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A NEW TENSORIAL CONSERVATION LAW FOR MAXWELL FIELDS ON THE KERR BACKGROUND

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Abstract

A new, conserved, symmetric tensor field for a source-free Maxwell test field on a four-dimensional spacetime with a conformal Killing–Yano tensor, satisfying a certain compatibility condition, is introduced. In particular, this construction works for the Kerr spacetime.

1. Introduction

In this paper, we consider the Maxwell equation for a real 2-form $F_{ab} = F_{[ab]}$,

$$\nabla^a F_{ab} = 0, \quad \nabla^a {*F}_{ab} = 0, \tag{1.1}$$

on a four-dimensional Lorentzian manifold $(\mathcal{M}, g_{ab})$. Recall that a conformal Killing–Yano tensor is a real 2-form $Y_{ab} = Y_{[ab]}$ satisfying

$$\nabla_{(a}Y_{b)c} = -\tfrac{1}{3}g_{ab}\nabla_d Y_c{}^d + \tfrac{1}{3}g_{(a|c|}\nabla^d Y_{b)d}. \tag{1.2}$$

Associated with Y_{ab} is the complex 1-form

$$\xi_a = \tfrac{1}{3}i\nabla_b Y_a{}^b - \tfrac{1}{3}\nabla_b {*Y}_a{}^b. \tag{1.3}$$

We say that Y_{ab} satisfies the aligned matter condition if the Ricci curvature and Y_{ab} satisfy

$$R_{(a}{}^c Y_{b)c} = 0, \quad R_{(a}{}^c {*Y}_{b)c} = 0. \tag{1.4}$$

Theorem 1.1. *Let Y_{ab} and F_{ab} be real 2-forms. Define the real 2-form Z_{ab} and the complex 1-form η_a by*

$$Z_{ab} = -\tfrac{4}{3}({*F})_{[a}{}^c Y_{b]c}, \tag{1.5}$$

$$\eta_a = -\tfrac{1}{2}\nabla_b Z_a{}^b - \tfrac{1}{2}i\nabla_b {*Z}_a{}^b, \tag{1.6}$$

and the real symmetric 2-tensor V_{ab} by

$$\begin{aligned} V_{ab} = {} & \eta_{(a}\bar{\eta}_{b)} - \tfrac{1}{2}g_{ab}\eta^c\bar{\eta}_c - \tfrac{1}{3}(\mathcal{L}_{Re\xi}F)_{(a}{}^c Z_{b)c} + \tfrac{1}{12}g_{ab}(\mathcal{L}_{Re\xi}F)^{cd}Z_{cd} \\ & + \tfrac{1}{3}(\mathcal{L}_{Im\xi}{*F})_{(a}{}^c Z_{b)c} - \tfrac{1}{12}g_{ab}(\mathcal{L}_{Im\xi}{*F})^{cd}Z_{cd}, \end{aligned} \tag{1.7}$$

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where ξ_a *is given by equation* (1.3) *and* $\bar{\eta}_a$ *denotes the complex conjugate of* η_a*.*

If Y_{ab} *is a conformal Killing–Yano tensor satisfying the aligned matter condition* (1.4) *and* F_{ab} *satisfies the Maxwell equations* (1.1)*, then* V_{ab} *has vanishing divergence,* $\nabla^a V_{ab} = 0$*.*

Remark 1.2. 1) The vector field ξ^a is Killing, $\nabla_{(a}\xi_{b)} = 0$, if the aligned matter condition (1.4) holds, cf. equation (2.9) below.
2) If $\nabla^a Y_{ab} = 0$ then Y_{ab} is a Killing–Yano tensor and ξ_a is real. In this case, the first condition in (1.4) is trivially satisfied, and the last two terms of (1.7) vanish.
3) The Kerr family of stationary, rotating vacuum black hole metrics admit a Killing–Yano tensor. More generally, the Kerr–Newman family of stationary, rotating electro-vacuum black hole metrics admit a Killing–Yano tensor satisfying the aligned matter condition. See section 3 for further discussion.

Let

$$T_{ab} = -F_a{}^c F_{bc} + \tfrac{1}{4} g_{ab} F_{cd} F^{cd}$$

be the symmetric energy-momentum tensor for the Maxwell field. It is traceless and satisfies the dominant energy condition, i.e., $T_{ab}\mu^a\nu^b \geq 0$ for any future causal vectors μ^a, ν^b. Further, if F_{ab} satisfies the Maxwell equations, T_{ab} is conserved, $\nabla^a T_{ab} = 0$. Hence, the current

$$J_a = T_{ab}\nu^b \tag{1.8}$$

is conserved, $\nabla^a J_a = 0$, if ν^a is a conformal Killing field, $\nabla_{(a}\nu_{b)} - \frac{1}{4}\nabla_c\nu^c g_{ab} = 0$.

For the Maxwell field on Minkowski space, and more generally on spacetimes admitting conformal Killing–Yano tensors satisfying the aligned matter condition, there are non-classical conserved currents not equivalent[1] to any of the classical conserved energy-momentum currents of the form (1.8), see [**3**] and references therein. For the Maxwell field on Minkowski space, these include chiral currents constructed using the 20-dimensional family of conformal Killing–Yano tensors of Minkowski space. As shown by the authors [**4**], analogous conserved currents exist also on spacetimes with conformal Killing–Yano tensors satisfying the aligned matter condition.

In spite of the large literature on conformal Killing–Yano tensors, and the related conservation laws, the tensorial conservation law exhibited in Theorem 1.1 appears to be new, even in the Minkowski case.[2] The fact

[1] A conserved current J_a is a 1-form concomitant of the Maxwell field, satisfying $\nabla^a J_a = 0$. We say that J_a is equivalent to $\tilde{J}_a$ if $J_a - \tilde{J}_a = \nabla^b C_{ab}$ for some 2-form $C_{ab} = C_{[ab]}$.

[2] Observe, however, that the conserved currents one can generate from this tensor are equivalent to the currents constructed in [**3**] in the Minkowski case.

that the new higher order tensor concomitant V_{ab} is conserved also in the case of the Kerr and Kerr–Newman spacetimes makes it interesting from the point of view of the black hole stability problem, which in fact served as an important motivation for the investigation which led to its discovery. See section 3 below for further remarks.

At this point, we should mention that the symmetric tensor

$$B_{ab} = \nabla_d F_{bc} \nabla^d F_a{}^c - \tfrac{1}{4} g_{ab} \nabla_f F_{cd} \nabla^f F^{cd},$$

which arises as a trace of the 4-index Chevreton tensor, was shown by Bergqvist et al. [**7**] to be traceless and conserved for a Maxwell field on a Ricci flat spacetime. Like the conserved tensor V_{ab} introduced in this paper, the tensor B_{ab} introduced by Bergqvist et al. depends on the F_{ab} and is quadratic in first derivatives of F_{ab}. However, B_{ab} is traceless and fails to satisfy any positivity condition. In contrast to B_{ab}, the new tensor V_{ab} introduced here has trace $V^a{}_a = -\eta^a \bar{\eta}_a$ and its leading-order terms,

$$\eta_{(a} \bar{\eta}_{b)} - \tfrac{1}{2} g_{ab} \eta^c \bar{\eta}_c \tag{1.9}$$

satisfies the dominant energy condition. Here, the order of a term is defined as the total number of derivatives of the underlying field F_{ab} in the term. That (1.9) satisfies the dominant energy condition can be seen by comparing with the form of the standard energy-momentum for a scalar field, see also [**13**]. The remaining part of V_{ab} as defined in (1.7), which has no dominant property, is at most linear in first derivatives of F_{ab} and, hence, is a lower-order term. This leads one to expect that this can be dominated by the expression in (1.9). However, this property is subtle, requiring the construction of suitable equivalent currents which shall be discussed in forthcoming work [**4**].

We observe that although V_{ab} is conserved and has a leading-order term which satisfies the dominant energy condition, this does not give a canonical choice of a conserved current, i.e., divergence-free 1-form. In particular, if we consider the exterior of a rotating Kerr black hole, the vector field ξ^a is Killing but fails to be timelike in the ergoregion, and hence the current $J_a = V_{ab}\xi^b$ is conserved but is not necessarily timelike. In fact, even the leading-order term $(\eta_{(a}\bar{\eta}_{b)} - \frac{1}{2}\eta^c\bar{\eta}_c g_{ab})\xi^b$ of J_a fails to be timelike everywhere in the Kerr exterior.

Although the Maxwell equation (1.1) for F_{ab} and the conformal Killing-Yano condition (1.2) for Y_{ab} are conformally covariant, neither the aligned matter condition (1.4) for Y_{ab} nor the divergence-free property for V_{ab} are conformally covariant. For instance, one can see that (1.3) is not conformally invariant if the weight of Y_{ab} is chosen to be compatible with (1.2).

Theorem 1.1 relies on the notion of conformal Killing–Yano tensor which makes sense in all dimensions and all signatures. However, our proof of Theorem 1.1, which will be given in the next section, makes use of computations in the 2-spinor formalism. This formalism is particularly closely related to the four-dimensional Lorentzian setting. Thus, our method does not extend to higher dimensions, and it remains an open question of whether extensions of Theorem 1.1 for appropriate analogues of the Maxwell equations exist in those cases.

In the investigations leading to the main result, the *SymManipulator* package [**6**], developed by one of the authors (T.B.) for the Mathematica based symbolic differential geometry suite *xAct* [**11**], has played an essential role. SymManipulator makes it possible to systematically exploit decompositions in terms of irreducible representations of the spin group $\mathrm{SL}(2,\mathbb{C})$, and allows one to carry out investigations that are not feasible by hand.

In section 3, we show how the main result relates to the Teukolsky and Teukolsky–Starobinsky equations using The Geroch–Held–Penrose (GHP) formalism [**9**]. In this section we restrict the attention to Petrov type $\{2,2\}$ spacetimes, while still assume existence of a valence $(2,0)$ Killing spinor with aligned matter. This class includes the Kerr–Newman family of electro-vacuum spacetimes.

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2. Proof of Theorem 1.1

For the remainder of this paper, we will make use of the 2-spinor formalism, following the conventions of [**12**]. Since our considerations are local, we can assume without loss of generality that $(\mathcal{M}, g_{ab})$ is oriented and globally hyperbolic. This also implies that $\mathcal{M}$ is spin. Furthermore, we assume all objects to be smooth.

The spin group is $\mathrm{SL}(2,\mathbb{C})$ which has the inequivalent spinor representations $\mathbb{C}^2$ and $\bar{\mathbb{C}}^2$. Unprimed upper case Latin indices and their primed versions are used for sections of the corresponding spinor bundles, respectively. The correspondence between spinors and tensors makes it possible to translate all tensor expressions to spinor form. The action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}^2$ leaves invariant the spin metric $\epsilon_{AB} = \epsilon_{[AB]}$, which is used to raise and lower indices on spinors. The metric g_{ab} is related to ϵ_{AB} by $g_{ab} = \epsilon_{AB}\bar{\epsilon}_{A'B'}$. Let $\mathcal{S}_{k,l}$ denote the space of symmetric spinors with k unprimed indices and l primed indices.

There are symmetric spinors κ_{AB}, ϕ_{AB}, and Θ_{AB} such that

$$Y_{ab} = \tfrac{3}{2}i(\bar{\epsilon}_{A'B'}\kappa_{AB} - \epsilon_{AB}\bar{\kappa}_{A'B'}),$$
$$F_{ab} = \bar{\epsilon}_{A'B'}\phi_{AB} + \epsilon_{AB}\bar{\phi}_{A'B'},$$
$$Z_{ab} = \bar{\epsilon}_{A'B'}\Theta_{AB} + \epsilon_{AB}\overline{\Theta}_{A'B'}.$$

The normalization of Y_{ab} is chosen for convenience. Equations (1.1)–(1.7) become, respectively

$$\nabla^{A}{}_{A'}\phi_{AB} = 0, \tag{2.1}$$
$$\nabla_{(A|A'|}\kappa_{BC)} = 0, \tag{2.2}$$
$$\xi_{AA'} = \nabla^{B}{}_{A'}\kappa_{AB}, \tag{2.3}$$
$$\Phi_{(A}{}^{C}{}_{|A'B'|}\kappa_{B)C} = 0, \tag{2.4}$$
$$\Theta_{AB} = -2\kappa_{(A}{}^{C}\phi_{B)C}, \tag{2.5}$$
$$\eta_{AA'} = \nabla^{B}{}_{A'}\Theta_{AB}, \tag{2.6}$$

and

$$V_{ABA'B'} = \tfrac{1}{2}\eta_{AB'}\bar{\eta}_{A'B} + \tfrac{1}{2}\eta_{BA'}\bar{\eta}_{B'A} + \tfrac{1}{3}\Theta_{AB}(\hat{\mathcal{L}}_{\bar{\xi}}\bar{\phi})_{A'B'} + \tfrac{1}{3}\bar{\Theta}_{A'B'}(\hat{\mathcal{L}}_{\xi}\phi)_{AB}, \tag{2.7}$$

where $\hat{\mathcal{L}}_{\xi}$ is a conformally weighted Lie derivative on spinors, see equation (2.10) below.

The projection of the spinor covariant derivative $\nabla_{AA'}$ on symmetric spinors (which form the irreducible representations of the spin group $\mathrm{SL}(2,\mathbb{C})$) gives the following fundamental operators:

Definition 2.1 ([**5**, Definition 13])**.** Let the differential operators $\mathscr{D}_{k,l} : \mathcal{S}_{k,l} \to \mathcal{S}_{k-1,l-1}$, $\mathscr{C}_{k,l} : \mathcal{S}_{k,l} \to \mathcal{S}_{k+1,l-1}$, $\mathscr{C}^{\dagger}_{k,l} : \mathcal{S}_{k,l} \to \mathcal{S}_{k-1,l+1}$, and $\mathscr{T}_{k,l} : \mathcal{S}_{k,l} \to \mathcal{S}_{k+1,l+1}$ be defined by

$$(\mathscr{D}_{k,l}\varphi)_{A_1\dots A_{k-1}}{}^{A'_1\dots A'_{l-1}} = \nabla^{BB'}\varphi_{A_1\dots A_{k-1}B}{}^{A'_1\dots A'_{l-1}}{}_{B'},$$
$$(\mathscr{C}_{k,l}\varphi)_{A_1\dots A_{k+1}}{}^{A'_1\dots A'_{l-1}} = \nabla_{(A_1}{}^{B'}\varphi_{A_2\dots A_{k+1})}{}^{A'_1\dots A'_{l-1}}{}_{B'},$$
$$(\mathscr{C}^{\dagger}_{k,l}\varphi)_{A_1\dots A_{k-1}}{}^{A'_1\dots A'_{l+1}} = \nabla^{B(A'_1}\varphi_{A_1\dots A_{k-1}B}{}^{A'_2\dots A'_{l+1})},$$
$$(\mathscr{T}_{k,l}\varphi)_{A_1\dots A_{k+1}}{}^{A'_1\dots A'_{l+1}} = \nabla_{(A_1}{}^{(A'_1}\varphi_{A_2\dots A_{k+1})}{}^{A'_2\dots A'_{l+1})}.$$

The operators are called, respectively, the divergence, curl, curl-dagger, and twistor operators.

With respect to complex conjugation, the operators $\mathscr{D}, \mathscr{T}$ satisfy $\overline{\mathscr{D}_{k,l}} = \mathscr{D}_{l,k}$, $\overline{\mathscr{T}_{k,l}} = \mathscr{T}_{l,k}$, while $\overline{\mathscr{C}_{k,l}} = \mathscr{C}^{\dagger}_{l,k}$, $\overline{\mathscr{C}^{\dagger}_{k,l}} = \mathscr{C}_{l,k}$. In the following, we shall use the fundamental operators and their properties freely. Any covariant expression in spinors and their covariant derivatives can be written in terms of the fundamental operators using the following Lemma:

Lemma 2.2 ([**5**, Lemma 15])**.** *For any* $\varphi_{A_1\dots A_k}{}^{A'_1\dots A'_l} \in \mathcal{S}_{k,l}$*, we have the irreducible decomposition*

$$\begin{aligned}\nabla_{A_1}{}^{A'_1}\varphi_{A_2\dots A_{k+1}}{}^{A'_2\dots A'_{l+1}} &= (\mathscr{T}_{k,l}\varphi)_{A_1\dots A_{k+1}}{}^{A'_1\dots A'_{l+1}}\\ &\quad - \tfrac{l}{l+1}\bar{\epsilon}^{A'_1(A'_2}(\mathscr{C}_{k,l}\varphi)_{A_1\dots A_{k+1}}{}^{A'_3\dots A'_{l+1})}\\ &\quad - \tfrac{k}{k+1}\epsilon_{A_1(A_2}(\mathscr{C}^\dagger_{k,l}\varphi)_{A_3\dots A_{k+1})}{}^{A'_1\dots A'_{l+1}}\\ &\quad + \tfrac{kl}{(k+1)(l+1)}\epsilon_{A_1(A_2}\bar{\epsilon}^{A'_1(A'_2}(\mathscr{D}_{k,l}\varphi)_{A_3\dots A_{k+1})}{}^{A'_3\dots A'_{l+1})}.\end{aligned}$$

For example, the Maxwell equation and the Killing spinor equations take the form

$$(\mathscr{C}^\dagger_{2,0}\phi)_{AA'} = 0,$$

and

$$(\mathscr{T}_{2,0}\kappa)_{ABCA'} = 0,$$

respectively, in terms of the fundamental operators.

In the computations below we shall need some commutator relations satisfied by the fundamental operators, see [**5**, Lemma 18]. The following lemma gives the commutators which are relevant here.

Lemma 2.3. *Let* $\varphi_{AB} \in \mathcal{S}_{2,0}$*. The operators* $\mathscr{D}$*,* $\mathscr{C}$*,* $\mathscr{C}^\dagger$ *and* $\mathscr{T}$ *satisfy the following commutator relations:*

$$(\mathscr{D}_{1,1}\mathscr{C}^\dagger_{2,0}\varphi) = 0, \tag{2.8a}$$

$$(\mathscr{C}_{3,1}\mathscr{T}_{2,0}\varphi)_{ABCD} = 2\Psi_{(ABC}{}^{F}\varphi_{D)F}, \tag{2.8b}$$

$$(\mathscr{C}^\dagger_{3,1}\mathscr{T}_{2,0}\varphi)_{ABA'B'} = 2\Phi_{(A}{}^{C}{}_{|A'B'|}\varphi_{B)C} + \tfrac{2}{3}(\mathscr{T}_{1,1}\mathscr{C}^\dagger_{2,0}\varphi)_{ABA'B'}, \tag{2.8c}$$

$$(\mathscr{D}_{3,1}\mathscr{T}_{2,0}\varphi)_{AB} = 2\Psi_{ABCD}\varphi^{CD} - 8\Lambda\varphi_{AB} - \tfrac{4}{3}(\mathscr{C}_{1,1}\mathscr{C}^\dagger_{2,0}\varphi)_{AB}. \tag{2.8d}$$

Directly from the Killing spinor equation and the commutators (2.8a) and (2.8d) we get

$$(\mathscr{D}_{1,1}\xi) = 0, \tag{2.9a}$$

$$(\mathscr{T}_{1,1}\xi)_{ABA'B'} = -3\Phi_{(A}{}^{C}{}_{|A'B'|}\kappa_{B)C}. \tag{2.9b}$$

Hence, if the aligned matter condition is satisfied, $\xi^{AA'}$ is a Killing vector.

Given a conformal Killing vector $\xi^{AA'}$, we define a conformally weighted Lie derivative acting on a symmetric valence $(2s, 0)$ spinor field by [**5**, Definition 17]

$$\begin{aligned}\hat{\mathcal{L}}_\xi\varphi_{A_1\dots A_{2s}} &= \xi^{BB'}\nabla_{BB'}\varphi_{A_1\dots A_{2s}} + s\varphi_{B(A_2\dots A_{2s}}\nabla_{A_1)B'}\xi^{BB'}\\ &\quad + \tfrac{1-s}{4}\varphi_{A_1\dots A_{2s}}\nabla^{CC'}\xi_{CC'}.\end{aligned} \tag{2.10}$$

We shall now prove an auxiliary result on the derivatives of $\eta_{AA'}$, which will allow us to prove our main result.

Lemma 2.4. *Let* $\kappa_{AB} \in \mathcal{S}_{2,0}$ *satisfy the Killing spinor equation* (2.2) *and the aligned matter condition* (2.4)*, and let* $\xi_{AA'}$ *be given by* (2.3)*. If* $\phi_{AB} \in \mathcal{S}_{2,0}$ *satisfies the Maxwell equation* (2.1) *and* $\eta_{AA'}$ *is given by* (2.6)*, then*

$$(\mathscr{D}_{1,1}\eta) = 0, \tag{2.11a}$$

$$(\mathscr{C}_{1,1}\eta)_{AB} = \tfrac{2}{3}(\hat{\mathcal{L}}_\xi \phi)_{AB}, \tag{2.11b}$$

$$(\mathscr{C}^\dagger_{1,1}\eta)_{A'B'} = 0, \tag{2.11c}$$

$$\eta_{AA'}\xi^{AA'} = \kappa^{AB}(\hat{\mathcal{L}}_\xi \phi)_{AB}. \tag{2.11d}$$

Proof. Using the definition of the Lie derivative, the Maxwell equation and that $\xi^{AA'}$ is a Killing vector we get

$$(\hat{\mathcal{L}}_\xi \phi)_{AB} = \phi_{(A}{}^{C}(\mathscr{C}_{1,1}\xi)_{B)C} + \xi^{CA'}(\mathscr{T}_{2,0}\phi)_{ABCA'}. \tag{2.12}$$

The equation (2.11a) follows directly from the commutator relation (2.8a). Also using the commutators (2.8d), (2.8b) and the Killing spinor equation, we get

$$(\mathscr{C}_{1,1}\xi)_{AB} = (\mathscr{C}_{1,1}\mathscr{C}^\dagger_{2,0}\kappa)_{AB} = -6\Lambda\kappa_{AB} + \tfrac{3}{2}\Psi_{ABCD}\kappa^{CD}, \tag{2.13}$$

$$0 = \tfrac{1}{2}(\mathscr{C}_{3,1}\mathscr{T}_{2,0}\kappa)_{ABCD} = \Psi_{(ABC}{}^{F}\kappa_{D)F}. \tag{2.14}$$

Performing an irreducible decomposition of the contraction $\Psi_{ABCF}\kappa_D{}^F$, and using (2.13) and (2.14) we get

$$\Psi_{ABCF}\kappa_D{}^F = 3\Lambda\epsilon_{(A|D|}\kappa_{BC)} + \tfrac{1}{2}\epsilon_{(A|D|}(\mathscr{C}_{1,1}\xi)_{BC)}. \tag{2.15}$$

By using the definition of Θ_{AB}, the Leibniz rule, applying irreducible decompositions, and making use of the Killing spinor equation, the fact that $\xi_{AA'}$ is Killing, and the Maxwell equation, we find

$$\begin{aligned}(\mathscr{C}_{1,1}\eta)_{AB} &= (\mathscr{C}_{1,1}\mathscr{C}^\dagger_{2,0}\Theta)_{AB} \\ &= \kappa^{CD}(\mathscr{C}_{3,1}\mathscr{T}_{2,0}\phi)_{ABCD} + \tfrac{1}{2}\kappa_{(A}{}^{C}(\mathscr{D}_{3,1}\mathscr{T}_{2,0}\phi)_{B)C} \\ &\quad + \tfrac{4}{3}\phi_{(A}{}^{C}(\mathscr{C}_{1,1}\xi)_{B)C} + \tfrac{2}{3}\xi^{CA'}(\mathscr{T}_{2,0}\phi)_{ABCA'}.\end{aligned}$$

Applying the commutator relations (2.8d) and (2.8b) and making use of (2.15) now gives

$$\begin{aligned}(\mathscr{C}_{1,1}\eta)_{AB} &= \tfrac{2}{3}\phi_{(A}{}^{C}(\mathscr{C}_{1,1}\xi)_{B)C} + \tfrac{2}{3}\xi^{CA'}(\mathscr{T}_{2,0}\phi)_{ABCA'} \\ &= \tfrac{2}{3}(\hat{\mathcal{L}}_\xi\phi)_{AB},\end{aligned}$$

where (2.12) was used in the last step.

Proceeding in a fashion similar to the above, using the definitions of $\eta_{AA'}$ and Θ_{AB}, the Leibniz rule, applying irreducible decompositions, and making use of the Killing spinor equation, the fact that $\xi_{AA'}$ is Killing, and the Maxwell equation, we find

$$(\mathscr{C}^\dagger_{1,1}\eta)_{A'B'} = \kappa^{AB}(\mathscr{C}^\dagger_{3,1}\mathscr{T}_{2,0}\phi)_{ABA'B'}.$$

The commutator relation (2.8c) then gives

$$(\mathscr{C}^\dagger_{1,1}\eta)_{A'B'} = -2\Phi_{BCA'B'}\kappa^{AB}\phi_A{}^C,$$

and the aligned matter condition gives (2.11c).

Finally, expanding the definition of $\eta_{AA'}$, and using the Killing spinor equation and the Maxwell equation yields

$$\kappa^{BC}(\mathscr{T}_{2,0}\phi)_{ABCA'} = \eta_{AA'} + \tfrac{4}{3}\xi^B{}_{A'}\phi_{AB}. \tag{2.16}$$

Contracting (2.12) with κ_{AB} and using (2.16), (2.13), and (2.14) gives (2.11d). q.e.d.

The proof of the main theorem is now a matter of straightforward verification.

Proof of Theorem 1.1. From the Leibniz rule, we first find

$$\begin{aligned}\nabla^{BB'}V_{ABA'B'} = {}& \tfrac{1}{2}\bar{\eta}_{A'B}\nabla^{BB'}\eta_{AB'} + \tfrac{1}{2}\bar{\eta}_{B'A}\nabla^{BB'}\eta_{BA'} \\ & + \tfrac{1}{2}\eta_{AB'}\nabla^{BB'}\bar{\eta}_{A'B} + \tfrac{1}{2}\eta_{BA'}\nabla^{BB'}\bar{\eta}_{B'A} \\ & + \tfrac{1}{3}\Theta_{AB}\nabla^{BB'}(\hat{\mathcal{L}}_{\bar{\xi}}\bar{\phi})_{A'B'} + \tfrac{1}{3}\overline{\Theta}_{A'B'}\nabla^{BB'}(\hat{\mathcal{L}}_{\xi}\phi)_{AB} \\ & + \tfrac{1}{3}\nabla^{BB'}\Theta_{AB}(\hat{\mathcal{L}}_{\bar{\xi}}\bar{\phi})_{A'B'} + \tfrac{1}{3}\nabla^{BB'}\overline{\Theta}_{A'B'}(\hat{\mathcal{L}}_{\xi}\phi)_{AB}.\end{aligned}$$

This can be simplified by first observing that $\hat{\mathcal{L}}_\xi$ is a symmetry operator taking solutions of the Maxwell equation to solutions of the Maxwell equation, so $(\mathscr{C}^\dagger_{2,0}\hat{\mathcal{L}}_\xi\phi)_{AB} = 0$ and similarly for the complex conjugate. It can be further simplified by substituting the definition $\nabla^B{}_{A'}\Theta_{AB} = \eta_{AA'}$, cf. (2.6), to eliminate the derivative of Θ_{AB} terms. This yields

$$\begin{aligned}\nabla^{BB'}V_{ABA'B'} = {}& -\tfrac{1}{2}\bar{\eta}_{A'}{}^B(\mathscr{C}_{1,1}\eta)_{AB} - \tfrac{1}{2}\eta^B{}_{A'}(\mathscr{C}_{1,1}\bar{\eta})_{AB} \\ & - \tfrac{1}{2}\bar{\eta}^{B'}{}_A(\mathscr{C}^\dagger_{1,1}\eta)_{A'B'} - \tfrac{1}{2}\eta_A{}^{B'}(\mathscr{C}^\dagger_{1,1}\bar{\eta})_{A'B'} \\ & + \tfrac{1}{2}\bar{\eta}_{A'A}(\mathscr{D}_{1,1}\eta) + \tfrac{1}{2}\eta_{AA'}(\mathscr{D}_{1,1}\bar{\eta}) \\ & + \tfrac{1}{3}\eta_A{}^{B'}(\hat{\mathcal{L}}_{\bar{\xi}}\bar{\phi})_{A'B'} + \tfrac{1}{3}\bar{\eta}_{A'}{}^B(\hat{\mathcal{L}}_\xi\phi)_{AB}.\end{aligned}$$

The terms involving $(\mathscr{C}^\dagger_{1,1}\eta)_{A'B'}$ and $(\mathscr{C}_{1,1}\bar{\eta})_{AB}$ are zero by equation (2.11c). Those involving $(\mathscr{D}_{1,1}\eta)$ and $(\mathscr{D}_{1,1}\bar{\eta})$ are zero by equation (2.11a). Finally by equation (2.11b), the terms involving $(\mathscr{C}_{1,1}\eta)_{AB}$ and $(\mathscr{C}^\dagger_{1,1}\bar{\eta})_{A'B'}$ cancel with those involving $(\hat{\mathcal{L}}_\xi\phi)_{AB}$ and $(\hat{\mathcal{L}}_{\bar{\xi}}\bar{\phi})_{A'B'}$, respectively. This completes the result. q.e.d.

3. Further remarks on Kerr and Petrov type $\{2,2\}$ spacetimes

The stationary, asymptotically flat, vacuum Kerr spacetimes, and more generally the electro-vacuum Kerr–Newman spacetimes, have algebraic type $\{2,2\}$, i.e., the Weyl spinor Ψ_{ABCD} has two distinct, repeated, principal spinors o_A, ι_A which are unique up to a rescaling. The dyad o_A, ι_A is normalized by $o_A\iota^A = 1$. For the following discussion, recall that given a spin dyad o_A, ι_A, one defines for a symmetric spinor $\varpi_{A_1\cdots A_k}$ scalars ϖ_i by contracting i times with ι^A and $k-i$ times with o^A. This yields Weyl scalars Ψ_i, $i = 0, \ldots, 4$ and Maxwell scalars ϕ_i, $i = 0, 1, 2$. Similarly we use a subscript j' to denote j contractions with $\bar{\iota}^{A'}$, and the remaining primed indices with $\bar{o}^{A'}$. In a spacetime of type $\{2,2\}$ with principal dyad o_A, ι_A, it holds that

$$\Psi_{ABCD} = 6\Psi_2 o_{(A} o_B \iota_C \iota_{D)}, \tag{3.1}$$

and in this case it follows from (2.14) that any valence $(2,0)$ Killing spinor must be of the form

$$\kappa_{AB} = \zeta o_{(A} \iota_{B)}, \tag{3.2}$$

for some scalar ζ, and hence of the three scalars κ_i, $i = 0, 1, 2$, only $\kappa_1 = -\zeta/2$ is non-vanishing. If in addition the aligned matter condition holds, then the Ricci spinor $\Phi_{ABA'B'}$ must be of the form

$$\Phi_{ABA'B'} = 4\Phi_{11'} o_{(A} \iota_{B)} \bar{o}_{(A'} \bar{\iota}_{B')}. \tag{3.3}$$

If (t, r, θ, ϕ) are Boyer–Lindquist coordinates, then the Coulomb field, i.e., the unique static, regular Maxwell test field, on the Kerr–Newman spacetime takes the form

$$\phi_{AB} = \frac{1}{(r - ia\cos\theta)^2} o_{(A} \iota_{B)},$$

up to a rescaling by a constant. In particular the extreme components ϕ_0, ϕ_2 are zero. The background Maxwell field in the electro-vacuum Kerr–Newman spacetime is a constant multiple of this Coulomb field.

The Killing spinor κ_{AB} is

$$\kappa_{AB} = \tfrac{2}{3}(r - ia\cos\theta) o_{(A} \iota_{B)}, \tag{3.4}$$

which is, therefore, proportional to the background Maxwell field in the Kerr–Newman spacetime. Hence, by the Einstein equation, $\Phi_{ABA'B'}$ is proportional to $\kappa_{AB}\bar{\kappa}_{A'B'}$. It follows that the aligned matter condition holds in the Kerr–Newman spacetime.

The normalization in equation (3.4) is chosen so that $\xi^a = (\partial_t)^a$, where ξ_a is given by (2.3). In particular ξ_a is real, which exhibits the fact that the Kerr–Newman family admits a Killing–Yano tensor, as remarked above. In particular, we see that the tensor V_{ab} given by (2.7) is conserved. More generally, any vacuum type $\{2,2\}$ spacetime admits a Killing spinor of valence $(2,0)$, of the form (3.2) with ζ proportional to $\Psi_2^{-1/3}$. This shows that Theorem 1.1 applies in the class of vacuum type $\{2,2\}$ metrics.

3.1. The Teukolsky equations and V_{ab}. The Maxwell equations on a Kerr black hole imply the spin-1 Teukolsky equations for the extreme scalars, ϕ_0 and ϕ_2 [**15**]. This system has many properties in common with the spin-2 Teukolsky equations which arise from linearizing the

Einstein equations. Despite the fact that the so-called Teukolsky Master Equation (TME) have been known for more than 40 years, and have been the subject of much study, no boundedness or decay estimates are known for the Teukolsky equations for fields with non-zero spin, other than the mode stability result of Whiting [**17**].

Although the extreme scalars for Maxwell and linearized gravity satisfy the pair of decoupled Teukolsky equations, they are in fact related by the Teukolsky–Starobinsky Identities (TSI), see [**16**, **14**, **10**] and references therein. Further, it is known that from the Maxwell system in a vacuum type $\{2,2\}$ spacetime one may derive a set of three second order differential equations (still denoted TSI) relating the extreme Maxwell scalars ϕ_0, ϕ_2 [**8**, **1**]. Conversely, if a pair of extreme Maxwell scalars, ϕ_0 and ϕ_2, satisfy the TME and TSI, then there is a ϕ_1 such that the ϕ_i satisfy the Maxwell equations [**8**].

We shall now explain the relation between the TME-TSI system and the conservation property of V_{ab}. This will make it apparent that one may view V_{ab} as an analogue of an energy-momentum tensor for the TME-TSI system. The discussion here is in terms of type $\{2,2\}$ spacetimes admitting a valence $(2,0)$ Killing spinor with aligned matter.

Lemma 2.4 gives the relations

$$(\mathscr{C}^{\dagger}_{1,1}\mathscr{C}^{\dagger}_{2,0}\Theta)_{A'B'} = 0, \tag{3.5a}$$

$$(\mathscr{C}_{1,1}\mathscr{C}^{\dagger}_{2,0}\Theta)_{AB} = \tfrac{2}{3}(\hat{\mathcal{L}}_{\xi}\phi)_{AB}. \tag{3.5b}$$

The following lemma shows that the system (3.5) is a sufficient condition for the conservation property of V_{ab}.

Lemma 3.1. *Assume that $\varphi_{AB} \in \mathcal{S}_{2,0}$ satisfies the system*

$$(\mathscr{C}^{\dagger}_{1,1}\mathscr{C}^{\dagger}_{2,0}\varphi)_{A'B'} = 0, \tag{3.6a}$$

$$(\mathscr{C}_{1,1}\mathscr{C}^{\dagger}_{2,0}\varphi)_{AB} = \varpi_{AB}, \tag{3.6b}$$

for some $\varpi_{AB} \in \mathcal{S}_{2,0}$. Let

$$\varsigma_{AA'} = (\mathscr{C}^{\dagger}_{2,0}\varphi)_{AA'}, \tag{3.7}$$

and define the symmetric tensor $X_{ABA'B'}$ by

$$X_{ABA'B'} = \tfrac{1}{2}\varsigma_{AB'}\bar{\varsigma}_{A'B} + \tfrac{1}{2}\varsigma_{BA'}\bar{\varsigma}_{B'A} + \tfrac{1}{2}\bar{\varpi}_{A'B'}\varphi_{AB} + \tfrac{1}{2}\varpi_{AB}\bar{\varphi}_{A'B'}. \tag{3.8}$$

Then

$$\nabla^{BB'}X_{ABA'B'} = 0. \tag{3.9}$$

Proof. By applying the operator $\mathscr{C}^{\dagger}_{2,0}$ to (3.6b), commuting derivatives and using (3.6a), we get the integrability condition $(\mathscr{C}^{\dagger}_{2,0}\varpi)_{AA'} = 0$. With $\varsigma_{AA'}$ given by (3.7), we directly get

$$(\mathscr{D}_{1,1}\varsigma) = 0, \qquad (\mathscr{C}^{\dagger}_{1,1}\varsigma)_{A'B'} = 0, \qquad (\mathscr{C}_{1,1}\varsigma)_{AB} = \varpi_{AB}. \tag{3.10}$$

The proof of Theorem 1.1 then gives (3.9). q.e.d.

Remark 3.2. 1) No assumptions were made on the spacetime geometry in Lemma 3.1.

2) Lemma 3.1 shows that the fact that V_{ab} is conserved, $\nabla^a V_{ab} = 0$, follows from the *second order system* (3.5), which is a consequence of the first order Maxwell system in any spacetime with a valence $(2,0)$ Killing spinor with aligned matter.

If the spacetime is of type $\{2,2\}$, then κ_{AB} is of the form (3.2) and the components of Θ_{AB} are of the form

$$\Theta_0 = -2\kappa_1\phi_0, \qquad \Theta_1 = 0, \qquad \Theta_2 = 2\kappa_1\phi_2.$$

Thus, in this case, only the extreme components of ϕ_{AB} appear in Θ_{AB}, and hence in $\eta_{AA'}$. This is also true for the right hand side of equation (3.5b). To see this, we note that equation (2.11d) can be used to express $(\hat{\mathcal{L}}_\xi\phi)_{AB}$ in terms of $\eta_{AA'}$ and $(\hat{\mathcal{L}}_\xi\Theta)_{AB}$,

$$(\hat{\mathcal{L}}_\xi\phi)_{AB} = \frac{\kappa_{AB}\xi^{FF'}\eta_{FF'}}{(\kappa_{CD}\kappa^{CD})} + \frac{\kappa_{(A}{}^{F}(\hat{\mathcal{L}}_\xi\Theta)_{B)F}}{(\kappa_{CD}\kappa^{CD})}. \tag{3.11}$$

Here, also the relation $(\hat{\mathcal{L}}_\xi\Theta)_{AB} = -\kappa_{(A}{}^{C}(\hat{\mathcal{L}}_\xi\phi)_{B)C}$ was used. The above discussion shows that in a type $\{2,2\}$ spacetime, the system (3.5) can be written as a second order differential system for the pair of extreme Maxwell scalars ϕ_0, ϕ_2. In particular, we find that in a type $\{2,2\}$ spacetime, V_{ab} can be written solely in terms of the extreme components of ϕ_{AB}. This has two important consequences.

Firstly, in the Kerr–Newman spacetime the extreme components of the time-independent Coulomb solutions of the Maxwell field equations are zero, and hence the conserved tensor V_{ab} naturally excludes non-radiating solutions of the Maxwell equation. Secondly, as we shall explain below, the second order system of differential equations for Θ_{AB}, becomes the combined TME-TSI system. To see this, we shall write out equation (3.5) as a set of scalar equations, by projecting the equations on a principal dyad. We shall write the resulting system of equations in the GHP formalism [**9**, **12**].

A priori, equations (3.5) imply two sets of three equations. However, one of the three scalar equations implied by equation (3.6b) is redundant. To see this, we shall need the following technical lemma. This fact can also be seen by direct calculations in the GHP formalism.

Lemma 3.3. *Assume that* $(\mathcal{M}, g_{ab})$ *is a type* $\{2,2\}$ *spacetime which admits a valence* $(2,0)$ *Killing spinor* κ_{AB} *and assume that the aligned matter condition holds with respect to* κ_{AB}.

If φ_{AB} has the property $\kappa^{AB}\varphi_{AB} = 0$, then equation (3.6b) *with*

$$\varpi_{AB} = \frac{2\kappa_{AB}\xi^{FF'}(\mathscr{C}^\dagger_{2,0}\varphi)_{FF'}}{3(\kappa_{CD}\kappa^{CD})} + \frac{2\kappa_{(A}{}^{F}(\hat{\mathcal{L}}_\xi\varphi)_{B)F}}{3(\kappa_{CD}\kappa^{CD})} \tag{3.12}$$

is equivalent to

$$\kappa_{(A}{}^{C}(\mathscr{C}_{1,1}\mathscr{C}^\dagger_{2,0}\varphi)_{B)C} = -\tfrac{1}{3}(\hat{\mathcal{L}}_\xi\varphi)_{AB}. \tag{3.13}$$

Proof. We see that ϖ_{AB} consists of two pieces,

$$\kappa_{(A}{}^{C}\varpi_{B)C} = -\tfrac{1}{3}(\hat{\mathcal{L}}_\xi\varphi)_{AB} \text{ and } \kappa^{AB}\varpi_{AB} = \tfrac{2}{3}\xi^{AA'}(\mathscr{C}^\dagger_{2,0}\varphi)_{AA'}.$$

Correspondingly, (3.6b) can be split up into (3.13) and

$$\kappa^{AB}(\mathscr{C}_{1,1}\mathscr{C}^\dagger_{2,0}\varphi)_{AB} = \tfrac{2}{3}\xi^{AA'}(\mathscr{C}^\dagger_{2,0}\varphi)_{AA'},$$

which follows from $\kappa^{AB}\varphi_{AB} = 0$: In fact the gradient of $\kappa^{AB}\varphi_{AB} = 0$ is

$$0 = -\tfrac{2}{3}\xi^{B}{}_{A'}\varphi_{AB} - \tfrac{2}{3}\kappa_{AB}(\mathscr{C}^\dagger_{2,0}\varphi)^{B}{}_{A'} + \kappa^{BC}(\mathscr{T}_{2,0}\varphi)_{ABCA'}.$$

Taking a divergence of this and using the commutator (2.8d) gives

$$\begin{aligned}0 = {}& \tfrac{4}{3}\xi^{AA'}(\mathscr{C}^\dagger_{2,0}\varphi)_{AA'} - 2\kappa^{AB}(\mathscr{C}_{1,1}\mathscr{C}^\dagger_{2,0}\varphi)_{AB} \\ &+ \Psi_{ABCD}\kappa^{AB}\varphi^{CD} - 4\Lambda\kappa^{AB}\varphi_{AB}.\end{aligned}$$

The curvature terms drop out due to $\kappa^{AB}\varphi_{AB} = 0$ and that (2.14) implies that Ψ_{ABCD} is proportional to $\kappa_{(AB}\kappa_{CD)}$. q.e.d.

For a Petrov type $\{2,2\}$ spacetime admitting a valence $(2,0)$ Killing spinor with aligned matter, the GHP spin coefficients $\kappa, \kappa', \sigma, \sigma'$ are zero. From this fact and a direct calculation one obtains the following lemma:

Lemma 3.4. *For any type $\{2,2\}$ spacetime admitting a Killing spinor κ_{AB} with aligned matter we can write equations* (3.6a) *and* (3.13) *in the GHP formalism as follows.*

1) *The GHP form of equation* (3.6a) *is*

$$0 = -2\rho\,þ\,\varphi_2 + þ\,þ\,\varphi_2 - 2\tau'\,ð'\,\varphi_0 + ð'\,ð'\,\varphi_0, \tag{3.14a}$$

$$\begin{aligned}0 = {}& -\tau\,þ\,\varphi_2 + \tfrac{1}{2}\bar{\tau}'\,þ\,\varphi_2 + \tfrac{1}{2}\,þ\,ð\,\varphi_2 + \tfrac{1}{2}\bar{\tau}\,þ'\,\varphi_0 - \tau'\,þ'\,\varphi_0 \\ &+ \tfrac{1}{2}\,þ'\,ð'\,\varphi_0 - \rho\,ð\,\varphi_2 + \tfrac{1}{2}\bar{\rho}\,ð\,\varphi_2 + \tfrac{1}{2}\,ð\,þ\,\varphi_2 - \rho'\,ð'\,\varphi_0 \\ &+ \tfrac{1}{2}\bar{\rho}'\,ð'\,\varphi_0 + \tfrac{1}{2}\,ð'\,þ'\,\varphi_0,\end{aligned} \tag{3.14b}$$

$$0 = -2\rho'\,þ'\,\varphi_0 + þ'\,þ'\,\varphi_0 - 2\tau\,ð\,\varphi_2 + ð\,ð\,\varphi_2. \tag{3.14c}$$

2) *The GHP form of equation* (3.13) *is*

$$0 = -þ\,þ'\,\varphi_0 + \rho\,þ'\,\varphi_0 + \bar{\rho}\,þ'\,\varphi_0 + ð\,ð'\,\varphi_0 - \tau\,ð'\,\varphi_0 - \bar{\tau}'\,ð'\,\varphi_0, \tag{3.15a}$$

$$0 = -\rho'\,þ\,\varphi_2 - \bar{\rho}'\,þ\,\varphi_2 + þ'\,þ\,\varphi_2 + \bar{\tau}\,ð\,\varphi_2 + \tau'\,ð\,\varphi_2 - ð'\,ð\,\varphi_2. \tag{3.15b}$$

Remark 3.5. 1) We see from Lemma 3.4 that equation (3.6a) with scalar form (3.14) is equivalent to the TSI for Maxwell given in scalar form (with a different scaling) in [**1**, §5.4.2]. Similarly, in view of Lemma 3.3, the equation defined by (3.6b) with right hand side given by (3.12), is given in scalar form (3.15) which is the TME with a different scaling, cf. [**2**].

2) Equations (3.14)–(3.15) are a sufficient condition for the tensor X_{ab} given by (3.8) with ϖ_{AB} given by (3.12) to be conserved.

We conclude from this discussion that V_{ab} (or rather X_{ab} given by (3.8) with ϖ_{AB} given by (3.12)) can be thought of as an "energy-momentum tensor" for the $s = 1$ combined TME-TSI system, which corresponds to the scalar equations (3.14)–(3.15).

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