gamma^T &= \{\gamma(s; y, \nu); y \in \Gamma, s \in [0, \sigma_\Gamma^T(y))\}. \end{aligned}$$

Theorem 3.5. *Let $T > 0$ and let $\Gamma \subset \partial M$ be open. Then the Riemannian manifold (Γ, g) , the Hermitian vector bundle $E|_\Gamma$ and Λ_Γ^{2T} determine (M_Γ^T, g) .*

Proof. By combining Corollary 3.2 and Lemmas 2.5 and 3.3 we can determine the relation

$$(25) \quad \{(\Sigma, s, h); M(\Sigma, s) \subset M(\Gamma, h)\},$$

for any open $\Sigma \subset \Gamma$, $s \in (0, T]$ and a function h of form (21). This relation determines σ_Γ^T and the Riemannian manifold (M_Γ^T, g) by using the purely geometric method described in Sections 4.2–4.4 of [28]. Note that the relations with $M(\Gamma, h)$ replaced by the union of two domains of influence are obtained by using piecewise continuous functions h as in [28, Lem. 6], and that also the two limiting arguments in the proof of [28, Prop. 2] are needed. q.e.d.

3.3. Reconstruction of the connection. Our reconstruction method is based on a use of sequences of sources $(f_j)_{j=1}^\infty$ such that $\text{supp}(Wf_j)$ converges to a point.

Lemma 3.6. *Let $\Gamma_1, \Gamma_2 \subset \partial M$ be open and $r_1, r_2 > 0$. Suppose that for a sequence $(f_j)_{j=1}^\infty \subset C_0^\infty((T - r_1, T) \times \Gamma_1; E)$ the sequence $(Wf_j)_{j=1}^\infty$ converges weakly to a function $\phi \in L^2(M; E)$, and that*

$$\langle Wf_j, Wh \rangle_{L^2(M; E)} \rightarrow 0, \quad h \in C_0^\infty((T - r_2, T) \times \Gamma_2; E).$$

Then $\text{supp}(\phi) \subset M(\Gamma_1, r_1) \setminus M(\Gamma_2, r_2)^{\text{int}}$.

Proof. The lemma follows immediately from the density of the set (19). q.e.d.

Lemma 3.7. *Let $T > 0$, $\Gamma \subset \partial M$ be open, and let $x \in \Gamma \cup M^{\text{int}}$ satisfy $d_g(x, \Gamma) < T$. Then there are functions $h_\ell \in C_0^\infty((0, 2T) \times \Gamma; E)$ such that $Wh_\ell(x)$, $\ell = 1, \dots, n$, form an orthonormal basis of the fiber E_x of E at x .*

Proof. If $x \in \Gamma$, then $Wh(x) = h(T, x)$ and the claim clearly holds in this case. Suppose now that $x \in M^{\text{int}}$. It is enough to show that the fiber E_x is spanned by the vectors

$$Wh(x), \quad h \in C_0^\infty((0, T) \times \Gamma; E).$$

In order to show this it is enough to show that if $e \in E_x$ and

$$(26) \quad \langle e, Wh(x) \rangle_E = 0, \quad h \in C_0^\infty((0, T) \times \Gamma; E),$$

then $e = 0$.

We recall that the adjoint of W is given by $W^*\phi = \nabla_\nu v|_{(0, T) \times \partial M}$, where v is the solution of (17), and that the continuity (18) holds. We choose $\phi = e\delta_x$. The restriction $W^*\phi|_{(0, T) \times \Gamma} = \nabla_\nu v|_{(0, T) \times \Gamma}$ vanishes by (26), and $v|_{(0, T) \times \Gamma}$ vanishes by the boundary condition in (17). We extend v on the time interval $(0, 2T)$ by the odd reflection with respect to $t = T$, and denote the extension still by v . The extension satisfies $(\partial_t^2 + P + V)v = 0$ on $(0, 2T) \times M$. Theorem 2.3 implies that $e = 0$.
 q.e.d.

Lemma 3.8. *Let $\Gamma \subset \partial M$ be open, let $T > 0$, and let $e : M \rightarrow E$ be a section of E . Let $U \subset M^{\text{int}} \cup \Gamma$ be open in M and suppose also that $U \subset M(\Gamma, T)$. Suppose, furthermore, that $x \mapsto \langle e(x), Wh(x) \rangle_E$ is smooth on U for all $h \in C_0^\infty((0, 2T) \times \Gamma; E)$. Then e is smooth on U .*

Proof. Let $x \in U$, and let us choose $h_\ell, \ell = 1, \dots, n$, as in Lemma 3.7. Then the functions Wh_ℓ form a smooth frame near x , and the representation of e in this frame is smooth.
 q.e.d.

We recall that $|X|$ denotes the Riemannian volume of a measurable set $X \subset M$, and that the set M_Γ is defined by (8).

Lemma 3.9. *Let $\Gamma \subset \partial M$ be open. Let $x \in M_\Gamma$, and let $y \in \Gamma$ and $s \in [0, \sigma_\Gamma(y))$ satisfy $\gamma(s; y, \nu) = x$. Define $s_k = s + 1/k$,*

$$Y_k = \{\tilde{y} \in \Gamma; d_g(\tilde{y}, y) < 1/k\}, \quad X_k = M(Y_k, s_k) \setminus M(\Gamma, s).$$

Suppose that a double sequence $\Phi = (f_{jk})_{j,k=1}^\infty$ of functions in the space $C_0^\infty((T - s_k, T) \times Y_k; E)$ satisfies the following

- (i) *For each $k = 1, 2, \dots$, the sequence $(Wf_{jk})_{j=1}^\infty$ converges weakly in $L^2(M; E)$ to a function supported in X_k .*
- (ii) *There is $C > 0$ such that*

$$\|Wf_{jk}\|_{L^2(M; E)} \leq C|X_k|^{-1/2}, \quad j, k = 1, 2, \dots$$

- (iii) *The limit $\lim_{k \rightarrow \infty} \lim_{j \rightarrow \infty} \langle Wf_{jk}, Wh \rangle_{L^2(M; E)}$ exists for any function h in the space $C_0^\infty((0, 2T) \times \Gamma; E)$.*

Then there is a vector $e(x; \Phi) \in E_x$ that depends on x and Φ such that

$$(27) \quad \lim_{k \rightarrow \infty} \lim_{j \rightarrow \infty} \langle Wf_{jk}, \phi \rangle_{L^2(M; E)} = \langle e(x; \Phi), \phi(x) \rangle_E, \quad \phi \in C^\infty(M; E).$$

Note that we allow here the case $x \in \Gamma$, i.e., $s = 0$.

Proof. By Lemma 3.7 there are h_ℓ such that $Wh_\ell(x), \ell = 1, \dots, n$, form an orthonormal basis of E_x . Let us write $b_\ell = Wh_\ell$ and denote the weak limit of $(Wf_{jk})_{j=1}^\infty$ by u_k . We choose local coordinates \tilde{x} in a neighborhood $U \subset M$ of x , and suppose that k is large enough so that

$X_k \subset U$ and that the sections $b_\ell(\tilde{x})$ form a basis in $E_{\tilde{x}}$ for all $\tilde{x} \in X_k$. Let $\phi \in C^\infty(M; E)$ and write $\phi(\tilde{x}) = c^\ell b_\ell(\tilde{x}) + (x^p - \tilde{x}^p)\psi_p(\tilde{x})$, where $c^\ell \in \mathbb{C}$ and $\psi_p \in C^\infty(U; E)$, $p = 1, \dots, m$. Then

$$(28) \quad \langle u_k, \phi \rangle_{L^2(M; E)} = \overline{c^\ell} \langle u_k, b_\ell \rangle_{L^2(M; E)} + R_k,$$

where the remainder term satisfies

$$\begin{aligned} |R_k| &\leq m \max_{p=1, \dots, m} \|\psi_p\|_{C(U)} \operatorname{diam}(X_k) \int_{X_k} |u_k(\tilde{x})|_E d\tilde{x} \\ &\leq m \max_{p=1, \dots, m} \|\psi_p\|_{C(U)} \operatorname{diam}(X_k) \|u_k\|_{L^2(M; E)} |X_k|^{1/2}. \end{aligned}$$

Note that $\operatorname{diam}(X_k) \rightarrow 0$ since $X_k \supset X_{k+1}$ and $X_k \rightarrow x$ as $k \rightarrow \infty$. Thus, (ii) implies that $R_k \rightarrow 0$. By (iii) the limits

$$a^\ell = \lim_{k \rightarrow \infty} \langle u_k, b_\ell \rangle_{L^2(M; E)}, \quad \ell = 1, \dots, n,$$

exist. We set $e = a^\ell b_\ell(x)$. Then

$$\lim_{k \rightarrow \infty} \langle u_k, \phi \rangle_{L^2(M; E)} = \overline{c^\ell} \lim_{k \rightarrow \infty} \langle u_k, b_\ell \rangle_{L^2(M; E)} = \sum_{\ell=1}^n a^\ell \overline{c^\ell} = \langle e, \phi(x) \rangle_E. \quad \text{q.e.d.}$$

Lemma 3.10. *Let $\Gamma \subset \partial M$ be open, let $x \in M_\Gamma$ and let $e \in E_x$. Then there is a double sequence $\Phi = (f_{jk})_{j,k=1}^\infty$ that satisfies the conditions of Lemma 3.9, and furthermore, $e(x; \Phi) = e$ where $e(x; \Phi)$ is as in (27).*

Proof. Let $\tilde{e} \in C^\infty(M; E)$ satisfy $\tilde{e}(x) = e$. By Lemma 2.4 there is a double sequence $\Phi = (f_{jk})_{j,k=1}^\infty$ of functions in $C_0^\infty((T - s_k, T) \times Y_k; E)$ such that $(Wf_{jk})_{j=1}^\infty$ converges to the function $u_k = |X_k|^{-1} 1_{X_k} \tilde{e}$. We recall that 1_{X_k} is the indicator function of the set X_k and $|X_k|$ is its volume. Moreover, u_k satisfies $\|u_k\|_{L^2(M; E)} \leq |X_k|^{-1/2} \|\tilde{e}\|_{L^\infty(M; E)}$ and, for a function $\phi \in C^\infty(M; E)$,

$$\langle u_k, \phi \rangle_{L^2(M; E)} = \frac{1}{|X_k|} \int_{X_k} \langle \tilde{e}(\tilde{x}), \phi(\tilde{x}) \rangle_E d\tilde{x} \rightarrow \langle e, \phi(x) \rangle_E,$$

where \tilde{x} are local coordinates on X_k .

q.e.d.

Theorem 3.11. *Let $T > 0$, let $\Gamma \subset \partial M$ be open and suppose that the vector bundle $E|_\Gamma$ is trivial. Then the Riemannian manifold (M_Γ^T, g) , where M_Γ^T is defined in (24), the Hermitian vector bundle $E|_\Gamma$ and the restricted Dirichlet-to-Neumann map Λ_Γ^{2T} determine the Hermitian vector bundle $E|_{M_\Gamma^T}$, the connection ∇ and the potential V on $E|_{M_\Gamma^T}$.*

Proof. We choose for each $x \in M_\Gamma^T$ a double sequence $\Phi^x = (f_{jk}^x)_{j,k=1}^\infty$ satisfying conditions (i)–(iii) of Lemma 3.9. Observe that, by combining Corollary 3.2 and Lemma 3.6, we can determine if condition (i) of Lemma 3.9 is valid, while conditions (ii) and (iii) can be verified by using Lemma 3.1 alone. We use Lemma 3.1 once again to compute the inner

products $\langle e(x; \Phi^x), Wh(x) \rangle_E$ for $h \in C_0^\infty((0, 2T) \times \Gamma; E)$. Next we will impose some further conditions on the choice of the double sequences Φ^x .

First, we choose the double sequences $\Phi^x, x \in M_\Gamma^T$ so that the functions

$$(29) \quad x \mapsto \langle e(x; \Phi^x), Wh(x) \rangle_E, \quad h \in C_0^\infty((0, 2T) \times \Gamma; E),$$

are smooth in M_Γ^T . Then Lemma 3.8 implies that $e(x) = e(x; \Phi^x)$ is a smooth section of the vector bundle $E|_{M_\Gamma^T}$.

Second, we pick an orthonormal frame $\mathcal{B} = (b_\ell)_{\ell=1}^n$ of $E|_\Gamma$ and choose double sequences $\Phi_\ell^x = (f_{jk,\ell}^x)_{j,k=1}^\infty, \ell = 1, \dots, n$, so that the corresponding smooth sections $e_\ell(x) = e(x; \Phi_\ell^x)$ satisfy,

$$\langle e_\ell(x), Wh(x) \rangle_E = \langle b_\ell(x), h(T, x) \rangle_E, \quad x \in \Gamma, \quad h \in C_0^\infty((0, 2T) \times \Gamma; E).$$

This condition implies that $e_\ell = b_\ell$ on Γ .

Our next goal is to choose Φ_ℓ^x so that the corresponding sections e_ℓ form an orthonormal frame also on the set $M_0 = M_\Gamma^T \cap M^{\text{int}}$. To this end, we observe that the vector bundle $E|_{M_\Gamma^T}$ is trivial. This follows from [20, Th. 4.2.4], since the identity map on M_Γ^T is smoothly homotopic with the map $(s, y) \mapsto (0, y)$ in coordinates (6).

Let $x \in M_0$, and choose a cut off function $\chi \in C_0^\infty(M_0)$ such that $\chi(x) = 1$. As the functions (29) and the geometry (M_Γ^T, g) are known, we can compute the limits

$$(30) \quad \lim_{k \rightarrow \infty} \lim_{j \rightarrow \infty} \langle \chi e_\kappa, W f_{jk,\ell}^x \rangle_{L^2(M; E)} = \langle e_\kappa(x), e_\ell(x) \rangle_E, \quad \kappa, \ell = 1, \dots, n,$$

where the equality follows from Lemma 3.9. Hence, we can choose the double sequences Φ_ℓ^x so that $\mathcal{E} = (e_\ell)_{\ell=1}^n$ forms an orthonormal frame on M_0 . Note that Lemma 3.10 implies that for any frame on M_0 there are double sequences $\Phi_\ell^x, \ell = 1, \dots, n, x \in M_0$, such that the corresponding functions e_ℓ coincide with the frame.

Now $(x, a) \mapsto a^\ell e_\ell(x)$, where $a = (a^\ell)_{\ell=1}^n \in \mathbb{C}^n$ and $x \in M_\Gamma^T$, is a trivialization of $E|_{M_\Gamma^T}$, and the Hermitian inner product is given by

$$\left\langle a^\ell e_\ell(x), c^\kappa e_\kappa(x) \right\rangle_E = \sum_{\ell=1}^n a^\ell \overline{c^\ell}, \quad a, c \in \mathbb{C}^n, \quad x \in M_\Gamma^T,$$

on this trivialization.

Let us write $u^h = u$ for the solution of (2) with $f = h$. The functions (29) determine the representation of

$$(31) \quad Wh(x) = u^h(t, x), \quad t = T, \quad x \in M_0, \quad h \in C_0^\infty((0, 2T) \times \Gamma; E),$$

in the frame \mathcal{E} . To avoid cumbersome notation, we will not make explicit distinction between the functions (31) and their representation until Section 4.2.

Observe that the wave equation (2) is translation invariant in time in the sense that $u^h(t - s, \cdot) = u^{\tilde{h}}(t, \cdot)$ where $\tilde{h}(t, \cdot) = h(t - s, \cdot)$ and $s \geq 0$. Thus, the functions (31) are determined also for $t \in (0, T)$. We differentiate twice in time and obtain the functions

$$(P + V)u^h(t, x), \quad t \in (0, T), \quad x \in M_0, \quad h \in C_0^\infty((0, 2T) \times \Gamma; E).$$

Let $\phi \in C_0^\infty(M_0; E)$. We can compute the inner products

$$\left\langle (P + V)u^h(T), \phi \right\rangle_{L^2(M; E)} = \langle Wh, (P + V)\phi \rangle_{L^2(M; E)},$$

for $h \in C_0^\infty((0, 2T) \times \Gamma; E)$. As the functions (31) are known and dense in $L^2(M_0; E)$, we can determine $(P + V)\phi$ on M_0 .

Let $x \in M_0$, $\ell = 1, \dots, n$ and $k = 1, \dots, m$. We choose $\phi = \phi_\ell^k$ such that $\phi(x) = 0$ and $\partial_j \phi(x) = \delta_j^k e_\ell$ for $j = 1, \dots, m$. As the metric tensor is known near x , we can compute $d^*d\phi$ at x . Thus, we can recover the first order term in $(P + V)\phi$ at x . By (5), this is

$$-2(A, d\phi)(x) = -2g^{ik}(x)A_i e_\ell(x),$$

and therefore, A can be determined. Finally, A and g determine P , and we can determine V by $V = P + V - P$. q.e.d.

3.4. Reconstruction of ∇ when (M, g) is known and E is trivial. We will show next that Corollary 1.2 follows from the above local reconstruction step, that is, from the proof of Theorem 3.11.

Corollary 3.12. *Suppose that (M, g) is known, E is the trivial bundle $M \times \mathbb{C}^n$, and that $T > \max_{x \in M} d_g(x, \partial M)$. Let d_A be a Hermitian connection on E . Then the Dirichlet-to-Neumann map $\Lambda_{\partial M; A}^{2T}$ determines the orbit*

$$\mathcal{O}(A) = \{U^{-1}AU + U^{-1}dU; U : M \rightarrow U(n), U|_{\partial M} = Id\}.$$

Proof. Let b_1, \dots, b_n be the standard basis of \mathbb{C}^n and let \mathcal{B} be the corresponding constant frame of E . Let \mathcal{E} be the orthonormal frame of $E|_{M_{\partial M}}$ chosen in the proof of Theorem 3.11. We recall that \mathcal{E} can be enforced to satisfy $\mathcal{E} = \mathcal{B}$ on ∂M .

We have $M_{\partial M} = M \setminus N$ where the cut locus N is of measure zero, see, e.g., [6]. In particular, $M_{\partial M}$ is dense in M . We know the representation of the functions Wh , $h \in C_0^\infty((0, 2T) \times \partial M; E)$, in the frame \mathcal{E} , see (31) above. Let us impose the further condition on the choice of Φ_ℓ^x in the proof of Theorem 3.11 that the representation of $Wh(x)$ in the frame \mathcal{E} is smooth in $M = \overline{M_{\partial M}}$ for all $h \in C_0^\infty((0, 2T) \times \partial M; E)$. Then Lemma 3.8 implies that \mathcal{E} gives a smooth frame for the whole vector bundle E .

There is a smooth transition function $U : M \rightarrow U(n)$ between the two frames \mathcal{E} and \mathcal{B} , and $U = Id$ on ∂M . Moreover, we can reconstruct the

representation of d_A in the frame \mathcal{E} . Let us denote the representation by $d_{\tilde{A}}$. Then

$$\tilde{A} = U^{-1}AU + U^{-1}dU,$$

and, hence, we can determine the orbit $\mathcal{O}(\tilde{A}) = \mathcal{O}(A)$. q.e.d.

Suppose now that d_A and d_B are two Hermitian connections on E , and that the assumptions of Corollary 1.2 are satisfied. Then the above corollary implies that $\mathcal{O}(A) = \mathcal{O}(B)$, and we have shown Corollary 1.2.

4. Global reconstruction

In this section, we show how to recover globally the coefficient of $P + V$, up to the gauge invariances, by iterating the local reconstruction step and by continuing the data Λ_S^{2T} inside the region that we have already reconstructed. We will begin by giving a brief outline of the iterative scheme. The data Λ_S^{2T} can be viewed as a model of measurements with sources and receivers on Γ . To initialize the iteration, we choose a small ball B_0 in the region where the coefficients of $P + V$ are already known from the local reconstruction step in the previous section. Then we use unique continuation to recover data modeling measurements with sources on Γ and receivers on B_0 , and also with both sources and receivers on B_0 . Then we repeat the local reconstruction step for the data with sources and receivers on B_0 , and recover the coefficients of $P + V$ on a larger ball B containing B_0 . Using unique continuation again, we recover the data with sources and receivers on a small ball B_1 in B , and also the data with sources on Γ and the receivers on B_1 . Iterating this alternating procedure, we can cover M with small patches where the coefficients of $P + V$ are known. The data with sources on Γ and the receivers on B_0, B_1, \dots , is then used to glue the patches together.

4.1. Continuation of the data. For $T > 0$ and open sets $B \subset M$ and $\Gamma \subset \partial M$, we define the map

$$L_{\Gamma, B}^T f = u|_{(0, T) \times B}, \quad f \in C_0^\infty((0, T) \times \Gamma; E),$$

where u is the solution of (2). Moreover, for open $B \subset M^{\text{int}}$, we define the map

$$L_B^T F = u|_{(0, T) \times B}, \quad F \in C_0^\infty((0, T) \times B; E),$$

where u is the solution of

$$(32) \quad \begin{aligned} (\partial_t^2 + P + V)u(t, x) &= F, & (0, \infty) \times M, \\ u|_{(0, \infty) \times \partial M} &= 0, \\ u|_{t=0} = \partial_t u|_{t=0} &= 0. \end{aligned}$$

We write $B(x, \epsilon) = \{y \in M; d_g(y, x) < \epsilon\}$ for $x \in M$ and $\epsilon > 0$.

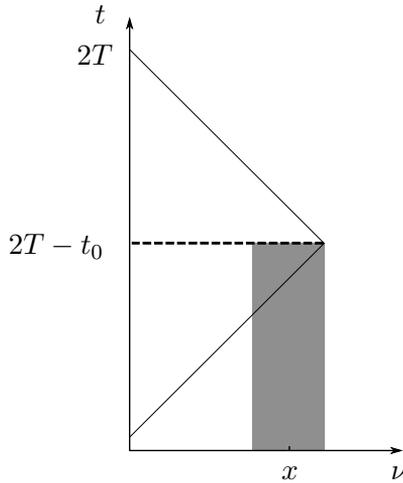


Figure 1. A schematic of the unique continuation argument in the proof of Lemma 4.1. The origin represents the set Γ , and the gray area is the cylinder $(0, 2T - t_0) \times B$. In order to recover u on $\{2T - t_0\} \times B$, data $(u, \nabla_\nu u)$ is needed on the cylinder $I \times \Gamma$ where $I = (2T - 2t_0, 2T)$. We may translate the interval I to cover the whole gray cylinder.

Lemma 4.1. *Let $T > 0$, $\Gamma \subset \partial M$ be open and let $x \in M_\Gamma^T$. Define $s = d_g(x, \Gamma)$, let $\epsilon \in (0, T - s)$ and define*

$$B = B(x, \epsilon), \quad t_0 = s + \epsilon.$$

Then Λ_Γ^{2T} and the structure (g, E, ∇, V) on M_Γ^T determine the map $L_{\Gamma, B \cap M_\Gamma^T}^{2T-t_0}$. Furthermore, if $B \subset M_\Gamma^T \cap M^{\text{int}}$ then they determine also the map $L_B^{2(T-t_0)}$.

Proof. Let $f \in C_0^\infty((0, 2T) \times \Gamma; E)$. We will next use unique continuation to determine $L_{\Gamma, B \cap M_\Gamma^T}^{2T-t_0} f$ given $P + V$ on M_Γ^T and $\Lambda_\Gamma^{2T} f$. Let us first extend the solution u of (2) by 0 to $(-\infty, 0) \times M$. We denote the distance function of (M_Γ^T, g) by \tilde{d}_g and observe that

$$\tilde{d}_g(x, \Gamma) = d_g(x, \Gamma), \quad x \in M_\Gamma^T,$$

by the definition of M_Γ , see (8). Let \tilde{u} be a solution of

$$(33) \quad (\partial_t^2 + P + V)\tilde{u} = 0, \quad (-\infty, 2T) \times M_\Gamma^T,$$

satisfying the boundary conditions

$$(34) \quad \tilde{u} = f \quad \text{and} \quad \nabla_\nu \tilde{u} = \Lambda_\Gamma^{2T} f \quad \text{on} \quad (-\infty, 2T) \times \Gamma.$$

Given $P + V$ on M_Γ^T and $\Lambda_\Gamma^{2T} f$, we can determine the set of functions

$$\mathbb{U}_f = \{\tilde{u} \in C^\infty((-\infty, 2T) \times M_\Gamma^T); \text{ (33) and (34) hold}\}.$$

Let $\tilde{u} \in \mathbb{U}_f$, and apply Theorem 2.3 on the function $w = \tilde{u} - u$ with M replaced by M_Γ^T and with suitable translations in the time variable, see Figure 1. This implies that $\tilde{u} = u$ on $(0, 2T - t_0) \times (B \cap M_\Gamma^T)$, and we have shown the first claim.

Let us now assume that $B \subset M_\Gamma^T \cap M^{\text{int}}$. We will reconstruct the map $L_B^{2(T-t_0)}$ in two steps that we outline before giving a detailed proof. Note that $L_{\Gamma, B}^{2T-t_0}$ can be interpreted as data with sources on Γ and receivers on B . We will first transpose $L_{\Gamma, B}^{2T-t_0}$ and obtain data with sources on B and receivers on Γ . Then we will use unique continuation to obtain data with both sources and receivers on B , that is, the map $L_B^{2(T-t_0)}$.

By taking the adjoint of $L_{\Gamma, B}^{2T-t_0}$ and conjugating it with the operator reversing the time on the interval $(0, 2T - t_0)$, we get the map

$$(35) \quad F \mapsto \nabla_\nu u : C_0^\infty((0, 2T - t_0) \times B; E) \rightarrow C^\infty((0, 2T - t_0) \times \Gamma; E),$$

where u is the solution of (32). We extend u by 0 to $(-\infty, 0) \times M$, and let \tilde{u} be a solution of

$$(\partial_t^2 + P + V)\tilde{u} = F, \quad (0, \infty) \times M_\Gamma^T,$$

satisfying $\tilde{u} = 0$ and $\nabla_\nu \tilde{u} = \nabla_\nu u$ on $(-\infty, 2T - t_0) \times \Gamma$. Then $w = \tilde{u} - u$ satisfies conditions of Theorem 2.3 with M again replaced by M_Γ^T , and therefore, $\tilde{u} = u$ on $(0, 2T - t_0 - t_0) \times B$. This implies the second claim. q.e.d.

We denote by SM the unit sphere bundle of M . Similarly to σ_Γ and σ_Γ^T , see (7) and (24), we define for $x \in M^{\text{int}}$, $\xi \in S_x M$ and $T > 0$,

$$\begin{aligned} \sigma_x(\xi) &= \sup\{t \in (0, \tau_x(\xi)]; d_g(\gamma(t; x, \xi), x) = t\}, \\ \tau_x(\xi) &= \sup\{t \in (0, \infty); \gamma(t; x, \xi) \in M^{\text{int}}\}, \end{aligned}$$

and $\sigma_x^T(\xi) = \min(\sigma_x(\xi), T)$. Moreover, we define

$$M_x^T = \{\gamma(t; x, \xi); \xi \in S_x M, t \in [0, \sigma_x^T(\xi)]\}.$$

Note that the injectivity radius inj_x at a point $x \in M^{\text{int}}$ satisfies

$$\text{inj}_x = \min_{\xi \in S_x M} \sigma_x(\xi).$$

Lemma 4.2. *Let $T > 0$, $x \in M^{\text{int}}$, $\epsilon \in (0, \text{inj}_x)$, and set $B = B(x, \epsilon)$. Then L_B^{2T} and the structure (g, E, ∇, V) on B determine the structure (g, E, ∇, V) on $M_x^{T+\epsilon}$.*

Proof. We define $\tilde{M} = M \setminus B$ and consider the wave equation

$$(36) \quad \begin{aligned} (\partial_t^2 + P + V)\tilde{u} &= 0, & (0, \infty) \times \tilde{M}, \\ \tilde{u}|_{(0, \infty) \times \partial B} &= f, \quad \tilde{u}|_{(0, \infty) \times \partial M} = 0, \\ \tilde{u}|_{t=0} &= \partial_t \tilde{u}|_{t=0} = 0. \end{aligned}$$

We will show that L_B^{2T} determines the restricted Dirichlet-to-Neumann map $\Lambda_{\partial B}^{2T}$ of \tilde{M} , that is, the map

$$\Lambda_{\partial B}^{2T} f = \nabla_\nu \tilde{u}|_{(0, 2T) \times \partial B}, \quad f \in C_0^\infty((0, 2T) \times \partial B; E),$$

where \tilde{u} is the solution of (36). Let $f \in C_0^\infty((0, 2T) \times \partial B; E)$ and extend the solution of (36) smoothly into $(0, \infty) \times B$ keeping the notation \tilde{u} for the extension. Then \tilde{u} satisfies (32) with $\tilde{F} = (\partial_t^2 + P + V)\tilde{u}$, and \tilde{F} belongs to

$$\mathcal{C} = \{F \in C^\infty((0, \infty) \times M; E); \text{supp}(F) \subset (0, \infty] \times \bar{B}\}.$$

Observe that L_B^{2T} has a unique extension as an operator on $L^2((0, 2T) \times B; E)$. By using this extension, we can determine the set

$$\mathbb{F}_f = \{F \in \mathcal{C}; L_B^{2T} F|_{(0, 2T) \times \partial B} = f\}.$$

Since the solution of (36) is unique, it holds for $F \in \mathbb{F}_f$ that

$$\nabla_\nu L_B^{2T} F|_{(0, 2T) \times \partial B} = \nabla_\nu \tilde{u}|_{(0, 2T) \times \partial B}.$$

We have shown that the map L_B^{2T} determines the map $\Lambda_{\partial B}^{2T}$.

We denote by $\sigma_{\partial B}$ the cut distance on the manifold \tilde{M} defined analogously to (7) and define $\sigma_{\partial B}^T(y) = \max(\sigma_{\partial B}(y), T)$, $y \in \partial B$. Note that the vector bundle $E|_{\partial B}$ is trivial, in fact, E is trivial over M_x^T due to its contractibility via the radial geodesics emanating from x . We apply Theorems 3.5 and 3.11 with $M = \tilde{M}$ and $\Gamma = \partial B$. This gives us the structure (g, E, ∇, V) on

$$\tilde{M}_{\partial B}^T = \{\gamma(s; y, \nu); y \in \partial B, s \in [0, \sigma_{\partial B}^T(y))\}.$$

Note that $\sigma_x(\xi) = \sigma_{\partial B}(y) + \epsilon$, where $y = \gamma(\epsilon; x, \xi)$, and therefore, $M_x^{T+\epsilon} = B \cup \tilde{M}_{\partial B}^T$. q.e.d.

Lemma 4.3. *Let $T_0, \epsilon_0 > 0$, $x_0 \in M^{\text{int}}$, and define $B_0 = B(x_0, \epsilon_0)$ and $M_0 = M_{x_0}^{T_0}$. Let $x \in M_0 \setminus B_0$ and define $s = d_g(x, x_0)$. Let $T > 0$ and let $\epsilon \in (0, \text{inj}_x)$ satisfy*

$$\epsilon < d_g(x, \partial M_0), \quad \epsilon < T - s + \epsilon_0.$$

Define $B_1 = B(x, \epsilon)$ and $t_1 = s + \epsilon - \epsilon_0$. Then $L_{B_0}^{2T}$ and the structure (g, E, ∇, V) on M_0 determine the map $L_{B_1}^{2(T-t_1)}$. Furthermore, for open $\Gamma \subset \partial M$, L_{Γ, B_0}^{2T} and the structure (g, E, ∇, V) on M_0 determine the map $L_{\Gamma, B_1}^{2T-t_1}$.

Proof. By the proof of Lemma 4.2, $L_{B_0}^{2T}$ determines $\Lambda_{\partial B_0}^{2T}$. It holds that $d_g(x, \partial B_0) = s - \epsilon_0$, and Lemma 4.1 shows that $\Lambda_{\partial B}^{2T}$ and the structure (g, E, ∇, V) on M_0 determine $L_{B_1}^{2(T-t_1)}$. Finally, L_{Γ, B_0}^{2T} determines $L_{\Gamma, B_1}^{2T-t_1}$ by a unique continuation argument similar to that in the proof of Lemma 4.1. q.e.d.

4.2. Gluing local reconstructions in the interior. In this section, we show the following theorem:

Theorem 4.4. *Let $S \subset \partial M$ be open and suppose that*

$$(37) \quad T > \max_{x \in M} d_g(x, S).$$

Then the Hermitian vector bundle $E|_S$ and the restricted Dirichlet-to-Neumann operator Λ_S^{2T} determine the smooth manifold M^{int} and the structure (g, E, ∇, V) on M^{int} .

Up to this point we have avoided writing all the isomorphisms explicitly, but in this section, the distinction between different representations is crucial. Let us choose an open cover \mathcal{G}_S of S consisting of small enough sets $\Gamma \subset S$ so that each Γ is a coordinate neighborhood in ∂M and that the vector bundle $E|_\Gamma$ is trivial. Then we may choose an open set $Y_\Gamma \subset \mathbb{R}^{m-1}$ and a unitary trivialization

$$(38) \quad \begin{array}{ccc} E & \xrightarrow{\phi_\Gamma} & Y_\Gamma \times \mathbb{C}^n \\ \downarrow & & \downarrow \\ \Gamma & \xrightarrow{\psi_\Gamma} & Y_\Gamma \end{array}$$

By a unitary trivialization we mean that the diagram (38) commutes, ϕ_Γ is a smooth bijection that is linear in fibers, and that the Hermitian structure is preserved, that is, $\phi_\Gamma^* \langle \cdot, \cdot \rangle_{\mathbb{C}^n} = \langle \cdot, \cdot \rangle_E$.

Starting from the representation of Λ_Γ^{2T} on the trivialization (38), the local reconstruction method in Section 3 determines the cut distance $\sigma_\Gamma^T : \Gamma \rightarrow (0, T)$, a metric tensor g_Γ on X_Γ^T , and a connection ∇_Γ and potential V_Γ on $X_\Gamma^T \times \mathbb{C}^n$, such that there is a unitary trivialization

$$(39) \quad \begin{array}{ccc} E & \xrightarrow{\tilde{\Phi}_\Gamma} & X_\Gamma^T \times \mathbb{C}^n \\ \downarrow & & \downarrow \\ M_\Gamma^T & \xrightarrow{\tilde{\Psi}_\Gamma} & X_\Gamma^T \end{array}$$

satisfying $g = \tilde{\Psi}_\Gamma^* g_\Gamma$, $\nabla = \tilde{\Phi}_\Gamma^* \nabla_\Gamma$ and $V = \tilde{\Phi}_\Gamma^* V_\Gamma$. Here

$$X_\Gamma^T = \{(s, y) \in \mathbb{R}^m; s \in [0, \sigma_\Gamma^T \circ \psi_\Gamma^{-1}(y)], y \in Y_\Gamma\}$$

is the representation of M_Γ^T in boundary normal coordinates, and the restriction of $\tilde{\Phi}_\Gamma$ on the vector bundle $E|_\Gamma$ coincides with ϕ_Γ . We recall that σ_Γ^T is defined by (24).

We will next iterate the procedure in Section 4.1. The initial step is the following:

1. Given Λ_Γ^{2T} and a representation of the structure (g, E, ∇, V) on M_Γ^T , that is, $g_\Gamma, X_\Gamma^T \times \mathbb{C}^n, \nabla_\Gamma$ and V_Γ , we choose $(s_0, y_0) \in X_\Gamma^T$ and $\epsilon_0 > 0$ such that

$$(40) \quad B_0 = B(z_0, \epsilon_0) \subset M_\Gamma^T \cap M^{\text{int}},$$

where $z_0 = \tilde{\Psi}_\Gamma^{-1}(s_0, y_0) \in M_\Gamma^T$.

We invoke Lemma 4.1 to reconstruct the representations of $L_{\Gamma, B_0}^{2T-t_0}$ and $L_{B_0}^{2(T-t_0)}$ on the trivialization (39). Here

$$(41) \quad t_0 = s_0 + \epsilon_0,$$

and we emphasize that we do not know the point $z_0 \in M$, only its representation (s_0, y_0) in the boundary normal coordinates.

We iterate Lemmas 4.2 and 4.3 as follows:

2. Given a representation of $L_{B_j}^{2(T-t_j)}$, where $B_j = B(z_j, \epsilon_j)$, we determine a representation of the structure (g, E, ∇, V) on the set $M_j = M_{z_j}^{T-t_j+\epsilon_j}$.
3. We choose $s_{j+1} > 0, \xi_{j+1} \in S_{z_j}M$ and $\epsilon_{j+1} > 0$ such that

$$B_{j+1} = B(z_{j+1}, \epsilon_{j+1}) \subset M_j,$$

where $z_{j+1} = \gamma(s_{j+1}; z_j, \xi_{j+1})$. Again, we do not know z_{j+1} , only its representation (s_{j+1}, ξ_{j+1}) in normal coordinates at z_j . Given representations of $L_{B_j}^{2(T-t_j)}$ and $L_{\Gamma, B_j}^{2T-t_j}$, we determine representations of $L_{B_{j+1}}^{2(T-t_{j+1})}$ and $L_{\Gamma, B_j}^{2T-t_{j+1}}$, where

$$(42) \quad t_{j+1} = t_j + s_{j+1} + \epsilon_{j+1} - \epsilon_j.$$

We terminate the iteration after repeating the steps 2 and 3 a finite number of times denoted by $N = 0, 1, 2, \dots$. Note that we must satisfy the condition $t_j < T$ in each step of the iteration.

If $N = 0$ then we do not need to satisfy the constraint (40). That is, we can use Lemma 4.1 to reconstruct a representation of $L_{\Gamma, B_0 \cap M_\Gamma^T}^{2T-t_0}$ where $B_0 = B(z_0, \epsilon_0), z_0 \in M_\Gamma^T$ and $\epsilon_0 \in (0, T - s_0)$. In particular, for $y_0 \in \Gamma$ and for small enough $\epsilon_0 > 0$ we can reconstruct a representation of $L_{\Gamma, C_0}^{2T-\epsilon_0}$ where

$$(43) \quad C_0 = \{\gamma(s; y, \nu); s \in (0, \epsilon_0), y \in B_\partial(y_0, \epsilon_0)\},$$

and $B_\partial(y_0, \epsilon_0) = \{y \in \partial M; d_g(y, y_0) < \epsilon_0\}$.

There are is a lot freedom in our iteration process. Namely, we can choose N , the points z_j and the radii ϵ_j freely within the constraints of the iteration. Let A_Γ denote the set of all choices that are allowed within the constraints of iteration when starting from $\Gamma \in \mathcal{G}_S$. We define also the disjoint union $A = \bigsqcup_{\Gamma \in \mathcal{G}_S} A_\Gamma$.

We denote by $B_\alpha = B_{N(\alpha)}$ the set chosen in the last invocation of step 3 in the iteration process $\alpha \in A_\Gamma$, and use analogous notation for other chosen quantities. The iteration gives us a metric tensor g_α , a connection ∇_α and a potential V_α such that there is a unitary trivialization

$$(44) \quad \begin{array}{ccc} E & \xrightarrow{\tilde{\Phi}_\alpha} & X_\alpha \times \mathbb{C}^n \\ \downarrow & & \downarrow \\ B_\alpha & \xrightarrow{\tilde{\Psi}_\alpha} & X_\alpha \end{array}$$

satisfying $g = \tilde{\Psi}_\alpha^* g_\alpha$, $\nabla = \tilde{\Phi}_\alpha^* \nabla_\alpha$ and $V = \tilde{\Phi}_\alpha^* V_\alpha$. Here X_α is the open ball of radius $\epsilon_{N(\alpha)}$ in \mathbb{R}^m with center at the origin, and $\tilde{\Psi}_\alpha$ gives normal coordinates at $z_{N(\alpha)}$. The iteration gives also the representation L_α of $L_{\Gamma, B_\alpha}^{2T-t_{N(\alpha)}}$ on the trivialization (44).

If the iteration is terminated immediately after the initial step (that is, $N(\alpha) = 0$) we allow B_α to be also of the form (43).

Let us show that the balls B_α , $\alpha \in A_\Gamma$, cover $M(\Gamma, T)^{\text{int}}$ and that they separate points:

- (G1) For all distinct $z, z' \in M(\Gamma, T)^{\text{int}}$ there are $\alpha, \beta \in A_\Gamma$ such that $z \in B_\alpha$, $z' \in B_\beta$ and $B_\alpha \cap B_\beta = \emptyset$.

Proof. Let $z \in M(\Gamma, T)^{\text{int}}$. Then there is a shortest path γ from $\bar{\Gamma}$ to z having length strictly less than T . The path γ can be perturbed to get a broken geodesic $\tilde{\gamma}$ from $y \in \Gamma$ to z having length strictly less than T . Moreover, $\tilde{\gamma}$ can be chosen so that it intersects ∂M only at its starting point y . Then the points $z_j, j = 1, \dots, N$, can be chosen along $\tilde{\gamma}$. Moreover, when z_0 is close to Γ and the radius ϵ_N is chosen small enough, we have $t_N < T$. Indeed, by (41) and (42),

$$t_N = \epsilon_N + s_0 + \sum_{j=1}^N s_j,$$

where $s_0 = d_g(z_0, \Gamma)$ and $s_j = d_g(z_j, z_{j-1})$. In particular, the balls B_α , $\alpha \in A$, form an open cover of $M(\Gamma, T)^{\text{int}}$.

Let $z' \in M(\Gamma, T)^{\text{int}}$ and suppose that $z' \neq z$. We may choose the radius ϵ_N small enough so that $\epsilon_N < d_g(z, z')/2$, and perform an analogous construction for z' . This gives us disjoint balls as claimed. q.e.d.

Note that the assumption (37) does not imply that $M(\Gamma, T) = M$ since Γ might be smaller than \mathcal{S} . However, it implies that the sets

$M(\Gamma, T)^{\text{int}}$, $\Gamma \in \mathcal{G}_S$, form an open cover of M^{int} , and therefore, the sets B_α , $\alpha \in A$, form an open cover of M^{int} by (G1). We will show next how to glue together the local representations of (g, E, ∇, V) on the sets B_α , $\alpha \in A$.

Lemma 4.5. *Let $T > 0$, $\Gamma \subset \partial M$ be open, and suppose that $B \subset M^{\text{int}}$ is open and satisfies $B \subset M(\Gamma, T)$. Let $h \in C_0^\infty(B; E)$ and $s \in (0, T)$. Then the maps Λ_Γ^{2T} and $L_{\Gamma, B}^{2T-s}$ together with the structure (g, E) on B determine the non-empty set*

$$(45) \quad \{(f_j)_{j=1}^\infty \subset C_0^\infty((0, 2T) \times \Gamma; E); \lim_{j \rightarrow \infty} Wf_j = h \text{ in } L^2(M; E)\}.$$

Proof. We expand the squared norm

$$\begin{aligned} & \|Wf_j - h\|_{L^2(M; E)}^2 \\ &= \langle Wf_j, Wf_j \rangle_{L^2(M; E)} - 2 \operatorname{Re} \langle Wf_j, h \rangle_{L^2(M; E)} + \langle h, h \rangle_{L^2(M; E)}, \end{aligned}$$

and observe that Λ_Γ^{2T} determines the first term on the right-hand side by Corollary 3.2, $L_{\Gamma, B}^{2T-s}$ and (g, E) on B determine the second term, and (g, E) on B determines the third term. To conclude we observe that Lemma 2.4 implies that the set (45) is non-empty. q.e.d.

Lemma 4.6. *Suppose that open $\mathcal{S} \subset \partial M$ and $T > 0$ satisfy (37). Let $x_1, x_2 \in M^{\text{int}}$. We have $x_1 = x_2$ if and only if for all sufficiently small $\epsilon > 0$ and any $h_1 \in C_0^\infty(B(x_1, \epsilon); E)$ there is $h_2 \in C_0^\infty(B(x_2, \epsilon); E)$ such that*

$$(46) \quad \langle h_1 - h_2, Wf \rangle_{L^2(M; E)} = 0, \quad f \in C_0^\infty((0, 2T) \times \mathcal{S}; E).$$

Proof. Let us suppose that $x_1 \neq x_2$. We choose small enough $\epsilon > 0$ so that the balls $B(x_j, \epsilon)$, $j = 1, 2$, are disjoint. We choose non-zero $h_1 \in C_0^\infty(B(x_1, \epsilon); E)$ and let $h_2 \in C_0^\infty(B(x_2, \epsilon); E)$ be arbitrary. Then $h_1 \neq h_2$ and Lemma 2.4 implies that there is $f \in C_0^\infty((0, 2T) \times \mathcal{S}; E)$ satisfying

$$\langle h_1 - h_2, Wf \rangle_{L^2(M; E)} \neq 0.$$

The other implication is trivial. q.e.d.

Lemmas 4.5 and 4.6 allow us to determine if two points $x_i \in X_{\alpha_i}$, $\alpha_i \in A_{\Gamma_i}$, $\Gamma_i \in \mathcal{G}_S$, $i = 1, 2$, satisfy

$$(47) \quad \tilde{\Psi}_{\alpha_1}^{-1}(x_1) = \tilde{\Psi}_{\alpha_2}^{-1}(x_2).$$

Indeed, let $\epsilon > 0$ be small, let \tilde{B}_i , $i = 1, 2$, be the geodesic ball in $(X_{\alpha_i}, g_{\alpha_i})$ with center x_i and radius ϵ , and let $\tilde{h}_i \in C_0^\infty(\tilde{B}_i; E)$. Then using Lemma 4.5, we can find sequences $(f_j^i)_{j=1}^\infty \subset C_0^\infty((0, 2T) \times \Gamma_i; E)$ such that $\lim_{j \rightarrow \infty} Wf_j^i = h_i$ where $h_i = \Phi_{\alpha_i}^* \tilde{h}_i$. Note that in order to

apply Lemma 4.5 it is enough to know Λ_Γ^{2T} and the representations L_{α_i} and g_{α_i} , $i = 1, 2$. By Corollary 3.2, we can compute

$$(48) \quad \lim_{j \rightarrow \infty} \langle Wf_j^1 - Wf_j^2, Wf \rangle_{L^2(M;E)} = \langle h_1 - h_2, Wf \rangle_{L^2(M;E)},$$

for all $f \in C_0^\infty((0, 2T) \times \mathcal{S}; E)$. Hence, we can use (46) to determine if (47) holds.

Equation (47) gives an equivalence relation on the disjoint union $\tilde{\mathcal{X}} = \bigsqcup_{\alpha \in A} X_\alpha$ and we denote by \mathcal{X} and $q : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$ the corresponding quotient space and the canonical map. Moreover, we define the set $\mathcal{U}_\alpha = q(X_\alpha) \subset \mathcal{X}$ and the restriction $q_\alpha = q|_{X_\alpha}$, $\alpha \in A$. We will show that \mathcal{X} is a smooth manifold:

- (G2) The maps $q_\alpha : X_\alpha \rightarrow \mathcal{U}_\alpha$ are bijective, and there is a unique Hausdorff topology and a complete atlas on \mathcal{X} such that each q_α^{-1} is a coordinate system.

As we can determine if x and x' are equivalent given the data Λ_Γ^{2T} , we see that the smooth structure of \mathcal{X} is determined. Let us show (G2) simultaneously with the following:

- (G3) Let us define a map $\Psi : M^{\text{int}} \rightarrow \mathcal{X}$ by $\Psi(z) = q \circ \tilde{\Psi}_\alpha(z)$ when $z \in B_\alpha$. Then Ψ is a well-defined diffeomorphism.

Proof of (G2) and (G3). Let $z \in M^{\text{int}}$. Then (G1) implies that there is $\alpha \in A$ such that $z \in B_\alpha$. If $z \in B_\beta$ also for $\beta \in A$, then $q(x) = q(x')$ where $x = \tilde{\Psi}_\alpha(z)$ and $x' = \tilde{\Psi}_\beta(z)$. Thus, Ψ is well-defined.

Note that the sets \mathcal{U}_α cover \mathcal{X} since the sets $X_\alpha = \tilde{\Psi}_\alpha(B_\alpha)$ cover $\tilde{\mathcal{X}}$. This implies that Ψ is surjective. Suppose that $\Psi(z) = \Psi(z')$ for some $z \in B_\alpha$ and $z' \in B_\beta$. Then $q(x) = q(x')$ where $x = \tilde{\Psi}_\alpha(z)$ and $x' = \tilde{\Psi}_\beta(z)$. Thus, $z = z'$ by the definition of q , and we have shown that Ψ is injective.

We define $\Psi_\alpha : B_\alpha \rightarrow \mathcal{U}_\alpha$ as the restriction $\Psi_\alpha = \Psi|_{B_\alpha}$. It is clearly bijective. Now $\Psi_\alpha = q_\alpha \circ \tilde{\Psi}_\alpha$ implies that $q_\alpha = \Psi_\alpha \circ \tilde{\Psi}_\alpha^{-1}$. Hence, the maps q_α are bijective. Moreover, if $\mathcal{U} = \mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$ then we have on $q_\alpha^{-1}(\mathcal{U})$ that

$$q_\beta^{-1} \circ q_\alpha = \tilde{\Psi}_\beta \circ \Psi^{-1} \circ \Psi \circ \tilde{\Psi}_\alpha^{-1} = \tilde{\Psi}_\beta \circ \tilde{\Psi}_\alpha^{-1},$$

and we see that $q_\beta^{-1} \circ q_\alpha$ is smooth on the open set $q_\alpha^{-1}(\mathcal{U}) = \tilde{\Psi}_\alpha(B_\alpha \cap B_\beta)$. We have shown that the conditions (1) and (2) of [30, Prop. 1.42] hold. To finish the proof of (G2) we need only to verify the separation condition (3) in [30, Prop. 1.42].

Let $p, p' \in \mathcal{X}$ be distinct. Then we have $z \neq z'$ where $z = \Psi^{-1}(p)$ and $z' = \Psi^{-1}(p')$. Let $\alpha, \beta \in A$ be as in (G1). Then \mathcal{U}_α and \mathcal{U}_β are disjoint sets containing p and p' respectively, since $\mathcal{U}_\alpha = \Psi(B_\alpha)$ and $\mathcal{U}_\beta = \Psi(B_\beta)$. Now (G2) follows from [30, Prop. 1.42].

To show that Ψ is smooth, it is enough to show that each $q_\alpha^{-1} \circ \Psi \circ \tilde{\Psi}_\alpha^{-1}$ is smooth. But this is simply the identity map on X_α . q.e.d.

Let us show that the metric tensors g_α can be glued together:

(G4) We have $(q_\alpha^{-1})^* g_\alpha = (\Psi^{-1})^* g$ on each \mathcal{U}_α .

Proof. We recall that $g = \tilde{\Psi}_\alpha^* g_\alpha$ on B_α . Thus, we have on \mathcal{U}_α that

$$(\Psi^{-1})^* g = (\tilde{\Psi}_\alpha \circ \Psi^{-1})^* g_\alpha = (\tilde{\Psi}_\alpha \circ \Psi_\alpha^{-1})^* g_\alpha = (q_\alpha^{-1})^* g_\alpha. \quad \text{q.e.d.}$$

Let us now turn to gluing of the vector bundles $X_\alpha \times \mathbb{C}^n$. Denote by $\mathcal{E}^\alpha = (e_\ell^\alpha)_{\ell=1}^n$ the constant frame on $X_\alpha \times \mathbb{C}^n$ corresponding to the standard basis of \mathbb{C}^n . Suppose that \mathcal{U}_α and \mathcal{U}_β intersect for some indices $\alpha, \beta \in A$, and write

$$X_{\alpha\beta} = q_\alpha^{-1}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta), \quad X_{\beta\alpha} = q_\beta^{-1}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta).$$

We define functions $h_1 = \Phi_\alpha^* \tilde{h}_1$ and $h_2 = \Phi_\beta^* \tilde{h}_2$, where

$$\tilde{h}_1 = 1_{X_{\alpha\beta}} e_\ell^\alpha \in L^2(X_\alpha; \mathbb{C}^n), \quad \tilde{h}_2 = 1_{X_{\beta\alpha}} a_\ell^\kappa e_\kappa^\beta \in L^2(X_\beta; \mathbb{C}^n).$$

Here $\ell, \kappa = 1, \dots, n$ and $a_\ell^\kappa \in C^\infty(X_{\beta\alpha})$. Analogously to the considerations preceding (48), we can choose two sequences of sources $(f_j^i)_{j=1}^\infty, i = 1, 2$, such that $(W f_j^i)_{j=1}^\infty$ converges to h_i , and determine if (46) holds. Suppose now that we have chosen $a_\ell^\kappa \in C^\infty(X_{\beta\alpha})$ so that (46) holds. We define $U_{\beta\alpha} = (a_\ell^\kappa)_{\kappa,\ell=1}^n$ on $X_{\beta\alpha}$. Moreover, we define an equivalence relation on $\tilde{\mathcal{X}} \times \mathbb{C}^n$ by

$$(49) \quad q(x) = q(x'), \quad \xi' = U_{\beta\alpha}(x')\xi,$$

where $x \in X_\alpha, x' \in X_\beta$ and $\xi, \xi' \in \mathbb{C}^n$. We have:

(G5) Equations (49) hold if and only if $\tilde{\Phi}_\alpha^{-1}(x, \xi) = \tilde{\Phi}_\beta^{-1}(x', \xi')$.

Proof. Observe that $x \in X_\alpha, x' \in X_\beta$ and $q(x) = q(x')$ imply that $x' \in X_{\beta\alpha}$. Therefore, the second equation in (49) is well-defined whenever the first one holds.

We write $B = B_\alpha \cap B_\beta$. Let $Z \in \pi_E^{-1}(B)$ where $\pi_E : E|_{M^{\text{int}}} \rightarrow M^{\text{int}}$ is the bundle projection, and take $z = \pi_E(Z)$. Moreover, denote by $Z_p = (Z_p^\ell)_{\ell=1}^n$ the representation of Z in the frame $\tilde{\Phi}_p^* e_\ell^p, p = \alpha, \beta$. Then, since h_1 and h_2 are smooth in B and satisfy (46), Lemma 2.4 implies that

$$Z = Z_\alpha^\ell \tilde{\Phi}_\alpha^* e_\ell^\alpha|_z = Z_\alpha^\ell \tilde{\Phi}_\beta^*(a_\ell^\kappa e_\kappa^\beta)|_z = Z_\alpha^\ell a_\ell^\kappa (\tilde{\Psi}_\beta(z)) \tilde{\Phi}_\beta^* e_\kappa^\beta|_z.$$

Hence, $Z_\beta = U_{\beta\alpha}(\tilde{\Psi}_\beta(z))Z_\alpha$.

Suppose that (49) holds, and define $Z = \tilde{\Phi}_\alpha^{-1}(x, \xi)$. Then $Z \in \pi_E^{-1}(B)$ and we have, using the above notation $z = \pi_E(Z)$ and $Z_p = (Z_p^\ell)_{\ell=1}^n$,

$p = \alpha, \beta$, that $\tilde{\Psi}_\alpha(z) = x$ and $Z_\alpha = \xi$. Moreover, $\tilde{\Phi}_\beta(Z) = (\tilde{\Psi}_\beta(z), Z_\beta)$ where $\tilde{\Psi}_\beta(z) = x'$ as $q(x) = q(x')$, and

$$Z_\beta = U_{\beta\alpha}(\tilde{\Psi}_\beta(z))Z_\alpha = U_{\beta\alpha}(x')\xi = \xi'.$$

On the other hand, if $Z = \tilde{\Phi}_\alpha^{-1}(x, \xi) = \tilde{\Phi}_\beta^{-1}(x', \xi')$, then $q(x) = q(x')$ and

$$\xi' = Z_\beta = U_{\beta\alpha}(\tilde{\Psi}_\beta(z))Z_\alpha = U_{\beta\alpha}(x')\xi. \quad \text{q.e.d.}$$

We denote by F the quotient space with respect to the equivalence (49) and by $Q : \tilde{\mathcal{X}} \times \mathbb{C}^n \rightarrow F$ the corresponding canonical map. Moreover, we define

$$(50) \quad \pi_F : F \rightarrow \mathcal{X} : \pi_F(Q(x, \xi)) = q(x), \quad (x, \xi) \in \tilde{X} \times \mathbb{C}^n,$$

and Q_α as the restriction of Q on $X_\alpha \times \mathbb{C}^n$, $\alpha \in A$. These maps define a smooth vector bundle structure:

(G6) The map π_F is a well-defined surjection and the maps

$$Q_\alpha : X_\alpha \times \mathbb{C}^n \rightarrow \pi_F^{-1}(\mathcal{U}_\alpha)$$

are bijective. There is a unique Hausdorff topology and a complete atlas on F such that each Q_α^{-1} is a coordinate system. The maps $\xi \mapsto Q_\alpha(x, \xi)$ are bijective from \mathbb{C}^n to $\pi_F^{-1}(\{q(x)\})$ for $x \in X_\alpha$ and $\alpha \in A$, and, if the fibers $\pi_F^{-1}(\{p\})$, $p \in \mathcal{X}$, are equipped with the vector space structure that is pulled back from \mathbb{C}^n via the inverses of these maps, then $\pi_F : F \rightarrow \mathcal{X}$ is a smooth vector bundle that is trivial on each \mathcal{U}_α .

Let us show (G6) simultaneously with the following:

(G7) Let us define a map $\Phi : E|_{M^{\text{int}}} \rightarrow F$ by $\Phi(Z) = Q \circ \tilde{\Phi}_\alpha(Z)$ when $Z \in \pi_E^{-1}(B_\alpha)$. Here π_E is the bundle projection $E|_{M^{\text{int}}} \rightarrow M^{\text{int}}$. Then Φ is a well-defined vector bundle isomorphism covering Ψ .

Proof of (G6) and (G7). Clearly π_F is a well-defined surjection. A proof that Φ is a well-defined bijection is essentially identical with the above proof that Ψ is a well-defined bijection, and we omit it.

Let $\alpha \in A$, $x \in X_\alpha$, and consider the map $Q_\alpha^x(\xi) = Q_\alpha(x, \xi)$. The definition of π_F implies that $Q_\alpha^x : \mathbb{C}^n \rightarrow F^x$ where $F^x = \pi_F^{-1}(\{q(x)\})$. Let us show that Q_α^x is surjective. Let $\beta \in A$ and $x' \in X_\beta$ satisfy $q(x') = q(x)$ and let $\xi' \in \mathbb{C}^n$. Then, if we choose $\xi = U_{\beta\alpha}(x')^{-1}\xi'$, we have $Q_\beta^{x'}(\xi') = Q_\alpha^x(\xi)$ due to (49). Thus, Q_α^x is surjective. The surjectivity implies that

$$Q(X_\alpha \times \mathbb{C}^n) = \bigcup_{x \in X_\alpha} Q_\alpha^x(\mathbb{C}^n) = \pi_F^{-1}(q(X_\alpha)) = \pi_F^{-1}(\mathcal{U}_\alpha).$$

We write $E_\alpha = \pi_E^{-1}(B_\alpha)$, $F_\alpha = \pi_F^{-1}(\mathcal{U}_\alpha)$, and define $\Phi_\alpha = \Phi|_{E_\alpha}$. The sets

$$\Phi(E_\alpha) = Q(X_\alpha \times \mathbb{C}^n) = F_\alpha, \quad \alpha \in A,$$

cover F , and $\Phi_\alpha : E_\alpha \rightarrow F_\alpha$ is bijective. The factorization $\Phi_\alpha = Q_\alpha \circ \tilde{\Phi}_\alpha$ implies that Q_α is bijective, and $Q_\beta^{-1} \circ Q_\alpha = \tilde{\Phi}_\beta \circ \tilde{\Phi}_\alpha^{-1}$ is smooth on the open set $Q_\alpha^{-1}(F_\alpha \cap F_\beta) = \tilde{\Phi}_\alpha(E_\alpha \cap E_\beta)$.

Let $p, p' \in F$, and define $z = \pi_E \circ \Phi^{-1}(p)$ and $z' = \pi_E \circ \Phi^{-1}(p')$. If $z \neq z'$ then we may choose $\alpha, \beta \in A$ as in (G1). Then E_α and E_β are disjoint, whence F_α and F_β are disjoint sets containing p and p' respectively. On the other hand, if $z = z'$ then there is $\alpha \in A$ such that $p, p' \in F_\alpha$. Now [30, Prop. 1.42] implies that F has a unique smooth manifold structure.

To show that π_F is smooth, it is enough to show that each $q_\alpha^{-1} \circ \pi_F \circ Q_\alpha$ is smooth. But this is simply the map $\pi_\alpha : X_\alpha \times \mathbb{C}^n \rightarrow X_\alpha$, $\pi_\alpha(x, \xi) = x$. A proof that Φ is smooth is essentially identical with the above proof that Ψ is smooth, and we omit it.

We define a vector space structure on F^x by pulling back the addition and scalar multiplication via $(Q_\alpha^x)^{-1} : F^x \rightarrow \mathbb{C}^n$. That is,

$$Q_\alpha^x(\xi) + cQ_\alpha^x(\eta) = Q_\alpha^x(\xi + c\eta), \quad \xi, \eta \in \mathbb{C}^n \ c \in \mathbb{C}.$$

Let us show that this does not depend on the choice of $x' \in q^{-1}(\{x\})$. Suppose that $F^x = F^{x'}$ for some $\beta \in A$ and $x' \in X_\beta$, and let $\xi', \eta' \in \mathbb{C}^n$. We choose $\xi = U_{\beta\alpha}(x')^{-1}\xi'$ and $\eta = U_{\beta\alpha}(x')^{-1}\eta'$. Then it holds that $Q_\beta^{x'}(\xi') = Q_\alpha^x(\xi)$, $Q_\beta^{x'}(\eta') = Q_\alpha^x(\eta)$ and $Q_\beta^{x'}(\xi' + c\eta') = Q_\alpha^x(\xi + c\eta)$ for all $c \in \mathbb{C}$.

Next let us construct local trivializations for F . We define

$$\rho : \tilde{\mathcal{X}} \times \mathbb{C}^n \rightarrow \mathcal{X} \times \mathbb{C}^n$$

by $\rho = q \otimes id$, that is, $\rho(x, \xi) = (q(x), \xi)$, and set $\rho_\alpha = \rho \circ Q_\alpha^{-1}$. Then $\rho_\alpha : F_\alpha \rightarrow \mathcal{U}_\alpha \times \mathbb{C}^n$ is a smooth bijection since $(q_\alpha^{-1} \otimes id) \circ \rho_\alpha \circ Q_\alpha$ is the identity on $X_\alpha \times \mathbb{C}^n$. Moreover, $\pi_F \circ \rho_\alpha^{-1}$ is the identity on \mathcal{U}_α , and, for $x \in X_\alpha$, the map $\xi \mapsto \rho_\alpha^{-1}(q(x), \xi)$ is Q_α^x . Thus, the maps ρ_α^{-1} , $\alpha \in A$, give local trivializations for F , and $\pi_F : F \rightarrow \mathcal{X}$ is a smooth vector bundle.

Let us show that Φ is a vector bundle homomorphism. We recall that $q_\alpha = \Psi \circ \tilde{\Psi}_\alpha^{-1}$, $q_\alpha^{-1} \circ \pi_F \circ Q_\alpha = \pi_\alpha$ and $Q_\alpha = \Phi_\alpha \circ \tilde{\Phi}_\alpha^{-1}$, where π_α is the projection on right in (44). Thus, we have

$$(51) \quad q_\alpha^{-1} \circ \pi_F \circ \Phi \circ \tilde{\Phi}_\alpha^{-1} = q_\alpha^{-1} \circ \pi_F \circ Q_\alpha = \pi_\alpha,$$

and, as the diagram (44) commutes, we have also

$$(52) \quad q_\alpha^{-1} \circ \Psi \circ \pi_E \circ \tilde{\Phi}_\alpha^{-1} = q_\alpha^{-1} \circ \Psi \circ \tilde{\Psi}_\alpha^{-1} \circ \pi_\alpha = \pi_\alpha.$$

Thus, $\pi_F \circ \Phi = \Psi \circ \pi_E$. Let $\alpha \in A$, $z \in B_\alpha$. Then Φ is linear from the fiber $\pi_E^{-1}(\{z\})$ to the fiber $\pi_F^{-1}(\{\Psi(z)\})$, since $(Q_\alpha^x)^{-1}$, $\Theta(\xi)^x = \tilde{\Phi}_\alpha^{-1}(x, \xi)$ and $(Q_\alpha^x)^{-1} \circ \Phi \circ \Theta = id$ are linear where $x = \tilde{\Psi}_\alpha(z)$ and the last equation follows from (51) and (52). Hence, Φ is a vector bundle homomorphism. As it is bijective, it is a vector bundle isomorphism.

q.e.d.

The connections ∇_α , potentials V_α and the Hermitian structures can be glued together:

$$(G8) \text{ On each } \pi_F^{-1}(\mathcal{U}_\alpha), (Q_\alpha^{-1})^*\nabla_\alpha = (\Phi^{-1})^*\nabla, (Q_\alpha^{-1})^*V_\alpha = (\Phi^{-1})^*V \text{ and } (Q_\alpha^{-1})^*\langle \cdot, \cdot \rangle_{\mathbb{C}^n} = (\Phi^{-1})^*\langle \cdot, \cdot \rangle_E.$$

A proof is essentially identical with the proof of (G4) and we omit it.

To summarize, we have shown that the following diagram

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & F \\ \downarrow & & \downarrow \\ M^{\text{int}} & \xrightarrow{\Psi} & \mathcal{X} \end{array}$$

gives an isomorphism of the structure (g, E, ∇, V) on M^{int} when \mathcal{X} is equipped with the metric tensor given by the gluing (G4) and F is equipped with the connection, the potential and the Hermitian structure given by the gluing (G8). This concludes the proof of Theorem 4.4.

Let us show that Φ extends to the accessible part \mathcal{S} of the boundary. If $\alpha \in A_\Gamma$, $\Gamma \in \mathcal{G}_\mathcal{S}$, corresponds to an iteration that is terminated immediately after the initial step, then we can use $B_\alpha = C_0$, where C_0 is of the form (43) and $\tilde{\Phi}_\alpha = \tilde{\Phi}_\Gamma|_{C_0}$. Thus, $Q_\alpha^{-1} \circ \Phi|_{B_\alpha} = \tilde{\Phi}_\Gamma|_{C_0}$ extends to $C_0 \cup B_\partial(y_0, \epsilon_0)$ and

$$(53) \quad Q_\alpha^{-1} \circ \Phi = \phi_\Gamma, \quad \text{on } B_\partial(y_0, \epsilon_0).$$

4.3. Extension to the inaccessible part of boundary. We will give a non-constructive proof that the structure (g, E, ∇, V) is determined up to the boundary, and this will conclude the proof of Theorem 1.1. To this end, let $(M_i, g_i, E_i, \nabla_i, V_i)$, $i = 1, 2$, be two structures as in Theorem 1.1. Let $\mathcal{S}_i \subset \partial M_i$ be open and nonempty, and suppose that there is an isomorphism between the induced Hermitian vector bundles on \mathcal{S}_i , $i = 1, 2$,

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi} & E_2 \\ \downarrow & & \downarrow \\ \mathcal{S}_1 & \xrightarrow{\psi} & \mathcal{S}_2 \end{array}$$

Note that we do not assume *a priori* that ψ is an isometry.

Let us choose an open cover $\mathcal{G}_{\mathcal{S}_1}$ of \mathcal{S}_1 as in the proof of Theorem 4.4. Then for each $\Gamma_1 \in \mathcal{G}_{\mathcal{S}_1}$ there is a unitary trivialization

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi_{\Gamma_1}} & Y_{\Gamma_1} \times \mathbb{C}^n \\ \downarrow & & \downarrow \\ \Gamma_1 & \xrightarrow{\psi_{\Gamma_1}} & Y_{\Gamma_1} \end{array}$$

We define $\Gamma_2 = \psi(\Gamma_1)$ and $\phi_{\Gamma_2} = \phi_{\Gamma_1} \circ \phi^{-1}$. Then $\phi_{\Gamma_2} : E_2|_{\Gamma_2} \rightarrow Y_{\Gamma_1} \times \mathbb{C}^n$ is a unitary trivialization, and, if ϕ intertwines the maps $\Lambda_{S_1}^{2T}$ and $\Lambda_{S_2}^{2T}$, then their representations on the respective trivializations coincide.

Theorem 4.4 implies that there is a Hermitian vector bundle $F \rightarrow \mathcal{X}$, that is equipped with a Hermitian connection $\tilde{\nabla}$ and a potential \tilde{V} , and whose base manifold \mathcal{X} is equipped with a Riemannian metric \tilde{g} , such that, for both $i = 1, 2$, there is a Hermitian vector bundle isomorphism $\Phi_i : E_i|_{M_i^{\text{int}}} \rightarrow F$, covering an isometry $\Psi_i : M_i^{\text{int}} \rightarrow \mathcal{X}$, such that $\nabla_i = \Phi_i^* \tilde{\nabla}$ and $V_i = \Phi_i^* \tilde{V}$. Hence, $\Phi_2^{-1} \circ \Phi_1$ gives an isomorphism between the structures $(g_i, E_i, \nabla_i, V_i)$ on M_i^{int} , $i = 1, 2$.

It follows from [31] that $\Psi = \Psi_2^{-1} \circ \Psi_1$ extends smoothly to the boundary ∂M_1 and (M_i, g_i) , $i = 1, 2$, are isometric via the extended Ψ . By considering the pullback bundle $\Psi^* E_2$, we can assume without loss of generality that $M_1 = M_2$. Thus, the following proposition implies that also the bundle isomorphism $\Phi = \Phi_2^{-1} \circ \Phi_1$ extends smoothly to the boundary.

Proposition 4.7. *Let $E_i \rightarrow M$, $i = 1, 2$, be two Hermitian vector bundles over a smooth manifold with boundary ∂M , and let ∇_i be a Hermitian connection on E_i , $i = 1, 2$. Suppose that there exists a Hermitian vector bundle isomorphism Φ between $E_1|_{M^{\text{int}}}$ and $E_2|_{M^{\text{int}}}$ such that it covers the identity and that $\Phi^* \nabla_2 = \nabla_1$ on M^{int} . Then Φ extends smoothly to ∂E_1 and the bundles and connections are isomorphic on M via the extended Φ .*

Proof. Fix a point $x \in \partial M$ and introduce coordinates

$$(x^1, \dots, x^m) \in W := [0, \varepsilon) \times (-\varepsilon, \varepsilon)^{m-1},$$

around x such that the boundary of M is given by $x^1 = 0$. Without loss of generality we may assume that the bundles E_1 and E_2 are trivial over these coordinates and that $\nabla_1 = d + A$, $\nabla_2 = d + B$. The bundle isomorphism Φ can be represented by a smooth $U(n)$ -valued function $u(x^1, \dots, x^m)$ defined for $x^1 > 0$ and such that

$$B = u^{-1} du + u^{-1} Au.$$

Consider the smooth map $u_A : W \rightarrow U(n)$ uniquely defined by solving the following parallel transport equation along the curves $x^1 \mapsto (x^1, \dots, x^m)$:

$$\begin{aligned} \frac{du_A}{dx^1} + A_{(x^1, \dots, x^m)}(\partial_{x^1})u_A &= 0, \\ u_A(0, x^2, \dots, x^m) &= Id. \end{aligned}$$

Consider a similar map $u_B : W \rightarrow U(n)$ associated to B . These two maps are convenient because, if we set

$$\tilde{A} = u_A^{-1} du_A + u_A^{-1} Au_A, \quad \tilde{B} = u_B^{-1} du_B + u_B^{-1} Bu_B,$$

then $\tilde{A}(\partial_{x^1}) = \tilde{B}(\partial_{x^1}) = 0$. For $x^1 > 0$ define $v = u_A^{-1}uu_B$. Then, a simple calculation shows that

$$\tilde{B} = v^{-1}dv + v^{-1}\tilde{A}v, \quad x^1 > 0.$$

This implies $dv(\partial_{x^1}) = 0$ and the map v is independent of x^1 . Hence, v smoothly extends to $x^1 = 0$ and, since $u = u_Avu_B^{-1}$, u is also smooth up to the boundary $x_1 = 0$. q.e.d.

In order to finish the proof of Theorem 1.1 we still need to show that $\Phi|_{S_1} = \phi$. Using the coordinate systems Q_α^{-1} on F corresponding to choices α as in (53), we see that $\Phi_i = \phi_{\Gamma_i}$ on Γ_i . Thus,

$$\Phi = \Phi_2^{-1} \circ \Phi_1 = \phi_{\Gamma_2}^{-1} \circ \phi_{\Gamma_1} = \phi,$$

on each $\Gamma_i \in \mathcal{G}_{S_1}$. This concludes the proof of Theorem 1.1.

5. Calderón problem for connections on a cylinder

The proof of Corollary 1.3 is based on a simple relation between the Dirichlet-to-Neumann map $\Lambda(\lambda)$ of the operator $-\partial_t^2 + P_0 - \lambda$ and that of the transversal operator P_0 defined analogously to $\Lambda(\lambda)$. That is, if $\lambda \in \mathbb{C} \setminus [\lambda_1, \infty)$ then we define

$$\Lambda_0(\lambda)h = (\nabla_0)_\nu u|_{\partial M_0}, \quad h \in C^\infty(\partial M_0; E_0),$$

where u is the solution of the equation

$$(P_0 - \lambda)u = 0 \text{ in } C, \quad u|_{\partial C} = f.$$

We consider an L^2 -space with a weight in the Euclidean direction,

$$L_\delta^2(C; E) = \{f \in L_{loc}^2(C; E); (1 + t^2)^{\delta/2}f \in L^2(C; E)\}, \quad \delta \in \mathbb{R},$$

and define the corresponding Sobolev spaces H_δ^s analogously to [8, Section 5]. Now we can formulate a relation between $\Lambda(\lambda)$ and $\Lambda_0(\lambda)$.

Proposition 5.1. *Let $\lambda \in \mathbb{C} \setminus [\lambda_1, \infty)$ and $\delta \in \mathbb{R}$. Then $\Lambda(\lambda)$ extends as a bounded linear map $\Lambda(\lambda) : H_\delta^{3/2}(\partial C; E) \rightarrow H_\delta^{1/2}(\partial C; E)$. Moreover, if $k \in \mathbb{R}$, then*

$$\Lambda_0(\lambda - k^2)h = e^{-kit}\Lambda(\lambda)(e^{ikt}h).$$

Note that if $h \in H^{3/2}(\partial M_0; E_0)$, then $e^{ikt}h \in H_\delta^{3/2}(\partial C; E)$ for any $\delta < -1/2$.

Proof. The proof that $\Lambda(\lambda)$ extends as claimed is analogous to the scalar case [8, Proposition 5.1] and we omit it. Let $h \in H^{3/2}(\partial M_0; E_0)$ and let $v_h \in H^2(M_0; E_0)$ solve

$$(P_0 - (\lambda - k^2))v_h = 0 \text{ in } M_0, \quad v_h|_{\partial M_0} = h.$$

Since $\lambda \notin [\lambda_1, \infty)$, the number $\lambda - k^2$ is not a Dirichlet eigenvalue of P_0 and there is a unique solution v_h . Set $f(t, x) = e^{ikt}h(x)$ and

$u(t, x) = e^{ikt}v_h(x)$. The function u is in $H_\delta^2(C; E)$ for any $\delta < -1/2$, and solves

$$(-\partial_t^2 + P_0 - \lambda)u = 0 \text{ in } C, \quad u|_{\partial C} = f.$$

Note that $-\partial_t^2 + P_0 = \nabla^* \nabla$, where $\nabla = \pi^* \nabla_0$ and $\pi : C \rightarrow M_0$ is the canonical projection. It follows that

$$\Lambda(\lambda)f = \nabla_\nu u|_{\partial C} = e^{ikt}(\nabla_0)_\nu v_h|_{\partial M_0} = e^{ikt}\Lambda_0(\lambda - k^2)h,$$

and the proposition is proved. q.e.d.

Proof of Corollary 1.3. Using that $C_0^\infty(\partial C; E)$ is dense in $H_\delta^{3/2}(\partial C; E)$ for all δ together with Proposition 5.1, we can determine the map

$$\Lambda_0(\lambda - k^2) : H^{3/2}(\partial M_0; E_0) \rightarrow H^{1/2}(\partial M_0; E_0),$$

for all $k \in \mathbb{R}$. Since $\mu \mapsto \Lambda_0(\mu)$ is a meromorphic map whose poles are contained in $\{\lambda_1, \lambda_2, \dots\}$, see, e.g., [23, Lemma 4.5], we can recover $\Lambda_0(\mu)$ for all $\mu \in \mathbb{C}$. This is equivalent to knowing the Dirichlet-to-Neumann map $\Lambda_{\partial M_0}^T$ for the wave operator $\partial_t^2 + P_0$ for any $T > 0$ [23, Chapter 4]. Thus, Theorem 1.1 implies that we can recover the structure $(M_0, g_0, E_0, \nabla_0)$ as claimed. q.e.d.

References

- [1] M. I. Belishev. An approach to multidimensional inverse problems for the wave equation. *Dokl. Akad. Nauk SSSR*, 297(3):524–527, 1987, MR924687, Zbl 0661.35084.
- [2] M. I. Belishev and Y. V. Kurylev. To the reconstruction of a Riemannian manifold via its spectral data (BC-method). *Comm. Partial Differential Equations*, 17(5–6):767–804, 1992, MR1177292, Zbl 0812.58094.
- [3] N. Berline, E. Getzler, and M. Vergne. *Heat kernels and Dirac operators*, volume 298 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1992, MR1215720, Zbl 1037.58015.
- [4] A. S. Blagoveščenskii. The local method of solution of the nonstationary inverse problem for an inhomogeneous string. *Trudy Mat. Inst. Steklov.*, 115:28–38, 1971, MR0307558.
- [5] R. Bosi, Y. Kurylev, and M. Lassas. Reconstruction and stability in Gel'fand's inverse interior spectral problem. *Preprint arXiv:1702.07937*, 2017.
- [6] I. Chavel. *Riemannian geometry*, volume 98 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2006, MR2229062, Zbl 1099.53001.
- [7] M. F. Dahl, A. Kirpichnikova, and M. Lassas. Focusing waves in unknown media by modified time reversal iteration. *SIAM J. Control Optim.*, 48(2):839–858, 2009, MR2486096, Zbl 1186.35238.
- [8] D. Dos Santos Ferreira, Y. Kurylev, M. Lassas, and M. Salo. The Calderón problem in transversally anisotropic geometries. *J. Eur. Math. Soc. (JEMS)*, 18(11):2579–2626, 2016, MR3562352, Zbl 1359.35224.

- [9] M. Eller, V. Isakov, G. Nakamura, and D. Tataru. Uniqueness and stability in the Cauchy problem for Maxwell and elasticity systems. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. XIV (Paris, 1997/1998)*, volume 31 of *Stud. Math. Appl.*, pages 329–349. North-Holland, Amsterdam, 2002, MR1936000, Zbl 1038.35159.
- [10] M. Eller and D. Toundykov. A global Holmgren theorem for multidimensional hyperbolic partial differential equations. *Appl. Anal.*, 91(1):69–90, 2012, MR2864692, Zbl 1235.35058.
- [11] G. Eskin. On non-abelian Radon transform. *Russ. J. Math. Phys.*, 11(4):391–408, 2004, MR2179889, Zbl 1186.43009.
- [12] G. Eskin. Inverse problems for Schrödinger equations with Yang-Mills potentials in domains with obstacles and the Aharonov-Bohm effect. *Institute of Physics Conference Series*, 12:23–32, 2005.
- [13] G. Eskin. Inverse problems for hyperbolic equations. In *Phase space analysis of partial differential equations*, volume 69 of *Progr. Nonlinear Differential Equations Appl.*, pages 107–116. Birkhäuser Boston, Boston, MA, 2006, MR2263209, Zbl 1125.35111.
- [14] G. Eskin. Inverse hyperbolic problems with time-dependent coefficients. *Comm. Partial Differential Equations*, 32(10–12):1737–1758, 2007, MR2372486, Zbl 1153.35081.
- [15] G. Eskin. Inverse problems for the Schrödinger equations with time-dependent electromagnetic potentials and the Aharonov-Bohm effect. *J. Math. Phys.*, 49(2):022105, 18, 2008, MR2392842, Zbl 1153.81355.
- [16] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998, MR162584, Zbl 1194.35001.
- [17] D. Finch and G. Uhlmann. The x-ray transform for a non-abelian connection in two dimensions. *Inverse Problems*, 17(4):695–701, 2001, MR1861511, Zbl 1004.53055.
- [18] C. Guillarmou, G. P. Paternain, M. Salo, and G. Uhlmann. The X-ray transform for connections in negative curvature. *Comm. Math. Phys.*, 343(1):83–127, 2016, MR3475662, Zbl 1343.53078.
- [19] A. Hassell and T. Tao. Upper and lower bounds for normal derivatives of Dirichlet eigenfunctions. *Math. Res. Lett.*, 9(2–3):289–305, 2002, MR1909646, Zbl 1014.58015.
- [20] M. W. Hirsch. *Differential topology*, volume 33 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994, MR1336822, Zbl 0356.57001.
- [21] L. Hörmander. *The analysis of linear partial differential operators. III*, volume 274 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1985, MR781536, Zbl 0601.35001.
- [22] H. Isozaki and Y. Kurylev. *Introduction to spectral theory and inverse problem on asymptotically hyperbolic manifolds*, volume 32 of *MSJ Memoirs*. Mathematical Society of Japan, Tokyo, 2014, MR3222614, Zbl 06317422.
- [23] A. Katchalov, Y. Kurylev, and M. Lassas. *Inverse boundary spectral problems*, volume 123 of *Monographs and Surveys in Pure and Applied Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2001, MR1889089, Zbl 1037.35098.
- [24] Y. Kurylev. An inverse boundary problem for the Schrödinger operator with magnetic field. *J. Math. Phys.*, 36(6):2761–2776, 1995, MR1331288, Zbl 0845.58051.

- [25] Y. Kurylev and M. Lassas. Gelf'and inverse problem for a quadratic operator pencil. *J. Funct. Anal.*, 176(2):247–263, 2000, MR1784415, Zbl 0968.47005.
- [26] Y. Kurylev and M. Lassas. Inverse problems and index formulae for Dirac operators. *Adv. Math.*, 221(1):170–216, 2009, MR2509324, Zbl 1163.35041.
- [27] I. Lasiecka, J.-L. Lions, and R. Triggiani. Nonhomogeneous boundary value problems for second order hyperbolic operators. *J. Math. Pures Appl. (9)*, 65(2):149–192, 1986, MR867669, Zbl 0631.35051.
- [28] M. Lassas and L. Oksanen. Inverse problem for the Riemannian wave equation with Dirichlet data and Neumann data on disjoint sets. *Duke Math. J.*, 163(6):1071–1103, 2014, MR3192525, Zbl 06298159.
- [29] L. Oksanen. Solving an inverse problem for the wave equation by using a minimization algorithm and time-reversed measurements. *Inverse Probl. Imaging*, 5(3):731–744, 2011, MR2825736, Zbl 1230.35145.
- [30] B. O'Neill. *Semi-Riemannian geometry*, volume 103 of *Pure and Applied Mathematics*. Academic Press Inc., New York, 1983, MR719023, Zbl 0531.53051.
- [31] R. S. Palais. On the differentiability of isometries. *Proc. Amer. Math. Soc.*, 8:805–807, 1957, MR0088000, Zbl 0084.37405.
- [32] G. P. Paternain, M. Salo, and G. Uhlmann. The attenuated ray transform for connections and Higgs fields. *Geom. Funct. Anal.*, 22(5):1460–1489, 2012, MR2989440, Zbl 1256.53021.
- [33] D. Tataru. Unique continuation for solutions to PDE's; between Hörmander's theorem and Holmgren's theorem. *Comm. Partial Differential Equations*, 20(5–6):855–884, 1995, MR1326909, Zbl 0846.35021.
- [34] M. E. Taylor. *Partial differential equations. I: Basic theory*, volume 115 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1996, MR1395148, Zbl 0869.35002.

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