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ON THE LOCAL EXTENSION OF THE FUTURE NULL INFINITY

JUNBIN LI & XI-PING ZHU

Abstract

We consider a characteristic problem of the vacuum Einstein equations with part of the initial data given on a future asymptotically flat null cone, and show that the solution exists uniformly around the null cone for general such initial data. Therefore, the solution contains a piece of the future null infinity. The initial data are not required to be small and the decaying condition is consistent with those in the works of [8] and [11].

1. Introduction

1.1. Introduction. As one of the major problem in mathematical relativity, the weak cosmic censorship states that the maximal developments of generic asymptotically flat initial data possess a complete future null infinity. There are some mathematical works towards the resolution of the conjecture.

Christodoulou and Klainerman [8] proved the nonlinear stability of Minkowski space-time in vacuum. Recall that a Cauchy initial data $(\Sigma, \bar{g}, \bar{k})$ is called strongly asymptotically flat (introduced in [8]) if the following holds near infinity:

$$\bar{g}_{ij} = \left(1 + \frac{2M}{r}\right)\delta_{ij} + o_4(r^{-3/2}),\ \bar{k}_{ij} = o_3(r^{-5/2}),$$

where $f = o_m(r^s)$ means $\partial^k f = o(r^{s-k})$ for all $k \leq m$. It is proved in [8] that for any strongly asymptotically flat Cauchy initial data sets which are sufficiently close to the constant time slices in Minkowski space-time, the maximal future developments are future geodesically complete and are approaching the Minkowski space-time in a suitable sense. In particular, the weak cosmic censorship holds in this class. If the strongly asymptotically flat data is not close to Minkowski, then the global existence type results are not expected. Nevertheless, Klainerman and Nicolò proved in [11] the global existence in the domain of dependence outside a compact set in the initial data set using a different foliation

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of the space-time. The space-time is constructed in a neighborhood of space-like infinity, and the information about the interior region of the initial data is not used.

In both proofs of [**8**] and [**11**], a key ingredient is the construction of an optical function u, which increases towards future and is normalized at infinity. The level sets C_u of u are outgoing null cones extending to infinity. The function u is called the retarded time. The space-time is then foliated by the null cones C_u's. Along these null cones, the geometry tends to being Minkowskian in a suitable rate. For example, the curvature components[1] , in the space-time behave like:

$$\begin{aligned}\alpha, \beta = o(r^{-7/2}),\ \rho = O(r^{-3}),\ \sigma = O(r^{-3}\tau_-^{-1/2}),\\ \underline{\beta} = O(r^{-2}\tau_-^{-3/2}),\ \underline{\alpha} = O(r^{-1}\tau_-^{-5/2})\end{aligned} \tag{1.1}$$

along the null cones. Here r denotes the area radius of the intersection of each C_u and the maximal time slice, and $\tau_-^2 = 1 + u^2$. In the work [**8**], u varies in $(-\infty, +\infty)$, and $u = -\infty$ denotes the space-like infinity, $u = +\infty$ denotes the time-like infinity. In the work [**11**], u varies in $(-\infty, -\lambda_0]$, $u = -\infty$ denotes the space-like infinity, and $\lambda_0 > 0$ is a sufficiently large number. We remark that since λ_0 has been chosen sufficiently large, the theorem of [**11**] is essentially a small data result. In particular, the work [**11**] implies that for any strongly asymptotically flat Cauchy data $(\Sigma, \bar{g}, \bar{k})$, there exists a region Ω_{λ_0} (for example, we can choose Ω_{λ_0} to be the region bounded by the intersection of $C_{-\lambda_0}$ and Σ) with compact closure such that the boundary of its causal future $\partial J^+(\Omega_{\lambda_0})$ of Ω_{λ_0} in the maximal development (M, g) consists of future complete null geodesics. This property inspired Christodoulou to introduce the concept of "possessing a complete future null infinity" in [**4**]. Let (M, g) be the maximal future development of an asymptotically flat initial data $(\Sigma, \bar{g}, \bar{k})$ and Ω_{λ_0} be a region in Σ mentioned above. Then we say that (M, g) possesses a complete future null infinity if for any large $A > 0$, there exists some region Ω_0 in Σ such that Ω_0 contains Ω_{λ_0} and the boundary of its domain of dependence $\partial D^+(\Omega_0)$ satisfies the following property: the affine length of each of the future null geodesic generators of $\partial D^+(\Omega_0)$ is not less than A measured from $\partial D^+(\Omega_0) \bigcap \partial J^+(\Omega_{\lambda_0})$. Based on the discussions on the retarded time function above, one can also say that a space-time possessing a complete future null infinity means the space-time can be foliated by a family of future null cones C_u extending to infinity, with u being a retarded time function and varying in $[-\lambda_0, +\infty)$, and the weak cosmic censorship is equivalent to the global existence in retarded time.

[1]See the next section for the definitions.

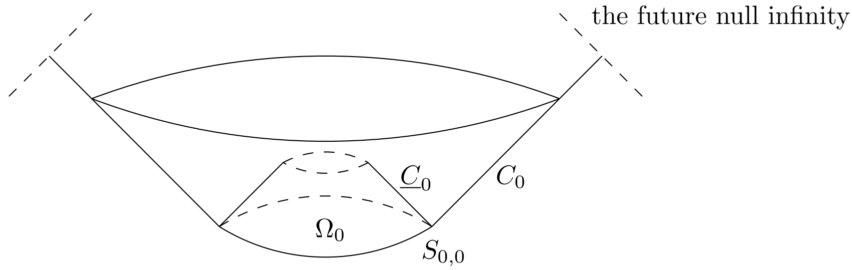

Figure 1. The characteristic initial value problem.

Therefore, it is natural to consider the characteristic-Cauchy mixed initial value problem, that the initial data are given on a three-dimensional disk Ω_0, and an asymptotically flat complete outgoing null cone C_0 rooted at $\partial\Omega_0$, on which the initial data is NOT assumed to be small. We remark that different from Ω_{λ_0} mentioned in the above paragraph, Ω_0 here is not necessarily a large enough region. Then the main goal of this paper is to extend the future null infinity towards the future, i.e., solve the solution to the vacuum Einstein equations around the initial asymptotically flat null cone in a uniform way. This is the local existence in retarded time.

A further remark should be made about the choice of the decay rate of the asymptotically flat initial null cone C_0. We work in the spacetimes arising from strongly asymptotically flat Cauchy data, and the corresponding null cones C_u behave like (1.1) at least for u sufficiently negative. It is reasonable to expect that if the weak cosmic censorship holds, for arbitrary u, the geometry on every single null cone C_u decays in the same manner, therefore, we can assume in this paper that the geometry on C_0 behaves like (1.1), by dropping the weight τ_-, that is,

$$\alpha, \beta = o(r^{-7/2}),\ \rho, \sigma = O(r^{-3}),\ \underline{\beta} = O(r^{-2}),\ \underline{\alpha} = O(r^{-1}). \tag{1.2}$$

We remark that the weight τ_- describes the decay along the future null infinity. We will omit the weight τ_- since we will only study a local piece of the future null infinity. We also remark that together with the condition (1.2), a list of the decay rates of the components of the null connection coefficients[2] on C_0 is also needed.

Note that we can solve the vacuum Einstein equations over Ω_0 first, then the boundary of the domain of dependence of Ω_0 can be served as a new incoming initial null hypersurface, we are then facing a characteristic problem with initial data given on two intersecting null cones[3] , as depicted in Figure 1. Now it is a good position to state a rough form of the main theorem:

[2]See the next section for the definitions.

[3]See Section 3.2 for a description on the characteristic initial data.

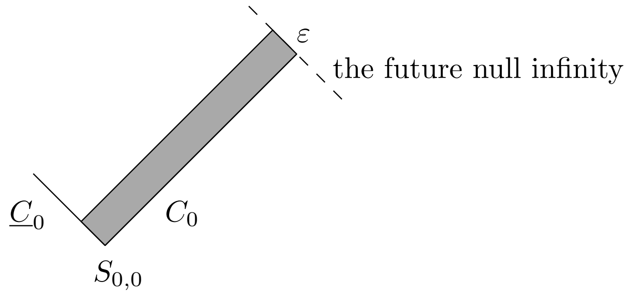

Figure 2. The region of existence in Theorem 1.

Theorem 1. *Consider the characteristic problem of the vacuum Einstein equations with the initial data given on an outgoing null cone C_0 and an incoming null cone $\underline{C}_0$. These two null cones intersect at a two-sphere, which we denote by $S_{0,0}$. Suppose that the null geodesic generators of C_0 are future complete and C_0 is foliated by the affine parameter s, whose level sets are spherical sections with area radius r. Suppose also that C_0 is asymptotically flat in the sense that, the curvature components on C_0 decay like*

$$\alpha, \beta = o(r^{-7/2}),\ \rho, \sigma = O(r^{-3}),\ \underline{\beta} = O(r^{-2}),\ \underline{\alpha} = O(r^{-1}),$$

up to the second order derivatives[4] *, and the connection coefficients decay like*

$$\widehat{\chi} = O(r^{-2}),\ \zeta = O(r^{-2}),\ \widehat{\underline{\chi}} = O(r^{-1}),$$
$$\mathrm{tr}\chi - \overline{\mathrm{tr}\chi} = O(r^{-2}),\ \mathrm{tr}\underline{\chi} = O(r^{-1}),$$

up to the third order derivatives. Then there exists a retarded time function u and an $\varepsilon > 0$ such that the solution of the vacuum Einstein equations exists in a double null foliation[5] *for $0 \leq \underline{u} < +\infty, 0 \leq u \leq \varepsilon$ where $\underline{u} = s$ on C_0. The choice of ε depends on appropriate weighted Sobolev norms of the connection coefficients and the curvature components. Therefore, the maximal future development contains a piece of the future null infinity.*

As mentioned above, the retarded time function u is understood as an optical function normalized at infinity, whose level sets are the future complete null cones. The decay rates listed above are the same as in [**8**] and [**11**] but NO smallness is assumed. The region of existence in the theorem is indicated by the dark grey region in Figure 2. The precise form of the theorem is in Theorem 2 in Section 3.1. Combined with the local existence over Ω_0, one direct corollary is:

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[4] The small o means that the decay of α, β is in $L^2(C_0)$ and the big O means that the decay of the other components is in L^∞_s relative to the affine parameter s.

[5] See the next section for the definition.

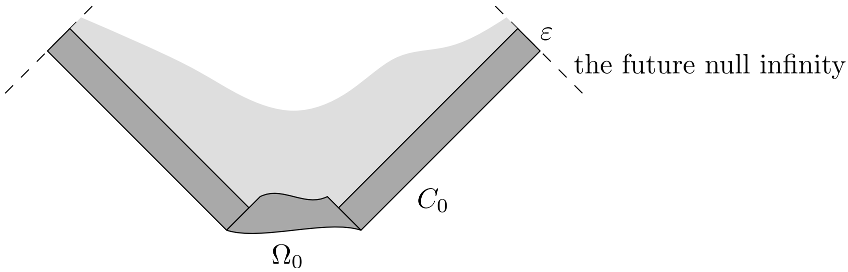

Figure 3. The region of existence in the Corollary.

Corollary. *Consider the characteristic-Cauchy mixed initial value problem of the vacuum Einstein equations with the initial data given on a three-dimensional disk Ω_0 and an outgoing null cone C_0 with future complete null geodesic generators. Ω_0 and C_0 intersect at the boundary $\partial\Omega_0$ of Ω_0 which is a two-sphere. Suppose that the initial connection coefficients and curvature components on C_0 decay as described in Theorem 1. Then the solution of the Einstein equations exists in a future neighborhood of the initial hypersurface $\Omega_0 \bigcup C_0$ in a uniform way, in the sense that there exists an $\varepsilon > 0$ such that the solution contains a part which can be foliated by a double null foliation for $0 \leq \underline{u} < +\infty$, $0 \leq u \leq \varepsilon$ where $\underline{u}$ restricted on C_0 coincides with its affine parameter and u is a retarded time function. Therefore, the maximal future development contains a piece of the future null infinity.*

The conclusion of this corollary is depicted in Figure 3. The dark grey region indicates the region of existence. The two rectangle regions are the part of the maximal development which can be foliated by the double null foliation for $0 \leq \underline{u} < +\infty$, $0 \leq u \leq \varepsilon$. The very light gray region indicates the maximal development.

The characteristic problem has one major advantage over the Cauchy initial value problem since the constraints on the data are ODEs rather than a system of PDEs. For example, it has been studied in the resolution of the weak cosmic censorship in the context of spherically symmetric Einstein-scalar equations. This was resolved by Christodoulou in a series of paper, see [**5**] and the references therein. In these works, the initial data consists a function $\alpha_0 = \frac{\partial(r\phi)}{\partial s}\big|_{C_0}$ where s is the affine parameter of the null generators of C_0, where C_0 is assumed to be a complete null cone from a point towards to the future null infinity. In addition, in the case of gravitational collapse, the global existence in retarded time for the asymptotically flat characteristic problem leads naturally to the event horizon, serving as the final piece of the solution. We remark that Christodoulou considered the characteristic problem with initial data given on a single null cone issued from a point, but not the characteristic-Cauchy mixed or characteristic problem with initial data given on two intersecting null cones. However, the ideas of the proof behind the different approaches are the same.

On the technical level, this work should be compared with the local existence results of the characteristic problem for general initial data, that were considered in [**20**] and [**13**]. Suppose that the initial data are given on two null cones C_0 and $\underline{C}_0$ that intersect at a two surface $S_{0,0}$. Rendall [**20**] showed[6] that for smooth initial data given on C_0 and $\underline{C}_0$, the solution of the vacuum Einstein equations exists in a neighborhood of their intersection $S_{0,0}$ to the future of two null cones. Luk [**13**] extended this result. He showed that if the initial data are given on C_0 for $0 \le \underline{u} \le I_1$ and $\underline{C}_0$ for $0 \le u \le I_2$, the solution, in fact, exists in a full neighborhood of two initial null cones to the future, i.e., $(\underline{u}, u) \in ([0, I_1] \times [0, \varepsilon]) \bigcup ([0, \varepsilon] \times [0, I_2])$, where $\varepsilon > 0$ is a small parameter depending on the size of the initial data. One can directly apply this result to the case when C_0 is complete. The solution then exists on a neighborhood of C_0 but not uniformly. Caciotta and Nicolò [**2**] [**3**] considered the initial outgoing null cones to be complete and asymptotically flat and data is assumed to be small additionally. In this case, the solution exists globally to the whole future of the two initial null cones, say $(\underline{u}, u) \in [0, +\infty) \times [0, I]$ where the constant I is the affine length of the null generators of $\underline{C}_0$.

There are some other works using characteristic setting to capture interesting mathematical and physical phenomena, by specifying some carefully designed initial data, such as in the recent breakthrough on the formation of black holes [**7**], [**12**], and on the impulsive gravitational waves in [**14**], [**15**].

Before concluding this subsection, we should mention here two closely related works. After the first version of this paper was finished, the authors were able to prove in [**17**], which was, however, published earlier, another version of Theorem 1 where only weaker decay conditions consistent with those in the work [**1**] by Bieri are assumed[7] . Such decay conditions are enough to guarantee the existence of Bondi mass by taking the limit of the Hawking mass of suitable sections of the outgoing null cones. Another work we should mention is the work [**9**] by Cabet–Chruściel–Wafo, whose first version was appeared at almost the same time as this paper. They proved (among many other things) that if C_0 is assumed to be smoothly conformally extendable across the boundary at future null infinity, then the solution contains a piece of a smooth future null infinity. The smoothness assumption allows them to use Friedrich's conformal Einstein equations and the problem at hand reduces to a finite

[6] In fact, Rendall considered the characteristic initial value problem of a general class of quasilinear wave equations including the Einstein equations.

[7] Different from the work [**1**], we used up to the third order derivatives of the curvature for simplicity.

problem. On the other hand, note that the decay condition (1.2) fails to satisfy the peeling properties, which is a consequence of the smoothness of the conformal compactification in order to define asymptotic flatness by Penrose. Christodoulou had explained in [**6**] (or see Dafermos [**10**]) the reason why the smoothness of the conformal compactification is too strong to include sufficiently many physically interesting Cauchy data.

1.2. Comments on the proof. The main tool of this paper is the energy estimates under a double null foliation. The technique of the energy estimates for the curvature was developed by Christodoulou and Klainerman in [**8**], using the Bel–Ronbinson tensor for the Weyl curvature of the space-time based on the contracted second Bianchi identities:

$$\nabla^\alpha \mathbf{R}_{\alpha\beta\gamma\delta} = 0.$$

Also, the Bianchi identities should be coupled with a group of the null structure equations which are used to estimate the connection coefficients in terms of the curvature. On the other hand, when solving the Einstein equations, one should first choose a suitable coordinate system or a foliation of the space-time in order to write down the estimates. The coordinate system or foliation should be chosen very carefully to capture some crucial properties of the solution. In the groundbreaking work [**8**], the space-time was foliated by two functions: a maximal time function t, whose level sets are maximal slices Σ_t representing the constant time slice, and an optical function u mentioned in the above subsection, whose level sets are future complete null cones C_u, which follows the gravitational waves radiating to infinity. The intersections of Σ_t and C_u are two spheres, and the equations will then be decomposed into ODEs along C_u and elliptic equations on the spheres and Σ_t. When considering a global problem, one should use some specific approximately Killing and conformal Killing vectorfields as multipliers and commutators in order to generate weighted energy estimates to capture the decay of different geometric quantities approaching the null infinity. This technique and procedure was also employed in [**11**]. Different from the work [**8**], the work [**11**] mainly employ the double null foliation, also introduced by Christodoulou, which means we foliate the space-time by two optical functions $\underline{u}$ and u, whose level sets are incoming null cones $\underline{C}_{\underline{u}}$ and outgoing null cones C_u and the intersections are two spheres. The equations in this case will be decomposed into ODEs along $\underline{C}_{\underline{u}}$, C_u, and elliptic equations on the spheres. The double null foliation is suitable for the problems concerning the null infinity, since the (future or past) null infinity can be viewed as the limit of the (incoming or outgoing) null cones. In the above two works, the outgoing null cones C_u, or the retarded function u, is not simply constructed by extending them from the initial hypersurface to infinity. We should construct u initiated from the future null infinity, requiring that u is the affine parameter at infinity. This is done by a limiting argument, and introducing a specific differential equation satisfied by u on the last slice. The corresponding foliation induced by u on the last slice is called the canonical foliation. In the current work, we show that in the case the geometric quantities are not assumed to be small, this method also works.

One another ingredient is that some special *reductive* structures in the Einstein equations are used to eliminate the high nonlinearity of the equations. In general, it means that for some semi-global problem, when no uniform smallness of the data is imposed, we can always write the Einstein equations down in some coordinate systems or foliations, and we will have some small parameter. Then the estimates can be divided in several steps, such that the highly coupled nonlinear terms can either be absorbed by the small parameter or can be controlled in the previous steps. The coupled nature is completely destroyed. This allows us to solve large data problem in a sufficient long but finite time and obtain interesting mathematical results related to physics. The gate to solving the large data problem was open in the work of the formation of trapped surface by Christodoulou [**7**]. In this work, even though the data prescribed is potentially very large for some components, it turns out that these components appear together with some other small components. The small parameter is still able to absorb the nonlinearity. After Christodoulou, Klainerman and Rodnianski extended this result in [**12**] by considering a larger class of the initial data. The current work should be compared with the semi-global existence results by Luk [**13**] which is mentioned in the above subsection, and by Luk and Rodnianski [**14**], [**15**]. In these cases, the data prescribed are bounded (in suitable norms) and the small parameter comes from the smallness of the existence region, which is the same in our current work.

The reductive structure is related to the null structure of Einstein equations, which corresponds to the null condition in nonlinear wave equations and ensures that when solving the equations up to infinity, the decay estimates are strong enough to make the space-time integral converges. The two structures are more or less the same thing in some cases. In the current work, they are considered simultaneously. The first example combining these two points is also given in Christodoulou's formation of trapped surface [**7**]. He solved the problem with initial data prescribed at the past null infinity. However, the notion of the canonical foliation need not to be introduced, because the data at infinity is given, but not to be solved.

A good application of Theorem 2 in this paper is also the work [**7**] of the formation of trapped surface. Based on the original setting in [**7**][8] , if we, in addition, assume the initial data given in $\underline{u} \in [0, \delta)$ is compactly

[8]The readers who are not familiar with the monograph [**7**] may first go to Section 3.2 for a simple introduction of the initial data.

supported and extend the initial data trivially (that is $\widehat{\chi} \equiv 0$), then we obtain the initial data given on a complete null cone C_{u_0}. With the incoming null cone $\underline{C}_\delta$ serving as the initial incoming cone, we have a new characteristic initial value problem with the initial data given on $C_{u_0} \bigcup \underline{C}_\delta$. We will prove, in Section 3.2, that the initial quantities on C_{u_0} for $\underline{u} \geq \delta$ satisfy the assumptions of Theorem 2. Consequently, we will know that the maximal future development has a piece of the future null infinity.

At last, we want to mention a technique of renormalized curvature estimates introduced in [**14**]. By defining two new quantities

$$\check{\rho} = \rho - \frac{1}{2}(\widehat{\chi}, \underline{\widehat{\chi}}), \quad \check{\sigma} = \sigma + \frac{1}{2}\widehat{\chi} \wedge \underline{\widehat{\chi}},$$

they can eliminate the component α in the null Bianchi equations $D\rho$, $D\sigma$, and then we can do energy estimate without knowing information about α. We find that this renormalization also works when considering an infinite problem. We will use this technique in proving Theorem 2 and the advantage is the decay of α is not needed. This may suggest that the decay of α is not quite relevant to the weak cosmic censorship.

1.3. Outline of the paper. The remainder of this paper is organized as follows. In Section 2, we will describe the double null foliation we will use in the proof. We also present, in this section, the definitions of the connection coefficients and curvature components, and the equations they satisfy. Also, some basic geometric lemmas are also given. In Section 3, we give the precise statement of out main theorem, and outline the main steps of the proof. We will also investigate the characteristic initial data in this section. The actual proof of the main theorem is then given in Section 4.

2. Preliminary

2.1. Basic geometric setup. We follow the geometric setup and notations in [**7**]. We use M to denote the underlying space-time (which will be the solution) and use g to denote the background 3+1 dimensional Lorentzian metric. We use ∇ to denote the Levi-Civita connection of the metric g.

Let $\underline{u}$ and u be two optical functions on M, that is

$$g(\nabla \underline{u}, \nabla \underline{u}) = g(\nabla u, \nabla u) = 0.$$

The space-time M is foliated by the level sets of $\underline{u}$ and u, respectively. Since the gradients of u and $\underline{u}$ are null, we call these two foliations together a double null foliation. We require the functions u and $\underline{u}$ increase towards the future. We use C_u to denote the outgoing null hypersurfaces which are the level sets of u and use $\underline{C}_{\underline{u}}$ to denote the incoming null hypersurfaces which are the level sets of $\underline{u}$. We denote the intersection $S_{\underline{u},u} = \underline{C}_{\underline{u}} \cap C_u$, which is a space-like two-sphere.

We define a positive function Ω by the formula $\Omega^{-2} = -2g(\nabla \underline{u}, \nabla u)$. We then define the normalized null pair (e_3, e_4) by $e_3 = -2\Omega\nabla\underline{u}$ and $e_4 = -2\Omega\nabla u$, and define one another null pair $\underline{L} = \Omega e_3$ and $L = \Omega e_4$. We remark that the flows generated by $\underline{L}$ and L preserve the double null foliation. On a given two sphere $S_{\underline{u},u}$ we choose a local orthonormal frame (e_1, e_2). We call (e_1, e_2, e_3, e_4) a *null frame*. As a convention, throughout the paper, we use capital Latin letters $A, B, C, \cdots$ to denote an index from 1 to 2, e.g., e_A denotes either e_1 or e_2.

We define ϕ to be a tangential tensorfield if ϕ is *a priori* a tensorfield defined on the space-time M and all the possible contractions of ϕ with either e_3 or e_4 are zeros. We use $D\phi$ and $\underline{D}\phi$ to denote the projection to $S_{\underline{u},u}$ of usual Lie derivatives $\mathcal{L}_L\phi$ and $\mathcal{L}_{\underline{L}}\phi$. The space-time metric g induces a Riemannian metric $\not{g}$ on $S_{\underline{u},u}$ and $\not{\epsilon}$ is the volume form of $\not{g}$ on $S_{\underline{u},u}$. We use $\not{d}$ and $\not{\nabla}$ to denote the exterior differential and covariant derivative (with respect to $\not{g}$) on $S_{\underline{u},u}$.

We recall the definitions of null connection coefficients. Roughly speaking, the following quantities are Christoffel symbols of ∇ according to the null frame (e_1, e_2, e_3, e_4):

$$\chi_{AB} = g(\nabla_A e_4, e_B), \quad \eta_A = -\frac{1}{2}g(\nabla_3 e_A, e_4), \quad \omega = \frac{1}{2}\Omega g(\nabla_4 e_3, e_4),$$

$$\underline{\chi}_{AB} = g(\nabla_A e_3, e_B), \quad \underline{\eta}_A = -\frac{1}{2}g(\nabla_4 e_A, e_3), \quad \underline{\omega} = \frac{1}{2}\Omega g(\nabla_3 e_4, e_3).$$

They are all tangential tensorfields. We also define $\chi' = \Omega^{-1}\chi$, $\underline{\chi}' = \Omega^{-1}\underline{\chi}$ and $\zeta = \frac{1}{2}(\eta - \underline{\eta})$. The trace of χ and $\underline{\chi}$ will play an important role in Einstein field equations and they are defined by $\mathrm{tr}\chi = \not{g}^{AB}\chi_{AB}$ and $\mathrm{tr}\underline{\chi} = \not{g}^{AB}\underline{\chi}_{AB}$. By definition, we can check directly the following identities $\not{d}\log\Omega = \frac{1}{2}(\eta + \underline{\eta})$, $D\log\Omega = \omega$, $\underline{D}\log\Omega = \underline{\omega}$.

We can also define the null components of the curvature tensor $\mathbf{R}$:

$$\alpha_{AB} = \mathbf{R}(e_A, e_4, e_B, e_4), \quad \beta_A = \frac{1}{2}\mathbf{R}(e_A, e_4, e_3, e_4),$$

$$\underline{\alpha}_{AB} = \mathbf{R}(e_A, e_3, e_B, e_3), \quad \underline{\beta}_A = \frac{1}{2}\mathbf{R}(e_A, e_3, e_3, e_4),$$

$$\rho = \frac{1}{4}\mathbf{R}(e_3, e_4, e_3, e_4), \quad \sigma = \frac{1}{4}\mathbf{R}(e_3, e_4, e_A, e_B)\not{\epsilon}^{AB}.$$

We then define several kinds of contraction of the tangential tensorfields, which are used in deriving the equations. For a symmetric tangential 2-tensorfield θ, we use $\widehat{\theta}$ and $\mathrm{tr}\theta$ to denote the trace-free part and trace of θ (with respect to $\not{g}$). If θ is trace-free, $\widehat{D}\theta$ and $\widehat{\underline{D}}\theta$ refer to the trace-free part of $D\theta$ and $\underline{D}\theta$. Let ξ be a tangential 1-form. We define some products and operators for later use. For the products, we define $(\theta_1, \theta_2) = \not{g}^{AC}\not{g}^{BD}(\theta_1)_{AB}(\theta_2)_{CD}$ and $(\xi_1, \xi_2) = \not{g}^{AB}(\xi_1)_A(\xi_2)_B$. This also leads to the following norms $|\theta|^2 = (\theta, \theta)$ and $|\xi|^2 = (\xi, \xi)$. We then define the contractions $(\theta \cdot \xi)_A = \theta_A{}^B\xi_B$, $(\theta_1 \cdot \theta_2)_{AB} = (\theta_1)_A{}^C(\theta_2)_{CB}$,

$\theta_1 \wedge \theta_2 = \not\epsilon^{AC}\not g^{BD}(\theta_1)_{AB}(\theta_2)_{CD}$ and $\xi_1 \widehat{\otimes} \xi_2 = \xi_1 \otimes \xi_2 + \xi_2 \otimes \xi_1 - (\xi_1, \xi_2)\not g$. The Hodge dual for ξ is defined by ${}^*\xi_A = \not\epsilon_A{}^C \xi_C$. For the operators, we define $\not{\mathrm{div}}\, \xi = \not\nabla^A \xi_A$, $\not{\mathrm{curl}}\, \xi_A = \not\epsilon^{AB} \not\nabla_A \xi_B$ and $(\not{\mathrm{div}}\, \theta)_A = \not\nabla^B \theta_{AB}$. We, finally, define a traceless operator $(\not\nabla \widehat{\otimes} \xi)_{AB} = (\not\nabla \xi)_{AB} + (\not\nabla \xi)_{BA} - \not{\mathrm{div}}\, \xi \not g_{AB}$.

2.2. Equations. The following is the first structure equations in the space-time written in a null frame (where K is the Gauss curvature of $S_{\underline{u},u}$):[9]

$$
\begin{aligned}
\widehat{D}\widehat{\chi}' = -\alpha, \quad D\mathrm{tr}\chi' &= -\frac{1}{2}\Omega^2(\mathrm{tr}\chi')^2 - \Omega^2|\widehat{\chi}'|^2, \\
\underline{\widehat{D}}\underline{\widehat{\chi}}' = -\underline{\alpha}, \quad \underline{D}\mathrm{tr}\underline{\chi}' &= -\frac{1}{2}\Omega^2(\mathrm{tr}\underline{\chi}')^2 - \Omega^2|\underline{\widehat{\chi}}'|^2, \\
D\eta &= \Omega(\chi \cdot \underline{\eta} - \beta), \\
\underline{D}\underline{\eta} &= \Omega(\underline{\chi} \cdot \eta + \underline{\beta}), \\
D\underline{\omega} &= \Omega^2(2(\eta, \underline{\eta}) - |\eta|^2 - \rho), \\
\underline{D}\omega &= \Omega^2(2(\eta, \underline{\eta}) - |\underline{\eta}|^2 - \rho), \\
K &= -\frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} + \frac{1}{2}(\widehat{\chi}, \underline{\widehat{\chi}}) - \rho, \\
\not{\mathrm{div}}\, \widehat{\chi}' &= \frac{1}{2}\not d\mathrm{tr}\chi' - \widehat{\chi}' \cdot \eta + \frac{1}{2}\mathrm{tr}\chi'\eta - \Omega^{-1}\beta, \\
\not{\mathrm{div}}\, \underline{\widehat{\chi}}' &= \frac{1}{2}\not d\mathrm{tr}\underline{\chi}' - \underline{\widehat{\chi}}' \cdot \underline{\eta} + \frac{1}{2}\mathrm{tr}\underline{\chi}'\underline{\eta} - \Omega^{-1}\underline{\beta}, \\
\not{\mathrm{curl}}\, \eta &= \sigma - \frac{1}{2}\widehat{\chi} \wedge \underline{\widehat{\chi}}, \\
\not{\mathrm{curl}}\, \underline{\eta} &= -\sigma + \frac{1}{2}\widehat{\chi} \wedge \underline{\widehat{\chi}}, \\
\widehat{D}(\Omega\underline{\widehat{\chi}}) &= \Omega^2(\not\nabla \widehat{\otimes} \underline{\eta} + \underline{\eta}\widehat{\otimes}\underline{\eta} + \frac{1}{2}\mathrm{tr}\chi\underline{\widehat{\chi}} - \frac{1}{2}\mathrm{tr}\underline{\chi}\widehat{\chi}), \\
D(\Omega\mathrm{tr}\underline{\chi}) &= \Omega^2(2\not{\mathrm{div}}\, \underline{\eta} + 2|\underline{\eta}|^2 - (\widehat{\chi}, \underline{\widehat{\chi}}) - \frac{1}{2}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} + 2\rho), \\
\underline{\widehat{D}}(\Omega\widehat{\chi}) &= \Omega^2(\not\nabla \widehat{\otimes} \eta + \eta\widehat{\otimes}\eta + \frac{1}{2}\mathrm{tr}\underline{\chi}\widehat{\chi} - \frac{1}{2}\mathrm{tr}\chi\underline{\widehat{\chi}}), \\
\underline{D}(\Omega\mathrm{tr}\chi) &= \Omega^2(2\not{\mathrm{div}}\, \eta + 2|\eta|^2 - (\widehat{\chi}, \underline{\widehat{\chi}}) - \frac{1}{2}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} + 2\rho), \\
\underline{D}\eta &= -\Omega(\underline{\chi} \cdot \eta + \underline{\beta}) + 2\not d\underline{\omega}, \\
D\underline{\eta} &= -\Omega(\chi \cdot \underline{\eta} - \beta) + 2\not d\omega.
\end{aligned}
$$

We also use the null frame to decompose the contracted second Bianchi identity $\nabla^\alpha \mathbf{R}_{\alpha\beta\gamma\delta} = 0$ into components. This leads the fol-

[9] See Chapter 1 of [7] for the derivation of these equations.

lowing null Bianchi equations:[10]

$$\begin{aligned}
\widehat{\underline{D}}\alpha-\frac{1}{2}\Omega\mathrm{tr}\underline{\chi}\alpha+2\underline{\omega}\alpha+\Omega\{-\nabla\!\!\!/\,\widehat{\otimes}\beta-(4\eta+\zeta)\widehat{\otimes}\beta+3\widehat{\chi}\rho+3^*\widehat{\chi}\sigma\}&=0,\\
\widehat{D}\underline{\alpha}-\frac{1}{2}\Omega\mathrm{tr}\chi\underline{\alpha}+2\omega\underline{\alpha}+\Omega\{\nabla\!\!\!/\,\widehat{\otimes}\underline{\beta}+(4\underline{\eta}-\zeta)\widehat{\otimes}\underline{\beta}+3\widehat{\underline{\chi}}\rho-3^*\widehat{\underline{\chi}}\sigma\}&=0,\\
D\beta+\frac{3}{2}\Omega\mathrm{tr}\chi\beta-\Omega\widehat{\chi}\cdot\beta-\omega\beta-\Omega\{\mathrm{div}\!\!\!/\,\alpha+(\underline{\eta}+2\zeta)\cdot\alpha\}&=0,\\
\underline{D}\underline{\beta}+\frac{3}{2}\Omega\mathrm{tr}\underline{\chi}\underline{\beta}-\Omega\widehat{\underline{\chi}}\cdot\underline{\beta}-\underline{\omega}\underline{\beta}+\Omega\{\mathrm{div}\!\!\!/\,\underline{\alpha}+(\eta-2\zeta)\cdot\underline{\alpha}\}&=0,\\
\underline{D}\beta+\frac{1}{2}\Omega\mathrm{tr}\underline{\chi}\beta-\Omega\widehat{\underline{\chi}}\cdot\beta+\underline{\omega}\beta-\Omega\{d\!\!\!/\rho+{}^*d\!\!\!/\sigma+3\eta\rho+3^*\eta\sigma+2\widehat{\chi}\cdot\underline{\beta}\}&=0,\\
D\underline{\beta}+\frac{1}{2}\Omega\mathrm{tr}\chi\underline{\beta}-\Omega\widehat{\chi}\cdot\underline{\beta}+\omega\underline{\beta}+\Omega\{d\!\!\!/\rho-{}^*d\!\!\!/\sigma+3\underline{\eta}\rho-3^*\underline{\eta}\sigma-2\widehat{\underline{\chi}}\cdot\beta\}&=0,\\
D\rho+\frac{3}{2}\Omega\mathrm{tr}\chi\rho-\Omega\{\mathrm{div}\!\!\!/\,\beta+(2\underline{\eta}+\zeta,\beta)-\frac{1}{2}(\widehat{\underline{\chi}},\alpha)\}&=0,\\
\underline{D}\rho+\frac{3}{2}\Omega\mathrm{tr}\underline{\chi}\rho+\Omega\{\mathrm{div}\!\!\!/\,\underline{\beta}+(2\eta-\zeta,\underline{\beta})+\frac{1}{2}(\widehat{\chi},\underline{\alpha})\}&=0,\\
D\sigma+\frac{3}{2}\Omega\mathrm{tr}\chi\sigma+\Omega\{\mathrm{cu}\!\!\!/\mathrm{l}\,\beta+(2\underline{\eta}+\zeta,{}^*\beta)-\frac{1}{2}\widehat{\underline{\chi}}\wedge\alpha\}&=0,\\
\underline{D}\sigma+\frac{3}{2}\Omega\mathrm{tr}\underline{\chi}\sigma+\Omega\{\mathrm{cu}\!\!\!/\mathrm{l}\,\underline{\beta}+(2\underline{\eta}-\zeta,{}^*\underline{\beta})+\frac{1}{2}\widehat{\chi}\wedge\underline{\alpha}\}&=0.
\end{aligned}$$

2.3. Hodge systems and commutation formulas. We will introduce the Hodge systems satisfied by the connection coefficients. We first define $\mu,\underline{\mu}$ to be

$$\begin{aligned}
\mu&=K+\frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi}-\mathrm{div}\!\!\!/\,\eta,\\
\underline{\mu}&=K+\frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi}-\mathrm{div}\!\!\!/\,\underline{\eta}.
\end{aligned}$$

And we also define $\kappa,\underline{\kappa}$ to be

$$\begin{aligned}
\kappa&=\Delta\!\!\!/\,\omega+\mathrm{div}\!\!\!/\,(\Omega\beta),\\
\underline{\kappa}&=\Delta\!\!\!/\,\underline{\omega}-\mathrm{div}\!\!\!/\,(\Omega\underline{\beta}).
\end{aligned}$$

Then we have[11]

Lemma 1. $(\Omega\widehat{\chi},\Omega\mathrm{tr}\chi)$ *satisfies*

$$\begin{aligned}
D(\Omega\mathrm{tr}\chi)&=-\frac{1}{2}(\Omega\mathrm{tr}\chi)^2-|\Omega\widehat{\chi}|^2+2\omega(\Omega\mathrm{tr}\chi),\\
\mathrm{div}\!\!\!/\,(\Omega\widehat{\chi})&=\frac{1}{2}d\!\!\!/(\Omega\mathrm{tr}\chi)+\Omega\widehat{\chi}\cdot\underline{\eta}-\frac{1}{2}\Omega\mathrm{tr}\chi\underline{\eta}-\Omega\beta.
\end{aligned}$$

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[10]See Proposition 1.2 of [7].
[11]See Chapter 6 of [7].

$(\Omega\widehat{\underline{\chi}}, \Omega\mathrm{tr}\underline{\chi})$ *satisfies*

$$\underline{D}(\Omega\mathrm{tr}\underline{\chi}) = -\frac{1}{2}(\Omega\mathrm{tr}\underline{\chi})^2 - |\Omega\widehat{\underline{\chi}}|^2 + 2\omega(\Omega\mathrm{tr}\underline{\chi}),$$
$$\mathrm{d}\!\!/\mathrm{iv}\,(\Omega\widehat{\underline{\chi}}) = \frac{1}{2}\mathrm{d}\!\!/(\Omega\mathrm{tr}\underline{\chi}) + \Omega\widehat{\underline{\chi}}\cdot\eta - \frac{1}{2}\Omega\mathrm{tr}\underline{\chi}\eta + \Omega\underline{\beta}.$$

(η, μ) *satisfies*

$$\begin{cases} \mathrm{d}\!\!/\mathrm{iv}\,\eta = -\rho + \frac{1}{2}(\widehat{\chi}, \widehat{\underline{\chi}}) - \mu, \\ \mathrm{cu}\!\!/\mathrm{rl}\,\eta = \sigma - \frac{1}{2}\widehat{\chi}\wedge\widehat{\underline{\chi}}, \end{cases}$$

$$\begin{aligned} D\mu = & -\Omega\mathrm{tr}\chi\mu - \frac{1}{2}\Omega\mathrm{tr}\chi\underline{\mu} - \frac{1}{4}\Omega\mathrm{tr}\underline{\chi}|\widehat{\chi}|^2 + \frac{1}{2}\Omega\mathrm{tr}\chi|\underline{\eta}|^2 \\ & + \mathrm{d}\!\!/\mathrm{iv}\,(2\Omega\widehat{\chi}\cdot\eta - \Omega\mathrm{tr}\chi\underline{\eta}). \end{aligned}$$

$(\underline{\eta}, \underline{\mu})$ *satisfies*

$$\begin{cases} \mathrm{d}\!\!/\mathrm{iv}\,\underline{\eta} = -\rho + \frac{1}{2}(\widehat{\chi}, \widehat{\underline{\chi}}) - \underline{\mu}, \\ \mathrm{cu}\!\!/\mathrm{rl}\,\underline{\eta} = -\sigma + \frac{1}{2}\widehat{\chi}\wedge\widehat{\underline{\chi}}, \end{cases}$$

$$\begin{aligned} \underline{D}\underline{\mu} = & -\Omega\mathrm{tr}\underline{\chi}\underline{\mu} - \frac{1}{2}\Omega\mathrm{tr}\underline{\chi}\mu - \frac{1}{4}\Omega\mathrm{tr}\chi|\widehat{\underline{\chi}}|^2 + \frac{1}{2}\Omega\mathrm{tr}\underline{\chi}|\eta|^2 \\ & + \mathrm{d}\!\!/\mathrm{iv}\,(2\Omega\widehat{\underline{\chi}}\cdot\underline{\eta} - \Omega\mathrm{tr}\underline{\chi}\eta). \end{aligned}$$

(ω, κ) *satisfies*

$$\Delta\!\!\!/\,\omega = \kappa - \mathrm{d}\!\!/\mathrm{iv}\,(\Omega\beta),$$
$$\underline{D}\kappa + \Omega\mathrm{tr}\underline{\chi}\kappa = -2(\Omega\widehat{\underline{\chi}}, \nabla\!\!\!/^{\,2}\omega) + m,$$

where

$$\begin{aligned} m = & -2(\mathrm{d}\!\!/\mathrm{iv}\,(\Omega\widehat{\underline{\chi}}), \mathrm{d}\!\!/\omega) + \frac{1}{2}\mathrm{d}\!\!/\mathrm{iv}\,(\Omega\mathrm{tr}\underline{\chi}\cdot\Omega\beta) - (\mathrm{d}\!\!/(\Omega^2), \mathrm{d}\!\!/\rho) \\ & + (\mathrm{d}\!\!/(\Omega^2), {}^*\mathrm{d}\!\!/\sigma) - \rho\Delta\!\!\!/\,(\Omega^2) + \Delta\!\!\!/\,(\Omega^2(2(\eta, \underline{\eta}) - |\underline{\eta}|^2)) \\ & + \mathrm{d}\!\!/\mathrm{iv}\,(\Omega^2(-\widehat{\underline{\chi}}\cdot\beta + 2\widehat{\chi}\cdot\underline{\beta} + 3\eta\rho + 3^*\eta\sigma)). \end{aligned}$$

$(\underline{\omega}, \underline{\kappa})$ *satisfies*

$$\Delta\!\!\!/\,\underline{\omega} = \underline{\kappa} + \mathrm{d}\!\!/\mathrm{iv}\,(\Omega\underline{\beta}),$$
$$D\underline{\kappa} + \Omega\mathrm{tr}\chi\underline{\kappa} = -2(\Omega\widehat{\chi}, \nabla\!\!\!/^{\,2}\underline{\omega}) + \underline{m},$$

where

$$\begin{aligned} \underline{m} = & -2(\mathrm{d}\!\!/\mathrm{iv}\,(\Omega\widehat{\chi}), \mathrm{d}\!\!/\underline{\omega}) + \frac{1}{2}\mathrm{d}\!\!/\mathrm{iv}\,(\Omega\mathrm{tr}\chi\cdot\Omega\underline{\beta}) - (\mathrm{d}\!\!/(\Omega^2), \mathrm{d}\!\!/\rho) \\ & - (\mathrm{d}\!\!/(\Omega^2), {}^*\mathrm{d}\!\!/\sigma) - \rho\Delta\!\!\!/\,(\Omega^2) + \Delta\!\!\!/\,(\Omega^2(2(\eta, \underline{\eta}) - |\eta|^2)) \\ & + \mathrm{d}\!\!/\mathrm{iv}\,(\Omega^2(\widehat{\chi}\cdot\underline{\beta} - 2\widehat{\underline{\chi}}\cdot\beta + 3\underline{\eta}\rho - 3^*\underline{\eta}\sigma)). \end{aligned}$$

We denote the first order elliptic operators (or Hodge operators) $\mathcal{D}_1, \mathcal{D}_2$, by[12]

$\mathcal{D}_1$: tangential one-form $\xi \mapsto$ a pair of functions $(\mathrm{d}\!\!/\mathrm{iv}\, \xi, \mathrm{cu}\!\!/\mathrm{rl}\, \xi)$;

$\mathcal{D}_2$: tangential symmetric trace-free $(0,2)$ type tensorfield θ

$\mapsto$ tangential one-form $\mathrm{d}\!\!/\mathrm{iv}\, \theta$.

It is easy to calculate the formal L^2 adjoint

${}^*\mathcal{D}_1$: a pair of functions $(f,g) \mapsto$ tangential one-form $-\mathrm{d}\!\!/ f + {}^*\mathrm{d}\!\!/ g$;

${}^*\mathcal{D}_2$: tangential one-form ξ

$\mapsto$ tangential symmetric trace-free $(0,2)$ type tensorfield $-\dfrac{1}{2}\nabla\!\!\!\!/\, \widehat{\otimes} \xi$.

We will denote any one of the above elliptic operators (or Hodge operators) and their formal L^2 adjoint by $\mathcal{D}, {}^*\mathcal{D}$.

We also need the following commutation formulas for the estimates for the derivatives[13] .

Lemma 2. *Given integer i and tangential tensorfield ϕ, we have*

$$[D, \nabla\!\!\!\!/^{\,i}]\phi = \sum_{j=1}^{i} \nabla\!\!\!\!/^{\,j}(\Omega\chi)\cdot\nabla\!\!\!\!/^{\,i-j}\phi,$$

$$[\underline{D}, \nabla\!\!\!\!/^{\,i}]\phi = \sum_{j=1}^{i} \nabla\!\!\!\!/^{\,j}(\Omega\underline{\chi})\cdot\nabla\!\!\!\!/^{\,i-j}\phi,$$

and

$$[\mathcal{D}, \nabla\!\!\!\!/^{\,i}]\phi = \sum_{j=1}^{i} \nabla\!\!\!\!/^{\,j-1}K\cdot\nabla\!\!\!\!/^{\,i-j}\phi,$$

$$[{}^*\mathcal{D}, \nabla\!\!\!\!/^{\,i}]\phi = \sum_{j=1}^{i} \nabla\!\!\!\!/^{\,j-1}K\cdot\nabla\!\!\!\!/^{\,i-j}\phi.$$

Here we use "$\cdot$" to represent an arbitrary contraction with the coefficients by $g\!\!\!/$ or $\epsilon\!\!/$. In addition, if ϕ is a function, then when $i = 1$, all commutators above are zero; when $i \geq 2$, all i's are replaced by $i-1$'s in above formulas.

2.4. Basic inequalities. We will introduce some frequently used basic inequalities, such as the Sobolev inequalities, the Poincaré inequalities, the Gronwall type inequalities, and the elliptic estimates for the Hodge systems.

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[12]See [**12**].

[13]See Chapter 4 of [**7**] for the first group. The second group can be derived directly by the definition of curvature.

For the Sobolev inequalities, we start from isoperimetric inequality: Given a function $f \in W^{1,1}(S_{\underline{u},u})$ and denoting by $\bar{f}$ the average of f on $S_{\underline{u},u}$, we have

$$\int_{S_{\underline{u},u}} (f-\bar{f})^2 \mathrm{d}\mu_{\not{g}} \leq I(S_{\underline{u},u}) \left(\int_{S_{\underline{u},u}} |\not{d} f| \mathrm{d}\mu_{\not{g}} \right)^2,$$

where $I(S_{\underline{u},u})$ is the isoperimetric constant. Based on the isoperimetric inequality, we have

Lemma 3 (Sobolev inequalities, see Section 5.2 of [7]). *Given a tangential tensorfield ϕ, we have for $q \in (2,+\infty)$,*

$$\|\phi\|_{L^q(S_{\underline{u},u})} \leq C\sqrt{\max\{I(S_{\underline{u},u}),1\}} \sum_{i=0}^{1} r^{-1+2/q} \|(r\not{\nabla})^i \phi\|_{L^2(S_{\underline{u},u})},$$

$$\|\phi\|_{L^\infty(S_{\underline{u},u})} \leq C\sqrt{\max\{I(S_{\underline{u},u}),1\}} \sum_{i=0}^{2} r^{-1} \|(r\not{\nabla})^i \phi\|_{L^2(S_{\underline{u},u})},$$

where $r = r(\underline{u},u)$ satisfies $4\pi r^2 = Area(S_{\underline{u},u})$ and C is a universal constant.

Applying the Hölder inequality to the right hand side of the isoperimetric inequality, we have:

Lemma 4 (Poincaré inequality). *Given a function f, we have*

$$\|f-\bar{f}\|_{L^2(S_{\underline{u},u})} \leq C\sqrt{I(S_{\underline{u},u})} \|(r\not{\nabla}) f\|_{L^2(S_{\underline{u},u})}.$$

We also need the following Gronwall type estimates:

Lemma 5 (Gronwall type estimates, see Chapter 4 of [7] or Chapter 4 of [11]). *Assume that on the outgoing null cone C_u, the parameter $\underline{u} \in [\underline{u}_0, \underline{u}_1] \subset [0,+\infty)$, $C^{-1}(\underline{u}+1) \leq r \leq C(\underline{u}+1)$ for some universal C, and*

$$\Omega|\widehat{\chi}| + \left|\Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi}\right| \leq Cr^{-3/2}.$$

Then, for an arbitrary $(0,s)$ type tangential tensorfield ϕ, $2 \leq q \leq +\infty$, and any real ν, we have

$$\begin{aligned}\|r^{s-\nu-2/q}\phi\|_{L^q(S_{\underline{u},u})} \leq & C_{q,\nu,s} \Big(\|r^{s-\nu-2/q}\phi\|_{L^q(S_{\underline{u}_0,u})} \\ & + \int_{\underline{u}_0}^{\underline{u}} \|r^{s-\nu-2/q}(D\phi - \frac{\nu}{2}\Omega\mathrm{tr}\chi\phi)\|_{L^q(S_{\underline{u}',u})} \mathrm{d}\underline{u}' \Big).\end{aligned}$$

It is also true if we replace the first term of the right hand side by taking value on $S_{\underline{u}_1,u}$ and the integral of the second term by integrating from $\underline{u}_1$ to $\underline{u}$.

Assume that on the incoming null cone $\underline{C}_u$ with the parameter $u \in [0, \varepsilon]$, we have

$$\Omega|\widehat{\underline{\chi}}| + \left|\Omega \mathrm{tr}\underline{\chi}\right| \leq C r^{-1}.$$

Then for an arbitrary tangential tensorfield ϕ, $2 \leq q \leq +\infty$, we have

$$\|\phi\|_{L^q(S_{\underline{u},u})} \leq C_q \left(\|\phi\|_{L^q(S_{\underline{u},0})} + \int_0^u \|\underline{D}\phi\|_{L^q(S_{\underline{u},u'})} \mathrm{d}u' \right).$$

Finally, we need the following elliptic estimates:

Lemma 6 (Elliptic estimates for Hodge systems, see Chapter 7 of [7]). *Assume that θ is a tangential symmetric trace-free $(0,2)$ type tensorfield with*

$$\mathrm{d}\!\!/\mathrm{iv}\, \theta = f,$$

where f is a tangential one-form. Then for $i \geq 1$, we have

$$\|(r\nabla\!\!\!\!/\,)^i \theta\|_{L^2(S_{\underline{u},u})} \leq C_K \left(\sum_{j=0}^{i-1} \|(r\nabla\!\!\!\!/\,)^j (rf)\|_{L^2(S_{\underline{u},u})} + \|\theta\|_{L^2(S_{\underline{u},u})} \right).$$

Given a tangential one-form ξ with

$$\mathrm{d}\!\!/\mathrm{iv}\, \xi = f, \quad \mathrm{cu}\!\!/\mathrm{rl}\, \xi = g,$$

we have for $i \geq 1$,

$$\begin{aligned} &\|(r\nabla\!\!\!\!/\,)^i \xi\|_{L^2(S_{\underline{u},u})} \\ \leq & C_K \left(\sum_{j=0}^{i-1} \|(r\nabla\!\!\!\!/\,)^j (rf)\|_{L^2(S_{\underline{u},u})} + \|(r\nabla\!\!\!\!/\,)^j (rg)\|_{L^2(S_{\underline{u},u})} + \|\xi\|_{L^2(S_{\underline{u},u})} \right). \end{aligned}$$

Here C_K depends on i, $r\|(r\nabla\!\!\!\!/\,)^{\leq \max\{i-2,0\}} K\|_{L^2(S_{\underline{u},u})}$ and the Sobolev constant (which depends on the isoperimetric constant $I(S_{\underline{u},u})$).

3. The main theorem and the structure of the proof

3.1. Definition of various quantities and the statement of the main theorem. We first define several norms of the connection coefficients and curvature components adapted to our problem. In many cases, we use Γ to denote one of the connection coefficients, say $\widehat{\chi}$, $\mathrm{tr}\chi$, $\widehat{\underline{\chi}}$, $\mathrm{tr}\underline{\chi}$, η, $\underline{\eta}$, ω and $\underline{\omega}$ up to a multiple by Ω. In fact, Γ will share all estimates with $\Omega\Gamma$ because Ω will be close to 1 up to all derivatives we use. We also use Γ to denote $\Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi}$ in some special cases. We emphasize that $\Omega\mathrm{tr}\chi$ and $\Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi}$ should be treated in a completely different way. We also use R to denote one of the curvature components β, ρ, σ, $\underline{\beta}$ and $\underline{R}$ to denote one of the curvature components ρ, σ, $\underline{\beta}$ and $\underline{\alpha}$. For the same component, that we use the underline or not depends on different roles the component plays in the equations. We assign a number p to every connection and curvature component to denote their expected decay rate about r:

$$\begin{aligned}(3.1)\qquad & p(\widehat{\chi}, \Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi}, \eta, \underline{\eta}) = 2,\ p(\widehat{\underline{\chi}}, \mathrm{tr}\chi, \mathrm{tr}\underline{\chi}, \underline{\omega}) = 1,\ p(\omega) = 3,\\ & p(\beta) = 4,\ p(\rho, \sigma) = 3,\ p(\underline{\beta}) = 2,\ p(\underline{\alpha}) = 1.\end{aligned}$$

We will use Γ_p, R_p and $\underline{R}_p$ to denote one of the quantities with p assigned. The above definitions also valid up to a multiple by Ω.

Let r be the area radius defined by $4\pi r(\underline{u}, u)^2 = Area(S_{\underline{u},u})$. For the null connection coefficients, we denote

$$\begin{aligned}\mathcal{O}^{0,\infty}[\Gamma_p] &= \sup_{\underline{u},u} \|r^p\Gamma_p\|_{L^\infty(S_{\underline{u},u})},\\ \mathcal{O}^{i,4}[\Gamma_p] &= \sup_{\underline{u},u} \|r^{p-1/2}(r\nabla\!\!\!\!/\,)^i\Gamma_p\|_{L^4(S_{\underline{u},u})} \text{ for } i \le 1,\\ \mathcal{O}^{i,2}[\Gamma_p] &= \sup_{\underline{u},u} \|r^{p-1}(r\nabla\!\!\!\!/\,)^i\Gamma_p\|_{L^2(S_{\underline{u},u})} \text{ for } i \le 2.\end{aligned}$$

We denote $\mathcal{O}[\Gamma]$ be the sum over all norms of Γ. We also use $\mathcal{O}^{0,\infty}$, $\mathcal{O}^{i,4}$, $\mathcal{O}^{i,2}$ and $\mathcal{O}$ to denote the corresponding sum over all Γ. For the curvature components, we denote

$$\begin{aligned}\mathcal{R}[R_p] &= \sup_{u} \sum_{i=0}^{2} \|r^{p-2}(r\nabla\!\!\!\!/\,)^i R_p\|_{L^2(C_u)},\\ \underline{\mathcal{R}}[\underline{R}_p] &= \sup_{\underline{u}} \sum_{i=0}^{2} \|r^{p-1}(r\nabla\!\!\!\!/\,)^i \underline{R}_p\|_{L^2(\underline{C}_{\underline{u}})},\end{aligned}$$

and $\mathcal{R}$, $\underline{\mathcal{R}}$ denote the sum over all components R or $\underline{R}$.

As discussed in the introduction, we consider the characteristic initial value problem of the vacuum Einstein equations with the initial data given on two null cones C_0 and $\underline{C}_0$ intersecting at a sphere $S_{0,0}$, where C_0 is complete towards future. Precisely, we will prove the following main theorem in this paper:

Theorem 2 (Main Theorem)**.** *Suppose that we have two intersected null cones C_0 and $\underline{C}_0$ where C_0 is an outgoing null cone extended to infinity and $\underline{C}_0$ an incoming null cone, and $S_0 = \underline{C}_0 \bigcap C_0$ is a two-sphere. Suppose also C_0 and $\underline{C}_0$ are foliated by the affine sections parametrized by two functions s and $\underline{s}$, respectively. Let $\Lambda(s)$ and $\lambda(s)$ be the larger and smaller eigenvalue of $r(s)^{-2}g\!\!\!/|_{S_{s,0}}$ relative to $r(0)^{-2}g\!\!\!/|_{S_{0,0}}$, and $r(s)$ be the area radius defined by $4\pi r(s)^2 = Area(S_{s,0})$. Suppose also that the initial data given on $\underline{C}_0 \bigcup C_0$ satisfy the following:*

$$C_{\lambda,\Lambda}^{-1} \le \lambda(s) \le \Lambda(s) \le C_{\lambda,\Lambda}, \quad C_r^{-1}(1+s) \le r(s) \le C_r(1+s),$$

$$\mathcal{O}_0 \triangleq \sup_s \sum_{\Gamma_p \ne \underline{\omega}} \left(\|r^p\Gamma_p\|_{L^\infty(S_{s,0})} + \sum_{i=0}^{1} \|r^{p-1/2}(r\nabla\!\!\!\!/\,)^i\Gamma_p\|_{L^4(S_{s,0})}\right.$$

$$+\sum_{i=0}^{2}\|r^{p-1}(r\nabla\!\!\!\!/\,)^i\Gamma_p\|_{L^2(S_{s,0})}\Bigg)$$

$$+\|(r\nabla\!\!\!\!/\,)^3(\underline{\eta},\widehat{\chi})\|_{L^2(C_0)}+\sup_s\|(r\nabla\!\!\!\!/\,)^3\mathrm{tr}\underline{\chi}\|_{L^2(S_{s,0})}+C_{\lambda,\Lambda}+C_r<\infty,$$

$$\mathcal{R}_0\triangleq\sum_{R_p}\sum_{i=0}^{2}\|r^{p-2}(r\nabla\!\!\!\!/\,)^iR_p\|_{L^2(C_0)}+\|\underline{D}^2\underline{\beta}\|_{L^2(C_0)}$$

$$+\sum_{i=0}^{1}\left(\|(r\nabla\!\!\!\!/\,)^i\underline{D}\underline{\beta}\|_{L^2(C_0)}+\sup_s\|r^{p-1}(r\nabla\!\!\!\!/\,)^iR_p\|_{L^2(S_{s,0})}\right)$$

$$+\sum_{i=0}^{1}\left(\sup_s\|r^{1/2}\underline{D\alpha},(r\nabla\!\!\!\!/\,)^i(r^{1/2}\underline{\alpha},r^{3/2}\underline{\beta})\|_{L^4(S_{s,0})}\right)<\infty,$$

$$\underline{\mathcal{R}}_0\triangleq\sum_{\underline{R}_p}\sum_{i=0}^{2}\|r^{p-1}(r\nabla\!\!\!\!/\,)^i\underline{R}_p\|_{L^2(\underline{C}_0)}$$

$$+\sum_{i=0}^{1}\|(r\nabla\!\!\!\!/\,)^i\underline{D\alpha}\|_{L^2(\underline{C}_0)}+\|\underline{D}^2\underline{\alpha}\|_{L^2(\underline{C}_0)}<\infty.$$

Then there exists an $\varepsilon>0$ depends on $\mathcal{O}_0$, $\mathcal{R}_0$, $\underline{\mathcal{R}}_0$ and a global optical function u such that the solution of the vacuum Einstein equations exists in a global double null foliation for $0\le\underline{u}<+\infty$, $0\le u\le\varepsilon$. In addition, $\Omega\to1$ along every C_u when $\underline{u}\to+\infty$ uniformly in u, which means that the function u on $\underline{C}_{\underline{u}}$ tends to the affine parameter of the null generators of $\underline{C}_{\underline{u}}$ when $\underline{u}\to+\infty$.

Remark 1. The last statement of the above theorem shows that the global optical function u can be viewed as the retarded time function.

Remark 2. Note that the initial quantities are written in the affine sections defined by s and $\underline{s}$. Because we are solving Einstein equations up to the future null infinity, the global optical function u restricted on $\underline{C}_0$ is not coincide with the affine parameter $\underline{s}$ in general. In constructing the space-time, the difference between two different foliations on $\underline{C}_0$ should be controlled. On the other hand, although the optical function $\underline{u}$ is constructed such that it coincides with s on C_0, but the lapse Ω may change. In other words, the vectorfield L restricted on C_0 and $\underline{L}'$ restricted on $\underline{C}_0$ are invariant in our whole construction, but L' restricted on C_0 and $\underline{L}$ restricted on $\underline{C}_0$ will change in general.

3.2. The characteristic initial data. Recall from [8] and [11] that for any strongly asymptotically flat Cauchy data, there exists a region Ω_{λ_0} with compact closure such that the boundary of the causal future of Ω_{λ_0} in the maximal development consists of complete null geodesic generators. The boundary of the causal future of Ω_{λ_0} satisfies the assumptions on the outgoing null cone of our Theorem 2.

In this subsection, we will give a brief description on how to specify the characteristic initial data on two intersected null cones and show that a trivial extension of the initial data in [**7**] satisfies the assumptions of Theorem 2 and can serve as a simple example of our setting. One can see the full details of specifying arbitrary characteristic initial data in [**7**], [**20**], [**13**], or [**2**] for the infinite cases.

The initial data can be specified as follows. Geometrically, the initial data set on the null cone C_0 consists of the conformal geometry of C_0, which means that given a family of spherical sections on C_0 (usually the affine sections) which are parameterized by a function s, one needs to specify a family of metrics $\widehat{\not g}(s)$ on the sections and then the actual metrics on the sections of C_0 are given by $\not g(s) = \phi^2(s)\widehat{\not g}(s)$ where the conformal factor $\phi(s)$ is determined by $\widehat{\not g}(s)$, see, for example, [**7**]. In practice, we will usually impose certain initial conditions on the shear $\widehat{\chi}(s)$, which is the derivative of the conformal geometry, see, for example, [**12**]. Similarly, the initial data on the null cone $\underline{C}_0$ also consist of the conformal geometry of $\underline{C}_0$ and we usually specify the shear $\widehat{\underline{\chi}}$. To ensure the well-posedness, we also need to specify the "full geometry" on the intersection of two null cones $S_{0,0} = C_0 \bigcap \underline{C}_0$. The full geometry consists of the metric $\not g$ induced on $S_{0,0}$ (but not only the conformal metric), the torsion ζ, and both expansions $\mathrm{tr}\chi$ and $\mathrm{tr}\underline{\chi}$.

Remark 3. Specifying initial data in such a way will cause the lost of derivatives. This is the nature of characteristic problem. For example, in order to ensure that the assumptions of Theorem 2 are true, the bounds for up to more than third derivatives of the shears $\widehat{\chi}$ or $\widehat{\underline{\chi}}$ are needed. However, only estimates for up to the third order derivatives of $\widehat{\chi}$ and $\underline{\chi}$ can be obtained.

We then want to show that a trivial extension of the initial data given in [**7**] satisfies the assumptions of Theorem 2. It is a good example where Theorem 2 is applicable, and the consequence is that the the maximal future development of such initial data will have a piece of the future null infinity, and a closed trapped surface. Recall that the initial data constructed in [**7**] is given on a null cone C_{u_0} from a point o where $u_0 < -1$ is a fixed number. Let $\underline{u}$ be the affine parameter on C_{u_0} with its value being u_0 at the point o. We assume that for $u_0 \leq \underline{u} \leq 0$, the initial data given on C_{u_0} is trivial, which means that the geometry is precisely the geometry of a null cone from a point in Minkowski spacetime. Let $\delta > 0$ be a small parameter, the initial data given in $0 \leq \underline{u} \leq \delta$ are the so-called short pulse data, of which the shear satisfies

$$|\delta^n |u_0|^m \not\nabla^m D^n \widehat{\chi}| \leq \delta^{-1/2}|u_0|^{-1} C_{m,n}.$$

The norm is taken by $\not g$ in L^∞.

We drop the weight u_0 because we are not considering the problem from the past null infinity. Remember the relation derived in Chapter 2 in [7] as follows:

$$\widehat{\chi} \sim \delta^{-1/2}, \mathrm{tr}\chi \sim 1,\ \zeta \sim \delta^{1/2},\ \widehat{\underline{\chi}} \sim \delta^{1/2},\ \mathrm{tr}\underline{\chi} \sim 1,$$
$$\alpha \sim \delta^{-3/2},\ \beta \sim \delta^{-1/2},\ \rho, \sigma \sim 1,\ \underline{\beta} \sim \delta,\ \underline{\alpha}, \underline{D\alpha} \sim \delta^{3/2}.$$

The above notation $\psi \sim \delta^r$ means that for $0 \leq \underline{u} \leq \delta$,

$$|\delta^n \nabla\!\!\!/^m D^n \psi| \leq \delta^r C_{m,n}.$$

In particular, the above relation holds for $\underline{u} = \delta$. We can also choose $\widehat{\chi}$ suitably such that $\mathrm{tr}\chi > 0$ at $\underline{u} = \delta$. We also remember the induced metrics on the spherical sections $\not g$ are expressed in the form

$$\not g(\underline{u}) = \phi^2(\underline{u}) \widehat{\not g}(\underline{u}),$$

where $\widehat{\not g}(\underline{u})$ has the same volume form for all $\underline{u}$. We know that for $0 \leq \underline{u} \leq \delta$,

$$\phi - 1 \sim \delta,\ |u_0|^{-2} \widehat{\not g} - \overset{\circ}{\not g} \sim \delta^{1/2},\ |u_0|^{-2} \not g - \overset{\circ}{\not g} \sim \delta^{1/2},$$

where $\overset{\circ}{\not g}$ is the standard metric on the unit sphere. The above relation, in particular, holds for $\underline{u} = \delta$.

We may assume that the shear imposed on $0 \leq \underline{u} < \delta$ has compact support such that we can smoothly extend $\widehat{\chi}$ to $\underline{u} > \delta$ trivially, i.e., $\widehat{\chi} \equiv 0$[14] . We then prove the following:

Proposition 1. *Extending the initial data on C_{u_0} as described above. Then the assumption of Theorem 2 on $\mathcal{O}_0$, $\mathcal{R}_0$ holds on the truncated cone C_{u_0} for $\underline{u} \geq \delta$.*

Remark 4. The incoming null cone $\underline{C}_\delta$ in the work of Christodoulou [7] will serve as the incoming null cone $\underline{C}_0$ in Theorem 2. We do not care about what happens for $0 \leq \underline{u} \leq \delta$.

Proof. The triviality of $\widehat{\chi}$ implies that $\widehat{\not g}(\underline{u}) \equiv \widehat{\not g}(\delta)$ for $\underline{u} > \delta$ and ϕ satisfies $D^2\phi = 0$. Recall the relation $D\phi = \phi \mathrm{tr}\chi/2$ and $\mathrm{tr}\chi(\delta) > 0$, we can solve ϕ as

$$\phi(\underline{u}) = \frac{\phi(\delta)\mathrm{tr}\chi(\delta)}{2}(\underline{u} - \delta) + \phi(\delta),$$

and $\phi > 0$ for all $\underline{u} > \delta$. Also, $\mathrm{tr}\chi$ satisfies $D\mathrm{tr}\chi = -\frac{1}{2}(\mathrm{tr}\chi)^2$ and can be solved as

$$\mathrm{tr}\chi(\underline{u}) = \frac{2}{\underline{u} - \delta + \frac{2}{\mathrm{tr}\chi(\delta)}},$$

[14] One can also do as in the last several pages in [7] without the compactly supported condition, but which is essentially the same.

which is positive for all $\underline{u} > \delta$. The above computation shows that C_{u_0} is a complete null cone without conjugate points and cut points along the null generators.

We then estimate r by using the equation $Dr = r\overline{\mathrm{tr}\chi}/2$. This equation follows from the variation of area:

$$D\int_{S_{\underline{u},u}} \mathrm{d}\mu_{\not{g}} = \int_{S_{\underline{u},u}} \Omega\mathrm{tr}\chi\mathrm{d}\mu_{\not{g}}.$$

We denote $a = \min_{S_{\delta,u_0}} \mathrm{tr}\chi(\delta)$, $A = \max_{S_{\delta,u_0}} \mathrm{tr}\chi(\delta)$, and $A \geq a > 0$. By the formula of $\mathrm{tr}\chi$, we have $2/(\underline{u}+2/a) \leq \mathrm{tr}\chi(\underline{u}) \leq 2/(\underline{u}+2/A)$ and this also holds for $\overline{\mathrm{tr}\chi}$. Therefore, we have

$$\log(\underline{u}-\delta+2/a) - \log(2/a) \leq \log r - \log r(\delta) \leq \log(\underline{u}+2/A) - \log(2/A),$$

which implies the condition on r with the constant depending on a and A.

It is not hard to estimate $\Lambda(\underline{u})$ and $\lambda(\underline{u})$ (the eigenvalues of $r(\underline{u})^{-2}\not{g}(\underline{u})$ with respect to $r(\delta)^{-2}\not{g}(\delta)$). Because along the null generators, the conformal geometry $\widehat{\not{g}}$ does not change, then

$$\Lambda(\underline{u}) = \lambda(\underline{u}) = [r(\delta)\phi(\underline{u})/r(\underline{u})\phi(\delta)]^2,$$

whose bound depends only on a, A and the maximal and minimal values of $\phi(\delta)$ on S_{δ,u_0}.

We begin to consider the connection coefficients and curvature components. For $\mathrm{tr}\chi$, we have

$$|\mathrm{tr}\chi| \leq Cr^{-1},\ |\mathrm{tr}\chi - \overline{\mathrm{tr}\chi}| \leq Cr^{-2},$$

by the relation between $\underline{u}$ and r, and C depends on a and A. For the derivatives of $\mathrm{tr}\chi$, we argue by induction on the order i of the derivatives. For $i = 1$, we consider the equation $D\not{d}\mathrm{tr}\chi = -\mathrm{tr}\chi\not{d}\mathrm{tr}\chi$ and apply the Gronwall type estimates:

$$|r^3\not{d}\mathrm{tr}\chi|(\underline{u}) \leq C|r^3\not{d}\mathrm{tr}\chi|_{\underline{u}=\delta} \leq C.$$

Note that the weight r^3 is related to the structure of the equation in an essential way. Now we assume $|(r\not{\nabla})^{i-1}(r\mathrm{tr}\chi)| \leq C_{i-1}$, and commute $\not{\nabla}^{i-1}$ with the equation $D\not{d}\mathrm{tr}\chi = -\mathrm{tr}\chi\not{d}\mathrm{tr}\chi$ to obtain

$$D\not{\nabla}^i\mathrm{tr}\chi = -\mathrm{tr}\chi\not{\nabla}^i\mathrm{tr}\chi + \sum_{j=1}^{i-1}\not{\nabla}^j\mathrm{tr}\chi\cdot\not{\nabla}^{i-j}\mathrm{tr}\chi.$$

The second term on the right hand side comes from the commutator $[D, \not{\nabla}^{i-1}]$ (see Lemma 2, and remember $\widehat{\chi} \equiv 0$). Again by the Gronwall type estimates, we can conclude

$$|(r\not{\nabla})^i(r\mathrm{tr}\chi)|(\underline{u}) \leq |(r\not{\nabla})^i(r\mathrm{tr}\chi)|_{S_{\delta,u_0}} \leq C_i.$$

We can then turn to $\zeta = \eta = -\eta$ on C_{u_0}. Combining the null Codazzi equations $\not{\mathrm{div}}\,\widehat{\chi} = \cdots$ and the equation for $D\eta$, we deduce that (see

Chapter 2 of [7])

$$D\zeta + \mathrm{tr}\chi\zeta = -\frac{1}{2}\not{d}\mathrm{tr}\chi.$$

Applying the Gronwall type estimates, we have

$$\begin{aligned}|r^3\zeta|(\underline{u}) \leq & C|r^3\zeta|_{S_{\delta,u_0}} + C\int_\delta^{\underline{u}} |r^3\not{d}\mathrm{tr}\chi|(\underline{u}')\mathrm{d}\underline{u}', \\ \leq & C|r^3\zeta|_{S_{\delta,u_0}} + Cr(\underline{u}).\end{aligned}$$

Dividing $r(\underline{u})$ on both sides leads to the desired estimate $|r^2\zeta|(\underline{u}) \leq C$. The derivatives of ζ are estimated in a similar way.

We consider the equation for the Gauss curvature K, (see Chapter 5 of [7])

$$DK + \mathrm{tr}\chi K = -\frac{1}{2}\not{\Delta}\,\mathrm{tr}\chi.$$

We can deduce that $|(r\not{\nabla})^i(r^2K)| \leq C_i$ by commuting derivatives and applying the Gronwall type estimates.

We turn to $\mathrm{tr}\underline{\chi}$ and $\widehat{\underline{\chi}}$. We combine the Gauss equation and the equation for $D\mathrm{tr}\underline{\chi}$ and obtain the following (see Chapter 2 of [7])

$$D\mathrm{tr}\underline{\chi} + \mathrm{tr}\chi\mathrm{tr}\underline{\chi} = -2K - 2\not{\mathrm{div}}\,\zeta + 2|\zeta|^2.$$

We then deduce that

$$\begin{aligned}|r^2\mathrm{tr}\underline{\chi}|(\underline{u}) \leq & C|r^2\mathrm{tr}\underline{\chi}|_{S_{\delta,u_0}} + C\int_\delta^{\underline{u}} (1 + r^{-1} + r^{-2})\mathrm{d}\underline{u}' \\ \leq & C|r^2\mathrm{tr}\underline{\chi}|_{S_{\delta,u_0}} + Cr(\underline{u}),\end{aligned}$$

and therefore $|r\mathrm{tr}\underline{\chi}|(\underline{u}) \leq C$. A similar argument gives the desired estimates on the derivatives of $\mathrm{tr}\underline{\chi}$. For $\widehat{\underline{\chi}}$, the equation we consider is (see Chapter 5 of [7])

$$D\widehat{\underline{\chi}} - \frac{1}{2}\mathrm{tr}\chi\widehat{\underline{\chi}} = -\not{\nabla}\widehat{\otimes}\zeta + \zeta\widehat{\otimes}\zeta.$$

We apply the Gronwall type estimates to obtain $|(r\not{\nabla})^i(r\widehat{\underline{\chi}})|(\underline{u}) \leq C_i$.

We remark that we do not need $\underline{\omega}$ here. In fact, if we consider the equation for $D\underline{\omega}$, we find that $\underline{\omega}$ need not to decay. The decaying condition on $\underline{\omega}$ is ensured by our construction of the canonical foliation.

We finally turn to the curvature components, which are easier to derive. We rely on the null Bianchi equations for $D\beta$, $D\rho$, $D\sigma$, $D\underline{\beta}$, $D\underline{\alpha}$ and $D\underline{D}\underline{\alpha}$. The last one comes from commuting $\underline{D}$ with the equation for $D\underline{\alpha}$. Remember $\widehat{\chi} \equiv 0$ and then $\alpha \equiv 0$. We can commute derivatives and apply the Gronwall type estimate, to obtain

$$|(r\not{\nabla})^i(r^4\beta, r^3\rho, r^3\sigma, r^2\underline{\beta}, r\underline{\alpha}, r\underline{D\alpha})| \leq C_i.$$

We should remark here that because $\widehat{\chi} = 0$ when $\underline{u} = \delta$, then by the null Codazzi equation we can obtain an improved estimate $\beta \sim 1$ when

$\underline{u} = \delta$ (originally $\beta \sim \delta^{-1/2}$). We remark that the trivial extension of the initial data on C_{u_0} to finite length is also used in [**16**] and such improvements are essential in the work [**16**], in which we have, in fact, obtained an even better improvement under an addition condition.

We have verified that the assumptions of Theorem 2 about $\mathcal{O}_0$ and $\mathcal{R}_0$ hold. Finally, we remark that the above estimates are done in C^k for any k and have stronger decay as compared to the assumptions in Theorem 2. q.e.d.

3.3. The structure of the proof. We will prove the above Main Theorem in the rest of the paper. First of all, we give the structure of the proof.

We begin the proof by defining

$$\mathcal{A}_{\varepsilon,\Delta} = \{c \geq 0 : c \text{ satisfies the following two properties} \\ \text{for two constants } \varepsilon, \Delta > 0\},$$

where ε is a small positive parameter and Δ is a positive large constant. They will be suitably chosen in the context depending only on $\mathcal{O}_0, \mathcal{R}_0, \underline{\mathcal{R}}_0$.

(1) The solution of the vacuum Einstein equations g exists in a double null foliation given by $(\underline{u}, u)$ for $0 \leq \underline{u} \leq c$, $0 \leq u \leq \varepsilon$, where $\underline{u}$ coincides with the affine function s on C_0 and $u|_{\underline{C}_c}$ is canonical, which means that the following equation holds on $\underline{C}_c$:

(3.2)
$$\overline{\log \Omega} = 0, \quad \Delta\!\!\!/ \log \Omega = \frac{1}{2}\mathrm{d}i\!\!\!/\mathrm{v}\, \underline{\eta} + \frac{1}{2}\left(\frac{1}{2}((\widehat{\chi}, \underline{\widehat{\chi}}) - \overline{(\widehat{\chi}, \underline{\widehat{\chi}})}) - (\rho - \overline{\rho})\right).$$

(2) Written in the double null foliation given by $(\underline{u}, u)$, $\mathcal{R}, \underline{\mathcal{R}} \leq \Delta$.

We will prove, for $\varepsilon > 0$ sufficiently small and Δ sufficiently large, $\underline{u}_* = \sup \mathcal{A}_{\varepsilon,\Delta} = +\infty$.

The proof is divided into following steps:

Step 1: We first construct on $\underline{C}_0$ a new function u which is canonical and such that the section $u = 0$ is the simply $S_{0,0} = C_0 \bigcap \underline{C}_0$ and u varies in $[0, \varepsilon]$ where ε depends on $\mathcal{O}_0$ and $\underline{\mathcal{R}}_0$. This shows that $\mathcal{A}_{\varepsilon,\Delta}$ is not empty. Then we are going to argue by contradiction and assume $\underline{u}_* = \sup \mathcal{A}_{\varepsilon,\Delta} < +\infty$.

Step 2: In this and the next step, we work in the space-time region $M_{\underline{u}_*,\varepsilon}$ which corresponds to $0 \leq \underline{u} \leq \underline{u}_*$, $0 \leq u \leq \varepsilon$. It is not hard to see $\underline{u}_* \in \mathcal{A}_{\varepsilon,\Delta}$. We will prove that, if $\varepsilon > 0$ is sufficiently small,

$$\mathcal{O}[\underline{\widehat{\chi}}, \mathrm{tr}\underline{\chi}, \underline{\eta}, \omega, \widehat{\chi}, \Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi}, \eta] \leq C(\mathcal{O}_0, \mathcal{R}_0),\ \mathcal{O}[\underline{\omega}] \leq C(\mathcal{O}_0, \mathcal{R}).$$

This is done in Section 4.2, Proposition 3.

Step 3: Under the conclusion of Step 2, we prove that, for $\varepsilon > 0$ sufficiently small,

$$\mathcal{R}, \underline{\mathcal{R}} \leq C(\mathcal{O}_0, \mathcal{R}_0, \underline{\mathcal{R}}_0).$$

This is done in Section 4.3, Proposition 6.

Step 4: We extend the solution g to $0 \leq \underline{u} \leq \underline{u}_* + \delta$, $0 \leq u \leq \varepsilon + \delta'$ for δ, δ' sufficiently small, such that it holds again $\mathcal{O}, \mathcal{R}, \underline{\mathcal{R}} \leq 2C(\mathcal{O}_0, \mathcal{R}_0, \underline{\mathcal{R}}_0)$. The extension of the space-time follows by [**13**]. We then construct on $\underline{C}_{\underline{u}_*+\delta}$ a new function u_δ for $0 \leq u_\delta \leq \varepsilon$ such that $u_\delta|_{\underline{C}_{\underline{u}_*+\delta}}$ is canonical, provided that $\varepsilon > 0$ and $\delta > 0$ are sufficiently small (δ may depend on ε and δ'). Then, by continuity, if $\delta > 0$ is sufficiently small, the new function u_δ can be extended inside up to $\underline{C}_0$ as an optical function, and we have the new double null foliation $0 \leq \underline{u} \leq \underline{u}_* + \delta$, $0 \leq u_\delta \leq \varepsilon$. In particular, the norms $\mathcal{R}_\delta, \underline{\mathcal{R}}_\delta$ expressed in the new foliation are bounded by some constant depending on $\mathcal{O}_0, \mathcal{R}_0, \underline{\mathcal{R}}_0$.

Now we can choose Δ sufficiently large such that $\underline{u}_* + \delta \in \mathcal{A}_{\varepsilon,\Delta}$ which leads to a contradiction to that $\underline{u}_* = \sup \mathcal{A}_{\varepsilon,\Delta} < +\infty$. Finally, we will show that we can construct a global retarded time function u and complete the proof.

4. Proof of the main theorem

4.1. Canonical foliation on initial null cone. We shall carry out Step 1 of the proof to construct canonical foliation on $\underline{C}_0$ in this subsection.

The construction of a canonical foliation on the last slice is carried out both in [**8**] and [**11**] (or [**3**]). The case for space-like hypersurface is considered in [**8**] and the case for incoming null hypersurface is considered in [**11**]. The full detail of the local existence of the canonical foliation on a null cone is given in [**18**]. For the sake of completeness, and because the setting is not exactly the same in our case, we give a proof here but do not carry out some detailed computations. The reader can also refer to Chapter 3 of [**19**] for the comparison between two different foliations and Chapter 7 for another argument.

We first define the following addition quantities which are needed in the construction of the canonical foliation:

$$\mathcal{R}[\underline{D}\underline{\beta}] = \sup_u \sum_{i=0}^{1} \|(r\nabla\!\!\!\!/\,)^i \underline{D}\underline{\beta}\|_{L^2(C_u)},$$

$$\underline{\mathcal{R}}[\underline{D\alpha}] = \sup_{\underline{u}} \sum_{i=0}^{1} \|(r\nabla\!\!\!\!/\,)^i \underline{D\alpha}\|_{L^2(\underline{C}_{\underline{u}})}, \quad \underline{\mathcal{R}}[\underline{D}^2\underline{\alpha}] = \sup_{\underline{u}} \|\underline{D}^2\underline{\alpha}\|_{L^2(\underline{C}_{\underline{u}})},$$

$$\mathcal{O}^2[\underline{D}\omega] = \sup_{\underline{u},u} \sum_{i=0}^{1} \|(r\nabla\!\!\!/)^i \underline{D}\omega\|_{L^2(S_{\underline{u},u})}, \quad \mathcal{O}[\underline{D}^2\underline{\omega}] = \sup_{\underline{u}} \|\underline{D}^2\underline{\omega}\|_{L^2(\underline{C}_{\underline{u}})}.$$

Now we work on an arbitrary incoming null cone $\underline{C}_{\underline{u}}$ with a background foliation given by a function $\lambda \geq 0$ and $\lambda = 0$ on $S_{\underline{u},0}$. We assume that on this null cone, we have:

$$\mathcal{O}, \underline{\mathcal{R}}, \underline{\mathcal{R}}[\underline{D}\alpha, \underline{D}^2\underline{\alpha}], \mathcal{O}^2[\underline{D}\omega], \mathcal{O}[\underline{D}^2\underline{\omega}] \leq C, \tag{4.1}$$

for some C. The norms above should be understood as the norm taken only on one single null cone $\underline{C}_{\underline{u}}$.

To define a new foliation, we use a function $W(s,\theta)$ defined on $[0,\varepsilon] \times S_{\underline{u},0}$ to represent a new foliation in the way that, the new foliation function ${}^{(W)}u$ is defined by the relation ${}^{(W)}u(W(s,\theta),\theta) = s$. Moreover, we fix a coordinate system (θ^1,θ^2) on $S_{\underline{u},0}$ and carry them to $\underline{C}_0$ by the null geodesic generators. Then each point on $\underline{C}_{\underline{u}}$ has the same θ^A coordinates in both the background foliation and the foliation given by ${}^{(W)}u$. Under the new foliation, we have also the new "lapse" function ${}^{(W)}\Omega$. W represents a foliation iff

$${}^{(W)}a(s,\theta) \triangleq \frac{\partial W}{\partial s}(s,\theta) = {}^{(W)}\Omega^2(W(s,\theta),\theta)\Omega^{-2}(W(s,\theta),\theta) > 0.$$

Under the new foliation, the following vectorfields

$$\begin{aligned} {}^{(W)}\underline{L} &= {}^{(W)}a\underline{L}, \\ {}^{(W)}L' &= \frac{1}{{}^{(W)}a}(L' + |\nabla\!\!\!/ W|^2\underline{L} + 2\nabla\!\!\!/^A W E_A), \\ {}^{(W)}E_A &= E_A + \nabla\!\!\!/_A W\underline{L}, \end{aligned} \tag{4.2}$$

also form a null frame with $g({}^{(W)}\underline{L}, {}^{(W)}\underline{L}) = g({}^{(W)}L', {}^{(W)}L') = 0$, $g({}^{(W)}\underline{L}, {}^{(W)}L') = -2$, and $[{}^{(W)}\underline{L}, {}^{(W)}E_A] = 0$ if $[\underline{L}, E_A] = 0$. In addition, ${}^{(W)}\underline{L}{}^{(W)}u = 1$. Here $\nabla\!\!\!/_A W = \partial_{\theta^A} W$. We can also compute the relation of the connection coefficients and curvature components in different foliations:

$$\begin{aligned} {}^{(W)}\Omega^{(W)}\chi =& \Omega^2[\chi' + 2\eta \otimes \nabla\!\!\!/ W + 2\nabla\!\!\!/ W \otimes \eta \\ & - 2(\Omega\underline{\widehat{\chi}} \cdot \nabla\!\!\!/ W) \otimes \nabla\!\!\!/ W - 2\nabla\!\!\!/ W \otimes (\Omega\underline{\widehat{\chi}} \cdot \nabla\!\!\!/ W) \\ & + 2\underline{\omega}\nabla\!\!\!/ W \otimes \nabla\!\!\!/ W - |\nabla\!\!\!/ W|^2\Omega\underline{\chi} + 2\nabla\!\!\!/^2 W], \\ {}^{(W)}\Omega^{-1(W)}\underline{\chi} =& \Omega^{-1}\underline{\chi}, \\ {}^{(W)}\underline{\eta} =& \underline{\eta} + \Omega\underline{\chi} \cdot \nabla\!\!\!/ W, \\ {}^{(W)}\rho =& \rho - 2(\nabla\!\!\!/ W, \underline{\beta}) + \underline{\alpha}(\nabla\!\!\!/ W, \nabla\!\!\!/ W). \end{aligned}$$

In the first formula, $\nabla\!\!\!/^2_{AB} W = \partial_{\theta^A}\partial_{\theta^B} W - \Gamma\!\!\!/^C_{AB}\partial_{\theta^C} W$ where $\Gamma\!\!\!/^C_{AB}$ are the Christoffel symbols of the spherical sections relative to the background foliation at $(W(s,\theta),\theta)$.

Suppose now that W represents a foliation. We consider a map $\mathcal{A}$, which is defined by

$$\mathcal{A}(W)(s,\theta)=\int_0^s {}^{(W_{\mathcal{A}})}\Omega^2(W(s',\theta),\theta)\Omega^{-2}(W(s',\theta),\theta)\mathrm{d}s',$$

where ${}^{(W_{\mathcal{A}})}\Omega$ is the solution of the equation

$$\begin{aligned}{}^{(W)}\Delta\!\!\!/ \log{}^{(W_{\mathcal{A}})}\Omega(W(s,\theta),\theta)&={}^{(W)}G(W(s,\theta),\theta),\\ {}^{(W)}\overline{\log{}^{(W_{\mathcal{A}})}\Omega}(s)&=0,\end{aligned}$$

where

$$\begin{aligned}{}^{(W)}G\triangleq&\frac{1}{2}{}^{(W)}\mathrm{di}\!\!/\mathrm{v}\,{}^{(W)}\underline{\eta}\\ &+\frac{1}{2}\left(\frac{1}{2}(({}^{(W)}\widehat{\chi},{}^{(W)}\underline{\widehat{\chi}})-{}^{(W)}\overline{({}^{(W)}\widehat{\chi},{}^{(W)}\underline{\widehat{\chi}})})-({}^{(W)}\rho-{}^{(W)}\overline{{}^{(W)}\rho})\right).\end{aligned}$$

We can express G in terms of geometric quantities related to background foliation and derivatives of W. We can compute (in a schematic form):

$$\begin{aligned}{}^{(W)}\mathrm{di}\!\!/\mathrm{v}\,{}^{(W)}\underline{\eta}=&\mathrm{di}\!\!/\mathrm{v}\,\underline{\eta}+\nabla\!\!\!/ W\cdot\Omega\underline{\chi}\cdot\underline{\eta}+\nabla\!\!\!/ W\cdot\underline{D}\underline{\eta}\\ &+\nabla\!\!\!/ W\cdot\nabla\!\!\!/(\Omega\underline{\chi})+\Omega\underline{\chi}\cdot\nabla\!\!\!/^2W+\nabla\!\!\!/ W\cdot\Omega\underline{\chi}\cdot\Omega\underline{\chi}\\ &+\nabla\!\!\!/ W\cdot\nabla\!\!\!/ W\cdot\underline{D}(\Omega\underline{\chi}),\\ ({}^{(W)}\widehat{\chi},{}^{(W)}\underline{\widehat{\chi}})=&(\widehat{\chi},\underline{\widehat{\chi}})-\Omega^2|\nabla\!\!\!/ W|^2|\underline{\widehat{\chi}}|^2\\ &+\Omega\underline{\widehat{\chi}}\cdot(\eta\cdot\nabla\!\!\!/ W+\underline{\omega}\nabla\!\!\!/ W\cdot\nabla\!\!\!/ W+\nabla\!\!\!/^2W),\\ {}^{(W)}\rho=&\rho-2\nabla\!\!\!/ W\cdot\underline{\beta}+\underline{\alpha}\cdot\nabla\!\!\!/ W\cdot\nabla\!\!\!/ W.\end{aligned}$$

For W which does not represent a foliation (but only a family of sections), the right hand sides of the above inequalities still make sense. Therefore, we can extend the map $\mathcal{A}$ to the case that W does not represent a foliation. Note that if W is a fixed point of the map $\mathcal{A}$, then we can estimate $\log\Omega$ directly and conclude that W represents a foliation, then the function ${}^{(W)}u$ given by W is canonical (see equation (3.2)).

We first carry out some calculations and then consider the behavior of $\mathcal{A}$ in suitable chosen function spaces.

For any W, we denote $S_{W(s)}$ or simply S_W to be the sections. We first compare two different families of sections W_1 and W_2. For fixed s, we consider a family of sections $W_t\triangleq tW_1(s,\theta)+(1-t)W_2(s,\theta)$ where $t\in[0,1]$. We will compute both sides of

$$\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{(W_t)}\Delta\!\!\!/\log{}^{((W_t)_{\mathcal{A}})}\Omega\right)=\frac{\mathrm{d}}{\mathrm{d}t}{}^{(W_t)}G.$$

We first compute the right hand side $\frac{\mathrm{d}}{\mathrm{d}t}{}^{(W_t)}G$. We denote $\widetilde{W} = W_1 - W_2$, then by the expression of ${}^{(W_t)}\rho$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}{}^{(W_t)}\rho =& \widetilde{W}\underline{D}\rho + 2\nabla\!\!\!/\,\widetilde{W}\cdot\underline{\beta} + 2\widetilde{W}\nabla\!\!\!/\,W_t\cdot\underline{D}\beta^\sharp \\ &+ 2\underline{\alpha}\cdot\nabla\!\!\!/\,\widetilde{W}\cdot\nabla\!\!\!/\,W_t + \widetilde{W}\underline{D\alpha}^\sharp\cdot\nabla\!\!\!/\,W_t\cdot\nabla\!\!\!/\,W_t.\end{aligned}$$

Note that we use $\underline{\beta}^\sharp$ and $\underline{\alpha}^\sharp$ to denote the corresponding contravariant tensorfields to $\underline{\beta}$ and $\underline{\alpha}$. The derivative $\underline{D}$ also applies to the metric $g\!\!\!/$. The addition term $\Omega\underline{\chi}$ arises, but since $\Omega\underline{\chi}$ is in L^∞, we do not need to care about it.

We also compute

$$\frac{\mathrm{d}}{\mathrm{d}t}{}^{(W_t)}\overline{{}^{(W_t)}\rho} = -{}^{(W_t)}\overline{\widetilde{W}\Omega\mathrm{tr}\underline{\chi}}{}^{(W_t)}\overline{{}^{(W_t)}\rho} + {}^{(W_t)}\overline{\frac{\mathrm{d}}{\mathrm{d}t}{}^{(W_t)}\rho + \widetilde{W}\Omega\mathrm{tr}\underline{\chi}{}^{(W_t)}\rho}.$$

The other terms of $\frac{\mathrm{d}}{\mathrm{d}t}{}^{(W_t)}G$ are computed in a similar way.

We then compute the left hand side

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left({}^{(W_t)}\Delta\!\!\!\!/\,\log{}^{((W_t)\mathcal{A})}\Omega\right)\\ =&{}^{(W_t)}\Delta\!\!\!\!/\left(\frac{\mathrm{d}}{\mathrm{d}t}\log{}^{((W_t)\mathcal{A})}\Omega\right) - \widetilde{W}\Omega\mathrm{tr}\underline{\chi}{}^{(W_t)}\Delta\!\!\!\!/\,\log{}^{((W_t)\mathcal{A})}\Omega\\ &- 2\widetilde{W}\Omega\underline{\widehat{\chi}}\cdot{}^{(W_t)}\nabla\!\!\!/^2\log{}^{((W_t)\mathcal{A})}\Omega - 2{}^{(W_t)}\mathrm{di}\!\!\!/\mathrm{v}\,(\widetilde{W}\Omega\underline{\widehat{\chi}})\cdot{}^{(W_t)}\nabla\!\!\!/\,\log{}^{((W_t)\mathcal{A})}\Omega\\ \triangleq&{}^{(W_t)}\Delta\!\!\!\!/\left(\frac{\mathrm{d}}{\mathrm{d}t}\log{}^{((W_t)\mathcal{A})}\Omega\right) - {}^{(W_t)}G_1.\end{aligned}$$

In the above expression, since ${}^{(W_t)}\Delta\!\!\!\!/\,\log{}^{((W_t)\mathcal{A})}\Omega = {}^{(W_t)}G$, we can estimate ${}^{(W_t)}\nabla\!\!\!/^2\log{}^{((W_t)\mathcal{A})}\Omega$ in $L^2(S_{W_t})$, by $\|{}^{(W_t)}G\|_{L^2(S_{W_t})}$, if we assume that for all W_t,

$$ {}^{(W)}r\|{}^{(W)}K\|_{L^2(S_{W_t})} \leq C, \tag{4.3}$$

and then the elliptic estimate applies. The expression ${}^{(W_t)}\mathrm{di}\!\!\!/\mathrm{v}\,(\widetilde{W}\Omega\underline{\widehat{\chi}})$ is of the form $\nabla\!\!\!/\,\widetilde{W}\cdot\Omega\underline{\widehat{\chi}}+\widetilde{W}\cdot(\nabla\!\!\!/\,(\Omega\underline{\widehat{\chi}})+\nabla\!\!\!/\,W_t\cdot\underline{D}(\Omega\underline{\widehat{\chi}}))$, and ${}^{(W_t)}\nabla\!\!\!/\,\log{}^{((W_t)\mathcal{A})}\Omega$ is also controlled in $L^p(S_{W_t})$ for all $p\in[2,+\infty)$, by $\|{}^{(W_t)}G\|_{L^2(S_{W_t})}$.

Also, we compute, by ${}^{(W_t)}\overline{\log{}^{((W_t)\mathcal{A})}\Omega} = 0$,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}{}^{(W_t)}\overline{\log{}^{((W_t)\mathcal{A})}\Omega} = {}^{(W_t)}\overline{\frac{\mathrm{d}}{\mathrm{d}t}\log{}^{((W_t)\mathcal{A})}\Omega + \widetilde{W}\Omega\mathrm{tr}\underline{\chi}\log{}^{((W_t)\mathcal{A})}\Omega}.$$

Therefore,

$$\begin{aligned}{}^{(W_t)}\Delta\!\!\!\!/\left(\frac{\mathrm{d}}{\mathrm{d}t}\log{}^{((W_t)\mathcal{A})}\Omega\right) =& \frac{\mathrm{d}}{\mathrm{d}t}{}^{(W_t)}G + {}^{(W_t)}G_1,\\ {}^{(W_t)}\overline{\frac{\mathrm{d}}{\mathrm{d}t}\log{}^{((W_t)\mathcal{A})}\Omega} =& -{}^{(W_t)}\overline{\widetilde{W}\Omega\mathrm{tr}\underline{\chi}\log{}^{((W_t)\mathcal{A})}\Omega},\end{aligned}$$

and we conclude by elliptic estimate that

$$\begin{aligned}
&\left\|{}^{(W_t)}(r\nabla\!\!\!/)^{\leq 2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\log{}^{((W_t)\mathcal{A})}\Omega\right)\right\|_{L^2(S_{W_t})}\\
\leq & C'\|{}^{(W_t)}r^2(\frac{\mathrm{d}}{\mathrm{d}t}{}^{(W_t)}G+{}^{(W_t)}G_1)\|_{L^2(S_{W_t})}\\
&+\|{}^{(W_t)}r^{2}\,{}^{(W_t)}\overline{\widetilde{W}\Omega\mathrm{tr}\underline{\chi}\log{}^{((W_t)\mathcal{A})}\Omega}\|_{L^2(S_{W_t})}.
\end{aligned}$$

Here the constant C' depends on the constants in the assumptions (4.1), (4.3) and the following: for $i=0,1,2$, any W and any function f, we assume

$$\begin{aligned}
&C^{-1}\|(r\nabla\!\!\!/)^i f(W(t,\cdot),\cdot)\|_{L^2(S_{\underline{u},0})}\\
&\leq\|{}^{(W)}(r\nabla\!\!\!/)^i f\|_{L^2(S_{W_t})}\leq C\|(r\nabla\!\!\!/)^i f(W(t,\cdot),\cdot)\|_{L^2(S_{\underline{u},0})}.
\end{aligned}\tag{4.4}$$

So, to compare two different foliations,

$$\begin{aligned}
&\|(r\nabla\!\!\!/)^{\leq 2}(\log{}^{((W_1)\mathcal{A})}\Omega(W_1(s,\cdot),\cdot)-\log{}^{((W_2)\mathcal{A})}\Omega(W_2(s,\cdot),\cdot))\|_{L^2(S_{\underline{u},0})}\\
=&\left\|(r\nabla\!\!\!/)^{\leq 2}\left(\int_0^1\frac{\mathrm{d}}{\mathrm{d}t}\log{}^{((W_t)\mathcal{A})}\Omega(W_t(\cdot),\cdot)\mathrm{d}t\right)\right\|_{L^2(S_{\underline{u},0})}\\
\leq & C'\int_0^1\left\|{}^{(W_t)}(r\nabla\!\!\!/)^{\leq 2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\log{}^{((W_t)\mathcal{A})}\Omega(W_t(\cdot),\cdot)\right)\right\|_{L^2(S_{W_t})}\mathrm{d}t\\
\leq & C'\int_0^1\Big(\|{}^{(W_t)}r^2(\frac{\mathrm{d}}{\mathrm{d}t}{}^{(W_t)}G+{}^{(W_t)}G_1)\|_{L^2(S_{W_t})}\\
&\qquad+\|{}^{(W_t)}r^{2}\,{}^{(W_t)}\overline{\widetilde{W}\Omega\mathrm{tr}\underline{\chi}\log{}^{((W_t)\mathcal{A})}\Omega}\|_{L^2(S_{W_t})}\Big)\,\mathrm{d}t.
\end{aligned}$$

We estimate

$$\begin{aligned}
&\|{}^{(W_t)}r^2\frac{\mathrm{d}}{\mathrm{d}t}{}^{(W_t)}G\|_{L^2(S_{W_t})}\\
\leq & C'|{}^{(W_t)}r\widetilde{W}|\Big(\|r\underline{D}(\mathrm{div}\!\!\!/\,\underline{\eta},(\widehat{\chi},\widehat{\underline{\chi}}),\rho)\|_{L^2(S_{(W_t)})}\\
&+\|r\Omega\mathrm{tr}\underline{\chi}(\mathrm{div}\!\!\!/\,\underline{\eta},(\widehat{\chi},\widehat{\underline{\chi}}),\rho)\|_{L^2(S_{(W_t)})}\\
&+\|{}^{(W_t)}r^{1/2}\nabla\!\!\!/ W_t\|_{L^4(S_{(W_t)})}\\
&\quad\times\|r^{3/2}(\underline{D}(\Omega\underline{\chi}\cdot\eta,\underline{D}\underline{\eta},\nabla\!\!\!/(\Omega\underline{\chi}),\Omega\underline{\chi}\cdot\Omega\underline{\chi},\Omega\underline{\chi}\cdot\eta,\underline{\beta})^\sharp,\\
&\quad\ r^{3/2}\nabla\!\!\!/(\Omega\underline{\chi}))\|_{L^4(S_{(W_t)})}\\
&+\|{}^{(W_t)}r^{3/2}|\nabla\!\!\!/ W_t|^2\|_{L^4(S_{(W_t)})}\\
&\quad\times\|r^{1/2}\underline{D}(\underline{D}(\Omega\underline{\chi}),|\Omega\widehat{\underline{\chi}}|^2,\Omega\underline{\chi}\underline{\omega},\underline{\alpha},\Omega\underline{\chi})^\sharp\|_{L^4(S_{(W_t)})}\\
&+\|{}^{(W_t)}r\nabla\!\!\!/^2W_t\|_{L^2(S_{(W_t)})}\|r\underline{D}(\Omega\underline{\chi})^\sharp\|_{L^\infty(S_{(W_t)})}\Big)\\
&+\|{}^{(W_t)}r^{3/2}\nabla\!\!\!/\widetilde{W}\|_{L^4(S_{(W_t)})}
\end{aligned}\tag{4.5}$$

$$
\begin{aligned}
&\times \|r^{1/2}(\Omega\underline{\chi}\cdot\eta, \underline{D}\eta, \nabla\!\!\!/(\Omega\underline{\chi}), \Omega\underline{\chi}\cdot\eta, \underline{\beta})\|_{L^4(S_{(W_t)})}\\
&+\|^{(W_t)}r^{7/4}\nabla\!\!\!/\widetilde{W}\|_{L^8(S_{(W_t)})}\|^{(W_t)}r^{3/4}\nabla\!\!\!/W_t\|_{L^8(S_{(W_t)})}\\
&\times \|r^{1/2}(\underline{D}(\Omega\underline{\chi}), |\Omega\widehat{\underline{\chi}}|^2, \Omega\underline{\chi}\underline{\omega}, \underline{\alpha}, \Omega\underline{\chi})\|_{L^4(S_{(W_t)})}\\
&+\|^{(W_t)}r^2\nabla\!\!\!/^2\widetilde{W}\|_{L^2(S_{(W_t)})}\|\Omega\underline{\chi}\|_{L^\infty(S_{(W_t)})}.
\end{aligned}
$$

We need to estimate the norms of the background quantities on the sphere $S_{(W_t)}$. For arbitrary section given by a function $Y = Y(\theta) \geq 0$, we consider a family of sections $Y_t(\theta) = tY(\theta)$ which connects the sphere S_Y and $S_{\underline{u},0}$. For an arbitrary tensorfield ψ, we compute

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{S_{Y_t}}|\psi|^4\mathrm{d}\mu_t &= \int_{S_{Y_t}} Y(\underline{D}(|\psi|^4)+\Omega\mathrm{tr}\underline{\chi}|\psi|^4)\mathrm{d}\mu_t\\
&\lesssim \int_{S_{Y_t}} Y|\psi|^4\mathrm{d}\mu_t + \int_{S_{Y_t}} Y|\psi|^3|\underline{D}\psi|\mathrm{d}\mu_t.
\end{aligned}
$$

We integrate the above inequality over $[0,1]$. Now, if we assume

$$
W(s,\theta) \leq \varepsilon_W, \tag{4.6}
$$

then we can choose ε_W small enough so that

$$
\begin{aligned}
&\int_{S_Y}|\psi|^4\mathrm{d}\mu\\
\lesssim&\int_{S_{\underline{u},0}}|\psi|^4\mathrm{d}\mu_0\\
&+\left(\int_0^1\int_{S_{Y_t}}Y|\psi|^6\mathrm{d}\mu_t\mathrm{d}t\right)^{1/2}\left(\int_0^1\int_{S_{Y_t}}Y|\underline{D}\psi|^2\mathrm{d}\mu_t\mathrm{d}t\right)^{1/2}\\
\lesssim&\|\psi\|^4_{L^4(S_{\underline{u},0})}+\|\psi\|^3_{L^6(\underline{C}_{\underline{u}})}\|\underline{D}\psi\|_{L^2(\underline{C}_{\underline{u}})}.
\end{aligned}
$$

The second inequality holds because the volume element $Y\mathrm{d}\mu_t\mathrm{d}t = \mathrm{d}\mu_{S_{\underline{u},u}}\mathrm{d}u$. Therefore, by the Sobolev inequality, we have

$$
\begin{aligned}
&\|\psi\|_{L^4(S_Y)}\\
\lesssim&\|\psi\|_{L^4(S_{\underline{u},0})}\\
&+\left(\|r^{-1/2}(r\nabla\!\!\!/)\psi\|_{L^2(\underline{C}_{\underline{u}})}+\|r^{-1/2}\psi\|_{L^2(\underline{C}_{\underline{u}})}\right)^{3/4}\|\underline{D}\psi\|^{1/4}_{L^2(\underline{C}_{\underline{u}})}.
\end{aligned} \tag{4.7}
$$

Using the above inequality, we can relate the norms of the background quantities on the sphere $S_{(W_t)}$ to the norms on $S_{\underline{u},0}$ and the norms of $\underline{D}$ derivative on $L^2(\underline{C}_{\underline{u}})$. One can check that the norms in (4.1) are enough to control the norms appearing in (4.5). Again if ε is small depending on C_1, we have

$$
\int_0^1\|^{(W_t)}r^2\frac{\mathrm{d}}{\mathrm{d}t}{}^{(W_t)}G\|_{L^2(S_{W_t})}\mathrm{d}t \leq C'\|\nabla\!\!\!/^{\leq 2}(W_1(s,\cdot)-W_2(s,\cdot))\|_{L^2(S_{\underline{u},0})}.
$$

Here C' depends on the constants in the assumptions (4.1), (4.3), (4.4) and the bound $\sup_s \|(r\nabla\!\!\!/)^{\leq 2} W_t(s,\cdot)\|_{L^2(S_{\underline{u},0})}$. The other terms are estimated in a similar way and we have

(4.8)

$$\begin{aligned}&\|(r\nabla\!\!\!/)^{\leq 2}(\log {}^{((W_1)_{\mathcal{A}})}\Omega(W_1(s,\cdot),\cdot) - \log {}^{((W_2)_{\mathcal{A}})}\Omega(W_2(s,\cdot),\cdot))\|_{L^2(S_{\underline{u},0})}\\ \leq & C'\|(r\nabla\!\!\!/)^{\leq 2}(W_1(s,\cdot) - W_2(s,\cdot))\|_{L^2(S_{\underline{u},0})}.\end{aligned}$$

Finally, we have the following:

Proposition 2. *Suppose that on incoming null cone $\underline{C}_{\underline{u}}$ with background foliation given by u, the assumptions (4.1), (4.3), (4.4), (4.6) hold. Then the inequality (4.8) holds for any two families of sections W_1 and W_2, if ε_W is sufficiently small.*

Remark 5. It is easy to see (4.3), (4.4) hold for some C depending on the bound in the assumption (4.1) and $\sup_s \|(r\nabla\!\!\!/)^{\leq 2} W_t(s,\cdot)\|_{L^2(S_{\underline{u},0})}$ if ε is sufficiently small.

We apply the above proposition on $\underline{C}_0$ to prove the existence of the canonical foliation on $\underline{C}_0$ if ε is sufficiently small. Given $\varepsilon > 0$, we define the function space $\mathcal{K}_{0,\varepsilon} \subset C([0,\varepsilon], H^2(S_{0,0}))$ to be the collection of functions W which satisfy

$$\begin{cases} W(0,\theta) = 0, W \geq 0,\\ \sup_s \|(r\nabla\!\!\!/)^{\leq 2} W(s,\cdot)\|_{L^2(S_{0,0})} \leq C_{\mathcal{K}_{0,\varepsilon}}.\end{cases}$$

We will choose $C_{\mathcal{K}_{0,\varepsilon}}$ sufficiently small such that the assumption (4.6) holds for suitable ε_W. Note that now the constant C' above depends on $C(\mathcal{O}_0, \mathcal{R}_0, \underline{\mathcal{R}}_0)$. We first simply take $W_1 = W$ and $W_2 = 0$. Note that the background foliation is the geodesic foliation, so $\Omega \equiv 1$, then by (4.8),

$$\begin{aligned}&\|(r\nabla\!\!\!/)^{\leq 2}(\mathcal{A}(W)(s,\cdot) - \mathcal{A}(0)(s,\cdot))\|_{L^2(S_{0,0})}\\ \lesssim & \int_0^s \|(r\nabla\!\!\!/)^{\leq 2}({}^{(W_{\mathcal{A}})}\Omega^2(W(s',\cdot),\cdot) - {}^{(0_{\mathcal{A}})}\Omega^2(0,\cdot))\|_{L^2(S_{0,0})}\mathrm{d}s\\ \lesssim & \varepsilon C(\mathcal{O}_0, \mathcal{R}_0, \underline{\mathcal{R}}_0, C_{\mathcal{K}_{0,\varepsilon}}) \sup_s \|(r\nabla\!\!\!/)^{\leq 2} W(s,\cdot)\|_{L^2(S_{0,0})}\\ \leq & \frac{1}{2} C_{\mathcal{K}_{0,\varepsilon}}.\end{aligned}$$

We remark that we should use (4.7) again here because we compare the bound of background lapse Ω on the sphere $S_{0,s}$ to the bound on $S_{W(s)}$. Also note that $\mathcal{A}(0)(s,\cdot) = s^{(0_{\mathcal{A}})}\Omega^2(0,\cdot)$, whose contribution to the above inequality can be bounded by $\frac{1}{4}C_{\mathcal{K}_{0,\varepsilon}}$ by choosing ε sufficiently small. Now we have verified $\mathcal{A}(\mathcal{K}_{0,\varepsilon}) \subset \mathcal{K}_{0,\varepsilon}$. To see $\mathcal{A} : \mathcal{K}_{0,\varepsilon} \to \mathcal{K}_{0,\varepsilon}$ is a contraction, we only need to apply (4.8) again for any two W_1 and W_2.

We then complete Step 1 of the proof.

4.2. Estimates for the connection coefficients. In this subsection, we derive estimates for the connection coefficients in terms of the curvature components and complete Step 2 of the proof. Remember that we work in the space-time with the double null coordinate $(\underline{u}, u)$ with $(\underline{u}, u) \in [0, \underline{u}_*] \times [0, \varepsilon]$, and u is canonical on $\underline{C}_{\underline{u}_*}$. Precisely, we will establish the following proposition.

Proposition 3. *Assume that $\mathcal{R}, \underline{\mathcal{R}} < \infty$. Then we have*

$$\mathcal{O}[\underline{\widehat{\chi}}, \mathrm{tr}\underline{\chi}, \underline{\eta}, \omega, \widehat{\chi}, \mathrm{tr}\chi, \Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi}, \eta] \leq C(\mathcal{O}_0, \mathcal{R}_0),\ \mathcal{O}[\underline{\omega}] \leq C(\mathcal{O}_0, \mathcal{R}).$$

We will prove this proposition in this subsection. We first introduce:
Bootstrap assumption (1): $\mathcal{O} \leq \Delta_1$,
Bootstrap assumption (2):

$$\sup_u \|(r\nabla\!\!\!\!/\,)^3(\underline{\eta}, \widehat{\chi}, r\omega), (r\nabla\!\!\!\!/\,)^2(r\underline{\mu}), (r\nabla\!\!\!\!/\,)(r^3\kappa)\|_{L^2(C_u)} \leq \Delta_2,$$

$$\sup_{\underline{u}} \|(r\nabla\!\!\!\!/\,)^3(r^{1/2}\eta, \underline{\widehat{\chi}}, \underline{\omega}), (r\nabla\!\!\!\!/\,)^2(r^{3/2}\mu), (r\nabla\!\!\!\!/\,)(r^2\underline{\kappa})\|_{L^2(\underline{C}_{\underline{u}})} \leq \Delta_2,$$

$$\sup_{\underline{u}, u} \|(r\nabla\!\!\!\!/\,)^3(r\mathrm{tr}\chi, \mathrm{tr}\underline{\chi})\|_{L^2(S_{\underline{u},u})} \leq \Delta_2.$$

We will prove the following two propositions. Then Proposition 3 follows directly by a bootstrap argument.

Proposition 4. *Assume that $\mathcal{R}, \underline{\mathcal{R}} < \infty$ and the bootstrap assumptions (1)(2) hold (in fact, we use only the bootstrap assumption (1) and $\sup_{\underline{u}} \|(r\nabla\!\!\!\!/\,)^3(r^{1/2}\eta, \underline{\omega})\|_{L^2(\underline{C}_{\underline{u}})} \leq \Delta_2$). Then for ε sufficiently small, we have*

$$\mathcal{O}[\underline{\widehat{\chi}}, \mathrm{tr}\underline{\chi}, \underline{\eta}, \omega, \widehat{\chi}, \mathrm{tr}\chi, \Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi}, \eta] \leq C(\mathcal{O}_0, \mathcal{R}_0),\ \mathcal{O}[\underline{\omega}] \leq C(\mathcal{O}_0, \mathcal{R}).$$

In particular, we can choose Δ_1 sufficiently large such that $\mathcal{O} \leq \frac{1}{2}\Delta_1$.

Proposition 5. *Assume that $\mathcal{R}, \underline{\mathcal{R}} < \infty$ and the bootstrap assumptions (1)(2) hold. Then for ε sufficiently small, and Δ_2 sufficiently large, the norms in the bootstrap assumption (2) are bounded by $\frac{1}{2}\Delta_2$.*

Proof of Proposition 4. We first establish the following lemma, which says that the geometric quantities share the same estimates up to a multiple by Ω and we need not to distinguish $\underline{u}$ and r:

Lemma 7. *If $\varepsilon > 0$ is sufficiently small depending on Δ_1, Δ_2, then*

$$\|(r\nabla\!\!\!\!/\,)^{\leq 1}\log\Omega\|_{L^\infty(S_{\underline{u},u})} + \|r^{-1/2}(r\nabla\!\!\!\!/\,)^2\log\Omega\|_{L^4(S_{\underline{u},u})}$$
$$+ \|r^{-1}(r\nabla\!\!\!\!/\,)^3\log\Omega\|_{L^2(S_{\underline{u},u})} \leq r^{-1}C(\mathcal{O}_0, \mathcal{R}_0).$$

In particular, we have $C(\mathcal{O}_0, \mathcal{R}_0)^{-1} \leq r(\Omega - 1) \leq C(\mathcal{O}_0, \mathcal{R}_0)$. Also, we have

$$C(\mathcal{O}_0)^{-1}(1 + \underline{u}) \leq r \leq C(\mathcal{O}_0)(1 + \underline{u}).$$

Proof. Recall that $u|_{\underline{C}_{\underline{u}_*}}$ is canonical. That is, on $S_{\underline{u}_*,0}$, Ω satisfies the equation

$$\overline{\log\Omega}=0,\quad \Delta\!\!\!/\ \log\Omega=\frac{1}{2}\mathrm{div}\!\!\!/\,\underline{\eta}+\frac{1}{2}\left(\frac{1}{2}((\widehat{\chi},\underline{\widehat{\chi}})-\overline{(\widehat{\chi},\underline{\widehat{\chi}})})-(\rho-\overline{\rho})\right).$$

The expression on the right hand side is invariant, in particular, does not depend on Ω. Therefore, by L^2 elliptic estimate and Sobolev inequalities (which are valid because we are working on initial null cone C_0), we have

$$\begin{aligned}&\|(r\nabla\!\!\!/)^{\leq 1}\log\Omega\|_{L^\infty(S_{\underline{u}_*,0})}+\|r^{-1/2}(r\nabla\!\!\!/)^2\log\Omega\|_{L^4(S_{\underline{u}_*,0})}\\&+\|r^{-1}(r\nabla\!\!\!/)^3\log\Omega\|_{L^2(S_{\underline{u}_*,0})}\leq r^{-1}|_{S_{\underline{u}_*,0}}C(\mathcal{O}_0,\mathcal{R}_0).\end{aligned}$$

Because we do not change the foliation $\underline{u}$ on C_0, Ω is extended as a constant along every null generator of C_0. The above estimates then hold along the whole C_0.

We introduce an auxiliary bootstrap assumption: $(4C_r)^{-1}(1+\underline{u})\leq r\leq 4C_r(1+\underline{u})$. Then the conclusion about $\|\log\Omega\|_{L^\infty(S_{\underline{u},u})}$ follows by integrating the equation $\underline{D}\log\Omega=\underline{\omega}$:

$$|\log\Omega|\leq|\log\Omega|_{S_{\underline{u},0}}|+\int_0^u|\underline{\omega}|\mathrm{d}u'\leq r^{-1}|_{S_{\underline{u}_*,0}}C(\mathcal{O}_0,\mathcal{R}_0)+\varepsilon(\min_{\underline{C}_{\underline{u}}}r)^{-1}\Delta_1,$$

and then multiplying both sides by r and choosing ε sufficiently small.

We now go to the estimate on r. This follows directly from the equation $\underline{D}r=r\overline{\Omega\mathrm{tr}\underline{\chi}}/2$ and the estimate on Ω derived above. Therefore, the auxiliary bootstrap assumption above can be improved and actually holds. The equation $\underline{D}r=r\overline{\Omega\mathrm{tr}\underline{\chi}}/2$ follows from the variation of area:

$$\underline{D}\int_{S_{\underline{u},u}}\mathrm{d}\mu_{g\!\!\!/}=\int_{S_{\underline{u},u}}\Omega\mathrm{tr}\underline{\chi}\mathrm{d}\mu_{g\!\!\!/}.$$

We then go back to $\log\Omega$ and its derivatives. It is obvious now the conclusion about $\log\Omega$ is true because the auxiliary bootstrap assumption is, in fact, true. We can then commute $\nabla\!\!\!/$ three times to the equation $\underline{D}\log\Omega=\underline{\omega}$ and then apply the Gronwall type inequality to obtain the conclusion. q.e.d.

The bootstrap assumption (1), the above lemma together with the bounds of $\Lambda(s)$, $\lambda(s)$ ensure the validity in the space-time of the basic inequalities, and the conclusions of Lemma 3, Lemma 4 and Lemma 5 in Section 2.4, which are used frequently. The constants will depend on $\mathcal{O}_0$ if ε is sufficiently small. We will not prove them here. They can be found in Chapter 4, 5 and 7 in [7] or in [**13**] (under a slightly stronger assumption).

Now we turn to the estimates for the connection coefficients. As in the first step, we consider the structure equations for $\underline{D}\Gamma$ where $\Gamma\in$

$\{\underline{\widehat{\chi}}', \mathrm{tr}\underline{\chi}', \underline{\eta}, \omega\}$, which can be written in the following form:

$$\underline{D}\Gamma_p = \underline{R}_{p'} + \sum \Gamma_{p_1}\Gamma_{p_2},$$

where $\underline{R} \in \{\rho, \underline{\beta}, \underline{\alpha}\}$. Note that in the above equations, $p \leq \min\{p', p_1 + p_2\}$. Then applying the Gronwall type estimates, we have

$$\|r^p\Gamma_p\|_{L^\infty(S_{\underline{u},u})} \lesssim \|r^p\Gamma_p\|_{L^\infty(S_{\underline{u},0})} + \int_0^u \|r^p\underline{R}_{p'}\|_{L^\infty(S_{\underline{u},u'})}\mathrm{d}u' + \int_0^u \|r^p\Gamma_{p_1}\Gamma_{p_2}\|_{L^\infty(S_{\underline{u},u'})}\mathrm{d}u'.$$

Therefore, by applying the Sobolev inequality to the curvature term,

$$\mathcal{O}^{0,\infty}[\underline{\widehat{\chi}}, \mathrm{tr}\underline{\chi}, \underline{\eta}, \omega] \lesssim \mathcal{O}_0 + \varepsilon^{1/2}\underline{\mathcal{R}} + \varepsilon(\Delta_1)^2,$$

which implies that for ε sufficiently small, $\mathcal{O}^{0,\infty}[\underline{\widehat{\chi}}, \mathrm{tr}\underline{\chi}, \underline{\eta}, \omega] \leq C(\mathcal{O}_0)$.

We then commute $\nabla\!\!\!/$ to the structure equations, for $i \leq 2$

$$\underline{D}\nabla\!\!\!/^{\,i}\Gamma_p = \nabla\!\!\!/^{\,i}\underline{R}_{p'} + \sum \nabla\!\!\!/^{\,i}(\Gamma_{p_1}\Gamma_{p_2}) + \sum_{j=1}^{i} \nabla\!\!\!/^{\,j}(\Omega\underline{\chi})\nabla\!\!\!/^{\,i-j}\Gamma_p.$$

The last term comes from the commutator $[\underline{D}, \nabla\!\!\!/^{\,i}]$, see Lemma 2. We estimate the nonlinear terms on the right in the following way:

$$\begin{aligned}
&\|r^p(r\nabla\!\!\!/)(\Gamma_{p_1}\Gamma_{p_2})\|_{L^4(S_{\underline{u},u})}\\
\lesssim&\|r^{p_1}\Gamma_{p_1}\|_{L^\infty(S_{\underline{u},u})}\|r^{p_2}(r\nabla\!\!\!/)\Gamma_{p_2}\|_{L^4(S_{\underline{u},u})}\\
&+ \|r^{p_2}(r\nabla\!\!\!/)\Gamma_{p_1}\|_{L^4(S_{\underline{u},u})}\|r^{p_2}\Gamma_{p_2}\|_{L^\infty(S_{\underline{u},u})},
\end{aligned}$$

and

$$\begin{aligned}
&\|r^p(r\nabla\!\!\!/)^2(\Gamma_{p_1}\Gamma_{p_2})\|_{L^2(S_{\underline{u},u})}\\
\lesssim&\|r^{p_1}\Gamma_{p_1}\|_{L^\infty(S_{\underline{u},u})}\|r^{p_2}(r\nabla\!\!\!/)^2\Gamma_{p_2}\|_{L^2(S_{\underline{u},u})}\\
&+ \|r^{p_2}(r\nabla\!\!\!/)^2\Gamma_{p_1}\|_{L^2(S_{\underline{u},u})}\|r^{p_2}\Gamma_{p_2}\|_{L^\infty(S_{\underline{u},u})}\\
&+ \|r^{p_1}(r\nabla\!\!\!/)\Gamma_{p_1}\|_{L^4(S_{\underline{u},u})}\|r^{p_2}(r\nabla\!\!\!/)\Gamma_{p_2}\|_{L^4(S_{\underline{u},u})}.
\end{aligned}$$

The term coming from the commutator is treated in the same way. Therefore, by applying the Gronwall type estimates,

$$\begin{aligned}
&\mathcal{O}^{1,4}, \mathcal{O}^{2,2}[\underline{\widehat{\chi}}, \mathrm{tr}\underline{\chi}, \underline{\eta}, \omega]\\
\lesssim&\mathcal{O}_0 + \varepsilon^{1/2}\underline{\mathcal{R}} + \varepsilon(\mathcal{O}^{0,\infty}\mathcal{O}^{2,2} + (\mathcal{O}^{1,4})^2) \leq \mathcal{O}_0 + \varepsilon^{1/2}\underline{\mathcal{R}} + \varepsilon(\Delta_1)^2,
\end{aligned}$$

choosing ε sufficiently small yields $\mathcal{O}^{1,4}, \mathcal{O}^{2,2}[\underline{\widehat{\chi}}, \mathrm{tr}\underline{\chi}, \underline{\eta}, \omega] \leq C(\mathcal{O}_0)$.

We then consider the structure equations for $\underline{D}\Gamma$ for $\Gamma \in \{\Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi}, \Omega\widehat{\chi}, \Omega\mathrm{tr}\chi, \eta\}$, which can be written in the following form:

$$\underline{D}\Gamma_p = \underline{R}_{p'} + \nabla\!\!\!/\,\Gamma''_{p''} + \sum \Gamma_{p_1}\Gamma_{p_2},$$

where $\underline{R} \in \{\rho, \underline{\beta}\}$, $\Gamma'' \in \{\eta, \underline{\omega}\}$ and $p \leq \min\{p', p'', p_1 + p_2\}$. There are no essential differences in the estimate procedure. But note that in order to estimate the bounds in a correct regularity, we need the bound $\|(r\nabla\!\!\!/)^3(r^{1/2}\eta, \underline{\omega})\|_{L^2(\underline{C}_{\underline{u}})}$. Therefore, we have

$$\mathcal{O}[\Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi}, \Omega\widehat{\chi}, \Omega\mathrm{tr}\chi, \eta] \leq C(\mathcal{O}_0).$$

Remark 6. We should make a remark here that the initial norms of the connection coefficients on C_0 are not exactly the same as in the assumptions of Theorem 2, because the optical function u depends on which is the last slice and then the vectorfield L' is not invariant on C_0 in the bootstrap argument. However, the vectorfield L is invariant because the foliation on C_0 does not change. This ensures that the connection coefficients $\underline{\chi}', \underline{\eta}, \Omega\chi$ and the covariant derivative $\nabla\!\!\!/$ do not change and ω keeps zero on C_0. And the difference between the second order derivatives of $\eta = -\underline{\eta} + 2d\!\!\!/\log\Omega$ on C_0 and the one appearing in the assumptions of Theorem 2 is controlled by up to third order derivatives of $\log\Omega$ on C_0.

Finally, we consider the structure equation for $D\underline{\omega}$:

$$D\underline{\omega} = \Omega^2(2(\eta, \underline{\eta}) - |\eta|^2 - \rho).$$

This equation should be integrated initiated from $\underline{C}_{\underline{u}_*}$. Recall the canonical foliation equation on $\underline{C}_{\underline{u}_*}$ (3.2). Since $d\!\!\!/\log\Omega = \frac{1}{2}(\eta + \underline{\eta})$, it can be written in the following form:

$$\mathrm{div}\!\!\!/\,\eta = \frac{1}{2}((\widehat{\chi}, \underline{\widehat{\chi}}) - \overline{(\widehat{\chi}, \underline{\widehat{\chi}})}) - (\rho - \overline{\rho}). \tag{4.9}$$

We commute $\underline{D}$ with the equation for $\mathrm{div}\!\!\!/\,\eta$ above,

$$\begin{aligned}&\mathrm{div}\!\!\!/\,(-\Omega(\underline{\chi}\cdot\eta + \underline{\beta}) + 2d\!\!\!/\underline{\omega}) - 2\mathrm{div}\!\!\!/\,(\Omega\underline{\widehat{\chi}}\cdot\eta) - \Omega\mathrm{tr}\underline{\chi}\mathrm{div}\!\!\!/\,\eta\\ =&\underline{D}(\frac{1}{2}((\widehat{\chi}, \underline{\widehat{\chi}}) - \overline{(\widehat{\chi}, \underline{\widehat{\chi}})}) - (\rho - \overline{\rho})).\end{aligned}$$

We plug in the equations for $\underline{D}(\Omega\widehat{\chi})$, $\underline{D}\underline{\widehat{\chi}}'$ and $\underline{D}\rho$, and denote $\check{\rho} = \rho - \frac{1}{2}(\widehat{\chi}, \underline{\widehat{\chi}})$. We deduce the equation for $\Delta\!\!\!/\,\underline{\omega}$:

$$\begin{aligned}2\Delta\!\!\!/\,\underline{\omega} =&2\mathrm{div}\!\!\!/\,(\Omega\underline{\beta}) + \mathrm{div}\!\!\!/\,(3\Omega\underline{\widehat{\chi}}\cdot\eta + \frac{1}{2}\Omega\mathrm{tr}\underline{\chi}\eta)\\ &+ \Omega\mathrm{tr}\underline{\chi}\mathrm{div}\!\!\!/\,\eta + (F - \overline{F} + \overline{\Omega\mathrm{tr}\underline{\chi}\check{\rho}} - \overline{\Omega\mathrm{tr}\underline{\chi}}\cdot\overline{\check{\rho}}),\end{aligned} \tag{4.10}$$

where

$$F = \frac{3}{2}\Omega\mathrm{tr}\underline{\chi}\check{\rho} - (d\!\!\!/\Omega, \underline{\beta}) + \{(2\eta - \zeta, \underline{\beta}) - \frac{1}{2}(\underline{\widehat{\chi}}, \nabla\!\!\!/\widehat{\otimes}\eta + \eta\widehat{\otimes}\eta) + \frac{1}{4}\mathrm{tr}\chi|\underline{\widehat{\chi}}|^2\}.$$

In addition, $\overline{\underline{\omega}} = -\overline{\Omega\mathrm{tr}\underline{\chi}\log\Omega}$ by $\overline{\log\Omega} = 0$.

We are going to estimate the norm $\|\nabla\!\!\!/^2\underline{\omega}\|_{L^2(S_{\underline{u}_*,u})}$ on the last slice in terms of the norms on C_0 provided that ε is sufficiently small. This is done by considering equation (4.10). We need the following lemma.

Lemma 8. *For $i \leq 1$ and ε sufficiently small,*

$$\|(r\nabla\!\!\!/)^i(r^2\rho, r^2\sigma, r\underline{\beta}, rK)\|_{L^2(S_{\underline{u},u})} \leq C(\mathcal{R}_0, \mathcal{O}_0).$$

Proof of Lemma 8. We consider the null Bianchi equations for $\underline{D}\rho$, $\underline{D}\sigma$ and $\underline{D}\underline{\beta}$ and commute $\nabla\!\!\!/$ with them. We can estimate by the Gronwall type estimates and Sobolev inequalities

$$\begin{aligned}
&\|(r\nabla\!\!\!/)^i(r^2\rho, r^2\sigma, r\underline{\beta})\|_{L^2(S_{\underline{u}_*,u})} \\
\lesssim&\|(r\nabla\!\!\!/)^i(r^2\rho, r^2\sigma, r\underline{\beta})\|_{L^2(S_{\underline{u}_*,0})} \\
&+\mathcal{O}\int_0^u \sum_{i=0}^2 \|(r\nabla\!\!\!/)^i(r^2\rho, r^2\sigma, r\underline{\beta}, \underline{\alpha})\|_{L^2(S_{\underline{u}_*,u'})} \mathrm{d}u' \\
\lesssim&\mathcal{R}_0 + \varepsilon^{1/2}\Delta_1\underline{\mathcal{R}} \lesssim C(\mathcal{R}_0).
\end{aligned}$$

The estimate for K comes from the Gauss equation $K + \frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} - \frac{1}{2}(\widehat{\chi}, \widehat{\underline{\chi}}) = -\rho$. q.e.d.

We compute

$$\begin{aligned}
\int_{S_{\underline{u},u}} |\nabla\!\!\!/\,\underline{\omega}|^2 = -\int_{S_{\underline{u},u}} \Delta\!\!\!/\,\underline{\omega}(\underline{\omega} - \overline{\underline{\omega}}) &\lesssim \|\Delta\!\!\!/\,\underline{\omega}\|_{L^2(S_{\underline{u},u})}\|\underline{\omega} - \overline{\underline{\omega}}\|_{L^2(S_{\underline{u},u})} \\
&\lesssim \|\Delta\!\!\!/\,\underline{\omega}\|_{L^2(S_{\underline{u},u})}\|\nabla\!\!\!/\,\underline{\omega}\|_{L^2(S_{\underline{u},u})}.
\end{aligned}$$

The last inequality holds by the Poincaré inequality. Therefore, by equation (4.10)

$$\begin{aligned}
&\|(r\nabla\!\!\!/)\underline{\omega}\|_{L^2(S_{\underline{u}_*,u})} + \|\underline{\omega}\|_{L^2(S_{\underline{u}_*,u})} \\
\lesssim&\|r^2\Delta\!\!\!/\,\underline{\omega}\|_{L^2(S_{\underline{u}_*,u})} + \|\overline{\underline{\omega}}\|_{L^2(S_{\underline{u}_*,u})} \lesssim C(\mathcal{O}_0, \mathcal{R}_0).
\end{aligned}$$

To derive the estimate for $\nabla\!\!\!/^2\underline{\omega}$, by the elliptic estimate (Lemma 8 and 6), we have

$$\|(r\nabla\!\!\!/)^2\underline{\omega}\|_{L^2(S_{\underline{u}_*,u})} \lesssim C(\mathcal{O}_0, \mathcal{R}_0).$$

Then both $\|r\underline{\omega}\|_{L^\infty(S_{\underline{u}_*,u})}$ and $\|r^{1/2}(r\nabla\!\!\!/)^2\underline{\omega}\|_{L^4(S_{\underline{u}_*,u})}$ are bounded by the Sobolev inequalities.

We apply the Gronwall type estimates to $D\underline{\omega}$, we have

$$\begin{aligned}
\|\underline{\omega}\|_{L^\infty(S_{\underline{u},u})} &\lesssim \|\underline{\omega}\|_{L^\infty(S_{\underline{u}_*,u})} + \int_0^{\underline{u}} \|2(\eta, \underline{\eta}) - |\eta|^2 - \rho\|_{L^\infty(S_{\underline{u}',u})} \mathrm{d}\underline{u}' \\
&\lesssim r^{-1}(\underline{u}, u)C(\mathcal{O}_0, \mathcal{R}_0) + r^{-3/2}(\underline{u}, u)C(\mathcal{O}_0, \mathcal{R}_0, \mathcal{R}).
\end{aligned}$$

We will also commute $\nabla\!\!\!/$ with the equation for $D\underline{\omega}$. By a similar argument, we finally obtain $\mathcal{O}[\underline{\omega}] \leq C(\mathcal{O}_0, \mathcal{R})$. q.e.d.

Proof of Proposition 5. In this proof we will appeal to the Hodge systems introduced in Lemma 1. Note that under the assumption of Proposition 5, the conclusion of Proposition 4 holds and we will make use of it. Now we proceed by considering each component.

Estimate for $\nabla\!\!\!/^3\underline{\eta}$. We commute $\nabla\!\!\!/$ twice with the equation for $\underline{D}\underline{\mu}$ and estimate the right hand side in suitable norm. The nonlinear terms on the right hand side involving only the connection coefficients (before commuting $\nabla\!\!\!/$) are simply bounded by $\mathcal{O}$. The nonlinear terms involving the derivatives of the connection coefficients or curvature components (before commuting $\nabla\!\!\!/$) can be estimated by the Hölder and Sobolev inequalities as

$$\sum_{i=0}^{2}\|\nabla\!\!\!/^{2-i}(\Gamma\cdot\Psi)\|_{L^2(S_{\underline{u},u})}\lesssim\sum_{i\leq 2,j\leq 2}\|\nabla\!\!\!/^{i}\Gamma\|_{L^2(S_{\underline{u},u})}\|\nabla\!\!\!/^{j}\Psi\|_{L^2(S_{\underline{u},u})},$$

where Ψ refers to $\nabla\!\!\!/\Gamma'$ or R, $\underline{R}$. The first factor is bounded by $\mathcal{O}$, and after integrating, the second factor is bounded by $\mathcal{R}$, $\underline{\mathcal{R}}$ or the bootstrap assumption (2).

Precisely, the right hand side of the equation $\underline{D}\nabla\!\!\!/^2\underline{\mu}$, divided into three groups, can be bounded as follows:

$$\begin{aligned}
&I=r^3\|\nabla\!\!\!/^2(-\Omega\mathrm{tr}\underline{\chi}\underline{\mu}-\frac{1}{2}\Omega\mathrm{tr}\underline{\chi}\mu)\|_{L^2(S_{\underline{u},u})}\\
\lesssim& r^{-1}\mathcal{O}[\mathrm{tr}\underline{\chi}]\sum_{i=0}^{2}\left(\|(r\nabla\!\!\!/)^i(r\underline{\mu})\|_{L^2(S_{\underline{u},u})}+r^{-1/2}\|(r\nabla\!\!\!/)^i(r^{3/2}\mu)\|_{L^2(S_{\underline{u},u})}\right),\\
&II=r^3\|\nabla\!\!\!/^2(-\frac{1}{4}\Omega\mathrm{tr}\chi|\widehat{\underline{\chi}}|^2+\frac{1}{2}\Omega\mathrm{tr}\underline{\chi}|\eta|^2)\|_{L^2(S_{\underline{u},u})}\\
\lesssim& r^{-1}C(\mathcal{O}),\\
&III=r^3\|\nabla\!\!\!/^2(\mathrm{d}\!\!\!/\mathrm{iv}\,(2\Omega\widehat{\underline{\chi}}\cdot\underline{\eta}-\Omega\mathrm{tr}\underline{\chi}\eta))\|_{L^2(S_{\underline{u},u})}\\
\lesssim& r^{-2}\mathcal{O}[\underline{\eta}]\|(r\nabla\!\!\!/)^3(\Omega\widehat{\underline{\chi}})\|_{L^2(S_{\underline{u},u})}+r^{-1}\mathcal{O}[\widehat{\underline{\chi}}]\|(r\nabla\!\!\!/)^3\underline{\eta}\|_{L^2(S_{\underline{u},u})}\\
&+r^{-3/2}\mathcal{O}[\mathrm{tr}\underline{\chi}]\|(r\nabla\!\!\!/)^3(r^{1/2}\eta)\|_{L^2(S_{\underline{u},u})}\\
&+r^{-2}\mathcal{O}[\eta]\|(r\nabla\!\!\!/)^3(\Omega\mathrm{tr}\underline{\chi})\|_{L^2(S_{\underline{u},u})}+r^{-2}C(\mathcal{O}).
\end{aligned}$$

Similarly, the commutator can be estimated

$$r^3\|[\underline{D},\nabla\!\!\!/^2]\underline{\mu}\|_{L^2(S_{\underline{u},u})}\lesssim r^{-1}\mathcal{O}[\widehat{\underline{\chi}},\mathrm{tr}\underline{\chi}]\sum_{i=1}^{2}\|(r\nabla\!\!\!/)^i(r\underline{\mu})\|_{L^2(S_{\underline{u},u})}.$$

Then we apply the Gronwall type estimates to the equation for $\underline{D}\nabla\!\!\!/^2\underline{\mu}$,

$$\begin{aligned}
\|(r\nabla\!\!\!/)^2(r\underline{\mu})\|_{L^2(S_{\underline{u},u})}\lesssim&\|(r\nabla\!\!\!/)^2(r\underline{\mu})\|_{L^2(S_{\underline{u},0})}\\
&+\int_0^u(I+II+III+r^3\|[\underline{D},\nabla\!\!\!/^2]\underline{\mu}\|_{L^2(S_{\underline{u},u'})})\mathrm{d}u'.
\end{aligned}$$

We take the square of the above inequality and integrate over $[0,\underline{u}]$. Therefore, we estimate, by the Hölder inequality,

$$\int_0^{\underline{u}}\left[\int_0^u(I+II+r^3\|[\underline{D},\nabla\!\!\!/^2]\underline{\mu}\|_{L^2(S_{\underline{u},u'})})\mathrm{d}u'\right]^2\mathrm{d}\underline{u}'$$

$$\lesssim \mathcal{O}[\mathrm{tr}\underline{\chi}, \widehat{\underline{\chi}}]^2 \varepsilon \sum_{i=0}^{2} \left[\int_0^u r^{-2} \|(r\nabla\!\!\!/)^i (r\underline{\mu})\|^2_{L^2(C'_u)} \mathrm{d}u' \right.$$

$$\left. + \int_0^{\underline{u}} r^{-3} \|(r\nabla\!\!\!/)^i (r^{3/2}\mu)\|^2_{L^2(\underline{C}_{\underline{u}'})} \mathrm{d}\underline{u}' \right] + \varepsilon^2 r^{-1} C(\mathcal{O})$$

$$\lesssim \varepsilon r^{-1} C(\mathcal{O}, \mathcal{R}, \underline{\mathcal{R}}, \Delta_2).$$

III is estimated in a similar way

$$\int_0^{\underline{u}} \left[\int_0^u III \mathrm{d}u' \right]^2 \mathrm{d}\underline{u}'$$

$$\lesssim \mathcal{O}[\underline{\eta}]^2 \varepsilon \int_0^{\underline{u}} r^{-4} \|(r\nabla\!\!\!/)^3 (\Omega\widehat{\underline{\chi}})\|^2_{L^2(\underline{C}_{\underline{u}'})} \mathrm{d}\underline{u}'$$

$$+ \mathcal{O}[\mathrm{tr}\underline{\chi}]^2 \varepsilon \int_0^{\underline{u}} r^{-3} \|(r\nabla\!\!\!/)^3 (r^{1/2}\eta)\|^2_{L^2(\underline{C}_{\underline{u}'})} \mathrm{d}\underline{u}'$$

$$+ \mathcal{O}[\widehat{\underline{\chi}}]^2 \varepsilon \int_0^u r^{-2} \|(r\nabla\!\!\!/)^3 \underline{\eta}\|^2_{L^2(C_{u'})} \mathrm{d}u'$$

$$+ \mathcal{O}[\eta]^2 \varepsilon \int_0^{\underline{u}} \int_0^u r^{-4} \|(r\nabla\!\!\!/)^3 (\Omega\mathrm{tr}\underline{\chi})\|^2_{L^2(S_{\underline{u}',u'})} \mathrm{d}u' \mathrm{d}\underline{u}'$$

$$+ \varepsilon r^{-3} C(\mathcal{O})$$

$$\lesssim \varepsilon r^{-2} C(\mathcal{O}, \mathcal{R}, \underline{\mathcal{R}}, \Delta_2).$$

Therefore, for ε sufficiently small,

$$\|(r\nabla\!\!\!/)^2 (r\underline{\mu})\|_{L^2(C_u)} \lesssim C(\mathcal{O}_0, \mathcal{R}_0).$$

By the div-curl system for $\underline{\eta}$ we have

$$\|(r\nabla\!\!\!/)^3 \underline{\eta}\|_{L^2(S_{\underline{u},u})}$$

$$\lesssim \|(r\nabla\!\!\!/)^2 (r\rho)\|_{L^2(S_{\underline{u},u})} + \|(r\nabla\!\!\!/)^2 (r\underline{\mu})\|_{L^2(S_{\underline{u},u})} + r^{-1} C(\mathcal{O}).$$

The last term comes from estimating the lower order terms by Proposition 4. Consequently,

$$\|(r\nabla\!\!\!/)^3 \underline{\eta}\|_{L^2(C_u)} \lesssim C(\mathcal{O}_0, \mathcal{R}, \mathcal{R}_0).$$

The estimates for $\nabla\!\!\!/^3 \underline{\chi}$ and $\nabla\!\!\!/^3 \omega$ are almost the same and we only sketch it. In fact, the estimates for $\nabla\!\!\!/^3 \underline{\eta}$, $\nabla\!\!\!/^3 \underline{\chi}$ and $\nabla\!\!\!/^3 \underline{\omega}$ can be done simultaneously.

Estimates for $\nabla\!\!\!/^3 \widehat{\underline{\chi}}$ and $\nabla\!\!\!/^3 \mathrm{tr}\underline{\chi}$. We commute $\nabla\!\!\!/^3$ with the equation for $\underline{D}(\Omega\mathrm{tr}\underline{\chi})$ and we estimate the right hand side as follows,

$$\int_0^u \|(r\nabla\!\!\!/)^3 (-\frac{1}{2}(\Omega\mathrm{tr}\underline{\chi})^2 - \frac{1}{2}|\Omega\widehat{\underline{\chi}}|^2 + 2\underline{\omega}(\Omega\mathrm{tr}\underline{\chi}))$$

$$+ [\underline{D}, \nabla\!\!\!/^3](\Omega\mathrm{tr}\underline{\chi})\|_{L^2(S_{\underline{u},u'})} \mathrm{d}u'$$

$$\begin{aligned}&\lesssim\mathcal{O}[\underline{\widehat{\chi}},\mathrm{tr}\underline{\chi},\underline{\omega}]\left[\int_0^u\|(r\nabla\!\!\!\!/\,)^3(\Omega\mathrm{tr}\underline{\chi})\|_{L^2(S_{\underline{u},u})}\mathrm{d}u'\right.\\&\quad\left.+\varepsilon^{1/2}(\|(r\nabla\!\!\!\!/\,)^3(\Omega\underline{\widehat{\chi}})\|_{L^2(\underline{C}_{\underline{u}})}+\|(r\nabla\!\!\!\!/\,)^3\underline{\omega}\|_{L^2(\underline{C}_{\underline{u}})})\right]+\varepsilon C(\mathcal{O})\\&\lesssim\varepsilon^{1/2}C(\mathcal{O},\underline{\mathcal{R}},\Delta_2).\end{aligned}$$

Then for ε sufficiently small, by the Gronwall type estimates for the equation for $\underline{D}\nabla\!\!\!\!/\,^3(\Omega\mathrm{tr}\underline{\chi})$,

$$\|(r\nabla\!\!\!\!/\,)^3\mathrm{tr}\underline{\chi}\|_{L^2(S_{\underline{u},u})}\lesssim C(\mathcal{O}_0).$$

By the $\mathrm{d}\!\!\!/\mathrm{iv}$ equation for $\underline{\widehat{\chi}}$, we have

$$\begin{aligned}&\|(r\nabla\!\!\!\!/\,)^3(\Omega\underline{\widehat{\chi}})\|_{L^2(S_{\underline{u},u})}\\&\lesssim\|(r\nabla\!\!\!\!/\,)^2(r\Omega\underline{\beta})\|_{L^2(S_{\underline{u},u})}+\|(r\nabla\!\!\!\!/\,)^3(\Omega\mathrm{tr}\underline{\chi})\|_{L^2(S_{\underline{u},u})}+r^{-1}C(\mathcal{O}).\end{aligned}$$

Consequently,

$$\|(r\nabla\!\!\!\!/\,)^3(\Omega\underline{\widehat{\chi}})\|_{L^2(\underline{C}_{\underline{u}})}\lesssim C(\mathcal{O}_0,\underline{\mathcal{R}}).$$

Estimate for $\nabla\!\!\!\!/\,^3\omega$. We commute $\nabla\!\!\!\!/$ with the equation for $\underline{D}\kappa$, and estimate

$$\begin{aligned}&\int_0^{\underline{u}}\left[\int_0^u r^3\|\nabla\!\!\!\!/\,(-\Omega\mathrm{tr}\underline{\chi}\kappa-2(\Omega\underline{\widehat{\chi}},\nabla\!\!\!\!/\,^2\omega)+m)\|_{L^2(S_{\underline{u}',u'})}\mathrm{d}u'\right]^2\mathrm{d}\underline{u}'\\&\lesssim\mathcal{O}^2\varepsilon\left[\int_0^u r^{-2}[\sum_{i=0}^1\|(r\nabla\!\!\!\!/\,)^i(r^3\kappa)\|^2_{L^2(C'_u)}+r^2\|(r\nabla\!\!\!\!/\,)^3(r\omega)\|^2_{L^2(C'_u)}\right.\\&\qquad+r^2\|(r\nabla\!\!\!\!/\,)^3\underline{\eta}\|^2_{L^2(C'_u)}+\mathcal{R}[\beta]^2]\mathrm{d}u'\\&\quad+\int_0^{\underline{u}}r^{-2}[\|(r\nabla\!\!\!\!/\,)^3(\Omega\underline{\widehat{\chi}})\|^2_{L^2(\underline{C}_{\underline{u}'})}\\&\qquad\left.+r\|(r\nabla\!\!\!\!/\,)^3(r^{1/2}\eta)\|^2_{L^2(\underline{C}_{\underline{u}'})}+\underline{\mathcal{R}}[\rho,\sigma,\underline{\beta}]^2]\mathrm{d}\underline{u}'\right]\\&\lesssim\varepsilon C(\mathcal{O},\mathcal{R},\underline{\mathcal{R}},\Delta_2).\end{aligned}$$

Note that there are no terms arising from commutator $[\underline{D},\nabla\!\!\!\!/\,]$ because we only commute $\nabla\!\!\!\!/$ once and κ is a function. Applying the Gronwall type estimates, we have

$$\|(r\nabla\!\!\!\!/\,)(r^3\kappa)\|_{L^2(C_u)}\lesssim C(\mathcal{O}_0,\mathcal{R}).$$

By the Laplacian equation for ω, we have

$$\|(r\nabla\!\!\!\!/\,)^3(r\omega)\|_{L^2(S_{\underline{u},u})}\lesssim\|(r\nabla\!\!\!\!/\,)(r^3\kappa)\|_{L^2(S_{\underline{u},u})}+\|(r\nabla\!\!\!\!/\,)^2(r^2\Omega\beta)\|_{L^2(S_{\underline{u},u})}.$$

Consequently, for ε sufficiently small,

$$\|(r\nabla\!\!\!\!/\,)^3(r\omega)\|_{L^2(C_u)}\lesssim C(\mathcal{O}_0,\mathcal{R}).\tag{4.11}$$

Remark 7. We also remark here that, $\underline{\chi}'$, $\underline{\mu}$ and κ are the same as in the assumptions of Theorem 2 in the construction, and the initial norm of $\Omega\underline{\chi}$ differs from the one in the assumptions of Theorem 2 by up to third order derivatives of $\log\Omega$.

Estimates for $\nabla\!\!\!/^3\widehat{\chi}$ and $\nabla\!\!\!/^3\mathrm{tr}\chi$. The difference here from the above estimates is that we rely on the Hodge systems coupled with propagation equations along D direction instead of $\underline{D}$ direction above. As in estimating $\mathcal{O}$, integrating along L direction should start from the last slice. Therefore, we should consider first the value of $\nabla\!\!\!/(\Omega\mathrm{tr}\chi)$ on the last slice. By the definition of the canonical foliation, the equation for $\underline{D}(\Omega\mathrm{tr}\chi)$ is reduced to

$$\underline{D}(\Omega\mathrm{tr}\chi)+\frac{1}{2}\Omega\mathrm{tr}\underline{\chi}\Omega\mathrm{tr}\chi=\Omega^2(2|\eta|^2-\overline{(\widehat{\chi},\widehat{\underline{\chi}})}+2\overline{\rho}).$$

The term $\mathrm{d}\!\!/\mathrm{iv}\,\eta$ disappears and $\overline{\rho}$ will disappear if commuting $\nabla\!\!\!/$ on the equation, that is,

$$\begin{aligned}\underline{D}(\nabla\!\!\!/^3(\Omega\mathrm{tr}\chi))=&-\frac{1}{2}\nabla\!\!\!/^3(\Omega\mathrm{tr}\underline{\chi}\Omega\mathrm{tr}\chi)\\&+2\nabla\!\!\!/^3(\Omega^2(2|\eta|^2-\overline{(\widehat{\chi},\widehat{\underline{\chi}})}+2\overline{\rho}))+[\underline{D},\nabla\!\!\!/^2]\nabla\!\!\!/(\Omega\mathrm{tr}\chi).\end{aligned}$$

The right hand side does not involve the third order derivatives of $\widehat{\chi}$ and η, and, therefore, can be controlled by $\mathcal{O}$ and Δ_2 after integrating along $\underline{L}$. We have, by the Gronwall type estimates,

$$\begin{aligned}&\|(r\nabla\!\!\!/)^3(r\Omega\mathrm{tr}\chi)\|_{L^2(S_{\underline{u}_*,u})}\\\lesssim&\|(r\nabla\!\!\!/)^3(r\Omega\mathrm{tr}\chi)\|_{L^2(S_{\underline{u}_*,0})}\\&+\int_0^u[\mathcal{O}[\mathrm{tr}\chi]\|(r\nabla\!\!\!/)^3(\Omega\mathrm{tr}\underline{\chi})\|_{L^2(S_{\underline{u}_*,u'})}\\&\quad+r^{-1}\mathcal{O}[\mathrm{tr}\underline{\chi}]\|(r\nabla\!\!\!/)^3(r\Omega\mathrm{tr}\chi)\|_{L^2(S_{\underline{u}_*,u'})}\\&\quad+r^{-3/2}\mathcal{O}[\eta]\|(r\nabla\!\!\!/)^3(r^{1/2}\eta)\|_{L^2(S_{\underline{u}_*,u'})}]\mathrm{d}u'+r^{-1}C(\mathcal{O})\\\lesssim&C(\mathcal{O}_0)+\varepsilon^{1/2}\mathcal{O}(\Delta_2+C(\mathcal{O}_0))\lesssim C(\mathcal{O}_0),\end{aligned}$$

for ε sufficiently small.

Now we commute $\nabla\!\!\!/^3$ with the equation for $D(\Omega\mathrm{tr}\chi)$ and estimate the terms on the right as follows,

$$\begin{aligned}&\int_{\underline{u}}^{\underline{u}_*}r^4\|\nabla\!\!\!/^3[-\frac{1}{2}(\Omega\mathrm{tr}\chi)^2-\frac{1}{2}|\Omega\widehat{\chi}|^2+2\omega(\Omega\mathrm{tr}\chi)]\\&\qquad\qquad+\Omega\mathrm{tr}\chi\nabla\!\!\!/^3(\Omega\mathrm{tr}\chi)+[D,\nabla\!\!\!/^3](\Omega\mathrm{tr}\chi)\|_{L^2(S_{\underline{u}',u})}\mathrm{d}\underline{u}'\\\lesssim&\mathcal{O}[\Omega\mathrm{tr}\chi-\overline{\Omega\mathrm{tr}\chi},\widehat{\chi},\omega]\\&\times\int_{\underline{u}}^{\underline{u}_*}r^{-2}[\|(r\nabla\!\!\!/)^3(r\Omega\mathrm{tr}\chi)\|_{L^2(S_{\underline{u}',u})}+\|(r\nabla\!\!\!/)^3(r\Omega\widehat{\chi})\|_{L^2(S_{\underline{u}',u})}\end{aligned}$$

$$
\begin{aligned}
&+\|(r\nabla\!\!\!/)^3(r\omega)\|_{L^2(S_{\underline{u}',u})}+C(\mathcal{O})]\mathrm{d}\underline{u}'\\
\lesssim&\mathcal{O}\int_{\underline{u}}^{\underline{u}_*}r^{-2}[\|(r\nabla\!\!\!/)^3(r\Omega\mathrm{tr}\chi)\|_{L^2(S_{\underline{u}',u})}+\|(r\nabla\!\!\!/)^3(r\Omega\widehat{\chi})\|_{L^2(S_{\underline{u}',u})}]\mathrm{d}\underline{u}'\\
&+\left(\int_{\underline{u}}^{\underline{u}_*}r^{-4}\mathrm{d}\underline{u}'\right)^{1/2}\|(r\nabla\!\!\!/)^3(r\omega)\|_{L^2(C_u)}+r^{-1}C(\mathcal{O}).
\end{aligned}
$$

By the $\mathrm{d}\!\!/\mathrm{iv}$ equation for $\Omega\widehat{\chi}$, we have

$$
\begin{aligned}
&\|(r\nabla\!\!\!/)^3(\Omega\widehat{\chi})\|_{L^2(S_{\underline{u},u})}\\
\lesssim&\|(r\nabla\!\!\!/)^3(\Omega\mathrm{tr}\chi)\|_{L^2(S_{\underline{u},u})}+\|(r\nabla\!\!\!/)^2(r\Omega\beta)\|_{L^2(S_{\underline{u},u})}+r^{-1}C(\mathcal{O}).
\end{aligned}
$$

Therefore, by applying the Gronwall type estimates, combining the above inequality multiplied by r, we have

$$
\begin{aligned}
&\|(r\nabla\!\!\!/)^3(r\Omega\mathrm{tr}\chi)\|_{L^2(S_{\underline{u},u})}\\
\lesssim&\|(r\nabla\!\!\!/)^3(r\Omega\mathrm{tr}\chi)\|_{L^2(S_{\underline{u}_*,u})}\\
&+\mathcal{O}\int_{\underline{u}}^{\underline{u}_*}r^{-2}\|(r\nabla\!\!\!/)^3(r\Omega\mathrm{tr}\chi)\|_{L^2(S_{\underline{u}',u})}\mathrm{d}\underline{u}'\\
&+r^{-3/2}(\|(r\nabla\!\!\!/)^3(r\omega)\|_{L^2(C_u)}+\|(r\nabla\!\!\!/)^3(r\beta)\|_{L^2(C_u)})+r^{-1}C(\mathcal{O}).
\end{aligned}
$$

The first term on the right hand side can be absorbed by the Gronwall's inequality, thanks to the factor r^{-2}. Finally, we have the estimate

$$
\|(r\nabla\!\!\!/)^3(r\Omega\mathrm{tr}\chi)\|_{L^2(S_{\underline{u},u})}\lesssim C(\mathcal{O}_0,\mathcal{R}),
$$

and, consequently,

$$
\|(r\nabla\!\!\!/)^3(\Omega\widehat{\chi})\|_{L^2(C_u)}\lesssim C(\mathcal{O}_0,\mathcal{R}).
$$

Estimate for $\nabla\!\!\!/^3\eta$. Note that by the definition of the canonical foliation, $\mu=\overline{\mu}$ and, hence, $\nabla\!\!\!/^2\mu=0$. Then we commute $\nabla\!\!\!/^2$ with the equation for $D\mu$, and estimate the right hand side as follows,

$$
\begin{aligned}
&\int_0^u\Bigg[\int_{\underline{u}}^{\underline{u}_*}r^3\|\nabla\!\!\!/^2(-\Omega\mathrm{tr}\chi\mu-\frac{1}{2}\Omega\mathrm{tr}\chi\underline{\mu})\\
&\qquad+\Omega\mathrm{tr}\chi\nabla\!\!\!/^2\mu+[D,\nabla\!\!\!/^2]\mu\|_{L^2(S_{\underline{u}',u'})}\mathrm{d}\underline{u}'\Bigg]^2\mathrm{d}u'\\
\lesssim&\int_0^u r^{-1}\mathcal{O}[\mathrm{tr}\chi]^2\sum_{i=0}^{2}\|(r\nabla\!\!\!/)^i(r\underline{\mu})\|^2_{L^2(C'_u)}\mathrm{d}u'\\
&+r^{-1}\mathcal{O}[\Omega\mathrm{tr}\chi-\overline{\Omega\mathrm{tr}\chi},\widehat{\chi}]^2\int_{\underline{u}}^{\underline{u}_*}r^{-1}\sum_{i=0}^{1}\|(r\nabla\!\!\!/)^i(r\mu)\|^2_{L^2(\underline{C}_{\underline{u}'})}\mathrm{d}\underline{u}'\\
\lesssim&r^{-1}C(\mathcal{O},\mathcal{O}_0,\mathcal{R},\underline{\mathcal{R}}),
\end{aligned}
$$

$$\begin{aligned}
&\int_0^u \left[\int_{\underline{u}}^{\underline{u}_*} r^3 \|\nabla\!\!\!/^2(-\frac{1}{4}\Omega\mathrm{tr}\underline{\chi}|\widehat{\chi}|^2 + \frac{1}{2}\Omega\mathrm{tr}\chi|\underline{\eta}|^2)\|_{L^2(S_{\underline{u}',u'})}\mathrm{d}\underline{u}'\right]^2 \mathrm{d}u' \\
\lesssim&\varepsilon r^{-4}C(\mathcal{O}), \\
&\int_0^u \left[\int_{\underline{u}}^{\underline{u}_*} r^3 \|\nabla\!\!\!/^2(\mathrm{d}\!\!\!/\mathrm{iv}\,(2\Omega\widehat{\chi}\cdot\eta - \Omega\mathrm{tr}\chi\underline{\eta}))\|_{L^2(S_{\underline{u}',u'})}\mathrm{d}\underline{u}'\right]^2 \mathrm{d}u' \\
\lesssim&\mathcal{O}[\eta]^2\int_0^u r^{-3}\|(r\nabla\!\!\!/)^3(\Omega\widehat{\chi})\|^2_{L^2(C_{u'})}\mathrm{d}u' \\
&+\mathcal{O}[\mathrm{tr}\chi]^2\int_0^u r^{-1}\|(r\nabla\!\!\!/)^3\underline{\eta}\|^2_{L^2(C_{u'})}\mathrm{d}u' \\
&+\mathcal{O}[\widehat{\chi}]^2 r^{-1}\int_{\underline{u}}^{\underline{u}_*} r^{-2}\|(r\nabla\!\!\!/)^3\eta\|^2_{L^2(\underline{C}_{\underline{u}'})}\mathrm{d}\underline{u}' \\
&+\mathcal{O}[\underline{\eta}]^2\int_0^u\left[\int_{\underline{u}}^{\underline{u}_*} r^{-3}\|(r\nabla\!\!\!/)^3(r\Omega\mathrm{tr}\chi)\|^2_{L^2(S_{\underline{u}',u'})}\mathrm{d}\underline{u}'\right]^2\mathrm{d}u' \\
&+\varepsilon r^{-4}C(\mathcal{O}) \\
\lesssim&\mathcal{O}[\widehat{\chi}]^2 r^{-1}\int_{\underline{u}}^{\underline{u}_*} r^{-2}\|(r\nabla\!\!\!/)^3\eta\|^2_{L^2(\underline{C}_{\underline{u}'})}\mathrm{d}\underline{u}' + r^{-1}C(\mathcal{O},\mathcal{O}_0,\mathcal{R}).
\end{aligned}$$

We estimate $\|(r\nabla\!\!\!/)^3\eta\|^2_{L^2(\underline{C}_{\underline{u}'})}$ by the $\mathrm{d}\!\!\!/\mathrm{iv}$-$\mathrm{cu}\!\!\!/\mathrm{rl}$ systems for η,

$$\begin{aligned}
&\|(r\nabla\!\!\!/)^3\eta\|_{L^2(\underline{C}_{\underline{u}})} \\
\lesssim&\|(r\nabla\!\!\!/)^2(r\rho)\|_{L^2(\underline{C}_{\underline{u}})} + \|(r\nabla\!\!\!/)^2(r\mu)\|_{L^2(\underline{C}_{\underline{u}})} + \varepsilon^{1/2}r^{-1}C(\mathcal{O}).
\end{aligned}$$

Therefore, by applying the Gronwall type estimates and integrating the square of the resulting inequality along $\underline{D}$ direction,

$$\begin{aligned}
\|(r\nabla\!\!\!/)^2(r\mu)\|^2_{L^2(\underline{C}_{\underline{u}})} \lesssim&\mathcal{O}[\widehat{\chi}]^2 r^{-1}\int_{\underline{u}}^{\underline{u}_*} r^{-2}\|(r\nabla\!\!\!/)^3(r\mu)\|^2_{L^2(\underline{C}_{\underline{u}'})}\mathrm{d}\underline{u}' \\
&+ r^{-1}C(\mathcal{O},\mathcal{O}_*,\mathcal{O}_0,\mathcal{R},\underline{\mathcal{R}}).
\end{aligned}$$

Then, by the Gronwall's inequality, we have

$$\|(r\nabla\!\!\!/)^2(r\mu)\|^2_{L^2(\underline{C}_{\underline{u}})} \lesssim r^{-1}C(\mathcal{O},\mathcal{O}_0,\mathcal{R},\underline{\mathcal{R}}),$$

and, finally,

$$\|(r\nabla\!\!\!/)^2(r^{1/2}\eta)\|_{L^2(\underline{C}_{\underline{u}})} \lesssim C(\mathcal{O},\mathcal{O}_0,\mathcal{R},\underline{\mathcal{R}}).$$

Estimate for $\nabla\!\!\!/^3\underline{\omega}$. At the last step, we consider $\nabla\!\!\!/^3\underline{\omega}$. As above, we should consider first the value of $\nabla\!\!\!/\kappa$ on the last slice. This can be directly seen from equation (4.10), which is, in terms of $\underline{\kappa}$, on the last slice, where

$$\begin{aligned}
2\underline{\kappa} =&\mathrm{d}\!\!\!/\mathrm{iv}\,(3\Omega\widehat{\underline{\chi}}\cdot\eta + \frac{1}{2}\Omega\mathrm{tr}\underline{\chi}\eta) \\
&+\Omega\mathrm{tr}\underline{\chi}\mathrm{d}\!\!\!/\mathrm{iv}\,\eta + (F - \overline{F} + \overline{\Omega\mathrm{tr}\underline{\chi}}\check{\rho} - \overline{\Omega\mathrm{tr}\underline{\chi}}\cdot\overline{\check{\rho}}).
\end{aligned}$$

It is easy to see that the right hand side involves only the curvature components and the first order derivatives of the connection coefficients. By Lemma 8 and Proposition 4, $\|(r\nabla\!\!\!/)(r^2\underline{\kappa})\|_{L^2(S_{\underline{u}_*,u})} \lesssim C(\mathcal{O}_0,\mathcal{R}_0)$.

We estimate the right hand side of the equation for $D\nabla\!\!\!/\,\underline{\kappa}$ which is obtained by commuting $\nabla\!\!\!/$ with the equation for $D\underline{\kappa}$ as follows.

$$\begin{aligned}
&\int_0^u \left[\int_{\underline{u}}^{\underline{u}_*} r^2\| -\nabla\!\!\!/\,(\Omega\mathrm{tr}\chi)\underline{\kappa} - \nabla\!\!\!/\,(2(\Omega\widehat{\chi},\nabla\!\!\!/^{\,2}\underline{\omega}) + \underline{m})\|_{L^2(S_{\underline{u}',u'})}\mathrm{d}\underline{u}'\right]^2 \mathrm{d}u' \\
\lesssim&\mathcal{O}r^{-1}\int_{\underline{u}}^{\underline{u}^*}\Bigg[r^{-2}(\sum_{i=0}^{1}\|(r\nabla\!\!\!/)^i(r^2\underline{\kappa})\|^2_{L^2(\underline{C}_{\underline{u}'})} + \|(r\nabla\!\!\!/)^3\underline{\omega}\|^2_{L^2(\underline{C}_{\underline{u}'})} \\
&+\|(r\nabla\!\!\!/^{\,3})(r^{1/2})\eta\|_{L^2(S_{\underline{u},u})} + \underline{\mathcal{R}}[\rho,\sigma,\underline{\beta}]^2)\Bigg]\mathrm{d}u' \\
&+\int_0^u r^{-6}\|(r\nabla\!\!\!/)^3\underline{\eta}\|^2_{L^2(C_{u'})} \\
\lesssim&\mathcal{O}r^{-1}\int_{\underline{u}}^{\underline{u}^*} r^{-2}\|(r\nabla\!\!\!/)^3\underline{\omega}\|^2_{L^2(\underline{C}_{\underline{u}'})}\mathrm{d}\underline{u}' + r^{-2}C(\mathcal{O},\mathcal{O}_0,\mathcal{R},\underline{\mathcal{R}}).
\end{aligned}$$

By the Laplacian equation for $\underline{\omega}$, we have

$$\|(r\nabla\!\!\!/)^3\underline{\omega}\|_{L^2(\underline{C}_{\underline{u}})} \lesssim \|(r\nabla\!\!\!/)(r^2\underline{\kappa})\|_{L^2(\underline{C}_{\underline{u}})} + \|(r\nabla\!\!\!/)^2(r\Omega\underline{\beta})\|_{L^2(\underline{C}_{\underline{u}})}.$$

Applying the Gronwall type estimates to the equation for $\underline{D}\nabla\!\!\!/\,\kappa$ yields

$$\begin{aligned}
\|r^2\nabla\!\!\!/\,\underline{\kappa}\|^2_{L^2(\underline{C}_{\underline{u}})} \lesssim&\|r^2\nabla\!\!\!/\,\underline{\kappa}\|^2_{L^2(\underline{C}_{\underline{u}_*})} + r^{-1}\mathcal{O}\int_{\underline{u}}^{\underline{u}^*} r^{-2}\|(r\nabla\!\!\!/)(r^2\underline{\kappa})\|^2_{L^2(\underline{C}_{\underline{u}'})}\mathrm{d}\underline{u}' \\
&+ r^{-2}C(\mathcal{O},\mathcal{O}_0,\mathcal{R},\underline{\mathcal{R}}).
\end{aligned}$$

We multiply the above inequality by r and apply the Gronwall's inequality to $\|r^{5/2}\nabla\!\!\!/\,\underline{\kappa}\|_{L^2(\underline{C}_{\underline{u}})}$. Note that the factor in the integral becomes $r^{-3/2}$ and is integrable, therefore,

$$\|r^{5/2}\nabla\!\!\!/\,\underline{\kappa}\|^2_{L^2(\underline{C}_{\underline{u}})} \lesssim r^{-1}C(\mathcal{O},\mathcal{O}_0,\mathcal{R},\underline{\mathcal{R}}),$$

and, therefore,

$$\|(r\nabla\!\!\!/)^3\underline{\omega}\|_{L^2(\underline{C}_{\underline{u}})} \lesssim C(\mathcal{O},\mathcal{O}_0,\mathcal{R},\underline{\mathcal{R}}).$$

At last, we choose Δ_2 sufficiently large to complete the proof. q.e.d.

4.3. Estimates for the curvature components. We are going to complete Step 3 of the proof in this subsection. We will prove in this subsection the following proposition:

Proposition 6. *If ε is sufficiently small depending on $\mathcal{O}_0$, $\mathcal{R}_0$, $\underline{\mathcal{R}}_0$, then we have*

$$\mathcal{R} + \underline{\mathcal{R}} \leq C(\mathcal{O}_0,\mathcal{R}_0,\underline{\mathcal{R}}_0).$$

Proof. As discussed in the introduction, we apply the renormalization in [**14**] in the current work. We write $\check{\rho} = \rho - \frac{1}{2}(\widehat{\chi}, \underline{\widehat{\chi}})$ and $\check{\sigma} = \sigma + \frac{1}{2}\widehat{\chi} \wedge \underline{\widehat{\chi}}$. By direct computation, they satisfy the following renormalized Bianchi equations

$$\begin{aligned} D\check{\rho} + \frac{3}{2}\Omega\mathrm{tr}\chi\check{\rho} =& \Omega\{\mathrm{d}\!\!/\mathrm{iv}\,\beta + (2\underline{\eta} + \zeta, \beta) \\ & - \frac{1}{2}(\widehat{\chi}, \nabla\!\!\!\!/\,\widehat{\otimes}\underline{\eta} + \underline{\eta}\widehat{\otimes}\underline{\eta}) + \frac{1}{4}\mathrm{tr}\underline{\chi}|\widehat{\chi}|^2\}, \\ D\check{\sigma} + \frac{3}{2}\Omega\mathrm{tr}\chi\check{\sigma} =& \Omega\{\mathrm{cu}\!\!/\mathrm{rl}\,\beta + (2\underline{\eta} + \zeta, ^*\beta) + \frac{1}{2}\widehat{\chi} \wedge (\nabla\!\!\!\!/\,\widehat{\otimes}\underline{\eta} + \underline{\eta}\widehat{\otimes}\underline{\eta})\}. \end{aligned}$$

This calculation is similar to that in deriving equation (4.10). To couple with these two equations, the equation for $\underline{D}\beta$ should also be rewritten as

$$\begin{aligned} &\underline{D}\beta + \frac{1}{2}\Omega\mathrm{tr}\underline{\chi}\beta = \Omega\underline{\widehat{\chi}} \cdot \beta - \underline{\omega}\beta \\ &+ \Omega\{\mathrm{d}\!\!/\check{\rho} + {}^*\mathrm{d}\!\!/\check{\sigma} + 3\eta\rho + 3^*\eta\sigma + 2\widehat{\chi} \cdot \underline{\beta} + \frac{1}{2}(\mathrm{d}\!\!/(\widehat{\chi}, \underline{\widehat{\chi}}) -^* \mathrm{d}\!\!/(\widehat{\chi} \wedge \underline{\widehat{\chi}}))\}. \end{aligned}$$

To make the expression in a uniform way, in this subsection, we will denote

$$\underline{R}_1 = \frac{1}{\sqrt{2}}\underline{\alpha},\ \underline{R}_2 = -\underline{\beta},\ \underline{R}_3 = (-\check{\rho}, \check{\sigma}),$$

and denote

$$R_2 = -\underline{\beta},\ R_3 = (\rho, \sigma),\ R_4 = \beta.$$

Compared to our usual conventions in previous sections, the definition are essentially the same up to a constant multiple.

Now define $\mathfrak{D}$ acting on $\underline{R}_j$, say $(-\check{\rho}, \check{\sigma}), -\underline{\beta}, \underline{\alpha}$ and their derivatives, respectively, to be

$$\begin{aligned} \mathfrak{D}\nabla\!\!\!\!/^{\,i}(-\check{\rho}, \check{\sigma}) &= D\nabla\!\!\!\!/^{\,i}(-\check{\rho}, \check{\sigma}) + \frac{1}{2}\Omega\mathrm{tr}\chi\nabla\!\!\!\!/^{\,i}(-\check{\rho}, \check{\sigma}), \\ \mathfrak{D}\nabla\!\!\!\!/^{\,i}(-\underline{\beta}) &= D\nabla\!\!\!\!/^{\,i}(-\underline{\beta}) - \Omega\widehat{\chi} \cdot \nabla\!\!\!\!/^{\,i}(-\underline{\beta}), \\ \mathfrak{D}\nabla\!\!\!\!/^{\,i}\underline{\alpha} &= \widehat{D}\nabla\!\!\!\!/^{\,i}\underline{\alpha} - \frac{1}{2}\Omega\mathrm{tr}\chi\nabla\!\!\!\!/^{\,i}\underline{\alpha}. \end{aligned}$$

We also denote $\mathcal{D}$ to be one of the Hodge operators $\mathcal{D}_1$, $\mathcal{D}_2$, ${}^*\mathcal{D}_1$ and ${}^*\mathcal{D}_2$ (see the definition before Lemma 2). We also define $\mathcal{D}$ to be acting on the derivatives of the corresponding tensorfields in an obvious way. Then we denote ${}^*\mathcal{D}$ to be the corresponding $L^2(S_{\underline{u},u})$ formal adjoint.[15]

We collect three groups of the null Bianchi equations, say $\underline{D}\beta$-$D\check{\rho}$-$D\check{\sigma}$, $\underline{D}\rho$-$\underline{D}\sigma$-$D\beta$ and $\underline{D}\underline{\beta}$-$\widehat{\underline{D}}\underline{\alpha}$. By the commutation formulas for $[D, \nabla\!\!\!\!/^{\,i}]$, $[\underline{D}, \nabla\!\!\!\!/^{\,i}]$, $[\mathcal{D}, \nabla\!\!\!\!/^{\,i}]$ and $[{}^*\mathcal{D}, \nabla\!\!\!\!/^{\,i}]$ (see Lemma 2), we can write them in the following form. For $j = 1, 2, 3$, denoting $\nabla\!\!\!\!/^{\,-1}K = 0$,

[15] We abuse the notation here as compared to Lemma 2.

$$
\begin{aligned}
E^3_{j,i} &\triangleq \underline{D}\nabla\!\!\!/^{\,i} R_{j+1} - \Omega \mathcal{D}\nabla\!\!\!/^{\,i} \underline{R}_j \\
&= \sum\sum_{k=0}^{i} \left(\nabla\!\!\!/^{\,k}\Gamma_p \nabla\!\!\!/^{\,i-k} R_{j+1} + \nabla\!\!\!/^{\,k}(\Omega\Gamma_{p_1})\nabla\!\!\!/^{\,i-k}\underline{R}_{p_2} + \nabla\!\!\!/^{\,k-1} K \nabla\!\!\!/^{\,i-k}\underline{R}_j\right),\\
E^4_{j,i} &\triangleq \mathfrak{D}\nabla\!\!\!/^{\,i}\underline{R}_j + \frac{j-1}{2}\Omega\mathrm{tr}\chi\nabla\!\!\!/^{\,i}\underline{R}_j + \Omega^* \mathcal{D}\nabla\!\!\!/^{\,i} R_{j+1}\\
&= \sum\sum_{k=0}^{i} \Big(\nabla\!\!\!/^{\,k}\Gamma_p\nabla\!\!\!/^{\,i-k}\underline{R}_j + \nabla\!\!\!/^{\,k}(\Omega\Gamma_{p_1})\nabla\!\!\!/^{\,i-k} R_{p_2} + \nabla\!\!\!/^{\,i}(\Omega\Gamma_{p_1'})\nabla\!\!\!/^{\,i-k}\nabla\!\!\!/\,\Gamma_{p_2'}\\
&\qquad + \nabla\!\!\!/^{\,k}(\Omega\Gamma_{p_1''})\nabla\!\!\!/^{\,i-k}(\Gamma_{p_2''}\Gamma_{p_3''}) + \nabla\!\!\!/^{\,k-1}K\nabla\!\!\!/^{\,i-k}R_{j+1}\Big).
\end{aligned}
$$

With the null Bianchi equations written in the above form, we can refer to the exact form of the null Bianchi equations by direct counting to obtain the following lemma:

Lemma 9. *In the first equation, $j+1 \le p_1 + p_2$, and by definition of the numbers assigned, $p \ge 1$ automatically. In the second equation, we have $p \ge 2$, $j+2 \le \min\{p_1+p_2, p_1'+p_2'+1, p_1''+p_2''+p_3''\}$.*

We then compute

$$
\begin{aligned}
&\underline{D}(r^{2(i+j-1)}|\nabla\!\!\!/^{\,i}R_{j+1}|^2 \mathrm{d}\mu_{\not g})\\
=&r^{2(i+j-1)}(\Gamma[\mathrm{tr}\underline{\chi},\widehat{\underline{\chi}}]|\nabla\!\!\!/^{\,i}R_{j+1}|^2 + 2(\nabla\!\!\!/^{\,i}R_{j+1}, \underline{D}\nabla\!\!\!/^{\,i}R_{j+1}))\\
=&r^{2(i+j-1)}(\Gamma[\mathrm{tr}\underline{\chi},\widehat{\underline{\chi}}]|\nabla\!\!\!/^{\,i}R_{j+1}|^2 + 2(\nabla\!\!\!/^{\,i}R_{j+1}, \Omega\mathcal{D}\nabla\!\!\!/^{\,i}\underline{R}_j + E^3_{j,i})).
\end{aligned}
$$

We also compute carefully

$$
\begin{aligned}
&D(r^{2(i+j-1)}|\nabla\!\!\!/^{\,i}\underline{R}_j|^2\mathrm{d}\mu_{\not g})\\
=&(2(i+j-1)r^{2(i+j)-2}Dr|\nabla\!\!\!/^{\,i}\underline{R}_j|^2 - r^{2(i+j-1)}i\Omega\mathrm{tr}\chi|\nabla\!\!\!/^{\,i}\underline{R}_j|^2\\
&+ 2r^{2(i+j-1)}(\nabla\!\!\!/^{\,i}\underline{R}_j, \mathfrak{D}\underline{R}_i) + r^{2(i+j-1)}\Omega\widehat{\chi}\cdot\nabla\!\!\!/^{\,i}\underline{R}_j\cdot\nabla\!\!\!/^{\,i}\underline{R}_j)\mathrm{d}\mu_{\not g}\\
=&r^{2(i+j-1)}\Big((i+j-1)(\Omega\mathrm{tr}\chi - \overline{\Omega\mathrm{tr}\chi})|\nabla\!\!\!/^{\,i}\underline{R}_j|^2\\
&\qquad - 2(\nabla\!\!\!/^{\,i}\underline{R}_j, \Omega^*\mathcal{D}\nabla\!\!\!/^{\,i}R_{j+1} + \Omega\widehat{\chi}\cdot\nabla\!\!\!/^{\,i}R_j + E^4_{j,i})\Big).
\end{aligned}
$$

Summing up the above two identities, then integrating on the spacetime manifold $M_{\underline{u},u}$ which is enclosed by C_0, $\underline{C}_0$, $\underline{C}_{\underline{u}}$ and C_u, we have

$$
\begin{aligned}
&\int_{C_u} r^{2(i+j-1)}|\nabla\!\!\!/^{\,i}R_{j+1}|^2 + \int_{\underline{C}_{\underline{u}}} r^{2(i+j-1)}|\nabla\!\!\!/^{\,i}\underline{R}_j|^2\\
=&\int_{C_0} r^{2(i+j-1)}|\nabla\!\!\!/^{\,i}R_{j+1}|^2 + \int_{\underline{C}_0} r^{2(i+j-1)}|\nabla\!\!\!/^{\,i}\underline{R}_j|^2 + \iint_{M_{\underline{u},u}} 2\Omega^{-2}\tau_j^{(i)}\\
\lesssim&\int_{C_0} r^{2(i+j-1)}|\nabla\!\!\!/^{\,i}R_{j+1}|^2 + \int_{\underline{C}_0} r^{2(i+j-1)}|\nabla\!\!\!/^{\,i}\underline{R}_j|^2 + \iint_{M} |\tau_j^{(i)}|
\end{aligned}
$$

where $M = M_{\underline{u},\varepsilon}$ and

$$\begin{aligned}\tau_j^{(i)} =& r^{2(i+j-1)}\Bigg(\sum\sum_{k=0}^{i}\Big(\nabla\!\!\!/^{k}\Gamma_p\nabla\!\!\!/^{i-k}R_{j+1}\\ &\qquad + \nabla\!\!\!/^{k}(\Omega\Gamma_{p_1})\nabla\!\!\!/^{i-k}\underline{R}_{p_2} + \nabla\!\!\!/^{k-1}K\nabla\!\!\!/^{i-k}\underline{R}_j\Big), \nabla\!\!\!/^{i}R_{j+1}\Bigg)\\ &+ r^{2(i+j-1)}\Bigg(\sum\sum_{k=0}^{i}\Big(\nabla\!\!\!/^{k}\Gamma_p\nabla\!\!\!/^{i-k}\underline{R}_j + \nabla\!\!\!/^{k}(\Omega\Gamma_{p_1})\nabla\!\!\!/^{i-k}R_{p_2}\\ &\qquad + \nabla\!\!\!/^{i}(\Omega\Gamma_{p_1'})\nabla\!\!\!/^{i-k}\nabla\!\!\!/\,\Gamma_{p_2'}\\ &\qquad + \nabla\!\!\!/^{k}(\Omega\Gamma_{p_1''})\nabla\!\!\!/^{i-k}(\Gamma_{p_2''}\Gamma_{p_3''})\\ &\qquad + \nabla\!\!\!/^{k-1}K\nabla\!\!\!/^{i-k}R_{j+1}\Big), \nabla\!\!\!/^{i}\underline{R}_j\Bigg),\end{aligned}$$

with the numbers assigned to every component satisfying Lemma 9. Here we abuse the notations, that the p, p_1, p_2 in the first two lines are not the same as in the other three lines.

We divide the terms contained in $\tau_j^{(i)}$ in different types.

Type 1: Terms like $r^{2(i+j-1)}(\nabla\!\!\!/^{i}R_{j+1}\nabla\!\!\!/^{i_1}\Gamma_{p_1}\nabla\!\!\!/^{i_2}\underline{R}_{p_2})$ where $i_1+i_2 = i \leq 2$. We have $j+1 \leq p_1+p_2$, therefore, we estimate by the Hölder inequality,

$$\begin{aligned}&\iint_M r^{2(i+j-1)}|\nabla\!\!\!/^{i}R_{j+1}\nabla\!\!\!/^{i_1}\Gamma_{p_1}\nabla\!\!\!/^{i_2}\underline{R}_{p_2}|\\ \lesssim&\left(\int_0^u\int_{C_{u'}} r^{2(i+j-1)}|\nabla\!\!\!/^{i}R_{j+1}|^2\mathrm{d}u'\right)^{1/2}\\ &\times\left(\int_0^{\underline{u}} r^{-2}\int_{\underline{C}_{\underline{u}'}} r^{2(i_1+i_2+p_1+p_2-1)}|\nabla\!\!\!/^{i_1}\Gamma_{p_1}\nabla\!\!\!/^{i_2}\underline{R}_{p_2}|^2\mathrm{d}\underline{u}'\right)^{1/2}\\ \triangleq& I\cdot II \lesssim \varepsilon^{1/2}\mathcal{R}\cdot II.\end{aligned}$$

For $i_1 = 0$,

$$\begin{aligned}II^2 \lesssim& \sup_{\underline{u},u}(\|r^{i_1+p_1}\nabla\!\!\!/^{i_1}\Gamma_{p_1}\|^2_{L^\infty(S_{\underline{u},u})})\\ &\times\int_0^{\underline{u}} r^{-2}\int_{\underline{C}_{\underline{u}'}} r^{2(i_2+p_2-1)}|\nabla\!\!\!/^{i_2}\underline{R}_{p_2}|^2 \lesssim C(\mathcal{O}_0)\underline{\mathcal{R}}^2.\end{aligned}$$

For $i_1 = 1$, then $i_2 \leq 1$, applying the Sobolev inequality,

$$II^2 \lesssim \sup_{\underline{u},u}(r^{-1/2}\|r^{i_1+p_1}\nabla\!\!\!/^{i_1}\Gamma_{p_1}\|^2_{L^4(S_{\underline{u},u})})$$

$$\times \int_0^{\underline{u}} r^{-2} \mathrm{d}\underline{u}' \int_0^u \mathrm{d}u' \| r^{(i_2+p_2-1)} \nabla\!\!\!/^{\,i_2} \underline{R}_{p_2} \|^2_{L^4(S_{\underline{u}',u'})}$$
$$\lesssim \sup_{\underline{u},u} (r^{-1/2} \| r^{i_1+p_1} \nabla\!\!\!/^{\,i_1} \Gamma_{p_1} \|^2_{L^4(S_{\underline{u},u})})$$
$$\times \int_0^{\underline{u}} r^{-2} \mathrm{d}\underline{u}' \sum_{k=i_2}^{i_2+1} \int_{\underline{C}_{\underline{u}'}} | r^{2(k+p_2-1)} \nabla\!\!\!/^{\,k} \underline{R}_{p_2} |^2 \lesssim C(\mathcal{O}_0) \underline{\mathcal{R}}^2.$$

For $i_1 = 2$, then $i_2 = 0$, we also apply the Hölder and Sobolev inequalities,

$$II^2 \lesssim \sup_{\underline{u},u} (r^{-1} \| r^{i_1+p_1} \nabla\!\!\!/^{\,i_1} \Gamma_{p_1} \|^2_{L^2(S_{\underline{u},u})})$$
$$\times \int_0^{\underline{u}} r^{-2} \mathrm{d}\underline{u}' \sum_{k=i_2}^{i_2+2} \int_{\underline{C}_{\underline{u}'}} | r^{2(k+p_2-1)} \nabla\!\!\!/^{\,k} \underline{R}_{p_2} |^2 \lesssim C(\mathcal{O}_0) \underline{\mathcal{R}}^2.$$

There are also terms like $r^{2(i+j-1)}(\nabla\!\!\!/^{\,i_1} \Gamma_{p_1} \nabla\!\!\!/^{\,i_2} R_{p_2} \nabla\!\!\!/^{\,i} \underline{R}_j)$. We estimate, since $j + 2 \leq p_1 + p_2$,

$$\iint_M r^{2(i+j-1)} | \nabla\!\!\!/^{\,i_1} \Gamma_{p_1} \nabla\!\!\!/^{\,i_2} R_{p_2} \nabla\!\!\!/^{\,i} \underline{R}_j |$$
$$\lesssim \left(\int_0^{\underline{u}} r^{-2} \int_{\underline{C}_{\underline{u}'}} r^{2(i+j-1)} | \nabla\!\!\!/^{\,i} \underline{R}_j |^2 \mathrm{d}\underline{u}' \right)^{1/2}$$
$$\times \left(\int_0^u \int_{C_{u'}} r^{2(i_1+i_2+p_1+p_2-2)} | \nabla\!\!\!/^{\,i_1} \Gamma_{p_1} \nabla\!\!\!/^{\,i_2} R_{p_2} |^2 \mathrm{d}u' \right)^{1/2}$$
$$\lesssim \varepsilon^{1/2} C(\mathcal{O}_0) \mathcal{R} \underline{\mathcal{R}}.$$

Type 2: Terms like $r^{2(i+j-1)}(\nabla\!\!\!/^{\,i_1} \Gamma_p \nabla\!\!\!/^{\,i_2} R_{j+1} \nabla\!\!\!/^{\,i} R_{j+1})$ and terms like $r^{2(i+j-1)}(\nabla\!\!\!/^{\,i_1} \Gamma_p \nabla\!\!\!/^{\,i_2} \underline{R}_j \nabla\!\!\!/^{\,i} \underline{R}_j)$ with $i_1 + i_2 = i \leq 2$. Because $p \geq 0$ in the first case, we estimate

$$\iint_M r^{2(i+j-1)} | \nabla\!\!\!/^{\,i_1} \Gamma_p \nabla\!\!\!/^{\,i_2} R_{j+1} \nabla\!\!\!/^{\,i} R_{j+1} |$$
$$\lesssim \sup_{\underline{u},u} (r^{-2/q} \| (r \nabla\!\!\!/)^{i_q} \Gamma \|_{L^q(S_{\underline{u},u})}) \sum_{k=0}^{2} \int_0^u \int_{C_{u'}} r^{2(k+j-1)} | \nabla\!\!\!/^{\,k} \beta |^2 \mathrm{d}u'$$
$$\lesssim \varepsilon C(\mathcal{O}_0, \mathcal{R}) \mathcal{R}^2,$$

where $i_q = 2, 1, 0$ for $q = 2, 4, \infty$, respectively. Note that $\underline{\omega}$ is involved in this case and $\mathcal{O}$ depends on $\mathcal{R}$.

For the second case, only $\widehat{\chi}$, $\Omega \mathrm{tr} \chi - \overline{\Omega \mathrm{tr} \chi}$ and ω are involved as a Γ. Therefore, we estimate

$$\iint_M r^{2(i+j-1)} | (\nabla\!\!\!/^{\,i_1} \Gamma_p \nabla\!\!\!/^{\,i_2} \underline{R}_j \nabla\!\!\!/^{\,i} \underline{R}_j) |$$

$$\begin{aligned}
&\lesssim \sup_{\underline{u},u}(r^{-2/q}\|r^2(r\nabla\!\!\!/)^{i_q}\Gamma\|_{L^q(S_{\underline{u},u})}) \\
&\quad\times\sum_{k=0}^{2}\int_0^{\underline{u}'} r^{-2}\int_{\underline{C}_{\underline{u}'}} r^{2(k+j-1)}|\nabla\!\!\!/^k\underline{R}_j|^2\mathrm{d}\underline{u}' \\
&\lesssim C(\mathcal{O}_0)\sum_{k=0}^{2}\int_0^{\underline{u}'} r^{-2}\int_{\underline{C}_{\underline{u}'}} r^{2(k+j-1)}|\nabla\!\!\!/^k\underline{R}_j|^2\mathrm{d}\underline{u}'.
\end{aligned}$$

Type 3: Terms like $r^{2(i+j-1)}(\nabla\!\!\!/^{i_1}\Gamma_{p_1'}\nabla\!\!\!/^{i_2}\nabla\!\!\!/\Gamma_{p_2'}\nabla\!\!\!/^i\underline{R}_j)$ and terms like $r^{2(i+j-1)}(\nabla\!\!\!/^{i_1}(\Gamma_{p_1''})\nabla\!\!\!/^{i_2}(\Gamma_{p_2''}\Gamma_{p_3''})\nabla\!\!\!/^i\underline{R}_j)$ for $i_1+i_2=i\leq 2$. This type which arises from renormalization only appears when $j=3$. For the first case, we have $j+2\leq p_1'+p_2'+1$ and $\Gamma_{p_2'}$ contains only $\underline{\eta}$. Therefore, the estimate can be done as the second case of Type 1. For the second case, note that the connection coefficients Γ and Γ^2 have the same regularity. That is, Γ^2 can be estimated in $L^\infty(S)$, $\nabla\!\!\!/(\Gamma^2)=\Gamma\nabla\!\!\!/\Gamma$ can be estimated in $L^4(S)$, and $\nabla\!\!\!/^2(\Gamma^2)=\Gamma\nabla\!\!\!/^2\Gamma+(\nabla\!\!\!/\Gamma)^2$ can be estimated in $L^2(S)$. Therefore, $\nabla\!\!\!/^{i_2}(\Gamma_{p_2''}\Gamma_{p_3''})$ can always be estimated in different norms on $S_{\underline{u},u}$ and get addition $\varepsilon^{1/2}$ factor after integrating on M.

Type 4: Terms like $r^{2(i+j-1)}(\nabla\!\!\!/^{i_1}K\nabla\!\!\!/^{i_2}\underline{R}_j\nabla\!\!\!/^iR_{j+1})$ and terms like $r^{2(i+j-1)}(\nabla\!\!\!/^{i_1}K\nabla\!\!\!/^{i_2}R_{j+1}\nabla\!\!\!/^i\underline{R}_j)$ for $i_1+i_2=i-1\leq 1$. By Lemma 8, $\|(r\nabla\!\!\!/)^{i_1}(rK)\|_{L^2(S_{\underline{u},u})}\leq C(\mathcal{R}_0,\mathcal{O}_0)$ and, hence, $\|r^{3/2}K\|_{L^4(S_{\underline{u},u})}\leq C(\mathcal{R}_0,\mathcal{O}_0)$. Therefore,

$$\begin{aligned}
&\iint_M r^{2(i+j-1)}|\nabla\!\!\!/^{i_1}K\nabla\!\!\!/^{i_2}\underline{R}_j\nabla\!\!\!/^iR_{j+1}| \\
\lesssim&\left(\int_0^u\int_{C_{u'}} r^{2(i+j-1)}|\nabla\!\!\!/^iR_{j+1}|^2\mathrm{d}u'\right)^{1/2} \\
&\times\left(\int_0^{\underline{u}} r^{-2}\int_{\underline{C}_{\underline{u}'}} r^{2(i_1+i_2+j+1)}|\nabla\!\!\!/^{i_1}K\nabla\!\!\!/^{i_2}\underline{R}_j|^2\mathrm{d}\underline{u}'\right)^{1/2} \\
\lesssim&\varepsilon^{1/2}C(\mathcal{R}_0,\mathcal{O}_0)\underline{\mathcal{R}}\mathcal{R},
\end{aligned}$$

and the second case can be treated similarly.

Then, by choosing ε sufficiently small,

$$\begin{aligned}
&\sum_{i=0}^{2}\left(\int_{C_u} r^{2(i+j-1)}|\nabla\!\!\!/^iR_{j+1}|^2+\int_{\underline{C}_{\underline{u}}} r^{2(i+j-1)}|\nabla\!\!\!/^i\underline{R}_j|^2\right) \\
\leq&\sum_{i=0}^{2}\left(\int_{C_0} r^{2(i+j-1)}|\nabla\!\!\!/^iR_{j+1}|^2+\int_{\underline{C}_0} r^{2(i+j-1)}|\nabla\!\!\!/^i\underline{R}_j|^2\right)
\end{aligned}$$

$$+ C(\mathcal{O}_0) \sum_{k=0}^{2} \int_0^{\underline{u}} r^{-2} \int_{\underline{C}_{\underline{u}'}} r^{2(k+j-1)} |\nabla\!\!\!\!/^{\,k} \underline{R}_j|^2 \mathrm{d}\underline{u}' + 1.$$

Because the factor r^{-2} is integrable and $\mathcal{O}_0$ is independent of $\underline{\mathcal{R}}$, then we can apply the Gronwall inequality to complete the proof of the proposition. q.e.d.

Remark 8. The function u restricted on $\underline{C}_0$ does not coincide with $\underline{s}$, the affine parameter on $\underline{C}_0$. Therefore, in the above estimate, the initial terms on $\underline{C}_0$

$$\int_{\underline{C}_0} r^{2(i+j-1)} |\nabla\!\!\!\!/^{\,i} \underline{R}_j|^2 = \int_{\underline{C}_0, 0 \le u \le \varepsilon} r^{2(i+j-1)} |\nabla\!\!\!\!/^{\,i} \underline{R}_j|^2 \mathrm{d}\mu_{g\!\!\!/_{\underline{u},u}} \mathrm{d}u$$

are not exactly the same as defined in $\underline{\mathcal{R}}_0$. Therefore, we need to estimate the initial terms by $\underline{\mathcal{R}}_0$. Suppose that the affine foliation given by $\underline{s}$ is the background foliation and the foliation given by u can be represented by a function $W(s,\theta)$[16] defined on $[0,\varepsilon] \times S^2$ as in Section 4.1. Observing that $\underline{s} = W(s,\theta)$ and $\partial_{\theta^A} W(s,\theta) = \nabla\!\!\!\!/_{A} \underline{s}\big|_{(\underline{s}=W(s,\theta),\theta)}$, it is convenient to estimate $\nabla\!\!\!\!/_{A} \underline{s}$ instead, where $\nabla\!\!\!\!/$ here refers to the angular covariant derivative relative to the foliation given by u. This is because $\nabla\!\!\!\!/\, \underline{s}$ is a geometric quantity defined on $\underline{C}_0$ while W is not defined on $\underline{C}_0$. From the relation

$$\underline{D} \underline{s} = \Omega^2,$$

we can estimate by Lemma 7 that

$$\|\nabla\!\!\!\!/^{\,\le 1} \underline{s}\|_{L^\infty(S_{0,u})} + \|\nabla\!\!\!\!/^{\,2} \underline{s}\|_{L^4(S_{0,u})} + \|\nabla\!\!\!\!/^{\,3} \underline{s}\|_{L^2(S_{0,u})} \le \varepsilon C(\mathcal{O}_0, \mathcal{R}_0),$$

where the weight r is dropped because the inequality is written on a single null cone $\underline{C}_0$ on which $r \approx 1$. Let ψ' be any geometric quantities like the connection coefficients and the curvature components, and their derivatives, written relative to the affine foliation. We estimate

$$\int_{\underline{C}_0, 0 \le u \le \varepsilon} |\psi'|^2 \mathrm{d}\mu_{g\!\!\!/_{\underline{u},u}} \mathrm{d}u$$
$$\lesssim \int_{\underline{C}_0, 0 \le \underline{s} \le \sup W} |\psi'|^2 \Omega^{-2} \mathrm{d}\mu_{S_{\underline{s}}} \mathrm{d}\underline{s} \lesssim \left(\int_{\underline{C}_0} |\psi'|^2 \right)'.$$

Here $S_{\underline{s}}$ refers to the affine sections and the last term (with the prime symbol) refers to the integral over $\underline{C}_0$ relative to the affine foliation as in the definition of $\underline{\mathcal{R}}_0$. Let $\nabla\!\!\!\!/'$ be the angular covariant derivative relative to the affine foliation, and $\underline{D}'$ be the projection of $\mathcal{L}_{\underline{L}'}$ on the affine sections, we will have, in a schematic form,

$$\nabla\!\!\!\!/\, \psi' = \nabla\!\!\!\!/' \psi' + \nabla\!\!\!\!/\, s \cdot (\underline{D}' \psi' + \underline{\chi}' \cdot \psi').$$

[16] Here s is not the affine parameter on C_0 but simply the first variable of W.

Combined with the estimates for $\nabla\!\!\!/\, \underline{s}$ derived above and $\underline{\chi}'$ derived in Section 4.2, which are in L^∞, we have

$$\begin{aligned}&\int_{\underline{C}_0, 0\leq u\leq\varepsilon} |\nabla\!\!\!/\, \psi'|^2 \mathrm{d}\mu_{\not{g}_{\underline{u},u}} \mathrm{d}u \\ \lesssim &\left(\int_{\underline{C}_0} |\nabla\!\!\!/\,' \psi'|^2\right)' + \varepsilon C(\mathcal{O}_0, \mathcal{R}_0) \left(\int_{\underline{C}_0} |\underline{D}' \psi'|^2 + |\psi'|^2\right)'.\end{aligned}$$

Taking one more $\nabla\!\!\!/$, by the estimates for $\nabla\!\!\!/^2 \underline{s}$ and $\nabla\!\!\!/\, \underline{\chi}'$ in $L^4(S_{\underline{u},u})$, and the Sobolev inequalities on $S_{\underline{u},u}$, we have

$$\begin{aligned}&\int_{\underline{C}_0, 0\leq u\leq\varepsilon} |\nabla\!\!\!/^2 \psi'|^2 \mathrm{d}\mu_{\not{g}_{\underline{u},u}} \mathrm{d}u \\ \lesssim &\left(\int_{\underline{C}_0} |\nabla\!\!\!/^{\prime 2} \psi'|^2\right)' + \varepsilon C(\mathcal{O}_0, \mathcal{R}_0) \sum_{i=0}^{2} \left(\int_{\underline{C}_0} |\nabla'^i \psi'|^2\right)',\end{aligned}$$

where ∇' refers to $\nabla\!\!\!/\,'$ or $\underline{D}'$. Now from the relations of the curvature components $\rho, \sigma, \underline{\beta}$ and $\underline{\alpha}$ between two foliations:

$$\begin{aligned}\rho =&\rho' - 2(\nabla\!\!\!/\, \underline{s}, \underline{\beta}') + \underline{\alpha}'(\nabla\!\!\!/\, \underline{s}, \nabla\!\!\!/\, \underline{s}),\\ \sigma =&\sigma' - 2\nabla\!\!\!/\, \underline{s} \wedge \underline{\beta}' + (\underline{\alpha}' \cdot \nabla\!\!\!/\, \underline{s}) \wedge \nabla\!\!\!/\, \underline{s},\\ \underline{\beta} =&\Omega(\underline{\beta}' - \underline{\alpha}' \cdot \nabla\!\!\!/\, \underline{s}),\\ \underline{\alpha} =&\Omega^2 \underline{\alpha}',\end{aligned}$$

where $\rho', \sigma', \beta', \alpha'$ refer to the curvature components in the affine foliation, and the estimates for $\nabla\!\!\!/^3 \underline{s}$ in $L^2(S_{\underline{u},u})$, we have

$$\begin{aligned}&\int_{\underline{C}_0, 0\leq u\leq\varepsilon} r^{2(i+j-1)} |\nabla\!\!\!/^i \underline{R}_j|^2 \mathrm{d}\mu_{\not{g}_{\underline{u},u}} \mathrm{d}u \\ \lesssim &\underline{\mathcal{R}}_0^2 + \varepsilon C(\mathcal{O}_0, \mathcal{R}_0) \\ &\times \left(\int_{\underline{C}_0} r^{2j} |\nabla\!\!\!/\,' \underline{D}' \underline{R}_j'|^2 + r^{2j-2}(|\underline{D}'^2 \underline{R}_j'|^2 + |\underline{D}' \underline{R}_j'|^2)\right)' + \varepsilon C(\mathcal{O}_0).\end{aligned}$$

The last term comes from the term $\widehat{\chi} \cdot \widehat{\underline{\chi}}$ which is the differences between ρ and $\check{\rho}$, σ and $\check{\sigma}$. For $\underline{\alpha}'$, the second term on the right hand side is contained in the definition of $\underline{\mathcal{R}}_0$. For ρ', σ' and $\underline{\beta}'$, we can use the null Bianchi equations to express their $\underline{D}'$ derivatives in terms of the $\nabla\!\!\!/\,'$ derivatives of $\underline{\beta}'$ and $\underline{\alpha}'$, which are also contained in the definition of $\underline{\mathcal{R}}_0$, and the lower order terms which are a product of one of the connection coefficients and one of $\rho', \underline{\beta}'$ or $\underline{\alpha}'$. The connection coefficients on $\underline{C}_0$ can also be estimated in terms of $\underline{\mathcal{R}}_0$ using the null structure equations along $\underline{D}'$ direction. Finally, by choosing ε sufficiently small, we achieve that

$$\sum_{\underline{R}_p} \int_{\underline{C}_0, 0\le u\le\varepsilon} r^{2(i+j-1)} |\nabla\!\!\!/^{\,i} \underline{R}_j|^2 \mathrm{d}\mu_{\not{g}_{\underline{u},u}} \mathrm{d}u \lesssim \underline{\mathcal{R}}_0^2 + 1,$$

which is the desired estimate.

The initial terms on C_0

$$\int_{C_0} r^{2(i+j-1)} |\nabla\!\!\!/^{\,i} R_{j+1}|^2$$

are also different from the terms in $\mathcal{R}_0$. However, it is much easier because the foliation given by $\underline{u}$ is the same as the foliation given by the affine parameter s on C_0, we will have

$$\beta = \Omega^{-1}\beta',\ \rho = \rho',\ \sigma = \sigma',\ \underline{\beta} = \Omega\underline{\beta}',\ \nabla\!\!\!/ = \nabla\!\!\!/'.$$

Then by Lemma 7, the estimates for Ω, we immediately have, for ε sufficiently small,

$$\int_{C_0} r^{2(i+j-1)} |\nabla\!\!\!/^{\,i} R_{j+1}|^2 \lesssim \mathcal{R}_0^2,$$

which is the desired estimate.

4.4. Canonical last slice. We will carry out the last step of the proof in this subsection. As introduced above, we first extend the solution to a larger region M_δ which corresponds to $0 \le \underline{u} \le \underline{u}_* + \delta$, $0 \le u \le \varepsilon + \delta'$. First of all, δ and δ' should be chosen such that all bounds derived in Section 4.2 and Section 4.3 are bounded by twice of their bounds in M. This is ensured simply by continuity because δ and δ' are allowed to depend on ε and $\underline{u}_*$. In the rest of this subsection, we will fix δ' and then choose δ sufficiently small such that we can construct a new canonical function u_δ varying from 0 to ε.

At this point, we hope to apply Proposition 2 to the new last slice $\underline{C}_{\underline{u}_*+\delta}$. Different from the initial slice $\underline{C}_0$, we should note that, first, we must estimate the bounds for $\underline{\mathcal{R}}[\underline{D\alpha}, \underline{D}^2\underline{\alpha}], \mathcal{O}^2[\underline{D\omega}], \mathcal{O}[\underline{D}^2\underline{\omega}]$ which are not estimated yet, second, on the null cone $\underline{C}_{\underline{u}_*+\delta}$, we have less room as compared to $\underline{C}_0$, to solve the canonical slice equation, because the background foliation u which is canonical on $\underline{C}_{\underline{u}_*}$ only varies from $[0, \varepsilon+\delta']$, and this is exactly the reason why we need the δ' to be positive.

So, we will prove the following as the first step:

Proposition 7. *If ε is sufficiently small depending on $\mathcal{O}_0$, $\mathcal{R}_0$, $\underline{\mathcal{R}}_0$, then we have*

$$\underline{\mathcal{R}}[\underline{D\alpha}, \underline{D}^2\underline{\alpha}], \mathcal{O}^2[\underline{D\omega}], \mathcal{O}[\underline{D}^2\underline{\omega}] \le C(\mathcal{O}_0, \mathcal{R}_0, \underline{\mathcal{R}}_0).$$

Remark 9. This proposition is proven in the origin region $M_{\underline{u}_*,\varepsilon}$. We can then choose δ and δ' sufficiently small such that this proposition still holds in the region $M_{\underline{u}_*+\delta,\varepsilon+\delta'}$ with a twice large bound.

Proof. For the norms about $\underline{\alpha}$, we rely on the null Bianchi equations for $\underline{D}\underline{\beta}$ and $\widehat{D}\underline{\alpha}$. We first commute $\underline{D}$ with the Bianchi equations:

$$
\begin{aligned}
&\widehat{D}\widehat{\underline{D}}\underline{\alpha} - \frac{1}{2}\Omega\mathrm{tr}\chi\widehat{\underline{D}}\underline{\alpha} + \Omega\nabla\!\!\!/\,\widehat{\otimes}\underline{D}\underline{\beta} \\
=&\frac{1}{2}\underline{D}(\Omega\mathrm{tr}\chi)\cdot\underline{\alpha} - 2\omega\widehat{\underline{D}}\underline{\alpha} - 2\underline{D}\omega\cdot\underline{\alpha} + [\widehat{D},\widehat{\underline{D}}]\underline{\alpha} - [\widehat{\underline{D}},\nabla\!\!\!/\,\widehat{\otimes}]\underline{\beta} \\
&-\Omega\underline{\omega}\nabla\!\!\!/\,\widehat{\otimes}\underline{\beta} - \underline{D}\left(\Omega\{(4\underline{\eta}-\zeta)\widehat{\otimes}\underline{\beta} + 3\underline{\widehat{\chi}}\rho - 3{}^*\underline{\widehat{\chi}}\sigma\}\right), \\
&\underline{D}\underline{D}\underline{\beta} - \Omega\underline{\widehat{\chi}}\cdot\underline{D}\underline{\beta} + \Omega\mathrm{d}\!\!\!/\mathrm{iv}\,\widehat{\underline{D}}\underline{\alpha} \\
=&\underline{D}(\Omega\underline{\widehat{\chi}})\cdot\underline{\beta} - [\widehat{\underline{D}},\mathrm{d}\!\!\!/\mathrm{iv}\,]\underline{\alpha} - \Omega\underline{\omega}\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\alpha} \\
&-\underline{D}\left(\frac{3}{2}\Omega\mathrm{tr}\underline{\chi}\underline{\beta} - \underline{\omega}\underline{\beta} + \Omega\{(\eta-2\zeta)\cdot\underline{\alpha}\}\right).
\end{aligned}
$$

As what we have done in the energy estimates, we can establish the following inequality:

$$
\int_{C_u}|\underline{D}\underline{\beta}|^2 + \int_{\underline{C}_{\underline{u}}}|\underline{D}\underline{\alpha}|^2 \lesssim \int_{C_0}|\underline{D}\underline{\beta}|^2 + \int_{\underline{C}_0}|\underline{D}\underline{\alpha}|^2 + \int_M|\tau_1^4|.
$$

The term τ_1^4 comes from $\underline{D}\underline{\alpha}$ times the right hand side of the first equation, plus $\underline{D}\underline{\beta}$ times the right hand side of the second equation. By commuting one more $\nabla\!\!\!/$, we also have

$$
\begin{aligned}
&\int_{C_u}|(r\nabla\!\!\!/)\underline{D}\underline{\beta}|^2 + \int_{\underline{C}_{\underline{u}}}|(r\nabla\!\!\!/)\underline{D}\underline{\alpha}|^2 \\
\lesssim&\int_{C_0}|(r\nabla\!\!\!/)\underline{D}\underline{\beta}|^2 + \int_{\underline{C}_0}|(r\nabla\!\!\!/)\underline{D}\underline{\alpha}|^2 + \int_M|\tau_1^{4(1)}|.
\end{aligned}
$$

The terms which are $\underline{D}$ applying to the connection coefficients are expressed directly using the null structure equations (except $\underline{D}\omega$). Therefore, using the bounds for $\mathcal{O}$, $\mathcal{R}$, $\underline{\mathcal{R}}$, Lemma 8, the Hölder and Sobolev inequalities, for $\varepsilon > 0$ sufficiently small depending on $\mathcal{O}_0$, $\mathcal{R}_0$ and $\underline{\mathcal{R}}_0$, we have

$$
\begin{aligned}
&\int_M|\tau_1^4| + \int_M|\tau_1^{4(1)}| \\
\lesssim&\varepsilon^{1/2}C(\mathcal{O}_0,\mathcal{R}_0,\underline{\mathcal{R}}_0) \\
&+\int_0^{\underline{u}} r^{-2}\mathcal{O}[\omega,\Omega\mathrm{tr}\chi-\overline{\Omega\mathrm{tr}\chi}]\int_{\underline{C}_{\underline{u}'}}\sum_{i=0}^{1}|(r\nabla\!\!\!/)^i\underline{D}\underline{\alpha}|^2\mathrm{d}\underline{u}' \\
&+C(\mathcal{O}_0,\mathcal{R}_0)\left(\int_0^{\underline{u}} r^{-2}\int_{\underline{C}_{\underline{u}'}}\sum_{i=0}^{2}|(r\nabla\!\!\!/)^i\underline{\alpha}|^2\mathrm{d}\underline{u}'\right)^{1/2} \\
&\qquad\times\left(\int_0^{\underline{u}} r^{-2}\int_{\underline{C}_{\underline{u}'}}\sum_{i=0}^{1}|(r\nabla\!\!\!/)^i\underline{D}\underline{\alpha}|^2\mathrm{d}\underline{u}'\right)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&+ \mathcal{O}[\mathrm{tr}\underline{\chi}, \underline{\omega}, \widehat{\underline{\chi}}] \int_0^u \int_{C_{u'}} \sum_{i=0}^{1} |(r\nabla\!\!\!/)^i \underline{D}\underline{\beta}|^2 \mathrm{d}u' \\
&+ \mathcal{O}[\eta, \underline{\eta}] \left(\int_0^u \int_{C_{u'}} \sum_{i=0}^{2} |(r\nabla\!\!\!/)^i \underline{D}\underline{\beta}|^2 \mathrm{d}u' \right)^{1/2} \\
&\qquad \times \left(\int_0^{\underline{u}} r^{-2} \int_{\underline{C}_{\underline{u}'}} \sum_{i=0}^{1} |(r\nabla\!\!\!/)^i \underline{D}\underline{\alpha}|^2 \mathrm{d}\underline{u}' \right)^{1/2}, \\
&+ O^2[\underline{D}\underline{\omega}] \left(\int_0^u \int_{C_{u'}} \sum_{i=0}^{2} |(r\nabla\!\!\!/)^i \underline{\beta}|^2 \mathrm{d}u' \right)^{1/2} \\
&\qquad \times \left(\int_0^u \int_{C_{u'}} \sum_{i=0}^{1} |(r\nabla\!\!\!/)^i \underline{D}\underline{\beta}|^2 \mathrm{d}u' \right)^{1/2},
\end{aligned}
$$

where the terms which are already estimated are collected in the first term on the right hand side. By the Gronwall's inequality, we have

$$
\begin{aligned}
&\mathcal{R}[\underline{D}\underline{\beta}]^2 + \underline{\mathcal{R}}[\underline{D}\underline{\alpha}]^2 \\
\lesssim&\mathcal{R}_0[\underline{D}\underline{\beta}]^2 + \underline{\mathcal{R}}_0[\underline{\alpha}]^2 \\
&+ C(\mathcal{O}_0, \mathcal{R}_0, \underline{\mathcal{R}}_0) \left(\underline{\mathcal{R}}[\underline{D}\underline{\alpha}] + \varepsilon^{1/2} + \varepsilon^{1/2} \mathcal{R}[\underline{D}\underline{\beta}] \underline{\mathcal{R}}[\underline{D}\underline{\alpha}] \right) \\
&+ \varepsilon O^2[\underline{D}\underline{\omega}] \mathcal{R}[\underline{\beta}] \mathcal{R}[\underline{D}\underline{\beta}].
\end{aligned} \tag{4.12}
$$

We remark that the initial terms should be treated in the same manner as in Remark 8.

Therefore, we should first estimate $O^2[\underline{D}\underline{\omega}]$. We commute $\nabla\!\!\!/^i \underline{D}$ with the structure equation for $D\underline{\omega}$ for $i \leq 1$:

$$
D\nabla\!\!\!/^i \underline{D}\underline{\omega} = \nabla\!\!\!/^i \underline{D}(\Omega^2(2(\eta, \underline{\eta}) - |\eta|^2 - \rho)) + [D, \nabla\!\!\!/^i \underline{D}]\underline{\omega}.
$$

We should integrate this equation from the last slice $\underline{C}_{\underline{u}_*}$. Commuting $\underline{D}$ with equation (4.10) which is written on $\underline{C}_{\underline{u}_*}$:

$$
\begin{aligned}
2\Delta\!\!\!/ \, \underline{D}\underline{\omega} =& \underline{D}\bigg(2\mathrm{div}\!\!\!/\,(\Omega\underline{\beta}) + \mathrm{div}\!\!\!/\,(3\Omega\widehat{\underline{\chi}} \cdot \eta + \frac{1}{2}\Omega\mathrm{tr}\underline{\chi}\eta) \\
&\qquad + \Omega\mathrm{tr}\underline{\chi}\mathrm{div}\!\!\!/\,\eta + (F - \overline{F} + \overline{\Omega\mathrm{tr}\underline{\chi}\check{\rho}} - \overline{\Omega\mathrm{tr}\underline{\chi}} \cdot \overline{\check{\rho}}) \bigg) \\
&+ 2[\Delta\!\!\!/\,, \underline{D}]\underline{\omega}, \\
\overline{\underline{D}\underline{\omega}} =& \overline{\Omega\mathrm{tr}\underline{\chi}\underline{\omega}} - \overline{\Omega\mathrm{tr}\underline{\chi}}\overline{\underline{\omega}} - \overline{\underline{D}\Omega\mathrm{tr}\underline{\chi}\log\Omega}.
\end{aligned}
$$

We should pay special attention to the right hand side. Note that, by the null Bianchi equations for $\underline{D}\underline{\beta}$, up to the lower order terms, $\underline{D}\mathrm{div}\!\!\!/\,(\Omega\underline{\beta}) \sim \mathrm{div}\!\!\!/\,\mathrm{div}\!\!\!/\,\underline{\alpha}$, which is not in $L^2(S_{\underline{u},u})$ but only in $L^2(\underline{C}_{\underline{u}})$. But note that we can control $\|(r\nabla\!\!\!/)^i\underline{\alpha}\|_{L^4(S_{\underline{u},u})}$ for $i \leq 1$ using an analogue of Proposition 10.2 in [7] or a special case for $Y = u$ in (4.7), as:

$$\|(r\nabla\!\!\!/)^i\underline{\alpha}\|_{L^4(S_{\underline{u},u})} \lesssim \mathcal{R}_0 + C(\underline{\mathcal{R}}[\alpha], \underline{\mathcal{R}}[\underline{D}\alpha]).$$

Therefore, we should be able to control $\|(r\nabla\!\!\!/)^i\underline{D\omega}\|_{L^4(S_{\underline{u},u})}$. In this paper, we only control $\|(r\nabla\!\!\!/)^i\underline{D\omega}\|_{L^2(S_{\underline{u},u})}$ for simplicity, which is enough for our work.

To do this, we denote $\Delta\!\!\!/\,\underline{D\omega} = -\Omega \mathrm{d}\!\!\!/\mathrm{iv}\, \mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\alpha} + G_{(3)}$ on the last slice $\underline{C}_{\underline{u}_*}$. It is direct to verify that $\|r^2G_{(3)}\|_{L^2(S_{\underline{u},u})} \lesssim C(\mathcal{O}_0, \mathcal{R}_0)$. Then we compute

$$\begin{aligned}\int_{S_{\underline{u}_*,u}} |\nabla\!\!\!/\,\underline{D\omega}|^2 =& -\int_{S_{\underline{u}_*,u}} \Delta\!\!\!/\,\underline{D\omega}(\underline{D\omega} - \overline{\underline{D\omega}})\\ =& \int_{S_{\underline{u}_*,u}} (-\mathrm{d}\!\!\!/\mathrm{iv}\,\mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\alpha} + G_{(3)})(\underline{D\omega} - \overline{\underline{D\omega}})\\ \lesssim& \|G_{(3)}\|_{L^2(S_{\underline{u}_*,u})}\|\nabla\!\!\!/\,\underline{D\omega}\|_{L^2(S_{\underline{u}_*,u})} + \int_{S_{\underline{u}_*,u}} \mathrm{d}\!\!\!/\mathrm{iv}\,\underline{\alpha}\cdot\nabla\!\!\!/\,\underline{D\omega}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{i=0}^{1}\|(r\nabla\!\!\!/)^i\underline{D\omega}\|_{L^2(S_{\underline{u}_*,u})} \lesssim& \|r^2G_{(3)}\|_{L^2(S_{\underline{u}_*,u})} + \|(r\nabla\!\!\!/)\underline{\alpha}\|_{L^2(S_{\underline{u}_*,u})}\\ \lesssim& C(\mathcal{O}_0, \mathcal{R}_0, \underline{\mathcal{R}}_0, \underline{\mathcal{R}}[\underline{D}\alpha]).\end{aligned}$$

Then we apply the Gronwall type estimates to the equations for $D\nabla\!\!\!/^i\underline{D\omega}$ for $i \leq 1$, and conclude that the above estimate hold for all $\underline{u}$.

Substituting this estimate back to (4.12), we have

$$\mathcal{R}[\underline{D}\underline{\beta}] + \underline{\mathcal{R}}[\underline{D}\alpha] \lesssim C(\mathcal{O}_0, \mathcal{R}_0, \underline{\mathcal{R}}_0).$$

This in turn gives the desired estimate of $\|(r\nabla\!\!\!/)^i\underline{D\omega}\|_{L^2(S_{\underline{u},u})}$ for $i \leq 1$. By the Sobolev inequality, we can also bound $\|r^{1/2}\underline{D\omega}\|_{L^4(S_{\underline{u},u})}$.

In order to estimate $\underline{\mathcal{R}}[\underline{D}^2\underline{\alpha}]$, we commute $\underline{D}^2$ with the null Bianchi equations for $\widehat{D}\underline{\alpha}$ and $\underline{D}\underline{\beta}$. We can do the estimates exactly as above, provided that we have the estimate for $\mathcal{O}[\underline{D}^2\underline{\omega}]$. This can be done as the above estimates for $\mathcal{O}^2[\underline{D\omega}]$, by commuting $\underline{D}^2$ with the equation for $\underline{D\omega}$ and equation (4.10). Note that in this case, when considering the Laplacian equation for $\underline{D}^2\underline{\omega}$ on the last slice $\underline{C}_{\underline{u}_*}$, the highest order term of the is $\mathrm{d}\!\!\!/\mathrm{iv}\,\mathrm{d}\!\!\!/\mathrm{iv}\,(\underline{D}\alpha)$, which has no estimates. But we have controlled $\mathrm{d}\!\!\!/\mathrm{iv}\,(\underline{D}\alpha)$ in $L^2(\underline{C}_{\underline{u}})$. Therefore, by a similar argument, we can first control $\|\underline{D}^2\underline{\omega}\|_{L^2(\underline{C}_{\underline{u}_*})}$ and then integrate the equation for $D\underline{D}^2\underline{\omega}$ to obtain the estimate of $\|\underline{D}^2\underline{\omega}\|_{L^2(\underline{C}_{\underline{u}})}$ for all $\underline{u}$. q.e.d.

Now we consider the function space

$$\mathcal{K} = \mathcal{K}_{\underline{u}_*+\delta,\varepsilon+\delta'} \subset C([0,\varepsilon], H^2(S_{\underline{u}_*+\delta,0})),$$

such that

$$\begin{cases} W(0,\theta) = 0, \\ 0 \leq W(s,\theta) \leq \varepsilon + \delta', \\ \sup_s \|(r\nabla\!\!\!/)^{\leq 2}(W(s,\cdot) - s)\|_{L^2(S_{\underline{u}_*+\delta,0})} \leq \varepsilon_{\mathcal{K}}, \end{cases} \tag{4.13}$$

for some small number $\varepsilon_{\mathcal{K}}$ to be fixed. $\mathcal{K}$ is close in $C([0,\varepsilon], H^2(S_{\underline{u}_*+\delta,0}))$. We remark that the second condition ensures that there is enough room such that the new foliation ${}^{(W)}u$ given by $W \in \mathcal{K}$ also varies from 0 to ε. This is exactly why we need δ to be sufficiently small depending on δ'. Obviously we can first choose $\varepsilon_{\mathcal{K}}$ small enough such that the second condition is ensured. Also note that by our definition of $\mathcal{K}$, W need not to represent a foliation, but only a family of sections parameterized by s. As in the first step, we will prove

Proposition 8. *For ε sufficiently small depending on $\mathcal{O}_0$, $\mathcal{R}_0$, $\underline{\mathcal{R}}_0$, and δ sufficiently small (may depending on ε and δ'), $\mathcal{A}(\mathcal{K}) \subset \mathcal{K}$ and $\mathcal{A}$ is a contraction in $\mathcal{K} \subset C([0,\varepsilon], H^2(S_{\underline{u}_*+\delta,0}))$.*

Proof. It is not hard to see the assumptions of Proposition 2 hold for ε sufficiently small depending on $\mathcal{O}_0$, $\mathcal{R}_0$, $\underline{\mathcal{R}}_0$. We use (4.8) for $W_1 = W$ and $W_2 = s$. We recall that $\underline{C}_{\underline{u}_*}$ is a canonical last slice with the foliation given by the origin u, then for any $\varepsilon_* > 0$, by continuity, we can choose δ sufficiently small, such that

$$\begin{aligned} &\|(r\nabla\!\!\!/)^{\leq 2}(\log {}^{(W_{\mathcal{A}})}\Omega(W(s,\cdot),\cdot) - \log \Omega(s,\cdot))\|_{L^2(S_{\underline{u}_*+\delta,0})} \\ \lesssim& C(\mathcal{O}_0, \mathcal{R}_0, \underline{\mathcal{R}}_0)\|(r\nabla\!\!\!/)^{\leq 2}(W(s,\cdot) - s)\|_{L^2(S_{\underline{u}_*+\delta,0})} + \varepsilon_*. \end{aligned}$$

Then we integrate s from 0 to 1,

$$\begin{aligned} &\|(r\nabla\!\!\!/)^{\leq 2}(\mathcal{A}(W)(s,\cdot) - s)\|_{L^2(S_{\underline{u}_*+\delta,0})} \\ \lesssim& \int_0^s \|(r\nabla\!\!\!/)^{\leq 2}({}^{(W_{\mathcal{A}})}\Omega^2(W(s',\cdot),\cdot)\Omega^{-2}(W(s',\cdot),\cdot) - 1)\|_{L^2(S_{\underline{u}_*+\delta,0})} \mathrm{d}s \\ \lesssim& \varepsilon C(\mathcal{O}_0, \mathcal{R}_0, \underline{\mathcal{R}}_0) \sup_s \|(r\nabla\!\!\!/)^{\leq 2}(W(s,\cdot) - s)\|_{L^2(S_{\underline{u}_*+\delta,0})} + \varepsilon\varepsilon_*. \end{aligned}$$

The inequality (4.7) should also be used here because we compare the bound of the background lapse Ω on the sphere $S_{\underline{u}_*+\delta,s}$ to the bound on $S_{W(s)}$. We choose ε small depending only on $\mathcal{O}_0$, $\mathcal{R}_0$, $\underline{\mathcal{R}}_0$, then choose ε_* sufficiently small depending on $\varepsilon_{\mathcal{K}}$ and ε, such that

$$\sup_s \|(r\nabla\!\!\!/)^{\leq 2}(W(s,\cdot) - s)\|_{L^2(S_{\underline{u}_*+\delta,0})} \leq \varepsilon_{\mathcal{K}}.$$

This implies $\mathcal{A}$ maps $\mathcal{K}$ into itself.

Also, we can use (4.8) to conclude that,

$$\begin{aligned} &\|(r\nabla\!\!\!/)^{\leq 2}(\mathcal{A}(W_1)(s,\cdot) - \mathcal{A}(W_2)(s,\cdot))\|_{L^2(S_{\underline{u}_*+\delta,0})} \\ \lesssim& \int_0^s \|(r\nabla\!\!\!/)^{\leq 2}({}^{(W_{\mathcal{A}})}\Omega^2(W_1(s',\cdot))\Omega^{-2}(W_1(s',\cdot),\cdot) \end{aligned}$$

$$
\begin{aligned}
&- {}^{(W_{\mathcal{A}})}\Omega((W_2,\cdot),\cdot)\Omega^{-2}(W_2(s,\cdot),\cdot)\|_{L^2(S_{\underline{u}_*+\delta,0})}\mathrm{d}s\\
\lesssim&\varepsilon C(\mathcal{O}_0,\mathcal{R}_0,\underline{\mathcal{R}}_0)\sup_s\|(r\nabla\!\!\!/\,)^{\leq 2}(W(s,\cdot)-s)\|_{L^2(S_{\underline{u}_*+\delta,0})}.
\end{aligned}
$$

By choosing ε sufficiently small depending only on $\mathcal{O}_0$, $\mathcal{R}_0$, $\underline{\mathcal{R}}_0$, then $\mathcal{A}:\mathcal{K}\to\mathcal{K}$ is a contraction and this completes the proof. q.e.d.

To proceed further, we are going to estimate the difference between two optical functions u_1 and u_2 which are constructed from the canonical foliations on $\underline{C}_{\underline{u}_1}$ and $\underline{C}_{\underline{u}_2}$, respectively, and $\underline{u}_1\leq\underline{u}_2$. The corresponding geometric quantities are labeled by lower index 1 or 2. In the past of $\underline{C}_{\underline{u}_1}$, the null frames are related by

$$
\begin{aligned}
\underline{L}_2'&=\underline{L}_1',\\
L_2&=L_1+(-\frac{1}{2}g(L_1',L_2))\underline{L}_1+2(-\frac{1}{2}g(L_1,L_2))^{1/2}\sigma^A(E_1)_A,\\
(E_2)_A&=(E_1)_A+\Omega_1^{-1}(-\frac{1}{2}g(L_1',L_2))^{1/2}\sigma_A\underline{L}_1.
\end{aligned}
$$

Here σ^1, σ^2 satisfy $\sigma^A\sigma^Bg((E_1)_A,(E_1)_B)=1$. This relation is exactly the relation (4.2).

Along the null generators of $(C_2)_{u_2}$, using the above relation, we compute

$$
\begin{aligned}
&D_2(u_1-u_2)\\
=&[L_1+(-\frac{1}{2}g(L_1',L_2))\underline{L}_1+2(-\frac{1}{2}g(L_1,L_2))^{1/2}\sigma^A(E_1)_A](u_1)\\
=&-\frac{1}{2}g(L_1',L_2),
\end{aligned}
\tag{4.14}
$$

and

$$
\underline{D}_2(u_1-u_2)=\Omega_2^2\Omega_1^{-2}-1.
\tag{4.15}
$$

We should consider $g(L_1',L_2)$ to estimate u_1-u_2. Then we compute

$$
\begin{aligned}
&D_2g(L_1',L_2)\\
=&\omega_2g(L_1',L_2)+g(\nabla_{L_2}L_1',L_2)\\
=&\omega_2g(L_1',L_2)-2g(L_1',L_2)\sigma^A\sigma^B\Omega_1(\widehat{\chi}_1)_{AB}-g(L_1',L_2)\Omega_1(\mathrm{tr}\chi)_1\\
&-4g(L_1',L_2)(-\frac{1}{2}g(L_1,L_2))^{1/2}\sigma^A(\eta_1)_A-g(L_1',L_2)^2\underline{\omega}_1,
\end{aligned}
\tag{4.16}
$$

and

$$
\begin{aligned}
&\underline{D}_2g(L_1',L_2)\\
=&-2\Omega_1^{-2}\Omega_2^2\underline{\omega}_1g(L_1',L_2)+4\Omega_1^{-2}\Omega_2^2(-g(L_1,L_2))^{1/2}(\sigma_1^A(\eta_1)_A+\sigma_2^A(\eta_2)_A).
\end{aligned}
\tag{4.17}
$$

Here σ_1 and σ_2 satisfy the same property as σ.

Now we go back to the canonical foliation given by u_δ on the new last slice $\underline{C}_{\underline{u}_*+\delta}$, which we have constructed above. The origin optical function u which is canonical on $\underline{C}_{\underline{u}_*}$ is also extended up to $\underline{C}_{\underline{u}_*+\delta}$ and the difference between u and u_δ on $\underline{C}_{\underline{u}_*+\delta}$ is controlled by $\varepsilon_{\mathcal{K}}$, see (4.13). Note that we can choose $\varepsilon_{\mathcal{K}}$ arbitrarily small by setting δ sufficiently small.

Our goal is to extend u_δ as an optical function back to the initial null cone $\underline{C}_0$. This means that both u_δ and u satisfy the above system of equations (4.14)–(4.17) with $u_1 = u$ and $u_2 = u_\delta$. With the "initial data" on $\underline{C}_{\underline{u}_*+\delta}$ which are close to each other, we can conclude that the extension of u_δ to $\underline{C}_0$ can be done. This is essentially the continuity of equations (4.14) and (4.16). Again by continuity, the curvature norms $\mathcal{R}_\delta$, $\underline{\mathcal{R}}_\delta$ expressed in the new foliation $(\underline{u}, u_\delta)$ is bounded by twice the norms $\mathcal{R}, \underline{\mathcal{R}}$ in the old foliation $(\underline{u}, u)$. Therefore, we complete Step 4 of the proof.

4.5. Global optical function and recovering the full decay. Finally, we will construct the global retarded time function u to complete the proof. We also appeal to equations (4.14)–(4.17). Let u_1 and u_2 be the optical functions from the canonical foliations on $\underline{C}_{\underline{u}_1}$ and $\underline{C}_{\underline{u}_2}$, respectively, and $\underline{u}_1 \le \underline{u}_2$. The space-time is constructed by sending $\underline{u}_1$ to $+\infty$ and we will consider the convergence of the function $\underline{u}_1$. Therefore, we need to compare $\underline{u}_1$ to $\underline{u}_2$ for any $\underline{u}_2 \ge \underline{u}_1$.

We first consider (4.17) on $\underline{C}_{\underline{u}_1}$. Because $L_1 = L_2$ on C_0, then $g(L_1', L_2) = 0$ on C_0. We assume on $\underline{C}_{\underline{u}_1}$, $|r_2^4(\underline{u}_1) g(L_1', L_2)| \lesssim \Delta_4$[17] . Then by integrating (4.17), we have

$$|r_2^4(\underline{u}_1) g(L_1', L_2)| \lesssim \varepsilon r_2^{-1}(\underline{u}_1) C(\mathcal{O}_0, \mathcal{R}_0)\Delta_4 + \varepsilon C(\mathcal{O}_0, \mathcal{R}_0)\Delta_4^{1/2}.$$

By choosing ε sufficiently small, we have $|r_2^4(\underline{u}_1) g(L_1', L_2)| \lesssim \frac{1}{2}\Delta_4$. Therefore, we have deduce that $|r_2^4(\underline{u}_1) g(L_1', L_2)| \lesssim C(\mathcal{O}_0, \mathcal{R}_0)$ on $\underline{C}_{\underline{u}_1}$.

Now consider equation (4.16) in the past of $\underline{C}_{\underline{u}_1}$. We are going to apply the Gronwall type estimates to integrate this equation from $\underline{C}_{\underline{u}_1}$ to its past. The term $\omega_2 g(L_1', L_2) - 2g(L_1', L_2)\sigma^A \sigma^B \Omega_1 (\widehat{\chi}_1)_{AB}$ can be absorbed by the Gronwall's inequality because both ω_2 and $\widehat{\chi}_1$ decay[18] not slower than r_2^{-2}. On the other hand, by the structure equation for $\underline{D}\mathrm{tr}\chi$, we can estimate, in the foliations given by u_1, u_2,

$$|\Omega_i \mathrm{tr}\chi_i - \Omega_i \mathrm{tr}\chi_i|_{S_{\underline{u},0}}| \lesssim \varepsilon^{1/2} C(\mathcal{O}_0, \mathcal{R}_0) \frac{1}{r_i^{-2}},$$

for $i = 1, 2$. By construction $\Omega_1 \mathrm{tr}\chi_1|_{S_{\underline{u},0}} = \Omega_2 \mathrm{tr}\chi_2|_{S_{\underline{u},0}}$, we can estimate

$$|\Omega_1 \mathrm{tr}\chi_1 - \Omega_2 \mathrm{tr}\chi_2| \lesssim C(\mathcal{O}_0, \mathcal{R}_0) \frac{1}{r_2^{-2}}.$$

[17] r_2 should also depend on u_2 but we only emphasize the dependence on $\underline{u}_1$ here.
[18] Recall that both r_1 and r_2 are equivalent to $(1 + \underline{u})$.

Therefore, replacing the term $\Omega_1\mathrm{tr}\chi_1$ by $\Omega_2\mathrm{tr}\chi_2$ will add terms decaying not slower than r_2^{-2}, which can be absorbed by the Gronwall's inequality. We are in the position to apply the Gronwall type estimates,

$$\begin{aligned}&|r_2^2(\underline{u})g(L_1', L_2)|\\ \lesssim&|r_2^2(\underline{u}_1)g(L_1', L_2)|_{\underline{C}_{\underline{u}_1}}\\ &+\int_{\underline{u}}^{\underline{u}_1} r_2^2(\underline{u}')|(-g(L_1', L_2))^{3/2}\sigma^A(\eta_1)_A - g(L_1', L_2)^2\underline{\omega}_1|\mathrm{d}\underline{u}'.\end{aligned}$$

We argue by bootstrap again. We assume $|r_2^2(\underline{u})g(L_1', L_2)| \leq \Delta_5 r_2^{-2}(\underline{u}_1)$. Taking into account that $|r_2^4(\underline{u}_1)g(L_1', L_2)| \lesssim C(\mathcal{O}_0, \mathcal{R}_0)$ on $\underline{C}_{\underline{u}_1}$, we have

$$\begin{aligned}&|r_2^2(\underline{u})g(L_1', L_2)|\\ \lesssim&(r_2^{-2}(\underline{u}_1) + (\Delta_5^{3/2} r_2^{-3}(\underline{u}_1) + \Delta_5^2 r_2^{-4}(\underline{u}_1))r_2^{-2}(0))C(\mathcal{O}_0, \mathcal{R}_0).\end{aligned}$$

Now we can choose $r_2^{-2}(\underline{u}_1)$ sufficiently small, i.e., $\underline{u}_1$ sufficiently large, and Δ_5 sufficiently large, such that $|r_2^2(\underline{u})g(L_1', L_2)| \lesssim \frac{1}{2}\Delta_5 r_2^{-2}(\underline{u}_1)$. Then we conclude that

$$|r_2^2(\underline{u})g(L_1', L_2)| \lesssim C(\mathcal{O}_0, \mathcal{R}_0) r_2^{-2}(\underline{u}_1).$$

We go back to equations (4.15) and (4.14). By (4.15) on $\underline{C}_{\underline{u}_1}$, and $u_1 = u_2$ on C_0, we conclude that $|u_1 - u_2| \lesssim C(\mathcal{O}_0, \mathcal{R}_0) r_2^{-1}(\underline{u}_1)$. Then by (4.14) in the past of $\underline{C}_{\underline{u}_1}$, we have

$$\begin{aligned}|u_1 - u_2| \lesssim&|u_1 - u_2|_{\underline{C}_{\underline{u}_1}} + \int_{\underline{u}}^{\underline{u}_1} |g(L_1', L_2)|\mathrm{d}\underline{u}'\\ \lesssim&C(\mathcal{O}_0, \mathcal{R}_0) r_2^{-1}(\underline{u}_1),\end{aligned}$$

which means that, as $\underline{u}_1$ tends to infinity, u_1 will converge to a global function u.

It is direct to check the convergence of the derivatives of u_1. Now consider $\nabla(u_1 - u_2) = -\frac{1}{2}(L_1' - L_2')$. Because $u_1 - u_2 \to 0$, we have $\Omega_1 - \Omega_2 \to 0$. Also, the above computation shows that $g(L_1', L_2) \to 0$. By the relation between L_1 and L_2, we have $L_1' - L_2' \to 0$ and we conclude $L_1' = -2\nabla u_1 \to L' = -2\nabla u$ as $\underline{u}_1 \to +\infty$. In particular, $|\nabla u| = 0$ and u is a global optical function.

To verify that, under the global double null foliation $(\underline{u}, u)$, the curvature norms $\mathcal{R}, \underline{\mathcal{R}} \leq C(\mathcal{O}_0, \mathcal{R}_0, \underline{\mathcal{R}}_0)$, we need to show that the norms written in $(\underline{u}, u_1)$, $\mathcal{R}_1$, $\underline{\mathcal{R}}_1$ converge to $\mathcal{R}$ and $\underline{\mathcal{R}}$. To do this, it is sufficient to show that $\nabla\!\!\!/_2^i g(L_1', L_2) \to 0$ as $\underline{u}_1 \to +\infty$ for $i \leq 2$ in suitable norms. The case $i = 0$ is proved just above. For the case $i = 1$ we compute

$$(E_2)_A g(L_1', L_2) = g(\nabla_{(E_2)_A} L_1', L_2) + g(L_1', \nabla_{(E_2)_A} L_2).$$

For the first term on the right hand side, we express $(E_2)_A$ and L_2 in terms of $(E_1)_A$, $\underline{L}_1$ and L_1. Then the first term on the right hand side equals to the sum of the form $\Gamma_1 \cdot g(L_1', L_2)^\nu$ with the power $\nu \geq 1$. For the second term on the right hand side, we express $\nabla_{(E_2)_A} L_2 = \Omega_2(\chi_2)_A^C (E_2)_C + \Omega_2(\underline{\eta}_2)_A L_2$, and then express the E_2 and L_2 in terms of E_1, L_1 and $\underline{L}_1$. Therefore, the second term on the right hand side equals to the sum of $\Gamma_2 \cdot g(L_1', L_2)^\nu$ with $\nu \geq 1$. We already have $g(L_1', L_2) \to 0$ and $E_2 \to E$ where E is tangent to the sphere $S_{\underline{u},u}$, therefore, $\nabla\!\!\!/_2 g(L_1', L_2) \to 0$ as $\underline{u}_1 \to +\infty$. The second order derivatives are done in a similar way, involving one order derivative of the connection coefficients. Then $\nabla\!\!\!/_2^2 g(L_1', L_2) \to 0$ in $L^4(S_{\underline{u},u})$.

At last, if we denote

$$\begin{aligned}
\mathcal{R}[\alpha] &= \sup_u \|(r\nabla\!\!\!/)^{\leq 2}(r^2\alpha)\|_{L^2(C_u)}, \\
\underline{\mathcal{R}}[\beta] &= \sup_{\underline{u}} \|(r\nabla\!\!\!/)^{\leq 2}(r^2\beta)\|_{L^2(\underline{C}_{\underline{u}})}, \\
\mathcal{R}[D\alpha] &= \sup_u \|(r\nabla\!\!\!/)^{\leq 1}(r^3 D\alpha)\|_{L^2(C_u)}, \\
\underline{\mathcal{R}}[D\beta] &= \sup_{\underline{u}} \|(r\nabla\!\!\!/)^{\leq 1}(r^3 D\beta)\|_{L^2(\underline{C}_{\underline{u}})}, \\
\mathcal{R}[D^2\alpha] &= \sup_u \|r^4 D^2\alpha\|_{L^2(C_u)}, \\
\underline{\mathcal{R}}[D^2\beta] &= \sup_{\underline{u}} \|r^4 D^2\beta\|_{L^2(\underline{C}_{\underline{u}})}.
\end{aligned}$$

We denote $\mathcal{R}_0[\alpha, D\alpha, D^2\alpha]$ to be the corresponding norms taken on C_0 and $\underline{\mathcal{R}}_0[\beta, D\beta, D^2\beta]$ to be the corresponding norms taken on $\underline{C}_0$. We can show that if, in addition,

$$\mathcal{R}_0[\alpha, D\alpha, D^2\alpha] + \underline{\mathcal{R}}_0[\beta, D\beta, D^2\beta] < \infty,$$

then $\mathcal{R}[\alpha, D\alpha, D^2\alpha], \underline{\mathcal{R}}[\beta, D\beta, D^2\beta] \leq C(\mathcal{O}_0, \mathcal{R}_0, \underline{\mathcal{R}}_0)$, where we have included the above two groups of initial norms into the definition of $\mathcal{R}_0, \underline{\mathcal{R}}_0$. Such a decaying condition on α comes from the works of Christodoulou–Klainerman [**8**] and Klainerman–Nicolò [**11**], which deal with the so-called strongly asymptotically flat Cauchy data.

We firstly consider the case for α itself. For the angular derivatives $\nabla\!\!\!/$, the proof is similar. We use the null Bianchi equations for $\underline{D}\alpha$ and $D\beta$ to obtain

$$\int_{C_u} |r^2\alpha|^2 + \int_{\underline{C}_{\underline{u}}} 2|r^2\beta|^2 \lesssim \int_{C_0} |r^2\alpha|^2 + \int_{\underline{C}_0} 2|r^2\beta|^2 + \int_M r^4|\tau_4^{(0)}|.$$

We estimate

$$\begin{aligned}
\int_M r^4|\tau_4^{(0)}| \lesssim \int_M r^4 |&\Gamma[\mathrm{tr}\underline{\chi}, \underline{\omega}] \cdot \alpha \cdot \alpha + \Gamma[\eta, \underline{\eta}] \cdot \beta \cdot \alpha \\
&+ \widehat{\chi} \cdot R[\rho, \sigma] \cdot \alpha + \Gamma[\mathrm{tr}\chi, \omega] \cdot \beta \cdot \beta|.
\end{aligned}$$

To estimate this term, we put all the curvature components in suitably weighted $L^2(C_u)$ norm and obtain

$$\int_M r^4|\tau_4^{(0)}| \lesssim \varepsilon\mathcal{O}(\mathcal{R}[\alpha]^2 + \mathcal{R}[\beta,\rho,\sigma]\mathcal{R}[\alpha] + \mathcal{R}[\beta]^2).$$

Choosing ε sufficiently small we can derive the desired bound for $\mathcal{R}[\alpha]$ and then the bound for $\underline{\mathcal{R}}[\beta]$.

Remark 10. Though the proof is quite direct we also make some remarks on this estimate. We find that the decay rates of α and β are the same, and the decay rate implied by the norm on incoming null cone $\underline{\mathcal{R}}[\beta]$ is weaker than that implied by the norm on outgoing null cone $\mathcal{R}[\beta]$, which is not the case for other components $\rho, \sigma, \underline{\beta}$. Therefore, unlike the proof of Proposition 6, though $\mathrm{tr}\chi$ does appear in the error terms as $\mathrm{tr}\chi|\beta|^2$, we can also estimate this term by using $\mathcal{R}[\beta]$ instead of $\underline{\mathcal{R}}[\beta]$.

We then go to $D\alpha$. We commute D with the null Bianchi equations for $\underline{\widehat{D}}\alpha$-$D\beta$, and obtain

$$\begin{aligned}
&\underline{\widehat{D}}\widehat{D}\alpha - \Omega\nabla\!\!\!/\,\widehat{\otimes}D\beta \\
=&[\underline{\widehat{D}}, \widehat{D}]\alpha + [\widehat{D}, \nabla\!\!\!/\,\widehat{\otimes}]\beta + \Omega\omega\nabla\!\!\!/\,\widehat{\otimes}\beta \\
&- \widehat{D}\left(\Omega\{-\frac{1}{2}\mathrm{tr}\underline{\chi}\alpha - 2\underline{\omega}\alpha - (4\eta+\zeta)\widehat{\otimes}\beta + 3\widehat{\chi}\rho + 3^*\widehat{\chi}\sigma\}\right), \\
&DD\beta - \Omega\widehat{\chi}\cdot D\beta - \Omega\mathrm{d}\!\!\!/\mathrm{iv}\,\widehat{D}\alpha \\
=&-\frac{3}{2}\Omega\mathrm{tr}\chi D\beta - \frac{3}{2}D(\Omega\mathrm{tr}\chi)\beta + D(\Omega\widehat{\chi})\cdot\beta \\
&+ [\widehat{D}, \mathrm{d}\!\!\!/\mathrm{iv}]\alpha + \Omega\omega\mathrm{d}\!\!\!/\mathrm{iv}\,\alpha + D\left(\omega\beta + \Omega\{(\underline{\eta}+2\zeta)\cdot\alpha\}\right).
\end{aligned}$$

The terms which are D applying to the connection coefficients (except $D\omega$) are expressed directly by the null structure equations, and $D\omega$ is estimated by commuting D with the equation for $\underline{D}\omega$[19] . We can then complete the estimates by energy methods as above. We should note that, the first term on the right hand side of the second equation $-\frac{3}{2}\Omega\mathrm{tr}\chi D\beta$ suggests the weight function r^3, and β which appears in the second term $-\frac{3}{2}D(\Omega\mathrm{tr}\chi)\beta$ should be estimated by using $\mathcal{R}[\beta]$ instead $\underline{\mathcal{R}}[\beta]$ to gain enough decay, see Remark 10. There is another approach to the term $\frac{3}{2}\Omega\mathrm{tr}\chi D\beta$. According to the null Bianchi equation, $D\beta \sim \mathrm{d}\!\!\!/\mathrm{iv}\,\alpha$ and can be estimated by using $\mathcal{R}[\alpha]$ up to the lower order terms. This gains enough decay in the energy estimates. The estimate for $\nabla\!\!\!/\,D\alpha$-$\nabla\!\!\!/\,D\beta$ and then $D^2\alpha$-$D^2\beta$ are similar.

[19]Unlike in the proof of Proposition 7, the estimate for $D\omega$ is not coupled with $D\alpha$ because we simply integrate the equation from initial null cone C_0.

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Department of Mathematics
Sun Yat-sen University
Guangzhou
China
E-mail address: lijunbin@mail.sysu.edu.cn

Department of Mathematics
Sun Yat-sen University
Guangzhou
China
E-mail address: stszxp@mail.sysu.edu.cn