

THE COHOMOLOGICAL CREPANT RESOLUTION CONJECTURE FOR THE HILBERT-CHOW MORPHISMS

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Abstract

We prove that Ruan's Cohomological Crepant Resolution Conjecture holds for the Hilbert-Chow morphisms. There are two main ideas in the proof. The first one is to use the representation theoretic approach proposed in [QW] which involves vertex operator techniques. The second is to prove certain universality structures about the 3-pointed genus-0 extremal Gromov-Witten invariants of the Hilbert schemes by using the indexing techniques from [LiJ], the product formula from [Beh2] and the co-section localization from [KL1, KL2, LL]. We then reduce Ruan's Conjecture from the case of an arbitrary surface to the case of smooth projective toric surfaces which has already been proved in [Che].

1. Introduction

In [ChR], Chen and Ruan defined the orbifold cohomology ring $H_{\text{CR}}^*(Z)$ for an orbifold Z . Motivated by orbifold string theory from physics, Ruan [Ruan] proposed the Cohomological Crepant Resolution Conjecture. It eventually evolved into the Crepant Resolution Conjecture after the work of Bryan-Graber, Coates-Corti-Iritani-Tseng and Coates-Ruan [BG, CCIT, CoR]. Roughly speaking, assuming that an orbifold Z has a crepant resolution W , then the Crepant Resolution Conjecture predicts that the orbifold Gromov-Witten theory of Z is ring isomorphic (in the sense of analytic continuations, symplectic transformations and change of variables of type $q = -e^{i\theta}$) to the ordinary cohomology ring of W plus those quantum corrections on W which are related to curves contracted by the crepant resolution. We refer to [BG, Che, Coa] and the references there for other excellent examples confirming the Crepant Resolution Conjecture.

In this paper, we prove that Ruan's Cohomological Crepant Resolution Conjecture holds for the Hilbert-Chow morphisms. Let X be a smooth projective complex surface, and $X^{[n]}$ be the Hilbert scheme of

¹Partially supported by the grant FSGRF12SC10 and GRF602512.

²Partially supported by an NSF grant.

Received October, 11, 2013.

points in X . Sending an element in $X^{[n]}$ to its support in the symmetric product $X^{(n)}$, we obtain the Hilbert–Chow morphism $\rho_n : X^{[n]} \rightarrow X^{(n)}$, which is a crepant resolution of singularities. Let $H_{\rho_n}^*(X^{[n]})$ be the quantum corrected cohomology ring (see Sect. 4 for details).

Theorem 1.1. *Let X be a simply connected smooth projective surface. Then, Ruan’s Cohomological Crepant Resolution Conjecture holds for the Hilbert–Chow morphism ρ_n , i.e., the two rings $H_{\rho_n}^*(X^{[n]})$ and $H_{\text{CR}}^*(X^{(n)})$ are isomorphic.*

This theorem has been proved earlier when $n = 2, 3$ [ELQ, LQ], when K_X is trivial [FG, LS], and when X is a smooth toric surface [Che]. We also refer to [LQW4, MO, OP, QW, Zho] for discussions when X is quasi-projective.

There are two main ingredients in our proof of Theorem 1.1. The first one is the axiomatization approach originated from [Leh, LQW1] and formulated in [QW]. This approach involves Heisenberg algebra actions and vertex operator techniques pioneered in [Gro, Nak]. We recall that a graded Frobenius algebra over a field k is a finite dimensional graded vector space A with a graded associative multiplication $A \otimes A \rightarrow A$ and unit element 1_A together with a linear form $T : A \rightarrow k$ such that the induced bilinear form $\langle a, b \rangle := T(ab)$ is nondegenerate. For $k \geq 1$, the k -th co-product $\tau_{k*} : A \rightarrow A^{\otimes k}$ is defined by requiring $\langle \tau_{k*}(a), b_1 \otimes \cdots \otimes b_k \rangle = T(ab_1 \cdots b_k)$. Now the axiomatization in [QW] states that the algebra structure on each $A^{[n]}$ in a sequence of graded Frobenius algebras $A^{[n]}$ ($n \geq 0$) is determined if

- (A1) the direct sum $\bigoplus_n A^{[n]}$ affords the structure of the Fock space of a Heisenberg algebra modeled on $A := A^{[1]}$.
- (A2) There exists a sequence of elements $\tilde{G}_k(\alpha, n) \in A^{[n]}$ depending on $\alpha \in A$ (linearly) and a non-negative integer k . Define the operators $\tilde{\mathfrak{G}}_k(\alpha)$ on $\bigoplus_n A^{[n]}$ which act on the component $A^{[n]}$ via multiplication by $\tilde{G}_k(\alpha, n) \in A^{[n]}$. The operators $\tilde{\mathfrak{G}}_k(\alpha)$ and the Heisenberg generators satisfy:

$$(1.1) \quad \tilde{\mathfrak{G}}_1(1_A) = -\frac{1}{6} : \mathfrak{a}^3 :_0 (\tau_{3*} 1_A),$$

$$(1.2) \quad [\tilde{\mathfrak{G}}_k(\alpha), \mathfrak{a}_{-1}(\beta)] = \frac{1}{k!} \mathfrak{a}_{-1}^{\{k\}}(\alpha\beta),$$

where $: \mathfrak{a}^3 :_0$ is the zero mode in the normally ordered product $: \mathfrak{a}^3 :$, and $\mathfrak{a}_{-1}^{\{k\}}(\alpha)$ denotes the k -th derivative with $\mathfrak{a}_{-1}^{\{0\}}(\alpha) = \mathfrak{a}_{-1}(\alpha)$ and $\mathfrak{a}_{-1}^{\{k\}}(\alpha) = [\tilde{\mathfrak{G}}_1(1_A), \mathfrak{a}_{-1}^{\{k-1\}}(\alpha)]$ for $k \geq 1$.

When (A1) and (A2) are satisfied, the algebra $A^{[n]}$ is generated by the elements

$$\tilde{G}_k(\alpha, n) \in A^{[n]}, \quad \alpha \in A, k \geq 0.$$

In addition, the product is determined by (1.1) and (1.2). On one hand, with $A^{[n]} = H_{\text{CR}}^*(X^{(n)})$ (viewed as an algebra over \mathbb{C}), the results in [QW] (see Theorem 3.1 below) indicate that (A1) and (A2) hold for the rings $H_{\text{CR}}^*(X^{(n)})$. On the other hand, by [Gro, Nak] and [LL], the rings $A^{[n]} = H_{\rho_n}^*(X^{[n]}) (= H^*(X^{[n]}))$ as vector spaces also satisfy (A1) and (1.1). Moreover, using [Che], we prove that (1.2) holds when X is a smooth projective toric surface.

To prove that the rings $A^{[n]} = H_{\rho_n}^*(X^{[n]})$ satisfy (1.2) for an arbitrary surface X , our second main ingredient comes into play. It involves finer analysis of the virtual fundamental cycle using the method in [LiJ] and the co-section localization technique in [KL1, KL2, LL]. Let $X^{[n,d]}$ be the moduli space of 3-pointed genus-0 degree- d stable maps to $X^{[n]}$. By [LL], every stable map $(\varphi, C) \in X^{[n,d]}$ has a standard decomposition $\varphi = (\varphi_1, \dots, \varphi_l) \in X^{[n,d]}$ where the stable reduction φ_i^{st} is contained in $X^{[n_i, d_i]}$ for some n_i and d_i , $\rho_{n_i}(\text{Im}(\varphi_i)) = n_i x_i$, the points x_1, \dots, x_l are distinct, and $\varphi(p) = \sum_i \varphi_i(p)$ for all $p \in C$. We use the ideas from [LiJ] to index the support of $\rho_n(\text{Im}(\varphi)) = \sum_i n_i x_i \in X^{(n)}$. This is done by introducing the notion of 3-pointed genus-0 degree- δ α -stable maps to $X^{[n]}$, where $\alpha = (\alpha_1, \dots, \alpha_l)$ denotes a partition of the set $[n] = \{1, \dots, n\}$ and $\delta = (\delta_1, \dots, \delta_l)$ with δ_i 's being nonnegative integers. The set of such pairs (α, δ) with $\sum_i \delta_i = d$ is denoted by $\mathcal{P}_{[n],d}$. The techniques in [LiJ] and the product formula in [Beh2] for Gromov–Witten invariants enable us to express the virtual fundamental cycle $[X^{[n,d]}]^{\text{vir}}$ in terms of certain discrepancy cycles $[\Theta^{[\alpha, \delta]}]$, $(\alpha, \delta) \in \mathcal{P}_{[n],d}$. In fact, one of the key points in the paper is to study such decomposition of $[X^{[n,d]}]^{\text{vir}}$ as a sum of cycles indexed by the partition type of $\rho_n(\text{Im}(\varphi)) \in X^{(n)}$. However, this cannot be done on the moduli space $X^{[n,d]}$. The technique to overcome this impasse is to introduce the Hilbert scheme of α -points $X^{[n]}$ and an non-separated space $X^{[\leq n]}$ following [LiJ]. Then the cycle $\text{ev}_*([X^{[n,d]}]^{\text{vir}})$ is a sum of various $[\Theta^{[\alpha, \delta]}]$ in $(X^{[\leq n]})^3$. Even though the space $X^{[\leq n]}$ is not Hausdorff in analytic topology, all the operations involving $X^{[\leq n]}$ in this paper are all algebraic topological, such as pullbacks of cohomology classes and cap products, which are defined on any topological spaces. Combining with the co-section localization theory in [KL1, KL2, LL], pairings with $[\Theta^{[\alpha, \delta]}]$ can be studied via C^∞ -maps from X to the Grassmannians. For $d \geq 1$, we assemble those $[\Theta^{[\alpha, \delta]}]$, $(\alpha, \delta) \in \mathcal{P}_{[n],d}$ with $\delta_i > 0$ for every i into a homology class $\mathfrak{Z}_{n,d} \in H_*((X^{[n]})^3)$. Note that we are back to the original Hilbert scheme $X^{[n]}$. Now the structure of the 3-pointed genus-0 extremal Gromov–Witten invariants of $X^{[n]}$ is given by the following two theorems.

Theorem 1.2. *Let $A_1, A_2, A_3 \in H^*(X^{[n]})$ be Heisenberg monomial classes, and $\pi_{m,i}$ be the i -th projection on $(X^{[m]})^3$. Then, $\langle A_1, A_2, A_3 \rangle_{0,d\beta_n}$ is equal to*

$$(1.3) \quad \sum_{m \leq n} \sum_{\substack{A_{1,1} \circ A_{1,2} = A_1 \\ A_{2,1} \circ A_{2,2} = A_2 \\ A_{3,1} \circ A_{3,2} = A_3}} \langle A_{1,1}, A_{2,1}, A_{3,1} \rangle \cdot \left\langle \mathfrak{Z}_{m,d}, \prod_{i=1}^3 \pi_{m,i}^* A_{i,2} \right\rangle.$$

Theorem 1.3. *Let $A_1, A_2, A_3 \in H^*(X^{[n]})$ be Heisenberg monomial classes.*

(i) *If A_i contains a factor $\mathfrak{a}_{-j}(x)$ for some i , then*

$$\left\langle \mathfrak{Z}_{n,d}, \prod_{i=1}^3 \pi_{n,i}^* A_i \right\rangle = 0.$$

(ii) *For $1 \leq i \leq 3$, let $A_i = \mathfrak{a}_{-\lambda^{(i)}}(1_X) \mathfrak{a}_{-n_{i,1}}(\alpha_{i,1}) \cdots \mathfrak{a}_{-n_{i,u_i}}(\alpha_{i,u_i})|0$ where $u_i \geq 0$ and $|\alpha_{i,1}| = \dots = |\alpha_{i,u_i}| = 2$. Then,*

$$(1.4) \quad \left\langle \mathfrak{Z}_{n,d}, \prod_{i=1}^3 \pi_{n,i}^* A_i \right\rangle = \prod_{i=1}^3 \prod_{j=1}^{u_i} \langle K_X, \alpha_{i,j} \rangle \cdot p,$$

where p is a polynomial in $\langle K_X, K_X \rangle$ whose degree is at most $(n - \sum_{i,j} n_{i,j})/2$, and whose coefficients depend only on $d, n, \lambda^{(i)}, n_{i,j}$ (and hence are independent of the surface X and the classes $\alpha_{i,j}$).

We refer to Definition 2.9 for the operation \circ appearing in (1.3), and to Definition 2.2 for the notation $\mathfrak{a}_{-\lambda^{(i)}}(1_X)$ appearing in Theorem 1.3 (ii). Geometrically, we may think of the pairing

$$\left\langle \mathfrak{Z}_{m,d}, \prod_{i=1}^3 \pi_{m,i}^* A_{i,2} \right\rangle,$$

in (1.3) as the contributions of the non-constant components φ_i in the standard decomposition of $\varphi = (\varphi_1, \dots, \varphi_l) \in X^{[n,d]}$, while those constant components φ_i contribute to the factor $\langle A_{1,1}, A_{2,1}, A_{3,1} \rangle$ in (1.3).

Using Theorem 1.2 and Theorem 1.3, we are able to reduce the proof of (1.2) for $A^{[n]} = H_{\rho_n}^*(X^{[n]})$ from an arbitrary surface X to the case when X is a smooth projective toric surface. This proves (1.2) for $A^{[n]} = H_{\rho_n}^*(X^{[n]})$ and hence completes the proof of Theorem 1.1.

Finally, this paper is organized as follows. In Sect. 2, we review the Hilbert schemes of points on surfaces and Heisenberg algebras. In Sect. 3, we recall from [QW] the results regarding $H_{\text{CR}}^*(X^{(n)})$. In Sect. 4, we review Ruan’s Cohomological Crepant Resolution Conjecture. In Sect. 5, we prove Theorem 1.2 and Theorem 1.3. In Sect. 6, we verify (1.2) and Theorem 1.1.

Conventions: All the homology and cohomology groups are in \mathbb{C} -coefficients unless otherwise specified. For a subvariety Z of a smooth projective variety Y , we will use Z or $[Z]$ to denote the corresponding cycle/cohomology class, and use 1_Y to denote the fundamental cohomology class of Y . The symbol $A \cdot B$ denotes the cup product for $A, B \in H^*(Y)$. For $A_1, \dots, A_k \in H^*(Y)$, let $\langle A_1, \dots, A_k \rangle = \int_Y A_1 \cdots A_k$. By abuse of notation, for $A \in H_*(W)$ and $B \in H^*(W)$ of an arbitrary topological space W , $\langle A, B \rangle$ also stands for the natural pairing between the homology group and the cohomology group. For subsets A and B of W , $A \cap B \subset W$ stands for the intersection of the two subsets; for $A \in H_*(W)$ and $B \in H^*(W)$, $A \cap B \in H_*(W)$ denotes the cap product.

Acknowledgment. The authors thank Professor Jun Li for offering enormous helps and suggesting valuable ideas, without which this paper would be impossible to complete. In particular, the crucial Lemma 5.5, Lemma 5.9 and their proofs are due to him. The authors also thank Professors Wan Keng Cheong, Yongbin Ruan and Weiqiang Wang for stimulating discussions. In addition, the authors thank the referee for carefully reading the manuscript and for providing valuable suggestions which have greatly improved the exposition of the paper.

2. Hilbert schemes of points on surfaces

Let X be a smooth projective complex surface with the canonical class K_X and the Euler class e_X , and $X^{[n]}$ be the Hilbert scheme of points in X . An element in $X^{[n]}$ is represented by a length- n 0-dimensional closed subscheme ξ of X . It is well known that $X^{[n]}$ is smooth. For a subset $Y \subset X$, define

$$M_n(Y) = \{ \xi \in X^{[n]} \mid \text{Supp}(\xi) = \{x\} \text{ for some } x \in Y \}.$$

Let $\mathcal{Z}_n = \{ (\xi, x) \in X^{[n]} \times X \mid x \in \text{Supp}(\xi) \}$ be the universal codimension-2 subscheme of $X^{[n]} \times X$. Let p_1 and p_2 be the two projections of $X^{[n]} \times X$. Let

$$\mathbb{H}_X = \bigoplus_{n=0}^{+\infty} H^*(X^{[n]})$$

be the direct sum of total cohomology groups of the Hilbert schemes $X^{[n]}$.

For $m \geq 0$ and $n > 0$, let $Q^{[m,m]} = \emptyset$ and define $Q^{[m+n,m]}$ to be the closed subset:

$$\{ (\xi, x, \eta) \in X^{[m+n]} \times X \times X^{[m]} \mid \xi \supset \eta \text{ and } \text{Supp}(I_\eta/I_\xi) = \{x\} \}.$$

We recall Nakajima's definition of the Heisenberg operators [Nak]. Let $n > 0$. The linear operator $\mathbf{a}_{-n}(\alpha) \in \text{End}(\mathbb{H}_X)$ with $\alpha \in H^*(X)$ is defined by

$$\mathbf{a}_{-n}(\alpha)(a) = \tilde{p}_{1*}([Q^{[m+n,m]}] \cdot \tilde{\rho}^* \alpha \cdot \tilde{p}_2^* a),$$

for $a \in H^*(X^{[m]})$, where $\tilde{p}_1, \tilde{p}, \tilde{p}_2$ are the projections of $X^{[m+n]} \times X \times X^{[m]}$ to $X^{[m+n]}, X, X^{[m]}$ respectively. Define the linear operator $\mathfrak{a}_n(\alpha) \in \text{End}(\mathbb{H}_X)$ to be $(-1)^n$ times the operator obtained from the definition of $\mathfrak{a}_{-n}(\alpha)$ by switching the roles of \tilde{p}_1 and \tilde{p}_2 . We also set $\mathfrak{a}_0(\alpha) = 0$.

For $n > 0$ and a homogeneous class $\alpha \in H^*(X)$, let $|\alpha| = s$ if $\alpha \in H^s(X)$, and let $G_i(\alpha, n)$ be the component in $H^{|\alpha|+2i}(X^{[n]})$ of

$$G(\alpha, n) = p_{1*}(\text{ch}(\mathcal{O}_{Z_n}) \cdot p_2^* \text{td}(X) \cdot p_2^* \alpha) \in H^*(X^{[n]}),$$

where $\text{ch}(\mathcal{O}_{Z_n})$ denotes the Chern character of \mathcal{O}_{Z_n} and $\text{td}(X)$ denotes the Todd class. Set $G_i(\alpha, 0) = 0$. We extend the notion $G_i(\alpha, n)$ linearly to an arbitrary class $\alpha \in H^*(X)$. The Chern character operator $\mathfrak{G}_i(\alpha) \in \text{End}(\mathbb{H}_X)$ is defined to be the operator acting on the component $H^*(X^{[n]})$ by the cup product with $G_i(\alpha, n)$. It was proved in [LQW1] that the cohomology ring of $X^{[n]}$ is generated by the classes $G_i(\alpha, n)$ where $0 \leq i < n$ and α runs over a linear basis of $H^*(X)$. Let $\mathfrak{d} = \mathfrak{G}_1(1_X)$ where 1_X is the fundamental cohomology class of X . The operator \mathfrak{d} was first introduced in [Leh]. For a linear operator $\mathfrak{f} \in \text{End}(\mathbb{H}_X)$, define its derivative \mathfrak{f}' by $\mathfrak{f}' = [\mathfrak{d}, \mathfrak{f}]$. The k -th derivative $\mathfrak{f}^{(k)}$ is defined inductively by $\mathfrak{f}^{(k)} = [\mathfrak{d}, \mathfrak{f}^{(k-1)}]$.

Let $\mathfrak{a}_{m_1} \mathfrak{a}_{m_2}$: be $\mathfrak{a}_{m_1} \mathfrak{a}_{m_2}$ when $m_1 \leq m_2$ and $\mathfrak{a}_{m_2} \mathfrak{a}_{m_1}$ when $m_1 > m_2$. For $k \geq 1$, $\tau_{k*} : H^*(X) \rightarrow H^*(X^k)$ is the linear map induced by the diagonal embedding $\tau_k : X \rightarrow X^k$, and $\mathfrak{a}_{m_1} \cdots \mathfrak{a}_{m_k}(\tau_{k*}(\alpha))$ denotes $\sum_j \mathfrak{a}_{m_1}(\alpha_{j,1}) \cdots \mathfrak{a}_{m_k}(\alpha_{j,k})$ when $\tau_{k*} \alpha = \sum_j \alpha_{j,1} \otimes \cdots \otimes \alpha_{j,k}$ via the Künneth decomposition of $H^*(X^k)$.

The following is a combination of various theorems from [Nak, Gro, Leh, LQW1]. Our notations and convention of signs are consistent with [LQW2].

Theorem 2.1. *Let $k \geq 0, n, m \in \mathbb{Z}$ and $\alpha, \beta \in H^*(X)$. Then,*

- (i) *the operators $\mathfrak{a}_n(\alpha)$ satisfy a Heisenberg algebra commutation relation:*

$$[\mathfrak{a}_m(\alpha), \mathfrak{a}_n(\beta)] = -m \delta_{m,-n} \cdot \langle \alpha, \beta \rangle \cdot \text{Id}_{\mathbb{H}_X}.$$

The space \mathbb{H}_X is an irreducible module over the Heisenberg algebra generated by the operators $\mathfrak{a}_n(\alpha)$ with a highest weight vector $|0\rangle = 1 \in H^0(X^{[0]}) \cong \mathbb{C}$.

- (ii) $\mathfrak{G}_1(\alpha) = -\frac{1}{6} : \mathfrak{a}^3 :_0 (\tau_{3*} \alpha) - \sum_{n>0} \frac{n-1}{2} : \mathfrak{a}_n \mathfrak{a}_{-n} : (\tau_{2*}(K_X \alpha))$.
- (iii) $[\mathfrak{G}_k(\alpha), \mathfrak{a}_{-1}(\beta)] = \frac{1}{k!} \cdot \mathfrak{a}_{-1}^{(k)}(\alpha\beta)$.

The Lie brackets in Theorem 2.1 are understood in the super sense according to the parity of the degrees of the cohomology classes involved.

Also, Theorem 2.1 (i) implies that \mathbb{H}_X is linearly spanned by the cohomology classes $\mathbf{a}_{-n_1}(\alpha_1) \cdots \mathbf{a}_{-n_k}(\alpha_k)|0\rangle$ where $k \geq 0$ and $n_1, \dots, n_k > 0$. These classes are called *Heisenberg monomial classes*.

Definition 2.2. Let $\alpha \in H^*(X)$, and

$$\lambda = (\cdots (-2)^{m-2} (-1)^{m-1} 1^{m_1} 2^{m_2} \cdots)$$

be a *generalized partition* of the integer $n = \sum_i im_i$ whose part $i \in \mathbb{Z}$ has multiplicity m_i . Define $\ell(\lambda) = \sum_i m_i$, $|\lambda| = \sum_i im_i = n$, $s(\lambda) = \sum_i i^2 m_i$, $\lambda! = \prod_i m_i!$, and

$$\mathbf{a}_\lambda(\alpha) = \prod_i (\mathbf{a}_i(\alpha))^{m_i}, \quad \mathbf{a}_\lambda(\tau_*\alpha) = \left(\prod_i (\mathbf{a}_i)^{m_i} \right) (\tau_{\ell(\lambda)*}\alpha),$$

where $\prod_i (\mathbf{a}_i)^{m_i}$ is understood to be $\cdots \mathbf{a}_{-2}^{m-2} \mathbf{a}_{-1}^{m-1} \mathbf{a}_1^{m_1} \mathbf{a}_2^{m_2} \cdots$. A generalized partition becomes a *partition* in the usual sense if $m_i = 0$ for every $i < 0$. A partition λ of n is denoted by $\lambda \vdash n$.

The next three theorems were proved in [LQW3].

Theorem 2.3. Let $k \geq 0$, $n \in \mathbb{Z}$, and $\alpha \in H^*(X)$. Then, $\mathbf{a}_n^{(k)}(\alpha)$ is equal to

$$\begin{aligned} & (-n)^k k! \left(\sum_{\substack{\ell(\lambda)=k+1 \\ |\lambda|=n}} \frac{1}{\lambda!} \mathbf{a}_\lambda(\tau_*\alpha) - \sum_{\substack{\ell(\lambda)=k-1 \\ |\lambda|=n}} \frac{s(\lambda) - 1}{24\lambda!} \mathbf{a}_\lambda(\tau_*(e_X\alpha)) \right) \\ & + \sum_{\epsilon \in \{K_X, K_X^2\}} \sum_{\ell(\lambda)=k+1-|\epsilon|/2, |\lambda|=n} \frac{f_{|\epsilon|}(\lambda)}{\lambda!} \mathbf{a}_\lambda(\tau_*(\epsilon\alpha)), \end{aligned}$$

where all the numbers $f_{|\epsilon|}(\lambda)$ are independent of X and α .

Theorem 2.4. Let $k \geq 0$ and $\alpha \in H^*(X)$. Then, $\mathfrak{G}_k(\alpha)$ is equal to

$$\begin{aligned} & - \sum_{\ell(\lambda)=k+2, |\lambda|=0} \frac{1}{\lambda!} \mathbf{a}_\lambda(\tau_*\alpha) + \sum_{\ell(\lambda)=k, |\lambda|=0} \frac{s(\lambda) - 2}{24\lambda!} \mathbf{a}_\lambda(\tau_*(e_X\alpha)) \\ & + \sum_{\epsilon \in \{K_X, K_X^2\}} \sum_{\ell(\lambda)=k+2-|\epsilon|/2, |\lambda|=0} \frac{g_{|\epsilon|}(\lambda)}{\lambda!} \mathbf{a}_\lambda(\tau_*(\epsilon\alpha)), \end{aligned}$$

where all the numbers $g_{|\epsilon|}(\lambda)$ are independent of X and α .

Theorem 2.5. *Let $n \geq 1, k \geq 0$, and $\alpha \in H^*(X)$. Then, $G_k(\alpha, n)$ is equal to*

$$\begin{aligned} & \sum_{0 \leq j \leq k} \sum_{\substack{\lambda \vdash (j+1) \\ \ell(\lambda) = k-j+1}} \frac{(-1)^{|\lambda|-1}}{\lambda! \cdot |\lambda|!} \cdot \mathbf{1}_{-(n-j-1)} \mathbf{a}_{-\lambda}(\tau_* \alpha) |0\rangle \\ + & \sum_{0 \leq j \leq k} \sum_{\substack{\lambda \vdash (j+1) \\ \ell(\lambda) = k-j-1}} \frac{(-1)^{|\lambda|}}{\lambda! \cdot |\lambda|!} \cdot \frac{|\lambda| + s(\lambda) - 2}{24} \cdot \mathbf{1}_{-(n-j-1)} \mathbf{a}_{-\lambda}(\tau_*(e_X \alpha)) |0\rangle \\ + & \sum_{\substack{\epsilon \in \{K_X, K_X^2\} \\ 0 \leq j \leq k, \lambda \vdash (j+1) \\ \ell(\lambda) = k-j+1-|\epsilon|/2}} \frac{(-1)^{|\lambda|} g_{|\epsilon|}(\lambda + (1^{j+1}))}{\lambda! \cdot |\lambda|!} \cdot \mathbf{1}_{-(n-j-1)} \mathbf{a}_{-\lambda}(\tau_*(\epsilon \alpha)) |0\rangle, \end{aligned}$$

where $\mathbf{1}_{-(n-j-1)}$ denotes $\mathbf{a}_{-1}(1_X)^{n-j-1}/(n-j-1)!$ when $(n-j-1) \geq 0$ and is 0 when $(n-j-1) < 0$, the universal function $g_{|\epsilon|}$ is from Theorem 2.4, and $\lambda + (1^{j+1})$ is the partition obtained from λ by adding $(j+1)$ to the multiplicity of 1.

Lemma 2.6. $[\mathbf{a}_{n_1} \cdots \mathbf{a}_{n_k}(\tau_{k*} \alpha), \mathbf{a}_{m_1} \cdots \mathbf{a}_{m_s}(\tau_{s*} \beta)]$ is equal to

$$- \sum_{t=1}^k \sum_{j=1}^s n_t \delta_{n_t, -m_j} \cdot \left(\prod_{l=1}^{j-1} \mathbf{a}_{m_l} \prod_{1 \leq u \leq k, u \neq t} \mathbf{a}_{n_u} \prod_{l=j+1}^s \mathbf{a}_{m_l} \right) (\tau_{(k+s-2)*}(\alpha\beta)).$$

The above lemma was proved in [LQW2], and will be used implicitly in many proofs throughout the paper. The following geometric result was proved in [LQW5].

Proposition 2.7. *Let the classes $\alpha_1, \dots, \alpha_k \in \bigoplus_{i=1}^4 H^i(X)$ be respectively represented by the cycles $X_1, \dots, X_k \subset X$ in general position. Then, the Heisenberg monomial class*

$$\left(\prod_{i=1}^t \frac{\mathbf{a}_{-i}(1_X)^{s_i}}{s_i!} \right) \left(\prod_{j=1}^k \mathbf{a}_{-n_j}(\alpha_j) \right) |0\rangle$$

is represented by the closure of the subset consisting of the elements

$$(2.1) \quad \sum_{i=1}^t (\xi_{i,1} + \dots + \xi_{i,s_i}) + \sum_{j=1}^k \xi_j,$$

where $\xi_{i,m} \in M_i(x_{i,m})$ for some $x_{i,m} \in X$, $\xi_j \in M_{n_j}(x_j)$ for some $x_j \in X_j$, and all the points $x_{i,m}, 1 \leq i \leq t, 1 \leq m \leq s_i$ and $x_j, 1 \leq j \leq k$ are distinct.

Theorem 2.9 in [LQW4] expresses a Heisenberg monomial class in terms of a polynomial of the classes $G_k(\gamma, n)$. The following lemma is a special case.

Lemma 2.8. *Let $\lambda \vdash n_0$, $\alpha \in H^*(X)$ with $|\alpha| = 2$, and $m \geq 1$.*

- (i) *Then, the class $\mathbf{1}_{-(n-n_0)}\mathbf{a}_{-\lambda}(x)|0\rangle \in H^*(X^{[n]})$ can be written as a polynomial of the classes $G_k(x, n), k \geq 0$. Moreover, the coefficients and the integers k depend only on λ (hence, are independent of n and X);*
- (ii) *If the odd Betti numbers of the surface X are equal to zero, then*

$$\mathbf{1}_{-(n-n_0-m)}\mathbf{a}_{-\lambda}(x)\mathbf{a}_{-m}(\alpha)|0\rangle = \langle K_X, \alpha \rangle \cdot F_1(n) + \sum_i G_{k_i}(\alpha, n) \cdot F_{2,i}(n),$$

where $F_1(n)$ and $F_{2,i}(n)$ are polynomials of the classes $G_k(x, n), k \geq 0$. Moreover, the coefficients of $F_1(n), F_{2,i}(n)$ and the integers k, k_i depend only on λ and m (hence, are independent of n, α and X).

Proof. Follow from the proof of Theorem 2.9 in [LQW4] by setting $\mathcal{I} = \mathbb{C} \cdot x \subset H^*(X)$ and $\mathcal{I} = \mathbb{C} \cdot x + \mathbb{C} \cdot \alpha \subset H^*(X)$ respectively. q.e.d.

Next, we define some operations which will be used intensively.

Definition 2.9. Let $A = \mathbf{a}_{-n_1}(\alpha_1) \cdots \mathbf{a}_{-n_l}(\alpha_l)|0\rangle$ where $n_i > 0$.

- (i) If $B = \mathbf{a}_{-m_1}(\beta_1) \cdots \mathbf{a}_{-m_s}(\beta_s)|0\rangle$ with $m_1, \dots, m_s > 0$, then define
- (2.2) $A \circ B = \mathbf{a}_{-n_1}(\alpha_1) \cdots \mathbf{a}_{-n_l}(\alpha_l)\mathbf{a}_{-m_1}(\beta_1) \cdots \mathbf{a}_{-m_s}(\beta_s)|0\rangle$.
- (ii) We use the symbol $B \subset A$ if $B = \mathbf{a}_{-n_{i_1}}(\alpha_{i_1}) \cdots \mathbf{a}_{-n_{i_s}}(\alpha_{i_s})|0\rangle$ with $1 \leq i_1 < \dots < i_s \leq l$. In this case, we use A/B or AB^{-1} or $\frac{A}{B}$ to denote the cohomology class obtained from A by deleting the factors $\mathbf{a}_{-n_{i_1}}(\alpha_{i_1}), \dots, \mathbf{a}_{-n_{i_s}}(\alpha_{i_s})$.

3. The ring $H_{\text{CR}}^*(X^{(n)})$

For an orbifold Z , the ring $H_{\text{CR}}^*(Z)$ was defined by Chen and Ruan [ChR]. For a global orbifold M/G where M is a complex manifold with a finite group G action, the ring structure of $H_{\text{CR}}^*(M/G)$ was further clarified in [FG, Uri].

Next, let X be a closed complex manifold, and let $X^{(n)} = X^n/S_n$ be the n -th symmetric product of X . An explicit description of the ring structure of $H_{\text{CR}}^*(X^{(n)})$ has been obtained in [FG]. An alternative approach to the ring structure of $H_{\text{CR}}^*(X^{(n)})$ is given in [QW] via Heisenberg algebra actions. Put

$$\mathcal{F}_X = \bigoplus_{n=0}^{+\infty} H_{\text{orb}}^*(X^{(n)}).$$

In [QW], for $\alpha \in H^*(X)$ and $n \in \mathbb{Z}$, the Heisenberg operators $\mathfrak{p}_n(\alpha) \in \text{End}(\mathcal{F}_X)$ were defined via the restriction and induction maps. Moreover, for $k \geq 0$, the elements $O^k(\alpha, n) \in H_{\text{CR}}^*(X^{(n)})$ were introduced via

the Jucys–Murphy elements in the symmetric groups. Put $O_k(\alpha, n) = 1/k! \cdot O^k(\alpha, n)$. Let the operator $\mathfrak{D}_k(\alpha) \in \text{End}(\mathcal{F}_X)$ be the orbifold ring product with $O_k(\alpha, n)$ in $H_{\text{CR}}^*(X^{(n)})$ for every $n \geq 0$. The operator $\mathfrak{D}_1(1_X)$ plays the role of the boundary operator $\mathfrak{d} = \mathfrak{G}_1(1_X)$ for the Hilbert schemes. Define $\mathfrak{p}_m^{\{k\}}(\alpha)$ inductively by putting $\mathfrak{p}_m^{\{0\}}(\alpha) = \mathfrak{p}_m(\alpha)$ and $\mathfrak{p}_m^{\{k\}}(\alpha) = [\mathfrak{D}_1(1_X), \mathfrak{p}_m^{\{k-1\}}(\alpha)]$ for $k \geq 1$. The following result was proved in [QW].

Theorem 3.1. *Let X be a closed complex manifold. Then,*

- (i) *the operators $\mathfrak{p}_n(\alpha) \in \text{End}(\mathcal{F}_X)$ ($n \in \mathbb{Z}, \alpha \in H^*(X)$) generate a Heisenberg (super)algebra with commutation relations given by*

$$[\mathfrak{p}_m(\alpha), \mathfrak{p}_n(\beta)] = m\delta_{m,-n} \cdot \langle \alpha, \beta \rangle \cdot \text{Id}_{\mathcal{F}_X},$$

where $n, m \in \mathbb{Z}$, $\alpha, \beta \in H^(X)$, and \mathcal{F}_X is an irreducible representation of the Heisenberg algebra with the vacuum vector $|0\rangle = 1 \in H^*(pt) \cong \mathbb{C}$.*

- (ii) $\mathfrak{D}_1(1_X) = -\frac{1}{6} : \mathfrak{p}^3 :_0 (\tau_* 1_X)$. *In general, $\mathfrak{D}_k(\alpha)$ is equal to*

$$(-1)^k \cdot \left(\sum_{\ell(\lambda)=k+2, |\lambda|=0} \frac{1}{\lambda!} \mathfrak{p}_\lambda(\tau_* \alpha) + \sum_{\ell(\lambda)=k, |\lambda|=0} \frac{s(\lambda) - 2}{24\lambda!} \mathfrak{p}_\lambda(\tau_*(e_X \alpha)) \right).$$

- (iii) $[\mathfrak{D}_k(\alpha), \mathfrak{p}_{-1}(\beta)] = \frac{1}{k!} \mathfrak{p}_{-1}^{\{k\}}(\alpha\beta)$, *and both sides are equal to*

$$(-1)^k \cdot \left(\sum_{\substack{\ell(\lambda)=k+1 \\ |\lambda|=-1}} \frac{1}{\lambda!} \mathfrak{p}_\lambda(\tau_*(\alpha\beta)) + \sum_{\substack{\ell(\lambda)=k-1 \\ |\lambda|=-1}} \frac{s(\lambda) - 1}{24\lambda!} \mathfrak{p}_\lambda(\tau_*(e_X \alpha\beta)) \right).$$

Note that there is a fundamental sign difference in the two commutators of Theorems 2.1 (i) and Theorems 3.1 (i). Since $O_k(\alpha, n) = \mathfrak{D}_k(\alpha)\mathfrak{p}_{-1}(1_X)^n|0\rangle/n!$, we see from Theorem 3.1 (ii) that $O_k(\alpha, n)$ is equal to

$$(3.1) \quad (-1)^k \cdot \left(\sum_{\substack{0 \leq j \leq k, \lambda^+(j+1) \\ \ell(\lambda)=k-j+1}} \frac{1}{\lambda! \cdot |\lambda|!} \cdot \mathbf{1}_{-(n-j-1)} \mathfrak{p}_{-\lambda}(\tau_* \alpha)|0\rangle \right. \\ \left. + \sum_{0 \leq j \leq k} \sum_{\substack{\lambda^+(j+1) \\ \ell(\lambda)=k-j-1}} \frac{1}{\lambda! \cdot |\lambda|!} \cdot \frac{|\lambda| + s(\lambda) - 2}{24} \cdot \mathbf{1}_{-(n-j-1)} \mathfrak{p}_{-\lambda}(\tau_*(e_X \alpha))|0\rangle \right).$$

Moreover, as noted in [QW], the ring $H_{\text{CR}}^*(X^{(n)})$ is completely determined by Theorem 3.1 (i), the formula of $\mathfrak{D}_1(1_X)$ in Theorem 3.1 (ii), and Theorem 3.1 (iii). In particular, the ring $H_{\text{CR}}^*(X^{(n)})$ is generated by the classes $O_k(\alpha, n)$ where $k \geq 0$ and α runs over a fixed linear basis of $H^*(X)$.

4. Ruan’s Cohomological Crepant Resolution Conjecture

In this section, we briefly review the definition of Gromov–Witten invariants, and recall Ruan’s Cohomological Crepant Resolution Conjecture for the Hilbert–Chow morphisms.

Let Y be a smooth projective variety. For a fixed homology class $\beta \in H_2(Y, \mathbb{Z})$, let $\overline{\mathfrak{M}}_{g,k}(Y, \beta)$ be the coarse moduli space parameterizing all the stable maps $[\mu : (D; p_1, \dots, p_k) \rightarrow Y]$ such that $\mu_*[D] = \beta$ and the arithmetic genus of D is g . The i -th evaluation map $\text{ev}_i : \overline{\mathfrak{M}}_{g,k}(Y, \beta) \rightarrow Y$ is defined by $\text{ev}_i([\mu : (D; p_1, \dots, p_k) \rightarrow Y]) = \mu(p_i) \in Y$. It is known [FP, LT1, LT2, Beh1, BF] that $\overline{\mathfrak{M}}_{g,k}(Y, \beta)$ is projective and has a virtual fundamental cycle $[\overline{\mathfrak{M}}_{g,k}(Y, \beta)]^{\text{vir}} \in A_{d_0}(\overline{\mathfrak{M}}_{g,k}(Y, \beta))$ where $d_0 = -(K_Y \cdot \beta) + (\dim(Y) - 3)(1 - g) + k$. Let $\alpha_1, \dots, \alpha_k \in H^*(Y)$, and $\text{ev} = \text{ev}_1 \times \dots \times \text{ev}_k : \overline{\mathfrak{M}}_{g,k}(Y, \beta) \rightarrow Y^k$. Then, the k -pointed Gromov–Witten invariant is defined by

$$(4.1) \quad \langle \alpha_1, \dots, \alpha_k \rangle_{g,\beta} = \int_{[\overline{\mathfrak{M}}_{g,k}(Y, \beta)]^{\text{vir}}} \text{ev}^*(\alpha_1 \otimes \dots \otimes \alpha_k).$$

Next, let X be a smooth complex projective surface. Define the homology class

$$(4.2) \quad \beta_n = M_2(x_1) + x_2 + \dots + x_{n-1} \in H_2(X^{[n]}; \mathbb{Z}),$$

where x_1, \dots, x_{n-1} are fixed distinct points in X . An irreducible curve $C \subset X^{[n]}$ is contracted to a point by ρ_n if and only if $C \sim d\beta_n$ for some integer $d > 0$. Let q be a formal variable. For $w_1, w_2, w_3 \in H^*(X^{[n]})$, define a function of q :

$$\langle w_1, w_2, w_3 \rangle_{\rho_n}(q) = \sum_{d \geq 0} \langle w_1, w_2, w_3 \rangle_{0, d\beta_n} q^d.$$

Definition 4.1. The quantum corrected cohomology ring $H_{\rho_n}^*(X^{[n]})$ is the group $H^*(X^{[n]})$ together with the quantum corrected product $w_1 \cdot_{\rho_n} w_2$ defined by

$$(4.3) \quad \langle w_1 \cdot_{\rho_n} w_2, w_3 \rangle = \langle w_1, w_2, w_3 \rangle_{\rho_n}(-1).$$

CONJECTURE 4.2. (Ruan’s Cohomological Crepant Resolution Conjecture) The quantum corrected cohomology ring $H_{\rho_n}^*(X^{[n]})$ is ring isomorphic to $H_{\text{CR}}^*(X^{(n)})$.

Our idea to deal with Conjecture 4.2 is to use the axiomatization approach mentioned in the Introduction. On one hand, letting $A^{[n]} = H_{\text{CR}}^*(X^{(n)})$ and $\tilde{G}_k(\alpha, n) = O_k(\alpha, n)$, we see from Theorem 3.1 that both (A1) and (A2) in the Introduction hold for the rings $H_{\text{CR}}^*(X^{(n)})$. On the other hand, by [Gro, Nak], the rings $A^{[n]} = H_{\rho_n}^*(X^{[n]})$ also satisfy (A1) with $A = A^{[1]} = H^*(X)$. To deal with Axiom (A2) for $H_{\rho_n}^*(X^{[n]})$, we now define the elements $\tilde{G}_k(\alpha, n) \in H_{\rho_n}^*(X^{[n]})$.

Definition 4.3. Let $k \geq 0$ and $\alpha \in H^*(X)$. Define $\tilde{G}_k(\alpha, n) \in H_{\rho_n}^*(X^{[n]})$ to be

$$(4.4) \quad \sum_{0 \leq j \leq k} \sum_{\substack{\lambda^+(j+1) \\ \ell(\lambda)=k-j+1}} \frac{(-1)^{|\lambda|-1}}{\lambda! \cdot |\lambda|!} \cdot \mathbf{1}_{-(n-j-1)} \mathbf{a}_{-\lambda}(\tau_* \alpha) |0\rangle \\ + \sum_{0 \leq j \leq k} \sum_{\substack{\lambda^+(j+1) \\ \ell(\lambda)=k-j-1}} \frac{(-1)^{|\lambda|}}{\lambda! \cdot |\lambda|!} \cdot \frac{|\lambda| + s(\lambda) - 2}{24} \cdot \mathbf{1}_{-(n-j-1)} \mathbf{a}_{-\lambda}(\tau_*(e_X \alpha)) |0\rangle.$$

REMARK 4.4. We have $\tilde{G}_0(\alpha, n) = \mathbf{1}_{-(n-1)} \mathbf{a}_{-1}(\alpha) |0\rangle = G_0(\alpha, n)$ and

$$\tilde{G}_1(\alpha, n) = -\frac{1}{2} \mathbf{1}_{-(n-2)} \mathbf{a}_{-2}(\alpha) |0\rangle = G_1(\alpha, n).$$

In general, we see from Theorem 2.5 that the class $\tilde{G}_k(\alpha, n)$ consists of those terms in $G_k(\alpha, n)$ which do not contain the canonical divisor K_X .

Note from the definition of the operator $\tilde{\mathfrak{G}}_k(\alpha)$ on $\oplus_n H_{\rho_n}^*(X^{[n]})$ that

$$\langle \tilde{\mathfrak{G}}_k(\alpha) w_1, w_2 \rangle = \langle \tilde{G}_k(\alpha, n) \cdot_{\rho_n} w_1, w_2 \rangle = \langle \tilde{G}_k(\alpha, n), w_1, w_2 \rangle_{\rho_n} (-1),$$

for $w_1, w_2 \in H_{\rho_n}^*(X^{[n]})$. For convenience, we introduce the operator $\tilde{\mathfrak{G}}_k(\alpha; q)$ by

$$(4.5) \quad \langle \tilde{\mathfrak{G}}_k(\alpha; q) w_1, w_2 \rangle = \sum_{d \geq 0} \langle \tilde{G}_k(\alpha, n), w_1, w_2 \rangle_{0, d\beta_n} q^d.$$

In the rest of this section, let the surface X be simply connected. By Remark 4.4, $\tilde{G}_1(1_X, n) = G_1(1_X, n)$. Thus by [LL],

$$(4.6) \quad \tilde{\mathfrak{G}}_1(1_X) = -\frac{1}{6} : \mathbf{a}^3 :_0 (\tau_{3*} 1_X).$$

So (1.1) holds for the rings $H_{\rho_n}^*(X^{[n]})$ as well. To verify Ruan’s conjecture for ρ_n , it remains to show that (1.2) holds for $H_{\rho_n}^*(X^{[n]})$. For the right-hand-side of (1.2), we have the following which follows from (4.6) and the same proof of Theorem 2.3 (i.e., Theorem 4.4 in [LQW3]).

Lemma 4.5. *Let $k \geq 0$, $m \in \mathbb{Z}$, and $\alpha \in H^*(X)$. Then, $\mathbf{a}_m^{\{k\}}(\alpha)$ is equal to*

$$(-m)^k k! \left(\sum_{\substack{\ell(\lambda)=k+1 \\ |\lambda|=m}} \frac{1}{\lambda!} \mathbf{a}_\lambda(\tau_* \alpha) - \sum_{\substack{\ell(\lambda)=k-1 \\ |\lambda|=m}} \frac{s(\lambda) - 1}{24 \lambda!} \mathbf{a}_\lambda(\tau_*(e_X \alpha)) \right). \text{ q.e.d.}$$

Comparing with Theorem 2.3, we see that $\mathbf{a}_m^{\{k\}}(\alpha)$ consists of those terms in $\mathbf{a}_m^{(k)}(\alpha)$ which do not contain the canonical divisor K_X .

Lemma 4.6. *Let X be a smooth toric surface. Then (1.2) holds for $X^{[n]}$.*

Proof. Recall that \mathbb{P}^2 and the Hirzebruch surfaces \mathbb{F}_a are smooth toric surfaces, and admit $\mathbb{T} = (\mathbb{C}^*)^2$ -actions. By the Proposition in Subsection 2.5 of [Ful], X is obtained from \mathbb{P}^2 or \mathbb{F}_a by a succession of blow-ups at \mathbb{T} -fixed points.

Now let $\mathfrak{a}_m^{\mathbb{T}}(\alpha), H_{\rho_n}^{*,\mathbb{T}}(X^{[n]})$ and $\mathfrak{p}_m^{\mathbb{T}}(\alpha), H_{\text{CR}}^{*,\mathbb{T}}(X^{(n)})$ be the equivariant versions of $\mathfrak{a}_m(\alpha), H_{\rho_n}^*(X^{[n]})$ and $\mathfrak{p}_m(\alpha), H_{\text{CR}}^*(X^{(n)})$ respectively. By [Che], the equivariant version of Conjecture 4.2 holds for X , i.e., there exists a ring isomorphism

$$\Psi_n^{\mathbb{T}} : H_{\text{CR}}^{*,\mathbb{T}}(X^{(n)}) \rightarrow H_{\rho_n}^{*,\mathbb{T}}(X^{[n]})$$

sending $\sqrt{-1}^{n_1+\dots+n_s-s} \mathfrak{p}_{-n_1}^{\mathbb{T}}(\alpha_1) \cdots \mathfrak{p}_{-n_s}^{\mathbb{T}}(\alpha_s)|0$ to

$$\mathfrak{a}_{-n_1}^{\mathbb{T}}(\alpha_1) \cdots \mathfrak{a}_{-n_s}^{\mathbb{T}}(\alpha_s)|0.$$

Note that up to a scalar factor which depends only on the partition $\lambda = (n_1, \dots, n_s)$ and the tuple $\vec{\alpha} = (\alpha_1, \dots, \alpha_s)$, our notation

$$\mathfrak{p}_{-n_1}^{\mathbb{T}}(\alpha_1) \cdots \mathfrak{p}_{-n_s}^{\mathbb{T}}(\alpha_s)|0$$

coincides with the notation $\lambda(\vec{\alpha})$ used in [Che]. Also, our notation $\mathfrak{a}_m^{\mathbb{T}}(\alpha)$ coincides with the notation $\mathfrak{p}_m(\alpha)$ used in [Che]. The integer $n_1 + \dots + n_s - s$ is the age. Passing the map $\Psi_n^{\mathbb{T}}$ to the ordinary cohomology, we obtain a ring isomorphism

$$\Psi_n : H_{\text{CR}}^*(X^{(n)}) \rightarrow H_{\rho_n}^*(X^{[n]}),$$

which sends $\sqrt{-1}^{n_1+\dots+n_s-s} \mathfrak{p}_{-n_1}(\alpha_1) \cdots \mathfrak{p}_{-n_s}(\alpha_s)|0$ to

$$\mathfrak{a}_{-n_1}(\alpha_1) \cdots \mathfrak{a}_{-n_s}(\alpha_s)|0.$$

Using (3.1) and (4.4), we see that $\Psi_n(\sqrt{-1}^k O_k(\alpha, n)) = \tilde{G}_k(\alpha, n)$.

Next, let $A = \mathfrak{a}_{-n_1}(\alpha_1) \cdots \mathfrak{a}_{-n_s}(\alpha_s)|0 \in H^*(X^{[n-1]})$. By definition,

$$\begin{aligned} [\tilde{\mathfrak{G}}_k(\alpha), \mathfrak{a}_{-1}(\beta)]A &= \tilde{\mathfrak{G}}_k(\alpha)\mathfrak{a}_{-1}(\beta)A - \mathfrak{a}_{-1}(\beta)\tilde{\mathfrak{G}}_k(\alpha)A \\ &= \tilde{\mathfrak{G}}_k(\alpha, n) \cdot \mathfrak{a}_{-1}(\beta)A - \mathfrak{a}_{-1}(\beta)(\tilde{\mathfrak{G}}_k(\alpha, n-1) \cdot A). \end{aligned}$$

Put $P = \mathfrak{p}_{-n_1}(\alpha_1) \cdots \mathfrak{p}_{-n_s}(\alpha_s)|0$ and $a = n_1 + \dots + n_s - s$. Let \bullet denote the orbifold ring product. Then, $\Psi_n(\mathfrak{p}_{-1}(\beta)(\sqrt{-1}^k O_k(\alpha, n-1) \bullet \sqrt{-1}^a P))$ equals

$$\mathfrak{a}_{-1}(\beta)\Psi_n(\sqrt{-1}^k O_k(\alpha, n-1) \bullet \sqrt{-1}^a P) = \mathfrak{a}_{-1}(\beta)(\tilde{\mathfrak{G}}_k(\alpha, n-1) \cdot A),$$

and $\Psi_n(\sqrt{-1}^a \mathfrak{p}_{-1}(\beta)P) = \mathfrak{a}_{-1}(\beta)A$. So $[\tilde{\mathfrak{G}}_k(\alpha), \mathfrak{a}_{-1}(\beta)]A$ is equal to

$$\begin{aligned} &\Psi_n(\sqrt{-1}^k O_k(\alpha, n) \bullet \sqrt{-1}^a \mathfrak{p}_{-1}(\beta)P) \\ &- \Psi_n(\mathfrak{p}_{-1}(\beta)(\sqrt{-1}^k O_k(\alpha, n-1) \bullet \sqrt{-1}^a P)). \end{aligned}$$

Since $O_k(\alpha, n) \bullet \mathfrak{p}_{-1}(\beta)P = \mathfrak{D}_k(\alpha)\mathfrak{p}_{-1}(\beta)P$, we obtain

$$[\tilde{\mathfrak{G}}_k(\alpha), \mathfrak{a}_{-1}(\beta)]A = \sqrt{-1}^{k+a} \cdot \Psi_n([\mathfrak{D}_k(\alpha), \mathfrak{p}_{-1}(\beta)]P).$$

By Theorem 3.1 (iii), we conclude that $[\tilde{\mathfrak{G}}_k(\alpha), \mathfrak{a}_{-1}(\beta)]A$ is equal to

$$\begin{aligned} & \sqrt{-1}^{k+a} \cdot (-1)^k \cdot \Psi_n \left(\sum_{\substack{\ell(\lambda)=k+1 \\ |\lambda|=-1}} \frac{1}{\lambda!} \mathfrak{p}_\lambda(\tau_*(\alpha\beta))P \right. \\ & \left. + \sum_{\substack{\ell(\lambda)=k-1 \\ |\lambda|=-1}} \frac{s(\lambda)-1}{24\lambda!} \mathfrak{p}_\lambda(\tau_*(e_X\alpha\beta))P \right). \end{aligned}$$

Finally, by the definition of Ψ_n and Lemma 4.5, $[\tilde{\mathfrak{G}}_k(\alpha), \mathfrak{a}_{-1}(\beta)]A$ is equal to

$$\begin{aligned} & \sum_{\substack{\ell(\lambda)=k+1 \\ |\lambda|=-1}} \frac{1}{\lambda!} \mathfrak{a}_\lambda(\tau_*(\alpha\beta))A - \sum_{\substack{\ell(\lambda)=k-1 \\ |\lambda|=-1}} \frac{s(\lambda)-1}{24\lambda!} \mathfrak{a}_\lambda(\tau_*(e_X\alpha\beta))A \\ & = \frac{1}{k!} \mathfrak{a}_{-1}^{\{k\}}(\alpha\beta)A. \end{aligned}$$

Therefore, $[\tilde{\mathfrak{G}}_k(\alpha), \mathfrak{a}_{-1}(\beta)] = 1/k! \cdot \mathfrak{a}_{-1}^{\{k\}}(\alpha\beta)$. Hence (1.2) holds. q.e.d.

5. Extremal Gromov–Witten invariants of Hilbert schemes

In this section, we study the structure of extremal Gromov–Witten invariants of $X^{[n]}$ for a smooth projective surface X . We will use the ideas and approaches in [LiJ], and adopt many presentations, notations and results directly from [LiJ]. In addition, the product formula in [Beh2] and the co-section localization in [KL1, KL2, LL] for Gromov–Witten theory will play important roles. For convenience, we assume that X is simply connected.

5.1. Hilbert schemes of α -points and partial equivalence. In this subsection, we introduce some new spaces related to Hilbert schemes to provide a platform where, in the subsequent subsections, we can construct cycles $\mathfrak{Z}_{n,d} \in H_*((X^{[n]})^3)$ derived from various virtual cycles of moduli spaces of stable maps to these new spaces.

Let $Y \rightarrow T$ be a smooth family of projective surfaces over a smooth, projective base T . The relative Hilbert scheme of length- n 0-dimensional closed subschemes is denoted by $Y_T^{[n]}$. It is over T and for any $t \in T$, $Y_T^{[n]} \times_T \{t\} = (Y_t)^{[n]}$. Define its relative fiber product $Y_T^n = Y \times_T \cdots \times_T Y$ (n times), and its relative symmetric product $Y_T^{(n)} = Y_T^n / S_n$.

Let Λ be a finite set with $|\Lambda| = n$. We define $Y_T^{[\Lambda]} = Y_T^{[n]}$, $Y_T^{(\Lambda)} = Y_T^{(n)}$, and for accounting purpose, denote

$$Y_T^\Lambda = \{(x_a)_{a \in \Lambda} \mid x_a \in Y_t \text{ for some } t \in T\}.$$

Using the Hilbert–Chow morphism $\rho_\Lambda := \rho_n : Y_T^{[\Lambda]} \rightarrow Y_T^{(\Lambda)}$, we define the Hilbert scheme of Λ -points to be

$$(5.1) \quad Y_T^{[\Lambda]} = Y_T^{(\Lambda)} \times_{Y_T^{(\Lambda)}} Y_T^\Lambda.$$

These spaces $Y_T^{[\Lambda]}$ can be thought of as Hilbert schemes of ordered points.

Let \mathcal{P}_Λ be the set of partitions or equivalence relations on Λ . When $\alpha \in \mathcal{P}_\Lambda$ consists of l equivalence classes $\alpha_1, \dots, \alpha_l$, write $\alpha = (\alpha_1, \dots, \alpha_l)$. For such α , we form the *relative Hilbert scheme of α -points* as follows:

$$(5.2) \quad Y_T^{(\alpha)} = \prod_{i=1}^l Y_T^{(\alpha_i)}, \quad Y_T^{[\alpha]} = \prod_{i=1}^l Y_T^{[\alpha_i]}, \quad Y_T^{[\alpha]} = \prod_{i=1}^l Y_T^{[\alpha_i]},$$

where the products are taken relative to T . Note that $Y_T^{[\alpha]} = Y_T^{[\alpha]} \times_{Y_T^{(\alpha)}} Y_T^\Lambda$. The “indexing” morphism is defined to be the second projection

$$(5.3) \quad \text{in} : Y_T^{[\alpha]} \longrightarrow Y_T^\Lambda.$$

The spaces $Y_T^{[\alpha]}$ and $Y_T^{[\beta]}$ are birational. To make this precise, we first fix our convention on a partial ordering on \mathcal{P}_Λ . We agree

$$“\alpha \geq \beta” \iff “a \sim_\beta b \Rightarrow a \sim_\alpha b”.$$

Namely, $\alpha \geq \beta$ if β is finer than α . When $\beta = (\beta_1, \dots, \beta_r)$, we put

$$\alpha \wedge \beta = (\alpha_1 \cap \beta_1, \dots, \alpha_l \cap \beta_r),$$

which is the largest element among all that are less than or equal to both α and β . Note that \mathcal{P}_Λ contains a maximal and a minimal element. The maximal element is Λ consisting of a single equivalence class Λ ; the minimal element is 1^Λ whose equivalence classes are single element sets.

For $\alpha > \beta \in \mathcal{P}_\Lambda$, define

$$\Xi_\beta^\alpha = \{x \in Y_T^\Lambda \mid \exists a, b \in \Lambda \text{ so that } x_a = x_b, a \sim_\alpha b, a \not\sim_\beta b\}.$$

For $\alpha \neq \beta \in \mathcal{P}_\Lambda$, define $\Xi_\beta^\alpha = \Xi_{\alpha \wedge \beta}^\alpha \cup \Xi_{\alpha \wedge \beta}^\beta$. The discrepancy between $Y_T^{[\alpha]}$ and $Y_T^{[\beta]}$ (in $Y_T^{[\alpha]}$) and its complement are defined to be

$$(5.4) \quad \Xi_\beta^{[\alpha]} = Y_T^{[\alpha]} \times_{Y_T^\Lambda} \Xi_\beta^\alpha, \quad \text{and} \quad Y_\beta^{[\alpha]} = Y_T^{[\alpha]} - \Xi_\beta^{[\alpha]}.$$

More precisely, by Lemma 1.2 in [LiJ], there exists a functorial open embedding $\zeta_\alpha^\beta : Y_\beta^{[\alpha]} \rightarrow Y_T^{[\beta]}$ induced by the universal property of the respective moduli spaces such that $\text{Im}(\zeta_\alpha^\beta) = Y_\alpha^{[\beta]}$. Thus we obtain an isomorphism (equivalence) $\zeta_\alpha^\beta : Y_\beta^{[\alpha]} \xrightarrow{\cong} Y_\alpha^{[\beta]}$. We define

$$(5.5) \quad Y_T^{[\leq \alpha]} = \left(\prod_{\beta \leq \alpha} Y_T^{[\beta]} \right) / \sim,$$

where the equivalence is by identifying $Y_\gamma^{[\beta]} \subset Y_T^{[\beta]}$ and $Y_\beta^{[\gamma]} \subset Y_T^{[\gamma]}$ via ζ_β^γ for all $\beta, \gamma \leq \alpha$. Note that $Y_T^{[\leq \alpha]}$ is non-separated (except when $\alpha = 1^\Lambda$), and contains the spaces $Y_T^{[\leq \beta]}$, $\beta \leq \alpha$, as open subschemes.

Even though the non-separated space $Y_T^{[\leq \beta]}$ comes into the picture, in later subsections, we only perform standard algebraic topological operations on these non-Hausdorff spaces such as pull-backs of cohomology classes and cap products. These operations are allowed on any topological spaces (see [GH, Iv, Sp]).

5.2. Stable maps to Hilbert schemes of ordered points. We incorporate stable maps into the above constructions. This is motivated by the standard decompositions of stable morphisms introduced in [LL]. For $d \geq 0$, we let

$$Y_T^{[n,d]} := \overline{\mathfrak{M}}_{0,3}(Y_T^{[n]}, d\beta_n)$$

be the relative moduli space of 3-pointed genus-0 stable maps to $Y_T^{[n]}$ of class $d\beta_n$.

We study the standard decomposition of $[u, C] \in Y_T^{[n,d]}$. Given $[u, C] \in Y_T^{[n,d]}$, composed with the Hilbert–Chow morphism ρ_n , we obtain $\rho_n \circ u : C \rightarrow Y_T^{[n]}$. Since the fundamental class of $u(C)$ is a multiple of the null class β_n , and C is connected, $\rho_n \circ u$ is a constant map. We express $\rho_n \circ u(C) = \sum_{i=1}^l n_i x_i$, where $n_i \in \mathbb{N}_+$ such that $\sum n_i = n$, and x_i are distinct. With such data, for $p \in C$, we can decompose $u(p) = z_1(p) \cup \dots \cup z_l(p)$ such that $z_i(p) \in Y_T^{[n_i]}$, and $\rho_{n_i}(z_i(p)) = n_i x_i$. Because x_i are distinct, such decomposition is unique. We define

$$(5.6) \quad u_i : C \rightarrow Y_T^{[n_i]}, \quad u_i(p) = z_i(p).$$

Because of the uniqueness of the decomposition, one checks that u_i are morphisms; since $u_*[C] = d\beta_n$, we have $u_{i*}[C] = d_i \beta_{n_i}$ for some $d_i \geq 0$ such that $\sum d_i = d$. Using such data, we can define the Hilbert–Chow map from $Y_T^{[n,d]}$ to the weighted symmetric product of Y .

For the pair (n, d) , we define the weighted symmetric product of Y to be

$$Y_T^{(n,d)} = \left\{ \sum_{i=1}^l d_i [n_i x_i] \mid 1 \leq l \leq n, x_1, \dots, x_l \in Y_t \text{ distinct, for a } t \in T \right\}.$$

Here the formal summation $\sum d_i [n_i x_i]$ is subject to the constraints $d_i \in \mathbb{N}$, $\sum d_i = d$, $n_i \in \mathbb{N}_+$ and $\sum n_i = n$. Also, $[n_i x_i]$ represents the multiplicity- n_i 0-cycle supported at x_i , and d_i is its weight. Thus $d_i [n_i x_i] \neq [d_i n_i x_i]$ and $0[x_i]$ is non-trivial. Endow $Y_T^{(n,d)}$ with the obvious topology so that it is a stratified space such that the forgetful map $Y_T^{(n,d)} \rightarrow Y_T^{[n]}$ is continuous, proper and having finite fibers.

We define the Hilbert–Chow map:

$$(5.7) \quad \mathfrak{hc} : Y_T^{[n,d]} \longrightarrow Y_T^{(n,d)}, \quad [u] \mapsto \sum_{i=1}^l d_i [n_i x_i],$$

where (d_i, n_i, x_i) are data associated to (u_i) from (5.6). Define $\mathfrak{hc}_1 : Y_T^{[n,d]} \rightarrow Y_T^{(n)}$ to be the composite of \mathfrak{hc} with the forgetful map $Y_T^{(n,d)} \rightarrow Y_T^{(n)}$. For a finite set Λ (of order n), define

$$(5.8) \quad \begin{aligned} Y_T^{[\Lambda,d]} &= Y_T^{[n,d]}, \\ Y_T^{[\Lambda,d]} &= Y_T^{[\Lambda,d]} \times_{Y_T^{(n)}} Y_T^\Lambda = Y_T^{[n,d]} \times_{Y_T^{(n)}} Y_T^\Lambda. \end{aligned}$$

To simplify notations, the composition of $Y_T^{[\Lambda,d]} \rightarrow Y_T^{[\Lambda,d]}$ and $\mathfrak{hc}_1 : Y_T^{[\Lambda,d]} \rightarrow Y_T^{(n)}$ will again be denoted by \mathfrak{hc}_1 .

Definition 5.1. We call (α, δ) a *weighted partition* of Λ if $\alpha = (\alpha_1, \dots, \alpha_l) \in \mathcal{P}_\Lambda$ and $\delta = (\delta_1, \dots, \delta_l)$, $\delta_i \geq 0$ for every i . We define $\sum_i \delta_i$ to be the *total weight* of (α, δ) . For (Λ, d) , we denote by $\mathcal{P}_{\Lambda,d}$ the set of all weighted partitions of Λ with total weight d . We say that $(\alpha, \delta) \geq (\beta, \eta)$ if $\alpha \geq \beta$ and $\sum_{\beta_i \subset \alpha_j} \eta_i = \delta_j$ for every j .

For $(\alpha, \delta) \in \mathcal{P}_{\Lambda,d}$, define

$$Y_T^{[\alpha,\delta]} = Y_T^{[\alpha_1,\delta_1]} \times_T \dots \times_T Y_T^{[\alpha_l,\delta_l]}.$$

We form the relative moduli space of 3-pointed genus-0 degree- δ α -stable morphisms to the Hilbert scheme of points:

$$(5.9) \quad Y_T^{[\alpha,\delta]} = Y_T^{[\alpha_1,\delta_1]} \times_T \dots \times_T Y_T^{[\alpha_l,\delta_l]}.$$

5.3. Birationality. The key result Lemma 5.3 provides the comparison between $Y_T^{[\alpha,\delta]}$ and $Y_T^{[\beta,\eta]}$, which will be used in later subsections for the comparison of normal cones for $Y_T^{[\alpha,\delta]}$ and $Y_T^{[\beta,\eta]}$.

For $(\alpha, \delta) > (\beta, \eta)$,¹ the pair $Y_T^{[\alpha,\delta]}$ and $Y_T^{[\beta,\eta]}$ are “birational”. To make this more precise, we introduce some notations. Given an element

$$\xi = ([u, C], (y_a)) \in Y_T^{[\Lambda,d]} = Y_T^{[n,d]} \times_{Y_T^{(n)}} Y_T^\Lambda,$$

where $\mathfrak{hc}([u]) = \sum_{i=1}^l d_i [n_i x_i]$ and such that $\sum n_i x_i = \sum_a y_a$ (as 0-cycles in $Y_T^{(n)}$), we define a pair $(\mathfrak{a}(\xi), \mathfrak{d}(\xi)) \in \mathcal{P}_{\Lambda,d}$ by

$$\mathfrak{a}(\xi) = (\mathfrak{a}_1, \dots, \mathfrak{a}_l), \quad \mathfrak{a}_i = \{a \in \Lambda \mid y_a = x_i\}; \quad \mathfrak{d}(\xi) = (d_1, \dots, d_l).$$

Definition 5.2. For $(\beta, \eta) \in \mathcal{P}_{\Lambda,d}$, we define

$$Y_{(\beta,\eta)}^{[\Lambda,d]} = \{\xi \in Y_T^{[\Lambda,d]} \mid (\mathfrak{a}(\xi), \mathfrak{d}(\xi)) \leq (\beta, \eta)\},$$

$$Y_{(\Lambda,d)}^{[\beta,\eta]} = \{(\xi_1, \dots, \xi_r) \in Y_T^{[\beta,\eta]} \mid \mathfrak{hc}_1(\xi_1), \dots, \mathfrak{hc}_1(\xi_r) \text{ mutually disjoint}\}.$$

¹Without further mentioning $\alpha = (\alpha_1, \dots, \alpha_l)$ and $\beta = (\beta_1, \dots, \beta_r)$.

For $(\beta, \eta) \leq (\alpha, \delta)$, we define (as fiber products over T)

$$Y_{(\beta, \eta)}^{[\alpha, \delta]} = \prod_{i=1}^l Y_{(\beta \cap \alpha_i, \eta \cap \delta_i)}^{[\alpha_i, \delta_i]} \quad \text{and} \quad Y_{(\alpha, \delta)}^{[\beta, \eta]} = \prod_{i=1}^l Y_{(\alpha_i, \delta_i)}^{[\beta \cap \alpha_i, \eta \cap \delta_i]}.$$

Lemma 5.3. *For $(\alpha, \delta) > (\beta, \eta)$, we have a natural, proper surjective morphism*

$$(5.10) \quad \zeta_{\alpha, \delta}^{\beta, \eta} : Y_{(\beta, \eta)}^{[\alpha, \delta]} \longrightarrow Y_{(\alpha, \delta)}^{[\beta, \eta]}.$$

Proof. By definition, we only need to prove the case $(\alpha, \delta) = (\Lambda, d)$. Let $\xi = ([u, C, p_i], (y_a)) \in Y_{(\beta, \eta)}^{[\Lambda, d]}$, with $\mathfrak{hc}([u]) = \sum_{i=1}^l d_i [n_i x_i]$. Let $u_i : C \rightarrow Y_T^{[n_i]}$ be as in (5.6). Denote $\mathfrak{a}(\xi) = (\mathfrak{a}_1, \dots, \mathfrak{a}_l)$ and $\mathfrak{d}(\xi) = (d_1, \dots, d_l)$. Since $\xi \in Y_{(\beta, \eta)}^{[\Lambda, d]}$, we have $(\mathfrak{a}(\xi), \mathfrak{d}(\xi)) \leq (\beta, \eta)$. Thus we can form

$$u_{\beta_i} : C \longrightarrow Y_T^{[n_i]}; \quad u_{\beta_i}(p) = \cup_{\mathfrak{a}_j \subset \beta_i} u_j(p) \in Y_T^{[n_i]}.$$

Because the degree of u_j is d_j , and $(\mathfrak{a}(\xi), \mathfrak{d}(\xi)) \leq (\beta, \eta)$, the degree of u_{β_i} is η_i . For $1 \leq i \leq r$, let $u_{\beta_i}^{\text{st}} : C_{\beta_i} \longrightarrow Y_T^{[n_i]}$ be the stabilization of $[u_{\beta_i}, C, p_i]$. Then $(u_{\beta_1}^{\text{st}}, \dots, u_{\beta_r}^{\text{st}}) \in Y_T^{[\beta, \eta]}$. It is routine to check that

$$\zeta_{\Lambda, d}^{\beta, \eta} : Y_{(\beta, \eta)}^{[\Lambda, d]} \longrightarrow Y_T^{[\beta, \eta]}; \quad ([u, C], (y_a)_\Lambda) \mapsto (u_{\beta_1}^{\text{st}}, \dots, u_{\beta_r}^{\text{st}})$$

defines a morphism. By the definition of $Y_{(\Lambda, d)}^{[\beta, \eta]}$, we have $\text{Im}(\zeta_{\Lambda, d}^{\beta, \eta}) \subset Y_{(\Lambda, d)}^{[\beta, \eta]}$.

We now show that $\text{Im}(\zeta_{\Lambda, d}^{\beta, \eta}) = Y_{(\Lambda, d)}^{[\beta, \eta]}$. Note that a closed point in $Y_{(\Lambda, d)}^{[\beta, \eta]}$ is an r -tuple (ξ_1, \dots, ξ_r) with $\xi_i \in Y_T^{[\beta_i, \eta_i]}$ such that

$$\mathfrak{hc}_1(\xi_1), \dots, \mathfrak{hc}_1(\xi_r)$$

are mutually disjoint. Let $\xi_i = [u_i, C_i, p_{i,j}]$. Since $[C_i, p_{i,j}]$ are 3-pointed genus-0 nodal curves, we can find a 3-pointed genus-0 $[C, p_j]$ and contraction morphisms $\phi_i : C \rightarrow C_i$ so that $\phi_i(p_j) = p_{i,j}$, $j = 1, 2, 3$. Since $\mathfrak{hc}_1(\xi_1), \dots, \mathfrak{hc}_1(\xi_r)$ are mutually disjoint, the assignment $p \mapsto u(p) = u_1 \circ \phi_1(p) \cup \dots \cup u_r \circ \phi_r(p) \in Y_T^{[n]}$ defines a morphism $u : C \rightarrow Y_T^{[n]}$. We let $\xi = [u, C, p_j]^{\text{st}}$ be its stabilization. Then $\xi \in Y_{(\beta, \eta)}^{[\Lambda, d]}$, and $\zeta_{\Lambda, d}^{\beta, \eta}(\xi) = (\xi_1, \dots, \xi_r)$. Hence $\text{Im}(\zeta_{\Lambda, d}^{\beta, \eta}) = Y_{(\Lambda, d)}^{[\beta, \eta]}$.

We check that $\zeta_{\Lambda, d}^{\beta, \eta}$ is proper. Let $s_0 \in S$ be a pointed smooth curve over T ; let $S^* = S - s_0$. Suppose ξ^* is an S^* -family in $Y_{(\beta, \eta)}^{[\Lambda, d]}$ so that $\zeta_{\Lambda, d}^{\beta, \eta}(\xi^*) = (\xi_1^*, \dots, \xi_r^*)$ extends to an S -family (ξ_1, \dots, ξ_r) , we need

to show that, possibly after a base change, ξ^* extends to ξ so that $\zeta_{\Lambda,d}^{\beta,\eta}(\xi) = (\xi_1, \dots, \xi_r)$.

Since $Y_T^{[\Lambda,d]}$ is T -proper, possibly after a base change, we can extend ξ^* to an S -family ξ in $Y_T^{[\Lambda,d]}$. Let ξ be given by $([u, C, p_j], (y_a))$, where each term implicitly is an S -family. Let $y_{\beta_i} = \sum_{a \in \beta_i} y_a : S \rightarrow Y_T^{(\beta_i)}$. By definition, $\xi(s_0) = \xi \times_S \{s_0\} \in Y_{(\beta,\eta)}^{[\Lambda,d]}$ if $y_{\beta_1}(s_0), \dots, y_{\beta_r}(s_0)$ are mutually disjoint. Since $\zeta_{\Lambda,d}^{\beta,\eta}(\xi^*) = (\xi_1^*, \dots, \xi_r^*)$, we have $y_{\beta_i}|_{S^*} = \mathfrak{h}\mathfrak{c}_1 \circ \xi_i^*$. Since $Y_T^{(n)}$ is separated, we have $y_{\beta_i}(s_0) = \mathfrak{h}\mathfrak{c}_1(\xi_i(s_0))$. Further, since $(\xi_1(s_0), \dots, \xi_r(s_0)) \in Y_{(\Lambda,d)}^{[\beta,\eta]}$, $\mathfrak{h}\mathfrak{c}_1(\xi_1(s_0)), \dots, \mathfrak{h}\mathfrak{c}_1(\xi_r(s_0))$ are mutually disjoint. This proves that $\xi(s_0) \in Y_{(\beta,\eta)}^{[\Lambda,d]}$. Then ξ lies in $Y_{(\beta,\eta)}^{[\Lambda,d]}$, and by the separatedness of $Y_T^{[\Lambda,d]}$, we have $\zeta_{\Lambda,d}^{\beta,\eta}(\xi) = (\xi_1, \dots, \xi_r)$. This proves the properness. q.e.d.

The morphism $\zeta_{\alpha,\delta}^{\beta,\eta}$ fits into a fiber diagram that will be crucial for our virtual cycle comparison. As we only need the case where $(\beta, \eta) < (\alpha, \delta)$ is derived by a single splitting, meaning that $r = l + 1$, we will state it in the case $(\alpha, \delta) = (\Lambda, d)$, and $(\beta, \eta) = ((\beta_1, \beta_2), (d_1, d_2))$.

We first introduce necessary notation, following Behrend [Beh2]. Given a semi-group $G = \mathbb{N}$ or \mathbb{N}^2 , we call a triple (C, p_j, τ) a pointed G -weighted nodal curve if (C, p_i) is a pointed nodal curve and τ is a map from the set of irreducible components of C to G . We say (C, p_j, τ) is stable if for any $C_0 \cong \mathbb{P}^1 \subset C$, either $\tau([C_0]) \neq 0$ or C_0 contains at least three special points of (C, p_j) . (A special point of (C, p_j) is either a node or a marked point.)

We denote by $\mathcal{M}_{0,3}(d)$ the Artin stack of *stable* total weights d \mathbb{N} -weighted 3-pointed genus-0 nodal curves. We denote by $\mathfrak{D}(d_1, d_2)$ the Artin stack of the data

$$\{(C, p_j, \tau) \rightarrow (C_1, p_{1,j}, \tau_1), (C, p_j, \tau) \rightarrow (C_2, p_{2,j}, \tau_2)\},$$

so that (C, p_j, τ) is a stable total weight (d_1, d_2) \mathbb{N}^2 -weighted 3-pointed genus-0 nodal curve, $(C_i, p_{i,j}, \tau_i) \in \mathcal{M}_{0,3}(d_i)$, and the two arrows induce isomorphisms $(C, p_j, \text{pr}_i \circ \tau)^{\text{st}} \cong (C_i, p_{i,j}, \tau_i)$, where $\text{pr}_i : \mathbb{N}^2 \rightarrow \mathbb{N}$ is the i -th projection. (See the diagram (3) in [Beh2] for details.) For $i = 1, 2$, define $\epsilon_i : \mathfrak{D}(d_1, d_2) \rightarrow \mathcal{M}_{0,3}(d_i)$ to be the map sending the element

$$\xi := \{(C, p_j, \tau) \rightarrow (C_1, p_{1,j}, \tau_1), (C, p_j, \tau) \rightarrow (C_2, p_{2,j}, \tau_2)\},$$

to $(C_i, p_{i,j}, \tau_i)$ which is indeed stable by the definition of the data ξ .

Lemma 5.4. *Let $\beta = (\beta_1, \beta_2)$ be a partition of length two, and let $\eta = (d_1, d_2)$ with $d = d_1 + d_2$. We have a Cartesian diagram*

$$\begin{array}{ccc}
 Y_{(\beta,\eta)}^{[\Lambda,d]} & \longrightarrow & Y_{(\Lambda,d)}^{[\beta,\eta]} \\
 \downarrow & & \downarrow \\
 \mathfrak{D}(d_1, d_2) & \xrightarrow{(\epsilon_1, \epsilon_2)} & \mathcal{M}_{0,3}(d_1) \times \mathcal{M}_{0,3}(d_2)
 \end{array}$$

Further, (ϵ_1, ϵ_2) is proper and birational.

Proof. The proof is a direct application of Proposition 5 in [Beh2] plus the definition of $Y_{(\beta,\eta)}^{[\Lambda,d]}$. Note that the second vertical arrow is induced by $Y_{(\Lambda,d)}^{[\beta,\eta]} \subset Y_T^{[\beta_1,d_1]} \times_T Y_T^{[\beta_2,d_2]}$ and the forgetful morphism $Y_T^{[\beta_i,d_i]} \rightarrow \mathcal{M}_{0,3}(d_i)$. q.e.d.

5.4. Virtual classes and comparison of normal cones. As $Y_T^n \rightarrow Y_T^{(n)}$ is a finite quotient map by a finite group, it is flat. So the tautological projection $\rho_{\alpha,\delta} : Y_T^{[\alpha,\delta]} \rightarrow Y_T^{[\alpha,\delta]}$ is flat as well. Let $[Y_T^{[\alpha,\delta]}]^{vir}$ be the virtual class of $Y_T^{[\alpha,\delta]}$. We define $[Y_T^{[\alpha,\delta]}]^{vir}$ to be the pullback of $[Y_T^{[\alpha,\delta]}]^{vir}$ via $\rho_{\alpha,\delta}$. Our goal is to inductively construct cycle representatives of the virtual classes of $Y_T^{[\alpha,\delta]}$ that are compatible via the comparison $\zeta_{\alpha,\delta}^{\beta,\eta}$.

We recall the construction of virtual cycles in [BF, LT1]. Let

$$(\mathbb{E}_{[\alpha,\delta]})^\vee \rightarrow \mathbb{L}_{Y_T^{[\alpha,\delta]}/T \times (\mathcal{M}_{0,3})^l}$$

be the standard perfect relative obstruction theory² of $Y_T^{[\alpha,\delta]} \rightarrow T \times (\mathcal{M}_{0,3})^l$; let $\mathbf{C}_{[\alpha,\delta]} \subset \mathbf{F}_{[\alpha,\delta]} := h^1/h^0(\mathbb{E}_{[\alpha,\delta]})$ be its intrinsic normal cone. To use analytic Gysin map, we put it in a vector bundle. Following [BF, LT1], we can find a vector bundle (locally free sheaf) $E_{[\alpha,\delta]}$ on $Y_T^{[\alpha,\delta]}$ and a surjection of bundle-stack $E_{[\alpha,\delta]} \rightarrow h^1/h^0(\mathbb{E}_{[\alpha,\delta]})$. Let $C_{[\alpha,\delta]} \subset E_{[\alpha,\delta]}$ be the flat pullback of $\mathbf{C}_{[\alpha,\delta]}$. Then $[Y_T^{[\alpha,\delta]}]^{vir} = 0_{E_{[\alpha,\delta]}}^! [C_{[\alpha,\delta]}]$, the image of the Gysin map of the zero-section of $E_{[\alpha,\delta]}$. Let $E_{[\alpha,\delta]}^* = \rho_{\alpha,\delta}^* E_{[\alpha,\delta]}$ where $\rho_{\alpha,\delta} : Y_T^{[\alpha,\delta]} \rightarrow Y_T^{[\alpha,\delta]}$ is the tautological projection, and $C_{[\alpha,\delta]} \subset E_{[\alpha,\delta]}^*$ be the flat pullback of $C_{[\alpha,\delta]}$ via $E_{[\alpha,\delta]}^* \rightarrow E_{[\alpha,\delta]}$. The virtual class of $Y_T^{[\alpha,\delta]}$ is equal to

$$[Y_T^{[\alpha,\delta]}]^{vir} = (\rho_{\alpha,\delta})^* [Y_T^{[\alpha,\delta]}]^{vir} = 0_{E_{[\alpha,\delta]}^*}^* [C_{[\alpha,\delta]}] \in H_*(|Y_T^{[\alpha,\delta]}|; \mathbb{Q}),$$

where $0_{E_{[\alpha,\delta]}^*}^*$ is the Gysin homomorphism of the zero section of $E_{[\alpha,\delta]}^*$, and $|Y_T^{[\alpha,\delta]}|$ is the coarse moduli space of $Y_T^{[\alpha,\delta]}$. Also, put $\mathbb{E}_{[\alpha,\delta]} = \rho_{\alpha,\delta}^* \mathbb{E}_{[\alpha,\delta]}$, and let $\mathbf{F}_{[\alpha,\delta]} = h^1/h^0(\mathbb{E}_{[\alpha,\delta]}) = \rho_{\alpha,\delta}^* \mathbf{F}_{[\alpha,\delta]}$ be the flat pullback. Let $\mathbf{C}_{[\alpha,\delta]} \subset \mathbf{F}_{[\alpha,\delta]}$ be the flat pullback of $\mathbf{C}_{[\alpha,\delta]}$ via $\mathbf{F}_{[\alpha,\delta]} \rightarrow \mathbf{F}_{[\alpha,\delta]}$.

²Here $\mathbb{E}_{[\alpha,\delta]}$ is a derived object locally presented as a two-term complex of locally free sheaves placed at $[0, 1]$.

We now compare the cycles $C_{[\alpha,\delta]}$ using $\zeta_{\alpha,\delta}^{\beta,\eta}$. The tricky part is that the vector bundles $E_{[\alpha,\delta]}$ are not comparable. Thus we will state the comparison using cycles in $\mathbf{F}_{[\alpha,\delta]}$, and later will use the obstruction sheaf for accounting purpose.

Lemma 5.5. *For pairs $(\alpha, \delta) > (\beta, \eta)$, we have canonical isomorphisms*

$$\varphi_{\alpha,\delta}^{\beta,\eta} : (\zeta_{\alpha,\delta}^{\beta,\eta})^* (\mathbf{F}_{[\beta,\eta]}|_{Y_{(\alpha,\delta)}^{[\beta,\eta]}}) \xrightarrow{\cong} \mathbf{F}_{[\alpha,\delta]}|_{Y_{(\beta,\eta)}^{[\alpha,\delta]}},$$

that satisfy the cocycle condition: we have $\varphi_{\alpha,\delta}^{\beta,\eta} \circ (\zeta_{\alpha,\delta}^{\beta,\eta})^* (\varphi_{\beta,\eta}^{\gamma,\varepsilon}) = \varphi_{\alpha,\delta}^{\gamma,\varepsilon}$ for any triple $(\alpha, \delta) > (\beta, \eta) > (\gamma, \varepsilon)$. Further, let $\tilde{\varphi}_{\alpha,\delta}^{\beta,\eta} : \mathbf{F}_{[\alpha,\delta]}|_{Y_{(\beta,\eta)}^{[\alpha,\delta]}} \rightarrow \mathbf{F}_{[\beta,\eta]}|_{Y_{(\alpha,\delta)}^{[\beta,\eta]}}$ be the projection induced by $\varphi_{\alpha,\delta}^{\beta,\eta}$, which is proper by Lemma 5.3. Then

$$(\tilde{\varphi}_{\alpha,\delta}^{\beta,\eta})^* [\mathbf{C}_{[\alpha,\delta]}|_{Y_{(\beta,\eta)}^{[\alpha,\delta]}}] = [\mathbf{C}_{[\beta,\eta]}|_{Y_{(\alpha,\delta)}^{[\beta,\eta]}}].$$

Proof. By induction, we only need to prove the case where $\ell(\beta) = \ell(\alpha) + 1$; by definition this follows from the case $(\alpha, \delta) = (\Lambda, d)$ and $\beta = (\beta_1, \beta_2)$ with $\eta = (d_1, d_2)$, which we suppose in the remainder of this proof.

Let $y = (y_a) \in Y_T^\Lambda$ be a closed point so that $y_{\beta_1} = \rho_{\beta_1}((y_a)_{a \in \beta_1}) \in Y_T^{(\beta_1)}$ and $y_{\beta_2} \in Y_T^{(\beta_2)}$ (defined similarly) are disjoint. We then form

$$V_i = Y_T^{[\beta_i]} \times_{Y_T^{(\beta_i)}} \{y_{\beta_i}\} \quad \text{and} \quad V = Y_T^{[\Lambda]} \times_{Y_T^{(\Lambda)}} \{\rho_\Lambda(y)\}.$$

Note that $y_{\beta_1} \cap y_{\beta_2} = \emptyset$ implies that $V_1 \times_T V_2 \subset Y_T^{[\beta]}$. Also, there exists a canonical isomorphism $\zeta_\beta^\Lambda : V_1 \times_T V_2 \rightarrow V$. Let \hat{V}_i (respectively, \hat{V}) be the formal completion of $Y_T^{[\beta_i]}$ (respectively, $Y_T^{[\Lambda]}$) along V_i (respectively, V). The isomorphism ζ_β^Λ induces

$$\hat{\zeta}_\beta^\Lambda : \hat{V}_1 \times_T \hat{V}_2 \longrightarrow \hat{V},$$

which is injective and smooth.

For notational simplicity, we denote $\overline{\mathfrak{M}}(\hat{V}_i) = \overline{\mathfrak{M}}_{0,3}(\hat{V}_i, d_i)$ with ι_2 in (5.11) being the tautological morphism induced by $\hat{V}_i \rightarrow Y_T^{[\beta_i]}$; we let

$$\overline{\mathfrak{M}}(\hat{V}_1 \times_T \hat{V}_2) = \overline{\mathfrak{M}}_{0,3}(\hat{V}_1 \times_T \hat{V}_2, (d_1, d_2)),$$

with ι_1 in (5.11) being the tautological morphism induced by $\hat{V}_1 \times_T \hat{V}_2 \rightarrow Y_T^{[\Lambda]}$.

We consider the following commutative diagram of arrows, where ϕ is defined by sending $[u, C, p_j] \in \overline{\mathfrak{M}}(\hat{V}_1 \times_T \hat{V}_2)$ to (ξ_1, ξ_2) with $\xi_i =$

$[\pi_i \circ u, C, p_j]^{\text{st}}$ for $\pi_i : \hat{V}_1 \times_T \hat{V}_2 \rightarrow \hat{V}_i$ the projection; ϕ' is induced by ϕ .

$$\begin{array}{ccc}
 Y_{(\beta, \eta)}^{[\Lambda, d]} & \xrightarrow{\zeta_{\Lambda, d}^{\beta, \eta}} & Y_{(\Lambda, d)}^{[\beta, \eta]} \\
 \uparrow \varphi_1 & & \uparrow \varphi_2 \\
 \overline{\mathfrak{M}}(\hat{V}_1 \times_T \hat{V}_2) \times_{Y_T^{(\beta)}} Y_T^\beta & \xrightarrow{\phi'} & (\overline{\mathfrak{M}}(\hat{V}_1) \times_T \overline{\mathfrak{M}}(\hat{V}_2)) \times_{Y_T^{(\beta)}} Y_T^\beta \\
 \downarrow \psi_1 & & \downarrow \psi_2 \\
 \overline{\mathfrak{M}}(\hat{V}_1 \times_T \hat{V}_2) & \xrightarrow{\phi} & \overline{\mathfrak{M}}(\hat{V}_1) \times_T \overline{\mathfrak{M}}(\hat{V}_2) \\
 \downarrow \iota_1 & & \downarrow \iota_2 \\
 Y_T^{[\Lambda, d]} & & Y_T^{[\beta, \eta]}
 \end{array}
 \tag{5.11}$$

We let $\mathbf{C}_1 \subset \mathbf{F}_1$ be the intrinsic normal cone in the bundle stack of the obstruction complex of the perfect relative obstruction theory of $\overline{\mathfrak{M}}(\hat{V}_1 \times_T \hat{V}_2) \rightarrow T \times \mathcal{M}_{0,3}$. Because $\hat{V}_1 \times_T \hat{V}_2 \rightarrow Y_T^{[\Lambda, d]}$ is injective and smooth, we have $\iota_1^*(\mathbf{C}_{[\Lambda, d]} \subset \mathbf{F}_{[\Lambda, d]}) = (\mathbf{C}_1 \subset \mathbf{F}_1)$. Since $\mathbf{C}_{[\Lambda, d]} \subset \mathbf{F}_{[\Lambda, d]}$ is the pullback of $\mathbf{C}_{[\Lambda, d]} \subset \mathbf{F}_{[\Lambda, d]}$, we conclude $\varphi_1^*(\mathbf{C}_{[\Lambda, d]} \subset \mathbf{F}_{[\Lambda, d]}) = \psi_1^*(\mathbf{C}_1 \subset \mathbf{F}_1)$.

Similarly, letting $\mathbf{C}_2 \subset \mathbf{F}_2$ be the intrinsic normal cone in the bundle stack of the obstruction complex of the perfect relative obstruction theory of $\overline{\mathfrak{M}}(\hat{V}_1) \times_T \overline{\mathfrak{M}}(\hat{V}_2) \rightarrow T \times (\mathcal{M}_{0,3})^2$, we have $\varphi_2^*(\mathbf{C}_{[\beta, \eta]} \subset \mathbf{F}_{[\beta, \eta]}) = \psi_2^*(\mathbf{C}_2 \subset \mathbf{F}_2)$. Since φ_1 and φ_2 are injective and smooth, since ϕ' is proper, since the top square is commutative, and since the image of φ_1 (respectively, of φ_2) covers $Y_{(\beta, \eta)}^{[\Lambda, d]}$ (respectively, $Y_{(\Lambda, d)}^{[\beta, \eta]}$) for y varying through Y_T^Λ satisfying $y_{\beta_1} \cap y_{\beta_2} = \emptyset$, to prove that $\mathbf{F}_{[\Lambda, d]} = (\zeta_{\Lambda, d}^{\beta, \eta})^* \mathbf{F}_{[\beta, \eta]}$ and $(\zeta_{\Lambda, d}^{\beta, \eta})^* [\mathbf{C}_{[\Lambda, d]}] = [\mathbf{C}_{[\beta, \eta]}]$, it suffices to show that we have the canonical isomorphism and identity

$$\mathbf{F}_1 \cong \phi^* \mathbf{F}_2 \quad \text{and} \quad \tilde{\phi}_* [\mathbf{C}_1] = [\mathbf{C}_2],
 \tag{5.12}$$

where $\tilde{\phi} : \mathbf{F}_1 \rightarrow \mathbf{F}_2$ is the induced projection. But this follows from the Cartesian square

$$\begin{array}{ccc}
 \overline{\mathfrak{M}}(\hat{V}_1 \times_T \hat{V}_2) & \xrightarrow{\phi} & \overline{\mathfrak{M}}(\hat{V}_1) \times_T \overline{\mathfrak{M}}(\hat{V}_2) \\
 \downarrow & & \downarrow \\
 T \times \mathfrak{D}(d_1, d_2) & \longrightarrow & T \times \mathcal{M}_{0,3}(d_1) \times \mathcal{M}_{0,3}(d_2),
 \end{array}$$

similar to the one stated in Lemma 5.4 (originally constructed in Proposition 5 of [Beh2]). Since the lower horizontal line is birational, and T is smooth and projective, by Theorem 5.0.1 in [Cos], we have the isomorphism and identities in (5.12). This proves the lemma. q.e.d.

5.5. Multi-sections and pseudo-cycle representatives. In this subsection, we use multi-sections to intersect the cycles $C_{[[\alpha, \delta]]}$ to obtain pseudo-cycle representatives of $[Y_T^{[[\alpha, \delta]]}]^{\text{vir}}$.

In the remainder of this section, we will work with analytic topology and smooth (C^∞) sections. Let V be a vector bundle over a DM stack W . In case W is singular, we stratify W into a union of smooth locally closed DM stacks $W = \coprod W_\alpha$, and use continuous sections that are smooth when restricted to each stratum W_α . Without further commenting, all sections used in this section are stratified sections; we denote the space of such sections by $\mathcal{C}(W, V)$. Also, we will use $|W|$ and $|V|$ to denote the coarse moduli of W and V .

We recall the notion of multi-sections, following [FO, LT2]. We first consider the case where $W = U/G$ is a quotient stack and V is a G -vector bundle on U . Let $S^n(V) \rightarrow U$ be the n -th symmetric product bundle of V . A *liftable multi-section* s of V (of multiplicity n) is a G -equivariant section $s \in \mathcal{C}(U, S^n(V))^G$ such that there are n sections $s_1, \dots, s_n \in \mathcal{C}(U, V)$ so that s is the image of (s_1, \dots, s_n) . For a multi-section $s \in \mathcal{C}(U, S^n(V))^G$ that is the image of (s_1, \dots, s_n) , we define its integer multiple $ms \in \mathcal{C}(U, S^{mn}(V))^G$ be the image of $(s_1, \dots, s_1, \dots, s_n, \dots, s_n)$, where each s_i is repeated m times. Given two multi-sections s and s' of multiplicities n and n' , we say that s and s' are equivalent, denoted by $s \approx s'$, if $n's = ns'$ as multi-sections.

In general, since W is a DM-stack, it can be covered by (analytic) open quotient stacks $U_\alpha/G_\alpha \subset W$, and the restriction $V|_{U_\alpha/G_\alpha} = V_\alpha/G_\alpha$ for G_α -vector bundles V_α on U_α . A *multi-section* s of V consists of an analytic open covering U_α/G_α of W and a collection of liftable multi-sections s_α of $V|_{U_\alpha/G_\alpha}$ so that for any pair (α, β) , the pullbacks of s_α and s_β to $U_\alpha \times_W U_\beta$ are equivalent. We denote the space of multi-sections of V by $\mathcal{C}_{\text{mu}}(W, V)$. (Thus multi-sections in this paper are always locally liftable.)

The space of multi-sections of V has the same extension property as the space of sections of a vector bundle on a manifold. The usual extension property of vector bundles on manifolds is proved by using the partition of unity and the addition structure of the vector bundles. For multi-sections, over a chart U_α/G_α , we define the sum of two (liftable) multi-sections s and s' (with lifting $(s_i)_{i=1}^n$ and $(s'_j)_{j=1}^m$, respectively) be the multiplicity nm multi-section that is the image of $s + s' = (s_i + s'_j)$. This local sum extends to sum of two multi-sections on W . Thus combined with the partition of unity of $|W|$, we conclude that the mentioned extension property holds for $\mathcal{C}_{\text{mu}}(W, V)$.

We also have the following transversality property. Given a closed integral substack $C \subset V$ and a multi-section $s \in \mathcal{C}_{\text{mu}}(W, V)$, we say that s intersects C transversally if there is a stratification of C so that each strata C_α of C lies over a strata of W , say $W_{\alpha'}$, and the section

$s|_{W_{\alpha'}}$ intersects C_{α} transversally, meaning that the local liftings of $s|_{W_{\alpha'}}$ intersect C_{α} transversally. Given a cycle $[C] = \sum n_i [C_i]$ with C_i closed integral algebraic substacks, we say s intersects $[C]$ transversally if it intersects each C_i transversally.

Lemma 5.6. *Let $p : W' \rightarrow W$ be a proper morphism of DM-stacks; let V be a vector bundle on W and $\tilde{p} : p^*V \rightarrow V$ be the induced projection. Suppose $[C'] \in Z_*(p^*V)$ is an algebraic cycle and $[C] = \tilde{p}_*[C']$. If $s \in \mathcal{C}_{\text{mu}}(W, V)$ intersects $[C]$ transversally, then $p^*s \in \mathcal{C}_{\text{mu}}(W', p^*V)$ intersects $[C']$ transversally.*

Proof. We pick stratifications $W = \coprod W_{\alpha}$ and $W' = \coprod W'_{\alpha}$ so that $p(W'_{\alpha}) = W_{\alpha}$ and $p_{\alpha} = p|_{W'_{\alpha}} : W'_{\alpha} \rightarrow W_{\alpha}$ are smooth. We then pick a stratification $C' = \coprod C'_{\beta}$ so that each C'_{β} lies over a stratum of W' , and that $\tilde{p}|_{C'_{\beta}} : C'_{\beta} \rightarrow \tilde{p}(C'_{\beta})$ is smooth. Therefore, by the definition of transversal to C , we are reduced to check when $p : W' \rightarrow W$ and $C' \rightarrow \tilde{p}(C')$ are smooth. In this case, the statement of the lemma holds by direct local coordinate checking. This proves the lemma. q.e.d.

We now construct pseudo-cycle representatives of the topological Gysin map

$$(5.13) \quad 0_V^! : Z_*V \longrightarrow H_*(|W|, \mathbb{Q}),$$

via intersecting with multi-sections [**FO**, **LT2**, **LT3**, **McD**, **Zin**].

We assume W is proper. Let $\pi : V \rightarrow W$ and $\bar{\pi} : |V| \rightarrow |W|$ be the projections. Given a closed integral algebraic substack $C \subset V$, we find a multi-section s of V so that it intersects C transversally. Let $k = 2(\text{rank } V - \dim C)$. By slightly perturbing s if necessary, we can assume that there is a closed (stratifiable) subset $R \subset |V|$ of $\dim_{\mathbb{R}} R \leq k - 2$ and an (analytic) open covering of W by quotient stacks U_{α}/G_{α} so that, letting $q_{\alpha} : V_{\alpha} \rightarrow |V|$ be the projections,

- 1) $s|_{U_{\alpha}/G_{\alpha}}$ are images of $s_{\alpha,1}, \dots, s_{\alpha,m_{\alpha}}$ in $\mathcal{C}(U_{\alpha}, V_{\alpha})$;
- 2) there are topological spaces $S_{\alpha,i}$ and proper embeddings $f_{\alpha,i} : S_{\alpha,i} \rightarrow V_{\alpha}$ such that
 - a) there are dense open subsets $S_{\alpha,i}^{\circ} \subset S_{\alpha,i}$ so that $S_{\alpha,i}^{\circ}$ are smooth manifolds and $f_{\alpha,i}|_{S_{\alpha,i}^{\circ}} : S_{\alpha,i}^{\circ} \rightarrow V_{\alpha}$ are smooth embeddings;
 - b) $s_{\alpha,i} \cap (C \times_V V_{\alpha} - q_{\alpha}^{-1}(R)) = f_{\alpha,i}(S_{\alpha,i}^{\circ})$;
 - c) $f_{\alpha,i}(S_{\alpha,i} - S_{\alpha,i}^{\circ}) \subset q_{\alpha}^{-1}(R)$.

Since $s \in \mathcal{C}_{\text{mu}}(W, V)$, by definition, $\sum_{i=1}^{m_{\alpha}} f_{\alpha,i}(S_{\alpha,i}^{\circ})$ is G_{α} -equivariant. Define

$$(5.14) \quad (C \cap s)|_{|V_{\alpha}|} = \frac{1}{m_{\alpha}} \left(\sum_{i=1}^{m_{\alpha}} f_{\alpha,i}(S_{\alpha,i}) \right) / G_{\alpha},$$

viewed as a sum of piecewise smooth k -dimensional \mathbb{Q} -currents away from a $(k - 2)$ -dimensional subset. Since $(s_{\alpha,i})$ are local lifts of a global

multi-section s , the \mathbb{Q} -currents (5.14) patch to form a piecewise smooth \mathbb{Q} -currents with vanishing boundary in $|V| - R$. We denote this current by $C \cap s$. Since $|W|$ is compact, the current $C \cap s$ defines a homology class in $H_k(|V|, R; \mathbb{Q}) = H_k(|V|; \mathbb{Q})$. Applying the projection $\bar{\pi} : |V| \rightarrow |W|$, we obtain the image \mathbb{Q} -current $\bar{\pi}(C \cap s)$ and its associated homology class $[\bar{\pi}(C \cap s)] \in H_k(|W|; \mathbb{Q})$. Following the topological construction of Gysin map of intersecting with the zero-section of V ,

$$0_V^! [C] = [\bar{\pi}(C \cap s)] \in H_*(|W|; \mathbb{Q})$$

is the image of $[C]$ under the topological Gysin map $0_V^!$. By the linearity of Gysin map, this defines the topological $0_V^!$ in (5.13). The current $\bar{\pi}(C \cap s)$ is called a *pseudo-cycle representative* of the Gysin map.

We now assume in addition that \mathcal{F} is a quotient sheaf $\phi : \mathcal{O}_W(V) \rightarrow \mathcal{F}$, and the cycle $[C] = \sum n_i [C_i] \in Z_* W$ has the property

- (P) for each C_i , and any closed $z \in W$ and $a \in \mathcal{F}|_z$, letting $\phi_z : V_z \rightarrow \mathcal{F}|_z$ be ϕ restricting to z , we have either $\phi_z^{-1}(a) \cap C_i = \emptyset$ or $\phi_z^{-1}(a) \cap C_i = \phi_z^{-1}(a)$.³

Definition 5.7. Two multi-sections s and s' of V are \mathcal{F} -equivalent, denoted by $s \sim_{\mathcal{F}} s'$, if for any $x \in W$, as \mathbb{Q} -zero-cycles, we have $(\phi_x)_*(s(x)) = (\phi_x)_*(s'(x))$. A multi-section of \mathcal{F} is an $\sim_{\mathcal{F}}$ equivalence class of multi-sections of V . We say a multi-section \mathbf{s} of \mathcal{F} intersects $C \subset V$ transversally if a representative s of \mathbf{s} intersects C transversally.

We comment that when C satisfies property (P), the notion that a multi-section of \mathcal{F} intersects C transversally is well-defined, after we pick the stratification of W so that \mathcal{F} restricts to each stratum is locally free, which we always assume in the remaining discussion.

We apply this discussion to $C_{[\alpha, \delta]} \subset E_{[\alpha, \delta]}$. Let $\mathcal{F}_{[\alpha, \delta]} = H^1(\mathbb{E}_{[\alpha, \delta]})$, a coherent sheaf on $Y_T^{[\alpha, \delta]}$, and let $\mathcal{F}_{[\alpha, \delta]} = \rho_{\alpha, \delta}^* \mathcal{F}_{[\alpha, \delta]}$, the pullback sheaf on $Y_T^{[\alpha, \delta]}$. (Note that $\mathcal{F}_{[\alpha, \delta]}$ is the obstruction sheaf of the relative obstruction theory of $Y_T^{[\alpha, \delta]}$.) Then $\mathcal{F}_{[\alpha, \delta]}$ is the quotient sheaf of $E_{[\alpha, \delta]}$ via

$$\phi_{[\alpha, \delta]} : E_{[\alpha, \delta]} \longrightarrow \mathbf{F}_{[\alpha, \delta]} = h^1/h^0(\rho_{\alpha, \delta}^* \mathbb{E}_{[\alpha, \delta]}) \longrightarrow H^1(\rho_{\alpha, \delta}^* \mathbb{E}_{[\alpha, \delta]}) = \mathcal{F}_{[\alpha, \delta]}.$$

Since $C_{[\alpha, \delta]}$ is the pullback of the cycle $\mathbf{C}_{[\alpha, \delta]}$ in $\mathbf{F}_{[\alpha, \delta]}$, the cycle $C_{[\alpha, \delta]}$ satisfies property (P) for the pair $E_{[\alpha, \delta]} \rightarrow \mathcal{F}_{[\alpha, \delta]}$. Thus we can speak of multi-sections \mathbf{s} of $\mathcal{F}_{[\alpha, \delta]}$ intersecting $C_{[\alpha, \delta]} \subset E_{[\alpha, \delta]}$ transversally.

In the future, we will call a multi-section of $\mathcal{F}_{[\alpha, \delta]}$ intersecting $C_{[\alpha, \delta]}$ transversally a *good multi-section*. Let $k_{[\alpha, \delta]}$ be the virtual dimension of $Y_T^{[\alpha, \delta]}$. For a good multi-section $\mathbf{s}_{[\alpha, \delta]}$ of $\mathcal{F}_{[\alpha, \delta]}$, we denote

$$D(\mathbf{s}_{[\alpha, \delta]}) = \bar{\pi}(C_{[\alpha, \delta]} \cap s_{[\alpha, \delta]}),$$

³As argued in [CL], this means that C is a pull back of a “substack” of \mathcal{F} .

where $s_{[\alpha,\delta]}$ is a representative of $\mathbf{s}_{[\alpha,\delta]}$, and $D(\mathbf{s}_{[\alpha,\delta]})$ is a piecewise smooth $k_{[\alpha,\delta]}$ -dimensional \mathbb{Q} -current away from a subset of dimension at most $k_{[\alpha,\delta]} - 2$. (Note that $D(\mathbf{s}_{[\alpha,\delta]})$ is independent of the choice of $s_{[\alpha,\delta]}$.) We denote

$$[D(\mathbf{s}_{[\alpha,\delta]})] \in H_{k_{[\alpha,\delta]}}(|Y_T^{[\alpha,\delta]}|; \mathbb{Q}),$$

the homology class it represents.

Applying the pseudo-cycle representative of Gysin maps, we obtain:

Proposition 5.8. *Given a good multi-section $\mathbf{s}_{[\alpha,\delta]}$ of $\mathcal{F}_{[\alpha,\delta]}$, we have*

$$[D(\mathbf{s}_{[\alpha,\delta]})] = [Y_T^{[\alpha,\delta]}]_{\text{vir}} \in H_*(|Y_T^{[\alpha,\delta]}|; \mathbb{Q}).$$

5.6. Comparison of virtual cycles. Our goal in this subsection is to compare the virtual cycles in terms of pseudo-cycle representatives. We will prove the analogue of Lemma 5.6 in [LiJ].

To begin with, we recall α -diagonals, their tubular neighborhoods, and the associated partitions from [LiJ]. For $\alpha \in \mathcal{P}_\Lambda$, we form the strict α -diagonal:

$$(5.15) \quad \Delta_\alpha = \Delta_\alpha^Y = \{x \in Y_T^\Lambda \mid a \sim_\alpha b \Rightarrow x_a = x_b\};$$

it is closed in Y_T^Λ and isomorphic to Y_T^l when $\alpha = (\alpha_1, \dots, \alpha_l)$. Fix a sufficiently small number $c > 0$ and a large real N , and pick a function $\epsilon : \mathcal{P}_\Lambda \rightarrow (0, c)$ whose values on any ordered pair $\alpha > \beta$ satisfy $\epsilon(\alpha) > N \cdot \epsilon(\beta)$. After fixing a Riemannian metric on Y , we define the ϵ -neighborhood of $\Delta_\alpha \subset Y_T^\Lambda$ to be

$$(5.16) \quad \Delta_{\alpha,\epsilon} = \Delta_{\alpha,\epsilon}^Y = \{x \in Y_T^\Lambda \mid \text{dist}(x, \Delta_\alpha) < \epsilon(\alpha)\}.$$

For a pair $\alpha \geq \beta$, we define $\Delta_{\beta,\epsilon}^\alpha = \cup_{\alpha \geq \gamma \geq \beta} \Delta_{\gamma,\epsilon}$ and $Q_{\beta,\epsilon}^\alpha = \Delta_{\beta,\epsilon} - \cup_{\alpha \geq \gamma > \beta} \Delta_{\gamma,\epsilon} = \Delta_{\beta,\epsilon} - \cup_{\alpha \geq \gamma > \beta} \Delta_{\gamma,\epsilon}$. Then, $Q_{\beta,\epsilon}^\alpha$ is a closed subset of $\Delta_{\beta,\epsilon}$. By Lemma 5.5 of [LiJ], if $\Delta_{\beta_1,\epsilon} \cap Q_{\beta_2,\epsilon}^\alpha \neq \emptyset$ for some $\beta_1, \beta_2 \leq \alpha$, then

$$(5.17) \quad \beta_1 \leq \beta_2.$$

It follows that $\Delta_{\beta,\epsilon}^\alpha = \prod_{\alpha \geq \gamma \geq \beta} Q_{\gamma,\epsilon}^\alpha$. In particular, for any α , by taking $\beta = 1^\Lambda$, we get $Y_T^\Lambda = \prod_{\gamma \leq \alpha} Q_{\gamma,\epsilon}^\alpha$. Further, letting $Q_{\beta,\epsilon}^{[\alpha,\delta]} = Y_T^{[\alpha,\delta]} \times_{Y_T^\Lambda} Q_{\beta,\epsilon}^\alpha$, we obtain $Y_T^{[\alpha,\delta]} = \prod_{\beta \leq \alpha} Q_{\beta,\epsilon}^{[\alpha,\delta]}$. Note that for fixed β with $\beta \leq \alpha$, we have $Q_{\beta,\epsilon}^{[\alpha,\delta]} \subset \prod_{(\beta,\eta) \leq (\alpha,\delta)} Y_{(\beta,\eta)}^{[\alpha,\delta]}$. Define $Q_{(\beta,\eta),\epsilon}^{[\alpha,\delta]} = Q_{\beta,\epsilon}^{[\alpha,\delta]} \cap Y_{(\beta,\eta)}^{[\alpha,\delta]}$ for $(\beta,\eta) \leq (\alpha,\delta)$. Then, we obtain a partition:

$$(5.18) \quad Y_T^{[\alpha,\delta]} = \prod_{(\beta,\eta) \leq (\alpha,\delta)} Q_{(\beta,\eta),\epsilon}^{[\alpha,\delta]}.$$

Lemma 5.9. *For sufficiently small ϵ , we can find a collection of good multi-sections $\mathbf{s}_{[\alpha,\delta]}$ of $\mathcal{F}_{[\alpha,\delta]}$ that satisfy the properties*

- (i) each $\mathbf{s}_{[\alpha,\delta]}$ intersects transversally with the cycle $C_{[\alpha,\delta]} \subset E_{[\alpha,\delta]}$;

(ii) for $(\beta, \eta) < (\alpha, \delta)$, the pseudo-cycles (as \mathbb{Q} -currents)

$$(\zeta_{\alpha, \delta}^{\beta, \eta})_* (D(\mathbf{s}_{[\alpha, \delta]}) \cap \mathcal{Q}_{(\beta, \eta), \epsilon}^{[\alpha, \delta]}) = D(\mathbf{s}_{[\beta, \eta]}) \cap \zeta_{\alpha, \delta}^{\beta, \eta}(\mathcal{Q}_{(\beta, \eta), \epsilon}^{[\alpha, \delta]}).$$

Proof. We follow the proof of [LiJ, Lemma 5.6] line by line, with $\mathcal{Q}_{\beta, \epsilon}^\alpha$ (respectively, s_α) in [LiJ, p. 2156] replaced by $\mathcal{Q}_{(\beta, \eta), \epsilon}^{[\alpha, \delta]}$ (respectively, $\mathbf{s}_{[\alpha, \delta]}$).

To carry the argument in [LiJ, p. 2156] through in the current situation, two modifications are necessary. The first is using multi-sections of $\mathcal{F}_{[\alpha, \delta]}$, etc. The two properties of sections we used in the proof of [LiJ, Lemma 5.6] are the existence of extensions and general position results. For multiple-sections, similar results hold as we have mentioned before.

The other is to choose multi-section $\mathbf{s}_{[\alpha, \delta]}|_{[\beta, \eta]}$ of $\mathcal{F}_{[\alpha, \delta]}|_{\mathcal{Q}_{(\beta, \eta), \epsilon}^{[\alpha, \delta]}}$ to be the pullback

$$\mathbf{s}_{[\alpha, \delta]}|_{[\beta, \eta]} = (\zeta_{\alpha, \delta}^{\beta, \eta})^*(\mathbf{s}_{[\beta, \eta]}|_{\zeta_{\alpha, \delta}^{\beta, \eta}(\mathcal{Q}_{(\beta, \eta), \epsilon}^{[\alpha, \delta]})}).$$

(Compare the construction of $s_\alpha|_\beta = s_\beta|_{\mathcal{Q}_{(\beta, \alpha)}^\alpha}$ in [LiJ, p. 2156].) Since $\mathcal{F}_{[\alpha, \delta]}|_{Y_{(\beta, \eta)}^{[\alpha, \delta]}} = (\zeta_{\alpha, \delta}^{\beta, \eta})^* \mathcal{F}_{[\beta, \eta]}$, such pullback is well-defined.

What we need to make sure is that the section $\mathbf{s}_{[\alpha, \delta]}|_{[\beta, \eta]}$ intersects transversally with the cycle $C_{[\alpha, \delta]}$; this is true, following Lemma 5.5 and Lemma 5.6. This completes the proof of the lemma. q.e.d.

5.7. Approximating virtual cycles. In this subsection, we define the pseudo-cycle $\Theta^{[\alpha, \delta]}$ and study its properties. The formula (5.21) below can be roughly thought of as a decomposition of the virtual cycle $\text{ev}_* [Y_T^{[\alpha, \delta]}]^\text{vir}$ as a sum of cycles $\Theta^{[\beta, \eta]}$ supported near α -diagonals. The ideal situation is that we have a similar decomposition for $\text{ev}_* [Y_T^{[n, d]}]^\text{vir}$ in $(Y_T^{[n]})^3$. Unfortunately, such a decomposition doesn't exist. However, the decomposition (5.21) works equally well as if we had a decomposition for $\text{ev}_* [Y_T^{[n, d]}]^\text{vir}$. This is carried out in subsections 5.11 and 5.12. In fact, the main reason for introducing Hilbert schemes of α -points $Y_T^{[\alpha]}$, non-separated spaces $Y_T^{[\leq \alpha]}$, and moduli spaces $Y_T^{[\alpha, \delta]}$ of α -stable maps to Hilbert schemes is to provide appropriate spaces where we can define $\Theta^{[\alpha, \delta]}$.

Let $(\beta, \eta) \leq (\alpha, \delta) \in \mathcal{P}_{\Lambda, d}$. Define

$$\phi_{\beta, \alpha} : Y_T^{[\beta]} \rightarrow Y_T^{[\leq \alpha]}, \quad \tilde{\phi}_{\beta, \alpha} : Y_T^{[\leq \beta]} \rightarrow Y_T^{[\leq \alpha]}$$

to be the open immersions induced from the construction (5.5). The evaluation map $\text{ev}_i : Y_T^{[\Lambda, d]} \rightarrow Y_T^{[\Lambda]}$ induces an evaluation map $Y_T^{[\alpha, \delta]} \rightarrow Y_T^{[\alpha]}$ which will be denoted again by ev_i . Let $\text{ev} = \text{ev}_1 \times \text{ev}_2 \times \text{ev}_3 :$

$Y_T^{[\alpha, \delta]} \rightarrow (Y_T^{[\alpha]})^3$. Since $\text{ev}_i : Y_T^{[\alpha, \delta]} = Y_T^{[\alpha, \delta]} \times_{Y_T^{(\alpha)}} Y_T^\Lambda \rightarrow Y_T^{[\alpha]} = Y_T^{[\alpha]} \times_{Y_T^{(\alpha)}} Y_T^\Lambda$ does not affect the factor Y_T^Λ , we have $\text{ev}(Y_T^{[\alpha, \delta]}) \subset (Y_T^{[\alpha]})^3 \times_{(Y_T^\Lambda)^3} \iota_\Lambda(Y_T^\Lambda)$ where

$$\iota_\Lambda : Y_T^\Lambda \rightarrow (Y_T^\Lambda)^3$$

is the diagonal embedding. Define the indexing morphism to be

$$(5.19) \quad \text{in} : \bigcup_{(\beta, \eta) \leq (\alpha, \delta)} (\phi_{\beta, \alpha}^3) \text{ev}(Y_T^{[\beta, \eta]}) \longrightarrow \iota_\Lambda(Y_T^\Lambda) \cong Y_T^\Lambda.$$

Definition 5.10. Define the pseudo-cycle $\Theta^{[\alpha, \delta]} \subset (Y_T^{[\leq \alpha]})^3$ inductively by

$$(5.20) \quad \Theta^{[\alpha, \delta]} = (\phi_{\alpha, \alpha}^3)_* \text{ev}_* D(\mathbf{s}_{[\alpha, \delta]}) - \sum_{(\beta, \eta) < (\alpha, \delta)} (\tilde{\phi}_{\beta, \alpha}^3)_* \Theta^{[\beta, \eta]}.$$

By Proposition 5.8, we obtain

$$(5.21) \quad (\phi_{\alpha, \alpha}^3)_* \text{ev}_* [Y_T^{[\alpha, \delta]}]^{\text{vir}} = \sum_{(\beta, \eta) \leq (\alpha, \delta)} (\tilde{\phi}_{\beta, \alpha}^3)_* [\Theta^{[\beta, \eta]}].$$

Further properties of the pseudo-cycles $\Theta^{[\alpha, \delta]}$ are contained in the next two lemmas which are the analogues of Lemmas 5.7 and 5.9 in [LiJ].

Lemma 5.11. $\Theta^{[\alpha, \delta]} \subset (Y_T^{[\leq \alpha]})^3 \times_{(Y_T^\Lambda)^3} \iota_\Lambda(\Delta_{\alpha, \epsilon})$ for sufficiently small $c > 0$.

Proof. We use induction on the order of $(\alpha, \delta) \in \mathcal{P}_{\Lambda, d}$. Assume that (α, δ) is minimal. Then for each i , we have either $(\alpha_i, \delta_i) = (1, 0)$, or $\alpha_i = 2$ and $\delta_i > 0$. So $Y_T^{[\alpha, \delta]} = Y_T^{[\alpha, \delta]} \times_{Y_T^\Lambda} \Delta_\alpha = Y_T^{[\alpha, \delta]} \times_{Y_T^\Lambda} \Delta_{\alpha, \epsilon}$. Thus, $\Theta^{[\alpha, \delta]} = (\phi_{\alpha, \alpha}^3)_* \text{ev}_* D(\mathbf{s}_{[\alpha, \delta]}) \subset (Y_T^{[\leq \alpha]})^3 \times_{(Y_T^\Lambda)^3} \iota_\Lambda(\Delta_{\alpha, \epsilon})$.

Next, we assume that our lemma is true for every (γ, ρ) with $(\gamma, \rho) < (\alpha, \delta)$. Recall that $Y_T^\Lambda = \coprod_{\beta \leq \alpha} Q_{\beta, \epsilon}^\alpha$ and $Q_{\alpha, \epsilon}^\alpha = \Delta_{\alpha, \epsilon}$. So to prove the lemma, it suffices to verify $\Theta^{[\alpha, \delta]} \cap ((Y_T^{[\leq \alpha]})^3 \times_{(Y_T^\Lambda)^3} \iota_\Lambda(Q_{\beta, \epsilon}^\alpha)) = \emptyset$ for every $\beta < \alpha$. By (5.20), this is equivalent to proving that the intersection

$$(5.22) \quad \begin{aligned} & (\phi_{\alpha, \alpha}^3)_* \text{ev}_* D(\mathbf{s}_{[\alpha, \delta]}) \cap ((Y_T^{[\leq \alpha]})^3 \times_{(Y_T^\Lambda)^3} \iota_\Lambda(Q_{\beta, \epsilon}^\alpha)) \\ &= \sum_{(\gamma, \rho) < (\alpha, \delta)} (\tilde{\phi}_{\gamma, \alpha}^3)_* \Theta^{[\gamma, \rho]} \cap ((Y_T^{[\leq \alpha]})^3 \times_{(Y_T^\Lambda)^3} \iota_\Lambda(Q_{\beta, \epsilon}^\alpha)). \end{aligned}$$

On one hand, if $(\gamma, \rho) < (\alpha, \delta)$, then $\Theta^{[\gamma, \rho]} \subset (Y_T^{[\leq \gamma]})^3 \times_{(Y_T^\Lambda)^3} \iota_\Lambda(\Delta_{\gamma, \epsilon})$ by induction. Thus, a nonempty $(\tilde{\phi}_{\gamma, \alpha}^3)_* \Theta^{[\gamma, \rho]} \cap ((Y_T^{[\leq \alpha]})^3 \times_{(Y_T^\Lambda)^3} \iota_\Lambda(Q_{\beta, \epsilon}^\alpha))$

forces $\Delta_{\gamma,\epsilon} \cap Q_{\beta,\epsilon}^\alpha \neq \emptyset$ which in turn implies $\gamma \leq \beta$ by (5.17). Therefore, the right-hand-side of (5.22) equals

$$(5.23) \quad \sum_{(\gamma,\rho) < (\alpha,\delta), \gamma \leq \beta} (\tilde{\phi}_{\gamma,\alpha}^3)_* \Theta^{[\gamma,\rho]} \cap ((Y_T^{[\leq \alpha]})^3 \times_{(Y_T^\Delta)^3} \iota_\Lambda(Q_{\beta,\epsilon}^\alpha))$$

$$= \sum_{(\beta,\eta) \leq (\alpha,\delta)} \sum_{(\gamma,\rho) \leq (\beta,\eta)} (\tilde{\phi}_{\beta,\alpha}^3)_* (\tilde{\phi}_{\gamma,\beta}^3)_* \Theta^{[\gamma,\rho]} \cap ((Y_T^{[\leq \alpha]})^3 \times_{(Y_T^\Delta)^3} \iota_\Lambda(Q_{\beta,\epsilon}^\alpha)).$$

Since $(\phi_{\beta,\beta}^3)_* \text{ev}_* D(\mathbf{s}_{[\beta,\eta]}) = \sum_{(\gamma,\rho) \leq (\beta,\eta)} (\tilde{\phi}_{\gamma,\beta}^3)_* \Theta^{[\gamma,\rho]}$, (5.23) is equal to

$$(5.24) \quad \sum_{(\beta,\eta) \leq (\alpha,\delta)} (\tilde{\phi}_{\beta,\alpha}^3)_* (\phi_{\beta,\beta}^3)_* \text{ev}_* D(\mathbf{s}_{[\beta,\eta]}) \cap ((Y_T^{[\leq \alpha]})^3 \times_{(Y_T^\Delta)^3} \iota_\Lambda(Q_{\beta,\epsilon}^\alpha))$$

$$= \sum_{(\beta,\eta) \leq (\alpha,\delta)} (\phi_{\beta,\alpha}^3)_* \text{ev}_* D(\mathbf{s}_{[\beta,\eta]}) \cap ((Y_T^{[\leq \alpha]})^3 \times_{(Y_T^\Delta)^3} \iota_\Lambda(Q_{\beta,\epsilon}^\alpha)).$$

Since $Q_{\beta,\epsilon}^\alpha = \Delta_{\beta,\epsilon} - \cup_{\alpha \geq \gamma > \beta} \Delta_{\gamma,\epsilon}^\alpha$, we see that $(\phi_{\beta,\alpha}^3)_* \text{ev}_* D(\mathbf{s}_{[\beta,\eta]}) \cap ((Y_T^{[\leq \alpha]})^3 \times_{(Y_T^\Delta)^3} \iota_\Lambda(Q_{\beta,\epsilon}^\alpha))$ is contained in $\phi_{\beta,\alpha}^3 \text{ev}(\zeta_{\alpha,\delta}^{\beta,\eta}(\mathcal{Q}_{(\beta,\eta),\epsilon}^{[\alpha,\delta]}))$. So (5.24) (hence the right-hand-side of (5.22)) equals

$$(5.25) \quad \sum_{(\beta,\eta) \leq (\alpha,\delta)} (\phi_{\beta,\alpha}^3)_* \text{ev}_* D(\mathbf{s}_{[\beta,\eta]}) \cap \phi_{\beta,\alpha}^3 \text{ev}(\zeta_{\alpha,\delta}^{\beta,\eta}(\mathcal{Q}_{(\beta,\eta),\epsilon}^{[\alpha,\delta]}))$$

$$= \sum_{(\beta,\eta) \leq (\alpha,\delta)} (\phi_{\beta,\alpha}^3)_* \text{ev}_*(D(\mathbf{s}_{[\beta,\eta]}) \cap \zeta_{\alpha,\delta}^{\beta,\eta}(\mathcal{Q}_{(\beta,\eta),\epsilon}^{[\alpha,\delta]})).$$

On the other hand, $(\phi_{\alpha,\alpha}^3)_* \text{ev}_* D(\mathbf{s}_{[\alpha,\delta]})$ is supported on

$$\cup_{(\gamma,\rho) \leq (\alpha,\delta)} \phi_{\alpha,\alpha}^3 \text{ev}(\mathcal{Q}_{(\gamma,\rho),\epsilon}^{[\alpha,\delta]}),$$

by (5.18). Moreover, $\phi_{\alpha,\alpha}^3 \text{ev}(\mathcal{Q}_{(\gamma,\rho),\epsilon}^{[\alpha,\delta]})$ is contained in $(Y_T^{[\leq \alpha]})^3 \times_{(Y_T^\Delta)^3} \iota_\Lambda(Q_{\gamma,\epsilon}^\alpha)$, and the subsets $\iota_\Lambda(Q_{\gamma,\epsilon}^\alpha), \gamma \leq \alpha$ are disjoint. So the left-hand-side of (5.22) is equal to

$$\sum_{(\beta,\eta) \leq (\alpha,\delta)} (\phi_{\alpha,\alpha}^3)_* \text{ev}_* D(\mathbf{s}_{[\alpha,\delta]}) \cap \phi_{\alpha,\alpha}^3 \text{ev}(\mathcal{Q}_{(\beta,\eta),\epsilon}^{[\alpha,\delta]})$$

$$= \sum_{(\beta,\eta) \leq (\alpha,\delta)} (\phi_{\alpha,\alpha}^3)_* \text{ev}_*(D(\mathbf{s}_{[\alpha,\delta]}) \cap \mathcal{Q}_{(\beta,\eta),\epsilon}^{[\alpha,\delta]})$$

$$= \sum_{(\beta,\eta) \leq (\alpha,\delta)} (\phi_{\beta,\alpha}^3)_* \text{ev}_*((\zeta_{\alpha,\delta}^{\beta,\eta})_*(D(\mathbf{s}_{[\alpha,\delta]}) \cap \mathcal{Q}_{(\beta,\eta),\epsilon}^{[\alpha,\delta]}))$$

$$= \sum_{(\beta,\eta) \leq (\alpha,\delta)} (\phi_{\beta,\alpha}^3)_* \text{ev}_*(D(\mathbf{s}_{[\beta,\eta]}) \cap \zeta_{\alpha,\delta}^{\beta,\eta}(\mathcal{Q}_{(\beta,\eta),\epsilon}^{[\alpha,\delta]})),$$

where we have used Lemma 5.9 (ii) in the last step. Combining with (5.25), we get (5.22). q.e.d.

Lemma 5.12. *Let $(\alpha, \delta) \in \mathcal{P}_{\Lambda, d}$ with $\alpha = (\alpha_1, \dots, \alpha_l)$. Then, $\Theta^{\llbracket \alpha, \delta \rrbracket} = \prod_{i=1}^l \Theta^{\llbracket \alpha_i, \delta_i \rrbracket}$ via the natural identification $(Y_T^{\llbracket \leq \alpha \rrbracket})^3 = \prod_{i=1}^l (Y_T^{\llbracket \leq \alpha_i \rrbracket})^3$.*

Proof. First of all, since $Y_T^{\llbracket \alpha, \delta \rrbracket} = \prod_{i=1}^l Y_T^{\llbracket \alpha_i, \delta_i \rrbracket}$, we have

$$(5.26) \quad D(\mathbf{s}_{\llbracket \alpha, \delta \rrbracket}) = \prod_{i=1}^l D(\mathbf{s}_{\llbracket \alpha_i, \delta_i \rrbracket}).$$

Next, to prove the lemma, we use induction on the size $|\Lambda|$ and on the order of $(\alpha, \delta) \in \mathcal{P}_{\Lambda, d}$. Assume that (α, δ) is minimal in $\mathcal{P}_{\Lambda, d}$. Then (α_i, δ_i) is minimal in $\mathcal{P}_{\alpha_i, \delta_i}$. By (5.20) and (5.26),

$$\Theta^{\llbracket \alpha, \delta \rrbracket} = (\phi_{\alpha, \alpha}^3)_* \text{ev}_* D(\mathbf{s}_{\llbracket \alpha, \delta \rrbracket}) = \prod_{i=1}^l (\phi_{\alpha_i, \alpha_i}^3)_* \text{ev}_* D(\mathbf{s}_{\llbracket \alpha_i, \delta_i \rrbracket}) = \prod_{i=1}^l \Theta^{\llbracket \alpha_i, \delta_i \rrbracket}.$$

In particular, the lemma holds for $|\Lambda| = 1$ (necessarily, $(\alpha, \delta) = (1, 0)$). Next, assume that $\Theta^{\llbracket \beta, \eta \rrbracket} = \prod_i \Theta^{\llbracket \beta_i, \eta_i \rrbracket}$ for every $(\beta, \eta) < (\alpha, \delta)$. By (5.26) and (5.20),

$$\begin{aligned} (\phi_{\alpha, \alpha}^3)_* \text{ev}_* D(\mathbf{s}_{\llbracket \alpha, \delta \rrbracket}) &= \prod_{i=1}^l (\phi_{\alpha_i, \alpha_i}^3)_* \text{ev}_* D(\mathbf{s}_{\llbracket \alpha_i, \delta_i \rrbracket}) \\ &= \prod_{i=1}^l \sum_{(\beta^{(i)}, \eta^{(i)}) \leq (\alpha_i, \delta_i)} (\tilde{\phi}_{\beta^{(i)}, \alpha_i}^3)_* \Theta^{\llbracket \beta^{(i)}, \eta^{(i)} \rrbracket} \\ &= \prod_{i=1}^l \Theta^{\llbracket \alpha_i, \delta_i \rrbracket} + \sum_{(\beta, \eta) < (\alpha, \delta)} (\tilde{\phi}_{\beta, \alpha}^3)_* \Theta^{\llbracket \beta, \eta \rrbracket} \end{aligned}$$

noting that induction has been used in the last step to handle those $\beta^{(i)}$ which have length greater than 1. Applying (5.20) again, we obtain the lemma. q.e.d.

5.8. Co-section localizations. We now apply the co-section localization techniques from [KL1, KL2, LL] to the constructions in the previous subsections. Let θ be a meromorphic section of $\mathcal{O}_X(K_X)$, and let D_0 and D_∞ be the vanishing and pole divisors of θ respectively. For simplicity, we assume that D_0 and D_∞ are smooth irreducible curves intersecting transversally. Let $X_\theta^{[n, d]}$ be the subset of $X^{[n, d]}$ consisting of those φ whose standard decomposition $(\varphi_1, \dots, \varphi_l)$ have the property that for each i , either φ_i is constant or the support of φ_i lies in $D_0 \cup D_\infty$. The meromorphic section θ induces a meromorphic section $\theta^{[n]}$ of $\Omega_{X^{[n]}}^2$. By [KL2, LL], we obtain the localized virtual fundamental cycle $[X^{[n, d]}]_{\text{loc}}^{\text{vir}} \in A_*(X_\theta^{[n, d]})$ of $X^{[n, d]}$ such that $\iota_* [X^{[n, d]}]_{\text{loc}}^{\text{vir}} = [X^{[n, d]}]^{\text{vir}}$

where ι_* is the map induced by the inclusion map $\iota : X_\theta^{[n,d]} \hookrightarrow X^{[n,d]}$. For simplicity of notations, we write $[X^{[n,d]}]_{\text{loc}}^{\text{vir}} = [X^{[n,d]}]^{\text{vir}}$.

The constructions in [KL2, LL] and Subsections 5.1–5.7 are canonical. Applying the constructions in [KL2, LL] to Subsections 5.1–5.7, we obtain localized cycles $[X^{[\alpha,\delta]}]_{\text{loc}}^{\text{vir}} \in H_*(X_\theta^{[\alpha,\delta]}; \mathbb{Q})$, $D(\mathfrak{s}_{[\alpha,\delta]})_{\text{loc}}$, and $\Theta_{\text{loc}}^{[\alpha,\delta]} \subset \cup_{(\beta,\eta) \leq (\alpha,\delta)} \phi_{\beta,\alpha}^3 \text{ev}(X_\theta^{[\beta,\eta]})$ with $[X^{[\alpha,\delta]}]_{\text{loc}}^{\text{vir}} = [X^{[\alpha,\delta]}]^{\text{vir}}$ and $[\Theta_{\text{loc}}^{[\alpha,\delta]}] = [\Theta^{[\alpha,\delta]}]$ in $H_*(X^{[\alpha,\delta]}; \mathbb{Q})$ and $H_*((X^{[\leq \alpha]})^3; \mathbb{Q})$ respectively. Here the subset $X_\theta^{[\alpha,\delta]} \subset X^{[\alpha,\delta]}$ is defined similarly as $X_\theta^{[n,d]} \subset X^{[n,d]}$.

5.9. Extensions of Heisenberg monomial classes. Let $(\beta, \eta) \in \mathcal{P}_{[n],d}$. To study the pairings with $(\tilde{\phi}_{\beta,[n]}^3)_*[\Theta^{[\beta,\eta]}]$, we need to extend the classes $(f^{[n]})^*w \in H^*(X^{[n]})$ from $X^{[n]}$ to $X^{[\leq n]}$, where $f^{[n]} : X^{[n]} \rightarrow X^{[n]}$ is the tautological map. Let $f^{[\beta]} = \prod_i f^{[|\beta_i|]}$.

Lemma 5.13. *Let $\alpha_i \in H^*(X)$ be homogeneous with $|\alpha_i| > 0$, and $\alpha_{i,j} = 1_X$. Let*

$$(5.27) \quad w = \left(\prod_{i=1}^t \prod_{j=1}^{s_i} \mathfrak{a}_{-i}(\alpha_{i,j}) \right) \left(\prod_{i=1}^k \mathfrak{a}_{-n_i}(\alpha_i) \right) |0\rangle \in H^*(X^{[n]}).$$

Then there exists a class $w^{[\leq n]} \in H^*(X^{[\leq n]})$ such that $(\phi_{[n],[n]})^*w^{[\leq n]} = (f^{[n]})^*w$, and that if $\beta = (\beta_1, \dots, \beta_l) \leq [n]$, then via the identification $X^{[\leq \beta]} = \prod_{i=1}^l X^{[\leq \beta_i]}$,

$$(5.28) \quad (\tilde{\phi}_{\beta,[n]})^*w^{[\leq n]} = \sum_{w_1 \circ \dots \circ w_l = w} \otimes_{i=1}^l w_i^{[\leq |\beta_i|]},$$

where each $w_i \in H^*(X^{[|\beta_i|]})$ is a Heisenberg monomial class.

Proof. We use induction on n . The lemma is trivially true when $n = 1$. In the following, assume that the lemma holds for all $X^{[m]}$ with $m < n$.

Let S be the set consisting of all the partitions β of $[n]$ with $\beta \neq [n]$. By (5.5), $X^{[\leq n]}$ is covered by the open subsets $\phi_{[n],[n]}(X^{[n]})$ and $\tilde{\phi}_{\beta,[n]}(X^{[\leq \beta]})$, $\beta \in S$. For each $\beta \in S$, define

$$w^{[\leq \beta]} = \sum_{w_1 \circ \dots \circ w_l = w} \otimes_{i=1}^l w_i^{[\leq |\beta_i|]} \in H^*(X^{[\leq \beta]}) = H^*(\tilde{\phi}_{\beta,[n]}(X^{[\leq \beta]})),$$

where since $|\beta_i| < n$, the existence of each class $w_i^{[\leq |\beta_i|]}$ comes from induction. Now we have the class $(f^{[n]})^*w$ on $\phi_{[n],[n]}(X^{[n]})$ and the class $w^{[\leq \beta]}$ on $\tilde{\phi}_{\beta,[n]}(X^{[\leq \beta]})$ for each $\beta \in S$. We claim that these classes can be glued into a class, denoted by $w^{[\leq n]}$, on $X^{[\leq n]}$. Indeed, by applying the Mayer–Vietoris sequence successively, we see that it suffices to prove the following:

- (A) the restrictions of $w^{\llbracket \leq \beta \rrbracket}$, $\beta \in S$ and $w^{\llbracket \leq \gamma \rrbracket}$, $\gamma \in S$ to the intersection $\tilde{\phi}_{\beta, [n]}(X^{\llbracket \leq \beta \rrbracket}) \cap \tilde{\phi}_{\gamma, [n]}(X^{\llbracket \leq \gamma \rrbracket})$ are equal;
- (B) the restrictions of $(f^{[n]})^*w$ and $w^{\llbracket \leq \gamma \rrbracket}$, $\gamma \in S$ to the intersection $\phi_{[n], [n]}(X^{\llbracket n \rrbracket}) \cap \tilde{\phi}_{\gamma, [n]}(X^{\llbracket \leq \gamma \rrbracket})$ are equal.

First of all, we verify (A). Let $\beta, \gamma = (\gamma_1, \dots, \gamma_r) \in S$ and $\beta \neq \gamma$. Then,

$$\tilde{\phi}_{\beta, [n]}(X^{\llbracket \leq \beta \rrbracket}) \cap \tilde{\phi}_{\gamma, [n]}(X^{\llbracket \leq \gamma \rrbracket}) \subset \tilde{\phi}_{\beta \wedge \gamma, [n]}(X^{\llbracket \leq \beta \wedge \gamma \rrbracket}).$$

Let $\beta_i \wedge \gamma = (\beta_i \cap \gamma_1, \dots, \beta_i \cap \gamma_r) \in \mathcal{P}_{\beta_i}$. Then $(\tilde{\phi}_{\beta \wedge \gamma, \beta})^*w^{\llbracket \leq \beta \rrbracket}$ is equal to

$$(\tilde{\phi}_{\beta \wedge \gamma, \beta})^* \sum_{w_1 \circ \dots \circ w_l = w} \otimes_{i=1}^l w_i^{\llbracket \leq |\beta_i| \rrbracket} = \sum_{w_1 \circ \dots \circ w_l = w} \otimes_{i=1}^l (\tilde{\phi}_{\beta_i \wedge \gamma, \beta_i})^* w_i^{\llbracket \leq |\beta_i| \rrbracket}.$$

Applying induction to the classes $w_i^{\llbracket \leq |\beta_i| \rrbracket}$, we see that

$$\begin{aligned} (5.29) \quad & (\tilde{\phi}_{\beta \wedge \gamma, \beta})^* w^{\llbracket \leq \beta \rrbracket} \\ &= \sum_{w_1 \circ \dots \circ w_l = w} \otimes_{i=1}^l \left(\sum_{w_{i,1} \circ \dots \circ w_{i,r} = w_i} \otimes_{j=1}^r w_{i,j}^{\llbracket \leq |\beta_i \cap \gamma_j| \rrbracket} \right) \\ &= \sum_{w_{1,1} \circ \dots \circ w_{l,r} = w} \otimes_{i=1}^l \otimes_{j=1}^r w_{i,j}^{\llbracket \leq |\beta_i \cap \gamma_j| \rrbracket}. \end{aligned}$$

It follows immediately that $(\tilde{\phi}_{\beta \wedge \gamma, \beta})^* w^{\llbracket \leq \beta \rrbracket} = (\tilde{\phi}_{\beta \wedge \gamma, \gamma})^* w^{\llbracket \leq \gamma \rrbracket}$.

Next, we verify (B). Note that $X^{\llbracket \leq \gamma \rrbracket}$ is covered by the open subsets $\phi_{\beta, \gamma}(X^{\llbracket \beta \rrbracket})$, $\beta \leq \gamma$, and $\phi_{[n], [n]}(X^{\llbracket n \rrbracket}) \cap \phi_{\beta, [n]}(X^{\llbracket \beta \rrbracket})$ is identified with the images of $X_{\beta}^{\llbracket n \rrbracket} \cong X_{[n]}^{\llbracket \beta \rrbracket}$. So it suffices to prove

$$(5.30) \quad (f^{[n]})^*w|_{X_{\beta}^{\llbracket n \rrbracket}} = (\zeta_{[n]}^{\beta})^*((\phi_{\beta, \gamma})^*w^{\llbracket \leq \gamma \rrbracket}|_{X_{[n]}^{\llbracket \beta \rrbracket}}).$$

To see this, represent each $\alpha_i \in H^*(X)$ by a cycle X_i such that X_1, \dots, X_k are in general position. By Proposition 2.7, the class $w_{[n]} := w / \prod_{i=1}^t s_i!$ is represented by the closure W of the subset consisting of elements of the form (2.1). Then, $(f^{[n]})^*w_{[n]}$ is represented by $(f^{[n]})^{-1}(W)$. By Proposition 2.7 again, the closure of $f^{\llbracket \beta \rrbracket}(\zeta_{[n]}^{\beta}((f^{[n]})^{-1}(W) \cap X_{\beta}^{\llbracket n \rrbracket}))$ in $X^{\llbracket \beta \rrbracket}$ represents the class

$$w_{\beta} := \sum_{w_1 \circ \dots \circ w_l = w} \left(\prod_{i=1}^t \prod_{j=1}^l \frac{1}{s_{i,j}!} \right) \cdot w_1 \otimes \dots \otimes w_l \in H^*(X^{\llbracket \beta \rrbracket}) \cong \bigotimes_{i=1}^l H^*(X^{\llbracket \beta_i \rrbracket}),$$

where each $w_j \in H^*(X^{\llbracket \beta_j \rrbracket})$ contains exactly $s_{i,j}$ copies of $\mathbf{a}_{-i}(1_X)$. Note that $\sum_{j=1}^l s_{i,j} = s_i$. Also, the class $(f^{\llbracket \beta \rrbracket})^*w_{\beta} \in H^*(X^{\llbracket \beta \rrbracket})$ is represented by the closure of $\zeta_{[n]}^{\beta}((f^{[n]})^{-1}(W) \cap X_{\beta}^{\llbracket n \rrbracket})$ in $X^{\llbracket \beta \rrbracket}$. So $(f^{[n]})^*w_{[n]}|_{X_{\beta}^{\llbracket n \rrbracket}} =$

$(\zeta_{[n]}^\beta)^*((f^{[\beta]})^*w_\beta|_{X^{[\beta]}})$. Note that for fixed integers $s_{i,j}$, the number of choices of w_1, \dots, w_l satisfying $w_1 \circ \dots \circ w_l = w$ is precisely equal to $\prod_{i=1}^t s_i! / \prod_{i=1}^t \prod_{k=1}^l s_{i,j}!$. Therefore, $(f^{[n]})^*w|_{X^{[n]}}$ is equal to

$$(5.31) \quad \begin{aligned} & (\zeta_{[n]}^\beta)^* \left((f^{[\beta]})^* \left(\prod_{i=1}^t s_i! \cdot w_\beta \right) \Big|_{X^{[\beta]}} \right) \\ &= (\zeta_{[n]}^\beta)^* \left((f^{[\beta]})^* \sum_{w_1 \circ \dots \circ w_l = w} w_1 \otimes \dots \otimes w_l \Big|_{X^{[\beta]}} \right). \end{aligned}$$

On the other hand, since $\phi_{\beta,\gamma} = \tilde{\phi}_{\beta,\gamma} \circ \phi_{\beta,\beta}$, we obtain from (5.29) that

$$\begin{aligned} (\phi_{\beta,\gamma})^*w^{[\leq \gamma]} &= (\phi_{\beta,\beta})^* \sum_{w_1 \circ \dots \circ w_l = w} \otimes_{i=1}^l w_i^{[\leq |\beta_i|]} \\ &= \sum_{w_1 \circ \dots \circ w_l = w} \otimes_{i=1}^l (\phi_{\beta_i,\beta_i})^*w_i^{[\leq |\beta_i|]} \\ &= \sum_{w_1 \circ \dots \circ w_l = w} \otimes_{i=1}^l (f^{[|\beta_i|]})^*w_i = (f^{[\beta]})^* \sum_{w_1 \circ \dots \circ w_l = w} \otimes_{i=1}^l w_i, \end{aligned}$$

where we have used induction in the third equality. Combining with (5.31) verifies (5.30).

Finally, the properties of the class $w^{[\leq n]} \in H^*(X^{[\leq n]})$ stated in the Lemma follow from the fact that $w^{[\leq n]}$ is the gluing of the class $(f^{[n]})^*w$ and the classes $w^{[\leq \beta]}, \beta \in S$. q.e.d.

Our next lemma says that even though the extension $w^{[\leq n]}$ may not be unique, it does not affect the pairings with $[\Theta^{[n,d]}]$. Recall that the tautological map $\rho_{\alpha,\delta} : X^{[\alpha,\delta]} \rightarrow X^{[\alpha,\delta]}$ is a finite map of degree $n!$.

Lemma 5.14. *Let $A_1, A_2, A_3 \in H^*(X^{[n]})$ be Heisenberg monomial classes. Then, the pairing*

$$\left\langle [\Theta^{[n,d]}], A_1^{[\leq n]} \otimes A_2^{[\leq n]} \otimes A_3^{[\leq n]} \right\rangle$$

is independent of the choices of $A_1^{[\leq n]}, A_2^{[\leq n]}, A_3^{[\leq n]}$.

Proof. Since $[X^{[n,d]}]^{vir} = \rho_{[n],d}^*[X^{[n,d]}]^{vir}$, we have

$$\begin{aligned} & \left\langle [X^{[n,d]}]^{vir}, \text{ev}^*(A_1 \otimes A_2 \otimes A_3) \right\rangle \\ &= \frac{1}{n!} \left\langle [X^{[n,d]}]^{vir}, \rho_{[n],d}^* \text{ev}^*(A_1 \otimes A_2 \otimes A_3) \right\rangle \\ &= \frac{1}{n!} \left\langle [X^{[n,d]}]^{vir}, \text{ev}^* \left(\bigotimes_{i=1}^3 (f^{[n]})^*A_i \right) \right\rangle. \end{aligned}$$

By Lemma 5.13 and (5.21), $\langle [X^{[n,d]}]^{vir}, \text{ev}^*(A_1 \otimes A_2 \otimes A_3) \rangle$ is equal to

$$\begin{aligned}
 (5.32) \quad & \frac{1}{n!} \left\langle [X^{[n,d]}]^{vir}, \text{ev}^* \left(\bigotimes_{i=1}^3 (\phi_{[n],[n]}^*)^* A_i^{[\leq n]} \right) \right\rangle \\
 &= \frac{1}{n!} \left\langle (\phi_{[n],[n]}^3)_* \text{ev}^* [X^{[n,d]}]^{vir}, \bigotimes_{i=1}^3 A_i^{[\leq n]} \right\rangle \\
 &= \frac{1}{n!} \sum_{(\alpha, \delta) \leq ([n], d)} \left\langle (\tilde{\phi}_{\alpha, [n]}^3)_* [\Theta^{[\alpha, \delta]}], \bigotimes_{i=1}^3 A_i^{[\leq n]} \right\rangle.
 \end{aligned}$$

Next, to prove the lemma, we use induction on n . When $n = 1$, the lemma is trivially true since $A_i^{[\leq n]} = A_i$. Assume that the lemma holds for all $X^{[m]}$ with $m < n$. Let $(\alpha, \delta) < ([n], d)$. By Lemma 5.12 and (5.28), $\langle (\tilde{\phi}_{\alpha, [n]}^3)_* [\Theta^{[\alpha, \delta]}], A_1^{[\leq n]} \otimes A_2^{[\leq n]} \otimes A_3^{[\leq n]} \rangle$ is equal to

$$\begin{aligned}
 (5.33) \quad & \left\langle [\Theta^{[\alpha, \delta]}], (\tilde{\phi}_{\alpha, [n]}^*)^* A_1^{[\leq n]} \otimes (\tilde{\phi}_{\alpha, [n]}^*)^* A_2^{[\leq n]} \otimes (\tilde{\phi}_{\alpha, [n]}^*)^* A_3^{[\leq n]} \right\rangle \\
 &= \sum_{\substack{A_{1,1} \circ \dots \circ A_{1,l} = A_1 \\ A_{2,1} \circ \dots \circ A_{2,l} = A_2 \\ A_{3,1} \circ \dots \circ A_{3,l} = A_3}} \prod_{i=1}^l \left\langle [\Theta^{[\alpha_i, \delta_i]}], A_{1,i}^{[\leq |\alpha_i|]} \otimes A_{2,i}^{[\leq |\alpha_i|]} \otimes A_{3,i}^{[\leq |\alpha_i|]} \right\rangle.
 \end{aligned}$$

Now our lemma follows from (5.32) and induction. q.e.d.

REMARK 5.15. Note that for $A \in H_k(W)$ and $B \in H^k(W)$ on a topological space W , the pairing $\langle A, B \rangle$ is the degree of the 0-cycle $A \cap B \in H_0(W)$. As 0-cycles, $(\phi_{[n],[n]}^3)_*(\text{ev}^*[D(\mathbf{S}_{[n,d]})] \cap ((f^{[n]})^* A_1 \otimes (f^{[n]})^* A_2 \otimes (f^{[n]})^* A_3))$ is equal to

$$\begin{aligned}
 & \sum_{(\alpha, \delta) \leq ([n], d)} (\tilde{\phi}_{\alpha, [n]}^3)_* \sum_{\substack{A_{1,1} \circ \dots \circ A_{1,l} = A_1 \\ A_{2,1} \circ \dots \circ A_{2,l} = A_2 \\ A_{3,1} \circ \dots \circ A_{3,l} = A_3}} \\
 & \prod_{i=1}^l \left([\Theta^{[\alpha_i, \delta_i]}] \cap (A_{1,i}^{[\leq |\alpha_i|]} \otimes A_{2,i}^{[\leq |\alpha_i|]} \otimes A_{3,i}^{[\leq |\alpha_i|]}) \right).
 \end{aligned}$$

Next, we extend the notation of Heisenberg monomial classes to a smooth family $Y \rightarrow T$ of quasi-projective surfaces.

Definition 5.16. Fix integers $s_1, \dots, s_t > 0$ with $\sum_i i s_i = n$. Define

$$(5.34) \quad w^Y = \prod_{i=1}^t \mathfrak{a}_{-i}^Y (1_X)^{s_i} |0\rangle \in H^*(Y_T^{[n]}),$$

to be the cohomology class represented by the cycle $\prod_{i=1}^t s_i! \cdot [W] \in A_*(Y_T^{[n]})$ where $W \subset Y_T^{[n]}$ is the closure of the subset consisting of

elements of the form

$$\sum_{i=1}^t (\xi_{i,1} + \dots + \xi_{i,s_i}) \in (Y_u)^{[n]}, \quad u \in T,$$

where $\xi_{i,m} \in M_i(x_{i,m})$ for some $x_{i,m} \in Y_u$, and all the points $x_{i,m}$ are distinct.

The following is similar to Lemma 5.13, and its proof is omitted.

Lemma 5.17. *Let $w = w^Y$ be as in (5.34). Then there exists $w^{[\leq n]} \in H^*(Y_T^{[\leq n]})$ such that $(\phi_{[n],[n]})^*w^{[\leq n]} = (f^{[n]})^*w$, and that if $\beta = (\beta_1, \dots, \beta_l) \leq [n]$, then $(\tilde{\phi}_{\beta,[n]})^*w^{[\leq n]} = \sum_{w_1 \circ \dots \circ w_l = w} \otimes_{i=1}^l w_i^{[\leq |\beta_i|]}$ via the identification $Y_T^{[\leq \beta]} = \prod_{i=1}^l Y_T^{[\leq \beta_i]}$.*

5.10. Normal slices and universal families. This subsection mainly provides a necessary set-up for the proof of the universality result Lemma 5.21 in the next subsection.

By Lemma 5.11, we have $\Theta^{[\alpha,\delta]} \subset (Y_T^{[\leq \alpha]})^3 \times_{(Y_T^\Delta)^3} \iota_\Lambda(\Delta_{\alpha,\epsilon})$. In this subsection, with $Y = X$ and $\alpha = [n]$, we will describe an analytic space, independent of ϵ , which contains $(X^{[\leq n]})^3 \times_{(X^n)^3} \iota_n(\Delta_{[n],\epsilon})$ whenever ϵ is sufficiently small.

To begin with, let $Y \rightarrow T$ be the total space of a rank-2 vector bundle, viewed as a smooth family of affine schemes. Define the fiber-wise averaging morphism

$$\mathbf{av} : Y_T^{(n)} \rightarrow Y; \quad \sum m_i[x_i] \in Y_t^{(n)} \mapsto \frac{1}{n} \sum m_i x_i \in Y_t, \quad t \in T.$$

Here $\sum m_i x_i$ is the sum using the fiber-wise linear structure of Y/T . Using $Y_T^n \rightarrow Y_T^{(n)}$ and $Y_T^{[n]} \rightarrow Y_T^{(n)}$, we obtain the averaging maps $\mathbf{av} : Y_T^n$ and $Y_T^{[n]} \rightarrow Y$. We define the relative Hilbert scheme of *centered α -points* to be

$$(5.35) \quad Y_{T,0}^{[\alpha]} = Y_T^{[\alpha]} \times_{\mathbf{av}, Y} 0_Y,$$

where $0_Y \subset Y$ is the zero-section of $Y \rightarrow T$.

Next, like in [LiJ], we need to express an open neighborhood of the diagonal $\Delta_{[2]} = \Delta_{[2]}^X \subset X \times X$ a vector bundle structure, using the first projection. As this is impossible in general, we will content to have a C^∞ -vector bundle structure. For this reason, we will again work with the analytic category. We will use differentiable map to mean a C^∞ -map; and an open subset will be open in analytic topology; we will use regular function and Zariski open subset to stand for their original meanings in algebraic geometry.

Consider the total space of the tangent bundle T_X , and its zero-section $0_X \subset T_X$. For an open $\mathcal{U} \subset X \times X$, we view it as a space over X

via (that induced by the first projection) $\text{pr}_1|_{\mathcal{U}} : \mathcal{U} \rightarrow X$. By Lemma 2.4 in [LiJ], there exists a diffeomorphism

$$(5.36) \quad \varphi : \mathcal{U} \longrightarrow \mathcal{V},$$

of a tubular neighborhood \mathcal{U} of $X_{[2]} \subset X \times X$ and a tubular neighborhood \mathcal{V} of $0_X \subset T_X$, both considered as fiber bundles over X , such that

- (A-i) restricting to each fiber $\mathcal{U}_x = (\text{pr}_1|_{\mathcal{U}})^{-1}(x)$, the map $\varphi_x = \varphi|_{\mathcal{U}_x} : \mathcal{U}_x \rightarrow \mathcal{V}_x$ is a biholomorphism,
- (A-ii) $\varphi_x(x) = 0 \in T_{X,x}$, and $d\varphi_x : T_{\mathcal{U}_x,x} \rightarrow T_{\mathcal{V}_x,0}$ is the identity map.

Since $\mathcal{V} \subset T_X$ (over X), we define

$$\mathcal{V}_X^{[\alpha]} = \{(\xi_1, \dots, \xi_l) \in (T_X)_X^{[\alpha]} \mid \text{Supp}(\xi_i) \in \mathcal{V}\}.$$

For \mathcal{U} over X , we define $\mathcal{U}_X^{[\alpha]} = \coprod_{x \in X} (\mathcal{U}_x)^{[\alpha]}$ endowed with the obvious smooth structure. By Lemma 2.5 in [LiJ], φ induces a differentiable isomorphism

$$(5.37) \quad \varphi^{[\alpha]} : \mathcal{U}_X^{[\alpha]} \longrightarrow \mathcal{V}_X^{[\alpha]},$$

as stratified spaces. Both $\mathcal{V}_X^{[\alpha]}$ and $\mathcal{U}_X^{[\alpha]}$ are bundles over X :

$$(5.38) \quad \mathcal{V}_X^{[\alpha]} \longrightarrow X \quad \text{and} \quad \mathcal{U}_X^{[\alpha]} \longrightarrow X.$$

The first is induced by the bundle $\mathcal{V} \subset T_X \rightarrow X$, and the second is via $(\mathcal{U}_x)^{[\alpha]} \mapsto \{x\}$. As $T_X \rightarrow X$ is a vector bundle, we obtain $(T_X)_{X,0}^{[\alpha]} \subset (T_X)_X^{[\alpha]}$ as in (5.35). Let $\mathcal{V}_{X,0}^{[\alpha]} = \mathcal{V}_X^{[\alpha]} \cap (T_X)_{X,0}^{[\alpha]}$, and let $U^{[\alpha]} \subset X^{[\alpha]}$ be the image of $\mathcal{V}_{X,0}^{[\alpha]}$ under the composition

$$\varrho_\alpha : \mathcal{V}_{X,0}^{[\alpha]} \xrightarrow{\subset} \mathcal{V}_X^{[\alpha]} \cong \mathcal{U}_X^{[\alpha]} \longrightarrow X^{[\alpha]} \times X \xrightarrow{\text{pr}_2} X^{[\alpha]},$$

where the first factor of $\mathcal{U}_X^{[\alpha]} \rightarrow X^{[\alpha]} \times X$ is induced by the inclusion $(\mathcal{U}_x)^{[\alpha]} \subset X^{[\alpha]}$, and the second is (5.38). By the Lemma 2.6 and Lemma 2.7 in [LiJ], after shrinking \mathcal{V} if necessary, $U^{[\alpha]}$ is an open neighborhood of $X^{[\alpha]} \times_{X^n} \Delta_{[n]} \subset X^{[\alpha]}$, and

$$(5.39) \quad \varrho_\alpha : \mathcal{V}_{X,0}^{[\alpha]} \longrightarrow U^{[\alpha]}$$

is a smooth isomorphism of stratified spaces fibered over $\Delta_{[n]}$, via the map

$$U^{[\alpha]} \subset \mathcal{U}_X^{[\alpha]} \longrightarrow X^{[\alpha]} \times X \xrightarrow{\text{pr}_2} X,$$

and preserves the partial equivalences of $\mathcal{V}_{X,0}^{[\alpha]}$ and $U^{[\alpha]}$. Note that

$$(5.40) \quad X^{[\leq n]} \times_{X^n} \Delta_{[n]} = \coprod_{\alpha \leq [n]} \phi_{[\alpha],[n]}(X^{[\alpha]} \times_{X^n} \Delta_{[n]}).$$

So $U^{[\leq n]} := \cup_{\alpha \leq [n]} \phi_{[\alpha],[n]}(U^{[\alpha]})$ is an open neighborhood of $X^{[\leq n]} \times_{X^n} \Delta_{[n]}$ in $X^{[\leq n]}$. Since ϵ is sufficiently small, $X^{[\leq n]} \times_{X^n} \Delta_{[n],\epsilon} \subset U^{[\leq n]}$. Thus,

$$(5.41) \quad (X^{[\leq n]})^3 \times_{(X^n)^3} \iota_n(\Delta_{[n],\epsilon}) \subset (U^{[\leq n]})^3$$

noting that by our convention, $(U^{[\leq n]})^3$ is a fibered product over $\Delta_{[n]}$. Since $\mathcal{V}_{X,0}^{[\alpha]} \subset (T_X)_{X,0}^{[\alpha]}$, we put $\mathcal{V}_{X,0}^{[\leq n]} = \cup_{\alpha \leq [n]} \phi_{[\alpha],[n]}(\mathcal{V}_{X,0}^{[\alpha]}) \subset (T_X)_{X,0}^{[\leq n]}$. Then, the smooth isomorphisms ϱ_α from (5.39) induces a smooth isomorphism

$$(5.42) \quad \varrho_{[[n]]} : \mathcal{V}_{X,0}^{[[\leq n]]} \rightarrow U^{[[\leq n]]},$$

of stratified spaces fibered over $X \cong \Delta_{[n]}$. Combining with (5.41), we have

$$(5.43) \quad \begin{aligned} ((T_X)_{X,0}^{[[\leq n]]})^3 &\supset (\mathcal{V}_{X,0}^{[[\leq n]]})^3 \xrightarrow{\varrho_{[[\leq n]]}^3} (U^{[[\leq n]]})^3 \\ &\supset (X^{[[\leq n]]})^3 \times_{(X^n)^3} \iota_n(\Delta_{[n],\epsilon}). \end{aligned}$$

To prove universality results later on, we pick a differentiable map

$$(5.44) \quad g : X \longrightarrow Gr = Gr(2, \mathbb{C}^N),$$

with $N \gg 0$ so that $T_X \cong g^*F$ as smooth vector bundles, where $F \rightarrow Gr$ is the total space of the universal quotient rank-2 bundle over Gr . Let $F_{Gr,0}^{[\alpha_i]} \rightarrow Gr$ be the associated relative Hilbert scheme of centered α_i -points. By Lemma 2.8 in [LiJ], g induces isomorphisms (as stratified spaces) of fiber bundles over X :

$$g^{\alpha_i} : (T_X)_{X,0}^{\alpha_i} \rightarrow g^*F_{Gr,0}^{\alpha_i} \quad \text{and} \quad g^{[\leq n]} : (T_X)_{X,0}^{[\leq n]} \rightarrow g^*F_{Gr,0}^{[\leq n]}.$$

5.11. Pairings with $[\Theta^{[n,d]}]$ when $d > 0$. Here we study more properties of $[\Theta^{[n,d]}]$ which are the crucial ingredients for the universality results of extremal Gromov–Witten invariants of $X^{[n]}$.

CONVENTION 5.18. Fix $d > 0$ and Heisenberg monomial classes

$$(5.45) \quad A_i = \mathbf{a}_{-\lambda(i)}(1_X) \mathbf{a}_{-n_{i,1}}(\alpha_{i,1}) \cdots \mathbf{a}_{-n_{i,u_i}}(\alpha_{i,u_i})|0 \in H^*(X^{[n]}),$$

where $1 \leq i \leq 3$, $u_i \geq 0$, and $|\alpha_{i,j}| > 0$. When $|\alpha_{i,j}| = 4$, we let $\alpha_{i,j} = x$ (the cohomology class of a point). Moreover, if $|\alpha_{i,j}| = 2$, then $\alpha_{i,j}$ can be represented by a Riemann surface intersecting transversally with $D_0 \cup D_\infty$. For simplicity, put $A^{[\leq n]} = A_1^{[\leq n]} \otimes A_2^{[\leq n]} \otimes A_3^{[\leq n]}$.

Our goal is to understand the pairing $\langle [\Theta^{[n,d]}], A^{[\leq n]} \rangle$ when $d > 0$.

Lemma 5.19. *Fix $d > 0$. Then, $\langle [\Theta^{[n,d]}], A^{[\leq n]} \rangle = 0$ if one of the following holds:*

- (i) $|\alpha_{i,j}| = 4$ for some (i, j) ;
- (ii) $|\alpha_{i,j}| = 2$ for two different pairs (i, j) .

Proof. (i) We begin with $d \geq 0$. Consider the 0-cycle $[\Theta_{\text{loc}}^{[n,d]}] \cap A^{[\leq n]}$ in $(X^{[\leq n]})^3$. Choose the point representation $x \in X$ of $\alpha_{i,j}$ such that $x \notin D_0 \cup D_\infty$. By Proposition 2.7, A_i can be represented by a cycle $W_i \subset X^{[n]}$ such that $x \in \text{Supp}(\xi_1)$ for every $\xi_1 \in W_i$. Thus for every ξ_2 contained in the 0-cycle $(\phi_{[n],[n]}^3)_*(\text{ev}_*[D(\mathbf{s}_{[n,d]})_{\text{loc}}] \cap ((f^{[n]})^*A_1 \otimes (f^{[n]})^*A_2 \otimes (f^{[n]})^*A_3))$, the point x is a component of $\text{in}(\xi_2)$ where in is from (5.19). By the localized version of Remark 5.15 and induction, we conclude that x is a component of $\text{in}(\xi)$ if ξ is contained in $[\Theta_{\text{loc}}^{[n,d]}] \cap A^{[\leq n]}$.

Now let $d > 0$. By the localized version of Lemma 5.11, we have

$$\Theta_{\text{loc}}^{[n,d]} \subset \left((X^{[\leq n]})^3 \times_{(X^\wedge)^3} \iota_\Lambda(\Delta_{[n],\epsilon}) \right) \cap \bigcup_{(\beta,\eta) \leq ([n],d)} \phi_{\beta,[n]}^3 \text{ev}(X_\theta^{[\beta,\eta]}).$$

Thus, since $d > 0$, if $\xi \in \Theta_{\text{loc}}^{[n,d]}$, then $\text{in}(\xi) \in \Delta_{[n],\epsilon}$ and $y \in D_0 \cup D_\infty$ for some component y of $\text{in}(\xi)$. Since ϵ is sufficiently small, we see from the previous paragraph that $[\Theta_{\text{loc}}^{[n,d]}] \cap A^{[\leq n]}$ is empty. Hence as pairings, $\langle [\Theta_{\text{loc}}^{[n,d]}], A^{[\leq n]} \rangle = \langle [\Theta_{\text{loc}}^{[n,d]}], A^{[\leq n]} \rangle = 0$.

(ii) Let $|\alpha_{i_1,j_1}| = |\alpha_{i_2,j_2}| = 2$ where $(i_1, j_1) \neq (i_2, j_2)$. Represent α_{i_1,j_1} and α_{i_2,j_2} by Riemann surfaces C_{i_1,j_1} and C_{i_2,j_2} respectively such that C_{i_1,j_1}, C_{i_2,j_2} and $D_0 \cup D_\infty$ are in general position. As in the proof of (i), we see that if $\xi \in [\Theta_{\text{loc}}^{[n,d]}] \cap A^{[\leq n]}$, then $\text{in}(\xi) \in \Delta_{[n],\epsilon}$ and the components of $\text{in}(\xi)$ contain three points $x_1 \in C_{i_1,j_1}, x_2 \in C_{i_2,j_2}$ and $x_3 \in D_0 \cup D_\infty$. This is impossible since ϵ is sufficiently small and $C_{i_1,j_1}, C_{i_2,j_2}, D_0 \cup D_\infty$ are in general position. So the 0-cycle $[\Theta_{\text{loc}}^{[n,d]}] \cap A^{[\leq n]}$ is empty. q.e.d.

Lemma 5.20. *Let $u_1 = 1, u_2 = u_3 = 0$, and $|\alpha_{1,1}| = 2$ in (5.45). Then, $\langle [\Theta_{\text{loc}}^{[n,d]}], A^{[\leq n]} \rangle = p \cdot \langle K_X, \alpha_{1,1} \rangle$ where p is a constant depending only on $n_{1,1}$ and the partitions $\lambda^{(i)}$.*

Proof. Represent $\alpha_{1,1}$ by a Riemann surface $C_{1,1}$ intersecting transversally with $D_0 \cup D_\infty$. Let

$$C_{1,1} \cap D_0 = \{x_1, \dots, x_{s_+}, x_{s_++1}, \dots, x_{s_++s_-}\},$$

and $C_{1,1} \cap D_\infty$ be

$$\{x_{s_++s_-+1}, \dots, x_{s_++s_-+t_+}, x_{s_++s_-+t_++1}, \dots, x_{s_++s_-+t_++t_-}\},$$

so that the points $x_1, \dots, x_{s_++s_-+t_++t_-}$ are distinct, the intersection of $C_{1,1}$ and D_0 at x_i for $1 \leq i \leq s_+$ (respectively, for $s_++1 \leq i \leq s_++s_-$) is equal to 1 (respectively, -1), and the intersection of $C_{1,1}$ and D_∞ at x_i for $s_++s_-+1 \leq i \leq s_++s_-+t_+$ (respectively, for $s_++s_-+t_++1 \leq i \leq s_++s_-+t_++t_-$) is equal to 1 (respectively, -1). So $s_++s_- = \langle D_0, \alpha_{1,1} \rangle$ and $t_++t_- = \langle D_\infty, \alpha_{1,1} \rangle$. Let $x_i \in X_i$ be a small analytic open neighborhood of x_i such that $X_1, \dots, X_{s_++s_-+t_++t_-}$ are mutually disjoint. As in the proof of Lemma 5.19 (i), we see that

the 0-cycle $[\Theta_{\text{loc}}^{[n,d]}] \cap A^{[\leq n]}$ is a disjoint union of $W_1, \dots, W_{s_++s_-+t_++t_-}$ such that $\text{in}(W_i) \subset (X_i)^n$ for every i . Let e_i be the contribution of each W_i to the pairing $\langle [\Theta_{\text{loc}}^{[n,d]}], A^{[\leq n]} \rangle$. Then,

$$\langle [\Theta^{[n,d]}], A^{[\leq n]} \rangle = \langle [\Theta_{\text{loc}}^{[n,d]}], A^{[\leq n]} \rangle = e_1 + \dots + e_{s_++s_-+t_++t_-}.$$

As in the proof of Lemma 4.3 in [LL], we conclude that each e_i can be computed from X_i so that $e_1 = \dots = e_{s_+} = -e_{s_++1} = \dots = -e_{s_++s_-}$ and $e_{s_++s_-+1} = \dots = e_{s_++s_-+t_+} = -e_{s_++s_-+t_++1} = \dots = -e_{s_++s_-+t_++t_-}$ depend only on $n_{1,1}$ and the partitions $\lambda^{(i)}$. Since $D_0 = K_X + D_\infty$,

$$\begin{aligned} (5.46) \quad \langle [\Theta^{[n,d]}], A^{[\leq n]} \rangle &= (s_+ - s_-)e_1 + (t_+ - t_-)e_{s_++s_-+1} \\ &= e_1 \cdot \langle D_0, \alpha_{1,1} \rangle + e_{s_++s_-+1} \cdot \langle D_\infty, \alpha_{1,1} \rangle \\ &= p \cdot \langle K_X, \alpha_{1,1} \rangle + p' \cdot \langle D_\infty, \alpha_{1,1} \rangle, \end{aligned}$$

where $p = e_1$ and $p' = e_1 + e_{s_++s_-+1}$. Note that for $m \gg 0$, there exists a meromorphic section θ_m of $\mathcal{O}_X(K_X)$ such that mD_∞ is the pole divisor of θ_m . By (5.46),

$$\langle [\Theta^{[n,d]}], A^{[\leq n]} \rangle = p \cdot \langle K_X, \alpha_{1,1} \rangle + p' \cdot \langle mD_\infty, \alpha_{1,1} \rangle,$$

for all $m \gg 0$. It follows that $p' = 0$ and $\langle [\Theta^{[n,d]}], A^{[\leq n]} \rangle = p \cdot \langle K_X, \alpha_{1,1} \rangle$. q.e.d.

Lemma 5.21. *Let $d > 0$ and $A_i = \mathbf{a}_{-\lambda^{(i)}}(1_X)|0\rangle$ for $i \in \{1, 2, 3\}$. Then, $\langle [\Theta^{[n,d]}], A^{[\leq n]} \rangle = p \cdot \langle K_X, K_X \rangle$ where the coefficient p is a constant depending only on the partitions $\lambda^{(i)}$.*

Proof. By Lemma 5.11, $\Theta^{[n,d]} \subset (X^{[\leq n]})^3 \times_{(X^n)^3} \iota_n(\Delta_{[n],\epsilon})$. Using (5.43) and the smooth isomorphism (5.42), we transport the 0-cycle $[\Theta^{[n,d]}] \cap A^{[\leq n]}$ in $(X^{[\leq n]})^3 \times_{(X^n)^3} \iota_n(\Delta_{[n],\epsilon})$ to the following 0-cycle in $(\mathcal{V}_{X,0}^{[\leq n]})^3 \subset ((T_X)_{X,0}^{[\leq n]})^3$:

$$(\varrho_{[\leq n]}^3)^*[\Theta^{[n,d]}] \cap (\varrho_{[\leq n]}^3)^*\left(A^{[\leq n]}|_{(X^{[\leq n]})^3 \times_{(X^n)^3} \iota_n(\Delta_{[n],\epsilon})}\right).$$

Note that these two 0-cycles have the same degree. So as pairings,

$$(5.47) \quad \langle [\Theta^{[n,d]}], A^{[\leq n]} \rangle = \left\langle (\varrho_{[\leq n]}^3)^*[\Theta^{[n,d]}], (\varrho_{[\leq n]}^3)^*\left(A^{[\leq n]}|_{(X^{[\leq n]})^3 \times_{(X^n)^3} \iota_n(\Delta_{[n],\epsilon})}\right) \right\rangle.$$

Let g from (5.44) be generic, and let $F \rightarrow Gr$ be the total space of the universal quotient rank-2 bundle over $Gr = Gr(2, \mathbb{C}^N)$. Let $\overline{T}_X \rightarrow X$ and $\overline{F} \rightarrow Gr$ be the projectifications of $T_X \rightarrow X$ and $F \rightarrow Gr$ respectively. Then the differentiable isomorphism $T_X \cong g^*F$ induces a differentiable isomorphism $\overline{T}_X \cong g^*\overline{F}$. Note that the top diagonal $\Delta_{[n]}^{F,0} := \Delta_{[n]}^F \cap F_{Gr,0}^n$ in $F_{Gr,0}^n$ is the 0-section of $F_{Gr,0}^n \rightarrow Gr$. Put $\Delta_{[n],\epsilon}^{F,0} = \Delta_{[n],\epsilon}^F \cap F_{Gr,0}^n$. Applying the previous constructions to the families $\overline{F} \rightarrow$

Gr and $\overline{T}_X \rightarrow X$ and adopting the proof of Lemma 6.1 in [LiJ], we conclude that there exists a cycle $\Theta_F^{[n,d]} \subset (F_{Gr,0}^{[\leq n]})^3 \times_{(F_{Gr,0}^n)^3} \iota_n(\Delta_{[n],\epsilon}^{F,0})$ such that

$$(5.48) \quad [\Theta_F^{[n,d]}] \in H_* \left((F_{Gr,0}^{[\leq n]})^3 \times_{(F_{Gr,0}^n)^3} \iota_n(\Delta_{[n]}^{F,0}) \right),$$

the intersection $\Theta_F^{[n,d]} \cap (((F_{Gr,0}^{[\leq n]})^3 \times_{(F_{Gr,0}^n)^3} \iota_n(\Delta_{[n],\epsilon}^{F,0})) \times_{Gr} X)$ is transversal, and

$$(5.49) \quad \begin{aligned} & (\varrho_{[\leq n]}^3)^{-1}([\Theta^{[n,d]}]) \\ &= \Theta_F^{[n,d]} \cap \left(((F_{Gr,0}^{[\leq n]})^3 \times_{(F_{Gr,0}^n)^3} \iota_n(\Delta_{[n],\epsilon}^{F,0})) \times_{Gr} X \right), \end{aligned}$$

via $((T_X)_{X,0}^{[\leq n]})^3 \times_{((T_X)_{X,0}^n)^3} \iota_n(\Delta_{[n],\epsilon}^{T_X,0}) \cong ((F_{Gr,0}^{[\leq n]})^3 \times_{(F_{Gr,0}^n)^3} \iota_n(\Delta_{[n],\epsilon}^{F,0})) \times_{Gr} X$. Thus,

$$(5.50) \quad (\varrho_{[\leq n]}^3)^*[\Theta^{[n,d]}] = [(\varrho_{[\leq n]}^3)^{-1}([\Theta^{[n,d]}])]$$

is a homology class supported on $((T_X)_{X,0}^{[\leq n]})^3 \times_{((T_X)_{X,0}^n)^3} \iota_n(\Delta_{[n]}^{T_X,0})$.

Let $A_i^{T_X} = \mathfrak{a}_{-\lambda(i)}^{T_X}(1_X)|0 \in H^*((T_X)_{X,0}^{[n]})$ be defined in Definition 5.16, and put

$$(A^{T_X,0})^{[\leq n]} = (A_1^{T_X,0})^{[\leq n]} \otimes (A_2^{T_X,0})^{[\leq n]} \otimes (A_3^{T_X,0})^{[\leq n]},$$

where $(A_i^{T_X,0})^{[\leq n]} \in H^*((T_X)_{X,0}^{[\leq n]})$ is the pull-back of $(A_i^{T_X})^{[\leq n]} \in H^*((T_X)_{X,0}^{[\leq n]})$ via the inclusion $(T_X)_{X,0}^{[\leq n]} \subset (T_X)_{X,0}^{[\leq n]}$. Let S denote the intersection

$$\begin{aligned} & \left(((T_X)_{X,0}^{[\leq n]})^3 \times_{((T_X)_{X,0}^n)^3} \iota_n(\Delta_{[n]}^{T_X,0}) \right) \\ & \cap (\varrho_{[\leq n]}^3)^{-1}((X^{[\leq n]})^3 \times_{(X^n)^3} \iota_n(\Delta_{[n],\epsilon})). \end{aligned}$$

Then, $S = (\varrho_{[\leq n]}^3)^{-1}((X^{[\leq n]})^3 \times_{(X^n)^3} \iota_n(\Delta_{[n]}))$. We claim that

$$(5.51) \quad (\varrho_{[\leq n]}^3)^* \left(A^{[\leq n]}|_{(X^{[\leq n]})^3 \times_{(X^n)^3} \iota_n(\Delta_{[n],\epsilon})} \right)|_S = (A^{T_X,0})^{[\leq n]}|_S,$$

i.e., $(\varrho_{[\leq n]}^3)^*(A^{[\leq n]}|_{(X^{[\leq n]})^3 \times_{(X^n)^3} \iota_n(\Delta_{[n]})}) = (A^{T_X,0})^{[\leq n]}|_S$. It suffices to prove that

$$(5.52) \quad \begin{aligned} & \varrho_{[\leq n]}^* \left(A_i^{[\leq n]}|_{X^{[\leq n]} \times_{X^n} \Delta_{[n]}} \right) \\ &= (A_i^{T_X,0})^{[\leq n]}|_{\varrho_{[\leq n]}^{-1}(X^{[\leq n]} \times_{X^n} \Delta_{[n]})}. \end{aligned}$$

Indeed, for every $\alpha \leq [n]$, we conclude from Lemma 5.13 and Lemma 5.17 that the same subvariety in $\varrho_{[\leq n]}^{-1}(\phi_{[\alpha],[n]}(X^{[\alpha]} \times_{X^n} \Delta_{[n]}))$ represents the cohomology classes $\varrho_{[\leq n]}^*(A_i^{[\leq n]}|_{\phi_{[\alpha],[n]}(X^{[\alpha]} \times_{X^n} \Delta_{[n]})})$ and

$$(A_i^{T_X,0})^{[\leq n]}|_{\varrho_{[\leq n]}^{-1}(\phi_{[\alpha],[n]}(X^{[\alpha]} \times_{X^n} \Delta_{[n]}))}.$$

Since $X^{\llbracket \leq n \rrbracket} \times_{X^n} \Delta_{[n]} = \coprod_{\alpha \leq [n]} \phi_{[\alpha], [n]}(X^{\llbracket \alpha \rrbracket} \times_{X^n} \Delta_{[n]})$, we obtain (5.52).

By (5.47), (5.51) and (5.50), as pairings, we have

$$\begin{aligned} \langle [\Theta^{\llbracket n, d \rrbracket}], A^{\llbracket \leq n \rrbracket} \rangle &= \langle (\varrho_{\llbracket \leq n \rrbracket}^3)^* [\Theta^{\llbracket n, d \rrbracket}], (A^{T_X, 0})^{\llbracket \leq n \rrbracket} |_S \rangle \\ &= \langle [(\varrho_{\llbracket \leq n \rrbracket}^3)^{-1}(\Theta^{\llbracket n, d \rrbracket})], (A^{T_X, 0})^{\llbracket \leq n \rrbracket} \rangle. \end{aligned}$$

Let $(g^{\llbracket \leq n \rrbracket})^3 : ((T_X)_{X, 0}^{\llbracket \leq n \rrbracket})^3 \rightarrow (F_{Gr, 0}^{\llbracket \leq n \rrbracket})^3 \times_{Gr} X$ be the isomorphism induced by g . By Lemma 5.17, $(A^{T_X, 0})^{\llbracket \leq n \rrbracket}$ can be taken to be

$$(g^{\llbracket \leq n \rrbracket})^{3*} ((A^{F, 0})^{\llbracket \leq n \rrbracket} |_{(F_{Gr, 0}^{\llbracket \leq n \rrbracket})^3 \times_{Gr} X}).$$

So

$$\begin{aligned} &\langle [\Theta^{\llbracket n, d \rrbracket}], A^{\llbracket \leq n \rrbracket} \rangle \\ &= \left\langle [(\varrho_{\llbracket \leq n \rrbracket}^3)^{-1}(\Theta^{\llbracket n, d \rrbracket})], (g^{\llbracket \leq n \rrbracket})^{3*} \left((A^{F, 0})^{\llbracket \leq n \rrbracket} |_{(F_{Gr, 0}^{\llbracket \leq n \rrbracket})^3 \times_{Gr} X} \right) \right\rangle. \end{aligned}$$

Combining with (5.49) and putting $W_\epsilon = (F_{Gr, 0}^{\llbracket \leq n \rrbracket})^3 \times_{(F_{Gr, 0}^{\llbracket \leq n \rrbracket})^3} \iota_n(\Delta_{[n], \epsilon}^{F, 0})$, we get

$$\begin{aligned} \langle [\Theta^{\llbracket n, d \rrbracket}], A^{\llbracket \leq n \rrbracket} \rangle &= \left\langle [(\varrho_{\llbracket \leq n \rrbracket}^3)^{-1}(\Theta^{\llbracket n, d \rrbracket})], (g_\epsilon^{\llbracket \leq n \rrbracket})^{3*} \left((A^{F, 0})^{\llbracket \leq n \rrbracket} |_{W_\epsilon} \right) \right\rangle \\ &= \left\langle ((g_\epsilon^{\llbracket \leq n \rrbracket})^3)_* [(\varrho_{\llbracket \leq n \rrbracket}^3)^{-1}(\Theta^{\llbracket n, d \rrbracket})], (A^{F, 0})^{\llbracket \leq n \rrbracket} |_{W_\epsilon} \right\rangle, \end{aligned}$$

where $(g_\epsilon^{\llbracket \leq n \rrbracket})^3 : ((T_X)_{X, 0}^{\llbracket \leq n \rrbracket})^3 \times_{((T_X)_{X, 0}^{\llbracket \leq n \rrbracket})^3} \iota_n(\Delta_{[n], \epsilon}^{T_X, 0}) \rightarrow W_\epsilon$ is the morphism induced by g . By (5.48), $[\Theta_F^{\llbracket n, d \rrbracket}]$ is supported on $W := (F_{Gr, 0}^{\llbracket \leq n \rrbracket})^3 \times_{(F_{Gr, 0}^{\llbracket \leq n \rrbracket})^3} \iota_n(\Delta_{[n]}^{F, 0})$. Therefore, by (5.49), we obtain

$$\begin{aligned} \langle [\Theta^{\llbracket n, d \rrbracket}], A^{\llbracket \leq n \rrbracket} \rangle &= \langle [\Theta_F^{\llbracket n, d \rrbracket}] \cap \pi^*(\text{PD}^{-1}[g(X)]), (A^{F, 0})^{\llbracket \leq n \rrbracket} |_W \rangle \\ &= \langle [\Theta_F^{\llbracket n, d \rrbracket}] \cap (A^{F, 0})^{\llbracket \leq n \rrbracket} |_W, \pi^*(\text{PD}^{-1}[g(X)]) \rangle \\ &= \left\langle \pi_* \left([\Theta_F^{\llbracket n, d \rrbracket}] \cap (A^{F, 0})^{\llbracket \leq n \rrbracket} |_W \right), \text{PD}^{-1}[g(X)] \right\rangle, \end{aligned}$$

where $\pi : W \rightarrow Gr$ is the tautological projection. Observe that W is in fact a disjoint union of Hausdorff spaces. Thus all the properties of Borel–Moore homology [Iv] such as properties of cap products [GH, Iv, Sp] can be used.

The Poincaré dual of $\pi_*([\Theta_F^{\llbracket n, d \rrbracket}] \cap (A^{F, 0})^{\llbracket \leq n \rrbracket} |_W)$ is a polynomial P in the Chern classes $c_i(F)$. Hence,

$$(5.53) \quad \langle [\Theta^{\llbracket n, d \rrbracket}], A^{\llbracket \leq n \rrbracket} \rangle = p \cdot \langle K_X, K_X \rangle + q \cdot \deg(e_X),$$

where p and q are constants depending only on the partitions $\lambda^{(i)}$.

Finally, it remains to prove that $q = 0$ in (5.53). To see this, choose the surface X such that $|K_X|$ contains a smooth divisor D . Let θ be a holomorphic section of $\mathcal{O}_X(K_X)$ such that the vanishing divisor of θ is $D = D_0$. By (5.50), $(\varrho_{\llbracket \leq n \rrbracket}^3)^*[\Theta_{\text{loc}}^{\llbracket n, d \rrbracket}]$ is a homology class supported on

$((T_X|_D)_{D,0}^{\llbracket \leq n \rrbracket})^3 \times ((T_X|_D)_{D,0}^n)^3 \iota_n(\Delta_{[n]}^{T_X|_D,0})$. Repeating the above argument and replacing $g : X \rightarrow Gr$ (respectively, $T_X \rightarrow X$) by $g|_D : D \rightarrow Gr$ (respectively, $T_X|_D \rightarrow D$), we get

$$\langle [\Theta^{\llbracket n,d \rrbracket}], A^{\llbracket \leq n \rrbracket} \rangle = p' \cdot \langle K_X, K_X \rangle = p \cdot \langle K_X, K_X \rangle + q \cdot \deg(e_X),$$

where p' depends only on the partitions $\lambda^{(i)}$. Since there exist two surfaces X with smooth $D \in |K_X|$ such that the pairs $(\langle K_X, K_X \rangle, \deg(e_X))$ are linearly independent, $p = p'$ and $q = 0$. q.e.d.

5.12. Proofs of Theorem 1.2 and Theorem 1.3. In this subsection, we introduce a new class $\mathfrak{Z}_{n,d} \in H_*((X^{[n]})^3)$ in terms of cycles $\Theta^{\llbracket \alpha, \delta \rrbracket}$ studied intensively in previous subsections. Note that now $\mathfrak{Z}_{n,d}$ is on the Hilbert scheme $(X^{[n]})^3$, not on the non-separated spaces $(X^{\llbracket \leq n \rrbracket})^3$.

Let $\mathcal{B} = \{\beta_1, \dots, \beta_b\}$ be a basis of $H^2(X)$. Then, $\{1_X, x, \beta_1, \dots, \beta_b\}$ is a basis of $H^*(X)$, and $H^*(X^{[n]})$ has a basis $\mathcal{B}^{[n]}$ consisting of

$$\mathbf{a}_{-\lambda}(1_X) \mathbf{a}_{-\mu}(x) \mathbf{a}_{-\nu^{(1)}}(\beta_1) \cdots \mathbf{a}_{-\nu^{(b)}}(\beta_b) |0\rangle,$$

where $|\lambda| + |\mu| + \sum_i |\nu^{(i)}| = n$. Via the Künneth decomposition, a basis of $H^*((X^{[n]})^3)$ consists of the elements $A_1 \otimes A_2 \otimes A_3 = \prod_{i=1}^3 \pi_{n,i}^* A_i$, where $A_1, A_2, A_3 \in \mathcal{B}^{[n]}$ and $\pi_{n,i}$ denotes the i -th projection $(X^{[n]})^3 \rightarrow X^{[n]}$.

Definition 5.22. (i) Let $d \geq 1$, and let $\mathcal{P}_{[n],d}^+$ be the subset of $\mathcal{P}_{[n],d}$ consisting of all the weighted partitions (α, δ) such that $\delta_i > 0$ for every i .

(ii) For $d \geq 1$, define the class $\mathfrak{Z}_{n,d} = \mathfrak{Z}_{n,d}^{\mathcal{B}} \in H_*((X^{[n]})^3)$ by putting

$$(5.54) \quad \left\langle \mathfrak{Z}_{n,d}, \prod_{i=1}^3 \pi_{n,i}^* A_i \right\rangle = \frac{1}{n!} \cdot \sum_{(\alpha, \delta) \in \mathcal{P}_{[n],d}^+} \left\langle (\tilde{\phi}_{\alpha, [n]}^3)_* [\Theta^{\llbracket \alpha, \delta \rrbracket}], A_1^{\llbracket \leq n \rrbracket} \otimes A_2^{\llbracket \leq n \rrbracket} \otimes A_3^{\llbracket \leq n \rrbracket} \right\rangle,$$

for the basis elements $A_1, A_2, A_3 \in \mathcal{B}^{[n]}$.

Next, we prove Theorem 1.2 and Theorem 1.3 which determine the structure of the 3-pointed genus-0 extremal Gromov–Witten invariants of $X^{[n]}$. Note from Theorem 1.3 that the class $\mathfrak{Z}_{n,d} = \mathfrak{Z}_{n,d}^{\mathcal{B}}$ is independent of the choice of the basis \mathcal{B} of $H^2(X)$. So from now on, the basis \mathcal{B} of $H^2(X)$ will be implicit in our presentation.

Proof of Theorem 1.2. By (5.32), $\langle A_1, A_2, A_3 \rangle_{0,d\beta_n}$ is equal to

$$(5.55) \quad \frac{1}{n!} \sum_{(\alpha, \delta) \leq ([n], d)} \left\langle (\tilde{\phi}_{\alpha, [n]}^3)_* [\Theta^{\llbracket \alpha, \delta \rrbracket}], A_1^{\llbracket \leq n \rrbracket} \otimes A_2^{\llbracket \leq n \rrbracket} \otimes A_3^{\llbracket \leq n \rrbracket} \right\rangle.$$

Define $\alpha^0 = \{(\alpha_i)_i \mid \delta_i = 0\}$, and let $(\alpha^0, 0)$ be the weighted partition such that all the weights are equal to 0. Let (α', δ') be the weighted

partition obtained from (α, δ) by deleting all the α_i and δ_i with $\delta_i = 0$. Let $|\alpha'| = m$, $\Lambda_{\alpha^0} = \prod_i (\alpha^0)_i$, and $\Lambda_{\alpha'} = \prod_i (\alpha')_i$. Then, $\alpha = (\alpha^0, 0) \amalg (\alpha', \delta')$, $|\alpha^0| = n - m$, and $[n] = \Lambda_{\alpha^0} \amalg \Lambda_{\alpha'}$. By (5.33),

$$\begin{aligned} & \langle (\tilde{\phi}_{\alpha, [n]}^3)_* [\Theta^{[\alpha, \delta]}], A_1^{[\leq n]} \otimes A_2^{[\leq n]} \otimes A_3^{[\leq n]} \rangle \\ &= \sum_{\substack{A_{1,1} \circ \dots \circ A_{1,l} = A_1 \\ A_{2,1} \circ \dots \circ A_{2,l} = A_2 \\ A_{3,1} \circ \dots \circ A_{3,l} = A_3}} \prod_{i=1}^l \langle [\Theta^{[\alpha_i, \delta_i]}], A_{1,i}^{[\leq |\alpha_i|]} \otimes A_{2,i}^{[\leq |\alpha_i|]} \otimes A_{3,i}^{[\leq |\alpha_i|]} \rangle \\ &= \sum_{\substack{A_{1,1} \circ A_{1,2} = A_1 \\ A_{2,1} \circ A_{2,2} = A_2 \\ A_{3,1} \circ A_{3,2} = A_3}} \langle (\tilde{\phi}_{\alpha^0, \Lambda_{\alpha^0}}^3)_* [\Theta^{[\alpha^0, 0]}], A_{1,1}^{[\leq (n-m)]} \otimes A_{2,1}^{[\leq (n-m)]} \otimes A_{3,1}^{[\leq (n-m)]} \rangle \\ & \quad \cdot \langle (\tilde{\phi}_{\alpha', \Lambda_{\alpha'}}^3)_* [\Theta^{[\alpha', \delta']}], A_{1,2}^{[\leq m]} \otimes A_{2,2}^{[\leq m]} \otimes A_{3,2}^{[\leq m]} \rangle. \end{aligned}$$

Put $\Lambda = \Lambda_{\alpha'}$. By (5.55), $\langle A_1, A_2, A_3 \rangle_{0, d\beta_n}$ is equal to

$$\begin{aligned} (5.56) \quad & \frac{1}{n!} \cdot \sum_{m \leq n} \sum_{\substack{A_{1,1} \circ A_{1,2} = A_1 \\ A_{2,1} \circ A_{2,2} = A_2 \\ A_{3,1} \circ A_{3,2} = A_3}} \sum_{\substack{\Lambda \subset [n] \\ |\Lambda| = m}} \sum_{(\alpha', \delta') \in \mathcal{P}_{\Lambda, d}^+} \\ & \sum_{\alpha^0 \in \mathcal{P}_{[n]-\Lambda}} \langle (\tilde{\phi}_{\alpha^0, [n]-\Lambda}^3)_* [\Theta^{[\alpha^0, 0]}], A_{1,1}^{[\leq (n-m)]} \otimes A_{2,1}^{[\leq (n-m)]} \otimes A_{3,1}^{[\leq (n-m)]} \rangle \\ & \quad \cdot \langle (\tilde{\phi}_{\alpha', \Lambda}^3)_* [\Theta^{[\alpha', \delta']}], A_{1,2}^{[\leq m]} \otimes A_{2,2}^{[\leq m]} \otimes A_{3,2}^{[\leq m]} \rangle. \end{aligned}$$

In particular, setting $d = 0$ in (5.56), we see that $\langle A_1, A_2, A_3 \rangle$ equals

$$\frac{1}{n!} \cdot \sum_{\alpha \in \mathcal{P}_{[n]}} \langle (\tilde{\phi}_{\alpha, [n]}^3)_* [\Theta^{[\alpha, 0]}], A_1^{[\leq n]} \otimes A_2^{[\leq n]} \otimes A_3^{[\leq n]} \rangle.$$

Therefore, by (5.56), $\langle A_1, A_2, A_3 \rangle_{0, d\beta_n}$ is equal to

$$\begin{aligned} & \frac{1}{n!} \cdot \sum_{m \leq n} \sum_{\substack{A_{1,1} \circ A_{1,2} = A_1 \\ A_{2,1} \circ A_{2,2} = A_2 \\ A_{3,1} \circ A_{3,2} = A_3}} \sum_{\substack{\Lambda \subset [n] \\ |\Lambda| = m}} \sum_{(\alpha', \delta') \in \mathcal{P}_{\Lambda, d}^+} (n-m)! \cdot \langle A_{1,1}, A_{2,1}, A_{3,1} \rangle \\ & \quad \cdot \langle (\tilde{\phi}_{\alpha', \Lambda}^3)_* [\Theta^{[\alpha', \delta']}], A_{1,2}^{[\leq m]} \otimes A_{2,2}^{[\leq m]} \otimes A_{3,2}^{[\leq m]} \rangle \\ &= \frac{1}{n!} \cdot \sum_{m \leq n} \sum_{\substack{A_{1,1} \circ A_{1,2} = A_1 \\ A_{2,1} \circ A_{2,2} = A_2 \\ A_{3,1} \circ A_{3,2} = A_3}} \sum_{(\alpha, \delta) \in \mathcal{P}_{[m], d}^+} \binom{n}{m} (n-m)! \cdot \langle A_{1,1}, A_{2,1}, A_{3,1} \rangle \\ & \quad \cdot \langle (\tilde{\phi}_{\alpha, [m]}^3)_* [\Theta^{[\alpha, \delta]}], A_{1,2}^{[\leq m]} \otimes A_{2,2}^{[\leq m]} \otimes A_{3,2}^{[\leq m]} \rangle. \end{aligned}$$

Using the definition of $\mathfrak{Z}_{m,d}$, we complete the proof.

q.e.d.

Proof of Theorem 1.3. Let

$$A_i = \mathbf{a}_{-\lambda^{(i)}}(1_X) \mathbf{a}_{-\mu^{(i)}}(x) \mathbf{a}_{-n_{i,1}}(\alpha_{i,1}) \cdots \mathbf{a}_{-n_{i,u_i}}(\alpha_{i,u_i})|0\rangle,$$

with $|\alpha_{i,j}| = 2$. By linearity, we may assume $\alpha_{i,j} \in \mathcal{B}$ for every i and j . By (5.54) and (5.33),

$$(5.57) \quad \left\langle \mathfrak{Z}_{n,d}, \prod_{i=1}^3 \pi_{n,i}^* A_i \right\rangle \\ = \frac{1}{n!} \cdot \sum_{(\alpha, \delta) \in \mathcal{P}_{[n],d}^+} \sum_{\substack{A_{1,1} \circ \cdots \circ A_{1,l} = A_1 \\ A_{2,1} \circ \cdots \circ A_{2,l} = A_2 \\ A_{3,1} \circ \cdots \circ A_{3,l} = A_3}} \prod_{i=1}^l \left\langle [\Theta^{[\alpha_i, \delta_i]}], \otimes_{j=1}^3 A_{j,i}^{[\leq |\alpha_i|]} \right\rangle.$$

So our theorem, except the degree of p in (ii), follows from Lemma 5.19, Lemma 5.20 and Lemma 5.21. To see the degree of p in (ii), consider a nonzero term in (5.57):

$$(5.58) \quad \prod_{i=1}^l \left\langle [\Theta^{[\alpha_i, \delta_i]}], A_{1,i}^{[\leq |\alpha_i|]} \otimes A_{2,i}^{[\leq |\alpha_i|]} \otimes A_{3,i}^{[\leq |\alpha_i|]} \right\rangle.$$

By Lemma 5.19 (ii), for each i in (5.58), the classes $A_{1,i}, A_{2,i}, A_{3,i}$ together contains at most one Heisenberg factor of the form $\mathbf{a}_{-n_{j,k}}(\alpha_{j,k})$. By Lemma 5.20 and Lemma 5.21, the degree of (5.58) as a monomial of $\langle K_X, K_X \rangle$ is equal to $|I|$ where I is the set consisting of the index $i \in \{1, \dots, l\}$ such that the classes $A_{1,i}, A_{2,i}, A_{3,i}$ together do not contain any Heisenberg factor of the form $\mathbf{a}_{-n_{j,k}}(\alpha_{j,k})$. Now for each $i \in I$, $|\alpha_i| \geq 2$ since $\delta_i \geq 1$. So we conclude that

$$|I| \leq \frac{1}{2} \sum_{i \in I} |\alpha_i| = \frac{1}{2} \left(n - \sum_{i \notin I} |\alpha_i| \right) \leq \frac{1}{2} \left(n - \sum_{j,k} n_{j,k} \right).$$

Hence the degree of p as a polynomial of $\langle K_X, K_X \rangle$ is at most $(n - \sum_{i,j} n_{i,j})/2$. q.e.d.

Corollary 5.23. *Let $d \geq 1$, and let $A_1, A_2, A_3 \in H^*(X^{[n]})$ be Heisenberg monomial classes.*

- (i) *If $A_1 = \mathbf{a}_{-1}(1_X)^{n-1} \mathbf{a}_{-1}(\alpha)|0\rangle$, then $\langle \mathfrak{Z}_{n,d}, \prod_{i=1}^3 \pi_{n,i}^* A_i \rangle = 0$.*
- (ii) *If $A_1 = \mathbf{a}_{-1}(1_X)^{n-1-|\lambda|} \mathbf{a}_{-1}(\alpha) \mathbf{a}_{-\lambda}(x)|0\rangle$ for some λ , then*

$$\langle A_1, A_2, A_3 \rangle_{0, d\beta_n} = 0.$$

Proof. (i) First of all, if $\alpha = x$, then $\langle \mathfrak{Z}_{n,d}, \prod_{i=1}^3 \pi_{n,i}^* A_i \rangle = 0$ by Theorem 1.3 (i).

Next, let $\alpha = 1_X$. Use induction on n . Since $d \geq 1$, the conclusion is trivially true when $n = 1$. Let $n > 1$. Recall that $1/n! \cdot A_1$ is the

fundamental class $1_{X^{[n]}}$ of $X^{[n]}$. By Theorem 1.2 and the Fundamental Class Axiom of Gromov–Witten theory,

$$\sum_{m=2}^n \sum_{\substack{A_{1,1} \circ A_{1,2} = A_1 \\ A_{2,1} \circ A_{2,2} = A_2 \\ A_{3,1} \circ A_{3,2} = A_3}} \langle A_{1,1}, A_{2,1}, A_{3,1} \rangle \cdot \left\langle \mathfrak{Z}_{m,d}, \prod_{i=1}^3 \pi_{m,i}^* A_{i,2} \right\rangle = 0.$$

Since $A_1 = \mathfrak{a}_{-1}(1_X)^n|0\rangle$, we have $A_{1,2} = \mathfrak{a}_{-1}(1_X)^m|0\rangle$. By induction, $\langle \mathfrak{Z}_{m,d}, \prod_{i=1}^3 \pi_{m,i}^* A_{i,2} \rangle = 0$ if $2 \leq m \leq n - 1$. It follows that $\langle \mathfrak{Z}_{n,d}, \prod_{i=1}^3 \pi_{n,i}^* A_i \rangle = 0$.

Now let $|\alpha| = 2$. By the Divisor Axiom of Gromov–Witten theory and $\langle A_1, \beta_n \rangle = 0$, we have $\langle A_1, A_2, A_3 \rangle_{0,d\beta_n} = 0$. Using an argument similar to the one in the previous paragraph, we conclude that $\langle \mathfrak{Z}_{n,d}, \prod_{i=1}^3 \pi_{n,i}^* A_i \rangle = 0$.

(ii) We compute $\langle A_1, A_2, A_3 \rangle_{0,d\beta_n}$ by using (1.3). Note that the class $A_{1,2}$ in (1.3) is equal to $\mathfrak{a}_{-1}(1_X)^m|0\rangle$, or is equal to $\mathfrak{a}_{-1}(1_X)^{m-1}\mathfrak{a}_{-1}(\alpha)|0\rangle$, or contains a factor $\mathfrak{a}_{-i}(x)$ for some $i > 0$. By (i) and Theorem 1.3 (i), we get $\langle A_1, A_2, A_3 \rangle_{0,d\beta_n} = 0$. q.e.d.

6. Proofs of (1.2) and Theorem 1.1

Let X be a simply connected smooth projective surface. Our goal in this section is to prove (1.2) and Theorem 1.1 for $A^{[n]} = H_{\rho_n}^*(X^{[n]})$. The proof of (1.2) is divided into three cases depending on the cohomology degree of the class α in (1.2) and leading to Proposition 6.3, Proposition 6.9 and Proposition 6.12. Assuming these three propositions, we now prove Theorem 1.1.

Proof of Theorem 1.1. The shift number of $\mathfrak{p}_{-n_1}(\alpha_1) \cdots \mathfrak{p}_{-n_s}(\alpha_s)|0\rangle$ is equal to $n_1 + \dots + n_s - s$. Define a linear isomorphism

$$(6.1) \quad \Psi : \mathcal{F}_X \rightarrow \mathbb{H}_X,$$

by sending $\sqrt{-1}^{n_1+\dots+n_s-s} \mathfrak{p}_{-n_1}(\alpha_1) \cdots \mathfrak{p}_{-n_s}(\alpha_s)|0\rangle$ to $\mathfrak{a}_{-n_1}(\alpha_1) \cdots \mathfrak{a}_{-n_s}(\alpha_s)|0\rangle$.

This induces a linear isomorphism $\Psi_n : H_{\text{CR}}^*(X^{(n)}) \rightarrow H^*(X^{[n]})$ for each n . Moreover, Ψ_1 is simply the identity map on the cohomology group of the surface X .

By (4.6), Proposition 6.3, Proposition 6.9 and Proposition 6.12, the two formulas (1.1) and (1.2) hold for $A^{[n]} = H_{\rho_n}^*(X^{[n]})$. By the proof of Theorem 2.4 (i.e., Theorem 4.7 in [LQW3]),

$$(6.2) \quad \begin{aligned} & \tilde{\mathfrak{G}}_k(\alpha) \\ &= - \sum_{\ell(\lambda)=k+2, |\lambda|=0} \frac{1}{\lambda!} \mathfrak{a}_\lambda(\tau_*\alpha) + \sum_{\ell(\lambda)=k, |\lambda|=0} \frac{s(\lambda) - 2}{24\lambda!} \mathfrak{a}_\lambda(\tau_*(e_X\alpha)). \end{aligned}$$

Combining this with Theorem 3.1 (ii), we check directly that

$$\begin{aligned} & \Psi_n \left(\sqrt{-1}^k O_k(\alpha, n) \bullet \sqrt{-1}^{n_1+\dots+n_s-s} \mathfrak{p}_{-n_1}(\alpha_1) \cdots \mathfrak{p}_{-n_s}(\alpha_s) |0\rangle \right) \\ &= \Psi_n \left(\sqrt{-1}^{k+n_1+\dots+n_s-s} \mathfrak{D}_k(\alpha) \mathfrak{p}_{-n_1}(\alpha_1) \cdots \mathfrak{p}_{-n_s}(\alpha_s) |0\rangle \right) \\ &= \tilde{\mathfrak{G}}_k(\alpha) \mathfrak{a}_{-n_1}(\alpha_1) \cdots \mathfrak{a}_{-n_s}(\alpha_s) |0\rangle \\ &= \tilde{G}_k(\alpha, n) \cdot \mathfrak{a}_{-n_1}(\alpha_1) \cdots \mathfrak{a}_{-n_s}(\alpha_s) |0\rangle, \end{aligned}$$

where $n_1 + \dots + n_s = n$. In particular, letting $s = n, n_1 = \dots = n_s = 1$ and $\alpha_1 = \dots = \alpha_s = 1_X$, we get $\Psi_n(\sqrt{-1}^k O_k(\alpha, n)) = \tilde{G}_k(\alpha, n)$. So

$$\begin{aligned} & \Psi_n \left(\sqrt{-1}^k O_k(\alpha, n) \bullet \sqrt{-1}^{n_1+\dots+n_s-s} \mathfrak{p}_{-n_1}(\alpha_1) \cdots \mathfrak{p}_{-n_s}(\alpha_s) |0\rangle \right) \\ &= \Psi_n \left(\sqrt{-1}^k O_k(\alpha, n) \right) \cdot \Psi_n \left(\sqrt{-1}^{n_1+\dots+n_s-s} \mathfrak{p}_{-n_1}(\alpha_1) \cdots \mathfrak{p}_{-n_s}(\alpha_s) |0\rangle \right). \end{aligned}$$

Since the classes $O_k(\alpha, n)$ with $k \geq 0, \alpha \in H^*(X)$ generate the ring $H_{\text{CR}}^*(X^{(n)})$, $\Psi_n : H_{\text{CR}}^*(X^{(n)}) \rightarrow H^*(X^{[n]})$ is a ring isomorphism. q.e.d.

REMARK 6.1. Using Heisenberg monomial classes, one checks that Ψ_n preserves the pairings on $H_{\text{CR}}^*(X^{(n)})$ and $H^*(X^{[n]})$.

In the next three subsections, we will verify (1.2) by proving Propositions 6.3, 6.9 and 6.12 used in the proof of Theorem 1.1. For simplicity, put $\langle w_1, w_2, w_3 \rangle_d = \langle w_1, w_2, w_3 \rangle_{0, d\beta_n}$. In addition, w_1, w_2 and w_3 will stand for Heisenberg monomial classes.

6.1. The case $\alpha = x$. We begin with a setup for the proof of (1.2) for arbitrary $\alpha, \beta \in H^*(X)$. To prove (1.2), it is equivalent to verify that

$$(6.3) \quad \langle [\tilde{\mathfrak{G}}_k(\alpha), \mathfrak{a}_{-1}(\beta)] w_1, w_2 \rangle = \frac{1}{k!} \langle \mathfrak{a}_{-1}^{\{k\}}(\alpha\beta) w_1, w_2 \rangle,$$

for $w_1 \in H_{\rho_n}^*(X^{[n-1]}) = H^*(X^{[n-1]})$ and $w_2 \in H_{\rho_n}^*(X^{[n]}) = H^*(X^{[n]})$. Put

$$(6.4) \quad \begin{aligned} & D_\beta^\alpha(w_1, w_2; q) \\ &:= \langle [\tilde{\mathfrak{G}}_k(\alpha; q), \mathfrak{a}_{-1}(\beta)] w_1, w_2 \rangle - \frac{1}{k!} \langle \mathfrak{a}_{-1}^{\{k\}}(\alpha\beta) w_1, w_2 \rangle, \end{aligned}$$

where k is omitted in $D_\beta^\alpha(w_1, w_2; q)$ as it will be clear from the context.

Lemma 6.2. *The difference $D_\beta^\alpha(w_1, w_2; q)$ is equal to*

$$\begin{aligned} & \sum_{\substack{0 \leq j \leq k, \lambda \vdash (j+1) \\ \ell(\lambda) = k-j+1}} \frac{(-1)^{|\lambda|-1}}{\lambda! \cdot |\lambda|!} \sum_{d \geq 1} \left(\langle \mathbf{1}_{-(n-j-1)} \mathfrak{a}_{-\lambda}(\tau_*\alpha) |0\rangle, \mathfrak{a}_{-1}(\beta) w_1, w_2 \rangle_d \right. \\ & \quad \left. - \langle \mathbf{1}_{-(n-j-2)} \mathfrak{a}_{-\lambda}(\tau_*\alpha) |0\rangle, w_1, \mathfrak{a}_{-1}(\beta)^\dagger w_2 \rangle_d \right) q^d \\ & + \sum_{\epsilon \in \{K_X, K_X^2\}} \sum_{\substack{\ell(\lambda) = k+1-|\epsilon|/2 \\ |\lambda| = -1}} \tilde{f}_{|\epsilon|}(\lambda) \cdot \langle \mathfrak{a}_\lambda(\tau_*(\epsilon\alpha\beta)) w_1, w_2 \rangle \end{aligned}$$

$$\begin{aligned}
 - \sum_{\substack{\epsilon \in \{K_X, K_X^2\} \\ 0 \leq j \leq k, \lambda^+(j+1) \\ \ell(\lambda) = k-j+1-|\epsilon|/2}} \tilde{g}_{|\epsilon|}(\lambda) \cdot \left(\langle \mathbf{1}_{-(n-j-1)} \mathbf{a}_{-\lambda}(\tau_*(\epsilon\alpha)) | 0 \rangle, \mathbf{a}_{-1}(\beta) w_1, w_2 \rangle \right. \\
 \left. - \langle \mathbf{1}_{-(n-j-2)} \mathbf{a}_{-\lambda}(\tau_*(\epsilon\alpha)) | 0 \rangle, w_1, \mathbf{a}_{-1}(\beta)^\dagger w_2 \rangle \right),
 \end{aligned}$$

where $\mathbf{a}_{-1}(\beta)^\dagger = -\mathbf{a}_1(\beta)$ is the adjoint operator of $\mathbf{a}_{-1}(\beta)$, and the functions $\tilde{f}_{|\epsilon|}(\lambda)$ and $\tilde{g}_{|\epsilon|}(\lambda)$ depend only on $k, |\epsilon|$ and λ .

Proof. By (4.5), $\langle [\tilde{\mathfrak{G}}_k(\alpha; q), \mathbf{a}_{-1}(\beta)] w_1, w_2 \rangle$ is equal to

$$\begin{aligned}
 (6.5) \quad & \langle \tilde{\mathfrak{G}}_k(\alpha; q)(\mathbf{a}_{-1}(\beta) w_1), w_2 \rangle - \langle \mathbf{a}_{-1}(\beta) \tilde{\mathfrak{G}}_k(\alpha; q)(w_1), w_2 \rangle \\
 & = \langle \tilde{\mathfrak{G}}_k(\alpha; q)(\mathbf{a}_{-1}(\beta) w_1), w_2 \rangle - \langle \tilde{\mathfrak{G}}_k(\alpha; q)(w_1), \mathbf{a}_{-1}(\beta)^\dagger w_2 \rangle \\
 & = \sum_{d \geq 0} \left(\langle \tilde{G}_k(\alpha, n), \mathbf{a}_{-1}(\beta) w_1, w_2 \rangle_d \right. \\
 & \quad \left. - \langle \tilde{G}_k(\alpha, n-1), w_1, \mathbf{a}_{-1}(\beta)^\dagger w_2 \rangle_d \right) q^d.
 \end{aligned}$$

If $d \geq 1$, then we see from (4.4) and Corollary 5.23 (ii) that

$$\begin{aligned}
 (6.6) \quad & \langle \tilde{G}_k(\alpha, n), \mathbf{a}_{-1}(\beta) w_1, w_2 \rangle_d \\
 & = \sum_{0 \leq j \leq k} \sum_{\substack{\lambda^+(j+1) \\ \ell(\lambda) = k-j+1}} \frac{(-1)^{|\lambda|-1}}{\lambda! \cdot |\lambda|!} \langle \mathbf{1}_{-(n-j-1)} \mathbf{a}_{-\lambda}(\tau_*\alpha) | 0 \rangle, \mathbf{a}_{-1}(\beta) w_1, w_2 \rangle_d.
 \end{aligned}$$

Similarly, if $d \geq 1$, then $\langle \tilde{G}_k(\alpha, n-1), w_1, \mathbf{a}_{-1}(\beta)^\dagger w_2 \rangle_d$ is equal to

$$(6.7) \quad \sum_{\substack{0 \leq j \leq k \\ \lambda^+(j+1) \\ \ell(\lambda) = k-j+1}} \frac{(-1)^{|\lambda|-1}}{\lambda! \cdot |\lambda|!} \langle \mathbf{1}_{-(n-j-2)} \mathbf{a}_{-\lambda}(\tau_*\alpha) | 0 \rangle, w_1, \mathbf{a}_{-1}(\beta)^\dagger w_2 \rangle_d.$$

Next, we study the two terms with $d = 0$ in (6.5). By (4.4) and Theorem 2.5, $\tilde{G}_k(\alpha, n)$ is equal to

$$G_k(\alpha, n) - \sum_{\substack{\epsilon \in \{K_X, K_X^2\} \\ 0 \leq j \leq k}} \sum_{\substack{\lambda^+(j+1) \\ \ell(\lambda) = k-j+1-|\epsilon|/2}} \tilde{g}_{|\epsilon|}(\lambda) \cdot \mathbf{1}_{-(n-j-1)} \mathbf{a}_{-\lambda}(\tau_*(\epsilon\alpha)) | 0 \rangle,$$

where $\tilde{g}_{|\epsilon|}(\lambda)$ depends only on $k, |\epsilon|$ and λ . By Theorem 2.1 (iii), Theorem 2.3 and Lemma 4.5,

$$\langle G_k(\alpha, n), \mathbf{a}_{-1}(\beta) w_1, w_2 \rangle - \langle G_k(\alpha, n-1), w_1, \mathbf{a}_{-1}(\beta)^\dagger w_2 \rangle$$

is equal to

$$\begin{aligned}
 & \langle G_k(\alpha, n) \cdot \mathbf{a}_{-1}(\beta) w_1, w_2 \rangle - \langle G_k(\alpha, n-1) \cdot w_1, \mathbf{a}_{-1}(\beta)^\dagger w_2 \rangle \\
 & = \langle \mathfrak{G}_k(\alpha) \mathbf{a}_{-1}(\beta) w_1, w_2 \rangle - \langle \mathbf{a}_{-1}(\beta) \mathfrak{G}_k(\alpha) w_1, w_2 \rangle \\
 & = \langle [\mathfrak{G}_k(\alpha), \mathbf{a}_{-1}(\beta)] w_1, w_2 \rangle = \frac{1}{k!} \langle \mathbf{a}_{-1}^{(k)}(\alpha) w_1, w_2 \rangle
 \end{aligned}$$

$$= \frac{1}{k!} \langle \mathbf{a}_{-1}^{\{k\}}(\alpha\beta)w_1, w_2 \rangle + \sum_{\substack{\epsilon \in \{K_X, K_X^2\} \\ \ell(\lambda) = k+1 - |\epsilon|/2 \\ |\lambda| = -1}} \tilde{f}_{|\epsilon|}(\lambda) \cdot \langle \mathbf{a}_\lambda(\tau_*(\epsilon\alpha\beta))w_1, w_2 \rangle.$$

Thus, $\langle \tilde{G}_k(\alpha, n), \mathbf{a}_{-1}(\beta)w_1, w_2 \rangle - \langle \tilde{G}_k(\alpha, n-1), w_1, \mathbf{a}_{-1}(\beta)^\dagger w_2 \rangle$ equals

$$(6.8) \quad \frac{1}{k!} \langle \mathbf{a}_{-1}^{\{k\}}(\alpha\beta)w_1, w_2 \rangle + \sum_{\substack{\epsilon \in \{K_X, K_X^2\} \\ \ell(\lambda) = k+1 - |\epsilon|/2 \\ |\lambda| = -1}} \tilde{f}_{|\epsilon|}(\lambda) \cdot \langle \mathbf{a}_\lambda(\tau_*(\epsilon\alpha\beta))w_1, w_2 \rangle \\ - \sum_{\substack{\epsilon \in \{K_X, K_X^2\} \\ 0 \leq j \leq k, \lambda^+(j+1) \\ \ell(\lambda) = k-j+1 - |\epsilon|/2}} \tilde{g}_{|\epsilon|}(\lambda) \cdot \left(\langle \mathbf{1}_{-(n-j-1)} \mathbf{a}_{-\lambda}(\tau_*(\epsilon\alpha))|0\rangle, \mathbf{a}_{-1}(\beta)w_1, w_2 \rangle \right) \\ - \langle \mathbf{1}_{-(n-j-2)} \mathbf{a}_{-\lambda}(\tau_*(\epsilon\alpha))|0\rangle, w_1, \mathbf{a}_{-1}(\beta)^\dagger w_2 \rangle.$$

Finally, our lemma follows from (6.5), (6.6), (6.7) and (6.8). q.e.d.

Now we deal with the simplest case when $\alpha = x$ and β is arbitrary.

Proposition 6.3. *If $\alpha = x$ is the cohomology class of a point, then (1.2) is true.*

Proof. By Corollary 5.23 (ii), every term in Lemma 6.2 is equal to zero. So $D_\beta^x(w_1, w_2; q) = 0$. Setting $q = -1$, we conclude immediately that (1.2) is true. q.e.d.

6.2. The case $|\alpha| = 2$. We begin with two lemmas about the structures of the intersections in $H^*(X^{[n]})$.

Lemma 6.4. *Let λ be a partition with $|\lambda| \leq n$. For $i = 1$ and 2, let*

$$(6.9) \quad w_i = \mathbf{a}_{-\lambda^{(i)}}(x) \mathbf{a}_{-\mu^{(i)}}(1_X) \mathbf{a}_{-n_{i,1}}(\alpha_{i,1}) \cdots \mathbf{a}_{-n_{i,u_i}}(\alpha_{i,u_i})|0\rangle,$$

where $|\alpha_{i,j}| = 2$ for all i and j . Then, $\langle \mathbf{a}_{-1}(1_X)^{n-|\lambda|} \mathbf{a}_{-\lambda}(x)|0\rangle, w_1, w_2 \rangle$ is equal to

$$(6.10) \quad \delta_{u_1, u_2} \cdot \sum_{\sigma \in \text{Perm}\{1, \dots, u_1\}} \prod_{i=1}^{u_1} \langle \alpha_{1,i}, \alpha_{2, \sigma(i)} \rangle \cdot p(\sigma),$$

where $p(\sigma)$ depends only on σ, n, λ and all the $\lambda^{(i)}, \mu^{(i)}, n_{i,j}$.

Proof. By Lemma 2.8 (i), $\mathbf{a}_{-1}(1_X)^{n-|\lambda|} \mathbf{a}_{-\lambda}(x)|0\rangle$ is a polynomial of the classes $G_k(x, n), k \geq 0$ whose coefficients are independent of X . In addition, the integers k involved depend only on λ . Note that

$$\langle G_{k_1}(x, n) \cdots G_{k_l}(x, n), w_1, w_2 \rangle = \langle \mathfrak{G}_{k_1}(x) \cdots \mathfrak{G}_{k_l}(x)w_1, w_2 \rangle$$

is equal to

$$\langle \mathbf{a}_{-\lambda^{(1)}}(x) \mathbf{a}_{-n_{1,1}}(\alpha_{1,1}) \cdots \mathbf{a}_{-n_{1,u_1}}(\alpha_{1,u_1}) \\ \cdot \mathfrak{G}_{k_1}(x) \cdots \mathfrak{G}_{k_l}(x) \mathbf{a}_{-\mu^{(1)}}(1_X)|0\rangle, w_2 \rangle.$$

So by Theorem 2.4 and Theorem 2.1 (i), $\langle G_{k_1}(x, n) \cdots G_{k_l}(x, n), w_1, w_2 \rangle$ equals

$$(6.11) \quad \delta_{u_1, u_2} \cdot \sum_{\sigma \in \text{Perm}\{1, \dots, u_1\}} \prod_{i=1}^{u_1} \langle \alpha_{1,i}, \alpha_{2,\sigma(i)} \rangle \cdot \tilde{p}(\sigma),$$

where $\tilde{p}(\sigma)$ depends only on $\sigma, n, k_1, \dots, k_l$ and all the $\lambda^{(i)}, \mu^{(i)}, n_{i,j}$. q.e.d.

Lemma 6.5. *Let $n_0 \geq 1$, $|\alpha| = 2$, and λ be a partition. Let w_1 and w_2 be given by (6.9). Then, $\langle \mathbf{1}_{-(n-|\lambda|-n_0)} \mathbf{a}_{-\lambda}(x) \mathbf{a}_{-n_0}(\alpha) | 0 \rangle, w_1, w_2 \rangle$ is equal to*

$$(6.12) \quad \begin{aligned} & \langle K_X, \alpha \rangle \cdot \delta_{u_1, u_2} \cdot \sum_{\sigma_1 \in \text{Perm}\{1, \dots, u_1\}} \prod_{i=1}^{u_1} \langle \alpha_{1,i}, \alpha_{2,\sigma_1(i)} \rangle \cdot p_1(\sigma_1) \\ & + \sum_{j=1}^{u_1} \langle \alpha, \alpha_{1,j} \rangle \cdot \delta_{u_1-1, u_2} \cdot \sum_{\sigma_2} \prod_{i \neq j} \langle \alpha_{1,i}, \alpha_{2,\sigma_2(i)} \rangle \cdot p_2(\sigma_2) \\ & + \sum_{j=1}^{u_2} \langle \alpha, \alpha_{2,j} \rangle \cdot \delta_{u_1, u_2-1} \cdot \sum_{\sigma_3} \prod_{i=1}^{u_1} \langle \alpha_{1,i}, \alpha_{2,\sigma_3(i)} \rangle \cdot p_3(\sigma_3), \end{aligned}$$

where σ_2 runs over all bijections $\{1, \dots, u_1\} - \{j\} \rightarrow \{1, \dots, u_2\}$, σ_3 runs over all bijections $\{1, \dots, u_1\} \rightarrow \{1, \dots, u_2\} - \{j\}$, and $p_1(\sigma_1)$ (respectively, $p_2(\sigma_2)$, $p_3(\sigma_3)$) depend only on σ_1 (respectively, σ_2 , σ_3), n, n_0, λ and all the $\lambda^{(i)}, \mu^{(i)}, n_{i,j}$.

Proof. By Lemma 2.8 (ii), $\mathbf{1}_{-(n-|\lambda|-n_0)} \mathbf{a}_{-\lambda}(x) \mathbf{a}_{-n_0}(\alpha) | 0 \rangle$ is equal to

$$\langle K_X, \alpha \rangle \cdot F_1(n) + \sum_i G_{k_i}(\alpha, n) \cdot F_{2,i}(n),$$

where $F_1(n)$ and $F_{2,i}(n)$ are polynomials of $G_k(x, n), k \geq 0$ whose coefficients are independent of n and α . Moreover, the integers k and k_i depend only on λ and n_0 . Thus,

$$(6.13) \quad \begin{aligned} & \langle \mathbf{1}_{-(n-|\lambda|-n_0)} \mathbf{a}_{-\lambda}(x) \mathbf{a}_{-n_0}(\alpha) | 0 \rangle, w_1, w_2 \rangle \\ & = \langle K_X, \alpha \rangle \cdot \langle F_1(n), w_1, w_2 \rangle + \sum_i \langle G_{k_i}(\alpha, n) \cdot F_{2,i}(n), w_1, w_2 \rangle. \end{aligned}$$

As in the proof of Lemma 6.4, $\langle F_1(n), w_1, w_2 \rangle$ is of the form

$$(6.14) \quad \delta_{u_1, u_2} \cdot \sum_{\sigma_1 \in \text{Perm}\{1, \dots, u_1\}} \prod_{i=1}^{u_1} \langle \alpha_{1,i}, \alpha_{2,\sigma_1(i)} \rangle \cdot \tilde{p}_{1,1}(\sigma_1),$$

where $\tilde{p}_{1,1}(\sigma_1)$ depends only on $\sigma_1, n, n_0, \lambda$ and all the $\lambda^{(i)}, \mu^{(i)}, n_{i,j}$. Also, $\langle G_{k_i}(\alpha, n) G_{s_1}(x, n) \cdots G_{s_l}(x, n), w_1, w_2 \rangle$ is equal to

$$\langle \mathfrak{G}_{s_1}(x) \cdots \mathfrak{G}_{s_l}(x) \mathfrak{G}_{k_i}(\alpha) w_1, w_2 \rangle.$$

By Theorem 2.4 and Lemma 2.6, $\langle G_{k_i}(\alpha, n)G_{s_1}(x, n) \cdots G_{s_l}(x, n), w_1, w_2 \rangle$ equals

$$\begin{aligned}
 (6.15) \quad & \langle K_X, \alpha \rangle \cdot \delta_{u_1, u_2} \cdot \sum_{\sigma_1 \in \text{Perm}\{1, \dots, u_1\}} \prod_{i=1}^{u_1} \langle \alpha_{1,i}, \alpha_{2, \sigma_1(i)} \rangle \cdot \tilde{p}_{1,2}(\sigma_1) \\
 & + \sum_{j=1}^{u_1} \langle \alpha, \alpha_{1,j} \rangle \cdot \delta_{u_1-1, u_2} \cdot \sum_{\sigma_2} \prod_{i \neq j} \langle \alpha_{1,i}, \alpha_{2, \sigma_2(i)} \rangle \cdot \tilde{p}_2(\sigma_2) \\
 & + \sum_{j=1}^{u_2} \langle \alpha, \alpha_{2,j} \rangle \cdot \delta_{u_1, u_2-1} \cdot \sum_{\sigma_3} \prod_{i=1}^{u_1} \langle \alpha_{1,i}, \alpha_{2, \sigma_3(i)} \rangle \cdot \tilde{p}_3(\sigma_3),
 \end{aligned}$$

where σ_2 runs over all the bijections $\{1, \dots, u_1\} - \{j\} \rightarrow \{1, \dots, u_2\}$, and σ_3 runs over all the bijections $\{1, \dots, u_1\} \rightarrow \{1, \dots, u_2\} - \{j\}$. Hence $\sum_i \langle G_{k_i}(\alpha, n) \cdot F_{2,i}(n), w_1, w_2 \rangle$ is of the form (6.15) as well. Combining with (6.13) and (6.14), we obtain (6.12). q.e.d.

Next, we introduce the notion of universal polynomials $P(K_X, S_1, S_2)$ in $\langle K_X, K_X \rangle$ of degree at most m and of type (u_1, u_2) , and prove a vanishing lemma.

Definition 6.6. Fix three integers $m, u_1, u_2 \geq 0$. Then a universal polynomial $P(K_X, S_1, S_2)$ in $\langle K_X, K_X \rangle$ of degree at most m and of type (u_1, u_2) is of the form

$$\begin{aligned}
 (6.16) \quad & \sum_{\substack{1 \leq j_1 < \dots < j_s \leq u_1 \\ 1 \leq l_1 < \dots < l_s \leq u_2}} \prod_{i \notin \{j_1, \dots, j_s\}} \langle K_X, \alpha_{1,i} \rangle \cdot \prod_{i \notin \{l_1, \dots, l_s\}} \langle K_X, \alpha_{2,i} \rangle \\
 & \cdot \sum_{\sigma \in \text{Perm}\{l_1, \dots, l_s\}} \prod_{i=1}^s \langle \alpha_{1, j_i}, \alpha_{2, \sigma(l_i)} \rangle \cdot p(j_1, \dots, j_s; l_1, \dots, l_s; \sigma),
 \end{aligned}$$

where $S_i = \{\alpha_{i,1}, \dots, \alpha_{i,u_i}\} \subset H^2(X)$, and $p(j_1, \dots, j_s; l_1, \dots, l_s; \sigma)$ is a polynomial in $\langle K_X, K_X \rangle$ whose degree is at most m and whose coefficients are independent of X and the classes $\alpha_{i,j}$.

Lemma 6.7. Fix $m, u_1, u_2 \geq 0$. Let $P(K_X, S_1, S_2)$ be a universal polynomial in $\langle K_X, K_X \rangle$ of degree at most m and of type (u_1, u_2) . Assume $P(K_X, S_1, S_2) = 0$ for every smooth projective toric surface X . Then $P(K_X, S_1, S_2) = 0$ for every smooth projective surface X .

Proof. Let $r \gg m + u_1 + u_2$, and let X_r be a smooth toric surface obtained from \mathbb{P}^2 as an r -fold blown-up. Let L_0 be a line in \mathbb{P}^2 , and let e_1, \dots, e_r be the exceptional divisors. Then, $K_{X_r} = -3L_0 + e_1 + \dots + e_r$. For fixed $j_1, \dots, j_s, l_1, \dots, l_s$ and σ , let

$$\begin{aligned}
 \{ \alpha_{1,i} \mid i \in \{1, \dots, u_1\} - \{j_1, \dots, j_s\} \} &= \{ -e_1, \dots, -e_{u_1-s} \}, \\
 \{ \alpha_{2,i} \mid i \in \{1, \dots, u_2\} - \{l_1, \dots, l_s\} \} &= \{ -e_{u_1-s+1}, \dots, -e_{u_1-s+u_2-s} \},
 \end{aligned}$$

and $\alpha_{1,j_i} = \alpha_{2,\sigma(l_i)} = e_{u_1-s+u_2-s+2i} - e_{u_1-s+u_2-s+2i-1}$ for $i = 1, \dots, s$. Then,

$$(6.17) \quad 0 = P(K_{X_r}, S_1, S_2) = (-2)^s \cdot p(j_1, \dots, j_s; l_1, \dots, l_s; \sigma),$$

by (6.16). It follows that $p(j_1, \dots, j_s; l_1, \dots, l_s; \sigma) = 0$ for all the surfaces X_r with $r \gg m + u_1 + u_2$. Since $p(j_1, \dots, j_s; l_1, \dots, l_s; \sigma)$ is a polynomial in $\langle K_{X_r}, K_{X_r} \rangle$ whose degree is at most m , we conclude that as polynomials, $p(j_1, \dots, j_s; l_1, \dots, l_s; \sigma) = 0$. Therefore, $P(K_X, S_1, S_2) = 0$ for every smooth projective surface X . q.e.d.

Our next lemma is about the structure of certain 3-pointed extremal Gromov–Witten invariants, and provides the motivation for Definition 6.6.

Lemma 6.8. *Let $d, n_0 \geq 1$ and $|\alpha| = 2$. Let w_1 and w_2 be given by (6.9). Then,*

$$\langle \mathbf{1}_{-(n-|\lambda|-n_0)} \mathbf{a}_{-\lambda}(x) \mathbf{a}_{-n_0}(\alpha) | 0 \rangle, w_1, w_2 \rangle_d = \langle K_X, \alpha \rangle \cdot P(K_X, S_1, S_2),$$

where $S_1 = \{\alpha_{1,1}, \dots, \alpha_{1,u_1}\}$, $S_2 = \{\alpha_{2,1}, \dots, \alpha_{2,u_2}\}$, and $P(K_X, S_1, S_2)$ is a universal polynomial in $\langle K_X, K_X \rangle$ of degree at most $(n - n_0)/2$ and of type (u_1, u_2) .

Proof. For simplicity, let $w_0 = \mathbf{1}_{-(n-|\lambda|-n_0)} \mathbf{a}_{-\lambda}(x) \mathbf{a}_{-n_0}(\alpha) | 0 \rangle$. Also, for $i = 1$ and 2 , let $\tilde{w}_i = \mathbf{a}_{-\mu^{(i)}}(1_X) \mathbf{a}_{-n_{i,1}}(\alpha_{i,1}) \cdots \mathbf{a}_{-n_{i,u_i}}(\alpha_{i,u_i}) | 0 \rangle$. We compute $\langle w_0, w_1, w_2 \rangle_d$ by using (1.3). Consider the following from (1.3):

$$(6.18) \quad \langle B_0, B_1, B_2 \rangle \cdot \left\langle \mathfrak{Z}_{m,d}, \pi_{m,1}^* \left(\frac{w_0}{B_0} \right) \cdot \pi_{m,2}^* \left(\frac{w_1}{B_1} \right) \cdot \pi_{m,3}^* \left(\frac{w_2}{B_2} \right) \right\rangle,$$

where $m \leq n$, $B_0, B_1, B_2 \in H^*(X^{[n-m]})$, $B_0 \subset w_0$, $B_1 \subset w_1$, and $B_2 \subset w_2$. By Theorem 1.3 (i) and Corollary 5.23 (i), such a term is nonzero only if $B_0 = \mathbf{a}_{-1}(1_X)^j \mathbf{a}_{-\lambda}(x) | 0 \rangle$ with $j \leq (n - |\lambda| - n_0)$, $B_1 = \mathbf{a}_{-\lambda^{(1)}}(x) \tilde{B}_1$ with $\tilde{B}_1 \subset \tilde{w}_1$, and $B_2 = \mathbf{a}_{-\lambda^{(2)}}(x) \tilde{B}_2$ with $\tilde{B}_2 \subset \tilde{w}_2$. In this situation, (6.18) can be rewritten as

$$(6.19) \quad \langle \mathbf{a}_{-1}(1_X)^j \mathbf{a}_{-\lambda}(x) | 0 \rangle, \mathbf{a}_{-\lambda^{(1)}}(x) \tilde{B}_1, \mathbf{a}_{-\lambda^{(2)}}(x) \tilde{B}_2 \rangle \cdot \left\langle \mathfrak{Z}_{m,d}, \pi_{m,1}^* \left(\frac{\mathbf{1}_{-(n-|\lambda|-n_0)} \mathbf{a}_{-n_0}(\alpha) | 0 \rangle}{\mathbf{a}_{-1}(1_X)^j | 0 \rangle} \right) \cdot \pi_{m,2}^* \left(\frac{\tilde{w}_1}{\tilde{B}_1} \right) \cdot \pi_{m,3}^* \left(\frac{\tilde{w}_2}{\tilde{B}_2} \right) \right\rangle.$$

Note that $\tilde{B}_1 = \mathbf{a}_{-\nu^{(1)}}(1_X) \mathbf{a}_{-n_{1,j_1}}(\alpha_{1,j_1}) \cdots \mathbf{a}_{-n_{1,j_s}}(\alpha_{1,j_s}) | 0 \rangle$ for some $1 \leq j_1 < \dots < j_s \leq u_1$ and some sub-partition $\nu^{(1)}$ of $\mu^{(1)}$ (i.e., every part of $\nu^{(1)}$ is a part of $\mu^{(1)}$). Similarly,

$$\tilde{B}_2 = \mathbf{a}_{-\nu^{(2)}}(1_X) \mathbf{a}_{-n_{2,l_1}}(\alpha_{2,l_1}) \cdots \mathbf{a}_{-n_{2,l_t}}(\alpha_{2,l_t}) | 0 \rangle,$$

for some $1 \leq l_1 < \dots < l_t \leq u_2$ and some sub-partition $\nu^{(2)}$ of $\mu^{(2)}$. By Lemma 6.4, the first factor in (6.19) equals

$$(6.20) \quad \delta_{s,t} \cdot \sum_{\sigma \in \text{Perm}\{l_1, \dots, l_s\}} \prod_{i=1}^s \langle \alpha_{1,j_i}, \alpha_{2,\sigma(l_i)} \rangle \cdot p_1(j_1, \dots, j_s; l_1, \dots, l_s; \sigma),$$

where p_1 is a number independent of the surface X and the classes $\alpha_{i,j}$. By Theorem 1.3, we see that the second factor in (6.19) is equal to

$$\begin{aligned} & \langle K_X, \alpha \rangle \cdot \prod_{i \notin \{j_1, \dots, j_s\}} \langle K_X, \alpha_{1,i} \rangle \\ & \cdot \prod_{i \notin \{l_1, \dots, l_s\}} \langle K_X, \alpha_{2,i} \rangle \cdot p_2(j_1, \dots, j_s; l_1, \dots, l_s; \sigma), \end{aligned}$$

where p_2 is a polynomial in $\langle K_X, K_X \rangle$ whose degree is at most $(m - n_0)/2 \leq (n - n_0)/2$, and whose coefficients are independent of the surface X and the classes $\alpha_{i,j}$. Combining this with (6.18), (6.19) and (6.20), we complete the proof of our lemma. q.e.d.

Proposition 6.9. *If $|\alpha| = 2$, then (1.2) is true.*

Proof. Recall that (1.2) is equivalent to (6.3), and the difference $D_\beta^\alpha(w_1, w_2; q)$ from (6.4) is computed by Lemma 6.2. Let w_1 and w_2 be given by (6.9). Let $u'_1 = \delta_{2,|\beta|} + u_1$ and $S_2 = \{\alpha_{2,1}, \dots, \alpha_{2,u_2}\}$. Let $S_1 = \{\alpha_{1,1}, \dots, \alpha_{1,u_1}\}$ if $|\beta| \neq 2$, and $S_1 = \{\beta, \alpha_{1,1}, \dots, \alpha_{1,u_1}\}$ if $|\beta| = 2$.

By Lemma 4.6 and Lemma 6.7, it suffices to prove that

$$(6.21) \quad D_\beta^\alpha(w_1, w_2; -1) = \langle K_X, \alpha \rangle \cdot P(K_X, S_1, S_2),$$

where $P(K_X, S_1, S_2)$ is a universal polynomial in $\langle K_X, K_X \rangle$ of degree at most $(n - 1)/2$ and of type (u'_1, u_2) . This follows if we can prove that

$$(6.22) \quad D_\beta^\alpha(w_1, w_2; q) = \langle K_X, \alpha \rangle \cdot \sum_{d \geq 0} P(K_X, S_1, S_2; d) q^d,$$

where every $P(K_X, S_1, S_2; d)$ is a universal polynomial in $\langle K_X, K_X \rangle$ of degree at most $(n - 1)/2$ and of type (u'_1, u_2) . We remark that d has been inserted into the notation $P(K_X, S_1, S_2; d)$ to emphasis its dependence on d .

In the following, we will show that the contribution of every term in Lemma 6.2 is of the form $P(K_X, S_1, S_2; d)$ for a suitable $d \geq 0$. Note that in $H^*(X^i)$,

$$\tau_{i*}(\alpha) = \alpha \otimes x \otimes \dots \otimes x + x \otimes \alpha \otimes x \otimes \dots \otimes x + \dots + x \otimes \dots \otimes x \otimes \alpha.$$

Thus, by Lemma 6.8, $\langle \mathbf{1}_{-(n-j-1)} \mathbf{a}_{-\lambda}(\tau_* \alpha) | 0 \rangle, \mathbf{a}_{-1}(\beta) w_1, w_2 \rangle_d$ is equal to

$$\langle K_X, \alpha \rangle \cdot P_1(K_X, S_1, S_2; d),$$

where $P_1(K_X, S_1, S_2; d)$ is a universal polynomial in $\langle K_X, K_X \rangle$ of degree at most $(n-1)/2$ and of type (u'_1, u_2) . Similarly, since $\mathbf{a}_{-1}(\beta)^\dagger w_2 = -\mathbf{a}_1(\beta)w_2$, we see from Theorem 2.1 (i) and Lemma 6.8 that

$$\langle \mathbf{1}_{-(n-j-2)} \mathbf{a}_{-\lambda}(\tau_* \alpha) | 0 \rangle, w_1, \mathbf{a}_{-1}(\beta)^\dagger w_2 \rangle_d = \langle K_X, \alpha \rangle \cdot P_2(K_X, S_1, S_2; d).$$

Next, we move to the term $\langle \mathbf{a}_\lambda(\tau_*(\epsilon\alpha\beta))w_1, w_2 \rangle$ in Lemma 6.2, where $\epsilon \in \{K_X, K_X^2\}$. Such a term is zero unless $\epsilon = K_X$ and $|\beta| = 0$. In this case, we may assume that $\beta = 1_X$. So let $\epsilon = K_X$ and $\beta = 1_X$. Then,

$$\begin{aligned} \langle \mathbf{a}_\lambda(\tau_*(\epsilon\alpha\beta))w_1, w_2 \rangle &= \langle K_X, \alpha \rangle \cdot \langle \mathbf{a}_\lambda(x)w_1, w_2 \rangle \\ &= \langle K_X, \alpha \rangle \cdot P_3(K_X, S_1, S_2; 0), \end{aligned}$$

by Theorem 2.1 (i), where $P_3(K_X, S_1, S_2; 0)$ is a universal polynomial in $\langle K_X, K_X \rangle$ of degree 0 (i.e., $\langle K_X, K_X \rangle$ does not appear) and of type (u'_1, u_2) .

Finally, $\tau_*(\epsilon\alpha)$ is zero unless $\epsilon = K_X$. Let $\epsilon = K_X$. By Lemma 6.4,

$$\begin{aligned} &\langle \mathbf{1}_{-(n-j-1)} \mathbf{a}_{-\lambda}(\tau_*(\epsilon\alpha)) | 0 \rangle, \mathbf{a}_{-1}(\beta)w_1, w_2 \rangle \\ &= \langle K_X, \alpha \rangle \cdot \langle \mathbf{1}_{-(n-j-1)} \mathbf{a}_{-\lambda}(x) | 0 \rangle, \mathbf{a}_{-1}(\beta)w_1, w_2 \rangle \\ &= \langle K_X, \alpha \rangle \cdot P_4(K_X, S_1, S_2; 0), \end{aligned}$$

where $P_4(K_X, S_1, S_2; 0)$ is a universal polynomial in $\langle K_X, K_X \rangle$ of degree 0 and of type (u'_1, u_2) . Similarly, since $\mathbf{a}_{-1}(\beta)^\dagger w_2 = -\mathbf{a}_1(\beta)w_2$,

$$\langle \mathbf{1}_{-(n-j-2)} \mathbf{a}_{-\lambda}(\tau_*(\epsilon\alpha)) | 0 \rangle, w_1, \mathbf{a}_{-1}(\beta)^\dagger w_2 \rangle$$

is equal to $\langle K_X, \alpha \rangle \cdot P_5(K_X, S_1, S_2; 0)$.

q.e.d.

6.3. The case $\alpha = 1_X$.

Lemma 6.10. *Let $d, n_0 \geq 1$. Let w_1 and w_2 be given by (6.9). Then,*

$$(6.23) \quad \langle \mathbf{1}_{-(n-|\lambda|-n_0)} \mathbf{a}_{-\lambda}(x) \mathbf{a}_{-n_0}(1_X) | 0 \rangle, w_1, w_2 \rangle_d = P(K_X, S_1, S_2),$$

where $S_1 = \{\alpha_{1,1}, \dots, \alpha_{1,u_1}\}$, $S_2 = \{\alpha_{2,1}, \dots, \alpha_{2,u_2}\}$, and $P(K_X, S_1, S_2)$ is a universal polynomial in $\langle K_X, K_X \rangle$ of degree at most $n/2$ and of type (u_1, u_2) .

Proof. This follows from the proof of Lemma 6.8 by replacing α by 1_X (and then by noticing that the factor $\langle K_X, \alpha \rangle$ there will not appear here). q.e.d.

Lemma 6.11. *Let $d \geq 1$ and $|\lambda| \leq n$. Let w_1 and w_2 be given by (6.9). Then,*

$$(6.24) \quad \langle \mathbf{1}_{-(n-|\lambda|)} \mathbf{a}_{-\lambda}(\tau_* 1_X) | 0 \rangle, w_1, w_2 \rangle_d = P(K_X, S_1, S_2),$$

where $S_1 = \{\alpha_{1,1}, \dots, \alpha_{1,u_1}\}$, $S_2 = \{\alpha_{2,1}, \dots, \alpha_{2,u_2}\}$, and $P(K_X, S_1, S_2)$ is a universal polynomial in $\langle K_X, K_X \rangle$ of degree at most $n/2$ and of type (u_1, u_2) .

Proof. For $i = 1$ and 2 , let

$$\tilde{w}_i = \mathbf{a}_{-\mu^{(i)}}(1_X)\mathbf{a}_{-n_{i,1}}(\alpha_{i,1}) \cdots \mathbf{a}_{-n_{i,u_i}}(\alpha_{i,u_i})|0\rangle.$$

Note that if the Künneth decomposition of $\tau_{2*}1_X \in H^*(X^2)$ is given by

$$\tau_{2*}1_X = x \otimes 1_X + 1_X \otimes x + \sum_j \gamma_{j,1} \otimes \gamma_{j,2},$$

where $|\gamma_{j,1}| = |\gamma_{j,2}| = 2$, then up to permutations of factors, a typical term in the Künneth decomposition of $\tau_{i*}1_X \in H^*(X^i)$ with $i \geq 3$ is either $x \otimes \cdots \otimes x \otimes 1_X$ or $x \otimes \cdots \otimes x \otimes \gamma_{j,1} \otimes \gamma_{j,2}$. In view of Lemma 6.10, it suffices to verify that

$$(6.25) \quad \sum_j \langle \mathbf{1}_{-\tilde{n}}\mathbf{a}_{-\tilde{\lambda}}(x)\mathbf{a}_{-n_1}(\gamma_{j,1})\mathbf{a}_{-n_2}(\gamma_{j,2})|0\rangle, w_1, w_2 \rangle_d = P_1(K_X, S_1, S_2),$$

where $\tilde{n} = n - |\tilde{\lambda}| - n_1 - n_2$, and $P_1(K_X, S_1, S_2)$ is a universal polynomial in $\langle K_X, K_X \rangle$ of degree at most $n/2$ and of type (u_1, u_2) . Let

$$w_0 = \mathbf{1}_{-\tilde{n}}\mathbf{a}_{-\tilde{\lambda}}(x)\mathbf{a}_{-n_1}(\gamma_{j,1})\mathbf{a}_{-n_2}(\gamma_{j,2})|0\rangle.$$

We see from (1.3) that to prove (6.25), it suffices to show that

$$(6.26) \quad \sum_j \langle B_0, B_1, B_2 \rangle \cdot \left\langle \mathfrak{Z}_{m,d}, \pi_{m,1}^* \left(\frac{w_0}{B_0} \right) \cdot \pi_{m,2}^* \left(\frac{w_1}{B_1} \right) \cdot \pi_{m,3}^* \left(\frac{w_2}{B_2} \right) \right\rangle$$

is equal to $P_2(K_X, S_1, S_2)$, where $m \leq n$, $B_0, B_1, B_2 \in H^*(X^{[n-m]})$, $B_0 \subset w_0$, $B_1 \subset w_1$, and $B_2 \subset w_2$. By Theorem 1.3 (i) and Corollary 5.23 (i), such a term is nonzero only if $B_1 = \mathbf{a}_{-\lambda^{(1)}}(x)\tilde{B}_1$ with $\tilde{B}_1 \subset \tilde{w}_1$, $B_2 = \mathbf{a}_{-\lambda^{(2)}}(x)\tilde{B}_2$ with $\tilde{B}_2 \subset \tilde{w}_2$, and $B_0 = \mathbf{a}_{-1}(1_X)^s\mathbf{a}_{-\tilde{\lambda}}(x)|0\rangle$ or $\mathbf{a}_{-1}(1_X)^s\mathbf{a}_{-\tilde{\lambda}}(x)\mathbf{a}_{-n_1}(\gamma_{j,1})|0\rangle$ or $\mathbf{a}_{-1}(1_X)^s\mathbf{a}_{-\tilde{\lambda}}(x)\mathbf{a}_{-n_2}(\gamma_{j,2})|0\rangle$ where $s \leq \tilde{n}$. In the following, we assume that (6.26) is nonzero. By symmetry, we need only to consider two cases for B_0 :

$$B_0 = \mathbf{a}_{-1}(1_X)^s\mathbf{a}_{-\tilde{\lambda}}(x)|0\rangle, \quad \text{or} \quad B_0 = \mathbf{a}_{-1}(1_X)^s\mathbf{a}_{-\tilde{\lambda}}(x)\mathbf{a}_{-n_1}(\gamma_{j,1})|0\rangle.$$

We begin with the case $B_0 = \mathbf{a}_{-1}(1_X)^s\mathbf{a}_{-\tilde{\lambda}}(x)|0\rangle$. Then (6.26) is

$$\sum_j \langle \mathbf{a}_{-1}(1_X)^s\mathbf{a}_{-\tilde{\lambda}}(x)|0\rangle, \mathbf{a}_{-\lambda^{(1)}}(x)\tilde{B}_1, \mathbf{a}_{-\lambda^{(2)}}(x)\tilde{B}_2 \rangle \cdot \left\langle \mathfrak{Z}_{m,d}, \pi_{m,1}^* \left(\frac{\mathbf{1}_{-\tilde{n}}\mathbf{a}_{-n_1}(\gamma_{j,1})\mathbf{a}_{-n_2}(\gamma_{j,2})|0\rangle}{\mathbf{a}_{-1}(1_X)^s|0\rangle} \right) \cdot \pi_{m,2}^* \left(\frac{\tilde{w}_1}{\tilde{B}_1} \right) \cdot \pi_{m,3}^* \left(\frac{\tilde{w}_2}{\tilde{B}_2} \right) \right\rangle.$$

Applying the same arguments as in the computation of (6.19), we conclude that the term (6.26) is equal to

$$\sum_j \langle K_X, \gamma_{j,1} \rangle \cdot \langle K_X, \gamma_{j,2} \rangle \cdot P_3(K_X, S_1, S_2),$$

where $P_3(K_X, S_1, S_2)$ is a universal polynomial in $\langle K_X, K_X \rangle$ of degree at most $(m - n_1 - n_2)/2$ and of type (u_1, u_2) . Note that for $\beta_1, \beta_2 \in H^2(X)$, we have

$$(6.27) \quad \sum_j \langle \beta_1, \gamma_{j,1} \rangle \cdot \langle \beta_2, \gamma_{j,2} \rangle = \langle \beta_1, \beta_2 \rangle.$$

Therefore, (6.26) is equal to $\langle K_X, K_X \rangle \cdot P_3(K_X, S_1, S_2)$ which is a universal polynomial in $\langle K_X, K_X \rangle$ of degree at most $m/2 \leq n/2$ and of type (u_1, u_2) .

Next, let $B_0 = \mathbf{a}_{-1}(1_X)^s \mathbf{a}_{-\tilde{\lambda}}(x) \mathbf{a}_{-n_1}(\gamma_{j,1})|0\rangle$. This time, (6.26) is

$$\begin{aligned} & \sum_j \langle \mathbf{a}_{-1}(1_X)^s \mathbf{a}_{-\tilde{\lambda}}(x) \mathbf{a}_{-n_1}(\gamma_{j,1})|0\rangle, \mathbf{a}_{-\lambda(1)}(x) \tilde{B}_1, \mathbf{a}_{-\lambda(2)}(x) \tilde{B}_2 \rangle \\ & \cdot \left\langle \mathfrak{Z}_{m,d}, \pi_{m,1}^* \left(\frac{\mathbf{1}_{-\tilde{n}} \mathbf{a}_{-n_2}(\gamma_{j,2})|0\rangle}{\mathbf{a}_{-1}(1_X)^s|0\rangle} \right) \cdot \pi_{m,2}^* \left(\frac{\tilde{w}_1}{\tilde{B}_1} \right) \cdot \pi_{m,3}^* \left(\frac{\tilde{w}_2}{\tilde{B}_2} \right) \right\rangle. \end{aligned}$$

Using Lemma 6.5, Theorem 1.3 and (6.27), we conclude that (6.26) is equal to $P_4(K_X, S_1, S_2)$ which is a universal polynomial in $\langle K_X, K_X \rangle$ of degree at most

$$(m - n_2)/2 + 1 \leq ((n - n_1) - n_2)/2 + 1 \leq n/2,$$

and of type (u_1, u_2) . This completes the proof of (6.24). q.e.d.

Proposition 6.12. *If $\alpha = 1_X$, then (1.2) is true.*

Proof. We adopt the same notations and approaches as in the proof of Proposition 6.9. By Lemma 4.6 and Lemma 6.7, it suffices to prove

$$(6.28) \quad D_\beta^{1_X}(w_1, w_2; -1) = P(K_X, S_1, S_2),$$

where $P(K_X, S_1, S_2)$ is a universal polynomial in $\langle K_X, K_X \rangle$ of degree at most $(n + 1)/2$ and of type (u'_1, u_2) . This follows if we can prove that

$$(6.29) \quad D_\beta^{1_X}(w_1, w_2; q) = \sum_{d \geq 0} P(K_X, S_1, S_2; d) q^d,$$

where $P(K_X, S_1, S_2; d)$ is a universal polynomial in $\langle K_X, K_X \rangle$ of degree at most $(n + 1)/2$ and of type (u'_1, u_2) . In the following, we will show that the contribution of every term in Lemma 6.2 is of the form $P(K_X, S_1, S_2; d)$ for a suitable $d \geq 0$.

First of all, when $d \geq 1$, we conclude from Lemma 6.11 that

$$\begin{aligned} & \langle \mathbf{1}_{-(n-j-1)} \mathbf{a}_{-\lambda}(\tau_* 1_X)|0\rangle, \mathbf{a}_{-1}(\beta) w_1, w_2 \rangle_d \\ & - \langle \mathbf{1}_{-(n-j-2)} \mathbf{a}_{-\lambda}(\tau_* 1_X)|0\rangle, w_1, \mathbf{a}_{-1}(\beta)^\dagger w_2 \rangle_d \end{aligned}$$

is equal to $P_1(K_X, S_1, S_2; d)$ which is a universal polynomial in $\langle K_X, K_X \rangle$ of degree at most $n/2$ and of type (u'_1, u_2) .

Next, consider

$$\langle \mathbf{a}_\lambda(\tau_*(\epsilon\alpha\beta)) w_1, w_2 \rangle = \langle \mathbf{a}_\lambda(\tau_*(\epsilon\beta)) w_1, w_2 \rangle,$$

from Lemma 6.2, where $\epsilon \in \{K_X, K_X^2\}$. It is zero unless $\epsilon = K_X^2$ and $\beta = 1_X$ (when $|\beta| = 0$, we let $\beta = 1_X$), or $\epsilon = K_X$ and $|\beta| = 2$, or $\epsilon = K_X$ and $\beta = 1_X$. If $\epsilon = K_X^2$ and $\beta = 1_X$, then

$$\begin{aligned} \langle \mathbf{a}_\lambda(\tau_*(\epsilon\beta))w_1, w_2 \rangle &= \langle K_X, K_X \rangle \cdot \langle \mathbf{a}_\lambda(x)w_1, w_2 \rangle \\ &= \langle K_X, K_X \rangle \cdot P_2(K_X, S_1, S_2; 0), \end{aligned}$$

by Theorem 2.1 (i), where $P_2(K_X, S_1, S_2; 0)$ is a universal polynomial in $\langle K_X, K_X \rangle$ of degree 0 and of type (u'_1, u_2) . If $\epsilon = K_X$ and $|\beta| = 2$, then

$$\begin{aligned} \langle \mathbf{a}_\lambda(\tau_*(\epsilon\beta))w_1, w_2 \rangle &= \langle K_X, \beta \rangle \cdot \langle \mathbf{a}_\lambda(x)w_1, w_2 \rangle \\ &= \langle K_X, \beta \rangle \cdot P_3(K_X, S_1, S_2; 0), \end{aligned}$$

which is a universal polynomial in $\langle K_X, K_X \rangle$ of degree 0 and of type (u'_1, u_2) . If $\epsilon = K_X$ and $\beta = 1_X$, then we obtain $\langle \mathbf{a}_\lambda(\tau_*(\epsilon\beta))w_1, w_2 \rangle = \langle \mathbf{a}_\lambda(\tau_*K_X)w_1, w_2 \rangle$ which again is a universal polynomial in $\langle K_X, K_X \rangle$ of degree 0 and of type (u'_1, u_2) .

Finally, let $\epsilon \in \{K_X, K_X^2\}$. We have $\tau_*(\epsilon\alpha) = \tau_*\epsilon$. Let I_ϵ be

$$\begin{aligned} &\langle \mathbf{1}_{-(n-j-1)}\mathbf{a}_{-\lambda}(\tau_*\epsilon)|0\rangle, \mathbf{a}_{-1}(\beta)w_1, w_2 \rangle \\ &- \langle \mathbf{1}_{-(n-j-2)}\mathbf{a}_{-\lambda}(\tau_*\epsilon)|0\rangle, w_1, \mathbf{a}_{-1}(\beta)^\dagger w_2 \rangle, \end{aligned}$$

from Lemma 6.2. When $\epsilon = K_X^2$, we see from Lemma 6.4 that

$$\begin{aligned} I_\epsilon &= \langle K_X, K_X \rangle \cdot \langle \mathbf{1}_{-(n-j-1)}\mathbf{a}_{-\lambda}(x)|0\rangle, \mathbf{a}_{-1}(\beta)w_1, w_2 \rangle \\ &\quad - \langle K_X, K_X \rangle \cdot \langle \mathbf{1}_{-(n-j-2)}\mathbf{a}_{-\lambda}(x)|0\rangle, w_1, \mathbf{a}_{-1}(\beta)^\dagger w_2 \rangle \\ &= \langle K_X, K_X \rangle \cdot P_4(K_X, S_1, S_2; 0), \end{aligned}$$

where $P_4(K_X, S_1, S_2; 0)$ is a universal polynomial in $\langle K_X, K_X \rangle$ of degree 0 and of type (u'_1, u_2) . When $\epsilon = K_X$, we see from Lemma 6.5 that $I_\epsilon = I_{K_X}$ is a universal polynomial in $\langle K_X, K_X \rangle$ of degree at most 1 and of type (u'_1, u_2) . q.e.d.

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