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ON REALIZATION OF TANGENT CONES OF HOMOLOGICALLY AREA-MINIMIZING COMPACT SINGULAR SUBMANIFOLDS

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Dedicated to Professor H. Blaine Lawson, Jr. on the occasion of his 75th birthday

Abstract

We show that every oriented area-minimizing cone in [**Law91**] can be realized as a tangent cone at a singular point of some homologically area-minimizing singular compact submanifold.

1. Introduction

Let C be a k-dimensional cone $\{tx : t \geq 0,\ x \in L\}$ over link $L = C \bigcap S^{N-1}(1)$ in an Euclidean space $(\mathbb{R}^N, g_E)$. It is called area-minimizing if the truncated cone $C_1 \triangleq C \bigcap \mathbf{B}^N(1)$ has least mass among all integral currents (see [**FF60**]) with boundary L.

Similarly, we say a d-closed compactly supported integral current in a Riemannian manifold is homologically area-minimizing if it has least mass in its homology class of integral currents.

A well-known result of Federer (Theorem 5.4.3 in [**Fed69**], also see Theorem 35.1 and Remark 34.6 (2) in Simon [**Sim83**]) asserts that a tangent cone at a point of an area-minimizing rectifiable current is itself area-minimizing. This paper studies its converse realization question by compact submanifolds ($\star$):

> *Can any area-minimizing cone be realized as a tangent cone at a point of some homologically area-minimizing* compact *singular submanifold?*

Through techniques of geometric analysis and Allard's regularity theorem, N. Smale found realizations for all strictly minimizing, strictly stable hypercones (see [**HS85**]) in [**Sma99**]. They are first examples of codimension one (closed) homological area-minimizers with singularities.

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Very recently, different realizations of many area-minimizing cones, including all homogeneous minimizing hypercones (completely classified by G. Lawlor [**Law91**], also see the partial classification in [**Law72**] and related discussions in [**Zha16**]) and special Lagrangian cones, by extending local calibration pairs were discovered in [**Zha17**].

However, in general the answer to ($\star$) is still far to be known. In this paper, we focus on the oriented area-minimizing cones gained in [**Law91**]. Lawlor studied when certain preferred bundle structure (somehow analogous to that in [**HS85**] for hypercones, but without the limitation to codimension one) of some angular neighborhood of a minimal cone can exist, and successfully added quite a few interesting new oriented area-minimizing cones (and non-orientable area-minimizing cones in the sense of modulo 2 as well, but we shall restrict ourselves to the oriented situation in this paper). In the oriented case, such bundle structure naturally induces a "calibration" of the cone that is singular in a set of codimension one and possibly also in the cone. In §5, we modify Lawlor's calibration in a bit larger angular neighborhood so that the resulting calibration is smooth in the cone (except at the origin).

By virtue of such calibrations, we obtain realizations.

Main Theorem. *Every oriented area-minimizing cone in* [**Law91**] *can be realized to* ($\star$).

Remark 1.1. To the author's knowledge, it covers all known minimizing hypercones. Our method here is different from that of N. Smale [**Sma99**]. It would be interesting to consider whether every minimizing hypercone can be realized. Two closely related questions raised by Simon, Hardt and Simon, respectively, are

Q1: Except trivial examples in low dimensions, are all minimizing hypercones strictly area-minimizing?

Q2: Is any non-trivial strictly area-minimizing hypercone always strictly stable?

Remark 1.2. Moreover, this answers affirmatively to ($\star$) for lots of area-minimizing cones of higher codimensions, for instance, a minimal cone C over a product of two or more spheres satisfying (1) $\dim(C) > 7$, or (2) $\dim(C) = 7$ with none of the spheres being a circle (see Theorem 5.1.1 in [**Law91**]). These cones do not split. Namely, they cannot be written as products of two or more area-minimizing cones of lower dimensions (vs. N. Smale [**Sma00**]).

The paper is organized as follows. In §2 our preferred model S of construction is introduced. By a monotonicity result of Allard, we get Lemma 3.1 which helps us, in §4, transform the global realization question to a local problem around S. Thus, we only need to construct a smooth metric $\hat{g}$ on some neighborhood W of S such that S is homologically area-minimizing in W.

In §5 realizations of oriented area-minimizing cones in [**Law91**] are constructed. The idea is the following. First, as mentioned above, we can obtain a "calibration", which is smooth in the cone except at the origin, and the cost is that the resulting form is discontinuous in the union of boundaries of two angular neighborhoods (Lawlor's is discontinuous in the boundary of one angular neighborhood). Then suitably extend the modified (local, singular and non-coflat) "calibrations" around two singular points p_1 and p_2 of S (see §2) to a "closed form" Φ in a neighborhood of S. Next, we create a smooth metric such that Φ becomes a "calibration" of S. By a mollification argument, the desired local homological area-minimality of S in $(W, \hat{g})$ can be achieved.

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2. Model of construction

Given a k-dimensional cone $C \subset \mathbb{R}^N$. As in [**Sma99**], consider

$$\Sigma_C \triangleq (C \times \mathbb{R}) \bigcap S^N(1)$$

in $\mathbb{R}^N \times \mathbb{R}$. Let M be an embedded oriented connected compact k-dimensional submanifold in some N-dimensional oriented compact manifold T with $[M] \neq [0] \in H_k(T; \mathbb{Z})$. Within smooth balls around a point of M and a regular point of Σ_C, respectively, one can connect T and $S^N(1)$, M and Σ_C simultaneously through one connected sum. Denote by X and S the resulting manifold and submanifold (singular at two points p_1 and p_2). Apparently $[S] \neq [0] \in H_k(X; \mathbb{Z})$.

3. Positive lower bound of mass

The lemma below will play a key role in §4. Readers who are unfamiliar with GMT may want to consult [**Fed69**] or [**Sim83**].

Lemma 3.1. *Let g be a metric on a compact manifold X, $W \Subset X$ an open domain with closure $\overline{W}$ forming a manifold with nonempty boundary $\partial\overline{W}$, and α a positive number. Then there exists $\beta = \beta_{\alpha, g|_{\overline{W}}} > 0$ such that every rectifiable current K in W with no boundary, vanishing generalized mean curvature vector field δK and at least one point in its support α away from $\partial\overline{W}$ has mass greater than β.*

Proof. By Nash's embedding theorem [**Nas56**], $(\overline{W}, g|_{\overline{W}})$ can be isometrically embedded through a map f into an Euclidean space $(\mathbb{R}^s, g_E)$. Then $f_{\#}K$ is a rectifiable current of $f(\overline{W})$. Denote the induced varifold by $V_{f_{\#}K}$. Since K has no boundary in W and δK vanishes, the norm of $\delta V_{f_{\#}K}$ in $\mathbb{R}^s$ is bounded from above a.e. by a constant A depending only on f.

Let $\overline{W_\alpha} = \{x \in W : \mathrm{dist}_g(x, \partial\overline{W}) \geq \alpha\}$, $2\mu \triangleq \mathrm{dist}_{g_E}(f(\overline{W_\alpha}), f(\partial\overline{W}))$. Note that the density of $V_{f_{\#}K}$ is a.e. at least one on the support set $\mathbf{spt}(f_{\#}K)$ of $f_{\#}K$. So there exists some point $p \in \mathbf{spt}(f_{\#}K) \bigcap f(W)$ with $\lambda \triangleq \mathrm{dist}_{g_E}(p, f(\partial\overline{W})) > \mu$ and density at least one.

By applying the following monotonicity result of Allard to A, p, μ and U the open λ-ball centered at p, we obtain our statement.

Theorem ([**All72**])**.** *Suppose $0 \leq A < \infty$, $p \in$ support of $\|V\|$, $V \in \mathbf{V}_m(U)$, where U is an open region of $\mathbb{R}^s$. If $0 < \mu <$ dist$_{g_E}(p, \mathbb{R}^s - U)$ and*

$$\|\delta V\| \mathbf{B}(p, r) \leq A \|V\| \mathbf{B}(p, r) \qquad \textit{whenever } 0 < r \leq \mu,$$

then $r^{-m}\|V\|\mathbf{B}(p, r) \exp Ar$ is nondecreasing in r for $0 < r \leq \mu$. q.e.d.

4. Reduction of $(\star)$ from global to local

The following theorem indicates that the essential difficulty of $(\star)$ comes from local. Hence, in §5 we make constructions on some neighborhood of S only.

Theorem 4.1. *Suppose S is homologically area-minimizing in $(U, \bar{g})$ where U is an open neighborhood of S and $\bar{g}$ is a smooth metric on U. Then there exists a smooth metric $\hat{g}$ on the compact manifold X such that S is homologically area-minimizing in $(X, \hat{g})$.*

Proof. Take open neighborhoods W, W' and W'' of S so that $W'' \Subset W' \Subset W \Subset U$ and the closure of W (W' and W'', respectively) is a manifold with nonempty boundary. Extend $\bar{g}$ to a metric $\tilde{g}$ on X with

$$\tilde{g}|_W = \bar{g}|_W.$$

Set $\alpha = \mathrm{dist}_{\tilde{g}}(\partial\overline{W'}, \partial\overline{W})$. Let β be the lower bound in Lemma 3.1 for α, domain $\overline{W'}^c$ and $\tilde{g}|_{\overline{W'}^c}$. Choose $\gamma = (t\beta^{-1}\mathrm{Vol}_{\tilde{g}}(S))^{-\frac{2}{k}} < 1$ for some large constant $t > 1$. Then construct $\hat{g}$ as follows.

$$\hat{g} = \begin{cases} \gamma\tilde{g} & \text{on } W'', \\ h\tilde{g} & \text{on } W'' \sim W', \\ \tilde{g} & \text{on } X \sim W', \end{cases} \tag{1}$$

where h is a smooth function on $\overline{W'} \sim W''$, no less than γ and equal to one near $\partial\overline{W'}$.

Now we show that S is homologically area-minimizing in $(X, \hat{g})$.

By the celebrated compactness result of Federer and Fleming [**FF60**] there exists an area-minimizing current T in $[S]$ with nonempty $\mathbf{spt}(T)$.

Case One: $\mathbf{spt}(T)$ is not contained in W. According to our construction, $\mathbf{M}(S) = \frac{\beta}{t} < \beta < \mathbf{M}(T)$ by Lemma 3.1. Contradiction.

Case Two: $\mathbf{spt}(T) \subset W$. By assumption and (1) S is homologically area-minimizing in $(W, \hat{g}|_W)$. As a result, S and T share the same mass. Hence, S is homologically area-minimizing in $(X, \hat{g})$. q.e.d.

Remark 4.2. $[S] \neq [0] \in H_k(X; \mathbb{Z})$ is crucial in our proof.

5. Proof of Main Theorem

5.1. Lawlor's idea. Let us briefly review the method in [**Law91**]. There Lawlor found oriented minimizing k-cones by constructing particular "calibrations" discontinuous in boundary $\mathfrak{B}$ of some open angular neighborhood $\mathcal{N}$ (to be explained below) for each of them. More precisely, let C be such a cone in $\mathbb{R}^N$. Then he constructed "calibrations" on $\mathcal{N}$ of form

$$\phi = d(f\varpi^*\psi).$$

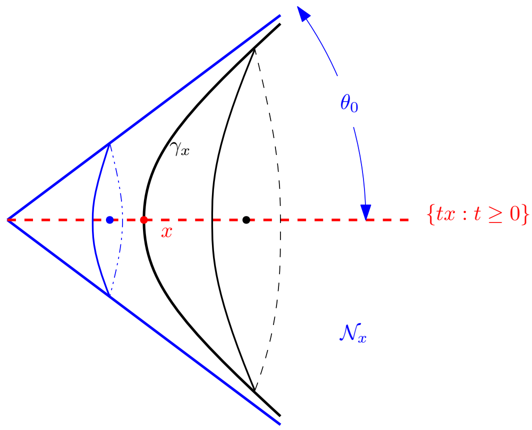
Here $\psi = \mathbf{r}\omega$, defined on $C \sim 0$, where ω is the unit volume form of the link L of C and $\mathbf{r}$ is the distance function in C to the origin; ϖ is the projection to nearest point on C; and f is C^2 in C and smooth elsewhere on $\mathcal{N}$, Lipschitzian in $\mathfrak{B}$ with value zero on $(\overline{\mathcal{N}})^c$.

The rough geometric idea is the following. By the beautiful characterization Theorem 2.1 of Hardt and Simon [**HS85**], in each side of an area-minimizing hypercone, there is a unique dilation-invariant foliation of minimal hypersurfaces. Therefore, the unit norm vector fields lead to a foliation of their integral curves on $\mathbb{R}^N \sim 0$, which is also invariant under dilation. It naturally induces an area-nonincreasing projection (for surfaces of codimension one) to the hypercone. Alternatively, Lawlor looked for similar structure in suitable angular neighborhood of a minimal cone instead of in the whole space $\mathbb{R}^N$. If the boundary of the neighborhood happens to be mapped to the origin under the projection, then one can smash everything outside the neighborhood to the origin. In this way an area-nonincreasing projection can be produced.

For cones of higher codimension, Lawlor considered the structure given by rotation of a suitable curve γ_x in each normal wedge

$$\mathcal{N}_x = \varpi^{-1}(\{tx : t > 0\}) \bigcap \mathcal{N},$$

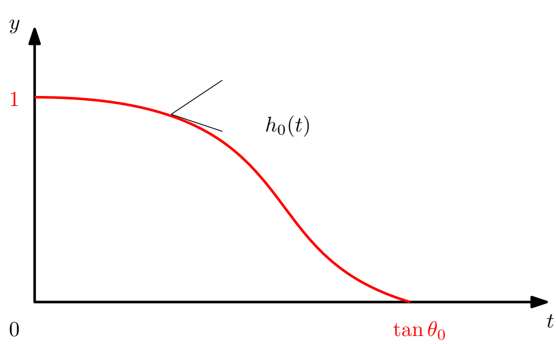
for $x \in L$. Positive homotheties of the surface rotated by γ_x should foliate $\mathcal{N}_x$ as illustrated below. Choose $f(q) = h(\tan\theta(q))$, where $\theta(q)$ is



the angle between $\overrightarrow{0q}$ and $\overrightarrow{0(\varpi(q))}$. Set $t = \tan\theta(q)$. Quite interestingly, by Section 2.4 in [**Law91**], the area-nonincreasing requirement on the projection along leaves (that can be given in terms of a first order ordinary differential inequality) turns out to be equivalent to the comass no larger than one ordinary differential inequality of ϕ:

$$(h - \frac{t}{k}h')^2 + (\frac{h'}{k})^2 \leq (p(t))^2, \qquad h(0) = 1, \tag{2}$$

where $p(t) = \inf_\nu \det(I - th^\nu_{ij}) = 1 + p_2 t^2 + \cdots$ with negative p_2. Here h^ν_{ij} means the second fundamental form for ν of L in $S^{N-1}(1)$ and ν runs among unit normals of the cone at x.



If, for each $x \in L$, (2) had a solution h_0 that reaches zero fastest, say at $t = \tan\theta_0(x)$, and these normal wedges of angle $\theta_0(x)$ do not intersect with each other, then their union $\mathcal{N}$ is the narrowest neighborhood of C with the desired area-nonincreasing projection. Under these assumptions, ϖ is well-defined in $\mathcal{N}$ and ϕ extends on $\mathbb{R}^N \sim 0$ by $d\tilde{\psi}$ where

$$\tilde{\psi} = \begin{cases} f\varpi^*\psi & \text{in} \quad \mathcal{N}, \\ 0 & \text{in} \quad \mathcal{N}^c. \end{cases}$$

Since ϕ is the exterior differential of Lipschitz form $\tilde{\psi}$, through mollifications one can show that cones with such calibrations are area-minimizing. In fact, the projection directions in $\mathcal{N}$ are pointwise perpendicular to the dual of ϕ and this implies that the projection is area-nonincreasing.

Making use of his method, Lawlor classified all homogeneous minimizing hypercones (Theorem 5.5.2), minimizing cones over products of spheres (Theorem 5.1.1), and minimizing coves over the matrix groups $\mathrm{O}(k)$, $\mathrm{SO}(k)$, $\mathrm{U}(k)$, $\mathrm{SU}(k)$ in $\mathbb{R}^{k^2}$ or $\mathbb{C}^{k^2}$ correspondingly (Section 5.4). Moreover, he characterized all isoparametric minimizing hypercones as well (Theorem 4.3.1).

5.2. Our modification on ϕ. We now adjust the form to be smooth in $C \sim 0$. The modification hinges on the following observations. Through point (t, y) for $0 < y < \sqrt{t^2+1}p(t)$, the inequality requirement in (2) becomes

$$\frac{k}{t^2+1}\left(ty - \sqrt{(t^2+1)p^2(t) - y^2}\right) \le h'(t) \le \frac{k}{t^2+1}\left(ty + \sqrt{(t^2+1)p^2(t) - y^2}\right).$$

Also note that

$$\frac{\partial}{\partial y}\left(ty - \sqrt{(t^2+1)p^2(t) - y^2}\right) = t + \frac{y}{\sqrt{(t^2+1)p^2(t) - y^2}} > 0. \tag{3}$$

Hence, in order to reach zero fastest, h_0 has to satisfy the O.D.E:

$$h'(t) = \frac{k}{t^2+1}\left(th - \sqrt{(t^2+1)p^2(t) - h^2}\right), \qquad h(0) = 1. \tag{4}$$

According to pages 42–45 of [**Law91**], $h_0(t) = 1 - a_{max}t^2 - bt^3 + \cdots$ near $t = 0$, where

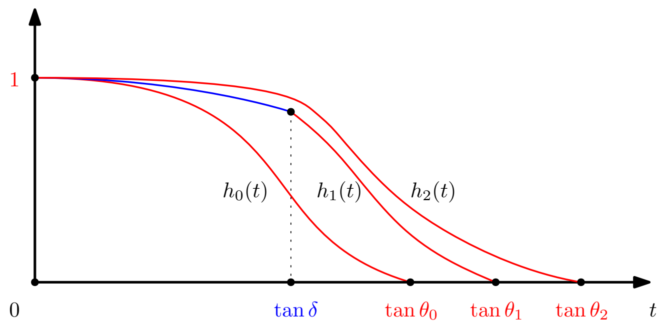
$$a_{max,min} = \frac{k}{4}\left(k - 2 \pm \sqrt{(k-2)^2 + 8p_2}\right)$$

is obtained by solving (4) for $h = 1 - at^2$ at the stage of second order:

$$4a^2 - 2ak(k-2) - 2k^2p_2 = 0;$$

and all other solutions to (4) are, locally, of form $1 - a_{min}t^2 + \cdots$.

By the numerical analysis that Lawlor used for a minimizing cone, there exists a solution h_2 with $h_2''(0) = -2a_{min}$ to (4) that reaches zero and the corresponding normal wedges of angle θ_2 do not intersect each other. Now we choose $a_{min} < a < a_{max}$ so that, for some small δ and



all $t \in (0, \tan\delta]$, $1 - at^2$ satisfies (2) and lies between h_0 and h_2. Then one can follow

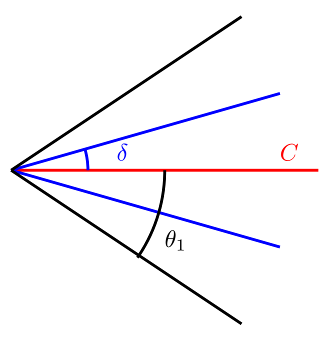
$$h'(t) = \frac{k}{t^2+1}\left(th - \sqrt{(t^2+1)p^2(t) - h^2}\right), \qquad h(\tan\delta) = 1 - a\tan\delta^2,$$

for $t \geq \tan\delta$ to get a piecewise smooth solution h_1 to (2) that stays below $h_2(t)$ and arrives at the t-*axis* before $t = \tan\theta_2$. This is guaranteed by (3).

Let f_1 be the function determined by h_1. Then it is smooth in $C \sim 0$ but Lipschitz in the boundaries $\mathfrak{B}_\delta$ and $\mathfrak{B}_{\theta_1}$ of the δ-angular and θ_1-angular neighborhoods of C. However, this is good enough. Via mollifications, the discontinuous $\phi_1 = d\tilde{\psi}_1$ where

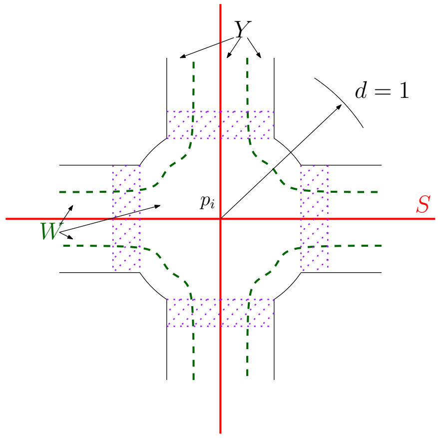
$$\tilde{\psi}_1 = \begin{cases} f_1\varpi^*\psi & \text{in} \quad \mathcal{N}(\theta_1) \sim \mathfrak{B}_\delta, \\ 0 & \text{in} \quad \mathcal{N}(\theta_1)^c \end{cases}$$

can serve as a calibration.



5.3. Construction of realization. Now we are ready to produce (local) realizations. Choose a metric g for our model in §2 such that

(i). balls $\mathbf{B}^g_{p_i}(1)$ of radius one centered at p_i are disjoint, and



(ii). local model $S \bigcap \mathbf{B}^g_{p_i}(1)$ in $(\mathbf{B}^g_{p_i}(1), g|_{\mathbf{B}^g_{p_i}(1)})$ is exactly $C \bigcap \mathbf{B}^N(1)$ in $(\mathbf{B}^N(1), g_E|_{\mathbf{B}^N(1)})$.

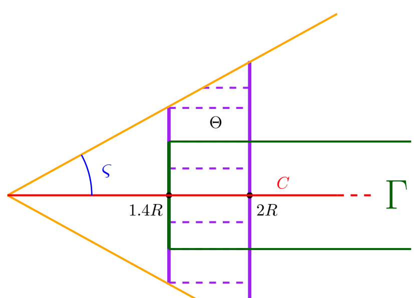
Our strategy is the following.

Step 1: We have ϕ_1 around p_1 and p_2, respectively. First, we glue them together to a form Φ in some neighborhood of S.

Step 2: Then we construct a smooth metric on a smaller neighborhood so that Φ serves as a calibration for S.

In this way a global realization can be gained based on §4.

Assume $0 < 3R < 1$. Let $\mathbf{r}$ be the distance to the origin in C and Θ an angular neighborhood of angle ς $(< \delta)$ over $C \bigcap \{1.4R < \mathbf{r} < 2R\}$ shown in the figure (also see the "dumbbell" picture below). Assume further that the middle part Γ of the dumbbell is sufficiently narrow so that the exponential map restricted to normals of $S \bigcap \Gamma$ has no focal points and, hence, ϖ is well-defined on Γ for g.



On $\Theta \bigcap \Gamma$, define $\mathbf{r} = \mathbf{r}(\varpi(\cdot))$ and

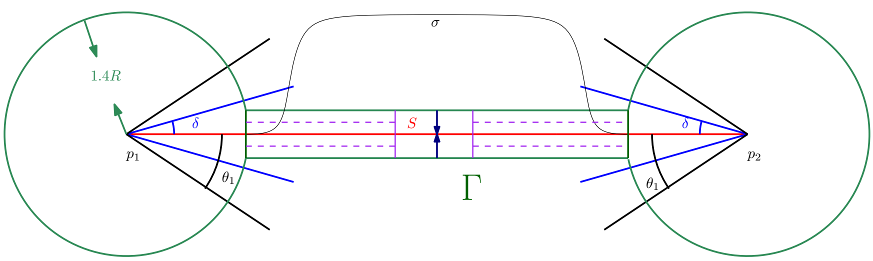
$$\Phi = d[\tau(\mathbf{r})(f_1 \varpi^* \psi) + (1 - \tau(\mathbf{r}))(\varpi^* \psi)], \tag{5}$$

where τ is a decreasing smooth function from value one to zero on $[1.4R, 2R]$ (short for $\mathbf{r}^{-1}([1.4R, 2R])$) with the support of $d\tau$ contained in $[1.6R, 1.7R]$. Note $df_1 = 0$ in $S \bigcap \Theta$ (not merely $df_1|_{S\bigcap\Theta} = 0$). Hence, in $S \bigcap \Theta$,

$$\Phi = d(\varpi^*\psi) = \varpi^*(d\psi),$$

with $d\psi$ equal to the unit volume form of S. Moreover, on $[1.7R, 2R] \bigcap \Gamma$ we have $\Phi = d(\varpi^*\psi) = \varpi^*(d\psi)$. Therefore, we can extend Φ in the dumbbell, by defining it to be the pullback of the unit volume form of S on the region $0.6R$ away from both ends of Γ, to a "form" (still called Φ) that is singular only in $\mathfrak{B}_\delta$ and $\mathfrak{B}_{\theta_1}$ in $(1.4R)$-balls centered at p_1 and p_2, respectively. Since fibers of $(\Gamma, S\bigcap\Gamma, \varpi)$ are perpendicular to $S \bigcap \Gamma$, it follows that, in $S \bigcap \Gamma$, $\Phi = \varpi^*$(the oriented unit volume form of $S \bigcap \Gamma$) is exactly the induced unit volume form of $S \bigcap \Gamma$ in the ambient space and, consequently, the comass function[1]

$$\|\Phi\|_g^* = 1, \qquad \text{in } \ S \bigcap \Gamma.$$



Next, we shall create a smooth metric in the dumbbell to make Φ a "calibration". By the relation $d(\varpi^*\psi) = \frac{1}{\mathbf{r}}\, d\mathbf{r} \wedge \varpi^*\psi$ and the smoothness of f_1 on $[1.4R, 2R]$, one can derive from (5) that

$$\Phi = A\; df_1 \wedge \varpi^*\psi + B\; d\mathbf{r} \wedge \varpi^*\psi, \tag{6}$$

for some smooth functions A and B. So Φ is, in fact, a simple form and $\|\Phi\|_g^*$ is smooth in Γ. Let us introduce $\sigma \geq 0$ in the picture which remains zero on $[1.4R, 1.5R]$ in both ends and keeps value one $0.2R$ away from both ends of Γ. (We glue forms in $[1.6R, 1.7R]$.) Then on $[1.4R, 1.6R]$ of both ends $(1-\sigma) + \sigma \cdot (\|\Phi\|_g^*)^{\frac{2}{k}} \geq (\|\Phi\|_g^*)^{\frac{2}{k}}$. Hence, with respect to

$$\bar{g} = [(1-\sigma) + \sigma \cdot (\|\Phi\|_g^*)^{\frac{2}{k}}]g,$$

we have on Γ the comass function

$$\|\Phi\|_{\bar{g}}^* \leq 1$$

and the equality holds in $\Gamma \bigcap S$ and also at where $\sigma = 1$.

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[1] We regard $\|\Phi\|_g^*$ as a pointwise function, namely $\|\Phi\|_g^*(x) = \|\Phi\|_{x,g}^*$ (see [**HL82a**, **HL82b**]).

Note $\bar{g} = g_E$ on $[1.4R, 1.5R]$ in both ends. Together with g_E on $(1.4R)$-balls around p_1 and p_2, it defines a smooth metric $\hat{g}$ in the dumbbell. Now we can show the homological area-minimality of S in a smaller neighborhood (W below under $\hat{g}|_W$) as follows. Take a smaller neighborhood Y of S where $Y \Subset$ the dumbbell and $(\overline{Y}, \hat{g}|_{\overline{Y}})$ forms a manifold with boundary. Isometrically embed $\overline{Y}$ into some Euclidean space $(\mathbb{R}^s, g_E)$ through F. By the compactness of $F(\overline{Y})$ there is $\tau_0 > 0$ such that the exponential map restricted to the τ-disk normal bundle $\mathfrak{D}$ over $F(Y)$ is a diffeomorphism. Denote by $\mathfrak{N}$ the image of $\mathfrak{D}$ and by π the induced projection. Choose an open neighborhood $W \Subset Y$ of S. Let λ_0= $\mathrm{dist}_{g_E}(\partial\overline{F(Y)}, \partial\overline{F(W)})$. Then mollify the Lipschitz form $\pi^*((F^{-1})^*(\Phi))$ with averaging radius $\epsilon < \epsilon_0 = \frac{1}{2}\min\{\lambda_0, \tau_0\}$ in the region $\{x \in \mathfrak{N} : \mathrm{dist}_{g_E}(x, \partial\overline{\mathfrak{N}}) \geq \epsilon_0\}$ of $\mathbb{R}^s$. Denote the generated smooth forms by $\tilde{\Phi}_\epsilon$ and set $\Phi_\epsilon = F^*(\tilde{\Phi}_\epsilon|_{F(W)})$. By commutativity of the exterior differentiation and mollification of Lipschitz forms in $\mathbb{R}^s$, it follows

$$d\Phi_\epsilon = 0.$$

By the compactness of $\overline{W}$, given $\delta > 0$, there exists ϵ such that, for any $0 < \epsilon' \leq \epsilon$, pointwise

$$\|\Phi_{\epsilon'}\|^*_{\hat{g}} \leq (1+\delta).$$

Note that ϵ can be further chosen for

$$\mathbf{M}(S) \leq (1+\delta)\int_S \Phi_{\epsilon'}, \tag{7}$$

where $\mathbf{M}$ means the mass functional (generalized volume functional). Thus, for an integral current T homologous to S in W,

$$\mathbf{M}(S) \leq (1+\delta)\int_S \Phi_\epsilon = (1+\delta)\int_T \Phi_\epsilon \leq (1+\delta)^2\mathbf{M}(T).$$

As δ goes to zero, the desired minimality follows and, hence, we complete the proof.

As a final remark, a similar argument can show that all Cheng's examples of homogeneous area-minimizing cones of codimension 2 in [**Che88**] (e.g., minimal cones over $\mathrm{U}(7)/\mathrm{U}(1) \times \mathrm{SU}(2)^3$ in $\mathbb{R}^{42}$, $\mathrm{Sp}(n) \times \mathrm{Sp}(3)/\mathrm{Sp}(1)^3 \times \mathrm{Sp}(n-3)$ in $\mathbb{R}^{12n}$ for $n \geq 4$, and $\mathrm{Sp}(4)/\mathrm{Sp}(1)^4$ in $\mathbb{R}^{27}$) and all the newly-discovered oriented area-minimizing cones in [**TZ**, **XYZ**] can be realized by our method.

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