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NONLOCAL s-MINIMAL SURFACES AND LAWSON CONES

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Abstract

The nonlocal s-fractional minimal surface equation for $\Sigma = \partial E$ where E is an open set in $\mathbb{R}^N$ is given by

$$H^s_\Sigma(p) := \int_{\mathbb{R}^N} \frac{\chi_E(x) - \chi_{E^c}(x)}{|x-p|^{N+s}}\, dx \;=\; 0 \quad \text{for all} \quad p \in \Sigma.$$

Here $0 < s < 1$, χ designates characteristic function, and the integral is understood in the principal value sense. The classical notion of minimal surface is recovered by letting $s \to 1$. In this paper we exhibit the first concrete examples (beyond the plane) of nonlocal $s-$minimal surfaces. When s is close to 1, we first construct a connected embedded s-minimal surface of revolution in $\mathbb{R}^3$, the **nonlocal catenoid**, an analog of the standard catenoid $|x_3| = \log(r+\sqrt{r^2-1})$. Rather than eventual logarithmic growth, this surface becomes asymptotic to the cone $|x_3| = r\sqrt{1-s}$. We also find a two-sheet embedded s-minimal surface asymptotic to the same cone, an analog to the simple union of two parallel planes.

On the other hand, for any $0 < s < 1$, $n, m \geq 1$, $s-$minimal Lawson cones $|v| = \alpha|u|$, $(u,v) \in \mathbb{R}^n \times \mathbb{R}^m$, are found to exist. In sharp contrast with the classical case, we prove their stability for small s and $n + m = 7$, which suggests that unlike the classical theory (or the case s close to 1), the regularity of s-area minimizing surfaces may not hold true in dimension 7.

1. Introduction

1.1. Fractional minimal surfaces. Phase transition models where the motion of the interface region is driven by curvature type flows arise in many applications. The standard flow by mean curvature of surfaces $\Sigma(t)$ in $\mathbb{R}^N$ is that in which the normal speed of each point $x \in \Sigma(t)$ is proportional to its mean curvature $H_{\Sigma(t)}(x) = \sum_{i=1}^{N-1} k_i(x)$ where the k_i's designate the principal curvatures, namely the eigenvalues of the second fundamental form. Evans [**14**] showed that standard mean curvature flow for level surfaces of a function can be recovered as the limit of a discretization scheme in time where heat flow $u_t - \Delta u = 0$

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of suitable initial data is used to connect consecutive time steps, which was introduced in [20]. When standard diffusion is replaced by that of the fractional Laplacian $u_t + (-\Delta)^{\frac{s}{2}} u = 0$ in order to describe long range, nonlocal interactions between points in the two distinct phases by a Levy process, Caffarelli and Souganidis [7], see also Imbert [17], found that for $1 \leq s < 2$ the flow by mean curvature is still recovered, while for $0 < s < 1$, the stronger nonlocal effect makes the surfaces evolve in normal velocity according to their *fractional mean curvature*, defined for a surface $\Sigma = \partial E$ where E is an open subset of $\mathbb{R}^N$ as

$$H^s_\Sigma(p) := \int_{\mathbb{R}^N} \frac{\chi_E(x) - \chi_{E^c}(x)}{|x-p|^{N+s}}\, dx \quad \text{for } p \in \Sigma. \tag{1.1}$$

Here χ denotes characteristic function, $E^c = \mathbb{R}^N \setminus E$ and the integral is understood in the principal value sense,

$$H^s_\Sigma(p) = \lim_{\delta \to 0} \int_{\mathbb{R}^N \setminus B_\delta(p)} \frac{\chi_E(x) - \chi_{E^c}(x)}{|x-p|^{N+s}}\, dx.$$

This quantity is well-defined provided that Σ is regular near p. It agrees with usual mean curvature in the limit $s \to 1$ by the relation

$$\lim_{s \to 1} (1-s)\, H^s_\Sigma(p) = c_N H_\Sigma(p), \tag{1.2}$$

see [17]. Stationary surfaces for the fractional mean curvature flow are naturally called fractional minimal surfaces. We say that Σ is an *s-minimal surface* in an open set Ω, if the surface $\Sigma \cap \Omega$ is sufficiently regular, and it satisfies the *nonlocal minimal surface equation*

$$H^s_\Sigma(p) = 0 \quad \text{for all } p \in \Sigma \cap \Omega. \tag{1.3}$$

For instance, it is clear by symmetry and definition (1.1) that a hyperplane is a s-minimal surface in $\mathbb{R}^N$ for all $0 < s < 1$. Similarly, the *Simons cone*

$$C^m_m = \{(u, v) \in \mathbb{R}^m \times \mathbb{R}^m \ / \ |v| = |u|\}$$

is a s-minimal surface in $\mathbb{R}^{2m} \setminus \{0\}$. As far as we know, no other explicit minimal surfaces in $\mathbb{R}^N$ have been found in the literature. The purpose of this paper is to exhibit a new class of non-trivial examples. The hyperplane is not just a minimal surface but also established in [6] to be *locally area minimizing* in a sense that we describe next.

Caffarelli, Roquejoffre and Savin introduced in [6] a nonlocal notion of surface area of $\Sigma = \partial E$ whose Euler–Lagrange equation corresponds to equation (1.3). For $0 < s < 1$, the s-perimeter of a measurable set $E \subset \mathbb{R}^N$ is defined as

$$\mathcal{I}_s(E) = \int_E \int_{E^c} \frac{dx\, dy}{|x-y|^{N+s}}.$$

The above quantity corresponds to a total interaction between points of E and E^c, where the interaction density $1/|x-y|^{N+s}$ is largest possible when the points $x \in E$ and $y \in E^c$ are both close to a given point of the boundary. $\mathcal{I}_s(E)$ has a simple representation in terms of the usual semi-norm in the fractional Sobolev space $H^{\frac{s}{2}}(\mathbb{R}^N)$. In fact,

$$\mathcal{I}_s(E) = [\chi_E]_{H^{\frac{s}{2}}(\mathbb{R}^N)} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\chi_E(x)-\chi_E(y))^2}{|x-y|^{N+s}}dx\,dy.$$

Alternatively, we can also write

$$\mathcal{I}_s(E) = [\chi_E]_{W^{s,1}(\mathbb{R}^N)} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{N+s}}dx\,dy.$$

If E is an open set and $\Sigma = \partial E$ is a smooth bounded surface we have that

$$(1-s)\mathcal{I}_s(E) \to c_N \mathcal{H}^{N-1}(\Sigma) = \int_{\mathbb{R}^N} |\nabla \chi_E|,$$

where the latter equality is classically understood in the sense of functions of bounded variation. $\mathcal{I}_s$ can also be achieved as the Γ-limit as $\varepsilon \to 0$ of the nonlocal Allen–Cahn phase transition functional $\int \frac{\varepsilon}{2}|\nabla^{\frac{s}{2}}u|^2 + \frac{1}{4\varepsilon}(1-u^2)^2$ along functions that ε-regularize $\chi_E - \chi_{E^c}$. See [**23**, **26**].

This nonlocal notion of perimeter is localized to a bounded open set Ω by taking away the contribution of points of E and E^c outside Ω, formally setting

$$\mathcal{I}_s(E,\Omega) = \int_E\int_{E^c} \frac{dx\,dy}{|x-y|^{N+s}} - \int_{E\cap\Omega^c}\int_{E^c\cap\Omega^c} \frac{dx\,dy}{|x-y|^{N+s}}.$$

This quantity makes sense, even if the last two terms above are infinite, by rewriting it in the form

$$\mathcal{I}_s(E,\Omega) = \int_{E\cap\Omega}\int_{E^c} \frac{dx\,dy}{|x-y|^{N+s}} + \int_{E\cap\Omega^c}\int_{E^c\cap\Omega} \frac{dx\,dy}{|x-y|^{N+s}}.$$

Again, if E is an open set with $\Sigma\cap\Omega$ smooth, $\Sigma = \partial E$. The usual notion of perimeter is recovered by the relation

$$\lim_{s\to 1}(1-s)\mathcal{I}_s(E,\Omega) = c_N\mathcal{H}^{N-1}(\Sigma\cap\Omega),$$

see [**9**]. Let h be a smooth function on Σ supported in Ω, and ν a normal vector field to Σ exterior to E. For a sufficiently small number t we let E_{th} be the set whose boundary ∂E_{th} is parametrized as

$$\partial E_{th} = \{x + th(x)\nu(x) \ / \ x \in \partial E\}.$$

The first variation of the perimeter along these normal perturbations yields precisely

$$\frac{d}{dt}\mathcal{I}_s(E_{th},\Omega)\Big|_{t=0} = -\int_\Sigma H^s_\Sigma h,$$

and this quantity vanishes for all such h if and only if (1.3) holds. Thus, $\Sigma = \partial E$ is an s-minimal surface in Ω if the first variation of perimeter for normal perturbations of E inside Ω is identically equal to zero.

If $\Sigma = \partial E$ is a nonlocal minimal surface the *second variation* of the s-perimeter in Ω can be computed as

$$\frac{d^2}{dt^2} Per_s(E_{th},\Omega)\Big|_{t=0} \;=\; -2\int_\Sigma \mathcal{J}^s_\Sigma[h]\,h, \tag{1.4}$$

see Appendix B. We call $\mathcal{J}^s_\Sigma[h]$ the *fractional Jacobi operator*. It is explicitly computed as

(1.5)
$$\mathcal{J}^s_\Sigma[h](p) = \int_\Sigma \frac{h(x)-h(p)}{|p-x|^{N+s}}dx + h(p)\int_\Sigma \frac{\langle \nu(p)-\nu(x),\nu(p)\rangle}{|p-x|^{N+s}}dx, \quad p\in\Sigma,$$

where the first integral is understood in a principal value sense. In agreement with formula (1.4), we say that an s-minimal surface Σ is *stable* in Ω if

$$-\int_\Sigma \mathcal{J}^s_\Sigma[h]\,h \;\ge\; 0 \quad \text{for all} \quad h\in C_0^\infty(\Sigma\cap\Omega).$$

Naturally we get the correspondence between this nonlocal operator and the usual Jacobi operator

$$\lim_{s\to 1}(1-s)\mathcal{J}^s_\Sigma[h] = c_N\mathcal{J}_\Sigma[h], \quad \mathcal{J}_\Sigma[h] = \Delta_\Sigma h + |A_\Sigma|^2 h, \tag{1.6}$$

where Δ_Σ is the Laplace–Beltrami operator and $|A_\Sigma|^2 = \sum_{i=1}^{N-1} k_i^2$ where the k_i are the principal curvatures.

A basic example of a stable fractional minimal surface $\Sigma = \partial E$ is a *fractional minimizing surface*. In [**6**] the existence of fractional perimeter-minimizing sets is proven in the following sense: let Ω be a bounded domain with Lipschitz boundary, and $E_0\subset\Omega^c$ a given set. Let $\mathcal{F}$ be the class of all sets F with $F\cap\Omega^c = E_0$. Then there exists a set $E\in\mathcal{F}$ with

$$\mathcal{I}_s(E,\Omega) \;=\; \inf_{F\in\mathcal{F}} \mathcal{I}_s(F,\Omega).$$

Moreover, $\partial E\cap\Omega$ is a $(N-1)$-dimensional set, which is a surface of class $C^{1,\alpha}$ except possibly on a singular set of Hausdorff dimension at most $N-2$. Minimizers E are proven to satisfy in a viscosity sense the fractional minimal surface equation (1.3). In fact, a hyperplane is minimizing in the above sense inside any bounded set. No other example of embedded smooth fractional minimal surface in $\mathbb{R}^N$ (minimizing or not) is known.

1.2. Axially symmetric s-minimal surfaces. After a plane, next in complexity in $\mathbb{R}^3$ is the *axially symmetric case*, namely the case of a surface of revolution around the x_3-axis. In the classical case, the minimal surface equation reduces to a simple ODE from which the catenoid C_1 is obtained:

$$C_1 = \{(x_1,x_2,x_3)\ /\ |x_3| = \log(r+\sqrt{r^2-1}), \quad r = \sqrt{x_1^2+x_2^2} > 1\}.$$

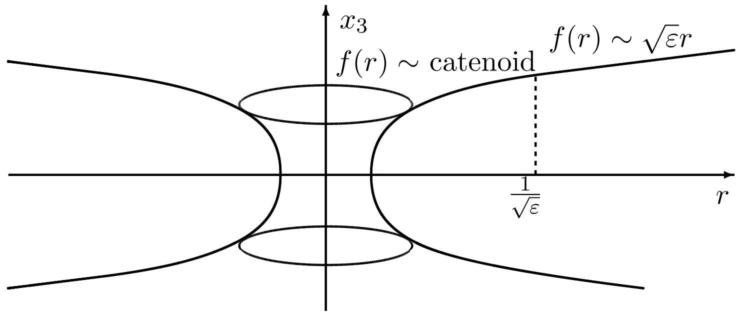

Figure 1. Fractional catenoid.

A main purpose of this paper is the construction of an axially symmetric s-minimal surface C_s for s close to 1 in such a way that $C_s \to C_1$ as $s \to 1$ on bounded sets. We call this surface the *fractional catenoid.* A striking feature of the surface of revolution C_s is that it becomes at main order as $r \to \infty$ a cone with small slope rather than having logarithmic growth. It is precisely in this feature where the strength of the nonlocal effect is felt.

Theorem 1. (The fractional catenoid) *For all $0 < s < 1$ sufficiently close to 1 there exists a connected surface of revolution C_s such that if we set $\varepsilon = (1 - s)$ then*

$$\sup_{x \in C_s \cap B(0,2)} \operatorname{dist}(x, C_1) \le c \frac{\sqrt{\varepsilon}}{|\log \varepsilon|},$$

and, for $r = \sqrt{x_1^2 + x_2^2} > 2$ the set C_s can be described as $|x_3| = f(r)$, where

$$f(r) = \begin{cases} \log(r + \sqrt{r^2 - 1}) + O\left(\frac{r\sqrt{\varepsilon}}{|\log \varepsilon|}\right) & \text{if} \quad r < \frac{1}{\sqrt{\varepsilon}}, \\ r\sqrt{\varepsilon} + O(|\log \varepsilon|) + O\left(\frac{r\sqrt{\varepsilon}}{|\log \varepsilon|}\right) & \text{if} \quad r > \frac{1}{\sqrt{\varepsilon}}. \end{cases}$$

The usual catenoid C_1 cannot be obtained by an area minimization scheme in expanding domains since it is linearly unstable, hence, non-minimizing, inside any sufficiently large domain. It is unlikely that C_s can be captured with a scheme based on the results in [**6**]. In fact, even worse, this is a highly unstable object compared with the classical case: there are elements in an approximate kernel of its s-Jacobi operator that change sign infinitely many times. The Morse index of C_s is infinite in any reasonable sense (unlike the standard catenoid, whose Morse index is one). Indeed, as we will see in Section 2, the equation $H^s_{C_s} = 0$ for $r >> \frac{1}{\sqrt{\varepsilon}}$, is well-approximated by the following equation for $f(r)$:

$$\Delta_{\mathbb{R}^2} f = \frac{\varepsilon}{f}. \tag{1.7}$$

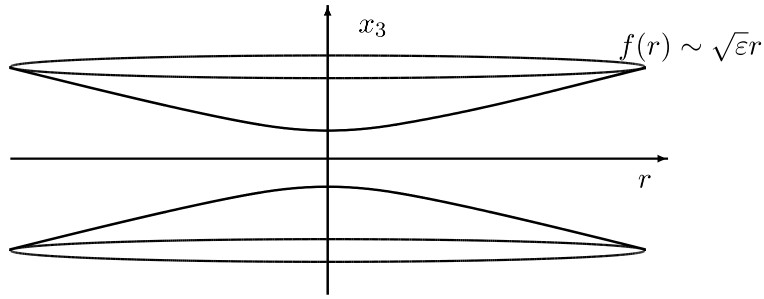

Figure 2. Two-sheet s-minimal surface.

Radial solutions to this problem are all asymptotic to the exact solution $f_0(r) = \sqrt{\varepsilon} r$. Hence, the linearized mean curvature, s-Jacobi operator is in correspondence with the Hardy operator $\Delta_{\mathbb{R}^2} + \frac{1}{r^2}$. The radial elements of the kernel of this operator oscillate infinitely many times.

As we have mentioned, a plane is an s-minimal surface for any $0 < s < 1$. In the classical scenario, so is the union of two parallel planes, say $x_3 = 1$ and $x_3 = -1$. This is no longer the case when $0 < s < 1$ since the nonlocal interaction between the two components deforms them and, in fact, equilibrium is reached when the two components diverge becoming cones. Our second results states the existence of a *two-sheet* nontrivial s-minimal surface D_s for s close to 1 where the components eventually become at main order the cone $x_3 = \pm r\sqrt{\varepsilon}$. As in the s-catenoid, the asymptotic profile of this surface is governed by equation (1.7), and, thus, we expect this to be a highly unstable object.

Theorem 2. (The two-sheet s-minimal surface) *For all $0 < s < 1$ sufficiently close to 1 there exists a two-component surface of revolution $D_s = D_s^+ \cup D_s^-$ such that if we set $\varepsilon = (1 - s)$ then $D_s^\pm$ is the graph of the radial functions $x_3 = \pm f(r)$ where f is a positive function of class C^2 with $f(0) = 1$, $f'(0) = 0$, and*

$$f(r) = \begin{cases} 1 + \frac{\varepsilon}{4} r^2 + O(\varepsilon r) & \text{if} \quad r < \frac{1}{\sqrt{\varepsilon}}, \\ r\sqrt{\varepsilon} + O(1) + O(\varepsilon r) & \text{if} \quad r > \frac{1}{\sqrt{\varepsilon}}. \end{cases}$$

As we shall discuss in Section 9, Theorem 2 can be generalized to the existence of a k-sheet axially symmetric s-minimal surface constituted by the union of the graphs of k radial functions $x_3 = f_j(r)$, $j = 1, \dots, k$, with

$$f_1 > f_2 > \cdots > f_k,$$

where asymptotically we have

$$f_j(r) = a_j r\sqrt{\varepsilon} + O(\varepsilon r) \quad \text{as } r \to +\infty. \tag{1.8}$$

Here the constants a_i are required to satisfy the constraints

$$a_1 > a_2 > \cdots > a_k, \quad \sum_{i=1}^{k} a_i = 0, \tag{1.9}$$

and the balancing conditions

$$a_i = 2\sum_{j\neq i} \frac{(-1)^{i+j+1}}{a_i - a_j}, \qquad \text{for all} \quad i = 1, \ldots, k. \tag{1.10}$$

A solution of the system (1.10) can be obtained by minimization of

$$E(a_1, \ldots, a_k) = \frac{1}{2}\sum_{i=1}^{k} a_i^2 + \sum_{i\neq j} (-1)^{i+j} \log(|a_i - a_j|)$$

in the set of k-tuples $a = (a_1, \ldots, a_k)$ that satisfy (1.9). If this minimizer or, more generally, a critical point a of E constrained to (1.9) is non-degenerate, in the sense that $D^2E(a)$ is non-singular, then an s-minimal surface with the required properties (1.8) can, indeed, be found. This condition is evidently satisfied by $a = (1, -1)$ when $k = 2$.

The method for the proofs of the above results relies on a simple idea of obtaining a good initial approximation Σ_0 to a solution of the equation $H^s_\Sigma = 0$. We do this in Section 2. Then we consider the surface perturbed normally by a small function h, Σ_h. As we will see, we can expand

$$H^s_{\Sigma_h} = H^s_{\Sigma_0} + \mathcal{J}^s_{\Sigma_0}[h] + N(h),$$

where $N(h)$ is at main order quadratic in h. In the classical case, $N(h)$ depends on first and second derivatives of h with various terms that can be qualitatively described (see [**18**]). We shall see in Section 4 that for our approximation Σ_0 the error $H^s_{\Sigma_0}$ is small in $\varepsilon = 1 - s$ and has suitable decay along the manifold. Then the problem is solved by a fixed point argument. To do so, we need to identify the functional spaces to set up the problem, that take into account the delicate issues of non-compactness and strong long range interactions. These spaces are such that a left inverse of $\mathcal{J}^s_{\Sigma_0}$ can be found with good transformation properties. We carry out these analysis in Sections 5, 6 and 7. The nonlinear operator $N(h)$ has a small Lipschitz dependence for the corresponding norms, as we establish in Section 8. This issue is especially delicate for $N(h)$, since it contains strongly singular integral nonlinear operators involving fractional derivatives up to the nearly second order. The transformation properties of these nonlinear terms have suitable analogs with those found by Kapouleas [**18**], but the proofs in the current situation are harder.

The procedure we set up in this paper, and the associated computations, apply in large generality, not just to the axially symmetric case.

For instance, most of the calculations actually apply to a general setting of finding as $s \to 1$ a connected surface with multiple ends that are eventually conic and satisfy relations (1.9), where the starting point is a multiple-logarithmic-end minimal surface. This paper sets the basis of the gluing arguments for the construction of fractional minimal surfaces, in a way similar that the paper [**18**] did for the construction by gluing methods of classical minimal and CMC surfaces.

1.3. Fractional Lawson cones. The pictures associated to Theorems 1 and 2 resemble that of "one-sheet" and "two-sheet" revolution hyperboloids, asymptotic to a cone $|x_3| = r\sqrt{1-s}$. It is reasonable to believe that a cone of this form, with aperture close to $\sqrt{1-s}$ is a fractional minimal surface with a singularity at the origin. We consider, more in general, for given $n, m \geq 1$, and $0 < s < 1$ the problem of finding a value $\alpha > 0$ such that the *Lawson cone*

$$C_\alpha = \{(u,v) \in \mathbb{R}^m \times \mathbb{R}^n \ / \ |v| = \alpha |u|\} \tag{1.11}$$

is a s-minimal surface in $\mathbb{R}^{m+n} \setminus \{0\}$. For the classical case $s = 1$ this is easy: since $\Sigma = C_\alpha$ is the zero level set of the function $g(u,v) = |v| - \alpha|u|$, for $(u,v) \in C_\alpha$ we have

$$H_\Sigma(u,v) = \operatorname{div}\left(\frac{\nabla g}{|\nabla g|}\right) = \frac{1}{\sqrt{1+\alpha^2}}\left[\frac{n-1}{|v|} - \alpha\frac{m-1}{|u|}\right],$$

and the latter quantity is equal to zero on Σ if and only if $n = m = 1$ and $\alpha = 1$ or

$$n \geq 2, \ m \geq 2, \quad \alpha = \sqrt{\frac{n-1}{m-1}}.$$

Following [**19**], we call this one the minimal Lawson cone C_m^n. For the fractional situation we have the following result which is proved in Section 10.

Theorem 3. (Existence of s-Lawson cones) *For any given $m \geq 1$, $n \geq 1$, $0 < s < 1$, there is a unique $\alpha = \alpha(s,m,n) > 0$ such that the cone C_α given by (1.11) is an s-fractional minimal surface. We call this $C_m^n(s)$ the s-Lawson cone.*

A notable different between classical and nonlocal cases is that in the latter, a nontrivial minimal cone in $\mathbb{R}^n$

$$C_1^{n-1}(s) \ = \ \left\{(x', x_n) \in \mathbb{R}^n \ / \ |x_n| = \alpha_n(s)|x'| \,\right\},$$

with $n \geq 3$ does exist. This is not true in the classical case. The bottomline is that when aperture becomes very large (α small), in the standard case mean curvature approaches 0, while the nonlocal interaction between the two pieces of the cone makes its fractional mean curvature go to $-\infty$. For $n = 2$, $C_1^2(s)$ is precisely the s-minimal cone that represents at main order the asymptotic behavior of the revolution

s-minimal surfaces of Theorems 1 and 2. Letting $\varepsilon = 1 - s \to 0$, we have, as suspected

$$\alpha_2(s) = \sqrt{\varepsilon} + O(\varepsilon),$$

so that the two halves of the minimal cone become planes. In the opposite limit, $s \to 0$, there is no collapsing. In fact, if $n \le m$ we have

$$\lim_{s\to 0} \alpha(s, m, n) = \alpha_0,$$

where $\alpha_0 > 0$ is the unique number α such that

$$\int_\alpha^\infty \frac{t^{n-1}}{(1+t^2)^{\frac{m+n}{2}}} dt - \int_0^\alpha \frac{t^{n-1}}{(1+t^2)^{\frac{m+n}{2}}} dt = 0.$$

An interesting analysis of asymptotics for the fractional perimeter $\mathcal{I}_s$ and associated s-minimizing surfaces as $s \to 0$ is contained in [**12**].

Minimal cones are important objects in the regularity theory of classical minimal surfaces and Bernstein type results for minimal graphs. Simons [**25**] proved that *no stable minimal cone exists in dimension $N \le 7$, except for hyperplanes.* This result implies that locally area minimizing surfaces must be smooth outside a closed set of Hausdorff dimension at most $N - 8$. He also proved that the cone C_4^4 (Simons' cone) was stable, and conjectured its minimizing character. This was proved in a deep work by Bombieri, De Giorgi and Giusti [**5**].

Savin and Valdinoci [**22**] proved the nonexistence of fractional minimizing cones in $\mathbb{R}^2$, which implies regularity of fractional minimizing surfaces except for a set of Hausdorff dimension at most $N-3$, thus, improving the original result in [**6**]. Figalli and Valdinoci [**15**] prove that, in every dimension, Lipschitz nonlocal minimal surfaces are smooth, see also [**2**]. Also, They extend to the nonlocal setting a famous theorem of De Giorgi stating that the validity of Bernstein's theorem as a consequence of the nonexistence of singular minimal cones in one dimension less.

In [**9**], Caffarelli and Valdinoci proved that regularity of non-local minimizers holds up to a $(N-8)$-dimensional set, whenever s is sufficiently close to 1. Thus, there remains a conspicuous gap between the best general regularity result found so far and the case s close to 1. Our second results concerns this issue. Its most interesting feature is that, in strong contrast with the classical case, when s is sufficiently close to zero, Lawson cones **are all stable** in dimension $N = 7$, which suggests that a regularity theory up to a $(N-7)$-dimensional set should be the best possible for general s.

Theorem 4. (Stability of s-Lawson cones) *There is a $s_0 > 0$ such that for each $s \in (0, s_0)$, all minimal cones $C_m^n(s)$ are unstable if $N = m + n \le 6$ and stable if $N = 7$.*

We will prove this result in Section 11.

Besides the results in [**25**, **5**], we remark that for $N > 8$ the cones C_m^n are all area minimizing. For $N = 8$ they are area minimizing if and only if $|m - n| \leq 2$. These facts were established by Lawson [**19**] and Simoes [**24**], see also [**21**, **10**, **3**, **11**].

The rest of this paper will be devoted to the proofs of Theorems 1–4. The proof of Theorem 2 is actually simpler than that of Theorem 1. We will concentrate on the proof of Theorem 1, explaining the variations needed for Theorem 2 in Section 9. We provide a detailed scheme of the proof of Theorem 1 in Section 2. There we shall isolate the main steps in the form of intermediate results which we prove in the subsequent sections. The proofs of Theorems 3 and 4 rely on explicit computations of singular integral quantities, and are carried out in Sections 10 and 11.

We leave for the Appendix self contained proofs of asymptotic formulas (1.2), (1.6) in Section A, and the computation of first and second variations of the s-perimeter in Section B.

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2. Scheme of the proof of Theorem 1

In this section, we shall outline the proof of Theorem 1, isolating the main steps whose proofs are delayed to later sections. We look for a set $E \subseteq \mathbb{R}^3$ with smooth $\Sigma = \partial E$ such that

$$H_\Sigma^s(x) := \int_{\mathbb{R}^3} \frac{\chi_E(y) - \chi_{E^c}(y)}{|x-y|^{3+s}}\, dy = 0, \quad \text{for all} \quad x \in \Sigma, \tag{2.1}$$

where $0 < s < 1$, $1 - s$ is small and the integral is understood in a principal value sense.

We look for E in the form of a solid of revolution around the x_3-axis. More precisely, let us represent points in space by $x = (x', x_3)$ with $x' \in \mathbb{R}^2$, and denote $r = |x'|$. We shall construct a first approximation for E of the form

$$E_0 = \{x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R} : |x'| < 1 \text{ or } (|x'| \geq 1 \text{ and } |x_3| > f(|x'|))\}, \tag{2.2}$$

where f is a positive and increasing function on $[1, \infty)$. The surface of revolution constituted by the boundary of E_0, $\Sigma_0 = \partial E_0$ will be a good approximation to a fractional minimal surface, namely of a solution of Equation (2.1), for a choice of a function $f(r)$ which makes E_0 coincide with the usual catenoid for $r < \frac{1}{\sqrt{\varepsilon}}$ and $\varepsilon = 1-s$. For larger r, the surface Σ_0 asymptotically becomes a cone of revolution, $f(r) \approx \sqrt{\varepsilon} r$. After this is done, we shall solve equation (2.1) as a small normal perturbation of Σ_0. To do so, we will develop a solvability theory for the corresponding linearized equation on which we will base a fixed point argument. As a matter of fact, for surfaces Σ close to Σ_0, we will see that Equation (2.1) reads at main order as

$$-2H_\Sigma(x) + \frac{\varepsilon}{|x_3|} = 0, \tag{2.3}$$

where $H_\Sigma(x)$ is the usual mean curvature of Σ at x.

For the construction of Σ_0 we take the standard catenoid parametrized as

$$|x_3| = f_C(r), \quad r = |x'| \geq 1,$$

where

$$f_C(r) = \log(r + \sqrt{r^2 - 1}), \quad r \geq 1. \tag{2.4}$$

If we describe $\Sigma = \partial E$ with E as in (2.2) and assume that for r large $f'(r)$ is small, then for large $x = (x', x_3) \in \Sigma$, $H_\Sigma(x) = \nabla \cdot \left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}} \right) \approx \Delta f$ and $\frac{\varepsilon}{|x_3|} = \frac{\varepsilon}{f}$, so (2.3) is approximated by

$$\Delta f = \frac{\varepsilon}{f}. \tag{2.5}$$

This motivates us to define $f_\varepsilon(r)$ as solution of the initial value problem

$$\begin{cases} f_\varepsilon'' + \dfrac{1}{r} f_\varepsilon' = \dfrac{\varepsilon}{f_\varepsilon}, \quad r > \varepsilon^{-\frac{1}{2}}, \\ f_\varepsilon(\varepsilon^{-\frac{1}{2}}) = f_C(\varepsilon^{-\frac{1}{2}}), \quad f_\varepsilon'(\varepsilon^{-\frac{1}{2}}) = f_C'(\varepsilon^{-\frac{1}{2}}). \end{cases} \tag{2.6}$$

Let

$$F_\varepsilon(r) := f_C(r) + \eta(r - \varepsilon^{-\frac{1}{2}})(f_\varepsilon(r) - f_C(r)), \quad r \geq 1,$$

where $\eta \in C^\infty(\mathbb{R})$ is a cut-off function with

$$\eta(t) = 0 \quad \text{for } t < 0, \quad \eta(t) = 1 \quad \text{for } t > 1. \tag{2.7}$$

We define the surface Σ_0 by

$$\Sigma_0 = \{ \, |x_3| = F_\varepsilon(r), r \geq 1 \, \}. \tag{2.8}$$

Then

$$\Sigma_0 = \partial E_0, \quad E_0 = \{ \, r < 1, \text{ or } r \geq 1 \text{ and } |x_3| \geq F_\varepsilon(r) \, \}.$$

Next we perturb the surface Σ_0 in the normal direction. For this, let $\nu_{\Sigma_0}(x)$ be the unit normal vector field on Σ_0 such that $\nu_3(x) x_3 \geq 0$. We consider a function h defined on Σ_0, and define

$$\Sigma_h = \{ \, x + h(x) \nu_{\Sigma_0}(x) \; / \; x \in \Sigma_0 \, \}.$$

If h is small in a suitable norm, then Σ_h is an embedded surface that can be written as $\Sigma_h = \partial E_h$ for a set E_h that is close to E_0. We can expand, for a point $x \in \Sigma_0$ and $x_h = x + h(x)\nu_{\Sigma_0}(x)$:

$$H^s_{\Sigma_h}(x_h) = H^s_{\Sigma_0}(x) + 2\mathcal{J}^s_{\Sigma_0}(h)(x) + N(h)(x), \tag{2.9}$$

where $\mathcal{J}^s_{\Sigma_0}$ is the *nonlocal Jacobi operator* given by

$$\mathcal{J}^s_{\Sigma_0}(h)(x) = \int_{\Sigma_0} \frac{h(y)-h(x)}{|x-y|^{3+s}}dy + h(x)\int_{\Sigma_0} \frac{\langle \nu_{\Sigma_0}(x) - \nu_{\Sigma_0}(y), \nu_{\Sigma_0}(x)\rangle}{|x-y|^{3+s}}dy,$$

for $x \in \Sigma_0$, and $N(h)$ is defined by equality (2.9).

The objective is then to find h such that

$$H^s_{\Sigma_0} + 2\mathcal{J}^s_{\Sigma_0}(h) + N(h) = 0. \tag{2.10}$$

We note that, assuming h is smooth and bounded,

$$\text{p.v.}\int_{\Sigma_0} \frac{h(y)-h(x)}{|x-y|^{3+s}}dy = \frac{1}{\varepsilon}\frac{\pi}{2}\Delta_{\Sigma_0} h(x) + O(1)$$

as $\varepsilon \to 0$, where Δ_{Σ_0} is the Laplace–Beltrami operator on Σ_0 (see Lemma A.2). Therefore, it is more convenient to rewrite (2.10) as

$$\varepsilon H^s_{\Sigma_0} + 2\varepsilon\mathcal{J}^s_{\Sigma_0}(h) + \varepsilon N(h) = 0 \quad \text{in } \Sigma_0.$$

It is natural to expect that h has linear growth, and, therefore, we will work with weighted Hölder norms allowing such behavior. For $0 < \alpha < 1$ and $\gamma \in \mathbb{R}$, we define norms for functions defined on Σ_0 or $\mathbb{R}^2$ as follows:

$$\begin{aligned}
[f]_{\gamma,\alpha} &= \sup_{x\neq y} \min(1+|x|, 1+|y|)^{\gamma+\alpha}\frac{|f(x)-f(y)|}{|x-y|^\alpha},\\
\|f\|_{\gamma,\alpha} &= \|(1+|x|)^\gamma f\|_{L^\infty} + [f]_{\gamma,\alpha},
\end{aligned}$$

and

$$\|h\|_* = \|(1+|x|)^{-1}h\|_{L^\infty} + \|\nabla h\|_{L^\infty} + \|(1+|x|)D^2h\|_{L^\infty} + [D^2h]_{1,\alpha}. \tag{2.11}$$

Then we look for a solution h of (2.10) with $\|h\|_* < \infty$ and measure $\varepsilon\mathcal{J}^s_{\Sigma_0}(h)$ in the norm

$$\|f\|_{1-\varepsilon,\alpha+\varepsilon} = \|(1+|x|)^{1-\varepsilon}f\|_{L^\infty} + [f]_{1-\varepsilon,\alpha+\varepsilon}. \tag{2.12}$$

More explicitly,

$$\begin{aligned}
\|f\|_{1-\varepsilon,\alpha+\varepsilon} &= \|(1+|x|)^{1-\varepsilon}f\|_{L^\infty}\\
&\quad + \sup_{x\neq y}\min(1+|x|, 1+|y|)^{1+\alpha}\frac{|f(x)-f(y)|}{|x-y|^{\alpha+\varepsilon}}.
\end{aligned}$$

An outline of the proof of Theorem 1 is the following. In Section 4, using estimates for f_ε obtained in Section 3, we will prove:

Proposition 2.1. *For $\varepsilon > 0$ sufficiently small we have*

$$\|\varepsilon H^s_{\Sigma_0}\|_{1-\varepsilon,\alpha+\varepsilon} \le \frac{C\varepsilon^{\frac{1}{2}}}{|\log\varepsilon|}. \tag{2.13}$$

The next result is about invertibility of the operator $\varepsilon\mathcal{J}^s_{\Sigma_0}$ on a weighted Hölder space.

Proposition 2.2. *There is a linear operator that to a function f on Σ_0 such that f is radially symmetric and symmetric with respect to $x_3 = 0$ with $\|f\|_{1-\varepsilon,\alpha+\varepsilon} < \infty$, gives a solution ϕ of*

$$\varepsilon\mathcal{J}^s_{\Sigma_0}(\phi) = f \quad \text{in } \Sigma_0.$$

Moreover, ϕ has the same symmetries as f and

$$\|\phi\|_* \le C\|f\|_{1-\varepsilon,\alpha+\varepsilon}. \tag{2.14}$$

The proof is given in Section 7, based on preliminaries in Sections 5 and 6.

In Section 8 we obtain the estimate

Proposition 2.3. *There is C independent of $\varepsilon > 0$ small such that for $\|h_i\|_* \le \sigma_0\varepsilon^{\frac{1}{2}}$, $i = 1, 2$ we have*

$$\varepsilon\|N(h_1) - N(h_2)\|_{1-\varepsilon,\alpha+\varepsilon} \le C\varepsilon^{-\frac{1}{2}}(\|h_1\|_* + \|h_2\|_*)\|h_1 - h_2\|_*. \tag{2.15}$$

Here $\sigma_0 > 0$ is small and fixed.

With these results we can give a

Proof of Theorem 1. We need a solution h to (2.10) which we look for in the Banach space

$$X = \{h \in C^{2,\alpha}_{loc}(\Sigma_0),\ \|h\|_* < \infty\},$$

with norm $\|\ \|_*$. Consider also the Banach space

$$Y = \{f \in C^{\alpha+\varepsilon}_{loc}(\Sigma_0),\ \|f\|_{1-\varepsilon,\alpha+\varepsilon} < \infty\},$$

with norm $\|\ \|_{1-\varepsilon,\alpha+\varepsilon}$. In both spaces we restrict functions to be axially symmetric and symmetric with respect to $x_3 = 0$.

Let T be the linear operator constructed in Proposition 2.2. Then we reformulate (2.10) as

$$2h = A(h) := T(-\varepsilon H^s_{\Sigma_0} - \varepsilon N(h)).$$

We claim that for $\varepsilon > 0$ small, A is a contraction on the ball

$$B = \{h \in X : \|h\|_* \le M\frac{\varepsilon^{\frac{1}{2}}}{|\log\varepsilon|}\},$$

if we choose M large. Indeed, for $h \in B$, by (2.13), (2.14) and (2.15)

$$\begin{aligned}\|A(h)\|_* &\le C\|\varepsilon H^s_{\Sigma_0}\|_{1-\varepsilon,\alpha+\varepsilon} + C\|\varepsilon N(h)\|_{1-\varepsilon,\alpha+\varepsilon}\\ &\le \frac{\varepsilon^{\frac{1}{2}}}{|\log\varepsilon|}(C + \frac{M^2}{|\log\varepsilon|}) \le M\frac{\varepsilon^{\frac{1}{2}}}{|\log\varepsilon|},\end{aligned}$$

if we take $M = 2C$ then let $\varepsilon > 0$ be small. Next, for h_1, $h_2 \in B$,

$$\|A(h_1) - A(h_2)\|_* \le C\varepsilon^{-\frac{1}{2}}(\|h_1\|_* + \|h_2\|_*)\|h_1 - h_2\|_*.$$

But $\varepsilon^{-\frac{1}{2}}(\|h_1\|_* + \|h_2\|_*) \le \frac{C}{|\log\varepsilon|}$ and so A is a contraction on B for $\varepsilon > 0$ small. q.e.d.

3. The ODE of the initial approximation

The purpose of this section is to analyze the solution $f_\varepsilon(r)$ of (2.6), which is used in the construction of the initial approximation. Thanks to (2.4) we have

$$\begin{cases} f_\varepsilon(\varepsilon^{-\frac{1}{2}}) = f_C(\varepsilon^{-\frac{1}{2}}) = \dfrac{1}{2}|\log\varepsilon| + \log 2 + O(\varepsilon), \\ f_\varepsilon'(\varepsilon^{-\frac{1}{2}}) = f_C'(\varepsilon^{-\frac{1}{2}}) = \sqrt{\varepsilon}(1 + O(\varepsilon)). \end{cases} \tag{3.1}$$

Note that $f_\varepsilon'(r) \ge 0$ so, in particular,

$$f_\varepsilon(r) \ge f_\varepsilon(\varepsilon^{-\frac{1}{2}}) \quad \text{for all } r \ge \varepsilon^{-\frac{1}{2}}. \tag{3.2}$$

Lemma 3.1. *We have*

$$C_1|\log\varepsilon| \le |f_\varepsilon(r)| \le C_2|\log\varepsilon|, \quad |f_\varepsilon'(r)| \le C\varepsilon^{\frac{1}{2}},$$

$$|f_\varepsilon''(r)| \le \frac{C}{r^2} + \frac{C\varepsilon}{|\log\varepsilon|^2},$$

for $\varepsilon^{-\frac{1}{2}} \le r \le |\log\varepsilon|\varepsilon^{-\frac{1}{2}}$.

Proof. We make the change of variables $f_\varepsilon(r) = |\log\varepsilon|\tilde{f}(\varepsilon^{\frac{1}{2}}r)$. Integrating the ODE satisfied by $\tilde{f}$ and using (3.2) the desired conclusion follows. q.e.d.

Now we study the asymptotic behavior of $f_\varepsilon(r)$ as $r \to \infty$. For this let us write

$$f_\varepsilon(r) = |\log\varepsilon| f_0^{(\varepsilon)}(\frac{\varepsilon^{\frac{1}{2}}}{|\log\varepsilon|}r), \qquad \text{for } r \ge \frac{1}{|\log\varepsilon|}, \tag{3.3}$$

for a new function $f_0^{(\varepsilon)}$. Then $f_0^{(\varepsilon)}$ satisfies

$$\Delta f_0^{(\varepsilon)} = \frac{1}{f_0^{(\varepsilon)}} \quad \text{for } r \ge \frac{1}{|\log\varepsilon|},$$

and from (3.1)

$$f_0^{(\varepsilon)}(\frac{1}{|\log\varepsilon|}) = \frac{1}{2} + \frac{\log 2}{|\log\varepsilon|} + O(\frac{\varepsilon}{|\log\varepsilon|}),$$

$$[f_0^{(\varepsilon)}]'(\frac{1}{|\log\varepsilon|}) = 1 + O(\varepsilon),$$

as $\varepsilon \to 0$.

Lemma 3.2. *For any $r_0 > 0$ there is a $C > 0$ such that for all $\varepsilon > 0$ sufficiently small we have*

$$|f_0^{(\varepsilon)}(r) - r| \le C, \qquad |[f_0^{(\varepsilon)}]'(r) - 1| \le \frac{C}{r},$$
$$|[f_0^{(\varepsilon)}]''(r)| \le \frac{C}{r},$$

for all $r \ge r_0$.

Proof. We make the change of variables

$$f_0^{(\varepsilon)}(r) = r\psi_\varepsilon(t), \qquad \text{where } r = e^t, \tag{3.4}$$

for $t \ge -\log|\log\varepsilon|$. Then $\psi_\varepsilon(t) > 0$ and satisfies the equation

$$\psi_\varepsilon'' + 2\psi_\varepsilon' + \psi_\varepsilon = \frac{1}{\psi_\varepsilon} \quad \text{for } t \ge -\log|\log\varepsilon|.$$

Using a standard Lyapunov functional for this autonomous equation we obtain that

$$|\psi_\varepsilon'(t)| + |\psi_\varepsilon(t) - 1| \le Ce^{-\delta t/2}, \quad \text{for all } t \ge 0,$$

with C and δ independent of ε. Linearizing around the equilibrium $\psi = 1$, phase plane analysis leads to

$$|\psi_\varepsilon'(t)| + |\psi_\varepsilon(t) - 1| \le Ce^{-t}, \quad \text{for all } t \ge 0, \tag{3.5}$$

and, hence, the lemma follows. q.e.d.

It will be useful for later purposes to also have estimates for the elements in the linearization of (3.3). Namely consider

$$\Delta z + \frac{1}{(f_0^{(\varepsilon)})^2(r)} z = 0, \quad \text{for } r \ge \frac{1}{|\log\varepsilon|}. \tag{3.6}$$

The function

$$\tilde{z}_1(r) = f_0^{(\varepsilon)} - r[f_0^{(\varepsilon)}]'(r) \tag{3.7}$$

satisfies (3.6), since equation (3.3) is invariant by the scaling $f_\lambda(r) = \frac{1}{\lambda} f(\lambda r)$, $\lambda > 0$. We may construct a second independent solution $\tilde{z}_2$ of (3.6) by solving this equation with initial conditions

$$\tilde{z}_2(r_0) = -\tilde{z}_1'(r_0), \qquad \tilde{z}_2'(r_0) = \tilde{z}_1(r_0).$$

Here $r_0 > 0$ is fixed.

Lemma 3.3. *Fix $r_0 > 0$. We have*

$$|\tilde{z}_i(r)| \le C, \quad |\tilde{z}_i'(r)| \le \frac{C}{r},$$

for all $r \ge r_0$, $i = 1, 2$.

Proof. In terms of ψ defined in (3.4), we may write

$$\tilde{z}_1(r) = -r\psi'(\log(r)),$$

so that the boundedness of $\tilde{z}_1$ is consequence of (3.5). For $\tilde{z}_2$, we may consider the equation

$$\phi'' + 2\phi' + 2\phi = g, \quad \text{for } t \geq \log(r_0),$$

with kernel given by $\zeta_1(t) = e^{-t}\cos(t)$, $\zeta_2(t) = e^{-t}\sin(t)$. Then we may express $\tilde{z}_2$ as a perturbation of the correct linear combination of ζ_1, ζ_2. q.e.d.

4. Approximate equation and error

The main result in this section is the proof of Proposition 2.1, namely the estimate

$$\|\varepsilon H^s_{\Sigma_0}\|_{1-\varepsilon,\alpha+\varepsilon} \leq \frac{C\varepsilon^{\frac{1}{2}}}{|\log\varepsilon|}.$$

For $x \in \Sigma_0$ we compute $H^s_{\Sigma_0}(x)$ by splitting

$$H^s_{\Sigma_0}(x) = \int_{\mathbb{R}^3} \frac{\chi_{E_0}(y) - \chi_{E_0^c}(y)}{|x-y|^{4-\varepsilon}}\, dy = I_i + I_o, \tag{4.1}$$

where

$$I_i = \int_{C_R(x)} \frac{\chi_{E_0}(y) - \chi_{E_0^c}(y)}{|x-y|^{4-\varepsilon}}\, dy, \qquad I_o = \int_{C_R(x)^c} \frac{\chi_{E_0}(y) - \chi_{E_0^c}(y)}{|x-y|^{4-\varepsilon}}\, dy$$

are inner and outer contributions respectively. The inner part is the integral on a cylinder $C_R(x)$ of radius R centered at x and the outer contribution the rest. We take R as a function of $x \in \Sigma_0$, $x = (x', F_\varepsilon(x'))$, defined by

$$R = (1 - \eta(|x'| - R_0))R_1 + \eta(|x'| - R_0)F_\varepsilon(|x'|), \tag{4.2}$$

where $R_0 > 0$ is fixed large, $R_1 > 0$ is a small constant and η is as in (2.7).

To define the cylinder, let Π_1, Π_2 be tangent vectors to Σ_0 at x, orthogonal and of length 1, and ν_{Σ_0} be the unit normal vector to Σ_0 oriented such that $\nu_{\Sigma_0}(x)x_3 > 0$. Introduce coordinates (t_1, t_2, t_3) in $\mathbb{R}^3$ by

$$(t_1, t_2, t_3) \mapsto t_1\Pi_1 + t_2\Pi_2 + t_3\nu_{\Sigma_0}.$$

Define the cylinder of center x, radius R and base plane the plane generated by Π_1, Π_2 as

$$C_R(x) = \{x + t_1\Pi_1 + t_2\Pi_2 + t_3\nu_{\Sigma_0}(x) : t_1^2 + t_2^2 < R^2, |t_3| < R\}.$$

For the computation of the inner integral, we represent the surface Σ_0 near x as the graph over its tangent plane at x. More precisely, if

$R_1 > 0$ in (4.2) is chosen small and $\|h\|_*$ is small, there is a function $g = g_x : B_R(0) \subset \mathbb{R}^2$ to $\mathbb{R}$ of class $C^{2,\alpha}$ such that

$$\Sigma_0 \cap C_R(x) = \{x + \Pi t + \nu_{\Sigma_0} g(t) : |t| < R\}, \tag{4.3}$$

where $t = (t_1, t_2)$ and

$$\Pi = [\Pi_1, \Pi_2].$$

Then

$$g(0) = 0, \quad \nabla g(0) = 0, \quad \Delta g(0) = 2H_{\Sigma_0}(x),$$

where H_{Σ_0} is the mean curvature of Σ_0 at x.

In the following statements we use the notation

$$[v]_{\alpha,D} = \sup_{x,y\in D,\, x\neq y} \frac{|v(x) - v(y)|}{|x-y|^\alpha}.$$

Lemma 4.1. *For $x \in \Sigma_0$ and $R = R(x)$ given by (4.2) we have*

$$I_i = -2\pi \frac{H_{\Sigma_0}(x) R^\varepsilon}{\varepsilon} + Rest_1, \tag{4.4}$$

where

$$|Rest_1| \le C[D^2 g]_{\alpha, B_R(0)} R^{1+\alpha-s} + C\|D^2 g\|^3_{L^\infty(B_R(0))} R^{3-s}. \tag{4.5}$$

Here C remains bounded as $s \to 1$ (i.e., $\varepsilon \to 0$).

The main contribution from the outer integral is given in the next result.

Lemma 4.2. *For $x = (x', F_\varepsilon(x')) \in \Sigma_0$ and $R = R(x)$ given by (4.2) we have*

$$|I_o| \le \frac{C}{R^{1-\varepsilon}}, \tag{4.6}$$

and if $|x'| \ge \varepsilon^{-\frac{1}{2}}$,

$$I_o = \frac{\pi}{R^{1-\varepsilon}} \left(1 + O(\varepsilon^{\frac{1}{2}})\right). \tag{4.7}$$

By (4.4) and (4.7) we see that the equation $H^s_{\Sigma_0}(x) = 0$ takes the form

$$-2H_{\Sigma_0}(x) + \frac{\varepsilon}{R} \approx 0,$$

which motivates (2.3).

Lemma 4.3. *Let $x \in \Sigma_0$, and write $x = (x', F_\varepsilon(x'))$, $r = |x'|$. There is $\delta_0 > 0$ and $g : B_\rho(0) \to \mathbb{R}$ of class $C^{2,\alpha}$ such that*

$$\Sigma_0 \cap C_\rho(x) = \{x + \Pi t + \nu g(t) : |t| < \rho\},$$

where $\rho = \delta_0 r$. In particular, g is well defined in $B_R(0)$ where R is defined in (4.2). Moreover, g satisfies

$$\|g\|_{L^\infty(B_R(0))} \leq \begin{cases} C\varepsilon^{\frac{3}{2}} r & \text{if } r \geq \delta|\log\varepsilon|\varepsilon^{-\frac{1}{2}}, \\ C\frac{\varepsilon^{\frac{1}{2}}|\log\varepsilon|}{r} & \text{if } \varepsilon^{-\frac{1}{2}} \leq r \leq \delta|\log\varepsilon|\varepsilon^{-\frac{1}{2}}, \\ C\frac{\log(r)^2}{r^2} & \text{if } r \leq \varepsilon^{-\frac{1}{2}}, \end{cases}$$

$$\|Dg\|_{L^\infty(B_R(0))} \leq \begin{cases} C\varepsilon^{\frac{1}{2}} & \text{if } r \geq \varepsilon^{-\frac{1}{2}}, \\ \frac{C}{r} & \text{if } R_0 \leq r \leq \varepsilon^{-\frac{1}{2}}, \end{cases}$$

$$\|D^2 g\|_{B_R(0)} \leq \begin{cases} \frac{C\varepsilon^{\frac{1}{2}}}{r} & \text{if } r \geq \varepsilon^{-\frac{1}{2}}, \\ \frac{C}{r^2} & \text{if } r \leq \varepsilon^{-\frac{1}{2}}, \end{cases} \qquad [D^2 g]_{\alpha, B_R} \leq \begin{cases} \frac{C\varepsilon^{\frac{1}{2}}}{r^{1+\alpha}} & \text{if } r \geq \varepsilon^{-\frac{1}{2}}, \\ \frac{C}{r^{2+\alpha}} & \text{if } r \leq \varepsilon^{-\frac{1}{2}}. \end{cases}$$

The proof of Lemma 4.3 follows from an application of the implicit function theorem.

Proof of Lemma 4.1. We compute

$$I_i = \int_{C_R(x)} \frac{\chi_{E_0}(y) - \chi_{E_0^c}(y)}{|x-y|^{4-\varepsilon}}\, dy = -2\int_{|t|<R}\int_0^{g(t)} \frac{1}{(|t|^2 + t_3^2)^{\frac{4-\varepsilon}{2}}} dt_3\, dt.$$

Let us decompose

$$I_i = I_{i,1} + I_{i,2} + I_{i,3},$$

where

$$I_{i,1} = -2\int_{|t|<R} \frac{\frac{1}{2}D^2 g(0)[t^2]}{|t|^{4-\varepsilon}}\, dt,$$

$$I_{i,2} = -2\int_{|t|<R} \frac{g(t) - \frac{1}{2}D^2 g(0)[t^2]}{|t|^{4-\varepsilon}}\, dt,$$

$$I_{i,3} = 2(4-\varepsilon)\int_{|t|<R} g(t)^2 \int_0^1 (1-\tau)\frac{\tau g(t)}{(|t|^2 + (\tau g(t))^2)^{\frac{6-\varepsilon}{2}}}\, d\tau\, dt,$$

and $D^2 g$ denotes the Hessian matrix of g. Then

$$I_{i,1} = -\pi\frac{\Delta g(0) R^\varepsilon}{\varepsilon} = -2\pi\frac{H_{\Sigma_0}(x) R^\varepsilon}{\varepsilon},$$

and we estimate

$$|I_{i,2}| \leq C[D^2 g]_{B_R(0),\alpha} R^{\alpha+\varepsilon}, \quad |I_{i,3}| \leq C\|D^2 g\|_{L^\infty}^3 R^{2+\varepsilon},$$

and (4.5) is proven. q.e.d.

Proof of Lemma 4.2. Let $x \in \Sigma_0$, $x = (x', F_\varepsilon(x'))$. We change variables $y = Rz$ and write $\tilde{x}_R = x/R$

$$\int_{C_R(x)^c} \frac{\chi_{E_0}(y) - \chi_{E_0^c}(y)}{|x-y|^{4-\varepsilon}}\, dy = \frac{1}{R^{1-\varepsilon}}\int_{C_1(\tilde{x}_R)^c} \frac{\chi_{E_0/R}(z) - \chi_{E_0^c/R}(z)}{|\tilde{x}_R - z|^{4-\varepsilon}}\, dz,$$

where $C_1(\tilde{x}_R)$ denotes the cylinder of radius 1 centered at $\tilde{x}_R$ and base plane given by the tangent plane to $\partial E_0/R$ at $\tilde{x}_R$. Then (4.6) follows since

$$\left|\int_{C_1(\tilde{x}_R)^c} \frac{\chi_{E_0/R}(z)-\chi_{E_0^c/R}(z)}{|\tilde{x}_R-z|^{4-\varepsilon}}\,dz\right| \le C.$$

To obtain the second estimate we first note that for any $\delta_0>0$ fixed,

$$\left|\int_{|\tilde{x}_R-z|\ge\delta_0\varepsilon^{-\frac{1}{2}}} \frac{\chi_{E_0/R}(z)-\chi_{E_0^c/R}(z)}{|\tilde{x}_R-z|^{4-\varepsilon}}\,dz\right| \le C\varepsilon^{\frac{1}{2}},$$

and, therefore, we need to prove

$$\left|\int_{C_1(\tilde{x}_R)^c,|\tilde{x}_R-z|\le\delta_0\varepsilon^{-\frac{1}{2}}} \frac{\chi_{E_0/R}(z)-\chi_{E_0^c/R}(z)}{|\tilde{x}_R-z|^{4-\varepsilon}}\,dz-\pi\right| \le C\varepsilon^{\frac{1}{2}}.$$

We note that

$$\int_{C_1(\tilde{x}_R)^c,|z-\tilde{x}_R|\le\delta_0\varepsilon^{-\frac{1}{2}}} \frac{\chi_{[|z_3|>1]}-\chi_{[|z_3|<1]}}{|z-\tilde{x}_R|^{4-\varepsilon}}\,dz=\pi+O(\varepsilon^{\frac{1}{2}})$$

(here $z=(z',z_3)$, $z'\in\mathbb{R}^2$, $e_3=(0,0,1)$). Indeed,

$$\begin{aligned}\int_{C_1(\tilde{x}_R)^c,|z-\tilde{x}_R|\le\delta_0\varepsilon^{-\frac{1}{2}}} &\frac{\chi_{[|z_3|>1]}-\chi_{[|z_3|<1]}}{|z-\tilde{x}_R|^{4-\varepsilon}}\,dz\\ &=\int_{|z-\tilde{x}_R|>1,|z-\tilde{x}_R|\le\delta_0\varepsilon^{-\frac{1}{2}}} \frac{\chi_{[|z_3|>1]}-\chi_{[|z_3|<1]}}{|z-\tilde{x}_R|^{4-\varepsilon}}\,dz,\end{aligned}$$

since by symmetry the difference of the two integrals is zero. Since

$$\int_{|z-\tilde{x}_R|\ge\delta_0\varepsilon^{-\frac{1}{2}}} \frac{\chi_{[|z_3|>1]}-\chi_{[|z_3|<1]}}{|z-\tilde{x}_R|^{4-\varepsilon}}\,dz=O(\varepsilon^{\frac{1}{2}}),$$

we get

$$\begin{aligned}\int_{C_1(\tilde{x}_R)^c,|z-\tilde{x}_R|\le\delta_0\varepsilon^{-\frac{1}{2}}} &\frac{\chi_{[|z_3|>1]}-\chi_{[|z_3|<1]}}{|z-\tilde{x}_R|^{4-\varepsilon}}\,dz\\ &=\int_{|z-\tilde{x}_R|>1} \frac{\chi_{[|z_3|>1]}-\chi_{[|z_3|<1]}}{|z-\tilde{x}_R|^{4-\varepsilon}}\,dz+O(\varepsilon^{\frac{1}{2}})\\ &=\pi+O(\varepsilon^{\frac{1}{2}}).\end{aligned}$$

Therefore,

$$\begin{aligned}&\left|\int_{C_1(\tilde{X}_R)^c,|\tilde{X}_R-Z|\le\delta_0\varepsilon^{-\frac{1}{2}}} \frac{\chi_{E_0/R}(Z)-\chi_{E_0^c/R}(Z)}{|\tilde{X}_R-Z|^{4-\varepsilon}}\,dZ-\pi\right| \le\\ &\left|\int_{C_1(\tilde{X}_R)^c,|\tilde{X}_R-Z|\le\delta_0\varepsilon^{-\frac{1}{2}}} \frac{\chi_{E_0/R}(Z)-\chi_{[|z_3|>1]}+\chi_{[|z_3|<1]}-\chi_{E_0^c/R}(Z)}{|\tilde{X}_R-Z|^{4-\varepsilon}}\,dZ\right|\\ &\quad+C\varepsilon^{\frac{1}{2}}.\end{aligned}$$

Note that the point $\tilde{x}_R$ has the form $\tilde{x}_R = (\frac{x'}{R}, 1)$. Inside the region $C_1(\tilde{x}_R)^c \cap \{z : |\tilde{x}_R - z| \le \delta_0 \varepsilon^{-\frac{1}{2}}\}$, ∂E_0 can be represented by

$$|z_3| = \frac{1}{R} F_\varepsilon(R|z'|).$$

As a consequence of Lemma 3.2 we have

$$|\frac{d}{dr}(\frac{1}{R} F_\varepsilon(Rr))| \le C\varepsilon^{\frac{1}{2}},$$

in $C_1(\tilde{x}_R)^c \cap \{z : |\tilde{x}_R - z| \le \delta_0 \varepsilon^{-\frac{1}{2}}\}$. Let us consider the upper part, namely $C_1(\tilde{x}_R)^c \cap \{z : |\tilde{x}_R - z| \le \delta_0 \varepsilon^{-\frac{1}{2}}\} \cap \{z_3 > 0\}$. Inside this region, the symmetric difference of the two sets E_0/R and $|z_3| > 1$ is contained in the cone

$$\tilde{x}_R + \{(z', z_3) \in \mathbb{R}^2 \times \mathbb{R} : |z'| \le \delta_0 \varepsilon^{-\frac{1}{2}}, |z_3| \le C\varepsilon^{\frac{1}{2}} |z'|\}.$$

Therefore, we can estimate

$$\begin{aligned} &\left| \int_{C_1(\tilde{x}_R)^c, |\tilde{x}_R - z| \le \delta_0 \varepsilon^{-\frac{1}{2}}, z_3 > 0} \frac{\chi_{E_0/R}(z) - \chi_{[|z_3|>1]} + \chi_{[|z_3|<1]} - \chi_{E_0^c/R}(z)}{|\tilde{x}_R - z|^{4-\varepsilon}} \, dz \right| \\ &\quad \le \int_{\frac{1}{10} \le |z'| \le \delta_0 \varepsilon^{-\frac{1}{2}}, |z_3| \le C\varepsilon^{\frac{1}{2}} |z|} \frac{1}{|z|^{4-\varepsilon}} dZ \le C\varepsilon^{\frac{1}{2}}. \end{aligned}$$

The integral over $C_1(\tilde{x}_R)^c \cap \{z : |\tilde{x}_R - z| \le \delta_0 \varepsilon^{-\frac{1}{2}}\} \cap \{z_3 < 0\}$ can be handled similarly. q.e.d.

Proof of Proposition 2.1. Let $x \in \Sigma_0$, $x = (x', F_\varepsilon(x'))$ where $|x'| \ge 1$. Let $R = R(x)$ be given by (4.2).

By (4.1), (4.4) we can write

$$\varepsilon H^s_{\Sigma_0}(x) = -2\pi H_{\Sigma_0} R^\varepsilon + \varepsilon Rest_1 + \varepsilon I_o.$$

Since Σ_0 is a minimal surface for $r = |x| \le \varepsilon^{-\frac{1}{2}}$, we have

$$\varepsilon H^s_{\Sigma_0}(x) = E_1 + E_2 + E_3 + E_4 + E_5, \tag{4.8}$$

where

$$\begin{aligned} E_1 &= \pi R^\varepsilon \eta_\varepsilon(-2H_{\Sigma_0} + \frac{\varepsilon}{R}), \\ E_2 &= -2\varepsilon \int_{|t|<R} \frac{g(t) - \frac{1}{2} D^2 g(0)[t^2]}{|t|^{4-\varepsilon}} \, dt, \\ E_3 &= \varepsilon 2(4-\varepsilon) \int_{|t|<R} g(t)^2 \int_0^1 (1-\tau) \frac{\tau g(t)}{(|t|^2 + (\tau g(t))^2)^{\frac{5+s}{2}}} \, d\tau \, dt, \\ E_4 &= \varepsilon I_o(1 - \eta_\varepsilon), \\ E_5 &= (\varepsilon I_o - \frac{\pi\varepsilon}{R^s})\eta_\varepsilon, \end{aligned}$$

and $\eta_\varepsilon(r) = \eta(r - \varepsilon^{-\frac{1}{2}})$ with η is the cut-off function (2.7). Here g is a function such that we have the representation of Σ_0 near X as the graph of g over the tangent plane of Σ_0 at X, as in (4.3).

We start with E_1. For $r \geq \varepsilon^{-\frac{1}{2}} + 1$, F_ε satisfies $\Delta F_\varepsilon = \frac{\varepsilon}{F_\varepsilon}$, so

$$E_1 = \pi F_\varepsilon^\varepsilon \left(\Delta F_\varepsilon (1 - \frac{1}{\sqrt{1 + (F_\varepsilon')^2}}) + \frac{(F_\varepsilon')^2 F_\varepsilon''}{(1 + (F_\varepsilon')^2)^{3/2}} \right).$$

But for this range $F_\varepsilon'(r) = O(\varepsilon^{\frac{1}{2}})$, $F_\varepsilon''(r) = O(\frac{\varepsilon^{\frac{1}{2}}}{r})$, $F_\varepsilon(r) \leq C\varepsilon^{\frac{1}{2}} r$ if $r \geq \delta\varepsilon^{-\frac{1}{2}} |\log\varepsilon|$ and $F_\varepsilon(r) \leq C|\log\varepsilon|$ if $\varepsilon^{-\frac{1}{2}} r \leq \delta\varepsilon^{-\frac{1}{2}} |\log\varepsilon|$, so

$$\sup_{r \geq \varepsilon^{-1/2}+1} r^{1-\varepsilon} |E_1| = O(\varepsilon^{\frac{3}{2}}), \quad \text{as } \varepsilon \to 0.$$

For $r \in [\varepsilon^{-\frac{1}{2}}, \varepsilon^{-\frac{1}{2}} + 1]$ we have $\Delta f_\varepsilon = O(\frac{\varepsilon}{|\log\varepsilon|})$, $\Delta f_C = O(\varepsilon^2)$, and so $(f_\varepsilon - f_C)' = O(\frac{\varepsilon}{|\log\varepsilon|})$, $f_\varepsilon - f_C = O(\frac{\varepsilon}{|\log\varepsilon|})$ in this region. Then for these r

$$\begin{aligned} -\Delta F_\varepsilon + \frac{\varepsilon}{F_\varepsilon} &= -\eta_\varepsilon \frac{\varepsilon}{f_\varepsilon} + \frac{\varepsilon}{\eta_\varepsilon f_\varepsilon + (1 - \eta_\varepsilon) f_C} \\ &\quad - (1 - \eta_\varepsilon)\Delta f_C - 2\eta_\varepsilon'(f_\varepsilon - f_C)' - \Delta\eta_\varepsilon (f_\varepsilon - f_C) \\ &= O(\frac{\varepsilon}{|\log\varepsilon|}). \end{aligned}$$

It follows that

$$\sup_{r \in [\varepsilon^{-\frac{1}{2}}, \varepsilon^{-\frac{1}{2}}+1]} r^{1-\varepsilon} |E_1| = O(\frac{\varepsilon^{\frac{1}{2}}}{|\log\varepsilon|}).$$

In a similar way, we obtain the bound

$$\sup_{r \geq \varepsilon^{-1/2}} r^{2-\varepsilon} |E_1'(r)| \leq C \frac{\varepsilon^{\frac{1}{2}}}{|\log\varepsilon|}.$$

From here, the desired estimate for the Hölder part of the norm, $[E_1]_{1-\varepsilon,\alpha+\varepsilon}$ readily follows.

Similar arguments can be used to obtain the same estimates for the remaining terms in decomposition (4.8). We omit the details. q.e.d.

5. Limit problem in Σ_0

We want to build a right inverse for the operator

$$L_0(h) = \Delta h + \frac{\varepsilon}{F_\varepsilon(r)^2} \eta_\varepsilon(r) h,$$

which arises as the linearization of the approximate problem (2.5). Here η_ε is any family of continuous cut-off functions with $\eta_\varepsilon(r) = 0$ for $r \leq \varepsilon^{-\frac{1}{2}}$ and $\eta_\varepsilon(r) = 1$ for $r \geq \delta|\log\varepsilon|\varepsilon^{-\frac{1}{2}}$, where $\delta > 0$ is a sufficiently small number.

We then consider the equation

$$L_0(\phi) = g, \quad \text{in } \mathbb{R}^2, \tag{5.1}$$

and work in the class of radial functions.

Proposition 5.1. *Let $1 \le \gamma < 2$. If $\varepsilon > 0$ is small there is $C > 0$ such that for g radially symmetric with $\|(1+|x|)^\gamma g\|_{L^\infty} < +\infty$ there exists a radially symmetric solution of (5.1) $\phi = T(g)$ with $\|(1+|x|)^{\gamma-2}\phi\|_{L^\infty} < +\infty$ that defines a linear operator of g with*

$$\||x|^{\gamma-2}\phi\|_{L^\infty} \ \le \ C\,\|(1+|x|)^\gamma g\|_{L^\infty},$$

and $\phi(0) = 0$.

Proof. Since all functions are radial, we have to solve

$$\phi'' + \frac{1}{r}\phi' + \frac{\varepsilon}{F_\varepsilon(r)^2}\eta_\varepsilon(r)\phi = g, \quad r > 0.$$

We solve this ODE with initial condition $\phi(0) = \phi'(0) = 0$. For a fixed small $\delta > 0$ and $r \le \delta|\log\varepsilon|\varepsilon^{-\frac{1}{2}}$ we directly obtain

$$(1+r)|\phi'(r)| + |\phi(r)| \le Cr^{2-\gamma}\|(1+|x|)^\gamma g\|_{L^\infty}.$$

Let us consider the range $r \ge r_1$ where $r_1 = \delta|\log\varepsilon|\varepsilon^{-\frac{1}{2}}$. We write the solution ϕ in terms of the elements of the kernel of the linear operator $\Delta + \frac{\varepsilon}{f_\varepsilon^2}$, which are given by

$$z_i(r) = \tilde z_i\Big(\frac{\varepsilon^{\frac{1}{2}}r}{|\log\varepsilon|}\Big), \quad r \ge \frac{\delta|\log\varepsilon|}{\varepsilon^{\frac{1}{2}}},$$

where $\tilde z_i$ is the functions introduced in (3.7). Using the estimates in Lemma 3.3 and the variation of parameters formula we obtain the desired estimate for ϕ for $r > r_1$. q.e.d.

6. Fractional exterior problem

In this section, we will construct a linear bounded operator that maps f defined on Σ_0 to ϕ defined also on Σ_0 with the property

$$\varepsilon\mathcal{J}^s_{\Sigma_0}(\phi)(x) = f(x) \quad \text{for } x \in \Sigma_0,\ |x| \ge R, \tag{6.1}$$

where $R > 0$ will be a large fixed constant.

Proposition 6.1. *If R is fixed large, there is a linear operator $f \mapsto \phi$ defined for radial, symmetric functions f on Σ_0 with $\|f\|_{1-\varepsilon,\alpha+\varepsilon} < \infty$, such that ϕ is radial, symmetric, satisfies (6.1) and*

$$\|\phi\|_* \le C\|f\|_{1-\varepsilon,\alpha+\varepsilon}.$$

Here the norms $\|\ \|_*$ and $\|\ \|_{1-\varepsilon,\alpha+\varepsilon}$ are the ones defined in (2.11), (2.12).

We will also need a version of this result for right hand sides with fast decay. Let $0 < \tau < 1$.

Proposition 6.2. *If R is fixed large, there is a linear operator $f \mapsto \phi$ defined for f radial, symmetric and $\||x|^{2+\tau-\varepsilon} f\|_{L^\infty(\Sigma_0)} < \infty$, such that ϕ is symmetric, satisfies (6.1) and*

$$\||x|^\tau \phi\|_{L^\infty(\Sigma_0)} \leq C\||x|^{2+\tau-\varepsilon} f\|_{L^\infty(\Sigma_0)}.$$

In order to prove Propositions 6.1 and 6.2 we study first

$$L_\varepsilon(\phi) + W_\varepsilon(r)\phi = f \quad \text{in } \mathbb{R}^2, \tag{6.2}$$

where

$$L_\varepsilon(\phi)(x) = \varepsilon \, \frac{2}{\pi} \text{p.v.} \int_{\mathbb{R}^2} \frac{\phi(y) - \phi(x)}{|x-y|^{4-\varepsilon}} \, dy, \tag{6.3}$$

and

$$W_\varepsilon(r) = \frac{\varepsilon}{F_\varepsilon(r)^{2-\varepsilon}} \eta_\varepsilon(r), \quad r = |x|,$$

where

$$\eta_\varepsilon(r) = \eta(\varepsilon^{-\frac{1}{2}} r - 1), \tag{6.4}$$

and η is a smooth cut-off function with $\eta(t) = 1$ for $t \geq 1$ and $\eta(t) = 0$ for $t \leq 0$.

We start with a version of Proposition 6.1 for problem (6.2).

Lemma 6.1. *There is a linear operator that given a radial function f in $\mathbb{R}^2$ such that $\|f\|_{1-\varepsilon,\alpha+\varepsilon} < \infty$ produces a radial solution ϕ of (6.2) with the property*

$$\|\phi\|_* \leq C\|f\|_{1-\varepsilon,\alpha+\varepsilon}. \tag{6.5}$$

Then norms are the ones defined in (2.11), (2.12) in the context of functions defined on $\mathbb{R}^2$.

For smooth bounded functions h, $L_\varepsilon(h)$ has the expansion

$$L_\varepsilon(h) = \Delta h(x) + O(\varepsilon) \quad \text{as } \varepsilon \to 0,$$

so equation (6.2) can be considered a perturbation of

$$\Delta h + W(x)h = g \quad \text{in } \mathbb{R}^2,$$

where

$$W(x) = \frac{\varepsilon}{F_\varepsilon(x)^2} \eta_\varepsilon(x).$$

The next lemma is a standard estimate for convolutions.

Lemma 6.2. *Assume $\gamma, \beta < 2$, $\gamma + \beta > 2$. Let $\|(1+|x|)^\gamma f\|_{L^\infty} < \infty$. Then*

$$\left| \int_{\mathbb{R}^2} \frac{1}{|x-y|^\beta} f(y) \, dy \right| \leq C\|(1+|x|)^\gamma f\|_{L^\infty} (1+|x|)^{2-\beta-\gamma}.$$

Lemma 6.3. *Let g be radial with $\|(1+|x|)^{\gamma-\varepsilon} g\|_{L^\infty} < \infty$ where $\gamma \in (1,2)$. Then for $\varepsilon > 0$ small (6.2) has a radial solution h depending linearly on g with $h(0) = 0$. Moreover,*

$$\|(1+|x|)^{\gamma-2} h\|_{L^\infty} \le C \|(1+|x|)^{\gamma-\varepsilon} g\|_{L^\infty}.$$

Proof. Instead of looking directly for a solution of (6.2) we will solve

$$D_r h(x) = c_{2,\varepsilon}\, \text{p.v.} \int_{\mathbb{R}^2} \frac{|x| - \langle y, \frac{x}{|x|}\rangle}{|x-y|^{2+\varepsilon}} (W_\varepsilon h - g)\, dy, \tag{6.6}$$

for a radial function h with $h(0) = 0$. Here D_r is the radial derivative.

The idea is that equation (6.2) is the same as $(-\Delta)^{1-\varepsilon/2} h + W_\varepsilon h = g$ and, hence, it makes sense to look for solutions as fixed points of $h(x) = c \int_{\mathbb{R}^2} \frac{1}{|x-y|^\varepsilon} (g(y) - W_\varepsilon(y) h(y))\, dy$. But we are looking for solutions with growth, and besides, we would like to treat this equation as a perturbation of the case $\varepsilon = 0$, so we choose instead to take a radial derivative. Note that for g radial the convolution $\int_{\mathbb{R}^2} \frac{1}{|x-y|^\varepsilon} g(y)\, dy$ is a radial function, and

$$D_r \int_{\mathbb{R}^2} \frac{1}{|x-y|^\varepsilon} g(y)\, dy = -\varepsilon \sum_{i=1}^2 \frac{x_i}{|x|} \int_{\mathbb{R}^2} \frac{x_i - y_i}{|x-y|^{2+\varepsilon}} g(y)\, dy.$$

This yields equation (6.6), for some appropriate constant $c_{2,\varepsilon}$.

In (6.6) the integral converges if $\|(1+|x|)^{\gamma-\varepsilon}(W_\varepsilon h - g)\|_{L^\infty} < \infty$ by Lemma 6.2. Equation (6.6) is equivalent to

$$D_r h - A_\varepsilon(h) = B_\varepsilon(g), \tag{6.7}$$

where

$$A_\varepsilon(h)(x) = c_{2,\varepsilon}\, \text{p.v.} \int_{\mathbb{R}^2} \frac{|x| - \langle y, \frac{x}{|x|}\rangle}{|x-y|^{2+\varepsilon}} W_\varepsilon(y) h(y)\, dy,$$
$$B_\varepsilon(g)(x) = -c_{2,\varepsilon}\, \text{p.v.} \int_{\mathbb{R}^2} \frac{|x| - \langle y, \frac{x}{|x|}\rangle}{|x-y|^{2+\varepsilon}} g(y)\, dy.$$

Let A_0 be the operator

$$A_0(h)(x) = c_2\, \text{p.v.} \int_{\mathbb{R}^2} \frac{|x| - \langle y, \frac{x}{|x|}\rangle}{|x-y|^2} W(y) h(y)\, dy.$$

Then (6.7) is equivalent to

$$D_r h - A_0(h) = A_\varepsilon(h) - A_0(h) + B_\varepsilon(g). \tag{6.8}$$

We claim that given ψ radial in $\mathbb{R}^2$ with $\|(1+r)^{\gamma-1}\psi\|_{L^\infty} < \infty$ we can find a radial solution h to

$$D_r h - A_0(h) = \psi \tag{6.9}$$

satisfying $h(0) = 0$ and

$$\|(1+r)^{\gamma-1} h'\|_{L^\infty} + \|r^{\gamma-2} h\|_{L^\infty} \le C \|(1+r)^{\gamma-1}\psi\|_{L^\infty}. \tag{6.10}$$

Indeed, we need to solve

$$h'(r) + \frac{1}{r}\int_0^r W(s)h(s)s\,ds = \psi(r) \quad \text{for all } r > 0.$$

Let

$$\tilde{\psi}(r) = \int_0^r \psi(s)\,ds, \quad \tilde{h}(r) = h(r) - \tilde{\psi}(r).$$

Then we look for $\tilde{h}$ satisfying

$$\tilde{h}'(r) + \frac{1}{r}\int_0^r W(s)\tilde{h}(s)s\,ds = -\frac{1}{r}\int_0^r W(s)\tilde{\psi}(s)s\,ds,$$

which we write as

$$\Delta\tilde{h} + W(r)\tilde{h}(r) = W(r)\tilde{\psi}(r), \quad 0 < r < \infty.$$

We solve this equation using Proposition 5.1 and obtain

$$\|(1+r)^{\gamma-1}\tilde{h}'\|_{L^\infty} + \|r^{\gamma-2}\tilde{h}\|_{L^\infty} \le C\|(1+r)^{\gamma-2}\tilde{\psi}\|_{L^\infty}.$$

Then $h = \tilde{h} + \tilde{\psi}$ satisfies (6.9), $h(0) = 0$ and estimate (6.10).

Let T denote the operator that to a radial function $\psi \in L^\infty(\mathbb{R}^2)$ gives the radial solution h to (6.9) just constructed, and note that by (6.10)

$$\|T(\psi)\|_a \le C\|(1+r)^{\gamma-1}\psi\|_{L^\infty}, \tag{6.11}$$

where

$$\|\varphi\|_a = \||x|^{\gamma-2}\varphi\|_{L^\infty} + \|(1+|x|)^{\gamma-1}\nabla\varphi\|_{L^\infty}.$$

We rewrite (6.8) as

$$h = T(A_\varepsilon(h) - A_0(h) + B_\varepsilon(g)) \tag{6.12}$$

in the space $X = \{h \in W^{1,\infty}_{loc}(\mathbb{R}^2) : h \text{ is radial}, \|h\|_a < \infty\}$ with norm $\|\ \|_a$.

We solve (6.12) by the contraction mapping principle. After some computation, we find that for some $b > 0$

$$|(A_\varepsilon(h) - A_0(h))(x)| \ \le \ \varepsilon^b(1+|x|)^{1-\gamma}\|h\|_a.$$

It follows that the map from X to itself given by $T(A_\varepsilon(h) - A_0(h) + B_\varepsilon(g))$ is a contraction for $\varepsilon > 0$ small, and, hence, it has a unique fixed point h. This fixed point satisfies

$$\|h\|_a \le C\|T(B_\varepsilon(g))\|_a \le C\|(1+r)^{\gamma-1}B_\varepsilon(g)\|_{L^\infty}$$

by (6.11). Using then Lemma 6.2 we find that

$$\|h\|_a \le C\|(1+|x|)^{\gamma-\varepsilon}g\|_{L^\infty},$$

and we check that this h, indeed, solves (6.2). q.e.d.

Proof of Lemma 6.1. The proof is based on the following apriori estimate for radial solutions h of (6.2) such that $\||x|^{-1}h\|_{L^\infty} < \infty$:

$$\||x|^{-1}h\|_{L^\infty} \le C\|(1+|x|)^{1-\varepsilon}g\|_{L^\infty}, \tag{6.13}$$

and we claim it holds if $\varepsilon > 0$ is sufficiently small.

We argue by contradiction, assuming that there are sequences $\varepsilon_i \to 0$, radial functions g_i, h_i solving (6.2) and satisfying

$$\||x|^{-1}h_i\|_{L^\infty} = 1, \quad \|(1+|x|)^{1-\varepsilon_i}g_i\|_{L^\infty} \to 0 \tag{6.14}$$

as $i \to \infty$. Let $x_i \in \mathbb{R}^2$ be such that

$$(1+|x_i|)^{-1}|h_i(x_i)| \ge \frac{1}{2}.$$

Assume first that x_i remains bounded and, up to a subsequence $x_i \to x$ as $i \to \infty$. The bounds (6.14) and standard estimates for L_ε, uniform as $\varepsilon \to 0$, show that h_i is bounded in $C^{1,\alpha}_{loc}$. Therefore, passing to a subsequence we find $h_i \to h$ locally uniformly in $\mathbb{R}^2$. Let $\varphi \in C_0^\infty(\mathbb{R}^2)$. Multiplying (6.2) by φ and integrating we find

$$\int_{\mathbb{R}^2} h_i L_{\varepsilon_i}(\varphi) + W_{\varepsilon_i} h_i \varphi_i = \int_{\mathbb{R}^2} g_i \varphi.$$

Taking the limit we find that h is harmonic in $\mathbb{R}^2$. But also $|h(x)| \ge \frac{1}{2}$, h is radial and $|h(r)| \le r$ for all $r \ge 0$, which is impossible.

Suppose that x_i is unbounded so that up to subsequence $r_i = |x_i| \to \infty$ as $i \to \infty$. Let

$$\tilde h_i(x) = \frac{1}{r_i} h(r_i x), \quad \tilde g_i(x) = r_i^{1-\varepsilon_i} g(r_i x),$$

so that

$$L_{\varepsilon_i}(\tilde h_i) + W_i(x)\tilde h_i = \tilde g_i \quad \text{in } \mathbb{R}^2,$$

where

$$W_i(x) = \frac{\varepsilon_i \eta_{\varepsilon_i}(r_i x) r_i^{2-\varepsilon_i}}{F_{\varepsilon_i}(r_i x)^{2-\varepsilon_i}}.$$

Also

$$\||x|^{-1}\tilde h_i\|_{L^\infty} = 1, \quad \||x|^{1-\varepsilon_i}\tilde g_i\|_{L^\infty} \to 0$$

as $i \to \infty$. Up to subsequence $\tilde h_i \to h$ locally uniformly in $\mathbb{R}^2$ and $x_i/r_i \to \hat x$. Moreover, $|h(\hat x)| \ge \frac{1}{2}$.

If $\varepsilon_i^{-\frac{1}{2}}|\log\varepsilon_i| r_i^{-1} \to \infty$ as $i \to \infty$ then $W_i(x) \to 0$ uniformly on compact sets and we reach a contradiction as before.

If $\varepsilon_i^{-\frac{1}{2}}|\log\varepsilon_i| r_i^{-1} \to R_0$, then $W_i(x) \to W(x)$ uniformly on compact sets where $W(x)$ is bounded for $|x| \le R_0$ and $W(x) = \frac{1}{|x|^2}$ for $|x| \ge R_0$. Then h solves

$$\Delta h + Wh = 0 \quad \text{in } \mathbb{R}^2,$$

with $|h(r)| \le r$ for all $r \ge 0$. This implies $h \equiv 0$, a contradiction.

Finally, if $\varepsilon_i^{-\frac{1}{2}}|\log\varepsilon_i|r_i^{-1}\to 0$, then h satisfies

$$\Delta h+\frac{1}{|x|^2}h=0\quad\text{in }\mathbb{R}^2\setminus\{0\},$$

with $|h(r)|\le r$ for all $r>0$. Again this implies that h is trivial.

Existence of a solution to (6.2) can be deduced from the solvability obtained in Lemma 6.3 and the apriori estimate (6.13), with an approximation argument. Namely, let g be radial with $\|(1+|x|)^{1-\varepsilon}g\|_{L^\infty}<\infty$ and η be a smooth cut-off function with $\eta(x)=1$ for $|x|\le 1$, $\eta(x)=0$ for $|x|\ge 2$. Thanks to Lemma 6.3 there is a radial solution h_n of (6.2) with right hand side $g\eta(x/n)$. By (6.13) we have $\|(1+|x|)^{-1}h_n\|_{L^\infty}\le C$ and by standard estimates h_n is bounded is $C^{1,\alpha}_{loc}$. Up to subsequence h_n converges to a solution h satisfying

$$\|(1+|x|)^{-1}h\|_{L^\infty}\le C\|(1+|x|)^{1-\varepsilon}g\|_{L^\infty}.$$

Finally, estimate (6.5) follows from a standard scaling argument and Schauder estimates for L_ε, which is $(-\Delta)^{\frac{1+s}{2}}$ up to constant, and which are uniform as $\varepsilon\to 0$. q.e.d.

Next we give a result analogous to Lemma 6.1 but for functions with fast decay.

Lemma 6.4. *There is a linear operator that given a radial function f in $\mathbb{R}^2$ such that $\|(1+|x|)^{2+\tau-\varepsilon}f\|_{L^\infty}<\infty$ produces a solution ϕ of (6.2) with the property*

$$\||x|^\tau\phi\|_{L^\infty}\le C\|(1+|x|)^{2+\tau-\varepsilon}f\|_{L^\infty}. \tag{6.15}$$

Proof. Let Y denote the space of radial functions in $\mathbb{R}^2$ satisfying $\||x|^\tau\phi\|_{L^\infty}<\infty$. We claim there exists $\phi\in Y$ that depends linearly on f satisfying

(6.16)
$$\nabla\phi(x)=c_{2,\varepsilon}\int_{\mathbb{R}^2}\left(\frac{x-y}{|x-y|^{2+\varepsilon}}-\frac{x}{|x|^{2+\varepsilon}}\right)\left(f(y)-\frac{\eta_\varepsilon(|y|)}{|y|^{2-\varepsilon}}\phi(y)\right)dy,$$

and the estimate (6.15). This function is the desired solution. Here $c_{2,\varepsilon}\to\frac{1}{2\pi}$ as $\varepsilon\to 0$.

Similar to Lemma 6.2 we have the following estimate. Assume $0<\beta<2$, $2<\gamma<3$ and $\gamma+\beta>2$. Let $\|(1+|x|)^\gamma f\|_{L^\infty}<\infty$. Then

$$\left|\int_{\mathbb{R}^2}\left(\frac{x-y}{|x-y|^{\beta+1}}-\frac{x}{|x|^{\beta+1}}\right)f(y)\,dy\right|\le C\|(1+|x|)^\gamma f\|_{L^\infty}|x|^{2-\beta-\gamma}.$$

Using this estimate with $\beta=1+\varepsilon$ we see that the integral (6.16) is well defined if $\|(1+|x|)^{2+\tau-\varepsilon}f\|_\infty<\infty$ and $\phi\in Y$.

We treat (6.16) as a perturbation of the case $\varepsilon = 0$. So first we consider the equation

$$\Delta\phi + \frac{\eta_\varepsilon}{r^2}\phi = f \quad \text{in } \mathbb{R}^2,$$

with η_ε as in (6.4), for which we want to construct a solution such that

$$\||x|^\tau \phi\|_{L^\infty(\mathbb{R}^2)} \le \|(1+|x|)^{2+\tau} f\|_{L^\infty(\mathbb{R}^2)}. \tag{6.17}$$

For $r \ge \varepsilon^{-\frac{1}{2}} + 1$ the equation is given by

$$\frac{1}{r}(r\phi')' + \frac{1}{r^2}\phi = f, \quad r \ge \varepsilon^{-\frac{1}{2}},$$

hence, we take ϕ of the form

$$\begin{aligned}\phi(r) = \cos(\log(r)) &\int_r^\infty \sin(\log(t)) t f(t) dt \\ &- \sin(\log(r)) \int_r^\infty \cos(\log(t)) t f(t) dt,\end{aligned}$$

for $r \ge \varepsilon^{-\frac{1}{2}} + 1$. From this formula we get directly

$$\sup_{r \ge \varepsilon^{-\frac{1}{2}}} r^\tau |\phi(r)| \le \|r^{2+\tau} f\|_{L^\infty}.$$

For $0 < r \le \varepsilon^{-\frac{1}{2}} + 1$ we define ϕ as the unique solution of the equation

$$\frac{1}{r}(r\phi')' + \frac{\eta_\varepsilon(r)}{r^2} = f, \quad r \le \varepsilon^{-\frac{1}{2}} + 1,$$

with initial conditions at $\varepsilon^{-\frac{1}{2}} + 1$ to make ϕ a global solution for $r \in (0, \infty)$. Note that

$$\phi(\varepsilon^{-\frac{1}{2}}) = O(\varepsilon^{\frac{\tau}{2}}), \quad \phi'(\varepsilon^{-\frac{1}{2}}) = O(\varepsilon^{\frac{1+\tau}{2}}).$$

Let $r_0 = \varepsilon^{-\frac{1}{2}}$. Then for $r \le r_0$ we can represent

$$\phi(r) = c_1 + c_2 \log(\frac{r}{r_0}) + \int_r^{r_0} \frac{1}{s} \int_s^{r_0} t f(t) dt ds,$$

where c_1, c_2 have to satisfy

$$c_1 = \phi(r_0) = O(\varepsilon^{\frac{\tau}{2}}), \quad c_2 = r_0 \phi'(r_0) = O(\varepsilon^{\frac{\tau}{2}}).$$

With this formula we can verify (6.17). The previous solution satisfies

$$\phi(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|x-y|} \left(f(y) - \frac{\eta_\varepsilon(|y|)}{|y|^2} \phi(y) \right) dy + A \log|x| + B,$$

where A, B depend on f and are such that $\phi(x) \to 0$ as $|x| \to \infty$. Therefore, for the gradient we have

$$\begin{aligned}\nabla\phi(x) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} \left(f(y) - \frac{\eta_\varepsilon(|y|)}{|y|^2} \phi(y) \right) dy + A \frac{x}{|x|^2} \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \left(\frac{x-y}{|x-y|^2} - \frac{x}{|x|^2} \right) \left(f(y) - \frac{\eta_\varepsilon(|y|)}{|y|^2} \phi(y) \right) dy. \qquad (6.18)\end{aligned}$$

Let $\phi = T(f)$ denote the operator that associates the function $\nabla\phi$ constructed above, so that, in particular, (6.17) and (6.18) hold. To find a solution of (6.16) it then suffices to find $\phi \in Y$ such that

$$\nabla\phi = T(B_\varepsilon(f) + A_0(\phi) - A_\varepsilon(\phi)),$$

where the operators B_ε, A_0, A_ε are defined as

$$B_\varepsilon(f)(x) = c_{2,\varepsilon}\int_{\mathbb{R}^2}\left(\frac{x-y}{|x-y|^{2+\varepsilon}} - \frac{x}{|x|^{2+\varepsilon}}\right) f(y)\,dy,$$
$$A_\varepsilon(\phi)(x) = c_{2,\varepsilon}\int_{\mathbb{R}^2}\left(\frac{x-y}{|x-y|^{2+\varepsilon}} - \frac{x}{|x|^{2+\varepsilon}}\right)\frac{\eta_\varepsilon(|y|)}{|y|^{2-\varepsilon}}\phi(y)\,dy,$$
$$A_0(\phi)(x) = c_{2,\varepsilon}\int_{\mathbb{R}^2}\left(\frac{x-y}{|x-y|^{2}} - \frac{x}{|x|^{2}}\right)\frac{\eta_\varepsilon(|y|)}{|y|^{2}}\phi(y)\,dy,$$

and ϕ is defined from $\nabla\phi$ by integration such that $\lim_{|x|\to\infty}\phi(x) = 0$ (here all functions are radial). Similarly, as in Lemma 6.3 we can show that for $\varepsilon > 0$ small the map from Y to Y given by $\phi \mapsto T(B_\varepsilon(f) + A_0(\phi) - A_\varepsilon(\phi))$ is a contraction. q.e.d.

For the proof of Proposition 6.1 we need an estimate of

$$a_\varepsilon(x) = \varepsilon\int_{\Sigma_0}\frac{1 - \langle\nu_{\Sigma_0}(y), \nu_{\Sigma_0}(y)\rangle}{|x-y|^{4-\varepsilon}}dy.$$

Lemma 6.5. *Let $x = (x', F_\varepsilon(x')) \in \Sigma_0$. Then*

$$a_\varepsilon(x) = \pi|A_{\Sigma_0}|^2|x'|^\varepsilon + O\Big(\frac{\varepsilon}{(1+|x|)^{2-\varepsilon}}\Big) + O\Big(\frac{\varepsilon}{\log(|x|)^{2-\varepsilon}}\Big)\chi_{|x|\le\varepsilon^{-\frac{1}{2}}}$$
$$+ \pi\frac{\varepsilon}{F_\varepsilon(x')^{2-\varepsilon}}(1+o(1))\chi_{|x|\ge\varepsilon^{-\frac{1}{2}}},$$

where $|A_{\Sigma_0}|$ is the norm of the second fundamental form of Σ_0 and $O()$, $o()$ are uniform x as $\varepsilon \to 0$.

For the proof, we locally represent the surface Σ_0 as a graph of a smooth function on a tangent plane at a given point, as given in Lemma 4.3. We omit the details.

Proof of Propositions 6.1 and 6.2. The idea is to reduce problem (6.1) to one in $\mathbb{R}^2$. Suppose that ϕ is a radial function on Σ_0, symmetric with respect to $x_3 = 0$ vanishing in $B_{2R}(0)$. Here $R > 0$ is large and fixed, to be chosen later. Since ϕ is symmetric with respect to $x_3 = 0$, we can define $\tilde\phi$ globally in $\mathbb{R}^2$ by

$$\tilde\phi(x) = \phi(x, \pm F_\varepsilon(x)), \quad |x| \ge R,$$

and $\tilde\phi = 0$ in $B_R(0)$. Let C_R be the cylinder

$$C_R = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 < R^2\}.$$

Then, for $X \in \Sigma_0$ of the form $X = (x, F_\varepsilon(x))$ with $|x| \ge R$, we have

$$
\begin{aligned}
&\text{p.v.} \int_{\Sigma_0 \setminus C_R} \frac{\phi(Y) - \phi(X)}{|Y - X|^{4-\varepsilon}}\, dY \\
&= \text{p.v.} \int_{\mathbb{R}^2 \setminus B_R} \frac{\tilde\phi(y) - \tilde\phi(x)}{(|x-y|^2 + (F_\varepsilon(x) - F_\varepsilon(y))^2)^{\frac{4-\varepsilon}{2}}} \sqrt{1 + |\nabla F_\varepsilon(y)|^2}\, dy \\
&\quad + \int_{\mathbb{R}^2 \setminus B_R} \frac{\tilde\phi(y) - \tilde\phi(x)}{(|x-y|^2 + (F_\varepsilon(x) + F_\varepsilon(y))^2)^{\frac{4-\varepsilon}{2}}} \sqrt{1 + |\nabla F_\varepsilon(y)|^2}\, dy.
\end{aligned}
$$

Then we find for $|X| \ge R$, $X = (x, F_\varepsilon(x))$,

$$
\text{p.v.} \int_{\Sigma_0} \frac{\phi(Y) - \phi(X)}{|Y - X|^{4-\varepsilon}}\, dY = \text{p.v.} \int_{\mathbb{R}^2} \frac{\tilde\phi(y) - \tilde\phi(x)}{|y - x|^{4-\varepsilon}}\, dy + b(x)\tilde\phi(x) + B_1(\tilde\phi)(x),
$$

where

$$
\begin{aligned}
b(x) &= \int_{B_R} \frac{1}{|x-y|^{4-\varepsilon}}\, dy - \int_{\Sigma_0 \cap C_R} \frac{1}{|(x, F_\varepsilon(x)) - Y|^{4-\varepsilon}}\, dY, \\
B_1(\tilde\phi)(x) &= \int_{\mathbb{R}^2 \setminus B_R} \left(\tilde\phi(y) - \tilde\phi(x)\right) \\
&\qquad \times \left(\frac{\sqrt{1 + |\nabla F_\varepsilon(y)|^2}}{(|x-y|^2 + (F_\varepsilon(x) - F_\varepsilon(y))^2)^{\frac{4-\varepsilon}{2}}} - \frac{1}{|x-y|^{4-\varepsilon}} \right) dy \\
&\qquad + \int_{\mathbb{R}^2 \setminus B_R} \frac{\tilde\phi(y) - \tilde\phi(x)}{(|x-y|^2 + (F_\varepsilon(x) + F_\varepsilon(y))^2)^{\frac{4-\varepsilon}{2}}} \sqrt{1 + |\nabla F_\varepsilon(y)|^2}\, dy.
\end{aligned}
$$

Let

$$
a_\varepsilon(X) = \varepsilon \int_{\Sigma_0} \frac{1 - \langle \nu_{\Sigma_0}(Y), \nu_{\Sigma_0}(X)\rangle}{|X - Y|^{3+s}} dY.
$$

Then (6.1) reads as

(6.19)
$$
L_\varepsilon(\tilde\phi) + \frac{\eta_\varepsilon}{|x|^{2-\varepsilon}} \tilde\phi(x) + \varepsilon B_1(\tilde\phi)(x) + (\varepsilon b(x) + a_\varepsilon - \frac{\eta_\varepsilon}{|x|^{2-\varepsilon}})\tilde\phi(x) = \tilde f(x),
$$

where $\tilde f(x) = f(x, F_\varepsilon(x))$ and L_ε is the operator (6.3). We look for $\tilde\phi$ of the form $\tilde\phi = \eta\varphi$, where η is a smooth radial cut-off function such that $\eta(x) = 1$ for $|x| \ge 3R$ and $\eta(x) = 0$ for $|x| \le 2R$. Then we ask that φ solves

(6.20)
$$
L_\varepsilon(\varphi) + \frac{\eta_\varepsilon}{|x|^{1-s}} \varphi + \varepsilon B_2(\varphi) + \eta(\varepsilon b(x) + a_\varepsilon - \frac{\eta_\varepsilon}{|x|^{1-s}})\varphi = \tilde f(x) \quad \text{in } \mathbb{R}^2,
$$

where

$$
B_2(\varphi)(x) = \varepsilon \tilde\eta(x) \int_{\mathbb{R}^2} \varphi(y) \frac{\eta(y) - \eta(x)}{|x-y|^{4-\varepsilon}}\, dy + \varepsilon \tilde\eta(x) B_1[\eta\varphi](x),
$$

and where $\tilde\eta$ is another radial smooth cut-off function such that $\tilde\eta(x) = 1$ for $|x| \geq 5R$, $\tilde\eta(x) = 0$ for $|x| \leq 4R$. If φ solves (6.20), then $\tilde\phi = \eta\varphi$ will satisfy (6.19) for $|x| \geq 5R$. Let T denote the operator constructed in Lemma 6.1, so that $\phi = T(f)$ is a radial solution to (6.2) satisfying the estimate (6.5). Then we rewrite (6.20) as the fixed point problem

$$\varphi = T(-\varepsilon B_2(\varphi) - \eta(\varepsilon b(x) + a_\varepsilon - \frac{\eta_\varepsilon}{|x|^{1-s}})\varphi + \tilde f).$$

We can apply the contraction mapping principle by the following estimates

$$\|\varepsilon B_2(\varphi)\|_{1-\varepsilon,\alpha} \leq o(1)\|\varphi\|_*,$$
$$\|\eta(\varepsilon b(x) + a_\varepsilon - \frac{\eta_\varepsilon}{|x|^{2-\varepsilon}})\varphi\|_{1-\varepsilon,\alpha} \leq o(1)\|\varphi\|_*,$$

where $o(1) \to 0$ as $\varepsilon \to 0$ and $R \to \infty$, which can be proved using Lemma 6.5.

The proof of Proposition 6.2 follows the same lines as the one of Proposition 6.1. q.e.d.

7. Linear theory

The purpose here is to construct a linear operator $f \mapsto \phi$ which gives a solution to the problem

$$\varepsilon \mathcal{J}^s_{\Sigma_0}(\phi) = f \quad \text{in } \Sigma_0, \tag{7.1}$$

where $\mathcal{J}^s_{\Sigma_0}$ is the nonlocal Jacobi operator

$$\mathcal{J}^s_{\Sigma_0}(\phi)(x) = \text{p.v.} \int_{\Sigma_0} \frac{\phi(y) - \phi(x)}{|x-y|^{4-\varepsilon}}\, dy + \phi(x) \int_{\Sigma_0} \frac{(\nu(x) - \nu(y)) \cdot \nu(x)}{|x-y|^{4-\varepsilon}} dy,$$

and Σ_0 is the surface defined in (2.8).

The main result is stated in Proposition 2.2, which we recall: there is a linear operator that to a function f on Σ_0 such that f is radially symmetric and symmetric with respect to $x_3 = 0$ with $\|f\|_{1-\varepsilon,\alpha+\varepsilon} < \infty$, gives a solution ϕ of (7.1). Moreover,

$$\|\phi\|_* \leq C\|f\|_{1-\varepsilon,\alpha+\varepsilon}.$$

The norms $\| \ \|_{1-\varepsilon,\alpha+\varepsilon}$ and $\| \ \|_*$ are defined in (2.12), (2.11).

As $\varepsilon \to 0$, Σ_0 approaches the standard catenoid $\mathcal{C}$ on compact sets, which can be described by the parametrization

$$y \in \mathbb{R} \mapsto \big(\sqrt{1+y^2}\cos(\theta), \sqrt{1+y^2}\sin(\theta), \log(y + \sqrt{1+y^2})\big),$$

with $y \in \mathbb{R}$, $\theta \in [0, 2\pi]$. Hence, for smooth bounded ϕ we have

$$\varepsilon \mathcal{J}^s_{\Sigma_0}(\phi) \to \frac{\pi}{2}(\Delta_{\mathcal{C}}\phi + |A|^2\phi),$$

uniformly over compact sets as $\varepsilon \to 0$, where $\Delta_{\mathcal{C}}$ is the Laplace–Beltrami operator and $|A|$ the norm of the second fundamental form of $\mathcal{C}$ (see Lemmas A.2 and A.4).

Let us recall the standard nondegeneracy property of the Jacobi operator $\Delta_{\mathcal{C}} + |A|^2$ on the catenoid. Linearly independent elements in its kernel are the functions

$$Z_1(y) = \frac{y}{\sqrt{y^2+1}}, \quad Z_2(y) = -1 + \frac{y}{\sqrt{y^2+1}} \log(y + \sqrt{y^2+1}). \tag{7.2}$$

The knowledge of these elements in the kernel of $\Delta_{\mathcal{C}} + |A|^2$, plus its explicit representation as a regular second order linear operator, see, for instance, [**1**] immediately yields

Lemma 7.1. *If ϕ is a bounded axially symmetric solution of $\Delta_{\mathcal{C}}\phi + |A|^2\phi = 0$ in $\mathcal{C}$ then $\phi = cZ_1$ for some $c \in \mathbb{R}$.*

Let

$$a_\varepsilon(x) = \varepsilon \int_{\Sigma_0} \frac{1 - \langle \nu_{\Sigma_0}(y), \nu_{\Sigma_0}(x)\rangle}{|x-y|^{3+s}} dy,$$

and

$$b_\varepsilon(x) = a_\varepsilon(x)\eta_\varepsilon(x),$$

where η_ε is smooth, radial, $\eta(x) = 0$ for $|x| \geq \varepsilon^{-\frac{1}{2}} + 1$, and $\eta(x) = 1$ for $|x| \leq \varepsilon^{-\frac{1}{2}}$.

Let us write

$$L_\varepsilon(\phi)(x) = \varepsilon \, \text{p.v.} \int_{\Sigma_0} \frac{\phi(y) - \phi(x)}{|x-y|^{4-\varepsilon}} dy,$$

and consider the equation

$$L_\varepsilon(\phi) + b_\varepsilon(x)\phi = f \quad \text{in } \Sigma_0. \tag{7.3}$$

We will consider from now only right hand sides $f : \Sigma_0 \to \mathbb{R}$ which are symmetric with respect to the plane $x_3 = 0$, and symmetric solutions ϕ.

Let $0 < \tau < 1$.

Proposition 7.1. *For $\varepsilon > 0$ small there is a linear operator that takes f symmetric with respect to x_3 with $\|y^{2+\tau-\varepsilon} f\|_{L^\infty} < \infty$ to a symmetric bounded solution ϕ of (7.3). Moreover,*

$$\|\phi\|_{L^\infty} \leq C\|y^{2+\tau-\varepsilon} f\|_{L^\infty},$$

$$\|(1+|y|)^{1+\tau}\nabla\phi\|_{L^\infty} \leq C\|y^{2+\tau-\varepsilon} f\|_{L^\infty}, \tag{7.4}$$

and $\lim_{|x|\to\infty} \phi(x)$ exists.

The counterpart of this result for the Jacobi operator $\Delta_{\mathcal{C}} + |A|^2$, without assuming any symmetry on f or ϕ is: if $\||y|^{2+\tau} f\|_{L^\infty} < \infty$ and $\int_{\mathcal{C}} fZ_1 = 0$, there is a bounded solution ϕ of

$$\Delta_{\mathcal{C}}\phi + |A|^2\phi = f \quad \text{in } \mathcal{C},$$

and this solution is unique except a constant times Z_1. Moreover, ϕ has limits at both ends, which have to coincide. In the nonlocal setting, to simplify we work with functions that are symmetric with respect to x_3, so in some sense the condition $\int_{\mathcal{C}} f Z_1 = 0$ is automatic.

For the existence part in Proposition 7.1 we study the truncated problem

$$\begin{cases} L_\varepsilon(\phi) + b_\varepsilon \phi = f \quad \text{in } \Sigma_0 \cap B_R(0), \\ \phi = 0 \quad \text{on } \Sigma_0 \setminus B_R(0). \end{cases} \tag{7.5}$$

Let

$$\sigma = \frac{1+s}{2} = 1 - \frac{\varepsilon}{2}.$$

Given in $f \in L^2(\Sigma_0 \cap B_R(0))$ there is a weak solution $\phi \in H^\sigma(\Sigma_0)$ of

$$\begin{cases} -L_\varepsilon(\phi) = f \quad \text{in } \Sigma_0 \cap B_R(0), \\ \phi = 0 \quad \text{on } \Sigma_0 \setminus B_R(0). \end{cases}$$

By weak solution we mean $\phi \in H^\sigma(\Sigma_0)$, $\phi = 0$ on $\Sigma_0 \setminus B_R(0)$ and

$$\int_{\Sigma_0} \int_{\Sigma_0} \frac{(\phi(y) - \phi(x))(\varphi(y) - \varphi(x))}{|x-y|^{2+2\sigma}} \, dy dx = \int_{\Sigma_0} f(x) \varphi(x) \, dx,$$

for all $\varphi \in H^\sigma(\Sigma_0)$ with $\varphi = 0$ in $\Sigma_0 \setminus B_R(0)$. This solution can be found by minimizing the functional

$$\frac{1}{4} \int_{\Sigma_0} \int_{\Sigma_0} \frac{(\phi(y) - \phi(x))^2}{|x-y|^{2+2\sigma}} \, dy dx - \int_{\Sigma_0} f(x) \phi(x) \, dx,$$

over the space $\{\phi \in H^\sigma(\Sigma_0) : \phi = 0 \quad \text{on } \Sigma_0 \setminus B_R(0)\}$. For f locally bounded and $\varepsilon > 0$ small (σ is close to 1), the solution belongs to $C^{1,\alpha}_{loc}$.

First we establish an apriori estimate for solutions of (7.5).

Lemma 7.2. *Suppose f is symmetric and $\| |y|^{2+\tau-\varepsilon} f \|_{L^\infty} < \infty$. There are $\varepsilon_0, R_0, C > 0$ such that for $0 < \varepsilon \le \varepsilon_0$, $R \ge R_0$, and any symmetric solution ϕ of (7.5) we have*

$$\|\phi\|_{L^\infty} \le C \| |y|^{2+\tau-\varepsilon} f \|_{L^\infty}.$$

Proof. If the conclusion fails, there are sequences $\varepsilon_n \to 0$, $R_n \to \infty$, ϕ_n solving (7.5) for some f_n such that

$$\|\phi_n\|_{L^\infty} = 1, \quad \| |y|^{2+\tau-\varepsilon_n} f_n \|_{L^\infty} \to 0$$

as $n \to \infty$. We show that for any $\rho > 0$ fixed

$$\sup_{\Sigma_0 \cap B_\rho(0)} |\phi_n| \to 0 \quad \text{as } n \to \infty.$$

If not, then passing to a subsequence, for some $x_n \in \Sigma_0 \cap B_\rho(0)$,

$$|\phi_n(x_n)| \ge \delta > 0.$$

By standard estimates, ϕ_n is bounded in C^α_{loc}. Hence, by passing to a new subsequence, $\phi_n \to \phi$ locally uniformly as $n \to \infty$. We pass to the limit in the weak formulation and obtain a bounded symmetric solution $\phi \not\equiv 0$ of

$$\Delta_{\mathcal{C}}\phi + |A|^2\phi = 0 \quad \text{in } \mathcal{C}.$$

But by Lemma 7.1 the only bounded solution is cZ_1, which is odd. Hence, $\phi \equiv 0$ and this is a contradiction.

We claim that

$$\|\phi_n\|_{L^\infty(\Sigma_0 \cap B_{R_n}(0))} \to 0$$

as $n \to \infty$, which is a contradiction.

Indeed, let $w = 1 - \delta|y|^{-\tau}$. One can check that

$$L_{\varepsilon_n}(w) \le -c_{\varepsilon_n}\delta|y|^{-\tau-2+\varepsilon_n},$$

for $|y| \ge \bar{R}$ where $\bar{R}$ is large and fixed and c_{ε_n} converges to a positive constant as $\varepsilon_n \to 0$. Next we choose $\delta > 0$ such that $\inf_{\Sigma_0 \cap B_{\bar{R}}(0)} w > 0$. We claim that

$$\phi_n \le C(\|\phi\|_{L^\infty(\Sigma_0 \cap B_{\bar{R}}(0))} + \||y|^{\tau+2-\varepsilon_n} f_n\|_{L^\infty})w \tag{7.6}$$

in $\Sigma_0 \cap (B_{R_n}(0) \setminus B_{\bar{R}}(0))$. Note that (7.6) holds for C large depending on ϕ_n because ϕ_n is bounded. The claim is that this holds for $C = C_0$ with

$$C_0 = \max\left(2(\inf_{\Sigma_0 \cap B_{\bar{R}}(0))} w)^{-1}, \sup \frac{|f_n|}{c_{\varepsilon_n}\delta|y|^{-\tau-2-+\varepsilon_n}}\right).$$

The comparison can be done by sliding. q.e.d.

Using the Fredholm alternative, we deduce the following result.

Lemma 7.3. *Suppose f is symmetric and $\||y|^{2+\tau-\varepsilon} f\|_{L^\infty} < \infty$. For $0 < \varepsilon \le \varepsilon_0$ and $R \ge R_0$ there is a unique symmetric solution ϕ of* (7.5).

Proof of Proposition 7.1. We fix $0 < \varepsilon \le \varepsilon_0$ for $R \ge R_0$ and let ϕ_R be the solution of (7.5). Then for a sequence $R_j \to \infty$, $\phi = \lim_{j\to\infty} \phi_{R_j}$ exists and is a solution of (7.3).

Estimate (7.4) is obtained by scaling and the gradient estimates of Caffarelli and Silvestre [8]. Finally, $\lim_{|x|\to\infty} \phi(x)$ exists because of (7.4). q.e.d.

We need a solvability theory with a constraint on the right hand side so that the solution decays. For this we consider the equation

$$L_\varepsilon(\phi) + b_\varepsilon\phi = f - cZ_2\eta_1 \quad \text{in } \Sigma_0, \tag{7.7}$$

where η_1 is a smooth radial symmetric cut-off function on Σ_0, such that $\eta_1(x) = 1$ for $|x| \le A_1$, $\eta_1(x) = 0$ for $|x| \ge A_1 + 1$ and A_1 is a fixed large constant. The function $Z_2\eta_1$ in the right hand side can be replaced by any f_0 with $f_0(x) = O(|x|^{-2-\tau+\varepsilon})$, $\int_{\Sigma_0} f_0 Z_2 \ne 0$.

Proposition 7.2. *There is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \le \varepsilon_0$ and any f symmetric with respect to x_3 with $\||y|^{2+\tau-\varepsilon} f\|_{L^\infty} < \infty$ there is a unique solution ϕ, c of (7.7) such that ϕ is symmetric and $\||y|^\tau \phi\|_{L^\infty} < \infty$. Moreover,*

$$\||y|^\tau \phi\|_{L^\infty} + |c| \le C\||y|^{2+\tau-\varepsilon} f\|_{L^\infty}.$$

Proof. First we prove existence. For this we let ϕ_0 be the solution of (7.3) constructed in Proposition 7.1 with right hand side $Z_2\eta_1$. Then $\lim_{|x|\to\infty} \phi_0(x) = \Lambda_\varepsilon$ exists. Testing equation (7.3) against Z_2 and integrating on Σ_0, after some computation we find that $\Lambda_0 = \lim_{\varepsilon\to 0} \Lambda_\varepsilon$ exists and it is strictly positive.

Now, we let $\hat\phi$ be the solution of (7.3) constructed in Proposition 7.1 with right hand side f. Then, subtracting off a suitable multiple of ϕ_0 from $\hat\phi$ we find a solution ϕ of Problem (7.7) for some c which is uniformly estimated thanks to estimate (7.4).

Let us prove uniqueness. Suppose that for a sequence $\varepsilon_n \to 0$ there is a nontrivial solution ϕ_n, c_n of (7.7) with $f = 0$. We can assume

$$\||y|^\tau \phi\|_{L^\infty} = 1. \tag{7.8}$$

To estimate c_n we multiply equation (7.3) by Z_2 and integrate on Σ_0, to find that

$$c_n \to 0 \quad \text{as } n \to \infty.$$

As in Lemma 7.2, $\phi_n \to 0$ uniformly on compact sets. Then by (7.8) there is a point $x_n \in \Sigma_0$ such that

$$(1 + |x_n|)^\tau |\phi_n(x_n)| \ge \frac{1}{2},$$

and $|x_n| \to \infty$. By scaling and translating we obtain a non-trivial ϕ satisfying

$$\Delta\phi = 0 \quad \text{in } \mathbb{R}^2 \setminus \{0\},$$

with

$$|\phi(x)| \le C|x|^{-\tau},$$

which is impossible. q.e.d.

Next we establish an a priori estimate for decaying solutions of (7.1). We do not expect solutions of this problem to decay, but that this will be the case if f satisfies a constraint. For this reason, instead of (7.1) we consider a projected equation

$$\varepsilon \mathcal{J}^s_{\Sigma_0}(\phi) = f - c f_0 \quad \text{in } \Sigma_0, \tag{7.9}$$

where f_0 is an appropriate function. For f_0 we can take almost any smooth function with compact support, but it will be important that

$$\int_{\Sigma_0} f_0 Z_2 \neq 0,$$

and that we have a solution ϕ_0 with $\|\phi_0\|_* < \infty$ of

$$\varepsilon \mathcal{J}^s_{\Sigma_0}(\phi_0) = f_0 \quad \text{in } \Sigma_0.$$

One possibility to achieve this is the following. Let $R > 0$ be the number given in Proposition 6.1. For $\rho > R$ let $\eta_\rho(x) = \eta(x/\rho)$ where η is a smooth radial cut-off function in $\mathbb{R}^3$, such that $\eta(x) = 1$ for $|x| \le 1$ and $\eta(x) = 0$ for $|x| \ge 2$. Let $f_\rho = Z_2 \eta_\rho$ and ϕ_ρ be the function constructed in Proposition 6.1. We recall that it satisfies

$$\varepsilon \mathcal{J}^s_{\Sigma_0}(\phi_\rho)(X) = f_\rho(X) \quad \text{for } X \in \Sigma_0,\ |X| \ge R,$$

and the estimate

$$\|\phi_\rho\|_* \le C \|f_\rho\|_{1-\varepsilon,\alpha+\varepsilon}.$$

Note that

$$\|f_\rho\|_{1-\varepsilon,\alpha+\varepsilon} \le C\rho \log(\rho),$$

and that since f_ρ is smooth, ϕ_ρ is also smooth. Using elliptic estimates we deduce that $\|\phi_\rho\|_{C^{2,\alpha}(B_R)} \le C\rho\log(\rho)$. Let

$$\tilde f_\rho = \varepsilon \mathcal{J}^s_{\Sigma_0}(\phi_\rho).$$

Then

$$\int_{\Sigma_0} \tilde f_\rho Z_2 = \int_{\Sigma_0 \cap B_R} \varepsilon \mathcal{J}^s_{\Sigma_0}(\phi_\rho) Z_2 + \int_{\Sigma_0 \setminus B_R} Z_2^2 \eta_\rho.$$

Since

$$\int_{\Sigma_0 \cap B_R} \varepsilon \mathcal{J}^s_{\Sigma_0}(\phi_\rho) Z_2 = O(\rho \log(\rho)), \quad \int_{\Sigma_0 \setminus B_R} Z_2^2 \eta_\rho = c\rho^2 \log(\rho)^2 (1+o(1))$$

as $\rho \to \infty$, where $c > 0$, we find that for $\rho > 0$ large

$$\int_{\Sigma_0} \tilde f_\rho Z_2 \ne 0.$$

We fix ρ large and take

$$\phi_0 = \phi_\rho, \qquad f_0 = \tilde f_\rho. \tag{7.10}$$

Lemma 7.4. *Assume $\||x|^{2+\tau-\varepsilon} f\|_{L^\infty(\Sigma_0)} < \infty$ and ϕ, c is a solution of (7.9) such that $\||x|^\tau \phi\|_{L^\infty(\Sigma_0)} < \infty$. If ε is small enough, then there is C independent of f, ϕ, c such that*

$$\||x|^\tau \phi\|_{L^\infty(\Sigma_0)} + |c| \le C \||x|^{2+\tau-\varepsilon} f\|_{L^\infty(\Sigma_0)}.$$

Proof. Assume by contradiction that there are sequences $\varepsilon_n \to 0$, ϕ_n, c_n solving (7.9) with right hand side f_n such that

$$\|(1+|x|)^\tau \phi_n\|_{L^\infty(\Sigma_0)} = 1, \quad \|(1+|x|)^{2+\tau-\varepsilon_n} f_n\|_{L^\infty(\Sigma_0)} \to 0$$

as $n \to \infty$. Recall that $\Sigma_0 = \Sigma_0(\varepsilon_n)$.

To estimate c_n, let Z_2 be given as in (7.2). We test equation (7.9) with $Z_2\eta_n$ where η_n is a smooth cut-off function such that $\eta_n(r) = 1$ for $r \leq R_n$ and $\eta_n(r) = 0$ for $r \geq 2R_n$, with $R_n \to \infty$ and

$$R_n << \varepsilon_n^{-\frac{1}{2}}.$$

We get

$$\begin{aligned}\varepsilon_n \int_{\Sigma_0(\varepsilon_n)} \phi_n(x) \int_{\Sigma_0(\varepsilon_n)} Z_2(y) \frac{\eta_n(y) - \eta_n(x)}{|x-y|^{4-\varepsilon_n}}\, dy\, dx \\ + \int_{\Sigma_0(\varepsilon_n)} \phi_n(y)\eta_n(y)\mathcal{J}_{\Sigma_0}(Z_2)(y)\, dy \\ = \int_{\Sigma_0(\varepsilon_n)} f_n Z_2 \eta_n - c_n \int_{\Sigma_0(\varepsilon_n)} f_0 Z_2 \eta_n.\end{aligned}$$

By a calculation

$$\varepsilon_n \int_{\Sigma_0(\varepsilon_n)} \phi_n(x) \int_{\Sigma_0(\varepsilon_n)} Z_2(y) \frac{\eta_n(y) - \eta_n(x)}{|x-y|^{4-\varepsilon_n}}\, dy\, dx \to 0$$

as $n \to \infty$, and

$$\int_{\Sigma_0(\varepsilon_n)} \phi_n(y)\eta_n(y)\mathcal{J}_{\Sigma_0}[Z_2](y)\, dy \to 0$$

as $n \to \infty$. It follows that

$$c_n \to 0 \quad \text{as } n \to \infty.$$

There is a point $x_n \in \Sigma_0(\varepsilon_n)$ such that

$$(1 + |x_n|)^\tau |\phi_n(x_n)| \geq \frac{1}{2}.$$

If x_n remains bounded, then up to subsequence $\phi_n \to \phi$ uniformly on compact sets of the catenoid $\mathcal{C}$ and ϕ is a nontrivial solution of

$$\Delta_{\mathcal{C}}\phi + |A|^2\phi = 0 \quad \text{on } \mathcal{C},$$

with $|\phi(x)| \leq (1+|x|)^{-\tau}$. By Lemma 7.1 ϕ must be zero, a contradiction.

Hence, x_n is unbounded. By scaling and translating we obtain a non-trivial ϕ satisfying

$$\Delta\phi + \frac{\tilde{\eta}}{r^2}\phi = 0 \quad \text{in } \mathbb{R}^2,$$

with

$$|\phi(x)| \leq C|x|^{-\tau},$$

where $0 \leq \tilde{\eta} \leq 1$ is a radial, non-decreasing function such that $\tilde{\eta} = 1$ for all $|x| \geq m$, where $m \geq 0$. For $r \geq m$ we get

$$\phi(r) = a\cos(\log(r)) + b\sin(\log(r)),$$

but then $a = b = 0$, so $\phi \equiv 0$, a contradiction. q.e.d.

Proof of Proposition 2.2. We want to solve (7.1) where f is radial and symmetric such that $\|f\|_{1-\varepsilon,\alpha+\varepsilon} < \infty$. First we reduce the problem to one where the right hand side has fast decay. Let $\bar\phi = \bar\phi(f)$ be the function constructed in Proposition 6.1 with right hand side f, namely $\bar\phi$ satisfies

$$\varepsilon \mathcal{J}^s_{\Sigma_0}(\bar\phi)(X) = f \quad X \in \Sigma_0, |X| \geq R,$$

where $R > 0$ is fixed in this proposition. Then we look for ϕ of the form $\phi = \phi_1 + \eta\bar\phi$ where $\eta \in C^\infty(\mathbb{R}^2)$ is a cut-off function such $\eta(x) = 1$ for $|x| \geq 2R$, $\eta(x) = 0$ for $|x| \leq R$. The function ϕ_1 then needs to satisfy

$$\varepsilon \mathcal{J}^s_{\Sigma_0}(\phi_1) = f_1 \quad \text{in } \Sigma_0,$$

where

$$f_1(x) = (1-\eta(x))f(x) - \varepsilon \int_{\Sigma_0} \bar\phi(y) \frac{\eta(y)-\eta(x)}{|y-x|^{4-\varepsilon}}\, dy.$$

Since the second term decays like $|x|^{-4+\varepsilon}$ as $|x| \to \infty$, f_1 has fast decay, meaning $\|(1+|x|)^{2+\tau-\varepsilon} f\|_{L^\infty(\Sigma_0)} < \infty$.

In the sequel, we assume that f is symmetric, radial with $\|(1+|x|)^{2+\tau-\varepsilon} f\|_{L^\infty(\Sigma_0)} < \infty$. First, we claim that it is possible to find a solution ϕ, c to (7.9), which depends linearly on f and such that

$$\|(1+|x|)^\tau \phi\|_{L^\infty} + |c| \leq C \|(1+|x|)^{2+\tau-\varepsilon} f\|_{L^\infty}.$$

We construct this solution by looking for it in the form

$$\phi = \varphi + \eta_0 \psi,$$

and we ask that

$$L_\varepsilon(\varphi) + b_\varepsilon \varphi = -[L_\varepsilon, \eta_0](\psi) + (1-\eta_0) f + c f_0 \quad \text{in } \Sigma_0, \tag{7.11}$$

$$L_\varepsilon(\psi) + a_\varepsilon \psi = -a_\varepsilon(1-\eta_\varepsilon)\varphi + f \quad \text{in } \Sigma_0 \setminus B_R(0). \tag{7.12}$$

Here

$$[L_\varepsilon, \eta](\psi) = L_\varepsilon(\eta_0\psi) - \eta_0 L_\varepsilon(\psi) = \varepsilon\, \text{p.v.} \int_{\Sigma_0} \psi(y) \frac{\eta_0(y) - \eta_0(x)}{|x-y|^{4-\varepsilon}}\, dy,$$

and R is the same as in Proposition 6.2. The smooth cut-off functions, η_0 and η_ε are radial in $\mathbb{R}^3$ and such that

$$\eta_0(x) = 0 \text{ for } |x| \leq R, \qquad \eta_0(x) = 1 \text{ for } |x| \geq 2R,$$

$$\eta_\varepsilon(x) = 1 \text{ for } |x| \leq \varepsilon^{-\frac12}, \qquad \eta_\varepsilon(x) = 0 \text{ for } |x| \geq \varepsilon^{-\frac12} + 1.$$

We rewrite this system as a fixed point problem as follows. Let Y be the space $Y = \{\varphi \in L^\infty(\Sigma_0) : \|(1+|x|)^\tau \varphi\|_{L^\infty} < \infty\}$ with the norm $\|\varphi\|_Y = \|(1+|x|)^\tau\varphi\|_{L^\infty}$. Given $\varphi \in Y$ we solve (7.12) using Proposition 6.2 and obtain a solution $\psi = \psi(\varphi)$. With this ψ we solve now problem (7.11) using Proposition 7.2 and obtain a solution $\tilde\varphi = \tilde\varphi(\varphi) \in Y$. Let $T(\varphi) = \tilde\varphi(\varphi)$ denote the operator defined in this way, so that $T : Y \to Y$ is an affine linear operator.

We claim that T is compact. Assume that φ_n is a bounded sequence in Y, and let ψ_n be the corresponding solution of (7.12). By Proposition 6.2 $\|\psi_n\|_Y \leq C$. Let $\tilde\varphi_n$, c_n be the solution of (7.11) with ψ replaced by ψ_n and c by c_n. We claim that up to subsequence $\tilde\varphi_n$ converges in Y. By standard regularity $\tilde\varphi_n$ is bounded in $C^{1,\alpha}_{loc}(\Sigma_0)$ (any $0<\alpha<1$). Then for a subsequence (denoted the same), $\tilde\varphi_n \to \tilde\varphi$ uniformly on compact sets of Σ_0 as $n\to\infty$. Let $\tau' \in (\tau,1)$. Then note that $[L_\varepsilon,\eta][\psi_n]$ and $(1-\eta_0)f + c_n f_0$ have fast decay uniform in ε, more precisely

$$\|(1+|x|)^{2+\tau'-\varepsilon}(-[L_\varepsilon,\eta_0](\psi_n) + (1-\eta_0)f + c_n f_0)\|_{L^\infty} \leq C.$$

By Proposition 7.2

$$\|(1+|x|)^{\tau'}\tilde\varphi_n\|_{L^\infty} \leq C,$$

and, hence, also $\|(1+|x|)^{\tau'}\tilde\varphi\|_{L^\infty} < \infty$. It follows that for any $r>0$

$$\limsup_{n\to\infty} \sup_{\Sigma_0\cap B_r(0)} (1+|x|)^{\tau}|\tilde\varphi_n - \varphi| = 0,$$

$$\limsup_{n\to\infty} \sup_{\Sigma_0\setminus B_r(0)} (1+|x|)^{\tau}|\tilde\varphi_n - \varphi| \leq C r^{\tau-\tau'},$$

so that $\limsup_{n\to\infty}\|\tilde\varphi_n - \varphi\|_Y \leq C r^{\tau-\tau'}$. Since r is arbitrary, $\|\tilde\varphi_n - \tilde\varphi\|_Y \to 0$ as $n\to\infty$. This proves that T is compact. By Lemma 7.4 and the Fredholm alternative there is a unique solution of the system (7.11), (7.12) and, hence, we find a unique solution ϕ to (7.9). Moreover,

$$\|(1+|x|)^{\tau}\phi\|_{L^\infty} + |c| \leq C\|(1+|x|)^{2+\tau-\varepsilon}f\|_{L^\infty},$$

by Lemma 7.4.

Finally, we solve (7.1) when $\|(1+|x|)^{2+\tau-\varepsilon}f\|_{L^\infty}<\infty$. For this let ϕ_0 be defined by (7.10). We look now for a solution ϕ of (7.1) of the form $\phi = \phi_1 + \alpha\phi_0$, where we want ϕ_1 to have fast decay. Then (7.1) is equivalent to

$$\varepsilon \mathcal{J}^s_{\Sigma_0}(\phi_1) = f - \alpha f_0.$$

Given $\alpha\in\mathbb{R}$, by the previous results we know that there exists $c_1 = c_1(\alpha)$ and $\phi_1 = \phi_1(\alpha)$ of fast decay solving

$$\varepsilon \mathcal{J}^s_{\Sigma_0}(\phi_1) = f - (\alpha + c_1(\alpha)) f_0.$$

We claim that it is possible to choose α such that $c_1(\alpha)=0$. For this, consider the function Z_2 of (7.2) and η a smooth cut-off function on Σ_0 such that $\eta(x)=1$ for $|x|\leq \tilde R$ and $\eta(x)=0$ for $|x|\geq 2\tilde R$ with $\tilde R$ such that $\tilde R\to\infty$ and $\varepsilon\tilde R^2\log(\tilde R)\to 0$. By the same calculation as in Proposition 7.2 we get

$$\varepsilon\int_{\Sigma_0}\phi_1(x)\int_{\Sigma_0} Z_2(y)\frac{\eta(y)-\eta(x)}{|x-y|^{4-\varepsilon}}\,dy\,dx + \int_{\Sigma_0}\phi_1(y)\eta(y)\mathcal{J}_{\Sigma_0}(Z_2)(y)\,dy$$

(7.13)

$$= \int_{\Sigma_0} f Z_2\eta - (\alpha + c_1(\alpha))\int_{\Sigma_0} f_0 Z_2 \eta.$$

For the first 2 terms, we have

$$\begin{aligned}&\left|\varepsilon\int_{\Sigma_0}\phi_1(x)\int_{\Sigma_0}Z_2(y)\frac{\eta(y)-\eta(x)}{|x-y|^{4-\varepsilon}}\,dy\,dx\right|\\&\quad=o(1)\|(1+|x|)^\tau\phi_1\|_{L^\infty}\\&\quad\le o(1)(\|(1+|x|)^{2+\tau-\varepsilon}f\|_{L^\infty}+|\alpha|),\end{aligned}$$

and

$$\begin{aligned}\left|\int_{\Sigma_0}\phi_1(y)\eta(y)\mathcal{J}_{\Sigma_0}(Z_2)(y)\,dy\right|&=o(1)\|(1+|x|)^\tau\phi_1\|_{L^\infty}\\&\le o(1)(\|(1+|x|)^{2+\tau-\varepsilon}f\|_{L^\infty}+|\alpha|),\end{aligned}$$

where $o(1)\to 0$ as $\tilde{R}\to\infty$ and $\varepsilon\to 0$. Then equation (7.13) for α is uniquely solvable if ε is small. q.e.d.

8. The nonlinear term

Consider h_1, h_2 defined on Σ_0 with $\|h_i\|_*\le\sigma_0\varepsilon^{\frac12}$, where $\sigma_0>0$ is a small constant. The main result in this section is the following estimate stated in Proposition 2.3:

$$\varepsilon\|N(h_1)-N(h_2)\|_{1-\varepsilon,\alpha+\varepsilon}\le C\varepsilon^{-\frac12}(\|h_1\|_*+\|h_2\|_*)\|h_1-h_2\|_*.$$

Note the “extra” $\varepsilon^{-\frac12}$ in the left hand side.

We rewrite the fractional mean curvature in the following way. For a point $x=(x',F_\varepsilon(x'))\in\Sigma_0$ let $x_h=x+\nu_{\Sigma_0}(x)h(x)$ and let $L_h(x)$ denote the half space defined by

$$L_h(x)=\{y\in\mathbb{R}^3:\langle y-x_h,\nu_{\Sigma_h}(x_h)\rangle\ge 0\},$$

where ν_{Σ_h} is the unit normal vector to ∂E_h pointing into E_h. Then

$$H^s_{E_h}(x_h)=2\int_{\mathbb{R}^3}\frac{\chi_{E_h}(y)-\chi_{L_h(x)}(y)}{|x_h-y|^{3+s}}\,dy,$$

which has the advantage that the integral is convergent.

To compute the previous integral restricted to a ball around x, let us represent Σ_h near this point as a graph over the tangent plane to Σ_0 at X. We start with r,θ polar coordinates for $x\in\mathbb{R}^2$, i.e., $x=(r\cos\theta,r\sin\theta)$ and let $\hat{r}=\frac{x'}{r}=(\cos\theta,\sin\theta)^T$, $\hat{\theta}=(-\sin\theta,\cos\theta)^T$. Given a point $x\in\Sigma_0$, $x=(x',F_\varepsilon(x'))$ we let

$$\Pi_1(x)=\frac{1}{\sqrt{1+F_\varepsilon'(x')^2}}\begin{bmatrix}\hat{r}\\F_\varepsilon'(x')\end{bmatrix},\quad\Pi_2(x)=\begin{bmatrix}\hat{\theta}\\0\end{bmatrix}\in\mathbb{R}^3,$$

$$\Pi=[\Pi_1,\Pi_2].$$

The unit normal vector to Σ_0 at X pointing up is then given by

$$\nu_{\Sigma_0}(X)=\frac{1}{\sqrt{1+F_\varepsilon'(x')^2}}\begin{bmatrix}-F_\varepsilon'(x')\hat{r}\\1\end{bmatrix}.$$

Then we consider coordinates $t=(t_1,t_2)$ and t_3 defined by

$$(t_1,t_2,t_3)\mapsto \Pi_1(x)t_1+\Pi_2(x)t_2+\nu_{\Sigma_0}(x)t_3.$$

Let

$$R_x=\delta|x|,$$

where $\delta>0$ is a small fixed constant, and let us define $t_0=t_0(x)$ such that $\Pi(x)t_0$ is the orthogonal projection of x onto the plane generated by $\Pi_1(x)$, $\Pi_2(x)$.

Using the implicit function theorem, given h on Σ_0 with $\|h\|_*\le \sigma_0\varepsilon^{\frac12}$, we can represent ∂E_h near $x_h=x+\nu_{\Sigma_0}(x)h(x)$ as

$$\Pi(x)t+\nu_{\Sigma_0}(x)g_h(t),\quad |t-t_0(x)|\le 2R_x,$$

where g_h is of class $C^{2,\alpha}$ in the ball $B_{4R_x}(t_0(x))$. We call G_x the operator defined by

$$g_h=G_x(h). \tag{8.1}$$

Let

$$\eta_x(t,t_3)=\eta\Big(\frac{|t-t_0(x)|}{R_x}\Big)\eta\Big(\frac{100|t_3|}{\varepsilon^{\frac12}|x|}\Big), \tag{8.2}$$

where $\eta\in C^\infty(\mathbb{R})$ is such that $\eta(s)=1$ for $s\le 1$ and $\eta(s)=0$ for $s\ge 2$. We also require $\eta'\le 0$.

Let us write

$$H^s_{\partial E_h}(x_h)=H_i(h)(x)+H_o(h)(x),$$

where

$$H_i(h)(x_h)=2\int_{\mathbb{R}^3}\eta_x(y-x_h)\frac{\chi_{E_h}(y)-\chi_{L_h(x)}(y)}{|x_h-y|^{3+s}}\,dy,$$
$$H_o(h)(x_h)=2\int_{\mathbb{R}^3}(1-\eta_x(y-x_h))\frac{\chi_{E_h}(y)-\chi_{L_h(x)}(y)}{|x_h-y|^{3+s}}\,dy.$$

Let us explain the choice of cut-off function (8.2). For this, let us write

$$D_{R_x}(x)=\{\Pi(x)t+x : t\in\mathbb{R}^2,\ |t-t_0(x)|<R_x\},$$

which is a 2-dimensional disk on the tangent plane to Σ_0 at x, centered at x, and of radius $R_x=\delta|x|$. Let us call

$$C(x)=\{\Pi(x)t+t_3\nu_{\Sigma_0}(x)+x : t\in\mathbb{R}^2,\ |t-t_0(x)|<R_x,\ |t_3|<\frac{\varepsilon^{\frac12}|x|}{100}\},$$

the cylinder with base the disk D_{R_x} and height $\varepsilon^{\frac12}|x|/100$, and

$$\tilde C(x)=\{\Pi(x)t+t_3\nu_{\Sigma_0}(x)+x : t\in\mathbb{R}^2,\ |t-t_0(x)|<2R_x,\ |t_3|<\frac{\varepsilon^{\frac12}|x|}{50}\},$$

which is a similar cylinder with twice the radius and height. The cut-off function (8.2) is zero outside the $\tilde C(x)$, while it is one on $C(x)$. Since we assume $\|h\|_* \le \sigma_0 \varepsilon^{\frac{1}{2}}$, we have $\|Dg_h\|_{L^\infty} = O(\varepsilon^{\frac{1}{2}})$ and then the set Σ_h separates from Σ_0 in the $\nu_{\Sigma_0}(x)$ direction an amount bounded by $O(\varepsilon^{\frac{1}{2}} 2R_x) = O(\delta\varepsilon^{\frac{1}{2}}|x|)$ over the disk $D_{2R_x}(x)$. By choosing $\delta << 100$ we achieve that the parts of Σ_h and the plane ∂L_h inside $\tilde{C}(x)$ are, in fact, contained in a cylinder with base $D_{2R_x}(x)$ but height $O(\delta\varepsilon^{\frac{1}{2}}|x|)$, which is much small than the height of $C(x)$.

We expand H_i, H_0

$$H_i(h)(x_h) = H_i(0)(x) + H_i'(0)(h)(x) + N_i(h)(x),$$
$$H_o(h)(x_h) = H_o(0)(x) + H_o'(0)(h)(x) + N_o(h)(x).$$

Estimate (2.15) will follow from similar estimates of $N_o(h)$ and $N_i(h)$, which we state in the next lemmas.

Lemma 8.1. *There is C independent of $\varepsilon > 0$ small such that for $\|h_i\|_* \le \sigma_0 \varepsilon^{\frac{1}{2}}$, $i = 1, 2$ we have*

$$\|N_i(h_1) - N_i(h_2)\|_{1-\varepsilon,\alpha+\varepsilon} \le \frac{C}{\varepsilon}(\|h_1\|_* + \|h_2\|_*)\|h_1 - h_2\|_*.$$

Lemma 8.2. *There is C independent of $\varepsilon > 0$ small such that for $\|h_i\|_* \le \sigma_0 \varepsilon^{\frac{1}{2}}$, $i = 1, 2$ we have*

$$\|N_o(h_1) - N_o(h_2)\|_{1-\varepsilon,\alpha+\varepsilon} \le \frac{C}{\varepsilon^{\frac{3}{2}}}(\|h_1\|_* + \|h_2\|_*)\|h_1 - h_2\|_*.$$

For the integral involved in H_i we can write

$$\begin{aligned} H_i(h)(x_h) = 2\int_{B_{2R_x}(0)} \frac{\eta(\frac{|t|}{R_x})}{|t|^{3-\varepsilon}} \Big(& \psi(\frac{\nabla g_h(t_0(x))t}{|t|}) \\ & - \psi(\frac{g_h(t+t_0(x)) - g_h(t_0(x))}{|t|})\Big) dt, \end{aligned}$$

where

$$\psi(s) = \int_0^s \frac{d\tau}{(1+\tau^2)^{\frac{4-\varepsilon}{2}}}.$$

For a given $C^{2,\alpha}$ function g defined on $B_{2R_x}(t_0(x))$ let

$$\begin{aligned} \tilde{H}_x(g) = 2\int_{B_{2R_x}(0)} \frac{\eta(\frac{|t|}{R_x})}{|t|^{3-\varepsilon}} \Big(& \psi(\frac{\nabla g(t_0(x))t}{|t|}) \\ & - \psi(\frac{g(t+t_0(x)) - g(t_0(x))}{|t|})\Big) dt, \end{aligned}$$

so that

$$H_i(h) = \tilde{H}_x(G_x(h)),$$

where G_x is the operator defined in (8.1).

For the expansion of $\tilde{H}_X$ it will be convenient to rewrite it as

$$\tilde{H}_X(g) = 2\int_0^1 \int_{\mathbb{R}^2} \frac{\eta(\frac{|z|}{R_X})}{|z|^{3-\varepsilon}} \psi'\left(A_t(g)\right) B(g)\, dzdt,$$

where

$$\begin{aligned} A_t(g)(X,z) &= t\frac{g(z+t_0(X)) - g(t_0(X))}{|z|} + (1-t)\frac{\nabla g(t_0(X))z}{|z|}, \\ B(g)(X,z) &= \frac{g(z+t_0(X)) - g(t_0(X)) - \nabla g(t_0(X))z}{|z|}. \end{aligned}$$

Note that

$$\begin{aligned} DH_i(h)[h_1] &= D\tilde{H}_X(G_X(h))[DG_X(h)[h_1]], \\ D^2H_i(h)[h_1,h_2] &= D^2\tilde{H}_X(G_X(h))[DG_X(h)[h_1], DG_X(h)[h_2]] \\ &\quad + D\tilde{H}_X(G_X(h))[D^2G_X(h)[h_1,h_2]], \end{aligned}$$

and

$$\begin{aligned} D\tilde{H}_X(g)[g_1] &= \int_{\mathbb{R}^2} \frac{\eta(\frac{|z|}{R_X})}{|z|^{3-\varepsilon}} \big[\psi''(A_t(g)(X,z))A_t(g_1)(X,z)B(g)(X,z) \\ &\qquad + \psi'(A_t(g)(X,z))B(g_1)(X,z)\big]\, dz, \\ D^2\tilde{H}_X(g)[g_1,g_2] & \\ = \int_0^1\int_{\mathbb{R}^2} &\frac{\eta(\frac{|z|}{R_X})}{|z|^{3-\varepsilon}} \big[\psi'''(A_t(g)(X,z))B(g)(X,z)A_t(g_1)(X,z)A_t(g_2)(X,z) \\ &\qquad + \psi''(A_t(g)(X,z))A_t(g_1)(X,z)B(g_2)(X,z) \\ &\qquad + \psi''(A_t(g)(X,z))A_t(g_2)(X,z)B(g_1)(X,z)\big]\, dzdt. \end{aligned}$$

For later computations we will need the following properties of DG_X, D^2G_X.

Lemma 8.3. *Let $\|h\|_*, \|h_1\|_*, \|h_2\|_* \le \sigma_0\varepsilon^{\frac{1}{2}}$, $X \in \Sigma_0$ and*

$$g = G_X(h), \quad g_i = DG_X(h)[h_i] \quad i=1,2, \quad \hat{g} = D^2G_X(h)[h_1,h_2].$$

Then

$$\|G_X(h)\|_b \le C,$$

where

$$\begin{aligned} \|g\|_b &= |X|^{-1}\|g\|_{L^\infty(B_X)} + \|\nabla g\|_{L^\infty(B_X)} \\ &\quad + |X|\|D^2 g\|_{L^\infty(B_X)} + |X|^{1+\alpha}[D^2 g]_{\alpha,B_X}, \end{aligned}$$

and $B_X = B_{2R_X}(t_0(X))$. *Also, for* $z \in B_X$*:*

$$|A_t(g)(X,z)| \le C\|h\|_*, \tag{8.3}$$

$$|B(g)(X,z)| \le C\frac{\|h\|_*}{|X|}|z|, \tag{8.4}$$

$$|A_t(g_i)(X,z)| \le C\|h_i\|_*, \tag{8.5}$$

$$|B(g_i)(X,z)| \le C\frac{\|h_i\|_*}{|X|}|z|. \tag{8.6}$$

These estimates follow, after some computation, from an application of the implicit function theorem.

Lemma 8.4. *Let* h*,* h_1*,* h_2 *be defined on* Σ_0 *with* $\|h\|_*, \|h_i\|_* \le \sigma_0 \varepsilon^{\frac{1}{2}}$*. Let* $X \in \Sigma_0$ *and*

$$g = G_X(h), \quad g_i = DG_X(h)[h_i] \quad i = 1,2, \quad \hat{g} = D^2 G_X(h)[h_1, h_2].$$

Then

$$\varepsilon |D\tilde{H}_X(g)[\hat{g}](X)| \le \frac{C}{|X|^{1-\varepsilon}}\|h_1\|_*\|h_2\|_*,$$
$$\varepsilon \left| D^2 \tilde{H}(g)[g_1, g_2](X) \right| \le \frac{C}{|X|^{1-\varepsilon}}\|h_1\|_*\|h_2\|_*.$$

Proof. Let us start with the first term in $D\tilde{H}_X(g)[g_1]$. Using (8.3), (8.5)

$$\begin{aligned}
&\left| \int_{\mathbb{R}^2} \frac{\eta(\frac{|z|}{R_X})}{|z|^{3-\varepsilon}} \psi''(A_t(g))A_t(g_1)B(g)\, dz \right| \\
&\le \|\psi''\|_{L^\infty}\|A_t(g_1)\|_{L^\infty} \int_{B_{2R_X}(0)} \frac{1}{|z|^{3-\varepsilon}}|B(g)|\, dz \\
&\le C\|h_1\|_* \int_{B_{2R_X}(0)} \frac{1}{|z|^{3-\varepsilon}}|B(g)|\, dz.
\end{aligned}$$

Then by (8.4)

$$\begin{aligned}
\int_{B_{2R_X}(0)} \frac{1}{|z|^{3-\varepsilon}}|B(g)|\, dz &\le \frac{\|h\|_*}{|X|}\int_{B_{2R_X}(0)} \frac{1}{|z|^{2-\varepsilon}}\, dz \\
&\le \frac{C}{|X|}\|h\|_* \frac{R_X^\varepsilon}{\varepsilon} \le \frac{C}{\varepsilon|X|^{1-\varepsilon}}\|h\|_*.
\end{aligned}$$

Therefore,

$$\left| \int_{\mathbb{R}^2} \frac{\eta(\frac{|z|}{R_X})}{|z|^{3-\varepsilon}} \psi''(A_t(g))A_t(g_1)B(g)\, dz \right| \le \frac{C}{\varepsilon|X|^{1-\varepsilon}}\|h_1\|_*.$$

For the second term observe that

$$\begin{aligned}\left|\int_{\mathbb{R}^2} \frac{\eta(\frac{|z|}{R_X})}{|z|^{3-\varepsilon}}\psi'(A_t(g))B(g_1)\,dz\right| &\leq C\int_{B_{2R_X}(0)} |A_t(g)B(g_1)|\,dz \\ &\leq \frac{C}{\varepsilon|X|^{1-\varepsilon}}\|g_1\|_b,\end{aligned}$$

which is obtained using (8.3) and (8.6).

For the first term in $D^2\tilde{H}_X(g)[g_1,g_2]$, we have, using (8.4) and (8.5),

$$\begin{aligned}&\left|\int_{\mathbb{R}^2} \frac{\eta(\frac{|z|}{R_X})}{|z|^{3-\varepsilon}}\psi'''(A_t(g))A_t(g_1)A_t(g_2)B(g)\,dz\right| \\ &\leq \|\psi'''\|_{L^\infty}\|A_t(g_1)\|_{L^\infty}\|A_t(g_2)\|_{L^\infty}\int_{B_{2R_X}(0)} \frac{1}{|z|^{3-\varepsilon}}|B(g)|\,dz \\ &\leq \frac{C}{\varepsilon|X|^{1-\varepsilon}}\|h_1\|_*\|h_2\|_*.\end{aligned}$$

Similarly, for the second and third terms

$$\begin{aligned}&\left|\int_{\mathbb{R}^2} \frac{\eta(\frac{|z|}{R_X})}{|z|^{3-\varepsilon}}\psi''(A_t(g))A_t(g_1)B(g_2)\,dz\right| \\ &\leq \|\psi''\|_{L^\infty}\|A_t(g_1)\|_{L^\infty}\int_{B_{2R_X}(0)} \frac{1}{|z|^{3-\varepsilon}}|B(g_2)|\,dz \\ &\leq \frac{C}{\varepsilon|X|^{1-\varepsilon}}\|h_1\|_*\|h_2\|_*.\end{aligned}$$

q.e.d.

Computations of the same kind as those in the above proof allow us to estimate the Hölder part of the norm $\|\ \|_{1-\varepsilon,\alpha+\varepsilon}$. We have:

Lemma 8.5. *Let $X_1 = (x_1, F_\varepsilon(x_1))$, $X_2 = (x_2, F_\varepsilon(x_2)) \in \Sigma_0$, be such that $|X_1| \leq |X_2|$ and $|X_1 - X_2| \leq \frac{1}{10}|X_1|$. Let*

$$\begin{aligned}g_{X_j} &= G_{X_j}(h) \quad j = 1,2, \\ g_{i,X_j} &= DG_{X_j}(h_0)[h_i] \quad i,j = 1,2.\end{aligned}$$

Then

$$\begin{aligned}&|D^2\tilde{H}_{X_1}(g_{X_1})[g_{1,X_1}, g_{2,X_1}] - D^2\tilde{H}_{X_2}(g_{X_2})[g_{1,X_2}, g_{2,X_2}]| \\ &\quad\leq \frac{C}{\varepsilon}(\|h_1\|_* + \|h_2\|_*)\|h_1 - h_2\|_* \frac{|X_1 - X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}}.\end{aligned}$$

Proof of Lemma 8.1. Write

$$\begin{aligned}N_i(h_1) - N_i(h_2) &= H_i(h_1) - H_i(h_2) - DH_i(0)[h_1 - h_2] \\ &= \int_0^1\int_0^1 D^2H_i(s(th_1 + (1-t)h_2)) \\ &\qquad \times [h_1 - h_2, th_1 + (1-t)h_2]\,dsdt.\end{aligned}$$

Using Lemma 8.4 we get

$$|N_i(h_1)(X) - N_i(h_2)(X)| \le \frac{C}{\varepsilon |X|^{1-\varepsilon}} \|h_1 - h_2\|_* (\|h_1\|_* + \|h_2\|_*).$$

By Lemma 8.5, if $|X_1 - X_2| \le \frac{1}{10}\min(|X_1|, |X_2|)$,

$$\begin{aligned} &|N_i(h_1)(X_1) - N_i(h_2)(X_1) - (N_i(h_1)(X_2) - N_i(h_2)(X_2))| \\ &\le \frac{C}{\varepsilon} \frac{|X_1 - X_2|^{\alpha+\varepsilon}}{\min(|X_1|, |X_2|)^{1+\alpha}} \|h_1 - h_2\|_* (\|h_1\|_* + \|h_2\|_*). \end{aligned}$$ q.e.d.

Proof of Lemma 8.2. By a direct and long computation we obtain

$$\varepsilon |D^2 H_o(h)[h_1, h_2](x)| \le \frac{C}{\varepsilon^{\frac{1}{2}} |x|^{1-\varepsilon}} \|h_1\|_* \|h_2\|_*,$$

for $x \in \Sigma_0$, and if $x_1, x_2 \in \Sigma_0$, $|x_1 - x_2| \le \frac{1}{10}|x_1|$, then

$$\begin{aligned} &\varepsilon |D^2 H_o(h)[h_1, h_2](x_1) - D^2 H_o(h)[h_1, h_2](x_2)| \\ &\le C \frac{|x_1 - x_2|^{\alpha+\varepsilon}}{\varepsilon^{\frac{1}{2}} |x_1|^{1+\alpha}} \|h_1\|_* \|h_2\|_*. \end{aligned}$$

Then the lemma follows as in the proof of Lemma 8.1. q.e.d.

9. Proof of Theorem 2 and multi-component fractional minimal surfaces

Proof of Theorem 2. The proof is essentially the same as for Theorem 1. This time we look for a set $E \subseteq \mathbb{R}^3$ of the form

$$E = \{(x', x_3) :\in \mathbb{R}^3 : |x_3| > f(x')\},$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is a positive radially symmetric function. We take as a first approximation

$$E_0 = \{(x', x_3) :\in \mathbb{R}^3 : |x_3| > f_\varepsilon(x')\},$$

where f_ε is the unique radial solution to

$$\Delta f_\varepsilon = \frac{\varepsilon}{f_\varepsilon}, \quad f_\varepsilon > 0, \quad \text{in } \mathbb{R}^2,$$

with $f_\varepsilon(0) = 1$. Then $f_\varepsilon(x) = f_1(\varepsilon^{\frac{1}{2}} x)$ where f_1 is the radial solution of $\Delta f = \frac{1}{f}$ with $f_1(0) = 1$. The same analysis of Section 3 applies to show that $f_1(r) = r + O(1)$ as $r \to \infty$ and one obtains the same estimates for f_ε as for F_ε. This leads to the estimate

$$\|\varepsilon H^s_{\Sigma_0}\|_{1-\varepsilon, \alpha+\varepsilon} \le C\varepsilon.$$

As before, we construct the surface Σ and the corresponding set E by perturbing the surface Σ_0 in the normal direction ν_{Σ_0} (it could also be done using vertical perturbations). That is, for a function h defined on Σ_0 (small with a suitable norm) we let

$$\Sigma_h = \{x + h(x)\nu_{\Sigma_0}(x) \ / \ x \in \Sigma_0\}.$$

As before, we are led to find h such that

$$H^s_{\Sigma_0} + 2\mathcal{J}^s_{\Sigma_0}(h) + N(h) = 0.$$

We solve for h in this equation using the contraction mapping principle, employing the same norms as in (2.11), (2.12). The solvability of the linearized problem

$$\varepsilon\mathcal{J}^s_{\Sigma_0}(h) = f \quad \text{in } \Sigma_0,$$

in weighted Hölder space and the estimates for $N(h)$ are very similar to the ones in Theorem 1. q.e.d.

We can also construct axially symmetric solutions with multiple layers. Suppose that

$$f_1 > f_2 > \ldots > f_k,$$

are radially symmetric functions on $\mathbb{R}^n$ and consider the surface Σ defined by

$$\Sigma = \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}^1 : x_{n+1} = f_i(x), \text{ for some } i\}.$$

It turns out that it is possible to choose the f_is in such a way that Σ is s-minimal for s close to 1.

Similar computations as in Section 4 yield that the f_i's should approximately satisfy the Toda-type system

$$\Delta f_i = c\varepsilon \sum_{j\neq i} \frac{(-1)^{i+j+1}}{f_i - f_j}, \quad 1, \ldots, k,$$

for some $c > 0$. Scaling out the factor $c\varepsilon$ we get the system

$$\Delta f_i = 2\sum_{j\neq i} \frac{(-1)^{i+j+1}}{f_i - f_j},$$

and look for a solution of the form

$$f_i = a_i f_0, \quad \Delta f_0 = \frac{1}{f_0}. \tag{9.1}$$

Then the a_i have to satisfy

$$a_i = 2\sum_{j\neq i} \frac{(-1)^{i+j+1}}{a_i - a_j}. \tag{9.2}$$

Note that $\sum_{i=1}^k f_i$ is harmonic and radially symmetric, so it is constant. Since $\sum f_i = f_0 \sum a_i$ is a constant we must have $\sum a_i = 0$.

A solution of the system (9.2) can be obtained by minimization of

$$E(a_1,\dots,a_k)=\frac{1}{2}\sum_{i=1}^{k}a_i^2+\sum_{i,j:i\neq j}(-1)^{i+j}\log(|a_i-a_j|),$$

subject to $\sum_{i=1}^k a_i=0$. Indeed, it is not hard to see that E attains a minimum over the set

$$\Lambda=\{(a_1,\dots,a_k)\in\mathbb{R}^k:\ a_1>a_2>\dots>a_k,\ a_j = -a_{k-j+1}\ \forall j\in\{1,\dots,k\}\}.$$

There is, however, a further restriction on a solution $a=(a_1,\dots,a_k)$ to (9.2) that we need to impose for our method to work, which is the nondegeneracy of a as a critical point of E. Indeed, the linearized operator around the approximate solution (9.1) is given by

$$\Delta\phi_i-2\sum_{j\neq i}(-1)^{i+j}\frac{\phi_i-\phi_j}{(f_i-f_j)^2}.$$

Let us write this operator acting on the vector $\Phi=(\phi_1,\dots,\phi_k)$ as

$$\Delta\Phi+\frac{1}{f_0^2}A\Phi,$$

where $A=(a_{ij})$ has entries

$$a_{ij}=\begin{cases}2\frac{(-1)^{i+j}}{(a_i-a_j)^2} & \text{if } i\neq j,\\ -2\sum_{k\neq i}\frac{(-1)^{i+k}}{(a_i-a_k)^2} & \text{if } i=j.\end{cases}$$

Note that $f_0\sim r$ as $r\to\infty$, so the linearized operator is asymptotic to

$$\Delta\Phi+\frac{1}{r^2}A\Phi,$$

as $r\to\infty$.

As done before, a natural space to find the solution Φ should involve norms allowing linear growth. We see that it is possible to find such solutions for a given right hand side of the form $\sim 1/r$ if the matrix A has no eigenvalue equal to -1, since otherwise, $\Phi(r)=rv$ with v an eigenvector of A associated to eigenvalue 1 would be in the kernel of the operator.

We note that

$$D^2_{a_i,a_k}E=\begin{cases}2(-1)^{i+k}\frac{1}{(a_i-a_k)^2} & \text{if } i\neq k,\\ 1-2\sum_{j\neq i}(-1)^{i+j}\frac{1}{(a_i-a_j)^2} & \text{if } i=k,\end{cases}$$

so that

$$D^2E=I+A.$$

At a local minimum of E, $D^2E\geq 0$ which means that eigenvalues of A are greater or equal than -1. If $(a_i,\dots,a_k)$ is a non degenerate local minimum of E then $D^2E>0$ and the eigenvalues of A are all greater than -1.

10. Existence of s-Lawson cones

Proof of Theorem 3. Let us write

$$E_\alpha = \{x = (y, z) \,:\, y \in \mathbb{R}^m,\, z \in \mathbb{R}^n,\, |z| > \alpha|y| \,\}, \tag{10.1}$$

so that $C_\alpha = \partial E_\alpha$.

Existence. We fix N, m, n with $N = m + n$, $n \le m$ and also fix $0 < s < 1$. If $m = n$ then C_1 is a minimal cone, since (1.1) is satisfied by symmetry. So we concentrate next on the case $n < m$.

Before proceeding we remark that for a cone C_α the quantity appearing in (1.1) has a fixed sign for all $p \in C_\alpha$, $p \neq 0$, since by rotation we can always assume that $p = rp_\alpha$ for some $r > 0$ where

$$p_\alpha = \frac{1}{\sqrt{1+\alpha^2}}(e_1^{(m)}, \alpha e_1^{(n)}),$$

with

$$e_1^{(m)} = (1, 0, \dots, 0) \in \mathbb{R}^m, \tag{10.2}$$

and, similarly, for $e_1^{(n)}$. Then we observe that

$$\text{p.v.} \int_{\mathbb{R}^N} \frac{\chi_{E_\alpha}(x) - \chi_{E_\alpha^c}(x)}{|x - rp_\alpha|^{N+s}}\, dx = \frac{1}{r^s} \text{p.v.} \int_{\mathbb{R}^N} \frac{\chi_{E_\alpha}(x) - \chi_{E_\alpha^c}(x)}{|x - p_\alpha|^{N+s}}\, dx.$$

Let us define

$$H(\alpha) = \text{p.v.} \int_{\mathbb{R}^N} \frac{\chi_{E_\alpha}(x) - \chi_{E_\alpha^c}(x)}{|x - p_\alpha|^{N+s}}\, dx, \tag{10.3}$$

and note that it is a continuous function of $\alpha \in (0, \infty)$.

Claim 1. *We have*

$$H(1) \le 0. \tag{10.4}$$

Indeed, write $y \in \mathbb{R}^m$ as $y = (y_1, y_2)$ with $y_1 \in \mathbb{R}^n$ and $y_2 \in \mathbb{R}^{m-n}$. Abbreviating $e_1 = e_1^{(n)} = (1, 0, \dots, 0) \in \mathbb{R}^n$ we rewrite

$$\begin{aligned} H(1) = \lim_{\delta \to 0} \int_{A_\delta} & \frac{1}{(|y_1 - \frac{1}{\sqrt{2}}e_1|^2 + |y_2|^2 + |z - \frac{1}{\sqrt{2}}e_1|^2)^{\frac{N+s}{2}}} \\ & - \lim_{\delta \to 0} \int_{B_\delta} \frac{1}{(|y_1 - \frac{1}{\sqrt{2}}e_1|^2 + |y_2|^2 + |z - \frac{1}{\sqrt{2}}e_1|^2)^{\frac{N+s}{2}}}, \end{aligned}$$

where

$$A_\delta = \{|z|^2 > |y_1|^2 + |y_2|^2,\ |y_1 - \frac{1}{\sqrt{2}}e_1|^2 + |y_2|^2 + |z - \frac{1}{\sqrt{2}}e_1|^2 > \delta^2\},$$

$$B_\delta = \{|z|^2 < |y_1|^2 + |y_2|^2,\ |y_1 - \frac{1}{\sqrt{2}}e_1|^2 + |y_2|^2 + |z - \frac{1}{\sqrt{2}}e_1|^2 > \delta^2\}.$$

But the first integral can be rewritten as

$$\int_{A_\delta} \frac{1}{(|y_1 - \frac{1}{\sqrt{2}}e_1|^2 + |y_2|^2 + |z - \frac{1}{\sqrt{2}}e_1|^2)^{\frac{N+s}{2}}}$$
$$= \int_{\tilde{A}_\delta} \frac{1}{(|y_1 - \frac{1}{\sqrt{2}}e_1|^2 + |y_2|^2 + |z - \frac{1}{\sqrt{2}}e_1|^2)^{\frac{N+s}{2}}},$$

where

$$\tilde{A}_\delta = \{|y_1|^2 > |z|^2 + |y_2|^2,\ |y_1 - \frac{1}{\sqrt{2}}e_1|^2 + |y_2|^2 + |z - \frac{1}{\sqrt{2}}e_1|^2 > \delta^2\}$$

(we just have exchanged y_1 by z and noted that the integrand is symmetric in these variables). But $\tilde{A}_\delta \subset B_\delta$ and so

$$\int_{\mathbb{R}^N\setminus B(p_1,\delta)} \frac{\chi_{E_1}(x) - \chi_{E_1^c}(x)}{|x-p_1|^{N+s}}\,dx$$
$$= -\int_{B_\delta\setminus\tilde{A}_\delta} \frac{1}{(|y_1 - \frac{1}{\sqrt{2}}e_1|^2 + |y_2|^2 + |z - \frac{1}{\sqrt{2}}e_1|^2)^{\frac{N+s}{2}}} \le 0.$$

This shows the validity of (10.4)*.*

Claim 2. *We have*

$$H(\alpha) \to +\infty \quad \text{as } \alpha \to 0. \tag{10.5}$$

Let $0 < \delta < 1/2$ be fixed and write

$$H(\alpha) = I_\alpha + J_\alpha,$$

where

$$I_\alpha = \int_{\mathbb{R}^N\setminus B(p_\alpha,\delta)} \frac{\chi_{E_\alpha}(x) - \chi_{E_\alpha^c}(x)}{|x-p_\alpha|^{N+s}}\,dx,$$
$$J_\alpha = p.v.\int_{B(p_\alpha,\delta)} \frac{\chi_{E_\alpha}(x) - \chi_{E_\alpha^c}(x)}{|x-p_\alpha|^{N+s}}\,dx.$$

With δ fixed

$$\lim_{\alpha\to 0} I_\alpha = \int_{\mathbb{R}^N\setminus B(p_\alpha,\delta)} \frac{1}{|x-p_0|^{N+s}}\,dx > 0. \tag{10.6}$$

For J_α we make a change of variables $x = \alpha\tilde{x} + p_\alpha$ and obtain

$$J_\alpha = p.v.\int_{B(p_\alpha,\delta)} \frac{\chi_{E_\alpha}(x) - \chi_{E_\alpha^c}(x)}{|x-p_\alpha|^{N+s}}\,dx \tag{10.7}$$
$$= \frac{1}{\alpha^s} p.v.\int_{B(0,\delta/\alpha)} \frac{\chi_{F_\alpha}(\tilde{x}) - \chi_{F_\alpha^c}(\tilde{x})}{|\tilde{x}|^{N+s}} d\tilde{x},$$

where $F_\alpha = \frac{1}{\alpha}(E_\alpha - p_\alpha)$. But

$$p.v.\int_{B(0,\delta/\alpha)} \frac{\chi_{F_\alpha}(\tilde{x}) - \chi_{F_\alpha^c}(\tilde{x})}{|\tilde{x}|^{N+s}} d\tilde{x} \to p.v\int_{\mathbb{R}^N} \frac{\chi_{F_0}(x) - \chi_{F_0^c}(x)}{|x|^{N+s}} dx$$

as $\alpha \to 0$ *where* $F_0 = \{x = (y, z) : y \in \mathbb{R}^m, z \in \mathbb{R}^n,\ |z + e_1^{(n)}| > 1\}$. *But writing* $z = (z_1, \ldots, z_n)$ *we see that*

$$
\begin{aligned}
p.v \int_{\mathbb{R}^N} \frac{\chi_{F_0}(x) - \chi_{F_0^c}(x)}{|x|^{N+s}} dx &\geq p.v \int_{\mathbb{R}^N} \frac{\chi_{[z_1>0 \ or \ z_1<-2]} - \chi_{[-2<z_1<0]}}{|x|^{N+s}} dx \\
&\geq \int_{\mathbb{R}^N} \frac{\chi_{[\ |z_1|>2\ ]}}{|x|^{N+s}} dx,
\end{aligned}
$$

and this number is positive. This and (10.7) *show that* $J_\alpha \to +\infty$ *as* $\alpha \to 0$ *and combined with* (10.6) *we obtain the desired conclusion.*

By (10.4), (10.5) and continuity we obtain the existence of $\alpha \in (0, 1]$ such that $H(\alpha) = 0$.

Uniqueness. Consider 2 cones C_{α_1}, C_{α_2} with $\alpha_1 > \alpha_2 > 0$, associated to solid cones E_{α_1} and E_{α_2}. We claim that there is a rotation R so that $R(E_{\alpha_1}) \subset E_{\alpha_2}$ (strictly) and that

$$
H(\alpha_1) = \text{p.v.} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_{R(E_{\alpha_1})}(x) - \chi_{R(E_{\alpha_1})^c}(x)}{|x - p_{\alpha_2}|^{N+s}}\, dx.
$$

Note that the denominator in the integrand is the same that appears in (10.3) for α_2 and then

$$
\begin{aligned}
H(\alpha_1) &= \text{p.v.} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_{R(E_{\alpha_1})}(x) - \chi_{R(E_{\alpha_1})^c}(x)}{|x - p_{\alpha_2}|^{N+s}}\, dx \\
&< \text{p.v.} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_{E_{\alpha_2}}(x) - \chi_{E_{\alpha_2}^c}(x)}{|x - p_{\alpha_2}|^{N+s}}\, dx = H(\alpha_2).
\end{aligned}
\tag{10.8}
$$

This shows that $H(\alpha)$ is decreasing in α and, hence, the uniqueness. To construct the rotation let us write as before $x = (y, z) \in \mathbb{R}^N$, with $y \in \mathbb{R}^m$, $z \in \mathbb{R}^n$, and $y = (y_1, y_2)$ with $y_1 \in \mathbb{R}^n$, $y_2 \in \mathbb{R}^{m-n}$ (we assume always $n \leq m$). Let us write the vector (y_1, z) in spherical coordinates of $\mathbb{R}^{2n}$ as follows

$$
y_1 = \rho \begin{bmatrix} \cos(\varphi_1) \\ \sin(\varphi_1)\cos(\varphi_2) \\ \sin(\varphi_1)\sin(\varphi_2)\cos(\varphi_3) \\ \vdots \\ \sin(\varphi_1)\sin(\varphi_2)\sin(\varphi_3)\ldots\sin(\varphi_{n-1})\cos(\varphi_n) \end{bmatrix},
$$

$$
z = \rho \begin{bmatrix} \sin(\varphi_1)\sin(\varphi_2)\sin(\varphi_3)\ldots\sin(\varphi_n)\cos(\varphi_{n+1}) \\ \vdots \\ \sin(\varphi_1)\sin(\varphi_2)\sin(\varphi_3)\ldots\sin(\varphi_{2n-2})\cos(\varphi_{2n-1}) \\ \sin(\varphi_1)\sin(\varphi_2)\sin(\varphi_3)\ldots\sin(\varphi_{2n-2})\sin(\varphi_{2n-1}) \end{bmatrix},
$$

where $\rho > 0$, $\varphi_{2n-1} \in [0, 2\pi)$, $\varphi_j \in [0, \pi]$ for $j = 1, \ldots, 2n-2$. Then

$$
|z|^2 = \rho^2 \sin(\varphi_1)^2 \sin(\varphi_2)^2 \ldots \sin(\varphi_n)^2, \quad |y_1|^2 + |z|^2 = \rho^2.
$$

The equation for the solid cone E_{α_i}, namely $|z| > \alpha_i|y|$, can be rewritten as

$$\rho^2 \sin(\varphi_1)^2 \sin(\varphi_2)^2 \dots \sin(\varphi_n)^2 > \alpha_i^2(|y_1|^2 + |y_2|^2).$$

Adding $\alpha_i^2|z|^2$ to both sides this is equivalent to

$$\sin(\varphi_1)^2 \sin(\varphi_2)^2 \dots \sin(\varphi_n)^2 > \sin(\beta_i)^2(1 + \frac{|y_2|^2}{\rho^2}),$$

where $\beta_i = \arctan(\alpha_i)$. We let $\theta = \beta_1 - \beta_2 \in (0, \pi/2)$, and define the rotated cone $R_\theta(E_{\alpha_1})$ by the equation

$$\sin(\varphi_1 + \theta)^2 \sin(\varphi_2)^2 \dots \sin(\varphi_n)^2 > \sin(\beta_1)^2(1 + \frac{|y_2|^2}{\rho^2}).$$

We want to show that $R_\theta(E_{\alpha_1}) \subset E_{\alpha_2}$. To do so, it suffices to prove that for any given $t \geq 1$, if φ satisfies the inequality $|\sin(\varphi + \theta)| > \sin(\beta_1)t$ then it also satisfies $|\sin(\varphi)| > \sin(\beta_2)t$. This in turn can be proved from the inequality

$$\arccos(\sin(\beta_1)t) + \theta < \arccos(\sin(\beta_2)t),$$

for $1 < t \leq \frac{1}{\sin(\beta_1)}$. For $t = 1$ we have equality by definition of θ. The inequality for $1 < t \leq \frac{1}{\sin(\beta_1)}$ can be checked by computing a derivative with respect to t. The strict inequality in (10.8) is because $R(E_{\alpha_1}) \subset E_{\alpha_2}$ strictly. q.e.d.

11. Stability and instability

We consider the nonlocal minimal cone $C_m^n(s) = \partial E_\alpha$ where E_α is defined in (10.1) and α is the one of Theorem 3. For $0 \leq s < 1$ we obtain a characterization of their stability in terms of constants that depend on m, n and s. For the case $s = 0$ we consider the limiting cone with parameter α_0 given in Proposition 11.2 below. Note that in the case $s = 0$ the limiting Jacobi operator $\mathcal{J}^0_{C_{\alpha_0}}$ is well defined for smooth functions with compact support.

For brevity, in this section, we write $\Sigma = C_m^n(s)$.

11.1. Characterization of stability. Recall that

$$\mathcal{J}^s_\Sigma[\phi](x) = \text{p.v.} \int_\Sigma \frac{\phi(y) - \phi(x)}{|y - x|^{N+s}} dy + \phi(x) \int_\Sigma \frac{1 - \langle \nu(x), \nu(y) \rangle}{|x - y|^{N+s}} dy,$$

for $\phi \in C_0^\infty(\Sigma \setminus \{0\})$. Let us rewrite this operator in the form

$$\mathcal{J}^s_\Sigma[\phi](x) = \text{p.v.} \int_\Sigma \frac{\phi(y) - \phi(x)}{|x - y|^{N+s}} dy + \frac{A_0(m, n, s)^2}{|x|^{1+s}} \phi(x),$$

where

$$A_0(m, n, s)^2 = \int_\Sigma \frac{\langle \nu(\hat{p}) - \nu(x), \nu(\hat{p}) \rangle}{|\hat{p} - x|^{N+s}} dx \geq 0,$$

and this integral is evaluated at any $\hat p \in \Sigma$ with $|\hat p| = 1$. We can think of $\mathcal{J}^s_\Sigma$ as analogous to the fractional Hardy operator $-(-\Delta)^{\frac{1+s}{2}}\phi + \frac{c}{|x|^{1+s}}\phi$ for which positivity is related to a fractional Hardy inequality with best constant, see Herbst [**16**]. This suggests that the positivity of $\mathcal{J}_\Sigma$ is related to the existence of β in an appropriate range such that $\mathcal{J}^s_\Sigma[|x|^{-\beta}] \le 0$, and it turns out that the best choice of β is $\beta = \frac{N-2-s}{2}$. This motivates the definition

$$H(m,n,s) = \text{p.v.} \int_\Sigma \frac{1 - |y|^{-\frac{N-2-s}{2}}}{|\hat p - y|^{N+s}} dy,$$

where $\hat p \in \Sigma$ is any point with $|\hat p| = 1$.

We have then the following Hardy inequality with best constant:

Proposition 11.1. *For any $\phi \in C_0^\infty(\Sigma \setminus \{0\})$ we have*

$$H(m,n,s) \int_\Sigma \frac{\phi(x)^2}{|x|^{1+s}} dx \le \frac{1}{2} \int_\Sigma \int_\Sigma \frac{(\phi(x) - \phi(y))^2}{|x-y|^{N+s}} dxdy, \tag{11.1}$$

and $H(m,n,s)$ is the best possible constant in this inequality.

As a result we have:

Corollary 11.1. *The cone $C^n_m(s)$ is stable if and only if $H(m,n,s) \ge A_0(m,n,s)^2$.*

Other related fractional Hardy inequalities have appeared in the literature, see, for instance, [**4**, **13**].

Proof of Proposition 11.1. Let us write $H = H(m,n,s)$ for simplicity. To prove the validity of (11.1) let $w(x) = |x|^{-\beta}$ with $\beta = \frac{N-2-s}{2}$ so that from the definition of H and homogeneity we have

$$\text{p.v.} \int_\Sigma \frac{w(y) - w(x)}{|y-x|^{N+s}} dy + \frac{H}{|x|^{1+s}} w(x) = 0 \quad \text{for all } x \in \Sigma \setminus \{0\}.$$

Now the same argument as in the proof of corollary B.1 shows that

$$\begin{aligned} &\frac{1}{2} \int_\Sigma \int_\Sigma \frac{(\phi(x) - \phi(y))^2}{|x-y|^{N+s}} dxdy \\ &\quad = \int_\Sigma \frac{H}{|x|^{1+s}} \phi(x)^2 dx + \frac{1}{2} \int_\Sigma \int_\Sigma \frac{(\psi(x) - \psi(y))^2 w(x) w(y)}{|x-y|^{N+s}} dxdy, \end{aligned} \tag{11.2}$$

for all $\phi \in C_0^\infty(\Sigma \setminus \{0\})$ with $\psi = \frac{\phi}{w} \in C_0^\infty(\Sigma \setminus \{0\})$.

Now let us show that H is the best possible constant in (11.1). Assume that

$$\tilde H \int_\Sigma \frac{\phi(x)^2}{|x|^{1+s}} dx \le \frac{1}{2} \int_\Sigma \int_\Sigma \frac{(\phi(x) - \phi(y))^2}{|x-y|^{N+s}} dxdy,$$

for all $\phi \in C_0^\infty(\Sigma \setminus \{0\})$. Using (11.2) and letting $\phi = w\psi$ with $\psi \in\in C_0^\infty(\Sigma \setminus \{0\})$ we then have

$$\tilde{H} \int_\Sigma \frac{w(x)^2\psi(x)^2}{|x|^{1+s}} dx$$
$$\leq H \int_\Sigma \frac{w(x)^2\psi(x)^2}{|x|^{1+s}} dx + \frac{1}{2} \int_\Sigma \int_\Sigma \frac{(\psi(x)-\psi(y))^2 w(x)w(y)}{|x-y|^{N+s}} dxdy.$$

For $R > 3$ let $\psi_R : \Sigma \to [0,1]$ be a radial function such that $\psi_R(x) = 0$ for $|x| \leq 1$, $\psi_R(x) = 1$ for $2 \leq |x| \leq 2R$, $\psi_R(x) = 0$ for $|x| \geq 3R$. We also require $|\nabla \psi_R(x)| \leq C$ for $|x| \leq 3$, $|\nabla \psi_R(x)| \leq C/R$ for $2R \leq |x| \leq 3R$. From a direct computation we find the estimates

$$a_0 \log(R) - C \leq \int_\Sigma \frac{w(x)^2\psi_R(x)^2}{|x|^{1+s}} dx \leq a_0 \log(R) + C,$$

where $a_0 > 0$, $C > 0$ are independent of R, while

$$\left| \int_\Sigma \int_\Sigma \frac{(\psi_R(x)-\psi_R(y))^2 w(x)w(y)}{|x-y|^{N+s}} dxdy \right| \leq C.$$

Letting then $R \to \infty$ we deduce that $\tilde{H} \leq H$. q.e.d.

11.2. Minimal cones for $s = 0$. Here we derive the limiting value $\alpha_0 = \lim_{s\to 0} \alpha_s$ where α_s is such that C_{α_s} is an s-minimal cone.

Proposition 11.2. *Assume that $n \leq m$ in (10.1), $N = m+n$. The number α_0 is the unique solution to*

$$\int_\alpha^\infty \frac{t^{n-1}}{(1+t^2)^{\frac{N}{2}}} dt - \int_0^\alpha \frac{t^{n-1}}{(1+t^2)^{\frac{N}{2}}} dt = 0.$$

Proof. We write $x = (y,z) \in \mathbb{R}^N$ with $y \in \mathbb{R}^m$, $z \in \mathbb{R}^n$. Let us assume in the rest of the proof that $n \geq 2$. The case $n = 1$ is similar. We evaluate the integral in (1.1) for the point $p = (e_1^{(m)}, \alpha e_1^{(n)})$ using spherical coordinates for $y = r\omega_1$ and $z = \rho\omega_2$ where $r, \rho > 0$ and

(11.3)
$$\omega_1 = \omega_1(\theta_1, \ldots, \theta_{m-1}) = \begin{bmatrix} \cos(\theta_1) \\ \sin(\theta_1)\cos(\theta_2) \\ \vdots \\ \sin(\theta_1)\sin(\theta_2)\ldots\sin(\theta_{m-2})\cos(\theta_{m-1}) \\ \sin(\theta_1)\sin(\theta_2)\ldots\sin(\theta_{m-2})\sin(\theta_{m-1}) \end{bmatrix},$$

and $\omega_2(\varphi_1, \ldots, \varphi_{n-1})$ defined similarly, where $\theta_j \in [0,\pi]$ for $j = 1, \ldots, m-2$, $\theta_{m-1} \in [0, 2\pi]$, $\varphi_j \in [0,\pi]$ for $j = 1, \ldots, n-2$, $\varphi_{n-1} \in [0, 2\pi]$. Then

$$|(y,z) - (e_1^{(m)}, \alpha e_1^{(n)})|^2 = r^2 + 1 - 2r\cos(\theta_1) + \rho^2 + \alpha^2 - 2\rho\alpha\cos(\varphi_1).$$

Assuming that $\alpha = \alpha_s > 0$ is such that C_{α_s} is an s-minimal cone, (1.1) yields the following equation for α

$$\text{p.v.} \int_0^\infty r^{m-1}(A_{\alpha,s}(r) - B_{\alpha,s}(r))dr = 0, \tag{11.4}$$

where

$$\begin{aligned} &A_{\alpha,s}(r) \\ &= \int_{r\alpha}^\infty \int_0^\pi \int_0^\pi \frac{\rho^{n-1}\sin(\theta_1)^{m-2}\sin(\varphi_1)^{n-2}}{(r^2+1-2r\cos(\theta_1)+\rho^2+\alpha^2-2\rho\alpha\cos(\varphi_1))^{\frac{N+s}{2}}} d\theta_1 d\varphi_1 d\rho, \\ &B_{\alpha,s}(r) \\ &= \int_0^{r\alpha} \int_0^\pi \int_0^\pi \frac{\rho^{n-1}\sin(\theta_1)^{m-2}\sin(\varphi_1)^{n-2}}{(r^2+1-2r\cos(\theta_1)+\rho^2+\alpha^2-2\rho\alpha\cos(\varphi_1))^{\frac{N+s}{2}}} d\theta_1 d\varphi_1 d\rho, \end{aligned}$$

which are well defined for $r \neq 1$. We get

$$A_{\alpha,s}(r) = c_{m,n} r^{-m-s} \int_\alpha^\infty \frac{t^{n-1}}{(1+t^2)^{\frac{N+s}{2}}} dt + O(r^{-m-s-1})$$

as $r \to \infty$ and this is uniform in s for $s > 0$ small. Here $c_{m,n} > 0$ is some constant. Similarly,

$$B_{\alpha,s}(r) = c_{m,n} r^{-m-s} \int_0^\alpha \frac{t^{n-1}}{(1+t^2)^{\frac{N+s}{2}}} dt + O(r^{-m-s-1}).$$

Then (11.4) takes the form

$$\begin{aligned} 0 &= \int_0^2 \dots dr + \int_2^\infty \dots dr \\ &= O(1) + C_s(\alpha) \int_2^\infty r^{-1-s} dr = O(1) + \frac{2^{-s}}{s} C_s(\alpha), \end{aligned}$$

where

$$C_s(\alpha) = \int_\alpha^\infty \frac{t^{n-1}}{(1+t^2)^{\frac{N+s}{2}}} dt - \int_0^\alpha \frac{t^{n-1}}{(1+t^2)^{\frac{N+s}{2}}} dt,$$

and $O(1)$ is uniform as $s \to 0$, because $0 < \alpha_s \leq 1$ by Theorem 3, and the only singularity in (11.4) occurs at $r = 1$. This implies that $\alpha_0 = \lim_{s\to 0} \alpha_s$ has to satisfy $C_0(\alpha_0) = 0$. q.e.d.

11.3. Proof of Theorem 4. In what follows we will obtain expressions for $H(m,n,s)$ and $A_0(m,n,s)^2$ for $m \geq 2$, $n \geq 1$, $0 \leq s < 1$. We always assume $m \geq n$. For the sake of generality, we will compute

$$C(m,n,s,\beta) = \text{p.v.} \int_\Sigma \frac{1-|x|^{-\beta}}{|\hat{p}-x|^{N+s}} dx,$$

where $\hat{p} \in \Sigma$, $|\hat{p}| = 1$, and $\beta \in (0, N-2-s)$, so that $H(m,n,s) = C(m,n,s,\frac{N-2-s}{2})$.

Let $x = (y, z) \in \Sigma$, with $y \in \mathbb{R}^m$, $z \in \mathbb{R}^n$. For simplicity in the next formulas we take $p = (e_1^{(m)}, \alpha e_2^{(n)})$ (see the notation in (10.2)), and $h(y, z) = |y|^{-\beta}$, so that

$$C(m, n, s, \beta) = (1 + \alpha^2)^{\frac{1+s}{2}} \text{p.v.} \int_\Sigma \frac{h(p) - h(x)}{|p - x|^{N+s}}\, dx.$$

Computation of $C(m, 1, s, \beta)$. Write $y = r\omega_1$, $z = \pm\alpha r$, with $r > 0$, $\omega_1 \in S^{m-1}$. Let us use the notation $\Sigma_\alpha^+ = \Sigma \cap [z > 0]$, $\Sigma_\alpha^- = \Sigma \cap [z < 0]$. Using polar coordinates $(\theta_1, \ldots, \theta_{m-1})$ for ω_1 as in (11.3) we have

$$|x - p|^2 = |r\theta_1 - e_1^{(m)}|^2 + \alpha^2 |r\theta_1 - e_1^{(m)}|^2 = r^2 + 1 - 2r\cos(\theta_1) + \alpha^2 (r - 1)^2,$$

for $x \in \Sigma_\alpha^+$ and

$$|x - p|^2 = |r\theta_1 - e_1^{(m)}|^2 + \alpha^2 |r\theta_1 - e_1^{(m)}|^2 = r^2 + 1 - 2r\cos(\theta_1) + \alpha^2 (r + 1)^2,$$

for $x \in \Sigma_\alpha^-$. Hence, with $h(y, z) = |y|^{-\beta}$

$$\text{p.v.} \int_\Sigma \frac{h(p) - h(x)}{|x - p|^{N+s}} dx = \sqrt{1 + \alpha^2} A_{m-2} \text{p.v.} \int_0^\infty (1 - r^{-\beta})(I_+(r) + I_-(r)) r^{N-2} dr, \tag{11.5}$$

where

$$I_+(r) = \int_0^\pi \frac{\sin(\theta_1)^{m-2}}{(r^2 + 1 - 2r\cos(\theta_1) + \alpha^2 (r - 1)^2)^{\frac{N+s}{2}}} d\theta_1,$$
$$I_-(r) = \frac{\sin(\theta_1)^{m-2}}{(r^2 + 1 - 2r\cos(\theta_1) + \alpha^2 (r + 1)^2)^{\frac{N+s}{2}}} d\theta_1,$$

and A_k denotes the area of the sphere $S^k \subseteq \mathbb{R}^{k+1}$. From (11.5) we obtain

$$C(m, 1, s, \beta) = (1 + \alpha^2)^{\frac{3+s}{2}} A_{m-2} \int_0^1 (r^{N-2} - r^{N-2-\beta} + r^s - r^{\beta+s})(I_+(r) + I_-(r)) dr. \tag{11.6}$$

Computation of $A_0(m, 1, s)^2$. A similar computation shows that

$$A_0(m, 1, s)^2 = (1 + \alpha^2)^{\frac{3+s}{2}} A_{m-2} \int_0^1 (r^{N-2} + r^s)(J_+(r) + J_-(r)) dr,$$

where

$$J_+(r) = \frac{\alpha^2}{1 + \alpha^2} \int_0^\pi \frac{(1 - \cos(\theta_1)) \sin(\theta_1)^{m-2}}{(r^2 + 1 - 2\cos(\theta_1) + \alpha^2 (r - 1)^2)^{\frac{N+s}{2}}} d\theta_1,$$
$$J_-(r) = \frac{1}{1 + \alpha^2} \int_0^\pi \frac{[2 + \alpha^2 - \alpha^2 \cos(\theta_1)) \sin(\theta_1)^{m-2}}{(r^2 + 1 - 2r\cos(\theta_1) + \alpha^2 (r + 1)^2)^{\frac{N+s}{2}}} d\theta_1.$$

Computation of $C(m,n,s,\beta)$ for $n \geq 2$. Similarly, we obtain

(11.7)

$$
\begin{aligned}
&C(m,n,s,\beta)\\
&= (1+\alpha)^{\frac{3+s}{2}} A_{m-2}A_{n-2}\int_0^1 (r^{N-2} - r^{N-2-\beta} + r^s - r^{\beta+s}) I(r)dr,
\end{aligned}
$$

where

$$
I(r) = \int_0^\pi\int_0^\pi \frac{\sin(\theta_1)^{m-2}\sin(\varphi_1)^{n-2}}{(r^2+1-2r\cos(\theta_1)+\alpha^2(r^2+1-2r\cos(\varphi_1)))^{\frac{N+s}{2}}} d\theta_1 d\varphi_1.
$$

Computation of $A_0(m,n,s)^2$ for $n \geq 2$. Similarly, we obtain

$$
A_0(m,n,s)^2 = (1+\alpha^2)^{\frac{3+s}{2}} A_{m-2}A_{n-2}\int_0^1 (r^{N-2}+r^s)J(r)dr,
$$

where

$$
\begin{aligned}
&J(r) = \frac{1}{1+\alpha^2}\\
&\times \int_0^\pi\int_0^\pi \frac{(1+\alpha^2-\alpha^2\cos(\theta_1)-\cos(\varphi_1))\sin(\theta_1)^{m-2}\sin(\varphi_1)^{n-2}}{(r^2+1-2r\cos(\theta_1)+\alpha^2(r^2+1-2r\cos(\varphi_1)))^{\frac{N+s}{2}}} d\theta_1 d\varphi_1.
\end{aligned}
$$

In Table 1 we show the values obtained for $H(m,n,0)$ and $A_0(m,n,0)^2$, divided by $(1+\alpha^2)^{\frac{3+s}{2}} A_{m-2}A_{n-2}$, from numerical approximation of the integrals. From these results we can say that for $s=0$, Σ is stable if $n+m=7$ and unstable if $n+m \leq 6$. The same holds for $s>0$ close to zero by continuity of the values with respect to s.

Remark 11.1. We see from formulas (11.6) and (11.7) that $C(m,n,s,\beta)$ is symmetric with respect to $\frac{N-2-s}{2}$ and is maximized for $\beta = \frac{N-2-s}{2}$.

Remark 11.2. One may conjecture that for $m=4$, $n=3$ there is s_0 such that the cone is stable for $0 \leq s \leq s_0$ and unstable for $s_0 < s < 1$.

Appendix A. Asymptotics

We prove convergence of geometric fractional quantities as $s \to 1$ ($\varepsilon = 1-s \to 0$). Let $\Sigma \subset \mathbb{R}^{n+1}$ be a smooth embedded hyper surface.

Lemma A.1. *Assume $\Sigma = \partial E$. Then for any $X \in \Sigma$*

$$
(1-s)\int_{\mathbb{R}^{n+1}} \frac{\chi_E(Y)-\chi_{E^c}(Y)}{|X-Y|^{n+1+s}}\, dY = -H_\Sigma(X) n\omega_n + O(1-s)
$$

as $s \to 1$, where $H_\Sigma(X) = \frac{\kappa_1+\ldots+\kappa_n}{n}$ is the mean curvature of Σ at X and ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Table 1. Values of $H(m,n,0)$ and $A_0(m,n,0)^2$ divided by $(1+\alpha^2)^{\frac{3+s}{2}}A_{m-2}A_{n-2}$

		n						
m		1	2	3	4	5	6	7
2	H	0.8140	1.0679					
	A_0^2	3.2669	2.3015					
3	H	1.1978	1.2346	0.3926				
	A_0^2	2.5984	1.7918	0.4463				
4	H	1.3968	1.3649	0.4477	0.1613			
	A_0^2	2.0413	1.5534	0.4288	0.1356			
5	H	1.5117	1.4570	0.4895	0.1845	0.06978		
	A_0^2	1.7332	1.3981	0.4118	0.1398	0.04849		
6	H	1.5833	1.5231	0.5215	0.2031	0.08013	0.03113	
	A_0^2	1.5318	1.2841	0.3955	0.1412	0.05173	0.01885	
7	H	1.6303	1.5719	0.5465	0.2182	0.08885	0.03583	0.01416
	A_0^2	1.3872	1.1951	0.3802	0.1409	0.05381	0.02051	0.007704

Proof. Let us fix $R>0$ and $X\in\Sigma$ and assume $X=0$ for simplicity. Let Σ_R be Σ intersected with the cylinder $B_R(0)\times(-R,R)$, $B_R(0)\subset\mathbb{R}^n$. After rotation, we describe Σ_R as the graph of $g: B_R(0)\to\mathbb{R}$ with

$$g(0)=0,\quad Dg(0)=0,$$

and assume E lies above Σ_R.

Note that

$$\int_{(B_R(0)\times(-R,R))^c}\frac{\chi_E(Y)-\chi_{E^c}(Y)}{|X-Y|^{n+1+s}}\,dY=O(1)$$

as $s\to 1$. We compute

$$\begin{aligned}I&=\int_{B_R(0)\times(-R,R)}\frac{\chi_E(Y)-\chi_{E^c}(Y)}{|X-Y|^{n+1+s}}\,dY\\&=-2\int_{B_R\subset\mathbb{R}^n}\int_0^{g(t)}\frac{1}{(|t|^2+t_3^2)^{\frac{n+1+s}{2}}}dt_3\,dt.\end{aligned}$$

A direct computation then shows that as $s\to 1$,

$$I=-\frac{\omega_n\Delta g(0)R^{1-s}}{1-s}+O(1)=-n\omega_n\frac{H_\Sigma(X)R^{1-s}}{(1-s)}+O(1).\quad\text{q.e.d.}$$

For the next results we assume that there is C such that for all $0<s<1$ and $X\in\Sigma$

$$\int_{Y\in\Sigma,|Y-X|\geq 1}\frac{1}{|X-Y|^{n+1+s}}\,dY\leq C.$$

Lemma A.2. *If h is $C^{2,\alpha}(\Sigma)$ and bounded,*

$$(1-s)p.v.\int_{\Sigma}\frac{h(Y)-h(X)}{|X-Y|^{n+1+s}}dY=\frac{\omega_n}{2}\Delta_{\Sigma}h(X)+O(1-s)$$

as $s\to 1$, where Δ_Σ is the Laplace–Beltrami operator on Σ and $\omega_n = \frac{area(S^{n-1})}{n}$ is the volume of the unit ball in $\mathbb{R}^n$.

For the proof we use the following computation.

Lemma A.3. *If $\phi\in C^{2,\alpha}(\overline{B}_R(0))$,*

$$(1-s)\int_{B_R\subset\mathbb{R}^n}\frac{\phi(t)-\phi(0)}{|t|^{n+1+s}}\,dt=\frac{\omega_n}{2}\Delta\phi(0)+O(1-s) \tag{A.1}$$

as $s\to 1$.

Proof. We expand

$$\phi(t)=\phi(0)+D\phi(0)t+\frac{1}{2}D^2\phi(0)[t^2]+O(|t|^{2+\alpha})\quad\text{as } t\to 0.$$

This gives as $s\to 1$:

$$\begin{aligned}\int_{B_R}\frac{\phi(t)-\phi(0)}{|t|^{n+1+s}}\,dt&=\frac{1}{2}\int_{B_R}\frac{D^2\phi(0)[t^2]}{|t|^{n+1+s}}\,dt+O(1)\\&=\frac{1}{2}\frac{area(S^{n-1})}{n}\frac{R^{1-s}}{1-s}\Delta\phi(0)+O(1).\quad\text{q.e.d.}\end{aligned}$$

Proof of Lemma A.2. Let us fix $R>0$ and $X\in\Sigma$ and assume $X=0$ for simplicity. Let Σ_R be Σ intersected with the cylinder $B_R(0)\times(-R,R)$, $B_R(0)\subset\mathbb{R}^n$. After rotation, we describe Σ_R as the graph of $g:B_R(0)\to\mathbb{R}$ with

$$g(0)=0,\quad Dg(0)=0.$$

Then

$$\int_{\Sigma_R^c}\frac{h(Y)-h(X)}{|X-Y|^{n+1+s}}dY=O(1)$$

as $s\to 1$. We have

$$\int_{\Sigma_R}\frac{h(Y)-h(X)}{|X-Y|^{n+1+s}}dY=\int_{B_R(0)}\frac{h(g(t))-h(g(0))}{(g(t)^2+|t|^2)^{\frac{n+1+s}{2}}}\sqrt{1+|Dg(t)|^2}\,dt.$$

The previous lemma also holds if ϕ depends on s and $\phi_s\to\phi$ in $C^{2,\alpha}$ as $s\to 1$. We apply (A.1) to

$$\phi_s(t)=\frac{h(g(t))-h(g(0))}{(\frac{g(t)^2}{|t|^2}+1)^{\frac{n+1+s}{2}}}\sqrt{1+|Dg(t)|^2},$$

and note that $\phi_s\to\phi$ as $s\to 1$, where

$$\phi(t)=\frac{h(g(t))-h(g(0))}{(\frac{g(t)^2}{|t|^2}+1)^{n+2}}\sqrt{1+|Dg(t)|^2},$$

and

$$\Delta\phi(0) = \sum_{i=1}^{n} D_i(h \circ g)(0) = \Delta_\Sigma h(0). \qquad \text{q.e.d.}$$

Lemma A.4. *Let ν be smooth choice of normal vector ν on Σ. Then*

$$(1-s)\int_\Sigma \frac{(\nu(x)-\nu(y))\cdot\nu(x)}{|x-y|^{n+1+s}}dy = \frac{\omega_n}{2}|A(x)|^2 + O(1)$$

as $s \to 1$, where $A(x)|^2 = \sum_{i=1}^n \kappa_i^2$ with $\kappa_1, \ldots, \kappa_n$ are the principal curvatures at x.

Proof. We apply Lemma A.2 with $h(y) = \nu(y)\cdot\nu(x) - 1$ and use that $\Delta_\Sigma h(x) = -|A(x)|^2$. q.e.d.

Appendix B. The Jacobi operator

In this section, we prove formula (1.4) and derive the formula for the nonlocal Jacobi operator (1.5).

Let $E \subset \mathbb{R}^N$ be an open set with smooth boundary and Ω be a bounded open set. Let ν be the unit normal vector field of $\Sigma = \partial E$ pointing to the exterior of E. Given $h \in C_0^\infty(\Omega\cap\Sigma)$ and t small, let E_{th} be the set whose boundary ∂E_{th} is parametrized as

$$\partial E_{th} = \{x + th(x)\nu(x) \ / \ x \in \partial E\},$$

with exterior normal vector close to ν.

Proposition B.1. *Let $\Sigma_{th} = \partial E_{th}$. For $p \in \Sigma$ fixed let $p_t = p + th(p)\nu(p) \in \Sigma_{th}$. Then for $h \in C^\infty(\Sigma)\cap L^\infty(\Sigma)$*

$$\frac{d}{dt}H^s_{\Sigma_{th}}(p_t)\Big|_{t=0} = 2\mathcal{J}^s_\Sigma[h](p). \tag{B.1}$$

Proposition B.2. *For $h \in C_0^\infty(\Omega\cap\Sigma)$*

$$\frac{d^2}{dt^2}Per_s(E_{th},\Omega)\Big|_{t=0} \;=\; -2\int_\Sigma \mathcal{J}^s_\Sigma[h]\,h - \int_\Sigma h^2 H H^s_\Sigma, \tag{B.2}$$

where $\mathcal{J}^s_\Sigma$ is the nonlocal Jacobi operator defined in (1.5), H is the classical mean curvature of Σ and H^s_Σ is the nonlocal mean curvature defined in (1.1).

In case that Σ is a nonlocal minimal surface in Ω we obtain formula (1.4).

A consequence of proposition B.1 is that entire nonlocal minimal graphs are stable.

Corollary B.1. *Suppose that $\Sigma = \partial E$ with*

$$E = \{(x', F(x')) \in \mathbb{R}^N : x' \in \mathbb{R}^{N-1}\}$$

is a nonlocal minimal surface. Then

$$-\int_{\Sigma} \mathcal{J}_{\Sigma}^{s}[h]\, h \geq 0 \quad \textit{for all} \quad h \in C_0^{\infty}(\Sigma). \tag{B.3}$$

The proof of Proposition B.2 goes along the same lines of Proposition B.1 so that we will only carry out the latter.

Proof of Proposition B.1. Let $\nu_t(x)$ denote the unit normal vector to ∂E_t at $x \in \partial E_t$ pointing outwards E_t. Note that $\nu(x) = \nu_0(x)$. Let L_t be the half space defined by $L_t = \{x : \langle x - p_t, \nu_t(p_t)\rangle > 0\}$. Then

$$H^s_{\Sigma_{th}}(p_t) = \int_{\mathbb{R}^N} \frac{\chi_{E_t}(x) - \chi_{L_t}(x) - \chi_{E^c}(x) + \chi_{L_t^c}(x)}{|x - p_t|^{N+s}}\, dx. \tag{B.4}$$

Note that the integral in (B.4) is well defined and

$$H^s_{\Sigma_{th}}(p_t) = 2\int_{\mathbb{R}^N} \frac{\chi_{E_t}(x) - \chi_{L_t}(x)}{|x - p_t|^{N+s}}\, dx.$$

For $\delta > 0$ let $\eta \in C^{\infty}(\mathbb{R}^N)$ be a radially symmetric cut-off function with $\eta(x) = 1$ for $|x| \geq 2$, $\eta(x) = 0$ for $|x| \leq 1$. Define $\eta_\delta(x) = \eta(x/\delta)$ and write

$$\int_{\mathbb{R}^N} \frac{\chi_{E_t}(x) - \chi_{L_t}(x)}{|x - p_t|^{N+s}}\, dx = f_\delta(t) + g_\delta(t),$$

where

$$f_\delta(t) = \int_{\mathbb{R}^N} \frac{\chi_{E_t}(x) - \chi_{L_t}(x)}{|x - p_t|^{N+s}} \eta_\delta(x - p_t)\, dx,$$

and $g_\delta(t)$ is the rest. After some computation we obtain

$$\begin{aligned} f'_\delta(0) = &\int_{\partial E} \frac{h(x) - h(p)}{|x - p|^{N+s}} \eta_\delta(x - p)\, dx \\ &+ h(p)\int_{\partial E} \frac{1 - \langle \nu(x), \nu(p)\rangle}{|x - p|^{N+s}} \eta_\delta(x - p)\, dx, \end{aligned}$$

and that $g'_\delta(t) \to 0$ as $\delta \to 0$, uniformly for t in a neighborhood of 0. Letting $\delta \to 0$ we find (B.1). q.e.d.

Proof of Corollary B.1. Invariance of the nonlocal minimal surface equation under translations in the N-th direction implies that the positive function $w = \langle \nu, e_N\rangle$ satisfies

$$\text{p.v.} \int_{\Sigma} \frac{w(y) - w(x)}{|y - x|^{N+s}} dy + w(x)A(x) = 0 \quad \text{for all } x \in \Sigma, \tag{B.5}$$

where

$$A(x) = \int_{\Sigma} \frac{\langle \nu(x) - \nu(y), \nu(x)\rangle}{|x - y|^{N+s}} dy.$$

As in the classical setting this implies that Σ is stable in the sense that (B.3) holds. Indeed, let $\phi \in C_0^\infty(\Sigma)$. Substituting $h = w\psi$ in the quadratic form (B.3) and using (B.5) we get

$$\begin{aligned}\frac{1}{2}\int_\Sigma\int_\Sigma \frac{(h(x)-h(y))^2}{|x-y|^{N+s}}dxdy \\ = \int_\Sigma A(x)h(x)^2dx + \frac{1}{2}\int_\Sigma\int_\Sigma \frac{(\psi(x)-\psi(y))^2 w(x)w(y)}{|x-y|^{N+s}}dxdy,\end{aligned}$$

and this shows (B.3). q.e.d.

Proof of Proposition B.2. Let

$$K_\delta(z) = \frac{1}{|z|^{N+s}}\eta_\delta(z),$$

where $\eta_\delta(x) = \eta(x/\delta)$ ($\delta > 0$) and $\eta \in C^\infty(\mathbb{R}^N)$ is a radially symmetric cut-off function with $\eta(x) = 1$ for $|x| \ge 2$, $\eta(x) = 0$ for $|x| \le 1$.

Consider

$$\begin{aligned}Per_{s,\delta}(E_{th},\Omega) = \int_{E_{th}\cap\Omega}\int_{\mathbb{R}^N\setminus E_{th}} K_\delta(x-y)\,dy\,dx \\ + \int_{E_{th}\setminus\Omega}\int_{\Omega\setminus E_{th}} K_\delta(x-y)dydx.\end{aligned} \tag{B.6}$$

We will show that $\frac{d^2}{dt^2}Per_{s,\delta}(E_{th},\Omega)$ approaches a certain limit $D_2(t)$ as $\delta \to 0$, uniformly for t in a neighborhood of 0 and that

$$D_2(0) = -2\int_\Sigma \mathcal{J}^s_\Sigma[h]\,h - \int_\Sigma h^2 H H^s_\Sigma.$$

First we need some extensions of ν and h to $\mathbb{R}^N$. To define them, let $K \subset \Sigma$ be the support of h and U_0 be an open bounded neighborhood of K such that for any $x \in U_0$, the closest point $\hat{x} \in \Sigma$ to x is unique and defines a smooth function of x. We also take U_0 smaller if necessary as to have $\overline{U}_0 \subset \Omega$. Let $\tilde{\nu} : \mathbb{R}^N \to \mathbb{R}^N$ be a globally defined smooth unit vector field such that $\tilde{\nu}(x) = \nu(\hat{x})$ for $x \in U_0$. We also extend h to $\tilde{h} : \mathbb{R}^N \to \mathbb{R}$ such that it is smooth with compact support contained in Ω and $\tilde{h}(x) = h(\hat{x})$ for $x \in U_0$. From now one we omit the tildes ($\tilde{}$) in the definitions of the extensions of ν and h. For t small $\bar{x} \mapsto \bar{x} + th(\bar{x})\nu(\bar{x})$ is a global diffeomorphism in $\mathbb{R}^N$. Let us write

$$u(\bar{x}) = h(\bar{x})\nu(\bar{x}) \quad \text{for } \bar{x} \in \mathbb{R}^N,$$
$$\nu = (\nu^1, \dots, \nu^N), \quad u = (u^1, \dots, u^N),$$

and let

$$J_t(\bar{x}) = J_{id+tu}(\bar{x})$$

be the Jacobian determinant of $id + tu$.

We change variables

$$x = \bar{x} + tu(\bar{x}), \quad y = \bar{y} + tu(\bar{y})$$

in (B.6)

$$
\begin{aligned}
Per_{s,\delta}(E_{th},\Omega) = &\int_{E\cap\phi_t(\Omega)}\int_{\mathbb{R}^N\setminus E} K_\delta(x-y)J_t(\bar{x})J_t(\bar{y})d\bar{y}d\bar{x},\\
&+\int_{E\setminus\phi_t(\Omega)}\int_{\phi_t(\Omega)\setminus E} K_\delta(x-y)J_t(\bar{y})d\bar{y}d\bar{x},
\end{aligned}
$$

where ϕ_t is the inverse of the map $\bar{x}\mapsto\bar{x}+tu(\bar{x})$.

Differentiating with respect to t:

$$
\begin{aligned}
&\frac{d}{dt}Per_{s,\delta}(E_{th},\Omega)\\
&\quad=\int_{E\cap\phi_t(\Omega)}\int_{\mathbb{R}^N\setminus E}\Big[\nabla K_\delta(x-y)(u(\bar{x})-u(\bar{y}))J_t(\bar{x})J_t(\bar{y})\\
&\qquad+K_\delta(x-y)(J_t'(\bar{x})J_t(\bar{y})+J_t(\bar{x})J_t'(\bar{y}))\Big]d\bar{y}d\bar{x}\\
&\qquad+\int_{E\setminus\phi_t(\Omega)}\int_{\phi_t(\Omega)\setminus E}\Big[\nabla K_\delta(x-y)(u(\bar{x})-u(\bar{y}))J_t(\bar{x})J_t(\bar{y})\\
&\qquad+K_\delta(x-y)(J_t'(\bar{x})J_t(\bar{y})+J_t(\bar{x})J_t'(\bar{y}))\Big]d\bar{y}d\bar{x},
\end{aligned}
$$

where

$$
J_t'(\bar{x})=\frac{d}{dt}J_t(\bar{x}).
$$

Note that there are no integrals on $\partial\phi_t(\Omega)$ for t small because u vanishes in a neighborhood of $\partial\Omega$.

Since the integrands in $\frac{d}{dt}Per_{s,\delta}(E_{th},\Omega)$ have compact support contained in $\phi_t(\Omega)$ (t small), we can write

$$
\begin{aligned}
\frac{d}{dt}Per_{s,\delta}(E_{th},\Omega)=&\int_E\int_{\mathbb{R}^N\setminus E}\Big[\nabla K_\delta(x-y)(u(\bar{x})-u(\bar{y}))J_t(\bar{x})J_t(\bar{y})\\
&+K_\delta(x-y)(J_t'(\bar{x})J_t(\bar{y})+J_t(\bar{x})J_t'(\bar{y}))\Big]d\bar{y}d\bar{x}.
\end{aligned}
$$

Differentiating once more, after some computation we get

$$
\begin{aligned}
&\frac{d^2}{dt^2}Per_{s,\delta}(E_{th},\Omega)\Big|_{t=0}\\
&\quad=2\int_{\partial E}\int_{\partial E}K_\delta(x-y)h(x)^2(\nu(x)\nu(y)-1)\,dy\,dx\\
&\qquad-2\int_{\partial E}h(x)\int_{\partial E}K_\delta(x-y)(h(y)-h(x))\,dydx\\
&\qquad-\int_{\partial E}h(x)^2H(x)\int_{\mathbb{R}^N}(\chi_E(y)-\chi_{E^c}(y))K_\delta(x-y)\,dy\,dx.
\end{aligned}
$$

Taking the limit as $\delta\to0$ we find (B.2). q.e.d.

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