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ON WEAKLY MAXIMAL REPRESENTATIONS OF SURFACE GROUPS

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Abstract

We introduce and study a new class of representations of surface groups into Lie groups of Hermitian type, called *weakly maximal* representations. We prove that weakly maximal representations are discrete and injective and we describe the structure of the Zariski closure of their image. Furthermore, we prove that the set of weakly maximal representations is a closed subset of the representation variety and describe its relation to other geometrically significant subsets of the representations variety.

1. Introduction

Given a compact oriented surface Σ of negative Euler characteristic, possibly with boundary, a general theme is to study the space of representations $\mathrm{Hom}(\pi_1(\Sigma), G)$ of the fundamental group $\pi_1(\Sigma)$ of Σ into a semisimple Lie group G, and, in particular, to distinguish subsets of geometric significance.

In recent years these studies led to the discovery of new subsets of representation varieties: Hitchin components [**37**, **17**, **38**, **29**, **30**], positive representations [**20**, **21**], maximal representations [**25**, **26**, **41**, **36**, **13**, **10**, **9**, **14**, **43**, **35**, **27**, **3**, **2**, **4**, **22**] and Anosov representations [**32**, **38**, **31**]. Even though these subsets are defined and investigated by very different methods, they exhibit several common properties, and their study is summarized under the terminology *higher Teichmüller theory.*

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Here we introduce a new class of representations of $\pi_1(\Sigma)$ into a Lie group of Hermitian type, the set of *weakly maximal representations*. We establish several results showing that weakly maximal representations are of geometric significance. We also construct various nontrivial examples of weakly maximal representations, establish a relation to orders on Lie groups, and discuss the relation of the set of weakly maximal representations with other subsets of the representation variety.

Weakly maximal representations with non-zero Toledo number also admit a geometric characterization. This geometric characterization is discussed in detail in the companion paper [**1**]. We show there that weakly maximal representations with non-zero Toledo number are representations that are order preserving in an appropriate sense with respect to an arbitrary bi-invariant continuous partial order on the cyclic covering of G. If the symmetric space associated to G is a Hermitian symmetric space of tube type, they are also order-preserving with respect to the order induced by the Kaneyuki causal ordering on the corresponding Shilov boundary. This leads to a particularly simple geometric characterization in the tube type case.

1.1. Structure theorems for weakly maximal representations. Recall that the Toledo invariant $\mathrm{T}(\rho)$ of a representation $\rho : \pi_1(\Sigma) \to G$ is defined as the evaluation, in an appropriate sense, of the pullback $\rho^*(\kappa_G^{\mathrm{b}})$ of the bounded Kähler class $\kappa_G^{\mathrm{b}} \in \mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R})$ on the relative fundamental class $[\Sigma, \partial\Sigma]$ (see § 2 for details). This definition in terms of bounded cohomology leads to a chain of inequalities

$$|\,\mathrm{T}(\rho)| \leq 2\|\rho^*(\kappa_G^{\mathrm{b}})\| \cdot |\chi(\Sigma)| \leq 2\|\kappa_G^{\mathrm{b}}\| \cdot |\chi(\Sigma)|\,,$$

where $\|\cdot\|$ denotes the canonical norm on the Banach spaces $\mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R})$ and $\mathrm{H}^2_{\mathrm{b}}(\pi_1(\Sigma), \mathbb{R})$, while $|\chi(\Sigma)|$ appears as (half of) the ℓ^1-simplicial norm of $[\Sigma, \partial\Sigma]$. *Maximal representations* are those for which both inequalities are equalities, while representations for which the second inequality is an equality are *tight homomorphisms*; these constitute a very interesting class of homomorphisms which can be defined and studied in much greater generality, see [**12**, **33**, **34**]. In this article we set ourselves the goal to analyze the structure of representations for which the first inequality is an equality.

Definition 1.1 ([**42**, Chapter 8, Definition 2.1])**.** A representation $\rho : \pi_1(\Sigma) \to G$ is *weakly maximal* if it satisfies the equality

$$\langle \rho^*(\kappa_G^{\mathrm{b}}), [\Sigma, \partial\Sigma]\rangle = \|\rho^*(\kappa_G^{\mathrm{b}})\|\|\Sigma, \partial\Sigma\|_1, \tag{1.1}$$

in the standard inequality of dual norms.

Remark 1.2. Notice that the left hand side of (1.1) is the Toledo invariant $\mathrm{T}(\rho)$, while the right hand side equals $2\|\rho^*(\kappa_G^{\mathrm{b}})\| \cdot |\chi(\Sigma)|$.

By definition a weakly maximal representation has non-negative Toledo number. Analogously one can define weakly minimal representations. Since the composition of a representation with an orientation reversing outer automorphism changes the sign of the Toledo number the theory is completely analogous.

Whereas for maximal representations the Toledo invariant is a fixed number independent of ρ, for weakly maximal representations the right hand side of the equality is $2\|\rho^*(\kappa_G^{\mathrm{b}})\| \cdot |\chi(\Sigma)|$ and depends on ρ. In particular, there is (a priori) no restriction on the Toledo invariant of a weakly maximal representation.

In order to get a better understanding of weakly maximal representations it is, therefore, important to give alternative characterizations. One such characterization is that a representation ρ is weakly maximal if and only if there exists $\lambda \geq 0$ such that

$$\rho^*(\kappa_G^{\mathrm{b}}) = \lambda\, \kappa_\Sigma^{\mathrm{b}}\,, \tag{1.2}$$

where $\kappa_\Sigma^{\mathrm{b}}$ if the bounded fundamental class of Σ (see (3.1)). This characterization was in effect first established in [**42**, Corollary 3.4] for compact surfaces and in [**13**, Cor. 4.15] in the general case; a different proof using Bavard's duality was later obtained in [**16**] (for a more thorough description of the relation with Calegari's work, see [**14**, Section 4.6]).

The constant λ appearing in (1.2) is related to the Toledo invariant by

$$\mathrm{T}(\rho) = \lambda\, |\chi(\Sigma)|\,.$$

When Σ is a surface without boundary the Toledo invariant is a characteristic number with values in $q_G^{-1}\mathbb{Z}$, where q_G is a natural number depending only on G (namely, q_G is the smallest integer such that $q_G\kappa_G^{\mathrm{b}}$ is an integral class, see Remark 6.5). On the other hand, when Σ is a surface with boundary, the Toledo invariant ranges over the whole closed interval $\big[-2\|\kappa_G^{\mathrm{b}}\| \cdot |\chi(\Sigma)|, 2\|\kappa_G^{\mathrm{b}}\| \cdot |\chi(\Sigma)|\big]$.

It is remarkable that for weakly maximal representations we can restrict the possible values of the Toledo-invariant by the following:

Theorem 1.3 (Rationality Theorem)**.** *There is a natural number ℓ_G depending only on G, such that for every weakly maximal representation $\rho : \pi_1(\Sigma) \to G$ we have $\mathrm{T}(\rho) \in \frac{|\chi(\Sigma)|}{\ell_G}\mathbb{Z}$. In particular,* T *takes only finitely many values on the set of weakly maximal representations.*

Remark 1.4. The integer ℓ_G depends in an explicit way on q_G and on the degree of non-integrality of the restriction $\kappa_G^{\mathrm{b}}|_H$ to various connected semisimple subgroups H of G, as well as on the cardinality of their center (see the proof of Corollary 6.4).

The Rationality Theorem is an essential ingredient in order to prove:

Theorem 1.5. *Let $\rho : \pi_1(\Sigma) \to G$ be a weakly maximal representation and* $\mathrm{T}(\rho) \neq 0$*. Then ρ is faithful with discrete image.*

The methods of proof for this theorem are very different from those used in the case of maximal representations [**13**] and give, in particular, a new and simpler proof of the injectivity and discreteness of maximal representations.

Whereas the Zariski closure for maximal representations is always reductive and subject to strong restrictions, the Zariski closure of the image of a weakly maximal representation can be quite wild. In particular, it need not be reductive (see § 9.6). However, we can establish a structure theorem describing the Zariski closure of weakly maximal representations as follows. Given a closed subgroup $L < G$, we show that there exists a unique maximal normal subgroup of L, that we call the *Kähler radical* $\mathrm{Rad}_{\kappa_G^{\mathrm{b}}}(L)$ of L, on which $\kappa_G^{\mathrm{b}}|_L$ vanishes. Since $\mathrm{Rad}_{\kappa_G^{\mathrm{b}}}(L)$ contains the solvable radical of L, the quotient $L/\mathrm{Rad}_{\kappa_G^{\mathrm{b}}}(L)$ is semisimple. We then establish the following as part of Theorem 7.2:

Theorem 1.6. *Let $\rho : \pi_1(\Sigma) \to G$ be a weakly maximal representation into a Lie group $G := \mathbf{G}(\mathbb{R})^\circ$ of Hermitian type, where $\mathbf{G}$ is a connected algebraic group defined over $\mathbb{R}$. Let $L := \mathbf{L}(\mathbb{R})$, where $\mathbf{L}$ is the Zariski closure of $\rho(\pi_1(\Sigma))$ in $\mathbf{G}$, and let $H = L/\,\mathrm{Rad}_{\kappa_G^{\mathrm{b}}}(L)$ be the quotient of L by its Kähler radical. Assume that* $\mathrm{T}(\rho) \neq 0$*. Then:*

1) *H is adjoint of Hermitian type and all of its simple factors are of tube type.*
2) *The composition $\pi_1(\Sigma) \to L \to H$ is faithful with discrete image.*

Recall that a Hermitian Lie group is of *tube type* if the associated symmetric space is of tube type, that is biholomorphic to $\mathbb{R}^n + iC$, for some $n \in \mathbb{N}$, where $C \subset \mathbb{R}^n$ is an open convex cone.

Remark 1.7. In the above Theorems 1.5 and 1.6 it is essential that the Toledo invariant is non-zero. However, the class of weakly maximal representations with $\mathrm{T}(\rho) = 0$ is also of interest. This is precisely the set where $\rho^*(\kappa_G^{\mathrm{b}}) = 0$. In the case when $G = \mathrm{PU}(1,n)$ such representations have been studied and characterized in [**8**] as representations that preserve a totally real subspace of complex hyperbolic space $\mathcal{H}^n_{\mathbb{C}}$.

1.2. Comparison with other classes of representations. Consider the representation variety $\mathrm{Hom}(\pi_1(\Sigma), G)$.

The subset of weakly maximal representations $\mathrm{Hom}_{wm}(\pi_1(\Sigma), G)$ decomposes as a disjoint union

$$\mathrm{Hom}_{wm}(\pi_1(\Sigma), G) = \mathrm{Hom}^*_{wm}(\pi_1(\Sigma), G) \sqcup \mathrm{Hom}_0(\pi_1(\Sigma), G),$$

of the set $\mathrm{Hom}^*_{wm}(\pi_1(\Sigma), G)$ of weakly maximal representations with non-zero Toledo invariant and the set of (weakly maximal) representations with vanishing Toledo invariant

$$\mathrm{Hom}_0(\pi_1(\Sigma), G) := \{\rho : \pi_1(\Sigma) \to G : \rho^*(\kappa_G^{\mathrm{b}}) = 0\}.$$

The latter contains $\mathrm{Hom}(\pi_1(\Sigma), L)$ for every closed subgroup $L < G$ for which $\kappa_G^{\mathrm{b}}|_L = 0$. In particular, representations in $\mathrm{Hom}_0(\pi_1(\Sigma), G)$ are not necessarily injective, and their images are not necessarily discrete (see Remark 1.7), while, according to Theorem 1.5, $\mathrm{Hom}^*_{wm}(\pi_1(\Sigma), G)$ is contained in the set $\mathrm{Hom}_{d,i}(\pi_1(\Sigma), G)$ of injective homomorphisms with discrete image. We prove (cf. Corollary 8.4):

Theorem 1.8. *The following:*

$$\begin{tikzcd} & & \mathrm{Hom}_{d,i}(\pi_1(\Sigma), G) \\ \mathrm{Hom}_{max}(\pi_1(\Sigma), G) \arrow[r, hook] & \mathrm{Hom}^*_{wm}(\pi_1(\Sigma), G) \arrow[ur, hook] \arrow[dr, hook] & \\ & & \mathrm{Hom}_{wm}(\pi_1(\Sigma), G) \end{tikzcd}$$

is a diagram of $\mathrm{Aut}(\pi_1(\Sigma))$*-invariant closed subsets of the representation variety* $\mathrm{Hom}(\pi_1(\Sigma), G)$.

If G is real algebraic, $\mathrm{Hom}_{max}(\pi_1(\Sigma), G)$ is a real semi-algebraic subset of $\mathrm{Hom}(\pi_1(\Sigma), G)$, [**13**, Corollary 14], but we do not have such precise information on the other sets appearing in Theorem 1.8.

A prominent role in higher Teichmüller theory is played by Anosov representations, a notion introduced by F. Labourie in his study of Hitchin representations [**38**], then studied for Hermitian Lie groups in [**9**, **10**] and in greater generality in [**32**]. The property of a representation to be Anosov is defined with respect to a parabolic subgroup $P < G$. In the context of weakly maximal representations the interest lies on representations $\rho : \pi_1(\Sigma) \to G$ which are Anosov with respect to the parabolic group Q which is the stabilizer in G of a point in the Shilov boundary $\check{S}$ of the bounded symmetric domain realization of $\mathcal{X}$ [**9**, **10**, **32**]. We will call such an Anosov representation *Shilov–Anosov*.

In the case in which $\partial\Sigma = \emptyset$, if we denote by $\mathrm{Hom}_{\check{S}\text{-}An}(\pi_1(\Sigma), G)$ the set of such representations and by $\mathrm{Hom}^*_{\check{S}\text{-}An}(\pi_1(\Sigma), G)$ the subset of Shilov–Anosov representations with positive Toledo invariant, we have

$$\begin{tikzcd} \mathrm{Hom}_{max}(\pi_1(\Sigma), G) \arrow[r, hook] & \mathrm{Hom}^*_{\check{S}\text{-}An}(\pi_1(\Sigma), G) \arrow[r, hook] & \mathrm{Hom}_{\check{S}\text{-}An}(\pi_1(\Sigma), G). \end{tikzcd}$$

Here $\mathrm{Hom}_{max}(\pi_1(\Sigma), G)$ is a union of components and $\mathrm{Hom}_{\check{S}\text{-}An}(\pi_1(\Sigma), G)$ is an open subset of $\mathrm{Hom}(\pi_1(\Sigma), G)$, [**9**, **10**, **32**].

On the other hand we prove in Corollary 8.5 the following:

Theorem 1.9. *If G is of tube type and $\partial\Sigma = \emptyset$, then*

$$\overline{\mathrm{Hom}_{\check{S}\text{-}An}(\pi_1(\Sigma), G)} \hookrightarrow \mathrm{Hom}_{wm}(\pi_1(\Sigma), G)\,.$$

If $G = \mathrm{Sp}(2n, \mathbb{R})$ with $n \geq 2$, we will see that $\mathrm{Hom}_{\check{S}\text{-}An}(\pi_1(\Sigma), G)$ is open but not closed. The study of the closure of the set of Anosov representations is a completely open problem. Since the set of weakly maximal representations is closed (see Corollary 8.3), it provides a natural framework in which to study limits of Shilov–Anosov representations. We refer the reader to § 9.5 for examples of weakly maximal representations into $\mathrm{Sp}(2n, \mathbb{R})$ for $n \geq 6$, that are not Shilov–Anosov, but are the limit of Shilov–Anosov representations.

For a closed surface Σ and a Hermitian group G of tube type we summarize the relation of various special subsets of the representation variety and results from [**38**, **12**] in the following diagram. Here we denote by $\mathrm{Hom}_{red}(\pi_1(\Sigma), G)$ the set of representations with reductive Zariski closure and by $\mathrm{Hom}_{Hitchin}(\pi_1(\Sigma), G)$ the Hitchin components in case G is locally isomorphic to $\mathrm{Sp}(2n, \mathbb{R})$ (and the empty set otherwise). Then we have the following inclusions:

$$\begin{tikzcd}
 & & \mathrm{Hom}_{\check{S}\text{-}An} \arrow[r, hook] & \mathrm{Hom}_{wm} & \\
\mathrm{Hom}_{Hitchin} \arrow[r, hook] & \mathrm{Hom}_{max} \arrow[r, hook] \arrow[dr, hook] & \mathrm{Hom}^*_{\check{S}\text{-}An} \arrow[r, hook] \arrow[u, hook] & \mathrm{Hom}^*_{wm} \arrow[r, hook] \arrow[u, hook] & \mathrm{Hom}_{d,i} \\
 & & \mathrm{Hom}_{tight} \arrow[r, hook] & \mathrm{Hom}_{red} &
\end{tikzcd}$$

2. The Toledo invariant

We recall here the definition of the Toledo invariant in a general context and indicate how the Milnor–Wood inequality follows from known isometric isomorphisms in bounded cohomology.

Let Σ be a compact oriented surface with (possibly empty) boundary $\partial\Sigma$. Let G be a locally compact group, $\kappa \in \mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R})$ a fixed continuous bounded class and $\rho : \pi_1(\Sigma) \to G$ a homomorphism. Recall that there is an isometric isomorphism

$$g_\Sigma : \mathrm{H}^2_{\mathrm{b}}(\pi_1(\Sigma), \mathbb{R}) \longrightarrow \mathrm{H}^2_{\mathrm{b}}(\Sigma, \mathbb{R})\,,$$

whose existence in general is due to Gromov [**28**]; in our case, the universal covering $\widetilde{\Sigma}$ is contractible and the existence and isometric property of g_Σ are easily established. Next, the inclusion of pairs $(\Sigma, \emptyset) \hookrightarrow (\Sigma, \partial\Sigma)$ gives rise to a map

$$j_{\partial\Sigma} : \mathrm{H}^2_{\mathrm{b}}(\Sigma, \partial\Sigma, \mathbb{R}) \longrightarrow \mathrm{H}^2_{\mathrm{b}}(\Sigma, \mathbb{R})\,,$$

where the left hand side refers to bounded relative cohomology. Since every connected component of $\partial\Sigma$ is a circle and hence has amenable fundamental group, the map $j_{\partial\Sigma}$ is an isometric isomorphism, [**5**, Theorem 1.2]. We hence define

$$\mathrm{T}_\kappa(\rho) := \langle (j_{\partial\Sigma})^{-1} g_\Sigma(\rho^*(\kappa)), [\Sigma, \partial\Sigma]\rangle\,,$$

where $[\Sigma, \partial\Sigma]$ is the relative fundamental class. Since g_Σ and $j_{\partial\Sigma}$ are isometries, we deduce that

$$|\,\mathrm{T}_\kappa(\rho)| \leq \|\rho^*(\kappa)\|\,\|[\Sigma, \partial\Sigma]\|_1\,,$$

where the second factor refers to the norm in relative ℓ^1-homology. Since $\|[\Sigma, \partial\Sigma]\|_1 = 2|\chi(\Sigma)|$ and the pullback is norm decreasing, then

$$|\,\mathrm{T}_\kappa(\rho)| \leq 2\|\rho^*(\kappa)\|\,|\chi(\Sigma)| \leq 2\|\kappa\|\,|\chi(\Sigma)|\,. \tag{2.1}$$

In view of (2.1), we give the following definition:

Definition 2.1. A representation $\rho : \pi_1(\Sigma) \to G$ is κ-*weakly maximal* if $\mathrm{T}_\kappa(\rho) = 2\|\rho^*(\kappa)\|\,|\chi(\Sigma)|$.

Remark 2.2. If G is of Hermitian type and $\kappa = \kappa_G^{\mathrm{b}}$ is the bounded Kähler class, then $\|\kappa_G^{\mathrm{b}}\| = \mathrm{rank}(G)/2$ (see [**13**, § 2.1]) and one obtains for the corresponding Toledo invariant $\mathrm{T}(\rho)$ the familiar Milnor–Wood inequality

$$|\,\mathrm{T}(\rho)| \leq \mathrm{rank}(G)\,|\chi(\Sigma)|\,.$$

This will be used in the examples in § 9, but otherwise never in the paper the explicit computation of the norm is used.

3. A Characterization of weakly maximal representations

In this section, we develop the general framework for the study of weakly maximal representations and establish some of their basic properties. In particular, we show that weak maximality of a representation $\rho : \pi_1(\Sigma) \to G$ is reflected in a relation between its lift to appropriate central extensions of $\pi_1(\Sigma)$ and G and canonically defined quasimorphisms on those central extensions (cf. Proposition 3.2).

3.1. The central extension of $\pi_1(\Sigma)$. Let h be any complete hyperbolic structure on the interior Σ° of Σ compatible with the fixed orientation and let $\rho_h : \pi_1(\Sigma) \to \mathrm{PU}(1,1)$ be the corresponding holonomy representation. Then

$$\kappa_\Sigma^{\mathrm{b}} := \rho_h^*(\kappa_{\mathrm{PU}(1,1)}^{\mathrm{b}}) \in \mathrm{H}_{\mathrm{b}}^2(\pi_1(\Sigma), \mathbb{R}) \tag{3.1}$$

is independent of the choice of h and is called the *(real) bounded fundamental class* of Σ, [**13**, § 8.2]. If $\widehat{\Gamma}$ denotes the central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widehat{\Gamma} \xrightarrow{p_\Sigma} \pi_1(\Sigma) \longrightarrow 0 \tag{3.2}$$

corresponding to the positive generator of $\mathrm{H}^2(\pi_1(\Sigma), \mathbb{Z})$ if $\partial\Sigma = \emptyset$ and $\widehat{\Gamma} = \pi_1(\Sigma)$ otherwise, then ρ_h lifts to a homomorphism $\widetilde{\rho}_h : \widehat{\Gamma} \to \widetilde{\mathrm{PU}(1,1)}$, where we think of the universal covering $\widetilde{\mathrm{PU}(1,1)}$ as contained in the group $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ of increasing homeomorphisms of the real line $\mathbb{R}$ commuting with integer translations. Denote by $\tau : \widetilde{\mathrm{PU}(1,1)} \to \mathbb{R}$ the Poincaré translation quasimorphism defined by $\tau(f) := \lim_{n\to\infty} \frac{f^n(x)-x}{n}$ for $x \in \mathbb{R}$. Then the composition $\tau\circ\widetilde{\rho}_h$ is a homogeneous quasimorphism which is $\mathbb{Z}$-valued

$$\tau \circ \widetilde{\rho}_h : \widehat{\Gamma} \to \mathbb{Z}\,, \tag{3.3}$$

since every element in the image of ρ_h has a fixed point. Moreover, the homogeneous quasimorphism τ corresponds in real bounded cohomology to the pullback, via the projection $p : \widetilde{\mathrm{PU}(1,1)} \to \mathrm{PU}(1,1)$, of the real bounded cohomology class $\kappa^{\mathrm{b}}_{\mathrm{PU}(1,1)}$

$$[d\tau] = p^*(\kappa^{\mathrm{b}}_{\mathrm{PU}(1,1)})\,, \tag{3.4}$$

and hence, again in real bounded cohomology,

$$[d(\tau\circ\widetilde{\rho}_h)] = p^*_\Sigma(\kappa^{\mathrm{b}}_\Sigma)\,. \tag{3.5}$$

While $\tau\circ\widetilde{\rho}_h$ depends on the hyperbolization ρ_h and on its lift $\widetilde{\rho}_h$, its restriction $(\tau\circ\widetilde{\rho}_h)|_\Lambda$ to $\Lambda := [\widehat{\Gamma},\widehat{\Gamma}]$ is independent of the above choices, (as the equivalence in Proposition 3.2 below shows).

3.2. The central extension for a locally compact group G. We now turn to the construction of the central extension (depending on a "rational" class) and of the associated quasimorphism for a general locally compact group. The relation with § 3.1 is outlined in § 3.3.

Definition 3.1. We say that a bounded cohomology class κ is *rational* if there is an integer $n \geq 1$ such that $n\kappa$ is representable by a bounded $\mathbb{Z}$-valued Borel cocycle.

Let κ be rational and let $c : G^2 \to \mathbb{Z}$ be a normalized bounded Borel cocycle representing $n\kappa$. Endow $G\times\mathbb{Z}$ with the group structure defined by the Borel map

$$(g_1, n_1)(g_2, n_2) := (g_1g_2, n_1 + n_2 + c(g_1, g_2)),$$

and let $G_{n\kappa}$ denote the Borel group $G\times\mathbb{Z}$ endowed with the unique compatible locally compact group topology [**39**]. This gives the topological central extension

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ i\ } G_{n\kappa} \xrightarrow{\ p_{n\kappa}\ } G \longrightarrow e\,. \tag{3.6}$$

Then $f'_{n\kappa}(g,m) = \frac{1}{n}m$ is a Borel quasimorphism such that $df'_{n\kappa}$ represents $p^*_{n\kappa}(\kappa)$. Its homogenization $f_{n\kappa} : G_{n\kappa} \to \mathbb{R}$ is a continuous homogeneous quasimorphism [**13**, Lemma 7.4] such that

$$[df_{n\kappa}] = p^*_{n\kappa}(\kappa)\,, \tag{3.7}$$

and

$$f_{n\kappa}(i(m)) = \frac{1}{n}m\,. \tag{3.8}$$

The homogeneous quasimorphisms $f_{n\kappa}$ are well defined in the following sense. Let $\zeta = n\kappa$ be the smallest integer multiple of κ representable by a $\mathbb{Z}$-valued bounded Borel cocycle and let $\ell \in \mathbb{Z}$, $\ell \neq 0$. Then the map $G \times \mathbb{Z} \to G \times \mathbb{Z}$ defined by $(g, m) \mapsto (g, \ell m)$ induces a continuous homomorphism that identifies G_ζ with a closed subgroup of finite index in $G_{\ell\zeta}$. Via this identification, the quasimorphism f_ζ is the restriction to G_ζ of $f_{\ell\zeta}$.

The following result describes the property of being weakly maximal in terms of quasimorphisms and will be essential in the sequel.

Proposition 3.2. *Let G be a locally compact group and $\kappa \in \mathrm{H}^2_{\mathrm{b}}(G, \mathbb{R})$ a rational class with $n\kappa$ integral. Let $\rho : \pi_1(\Sigma) \to G$ be a homomorphism and $\widetilde{\rho} : \widehat{\Gamma} \to G_{n\kappa}$ some lift, where $\widehat{\Gamma}$ and $G_{n\kappa}$ are defined, respectively, in* (3.2) *and* (3.6). *The following are equivalent:*

1) *ρ is κ-weakly maximal;*
2) *there exists $\lambda \geq 0$ such that $\rho^*(\kappa) = \lambda \kappa^{\mathrm{b}}_\Sigma$;*
3) *there exists $\lambda \geq 0$ and $\psi \in \mathrm{Hom}(\widehat{\Gamma}, \mathbb{R})$ such that*
$$f_{n\kappa} \circ \widetilde{\rho} = \lambda(\tau \circ \widetilde{\rho}_h) + \psi\,;$$
4) *there exists $\lambda \geq 0$ such that*
$$(f_{n\kappa} \circ \widetilde{\rho})|_\Lambda = \lambda(\tau \circ \widetilde{\rho}_h)|_\Lambda\,.$$

Remark 3.3. The constant $\lambda \geq 0$ that appears in Proposition 3.2 is nothing but $\lambda = \frac{\mathrm{T}_\kappa(\rho)}{|\chi(\Sigma)|}$, as it can be easily seen from the definition of the Toledo invariant in § 2.

Proof of Proposition 3.2. The equivalence of (1) and (2) is [**14**, Corollary 4.15]. In fact, the implication (2)⇒(1) is clear and we recall here the reverse implication for the convenience of the reader.

Let ρ_h be a complete finite area hyperbolization on Σ° so that $\Delta := \rho_h(\pi_1(\Sigma))$ is a lattice in $\mathrm{PU}(1,1)$. The space $\mathrm{H}^2_{\mathrm{b}}(\Delta, \mathbb{R})$ is isometrically isomorphic to the space of Δ-invariant alternating L^∞-cocycles on $(\partial\mathbb{D})^3$ and by [**14**, Proposition 4.13] we have a linear form $\mathrm{t}_{\mathrm{b}} : \mathrm{H}^2_{\mathrm{b}}(\Delta, \mathbb{R}) \to \mathbb{R}$ given by
$$\int_{\Delta\backslash\,\mathrm{PU}(1,1)} \alpha(gx, gy, gz)\, d\mu(g) = \frac{\mathrm{t}_{\mathrm{b}}(a)}{2}\, \mathrm{or}(x, y, z)\,,$$
where $\alpha : (\partial\mathbb{D})^3 \to \mathbb{R}$ corresponds to $a \in \mathrm{H}^2_{\mathrm{b}}(\Delta, \mathbb{R})$, $\mathrm{or} : (\partial\mathbb{D})^3 \to \{1, 0, -1\}$ is the orientation cocycle and μ is the invariant probability measure on $\Delta\backslash\,\mathrm{PU}(1,1)$. We obtained then in [**14**, Theorem 4.9] an analytic formula for $\mathrm{T}_\kappa(\rho)$ that reads
$$\mathrm{T}_\kappa(\rho) = \mathrm{t}_{\mathrm{b}}((\rho_h^*)^{-1}(\rho^*(\kappa)))|\chi(\Sigma)|\,.$$
If now ρ is weakly maximal, we have from the equality
$$\mathrm{T}_\kappa(\rho) = 2\|\rho^*(\kappa)\||\chi(\Sigma)|,$$

that

$$\mathrm{t_b}((\rho_h^*)^{-1}(\rho^*(\kappa))) = 2\|\rho^*(\kappa)\|,$$

and hence, if α corresponds to $\rho^*(\kappa)$, that

$$\int_{\Delta\backslash \mathrm{PU}(1,1)} \alpha(gx, gy, gz)\, d\mu(g) = \|\alpha\|_\infty \operatorname{or}(x, y, z)\,.$$

This implies that α is a positive multiple of the orientation cocycle and hence $\rho^*(\kappa) = \lambda\kappa_\Sigma^{\mathrm{b}}$ for some $\lambda \geq 0$.

Turning to the equivalence of (2) and (3), observe that, modulo modifying the lift $\widetilde{\rho}$ by a homomorphism with values in $\mathbb{Z} \hookrightarrow G_{n\kappa}$, the diagram

$$\begin{tikzcd} \widehat{\Gamma} \arrow[r, "\widetilde{\rho}"] \arrow[d, "p_\Sigma"'] & G_{n\kappa} \arrow[d, "p_{n\kappa}"] \\ \pi_1(\Sigma) \arrow[r, "\rho"] & G \end{tikzcd} \tag{3.9}$$

commutes. This, together with the relation (3.5) and the fact that p_Σ^* induces an isomorphism in real bounded cohomology, proves the equivalence of (2) and (3).

Finally, by taking coboundaries on both sides of the equality in (4), one obtains that the bounded classes $[d(f_{n\kappa}\circ\widetilde{\rho})]$ and $\lambda[d(\tau\circ\widetilde{\rho}_h)]$ coincide on Λ. Since $\widehat{\Gamma}/\Lambda$ is amenable, we deduce the equality in $\widehat{\Gamma}$, which in turns implies (3). q.e.d.

3.3. The Hermitian case. Assume that G is of Hermitian type and almost simple. We apply the above discussion to the bounded Kähler class κ_G^{b}, which is rational, [**7**, § 5]. If $\kappa = n\kappa_G^{\mathrm{b}}$ is any integer multiple represented by an integral cocycle, the topological central extension

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ i\ } G_\kappa \xrightarrow{\ p_\kappa\ } G \longrightarrow e$$

is not trivial. Since $\pi_1(G)$ is isomorphic to $\mathbb{Z}$ modulo torsion, there is a unique connected central $\mathbb{Z}$-extension $\widehat{G}$ and, as a result, the connected component of the identity $(G_\kappa)^\circ$ is isomorphic to $\widehat{G}$. We denote by $f_{\widehat{G}} : \widehat{G} \to \mathbb{R}$ the continuous homogeneous quasimorphism corresponding to f_κ under this isomorphism and hence obtain that

$$[df_{\widehat{G}}] = p^*(\kappa_G^{\mathrm{b}})\,, \tag{3.10}$$

where $p = p|_{(G_\kappa)^\circ} : \widehat{G} \to G$ is the projection. For example, if $G = \mathrm{PU}(1,1)$, the bounded Kähler class $\kappa_{\mathrm{PU}(1,1)}^{\mathrm{b}}$ is already integral as it is the image of the bounded Euler class e^{b} under the change of coefficients $\mathrm{H}^2_{\mathrm{cb}}(\mathrm{PU}(1,1),\mathbb{Z}) \to \mathrm{H}^2_{\mathrm{cb}}(\mathrm{PU}(1,1),\mathbb{R})$. Then $\widehat{G}$ is the universal covering $\widetilde{\mathrm{PU}(1,1)}$ and $f_{\widehat{G}}$ is the Poincaré translation quasimorphism.

4. A general framework for injectivity and discreteness

In this section, we give two results on weakly maximal representations: one concerns injectivity with general locally compact targets, and the other discreteness of the image in the case of Hermitian targets. In both cases we use the characterization of weakly maximal representations in § 3. However, while the characterization holds also if the Toledo invariant vanishes, both results in this section are obtained under the assumption that $\mathrm{T}_\kappa(\rho) \neq 0$. Discreteness requires in addition that $\mathrm{T}_\kappa(\rho)$ is rational. The rationality question will be further addressed in Section 6.

Proposition 4.1. *Let $\rho : \pi_1(\Sigma) \to G$ be κ-weakly maximal and assume that $\mathrm{T}_\kappa(\rho) \neq 0$. Then ρ is injective.*

Proof. Applying Proposition 3.2, we have that

$$f_\kappa \circ \widetilde{\rho} = \lambda(\tau \circ \widetilde{\rho}_h) + \psi,$$

with $\lambda = \frac{\mathrm{T}_\kappa(\rho)}{|\chi(\Sigma)|} \neq 0$. Since $p_\Sigma^{-1}(\ker\rho) = \widetilde{\rho}^{-1}(i(\mathbb{Z}))$, we deduce from $\lambda \neq 0$ that $\tau \circ \widetilde{\rho}_h$ is a homomorphism on $p_\Sigma^{-1}(\ker\rho)$. A formal argument using the central extension (3.2) restricted to $\ker\rho < \pi_1(\Sigma)$, the commutativity of the diagram

$$\begin{tikzcd} \widehat{\Gamma} \arrow[r, "\widetilde{\rho}_h"] \arrow[d, "p_\Sigma"'] & \widetilde{\mathrm{PU}(1,1)} \arrow[d, "p"] \\ \pi_1(\Sigma) \arrow[r, "\rho_h"] & \mathrm{PU}(1,1)\,, \end{tikzcd}$$

and (3.4) implies that

$$(\rho_h|_{\ker\rho})^*(\kappa^{\mathrm{b}}_{\mathrm{PU}(1,1)}) = 0\,.$$

Since this implies that $\rho_h(\ker\rho)$ is elementary (see next lemma), we conclude that $\ker\rho$ is trivial. q.e.d.

We provide a proof of the following easy lemma for ease of reference.

Lemma 4.2. *Let $\pi : \Delta \to \mathrm{PU}(1,1)$ be a homomorphism such that $\pi^*(\kappa^{\mathrm{b}}_{\mathrm{PU}(1,1)}) = 0$. Then $\pi(\Delta)$ is elementary, that is $\pi(\Delta)$ has a finite orbit in $\overline{\mathbb{D}}$.*

Proof. Since $\pi^*(\kappa^{\mathrm{b}}_{\mathrm{PU}(1,1)}) = 0$, we conclude from the exactness of the sequence

$$\mathrm{Hom}(\Delta, \mathbb{R}/\mathbb{Z}) \xrightarrow{\delta} \mathrm{H}^2_{\mathrm{b}}(\Delta, \mathbb{Z}) \longrightarrow \mathrm{H}^2_{\mathrm{b}}(\Delta, \mathbb{R})\ ,$$

that $\pi^*(e^{\mathrm{b}}) = \delta(\chi)$ for some homomorphism $\chi : \Delta \to \mathbb{R}/\mathbb{Z}$. In other words, if we consider χ as a homomorphism into the group of rotations in $\mathrm{PU}(1,1)$, we have that $\chi^*(e^{\mathrm{b}}) = \pi^*(e^{\mathrm{b}})$. In particular, $(\pi|_{[\Delta,\Delta]})^*(e^{\mathrm{b}}) =$

0. Hence, by Ghys' theorem [**23**, Theorem 6.6], $[\Delta,\Delta]$ has a fixed point in $\partial\mathbb{D}$. There are then two cases, depending on whether or not $[\Delta,\Delta]$ has infinitely many fixed points. In the first case $[\pi(\Delta),\pi(\Delta)]$ is trivial, that is $\pi(\Delta)$ is abelian. In the second case $\pi(\Delta)$ has a finite orbit in $\partial\mathbb{D}$. In either cases $\pi(\Delta)$ is elementary. q.e.d.

The following proposition gives a criterion to insure the discreteness of the image of a κ-weakly maximal representation. In Theorem 7.2 we are going to show that such criterion is always satisfied if κ is the bounded Kähler class.

Proposition 4.3. *Let G be a Lie group of Hermitian type and let $\kappa \in \mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})$ be a rational class. Assume that $\rho : \pi_1(\Sigma) \to G$ is κ-weakly maximal and that $\mathrm{T}_\kappa(\rho) \in \mathbb{Q}^\times$. Then ρ has discrete image.*

Remark 4.4. In the case in which $\partial\Sigma = \emptyset$, we have automatically that $\mathrm{T}_\kappa(\rho) \in \mathbb{Q}$, so that in this case the condition just reads $\mathrm{T}_\kappa(\rho) \neq 0$.

Proof. Let $\Gamma := \pi_1(\Sigma)$. It will be enough to show that $\rho([\Gamma,\Gamma])$ is discrete. In fact, $\rho([\Gamma,\Gamma])$ is a normal subgroup of $\rho(\Gamma)$ and, if discrete, also of $\overline{\rho(\Gamma)}$; thus $\rho([\Gamma,\Gamma])$ centralizes the connected component of the identity $\overline{\rho(\Gamma)}^\circ$ of $\overline{\rho(\Gamma)}$. If $\overline{\rho(\Gamma)}^\circ$ were not trivial, we could find $\rho(\gamma) \in \rho(\Gamma) \cap \overline{\rho(\Gamma)}^\circ$ with $\rho(\gamma) \neq e$: indeed $\overline{\rho(\Gamma)}^\circ$ is an open subgroup of $\overline{\rho(\Gamma)}$ and $\rho(\Gamma)$ is dense in $\overline{\rho(\Gamma)}$. But since, by Proposition 4.1, ρ is injective, this would imply that γ centralizes $[\Gamma,\Gamma]$. This is a contradiction, which shows that $\rho(\Gamma)$ is discrete.

To show the discreteness of $\rho([\Gamma,\Gamma])$, we retain the notation of § 3.3 and we define $L := \overline{\rho([\Gamma,\Gamma])}$ and $L' := p^{-1}(L)$. Observe that (3.6) and (3.9) imply that $p^{-1}(\rho[\Gamma,\Gamma]) = \widetilde{\rho}([\widehat{\Gamma},\widehat{\Gamma}])\ker p$. Since p is a covering, then $p^{-1}(\overline{\rho([\Gamma,\Gamma])}) = \overline{p^{-1}(\rho([\Gamma,\Gamma]))}$, so that

$$L' = \overline{\widetilde{\rho}([\widehat{\Gamma},\widehat{\Gamma}])\ker p}\,. \tag{4.1}$$

We apply now the implication (1)⇒(3) in Proposition 3.2 to obtain that

$$f_{n\kappa} \circ \widetilde{\rho} = \lambda(\tau \circ \widetilde{\rho}_h) + \psi\,,$$

where $\lambda = \frac{\mathrm{T}_\kappa(\rho)}{|\chi(\Sigma)|}$ and $\psi \in \mathrm{Hom}(\widehat{\Gamma},\mathbb{R})$. Since $\tau \circ \widetilde{\rho}$ is $\mathbb{Z}$-valued, we deduce that

$$f_{n\kappa}(\widetilde{\rho}([\widehat{\Gamma},\widehat{\Gamma}])) = \lambda\tau(\widetilde{\rho}_h([\widehat{\Gamma},\widehat{\Gamma}])) \subset \lambda\tau(\widetilde{\rho}_h(\widehat{\Gamma})) \subset \lambda\mathbb{Z}\,,$$

which, together with (3.8) implies that

$$f_{n\kappa}(\widetilde{\rho}([\widehat{\Gamma},\widehat{\Gamma}])\ker p) \subset \frac{1}{n}\mathbb{Z} + \lambda\mathbb{Z}\,.$$

Since $\lambda \in \mathbb{Q}$, then $\frac{1}{n}\mathbb{Z} + \lambda\mathbb{Z}$ is discrete in $\mathbb{R}$. Taking into account (4.1), this implies that $f_{n\kappa}(L') \subset \mathbb{R}$ is a discrete subset. Hence, if we denote by $(L')^\circ$ the identity component of L', then

$$f_{n\kappa}|_{(L')^\circ} = 0\,. \tag{4.2}$$

This implies easily that $\kappa|_{L^\circ}$ vanishes as a bounded real class: in fact, because of (4.2) and (3.7),

$$0 = df_{n\kappa}|_{(L')^\circ} = p^*(\kappa)|_{(L')^\circ} = (p|_{(L')^\circ})^*(\kappa|_{L^\circ}),$$

where we used that, since $p : G_{n\kappa} \to G$ is an open map, $p((L')^\circ) = L^\circ$.

Consider now the subgroup $\Delta := \rho^{-1}(\rho([\Gamma,\Gamma]) \cap L^\circ)$ of Γ. Then $(\rho|_\Delta)^*(\kappa|_{L^\circ}) = 0$ and hence, by hypothesis and definition of κ-weak maximality, $\lambda\kappa_\Sigma^{\mathrm{b}}|_\Delta = 0$. Since by hypothesis $\lambda \neq 0$, then it must be that $\kappa_\Sigma^{\mathrm{b}}|_\Delta = 0$. Thus $\rho_h|_\Delta : \Delta \to \mathrm{PU}(1,1)$ satisfies the hypotheses of Lemma 4.2 and hence is elementary. Since $\Delta \lhd \pi_1(\Sigma)$, this implies Δ is trivial and hence $\rho([\Gamma,\Gamma]) \cap L^\circ$ is trivial as well.

But since L is a Lie group (as a Lie subgroup of G), L° is open in L and $\rho([\Gamma,\Gamma]) \cap L^\circ$ is dense in L°. Hence L° is trivial, and thus L, and, consequently, $\rho([\Gamma,\Gamma])$, is discrete. q.e.d.

5. On the radical defined by a bounded class

In this section, given a locally compact group L and a bounded rational class $\kappa \in \mathrm{H}^2_{\mathrm{cb}}(L,\mathbb{R})$, we show the existence of a largest normal closed subgroup $\mathrm{Rad}_\kappa(L)$ on which the restriction of the class vanishes. We show, moreover, that the class κ comes from a bounded real class on the quotient $L/\mathrm{Rad}_\kappa(L)$, the radical of which is trivial. If L is a connected Lie group, the quotient $L/\mathrm{Rad}_\kappa(L)$ is adjoint semisimple without compact factors.

Proposition 5.1. *Let L be a locally compact second countable group and let $\kappa \in \mathrm{H}^2_{\mathrm{cb}}(L,\mathbb{R})$ be a rational class. There is a unique largest normal subgroup $N \lhd L$ with $\kappa|_N = 0$ which, in addition, is closed.*

This relies on the following:

Lemma 5.2. *Let $f : L \to \mathbb{R}$ be a continuous homogeneous quasimorphism.*

1) *There is a unique largest normal subgroup $N_1 \lhd L$ with $f|_{N_1} = 0$.*
2) *There is a unique largest normal subgroup $N_2 \lhd L$ on which f is a homomorphism.*

Both N_1 and N_2 are closed.

Proof. Clearly (2) implies (1) with $N_1 = \ker(f|_{N_2})$.

Let now M_1, M_2 be normal subgroups of L such that $f|_{M_i} : M_i \to \mathbb{R}$ is a homomorphism. For $m_1 \in M_1$ and $m_2 \in M_2$, let $\chi(m_1m_2) := f(m_1) + f(m_2)$. We claim that χ is well defined. Indeed, if $m_1m_2 = m_1'm_2'$ with $m_i, m_i' \in M_i$, then $(m_1')^{-1}m_1 = m_2'm_2^{-1} \in M_1 \cap M_2$. Thus $f((m_1')^{-1}m_1) = f(m_2'm_2^{-1})$, which implies, taking into account that f is a homomorphism on M_1 and M_2, that

$$-f(m_1') + f(m_1) = f(m_2') - f(m_2).$$

This shows that χ is well defined. Next we claim that χ is a homomorphism. If $m_i, m_i' \in M_i$, and since M_1 is normal in L, we have

$$\begin{aligned}\chi((m_1m_2)(m_1'm_2')) =&\chi(m_1(m_2m_1'm_2^{-1})m_2m_2')\\ =&f(m_1(m_2m_1'm_2^{-1})) + f(m_2m_2')\\ =&f(m_1) + f(m_2m_1'm_2^{-1}) + f(m_2) + f(m_2').\end{aligned}$$

Since f is a homogeneous quasimorphism, we have $f(m_2m_1'm_2^{-1}) = f(m_1')$, which implies that $\chi : M_1\, M_2 \to \mathbb{R}$ is a homomorphism. Since f is a quasimorphism, we have, in particular, that for all $m_i \in M_i$

$$|f(m_1m_2) - \chi(m_1m_2)| = |f(m_1m_2) - f(m_1) - f(m_2)| \le C\,,$$

for some constant C. Thus the homogeneous quasimorphism $f|_{M_1M_2}$ is at finite distance from the homomorphism χ and hence $f|_{M_1M_2} = \chi$.

This shows the existence of a unique largest normal subgroup $N_2 \lhd L$ on which f is a homomorphism. By continuity of f, the subgroup N_2 is closed. q.e.d.

Proof of Proposition 5.1. Let

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ i\ } L_{n\kappa} \xrightarrow{\ p\ } L \longrightarrow e$$

be the topological central extension in (3.6) and $f_{n\kappa} : L_{n\kappa} \to \mathbb{R}$ the continuous homogeneous quasimorphism such that $[df_{n\kappa}] = p^*(\kappa) \in \mathrm{H}^2_{\mathrm{cb}}(L_{n\kappa}, \mathbb{R})$. It is an easy verification that, given a subgroup $N < L$, the property $\kappa|_N = 0$ is equivalent to the property that $f_{n\kappa}|_{p^{-1}(N)}$ is a homomorphism. Together with Lemma 5.2, this concludes the proof of the proposition. q.e.d.

Definition 5.3. Let $\kappa \in \mathrm{H}^2_{\mathrm{cb}}(L, \mathbb{R})$ be a rational class. Denote by $\mathrm{Rad}_\kappa(L)$ the normal closed subgroup of L given by Proposition 5.1 and call it the *κ-radical* of L.

Corollary 5.4. *Let L be a locally compact second countable group and let $\kappa \in \mathrm{H}^2_{\mathrm{cb}}(L, \mathbb{R})$ be a rational class.*

1) $\mathrm{Rad}_\kappa(L) \supset \mathrm{Rad}_a(L)$, *where* $\mathrm{Rad}_a(L)$ *denotes the amenable radical of L.*
2) *Let*

$$\pi : L \to L/\mathrm{Rad}_\kappa(L) \tag{5.1}$$

denote the canonical projection. Then there is a unique class $u \in \mathrm{H}^2_{\mathrm{cb}}(L/\mathrm{Rad}_\kappa(L), \mathbb{R})$ with $\pi^*(u) = \kappa$. The restriction of u to any non-trivial closed normal subgroup of $L/\mathrm{Rad}_\kappa(L)$ does not vanish.

Proof. (1) follows from the fact that the bounded cohomology of an amenable group vanishes.

(2) The first assertion follows from the exactness of the short sequence

$$0 \longrightarrow \mathrm{H}^2_{\mathrm{cb}}(L/N, \mathbb{R}) \longrightarrow \mathrm{H}^2_{\mathrm{cb}}(L, \mathbb{R}) \longrightarrow \mathrm{H}^2_{\mathrm{cb}}(N, \mathbb{R}) \,,$$

where $N \lhd L$ is any closed normal subgroup [**15**, Theorem 4.1.1]. The second assertion follows from the first and the maximality of $\mathrm{Rad}_\kappa(L)$. q.e.d.

We denote by $\hat{\mathrm{H}}^2_{\mathrm{cb}}(L, \mathbb{Z})$ the cohomology of the complex of integer valued bounded Borel cochains on L. If $\varkappa \in \hat{\mathrm{H}}^2_{\mathrm{cb}}(L, \mathbb{Z})$, we denote by $\varkappa^{\mathbb{R}}$ the image of $\varkappa$ in $\mathrm{H}^2_{\mathrm{cb}}(L, \mathbb{R})$.

The above discussion applied to a general connected Lie group L has the following nice consequences:

Corollary 5.5. *Let L be a connected Lie group and $\varkappa \in \hat{\mathrm{H}}^2_{\mathrm{cb}}(L, \mathbb{Z})$. Then $H := L/\mathrm{Rad}_{\varkappa^{\mathbb{R}}}(L)$ is connected adjoint of Hermitian type and a direct product $H = H_1 \times \cdots \times H_n$ of simple non-compact factors.*

Proof. Since the quotient $L/\mathrm{Rad}_a(L)$ of L by its amenable radical is adjoint semisimple without compact factors, so is H (Corollary 5.4(1)). Let $H = H_1 \times \cdots \times H_n$ be the direct product decomposition into simple factors. Let $u \in \mathrm{H}^2_{\mathrm{cb}}(H, \mathbb{R})$ be such that $\pi^*(u) = \varkappa^{\mathbb{R}}$, where $\pi : L \to H$ is the projection in (5.1). According to Corollary 5.4(2), $u|_{H_j} \neq 0$ and hence $\mathrm{H}^2_{\mathrm{cb}}(H_j, \mathbb{R}) \neq 0$: thus H_j is of Hermitian type for every $1 \leq j \leq n$ and hence so is H. q.e.d.

The following definition will be needed in the next section:

Definition 5.6. Let L be a connected Lie group which admits a closed Levi factor and $\varkappa \in \hat{\mathrm{H}}^2_{\mathrm{cb}}(L, \mathbb{Z})$. A *$\varkappa$-Levi factor* of L is a connected semisimple subgroup S with finite center such that $\pi(S) = L/\mathrm{Rad}_\kappa(L)$, where π is as in (5.1), and $\ker(\pi|_S)$ is the center $Z(S)$ of S.

If L_0 is a Levi factor of L, a $\varkappa$-Levi factor of L is nothing but the product of the almost simple factors in L whose image via π is non-trivial.

6. Rationality questions

The class u in Corollary 5.4(2) is *a priori* only a real class. We show, in this section, that if L is a closed subgroup of a group G of Hermitian type and κ is the restriction of a bounded rational class on G, then u has nice integrality properties.

If, in particular, $\kappa = \kappa^{\mathrm{b}}_G|_L$ is the restriction of the bounded Kähler class, then the class u is the linear combination of the bounded Kähler classes of the individual simple factors of $L/\mathrm{Rad}_{\kappa^{\mathrm{b}}_G}(L)$ with rational coefficients whose denominators are bounded by an integer depending only on G.

If G is a Lie group of Hermitian type, $L < G$ a closed subgroup and $\varkappa \in \hat{\mathrm{H}}^2_{\mathrm{cb}}(G,\mathbb{Z})$, we denote by $\varkappa^{\mathbb{R}}_L$ the restriction of $\varkappa^{\mathbb{R}}$ to L.

Proposition 6.1. *Let G be a Lie group of Hermitian type and let $L < G$ be a closed connected subgroup that admits a closed Levi factor S. Let $\varkappa \in \hat{\mathrm{H}}^2_{\mathrm{cb}}(G,\mathbb{Z})$ and let $\pi : L \to H := L/\mathrm{Rad}_{\varkappa^{\mathbb{R}}_L}(L)$ denote the canonical projection. Let $p_j : H \to H_j$ be the projection onto the simple factors of H (see Corollary 5.5) and let $\varkappa_{H_j}$ be a generator of $\hat{\mathrm{H}}^2_{\mathrm{cb}}(H_j,\mathbb{Z})$, for $j = 1,\dots,n$. If $u \in \mathrm{H}^2_{\mathrm{cb}}(H,\mathbb{R})$ is such that $\pi^*(u) = \varkappa^{\mathbb{R}}_L$ (see Corollary 5.4(2)), then*

$$u = \sum_{j=1}^{n} \lambda_j p_j^*(\varkappa^{\mathbb{R}}_{H_j}), \tag{6.1}$$

with

$$\lambda_j \in \frac{1}{|Z(S)|}\mathbb{Z}\,, \tag{6.2}$$

where $Z(S)$ denotes the center of the $\varkappa$-Levi factor of L.

Proof. According to [**13**, Proposition 7.7 (3)], the set $\{p_j^*(\varkappa_{H_j}) : 1 \le j \le n\}$ is a basis of $\hat{\mathrm{H}}^2_{\mathrm{cb}}(H,\mathbb{Z})$ corresponding to the basis $\{p_j^*(\varkappa^{\mathbb{R}}_{H_j}) : 1 \le j \le n\}$ of $\mathrm{H}^2_{\mathrm{cb}}(H,\mathbb{R})$. It follows that:

$$u = \sum_{j=1}^{n} \lambda_j p_j^*(\varkappa^{\mathbb{R}}_{H_j})\,,$$

with $\lambda_j \in \mathbb{R}$.

In order to show that the λ_j are rational with an universal bound on the denominator, we consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}^2_{\mathrm{cb}}(H,\mathbb{R}) \arrow[r, "(\pi|_S)^*_{\mathbb{R}}"] \arrow[rr, bend left, "\pi^*"] & \mathrm{H}^2_{\mathrm{cb}}(S,\mathbb{R}) & \mathrm{H}^2_{\mathrm{cb}}(L,\mathbb{R}) \arrow[l, "\mathrm{Res}"'] \\
\hat{\mathrm{H}}^2_{\mathrm{cb}}(H,\mathbb{Z}) \arrow[u] \arrow[r, "(\pi|_S)^*_{\mathbb{Z}}"'] & \hat{\mathrm{H}}^2_{\mathrm{cb}}(S,\mathbb{Z}) \arrow[u] & \hat{\mathrm{H}}^2_{\mathrm{cb}}(L,\mathbb{Z})\,, \arrow[u] \arrow[l, "\mathrm{Res}"]
\end{tikzcd} \tag{6.3}$$

where Res is the restriction map in cohomology and the vertical arrows are the change of coefficients.

If $(\pi|_S)^*_{\mathbb{Z}}$ were surjective, again [**13**, Proposition 7.7 (3)], the commutativity of (6.3) and the fact that $(\pi|_S)^*_{\mathbb{R}}$ is an isomorphism would readily imply the integrality of the λ_j. This is, however, not necessarily true and the following lemma identifies explicitly the nature of the map $(\pi|_S)^*_{\mathbb{Z}}$.

Lemma 6.2. *Let $\omega : S \to H$ be a surjective homomorphism between connected semisimple Lie groups with finite center. Then the map*

$$\omega_{\mathbb{Z}}^* : \hat{\mathrm{H}}^2_{\mathrm{cb}}(H,\mathbb{Z}) \longrightarrow \hat{\mathrm{H}}^2_{\mathrm{cb}}(S,\mathbb{Z})$$

is injective and

$$\mathrm{Image}(\omega_{\mathbb{Z}}^*) \supset \ |\ker\omega|\,\hat{\mathrm{H}}^2_{\mathrm{cb}}(S,\mathbb{Z})\,,$$

where $|\ker\omega|$ denotes the cardinality of $\ker\omega$ and $\hat{\mathrm{H}}^2_{\mathrm{cb}}(S,\mathbb{Z})$ is considered as a $\mathbb{Z}$-module.

We postpone the proof of the lemma and use its conclusion with $\omega = \pi|_S$. If $Z(S) = \ker(\pi|_S)$ denotes the center of S, the same argument as above, applied to $|Z(S)|\varkappa_S^{\mathbb{R}} \in (\pi|_S)_{\mathbb{Z}}^*(\hat{\mathrm{H}}^2_{\mathrm{cb}}(H,\mathbb{Z}))$ shows that $|Z(S)|\,u$ is, in fact, in $\hat{\mathrm{H}}^2_{\mathrm{cb}}(H,\mathbb{Z})$ and hence its coordinates $|Z(S)|\,\lambda_j$ are integers, $j = 1,\dots,n$. q.e.d.

Proof of Lemma 6.2. If M is any connected semisimple Lie group with finite center and maximal compact K_M, Wigner's theorem asserts that the restriction map

$$\hat{\mathrm{H}}^2_{\mathrm{c}}(M,\mathbb{Z}) \longrightarrow \hat{\mathrm{H}}^2_{\mathrm{c}}(K_M,\mathbb{Z})$$

is an isomorphism [**44**]. This, together with the fact that the comparison map

$$\hat{\mathrm{H}}^2_{\mathrm{cb}}(M,\mathbb{Z}) \longrightarrow \hat{\mathrm{H}}^2_{\mathrm{c}}(M,\mathbb{Z})$$

is an isomorphism [**13**, Proposition 7.7], implies the isomorphism

$$j : \hat{\mathrm{H}}^2_{\mathrm{cb}}(M,\mathbb{Z}) \longrightarrow \hat{\mathrm{H}}^2_{\mathrm{c}}(K_M,\mathbb{Z})\,.$$

Using now the long exact sequence in cohomology associated to the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0\,,$$

and the fact that the terms of positive degree and real coefficients vanish, we obtain that the connecting homomorphism

$$\mathrm{Hom}_{\mathrm{c}}(K_M,\mathbb{R}/\mathbb{Z}) \xrightarrow{\delta_{K_M}} \hat{\mathrm{H}}^2_{\mathrm{c}}(K_M,\mathbb{Z})$$

is an isomorphism.

Given $\omega : S \to H$ as in the statement of the lemma, let $K_H < H$ be a maximal compact subgroup. Then $K_S := \omega^{-1}(K_H)$ is a maximal compact in S. One verifies then that the diagram

$$\begin{tikzcd}
\hat{\mathrm{H}}^2_{\mathrm{cb}}(H,\mathbb{Z}) \arrow[r, "\omega_{\mathbb{Z}}^*"] \arrow[d, "\delta_{K_H}^{-1}\circ j"'] & \hat{\mathrm{H}}^2_{\mathrm{cb}}(S,\mathbb{Z}) \arrow[d, "\delta_{K_S}^{-1}\circ j"] \\
\mathrm{Hom}_{\mathrm{c}}(K_H,\mathbb{R}/\mathbb{Z}) \arrow[r, "\omega^*"'] & \mathrm{Hom}_{\mathrm{c}}(K_S,\mathbb{R}/\mathbb{Z})\,,
\end{tikzcd}$$

where the vertical arrows are isomorphisms and the bottom one is the precomposition with $\omega|_{K_S}$, is commutative. Then the assertion of the lemma follows from the fact that $\ker\omega \subset K_S$ and that $\omega|_{K_S} : K_S \to K_H$ is surjective. q.e.d.

Recall that the bounded Kähler class $\kappa_G^{\mathrm{b}} \in \mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})$ of a Lie group G of Hermitian type is rational [**7**, § 5].

Definition 6.3. Let G be a Lie group of Hermitian type and $\kappa_G^{\mathrm{b}} \in \mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})$ its bounded Kähler class. The *Kähler radical* $\mathrm{Rad}_{\kappa_G^{\mathrm{b}}}(L)$ of a closed subgroup $L < G$ is the $\kappa_G^{\mathrm{b}}|_L$-radical of L.

If L admits a closed Levi factor, a *Kähler–Levi factor* of L is a $\kappa_G^{\mathrm{b}}|_L$-Levi factor.

Corollary 6.4. *Let G be a Lie group of Hermitian type and let $L < G$ be a closed connected subgroup that admits a closed Levi factor. Then the group $H := L/\mathrm{Rad}_{\kappa_G^{\mathrm{b}}}(L)$ is connected adjoint of Hermitian type, and admits a direct product decomposition $H = H_1 \times \cdots \times H_n$ into simple non-compact factors.*

Moreover, if $\pi : L \to H$ and $p_j : H \to H_j$ are the canonical projections, then there exists an integer $\ell_G \geq 1$ depending only on G such that

$$\kappa_G^{\mathrm{b}}|_L = \pi^* \left(\sum_{j=1}^n \nu_j p_j^*(\kappa_{H_j}^{\mathrm{b}}) \right) ,$$

where $\nu_j \in \frac{1}{\ell_G}\mathbb{Z}$.

Remark 6.5. Let H be a Lie group of Hermitian type. We denote by q_H the smallest integer such that there exists $\varkappa \in \mathrm{H}^2_{\mathrm{cb}}(H,\mathbb{Z})$ with $q_H\kappa_H^{\mathrm{b}} = \varkappa^{\mathbb{R}}$. If in addition H is simple and $\varkappa_H$ is a generator of $\mathrm{H}^2_{\mathrm{cb}}(H,\mathbb{Z})$, then there exists an integer m_H such that $\varkappa = m_H\varkappa_H$. It follows that:

$$\varkappa_H^{\mathbb{R}} = \frac{q_H}{m_H}\kappa_H^{\mathrm{b}} . \tag{6.4}$$

Proof of Corollary 6.4. We apply Proposition 6.1 to $\varkappa \in \mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{Z})$ such that $\varkappa^{\mathbb{R}} = q_G\kappa_G^{\mathrm{b}}$. Then (6.1) and (6.2), together with (6.4), show that

$$\kappa_G^{\mathrm{b}}|_L = \pi^* \left(\sum_{j=1}^n \nu_j p_j^*(\kappa_{H_j}^{\mathrm{b}}) \right) ,$$

with

$$\nu_j \in \frac{q_{H_j}}{q_G m_{H_j}|Z(S)|}\mathbb{Z} ,$$

where $|Z(S)|$ is the cardinality of the center of a Kähler–Levi factor S.

Since there are only finitely many possible conjugacy classes of connected semisimple subgroups of G, we obtain the result. q.e.d.

7. Structure of weakly maximal representations

In § 4 the discreteness of a κ-weakly maximal representation was proven under the assumption that the Toledo invariant $\mathrm{T}_\kappa(\rho)$ is rational. In this section, we prove that if κ is the bounded Kähler class, this is always the case and that the representation into the quotient of its Zariski closure by the Kähler radical is also discrete and injective. The definition and properties of the Kähler radical will be essential to show that the Toledo invariant of the representation into the quotient is also non-zero.

An interesting feature of the proof of the rationality of the Toledo invariant is that it depends upon showing first that the single factors of the quotient by the Kähler radical are of tube type.

Let G be a Lie group of Hermitian type and $\kappa_G^{\mathrm{b}} \in \mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})$ its bounded Kähler class. For ease of notation, we denote the Toledo invariant $\mathrm{T}_{\kappa_G^{\mathrm{b}}}$ by

$$\mathrm{T} : \mathrm{Hom}(\pi_1(\Sigma), G) \to \mathbb{R}\,.$$

Definition 7.1. A homomorphism $\rho : \pi_1(\Sigma) \to G$ is weakly maximal if

$$\mathrm{T}(\rho) = 2\|\rho^*(\kappa_G^{\mathrm{b}})\|\,|\chi(\Sigma)|\,,$$

that is if ρ is κ_G^{b}-weakly maximal in the sense of Definition 2.1.

Theorem 7.2. *Let $G = \mathbf{G}(\mathbb{R})^\circ$ be of Hermitian type, where $\mathbf{G}$ is a connected semisimple algebraic group defined over $\mathbb{R}$, and let $\rho : \pi_1(\Sigma) \to G$ be a weakly maximal homomorphism. Let L be the connected component of the real points of the Zariski closure of the image of ρ and let $\Gamma := \rho^{-1}(L)$. Assume that $\mathrm{T}(\rho) \neq 0$. Then:*

1) *the group $L/\mathrm{Rad}_{\kappa_G^{\mathrm{b}}}(L)$ is Hermitian of tube type;*
2) *there is an integer $\ell_G \geq 1$ depending only on G (see Corollary 6.4), such that*
$$\mathrm{T}(\rho) \in \frac{|\chi(\Sigma)|}{\ell_G}\mathbb{Z}\,;$$
3) *the composition*
$$\Gamma \xrightarrow{\rho|_\Gamma} L \xrightarrow{\pi} L/\mathrm{Rad}_{\kappa_G^{\mathrm{b}}}(L)$$
is injective with discrete image.

Using that in a surface group there are no finite normal subgroups, one obtains immediately:

Corollary 7.3. *Let $\rho : \pi_1(\Sigma) \to G$ be a weakly maximal homomorphism with $\mathrm{T}(\rho) \neq 0$. Then ρ is injective with discrete image.*

An important role in the proof is played by the generalized Maslov cocycle. Recall that if H is a connected Lie group of Hermitian type and

$\check{S}$ the Shilov boundary of the associated bounded symmetric domain, the *generalized Maslov cocycle* $\beta_{\check{S}} : \check{S}^3 \to \mathbb{R}$ is a bounded alternating H-invariant cocycle constructed by J.-L. Clerc in [**18**]. We will use it in two ways: on the one hand it represents the bounded Kähler class in a particular resolution useful to implement pullbacks in bounded cohomology (see (7.2)); on the other, through the relation (7.4) with the Hermitian triple product

$$\langle\langle \cdot, \cdot, \cdot \rangle\rangle : \check{S}^{(3)} \to \mathbb{R}^\times \backslash \mathbb{C}^\times ,$$

it gives a criterion to detect when H is of tube type, [**7**, **11**].

We refer the reader to [**6**, **7**] and [**11**, § 4.2] for details.

Proof of Theorem 7.2. We use heavily in this proof techniques developed in [**15**, **6**, **7**, **11**], to which we refer the reader for details.

According to Corollary 6.4, the group $H := L/\mathrm{Rad}_{\kappa^{\mathrm{b}}_G}(L) = H_1 \times \cdots \times H_n$ is of Hermitian type and $\kappa^{\mathrm{b}}_G|_L = \pi^*(u)$, where

$$u = \sum_{j=1}^{n} \nu_j p_j^*(k^{\mathrm{b}}_{H_j}) ,$$

and $p_j : H \to H_j$ and $\pi : L \to H$ are the canonical projections. We set aside for the moment that the ν_j are rational, and will pick it up towards the end of the proof.

Let $\xi := \pi \circ \rho|_\Gamma : \Gamma \to L \to H$ be the composition of π with $\rho|_\Gamma$. Observe that Γ is of finite index in $\pi_1(\Sigma)$. It follows that, since ρ is weakly maximal, that is $\rho^*(\kappa^{\mathrm{b}}_G) = \lambda\kappa^{\mathrm{b}}_\Sigma$ for $\lambda = \frac{\mathrm{T}(\rho)}{|\chi(\Sigma)|}$, then

$$\xi^*(u) = \lambda \kappa^{\mathrm{b}}_{\Sigma'} , \tag{7.1}$$

where $\Sigma' \to \Sigma$ is the finite covering corresponding to $\Gamma < \pi_1(\Sigma)$.

As usual, to realize the pullback ξ^* in bounded cohomology, we use boundary maps. According to [**6**], this is possible since if $(\mathcal{B}^\infty_{\mathrm{alt}}(\check{S}^\bullet))$ denotes the complex of bounded alternating Borel cocycles on $\check{S}$, then the class $[\beta_{\check{S}}]$ defined by the generalized Maslov cocycle corresponds to the bounded Kähler class κ^{b}_H under the canonical map

$$\mathrm{H}^\bullet(\mathcal{B}^\infty_{\mathrm{alt}}(\check{S}^\bullet)^H) \longrightarrow \mathrm{H}^\bullet_{\mathrm{cb}}(H, \mathbb{R}) . \tag{7.2}$$

Likewise, if $\check{S} = \check{S}_1 \times \cdots \times \check{S}_n$ is the decomposition into a product, where $\check{S}_j$ is the Shilov boundary of H_j and $\check{p}_j : \check{S} \to \check{S}_j$ is the projection, a standard cohomological argument shows that the diagram

$$\begin{array}{ccc} \mathrm{H}^\bullet_{\mathrm{cb}}(H_j, \mathbb{R}) & \xrightarrow{p_j^*} & \mathrm{H}^\bullet_{\mathrm{cb}}(H, \mathbb{R}) \\ \uparrow & & \uparrow \\ \mathrm{H}^\bullet(\mathcal{B}^\infty_{\mathrm{alt}}(\check{S}_j^\bullet)^{H_j}) & \xrightarrow[\check{p}_j^*]{} & \mathrm{H}^\bullet_{\mathrm{cb}}(\mathcal{B}^\infty_{\mathrm{alt}}(\check{S}^\bullet)^H) \end{array}$$

commutes. It follows that u is represented, again via (7.2), by the bounded Borel cocycle on $\check{S}^3$ defined by

$$(x,y,z)\longmapsto\sum_{j=1}^{n}\nu_j\beta_{\check{S}_j}(x_j,y_j,z_j)\,.$$

To recall the existence of the boundary map, endow the interior of Σ' with a complete hyperbolic metric of finite area so that Γ is identified with a lattice in $\mathrm{PU}(1,1)$. Since $\xi(\Gamma)$ is Zariski dense in H, there is a Γ-equivariant measurable map $\varphi:\partial\mathbb{D}\to\check{S}$. Because of the product structure of $\check{S}$, the map φ has the form

$$\varphi_1\times\cdots\times\varphi_n:\partial\mathbb{D}\longrightarrow\check{S}_1\times\cdots\times\check{S}_n\,,$$

where $\varphi_j:\partial\mathbb{D}\to\check{S}_j$ is measurable and equivariant with respect to $p_j\circ\xi$.

Consequently, according to [**6**] the pullback $\xi^*(u)$ is represented by the following bounded measurable Γ-invariant cocycle:

$$(x,y,z)\longmapsto\sum_{j=1}^{n}\nu_j\beta_{\check{S}_j}(\varphi_j(x),\varphi_j(y),\varphi_j(z))\,,$$

for almost all $(x,y,z)\in(\partial\mathbb{D})^3$.

Using now again that the bounded fundamental class of Σ is represented by $\beta_{\partial\mathbb{D}}$, it follows from (7.1) that:

$$\lambda\beta_{\partial\mathbb{D}}(x,y,z)=\sum_{j=1}^{n}\nu_j\beta_{\check{S}_j}(\varphi_j(x),\varphi_j(y),\varphi_j(z)),\tag{7.3}$$

for almost all $(x,y,z)\in(\partial\mathbb{D})^3$.

Recall now that the Hermitian triple product is an H-invariant real algebraic map on $(\check{S})^3$ whose relation with the Maslov cocycle is given by

$$\langle\langle x,y,z\rangle\rangle_{\check{S}}\equiv e^{i\pi d_{\check{S}}\beta_{\check{S}}(x,y,z)}\mod\mathbb{R}^\times\,,\tag{7.4}$$

where $d_{\check{S}}$ is an integer given in terms of the root system of H, [**11**]. We will use this relation for the Hermitian triple product on each of the H_j. To uniformize the exponents, since $\nu_j\in\mathbb{Q}^\times$, we can pick an integer $m\geq1$ such that $m\nu_j=n_jd_j$ for some $n_j\in\mathbb{Z}$, $1\leq j\leq n$. Multiplying (7.3) by m, exponentiating and using that $\beta_{\partial\mathbb{D}}$ takes only $\pm1/2$ as values, we obtain

$$e^{i\pi m\lambda(\pm1/2)}=\prod_{j=1}^{n}\langle\langle\varphi_j(x),\varphi_j(y),\varphi_j(z)\rangle\rangle_{\check{S}_j}^{n_j}\,,\tag{7.5}$$

for almost every $(x,y,z)\in(\partial\mathbb{D})^3$. If $x_j,y_j,z_j\in\check{S}$, we define now

$$R((x_j),(y_j),(z_j)):=\prod_{j=1}^{n}\langle\langle x_j,y_j,z_j\rangle\rangle_{\check{S}_j}^{n_j}\,.$$

Because of (7.5) the function R takes at most two values on $\varphi(\partial\mathbb{D})^3$; since the latter set is Zariski dense in $\check{S}^3$ and R is rational, it takes on at most two values on $\check{S}^3$. The factors $\langle\langle x_j, y_j, z_j\rangle\rangle_{\check{S}_j}^{n_j}$ are pairwise independent rational functions, therefore, each of these factors itself can take only finitely many values. As previously recalled, it follows from [**11**] that the corresponding groups are of tube type, thus showing assertion (1).

Furthermore, since $\beta_{\check{S}_j}$ takes only values in $\frac{1}{2}\mathbb{Z}$ [**19**, Theorem 4.3], it follows from (7.3) and Corollary 6.4 that $\lambda \in \frac{1}{\ell_G}\mathbb{Z}$. This shows assertion (2).

The third assertion follows from Proposition 4.1 (2) and Proposition 4.3. q.e.d.

8. Weakly maximal representations and relations with other representation varieties

In this section, we first show that the set of weakly maximal representations and the set of weakly maximal representations with non-zero Toledo number are closed. Then we examine the relationship of weakly maximal representations with Shilov–Anosov representations.

Returning to our general framework in § 3, let G be locally compact and second countable, Γ a discrete group and $\kappa \in \mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})$ a fixed class. We define now a (Hausdorff) topology on $\mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathbb{R})$ with respect to which the map

$$\begin{aligned} \mathrm{Hom}(\Gamma, G) &\to \mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathbb{R}), \\ \rho \quad &\longmapsto \rho^*(\kappa) \end{aligned} \tag{8.1}$$

will be continuous. To this purpose recall that $\mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathbb{R}) := \ker\delta^2/\operatorname{im}\delta^1$, where

$$0 \longrightarrow \mathbb{R} \longrightarrow \ell^\infty(\Gamma) \xrightarrow{\delta^1} \ell^\infty(\Gamma^2) \xrightarrow{\delta^2} \ell^\infty(\Gamma^3) \xrightarrow{\delta^3} \dots$$

denotes the inhomogeneous bar resolution. If we endow each $\ell^\infty(\Gamma^n)$ with the weak-$*$ topology as the dual of $\ell^1(\Gamma^n)$, then:

Lemma 8.1. $\mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathbb{R})$ *is a Hausdorff topological vector space with the quotient weak-$*$ topology.*

Proof. It is clear that $\ker\delta^2$ is weak-$*$ closed. Since $\mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathbb{R})$ is a Banach space, $\operatorname{im}\delta^1$ is closed and hence, by [**40**, Theorem 4.14], $\operatorname{im}\delta^1$ is weak-$*$ closed as well. q.e.d.

Lemma 8.2. *The function in* (8.1) *is continuous with respect to the weak-$*$ topology on* $\mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathbb{R})$.

Proof. Let $c_\kappa : G^2 \to \mathbb{R}$ be a bounded continuous inhomogeneous cocycle representing κ. Let $(\rho_n)_{n\geq 1}$ be a sequence of elements in $\mathrm{Hom}(\Gamma, G)$

converging to ρ. Setting

$$\begin{aligned} c_n(x,y) &:= c_\kappa(\rho_n(x),\rho_n(y))\,,\\ c(x,y) &:= c_\kappa(\rho(x),\rho(y))\,,\end{aligned}$$

we have that $c_n \in \ell^\infty(\Gamma^2)$ (respectively, $c \in \ell^\infty(\Gamma^2)$) represent $\rho_n^*(\kappa)$ (respectively, $\rho^*(\kappa)$). In addition

1) $c_n \to c$ pointwise on Γ^2, and
2) $\|c_n\|_\infty$ and $\|c\|_\infty$ are bounded by $\|c_\kappa\|_\infty$.

Then if $f \in \ell^1(\Gamma^2)$, we have that

1) $fc_n \to fc$ pointwise, and
2) $|f(x,y)c_n(x,y)| \le |f(x,y)|\,\|\kappa\|_\infty$,

then the Dominated Convergence Theorem implies that

$$\int_{\Gamma^2} fc_n \to \int_{\Gamma^2} fc$$

and shows, therefore, that $\rho_n^*(\kappa) \to \rho^*(\kappa)$ for the quotient topology in $\mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathbb{R})$. q.e.d.

If now $\Gamma = \pi_1(\Sigma)$, then

Corollary 8.3. *The set* $\mathrm{Hom}_{wm}(\pi_1(\Sigma),G)$ *of weakly maximal representations is closed in* $\mathrm{Hom}(\pi_1(\Sigma),G)$.

Proof. It follows from Proposition 3.2 that:

$$\mathrm{Hom}_{wm}(\pi_1(\Sigma),G) = \{\rho \in \mathrm{Hom}(\pi_1(\Sigma),G) : \rho^*(\kappa) = t\kappa_\Sigma^{\mathrm{b}} \text{ for some } t \in [0,\infty)\}\,.$$

Since $\mathrm{H}^2_{\mathrm{cb}}(\pi_1(\Sigma),\mathbb{R})$ with the weak-$*$ topology is Hausdorff, the subset $\{t\kappa_\Sigma^{\mathrm{b}} : t \in [0,\infty)\}$ is closed. The assertion follows then from the continuity of the map $\rho \mapsto \rho^*(\kappa)$. q.e.d.

Recall now that

$$\mathrm{Hom}^*_{wm}(\pi_1(\Sigma),G) := \{\rho : \pi_1(\Sigma) \to G : \rho \text{ is weakly maximal and } \mathrm{T}(\rho) \neq 0\}\,.$$

Corollary 8.4. *Let G be a Lie group of Hermitian type. Then the set of weakly maximal representations with non-zero Toledo invariant is closed in* $\mathrm{Hom}(\pi_1(\Sigma),G)$.

Proof. By Theorem 7.2(2) we have

$$\mathrm{Hom}^*_{wm}(\pi_1(\Sigma),G) = \left\{\rho \in \mathrm{Hom}_{wm}(\pi_1(\Sigma),G) : \mathrm{T}(\rho) \ge \frac{|\chi(\Sigma)|}{l_G}\right\}\,,$$

which implies the claim since $\rho \mapsto \mathrm{T}(\rho)$ is continuous, [**13**, Proposition 3.10], and $\mathrm{Hom}_{wm}(\pi_1(\Sigma),G)$ is closed. q.e.d.

Finally, we turn to the relation with Anosov representations. Let $\partial\Sigma = \emptyset$ and realize $\pi_1(\Sigma)$ as cocompact lattice Γ in $\mathrm{PU}(1,1)$ via a hyperbolization. If G is, say, simple of tube type with Shilov boundary $\check{S}$, then a Shilov–Anosov representation $\rho : \Gamma \to G$ implies the existence of a (unique) continuous equivariant map $\varphi : \partial\mathbb{D} \to \check{S}$ with the additional property that for every $x, y \in \partial\mathbb{D}$ with $x \neq y$, $\varphi(x)$ and $\varphi(y)$ are transverse. Let $(\partial\mathbb{D})^{3,+}$ be the connected set of distinct, positively oriented triples in $(\partial\mathbb{D})^3$ and $\check{S}^{(3)}$ the set of triples of pairwise transverse points in $\check{S}^3$. Then

$$\varphi \times \varphi \times \varphi : (\partial\mathbb{D})^{3,+} \to \check{S}^{(3)}$$

must send $(\partial\mathbb{D})^{3,+}$ into a connected component of $\check{S}^{(3)}$. Thus if $\beta_{\check{S}} : \check{S}^3 \to \mathbb{R}$ denotes the generalized Maslov cocycle, $\beta_{\check{S}} \circ \varphi^3$ is a multiple of the orientation cocycle. This and Proposition 3.2 imply that the set

$$\mathrm{Hom}^+_{\check{S}\text{-}An}(\pi_1(\Sigma), G)^\circ := \{\rho : \pi_1(\Sigma) \to G : \ \rho \text{ is Shilov–Anosov with } \mathrm{T}(\rho) \geq 0\}$$

is contained in $\mathrm{Hom}_{wm}(\pi_1(\Sigma), G)$. Taking into account that the latter is closed, we get

Corollary 8.5.

$$\overline{\mathrm{Hom}^+_{\check{S}\text{-}An}(\pi_1(\Sigma), G)} \subset \mathrm{Hom}_{wm}(\pi_1(\Sigma), G)\,. \tag{8.2}$$

9. Examples

In this section, we describe some examples of weakly maximal representations.

9.1. Maximal representations. Any maximal representation is a weakly maximal representation. Concrete examples of maximal representations are described in [**13**, **9**, **31**].

9.2. Embeddings of $\mathrm{SL}(2,\mathbb{R})$**.** Special examples of weakly maximal representations arise from embeddings of $\mathrm{SL}(2,\mathbb{R})$. Consider a faithful representation $\rho_0 : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{R})$ with discrete image, and let $\tau : \mathrm{SL}(2,\mathbb{R}) \to G$ be an injective homomorphism into a Lie group of Hermitian type G. Then the composition $\tau \circ \rho_0 : \pi_1(\Sigma) \to G$ is a weakly maximal representation.

More generally if H, G are Lie groups of Hermitian type, and $\tau : H \to G$ is a homomorphism such that $\tau^*(\kappa^{\mathrm{b}}_G)$ is a multiple of κ^{b}_H. Then the composition of a weakly maximal representation $\rho : \pi_1(\Sigma) \to H$ with τ is a weakly maximal representation into G.

9.3. Shilov–Anosov representations. We explained in § 8 that any Shilov–Anosov representation into a Hermitian Lie group G of tube type with non-negative Toledo number is weakly maximal. Here we just give an example of such a representation into $\mathrm{Sp}(2n,\mathbb{R})$. Let again

$\rho_i : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{R})$ $i = 1,\ldots,n$ be faithful representations with discrete image. Consider the embedding $\tau : \mathrm{SL}(2,\mathbb{R})^n \to \mathrm{Sp}(2n,\mathbb{R})$ corresponding to a maximal polydisc in the bounded symmetric domain realization of $\mathrm{Sp}(2n,\mathbb{R})$. Then $\rho := \tau \circ (\rho_1,\cdots,\rho_n) : \pi_1(\Sigma) \to \mathrm{Sp}(2n,\mathbb{R})$ is a Shilov–Anosov representation, and hence a weakly maximal representations. Since the set of Shilov–Anosov representations is open, any small deformation of such a representation is also weakly maximal. Similarly to the construction in [**9**] one can explicitly construct bending deformations of the representation ρ with Zariski dense image.

Note that ρ if maximal if and only if the representations ρ_i are orientation preserving for all $i = 1,\ldots,n$.

9.4. Canceling contributions. Let $G = G_1 \times \cdots \times G_n$ be a semisimple Lie group of Hermitian type. If $\rho : \pi_1(\Sigma) \to G$ is a maximal representation, then $\rho_i = p_i \circ \rho : \pi_1(\Sigma) \to G_i$, $i = 1,\cdots,n$ are maximal representations, where p_i denotes the projection onto the i-th factor.

In order to illustrate that this does not hold true for weakly maximal representations consider an arbitrary representation $\rho_a : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{R})$ and denote by $\overline{\rho_a} : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{R})$ the composition of ρ_a with the outer automorphism of $\mathrm{SL}(2,\mathbb{R})$ reversing the orientation. Let $\rho_0 : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{R})$ be a faithful representation with discrete image, as before. Then the representation $\rho = (\rho_0, \rho_a, \overline{\rho_a}) : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ is weakly maximal.

9.5. Limit of Shilov–Anosov representations. Let $\partial\Sigma = \emptyset$. We have seen in Corollary 8.5 that any representation that is the limit of Shilov–Anosov representations is weakly maximal. By choosing appropriately ρ_a in § 9.4 it is easy to construct representations that are weakly maximal but not Shilov–Anosov. While we do not know whether the containment in (8.2) is strict, we give here an example to show that the containment

$$\mathrm{Hom}^*_{\check{S}\text{-}An}(\pi_1(\Sigma), G) \subset \overline{\mathrm{Hom}^*_{\check{S}\text{-}An}(\pi_1(\Sigma), G)}$$

is strict, that is an example of a representation that is the limit of Shilov–Anosov representations but is not Shilov–Anosov.

To this purpose we consider the group $\mathrm{Sp}(2n,\mathbb{R})$ and we denote for simplicity by $\mathcal{R}_d(n)$ the set of representations $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(2n,\mathbb{R})$ with Toledo invariant equal to d and with $(\mathcal{R}_d(n))_{\check{S}\text{-}An} \subset \mathcal{R}_d(n)$ the subset of Shilov–Anosov representations.

If $d = (g-1)n$, that is if $\mathcal{R}_{(g-1)n}(n)$ consists of maximal representations, then by [**9**] there is the equality $(\mathcal{R}_{(g-1)n}(n))_{\check{S}\text{-}An} \equiv \mathcal{R}_{(g-1)n}(n)$. On the other hand, it is easy to construct a representation that has zero Toledo invariant but is not Shilov–Anosov (by taking for example $\rho_a \oplus \overline{\rho_a} : \pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \hookrightarrow \mathrm{Sp}(4,\mathbb{R})$, where ρ_a is any representation with dense image in $\mathrm{SL}(2,\mathbb{R})$). Since for $n \geq 3$ the set

$\mathcal{R}_0(n)$ is connected [**22**, Theorem 1.1 (1)] and $(\mathcal{R}_0(n))_{\check{S}\text{-}An}$ is open, this shows that

$$(\mathcal{R}_0(n))_{\check{S}\text{-}An} \subsetneq \overline{(\mathcal{R}_0(n))_{\check{S}\text{-}An}}\,.$$

We can use this fact as follows. Recall that the direct sum of symplectic vector spaces leads to a canonical homomorphism

$$\mathrm{Sp}(2n_1,\mathbb{R}) \times \mathrm{Sp}(2n_2,\mathbb{R}) \longrightarrow \mathrm{Sp}(2(n_1+n_2),\mathbb{R})\ ,$$

and, consequently, to a continuous map

$$\begin{aligned}\mathcal{R}_{d_1}(n_1) \times \mathcal{R}_{d_2}(n_2) &\to \mathcal{R}_{d_1+d_2}(n_1+n_2),\\ (\rho_1,\rho_2) &\mapsto \rho_1 \oplus \rho_2\,.\end{aligned}$$

Observe, moreover, that $\rho_1 \oplus \rho_2$ is Shilov–Anosov precisely if both ρ_1 and ρ_2 are. In particular, if $n_1 \geq 3$ we can consider a sequence $\rho_1^{(k)} \in (\mathcal{R}_0(n_1))_{\check{S}\text{-}An}$ whose limit ρ_1 is not Shilov–Anosov. Then for any $\rho_2 \in (\mathcal{R}_d(n_2))_{\check{S}\text{-}An}$, the sequence $\rho_1^{(k)} \oplus \rho_2$ converges to the representation $\rho_1 \oplus \rho_2$ that is not Shilov–Anosov and has Toledo invariant equal to d.

9.6. Weakly maximal representations with non-reductive Zariski closure. Let J_n be the $2n \times 2n$ matrix with block entries $J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We consider the subgroup of $\mathrm{Sp}(2(n+1),\mathbb{R})$ given by

$$Q = \left\{ \begin{pmatrix} A & b & 0 \\ 0 & 1 & 0 \\ c & d & 1 \end{pmatrix} : A \in \mathrm{Sp}(2n,\mathbb{R}) \text{ and } c = {}^t b J_n A \right\},$$

which is the semidirect product of $\mathrm{Sp}(2n,\mathbb{R})$ and the n-dimensional Heisenberg group H_n.

A map $\rho : \Gamma \to Q$ with entries $\pi(\gamma), b(\gamma), c(\gamma)$ and $d(\gamma)$ is a homomorphism if and only if $\pi : \Gamma \to \mathrm{Sp}(2n,\mathbb{R})$ is a homomorphism, $b : \Gamma \to \mathbb{R}^{2n}$ is a 1-cocycle (with the Γ-module structure on $\mathbb{R}^{2n}$ is given by π) and

$$d(\gamma_1\gamma_2) = {}^t b(\gamma_1) J_n \pi(\gamma_1) b(\gamma_2) + d(\gamma_1) + d(\gamma_2)\,. \tag{9.1}$$

To reinterpret (9.1), we observe that there is a bilinear symmetric form on $\mathrm{H}^1(\Gamma,\pi)$ with values in $\mathrm{H}^2(\Gamma,\mathbb{R})$ obtained by composing the cup product $\mathrm{H}^1(\Gamma,\pi) \times \mathrm{H}^1(\Gamma,\pi) \to \mathrm{H}^2(\Gamma,\pi\otimes\pi)$ with the projection on trivial coefficients given by the invariant symplectic form; denoting by Q_π the corresponding quadratic form, we have that

$$(\gamma_1,\gamma_2) \longmapsto {}^t b(\gamma_1) J_n \pi(\gamma_1) b(\gamma_2)$$

is a representative of $Q_\pi([b])$ and the existence of a function d satisfying (9.1) amounts to $Q_\pi([b]) = 0$.

Let now $\Gamma = \pi_1(S)$, where S is an oriented surface of genus g, and let $\rho_i : \Gamma \to \mathrm{Sp}(2n_i,\mathbb{R})$, where ρ_1 is in the Hitchin component of $\mathrm{Sp}(2n_1,\mathbb{R})$, while ρ_2 is the precomposition of a Hitchin representation into $\mathrm{Sp}(2n_2,\mathbb{R})$ with an orientation reversing automorphism of Γ.

Identifying $\mathrm{H}^2(\Gamma, \mathbb{R})$ with $\mathbb{R}$ by means of the chosen orientation, we have [**24**] that Q_{ρ_1} is positive definite while Q_{ρ_2} is negative definite. Choose $[b_i] \in \mathrm{H}^1(\Gamma, \rho_i)$ both non-zero such that

$$Q_{\rho_1}([b_1]) + Q_{\rho_2}([b_2]) = 0\,. \tag{9.2}$$

Set $\pi := \rho_1 \oplus \rho_2$, $b := b_1 + b_2$ and let d be a solution of (9.1), which exists by (9.2). Then $\rho : \pi_1(S) \to \mathrm{Sp}(2(n+1), \mathbb{R})$ is a weakly maximal representation with $\mathrm{T}(\rho) = n_1 - n_2$, and the real Zariski closure of its image is the semi direct product of $G_1 \times G_2 < \mathrm{Sp}(2n, \mathbb{R})$ with H_n; here G_i is the real Zariski closure of the image of ρ_i.

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