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PERIOD INTEGRALS AND THE RIEMANN–HILBERT CORRESPONDENCE

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Abstract

A tautological system, introduced in [**20**][**21**], arises as a regular holonomic system of partial differential equations that governs the period integrals of a family of complete intersections in a complex manifold X, equipped with a suitable Lie group action. A geometric formula for the holonomic rank of such a system was conjectured in [**5**], and was verified for the case of projective homogeneous space under an assumption. In this paper, we prove this conjecture in full generality. By means of the Riemann–Hilbert correspondence and Fourier transforms, we also generalize the rank formula to an arbitrary projective manifold with a group action.

1. Introduction

Let G be a connected algebraic group over a field k of characteristic zero. Let X be a projective G-variety and let $\mathcal{L}$ be a very ample G-linearized invertible sheaf over X which gives rise to a G-equivariant embedding

$$X \to \mathbb{P}(V),$$

where $V = \Gamma(X, \mathcal{L})^\vee$. Let $r = \dim V$. We assume that the action of G on X is locally effective, i.e. $\ker(G \to \mathrm{Aut}(X))$ is finite. Let $\mathbb{G}_m$ be the multiplicative group acting on V by homotheties. Let $\hat{G} = G \times \mathbb{G}_m$, whose Lie algebra is $\hat{\mathfrak{g}} = \mathfrak{g} \oplus ke$, where e acts on V by identity. We denote by $Z : \hat{G} \to \mathrm{GL}(V)$ the corresponding group representation, and $Z : \hat{\mathfrak{g}} \to \mathrm{End}(V)$ the corresponding Lie algebra representation. Note that under our assumptions, $Z : \hat{\mathfrak{g}} \to \mathrm{End}(V)$ is injective.

Let $\hat{\imath} : \hat{X} \subset V$ be the cone of X, defined by the ideal $I(\hat{X})$. Let $\beta : \hat{\mathfrak{g}} \to k$ be a Lie algebra homomorphism. Then a *tautological system* as defined in [**20**][**21**] is the cyclic D-module on $V^\vee$

$$\tau(G, X, \mathcal{L}, \beta) = D_{V^\vee}/D_{V^\vee}J(\hat{X}) + D_{V^\vee}(Z(\xi) + \beta(\xi), \xi \in \hat{\mathfrak{g}}),$$

where

$$J(\hat{X}) = \{\widehat{D} \mid D \in I(\hat{X})\}$$

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is the ideal of the commutative subalgebra $\mathbb{C}[\partial] \subset D_{V^\vee}$ obtained by the Fourier transform of $I(\hat{X})$ (see §A for the review of the Fourier transform and in particular (A.6) for the notation).

Given a basis $\{a_i\}$ of V, we have $Z(\xi) = \sum_{ij} \xi_{ij} a_i \partial_{a_j}$, where (ξ_{ij}) is the matrix representing ξ in the basis. Since the a_i are also linear coordinates on $V^\vee$, we can view $Z(\xi) \in \mathrm{Der}k[V^\vee] \subset D_{V^\vee}$. In particular, the identity operator $Z(e) \in \mathrm{End}V$ becomes the Euler vector field on $V^\vee$.

We recall the main motivation for studying tautological systems. Let X' be a compact complex manifold (not necessarily algebraic), such that the complete linear system of anticanonical divisors in X' is base point free. Let $\pi : \mathcal{Y} \to B := \Gamma(X', \omega_{X'}^{-1})_{sm}$ be the family of smooth CY hyperplane sections $Y_a \subset X'$, and let $\mathbb{H}^{top}$ be the Hodge bundle over B whose fiber at $a \in B$ is the line $\Gamma(Y_a, \omega_{Y_a}) \subset H^{n-1}(Y_a)$, where $n = \dim X'$. In [**21**], the period integrals of this family are constructed by giving a canonical trivialization of $\mathbb{H}^{top}$. Let $\Pi = \Pi(X')$ be the period sheaf of this family, i.e. the locally constant sheaf generated by the period integrals (Definition 1.1 [**21**].)

Let $V = \Gamma(X', \omega_{X'}^{-1})^\vee$, X be the image of the natural map $X' \to \mathbb{P}(V)$, and $\mathcal{L} = \mathcal{O}_X(1)$. Let G be a connected algebraic group acting on X.

Theorem 1.1. *The period integrals of the family $\pi : \mathcal{Y} \to B$ are solutions to*

$$\tau = \tau(G, X, \mathcal{L}, \beta_0),$$

where β_0 is the Lie algebra homomorphism which vanishes on $\mathfrak{g}$ and $\beta_0(e) = 1$.

This was proved in [**20**] for X' a partial flag variety, and in full generality in [**21**], where the result was also generalized to hyperplane sections of general type.

We note that when X' is a projective homogeneous manifold of a semisimple group G, in which case we have $X = X'$, τ is amenable to explicit descriptions. For example, one description says that the tautological system can be generated by the vector fields corresponding to the linear G action on $V^\vee$, and a twisted Euler vector field, together with a set of quadratic differential operators corresponding to the defining relations of X in $\mathbb{P}(V)$ under the Plücker embedding. The case where X is a Grassmannian has been worked out in detail [**20**]. Furthermore, when the middle primitive cohomology $H^n(X)_{prim} = 0$, it is also known that the system τ is complete, i.e. the solution sheaf coincides with the period sheaf [**5**].

We now return to a general tautological system τ. Applying an argument of [**16**], we find that if G acts on X by finitely many orbits, and if the character D-module on $\hat{G}$

$$\mathcal{L}_\beta := D_{\hat{G}} / D_{\hat{G}}(\xi + \beta(\xi),\ \xi \in \hat{\mathfrak{g}})$$

on $\hat{G}$ is regular singular, then τ is regular holonomic. See [**20**] Theorem 3.4(1). In this case, if $X = \sqcup_{l=1}^r X_l$ is the decomposition into G-orbits, then the singular locus of τ is contained in $\cup_{l=1}^r X_l^\vee$. Here $X_l^\vee \subset V^\vee$ is the conical variety whose projectivization $\mathbb{P}(X_l^\vee)$ is the projective dual to the Zariski closure of X_l in X. From now on we assume that G acts on X by finitely many orbits, and $\mathcal{L}_\beta$ is regular singular. Note that the latter assumption is always satisfied when G is reductive.

Let us now turn to the main problem studied in this paper. In the well-known applications of variation of Hodge structures in mirror symmetry, it is important to decide which solutions of our differential system come from period integrals. By Theorem 1.1, the period sheaf is a subsheaf of the solution sheaf of a tautological system. Thus an important problem is to decide when the two sheaves actually coincide, i.e. when τ is complete. If τ is not complete, how much larger is the solution sheaf relative to the period sheaf? From Hodge theory, we know that (see Proposition 6.3 [**5**]) the rank of the period sheaf is given by the dimension of the middle vanishing cohomology of the smooth hypersurface Y_a. Therefore, to answer those questions, it is clearly desirable to know precisely the holonomic rank of τ. For a brief overview of known results on these questions in a number of special cases, see Introduction in [**5**].

Conjecture 1.2. (Holonomic rank conjecture) Let X be an n-dimensional projective homogeneous space of a semisimple group G. The solution rank of $\tau = \tau(G, X, \omega_X^{-1}, \beta_0)$ at the point $a \in V^\vee$ is given by $\dim H_n(X - Y_a)$.

In [**5**], the following is proved:

Theorem 1.3. *Assume that the natural map*

$$\mathfrak{g} \otimes \Gamma(X, \omega_X^{-r}) \to \Gamma(X, T_X \otimes \omega_X^{-r})$$

is surjective for each $r \geq 0$. Then conjecture 1.2 *holds.*

In this paper, we prove this in full generality.

Theorem 1.4. *Conjecture* 1.2 *holds.*

This will be proved in §2. There are at least two immediate applications of this result. First we can now compute the solution rank for τ for *generic* $a \in V^\vee$.

Corollary 1.5. *The solution rank of τ at a smooth hyperplane section a is*

$$\dim H^n(X)_{prim} + \dim H^{n-1}(Y_a) - \dim H^{n+1}(X),$$

where the first term is the middle primitive cohomology of $X = G/P$ with $n = \dim X$.

The last two terms of the rank above can be computed readily in terms of the semisimple group G and the parabolic subgroup P by the Lefschetz hyperplane and the Riemann–Roch theorems. (See Example 2.4 [**20**].) A second application of Theorem 1.4 is to find certain exceptional points a in $V^\vee$ where the solution sheaf of τ degenerates "maximally".

Definition 1.1. A nonzero section $a \in V^\vee = \Gamma(X, \omega_X^{-1})$ is called a rank 1 point if the solution rank of τ at a is 1. In other words, $\mathrm{Hom}_{D_{V^\vee}}(\tau, \mathcal{O}_{V^\vee,a}) \simeq \mathbb{C}$.

Corollary 1.6. *Any projective homogeneous variety admits a rank 1 point.*

We will construct these rank 1 points in two explicit but different ways. The first, which works for $G = SL_l$, is a recursive procedure that produces such a rank 1 point by assembling rank 1 points from lower step flag varieties, starting from Grassmannians, and by repeatedly applying Theorem 1.4. The second way, which works for any semisimple group G, is by using a well-known stratification of the flag variety G/B to produce an open stratum in $X = G/P$ with a one-dimensional middle degree cohomology. The complement of this stratum is an anticanonical divisor, hence a rank 1 point of X by Theorem 1.4.

The geometric formula in Conjecture 1.2 appears to go well beyond the context of homogeneous spaces. Theorem 1.4 can be seen as a special case of the following much more general theorem. Consider a smooth projective G-variety X with $\mathcal{L} = \omega_X^{-1}$ very ample. Set $\tau = \tau(G, X, \mathcal{L}, \beta)$ and $V^\vee = \Gamma(X, \mathcal{L})$. We introduce some more notations. Let $\mathbb{L}^\vee$ be the total space of $\mathcal{L}$ and $\mathring{\mathbb{L}}^\vee$ be the complement of the zero section. Let

$$ev : V^\vee \times X \twoheadrightarrow \mathbb{L}^\vee, \quad (a, x) \mapsto a(x)$$

be the evaluation map, and $\mathbb{L}^\perp := \ker(ev)$. Finally let

$$\pi^\vee : U := V^\vee \times X - \mathbb{L}^\perp \to V^\vee.$$

Note that this is the complement of the universal family of hyperplane sections $\mathbb{L}^\perp \twoheadrightarrow V^\vee$, $(a, x) \mapsto a$. Put

$$D_{X,\beta} = (D_X \otimes k_\beta) \otimes_{U\hat{\mathfrak{g}}} k,$$

where k_β is the 1-dimensional $\hat{\mathfrak{g}}$-module given by the character β (see §A for the notations). Set $r := \dim(V)$. We now state the main result of this paper.

Theorem 1.7. *For $\beta(e) = 1$, there is a canonical isomorphism*

$$\tau \simeq H^0 \pi_+^\vee(\mathcal{O}_{V^\vee} \boxtimes D_{X,\beta})|_U.$$

Corollary 1.8. *Suppose G acts on X by finitely many orbits, and $k = \mathbb{C}$. Then the solution rank of τ at $a \in V^\vee$ is given by $\dim H_c^n(U_a,$ $\mathrm{Sol}(D_{X,\beta})|_{U_a})$, where $U_a = X - Y_a$.*

More generally, we have

Theorem 1.9. *For $\beta(e) \notin \mathbb{Z}_{\leq 0}$, and $\mathcal{L} = \omega_X^{-1}$, there is a canonical isomorphism*

$$\tau \simeq H^0 \pi_+^\vee ev^! (D_{\mathring{\mathbb{L}}^\vee, -\beta''})[1-r].$$

Here the Lie algebra homomorphism $\beta''\colon \hat{\mathfrak{g}} \to k$ is defined as

$$\beta''(\xi) = \mathrm{tr} Z(\xi) - \beta(\xi), \xi \in \mathfrak{g},\ \beta''(e) = 1 - \beta(e). \tag{1.1}$$

In addition to proving Conjecture 1.2 as a special case, Theorem 1.7 can also be used to derive the well-known formula for the solution rank of a GKZ system [**10**] at generic point a. But since Corollary 1.8 holds for arbitrary $a \in V^\vee$, it holds in particular for a corresponding the union of all T-invariant divisors in X (which is anticanonical). In this case, Theorem 1.7 implies that a is a rank 1 point – a result of [**12**] based on Gröbner basis theory but motivated by applications to mirror symmetry. Thus Theorem 1.7 and Corollary 1.8 interpolate a result of [**10**] and [**12**] by unifying the rank formula at generic point and at those exceptional rank 1 points, and at the same time, generalize them to an arbitrary G-variety.

Theorems 1.9 and 2.1 are clearly motivated by period integral problems in Calabi–Yau geometry. Equally important parallel problems for manifolds of general type have also been systematically studied [**9**][**21**]. In this paper, we develop the general type analogues of those two main theorems. Roughly speaking, ω_X^{-1} is replaced by an arbitrary very ample invertible sheaf $\mathcal{L}$ on X, and τ by a larger differential system defined on $\Gamma(X, \mathcal{L})^\vee \times \Gamma(X, \mathcal{L} \otimes \omega_X)^\vee$. This class of systems arise naturally from period integrals of general type hypersurfaces in X. The precise statements will be formulated and proved in §6 and §7.1.

As it is well-known, for the universal family of CY hypersurfaces in a given toric variety X, the GKZ system τ whose solutions include the period integrals of the family, is never complete. That is, its solution sheaf is always strictly larger than the period sheaf. While the period sheaf is by construction geometrical in nature, physicists have conjectured that the solution sheaf too has a purely geometrical origin. In fact, they have shown in some special cases that the solution sheaf of τ in this case are in fact integrals over topological chains with boundary on certain distinguished divisors in X [**2**]. It turns out that Theorems 2.1 and 6.2 can be used to prove precisely this statement in general. This will appear in our forthcoming paper [**14**], where we will also generalize the chain integral construction to a large class of G-varieties including toric varieties. We should also mention that some special cases of these chain integral solutions appear as an important ingredient, in the study of the arithmetic of Calabi–Yau varieties over finite fields [**8**].

We now outline the paper. In §2, we prove Theorem 1.7 and a number of its consequences, including Theorem 1.4. We also describe explicitly the "cycle-to-period" map $H_n(X-Y_a) \to \mathrm{Hom}_{D_{V^\vee}}(\tau, \mathcal{O}_{V^\vee,a})$ as a result of Theorem 1.7, and use it to answer a question recently communicated to us by S. Bloch. While §2 deals only with the case $\beta(e) = 1$, we remove this assumption in §§3–5. In §3, we study the !-fibers of τ, and describe some vanishing results at the special point $a = 0$. We describe the geometric set up in §4 for proving Theorem 1.9. The key step of the proof, involving an exact sequence for τ, is done in §5. In §6 and §7.1, we prove the general type analogues of Theorems 1.9 and 2.1. Finally, we apply our results to construct rank 1 points for partial flag varieties in the case $G = SL_l$ in §§9–10, and for general semisimple groups in §8. The appendix §A collects some standard facts on D-modules.

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2. CY hyperplane sections

We begin with Theorem 1.7: X is a G-variety with $\mathcal{L} = \omega_X^{-1}$ very ample, and $\beta(e) = 1$. This is in fact a special case of the more general Theorem 1.9 and therefore can be also obtained by the methods introduced in later sections. However, we decide to deal with this case first for several reasons. On the one hand, the proof given here is different from the later method and is more direct. On the other hand, the subcase when $\beta(\mathfrak{g}) = 0$, i.e. $\beta = \beta_0$, which is important to mirror symmetry, is already covered by Theorem 1.7.

Let $n = \dim X$. Let $U = V^\vee \times X - V(f)$, where $V(f) = \mathbb{L}^\perp$ is the universal hyperplane section, so that $U_a = X - V(f_a)$ where $V(f_a) = Y_a$, the zero locus of the section $f_a \equiv a \in V^\vee$. Let $\pi^\vee : U \to V^\vee$ denote the projection. The restriction of β to $\mathfrak{g}$ is still denoted by β when no confusion arises. Put $D_{X,\beta} = (D_X \otimes k_\beta) \otimes_{U\mathfrak{g}} k$. Note that if G acts on X by finitely many orbits, then $D_{X,\beta}$ is (G,β)-equivariant holonomic D-module on X (see Lemma A.5 and A.6), and therefore

$$\mathcal{N} := (\mathcal{O}_{V^\vee} \boxtimes D_{X,\beta})|_U$$

is a holonomic D-module on U.

Theorem 2.1. *Assume that $\beta(e) = 1$. Then there is a canonical isomorphism $\tau \simeq H^0\pi^\vee_+\mathcal{N}$.*

Corollary 2.2. *If $\beta(\mathfrak{g}) = 0$. There is a canonical surjective map*

$$\tau \to H^0\pi_+^\vee \mathcal{O}_U.$$

Proof. Note that there is always a surjective map $D_{X,0} = D_X/D_X\mathfrak{g} \to D_X/D_XT_X = \mathcal{O}_X$. The corollary follows from the fact that $\pi_+^\vee$ is right exact as $\pi^\vee : U \to V^\vee$ is affine. q.e.d.

We turn to the solution sheaf of τ via the Riemann–Hilbert correspondence. Assume G acts on X by finitely many orbits. Let us write $\mathcal{F} = \mathrm{Sol}(D_{X,\beta})$. This is a perverse sheaf on X.

Corollary 2.3. *Let $k = \mathbb{C}$ and $a \in V^\vee$. Then the solution rank of τ at a is given by* $\dim H_c^0(U_a, \mathcal{F}|_{U_a})$.

Proof. Denote $\mathrm{Sol}(\mathcal{N}) = \mathcal{G}$. According to the Riemann–Hilbert correspondence, $\mathrm{Sol}(\tau) = {}^pR^0\pi_!^\vee\mathcal{G}$, where ${}^pR^0\pi_!^\vee$ denotes the 0th perverse cohomology of $\pi_!^\vee$. Then the non-derived solution sheaf ${}^{cl}\,\mathrm{Sol}(\tau) = \mathrm{Hom}_{D_{V^\vee}}(\tau, \mathcal{O}_{V^\vee})$ is given by $H^{-r}({}^pR^0\pi_!^\vee\mathcal{G})$, the $(-r)$th (standard) sheaf cohomology of ${}^pR^0\pi_!^\vee\mathcal{G}$. However, as $R\pi_!^\vee\mathcal{G}$ lives in positive perverse degrees, $H^{-r}({}^pR^0\pi_!^\vee\mathcal{G}) = H^{-r}R\pi_!^\vee\mathcal{G} = R^{-r}\pi_!^\vee\mathcal{G}$. As $\mathcal{G} = \mathbb{C}[r] \boxtimes \mathcal{F}|_U$, the claim follows. q.e.d.

Remark 2.1. We will give more explicit descriptions of the perverse sheaf $\mathcal{F}$ in various situations later on. For example, in the case X is a homogeneous G-variety and $\beta(\mathfrak{g}) = 0$, then $\mathcal{F} = \mathbb{C}[n]$.

Now we prove Theorem 2.1. We will assume $\beta(\mathfrak{g}) = 0$ to simplify notations.

Let us write

$$\mathcal{R} := D_{V^\vee}/D_{V^\vee}J(\hat{X}), \tag{2.1}$$

which is a left $D_{V^\vee}$-module. Observe that for any $\xi \in \hat{\mathfrak{g}}$, $D_V I(\hat{X})Z^\vee(\xi) \subset D_V I(\hat{X})$, so $D_{V^\vee}J(\hat{X})Z(\xi) \subset D_{V^\vee}J(\hat{X})$. Therefore, $D_{V^\vee}J(\hat{X})$ can be regarded as a *right* $\hat{\mathfrak{g}}$-module, on which $\xi \in \hat{\mathfrak{g}}$ acts via the right multiplication by $Z(\xi)$. Accordingly, $\mathcal{R}$ is also a right $\hat{\mathfrak{g}}$-module. In addition, by definition we have

$$\tau = (\mathcal{R} \otimes k_\beta) \otimes_{\hat{\mathfrak{g}}} k, \tag{2.2}$$

where k_β is the 1-dimensional representation of $\hat{\mathfrak{g}}$ defined by β.

We now convert $\mathcal{R}$ to a left $\hat{\mathfrak{g}}$-module (cf. [**5**, §2]). Let $\{a_i\}$ be a basis of V and $\{a_i^*\}$ the dual basis. Observe that as $\mathcal{O}_{V^\vee}$-modules, one can write

$$\mathcal{R} \simeq \mathcal{O}_{V^\vee} \otimes S,$$

where

$$S = k[\partial_{a_i}]/J(\hat{X}) \simeq \mathcal{O}_V/I(\hat{X}) \tag{2.3}$$

is identified with the homogeneous coordinate ring of $\hat{X}$, and $\mathcal{O}_{V^\vee}$ acts on the first factor.[1] If we convert the right action of $\hat{\mathfrak{g}}$ on $\mathcal{R}$ described above to a left action α, then α will be the sum of the following two actions: the first is the action of $\hat{\mathfrak{g}}$ on the second factor through the dual representation $Z^\vee : \hat{\mathfrak{g}} \to \mathrm{End}V^\vee \to \mathrm{End}S$, which is denoted by α_1; to describe the second action α_2, observe that the natural multiplication map

$$(V \otimes V^\vee) \otimes (\mathcal{O}_{V^\vee} \otimes S) \to (\mathcal{O}_{V^\vee} \otimes S),$$

induces $V \otimes V^\vee \to \mathrm{End}\mathcal{R}$ and α_2 is via $Z^\vee : \hat{\mathfrak{g}} \to V \otimes V^\vee \to \mathrm{End}(\mathcal{R})$. Explicitly, if we write $a \otimes b \in \mathcal{O}_{V^\vee} \otimes S$, then

$$\alpha_1(\xi)(a \otimes b) = a \otimes Z^\vee(\xi)(b). \tag{2.4}$$

Let us write $Z^\vee(\xi) = -\sum_{ij} \xi_{ij} a_i \otimes a_j^*$. Then

$$\alpha_2(\xi)(a \otimes b) = -\sum_{ij} \xi_{ij} a a_i \otimes b a_j^*. \tag{2.5}$$

Let $f = \sum a_i \otimes a_i^* \in \mathcal{R}$, which can be regarded as the universal section of the line bundle $\mathcal{O}_{V^\vee} \boxtimes \mathcal{L}$ over $V^\vee \times X$. Recall that $U = V^\vee \times X - \mathbb{L}^\perp$. Then

$$\mathcal{O}_U = (\mathcal{O}_{V^\vee} \otimes S(\hat{X}))_{(f)}$$

is the homogeneous localization of $\mathcal{R}$ with respect to f, where the degree of $a \otimes b \in \mathcal{O}_{V^\vee} \otimes S(\hat{X})$ is the degree of b in the graded ring $S(\hat{X})$. As $\mathcal{L}^{-1} = \omega_X$, we can regard f^{-1} as a rational section of $\mathcal{O}_{V^\vee} \boxtimes \omega_X$, regular on U. Then $\mathcal{O}_U f^{-1}$ can be identified with the regular sections of $\mathcal{O}_{V^\vee} \boxtimes \omega_X$ over U. In other words,

$$\mathcal{O}_U f^{-1} \simeq \omega_{U/V^\vee} = (\mathcal{O}_{V^\vee} \boxtimes \omega_X)|_U. \tag{2.6}$$

Therefore, it is equipped with a $(D_{V^\vee} \boxtimes D_X^{op})|_U$ module structure (see [**6**, VI, §3] for the definition of right D_X module structure on ω_X). As $\mathfrak{g}$ maps to the vector fields on X, $\mathcal{O}_U f^{-1}$ is a $D_{V^\vee} \times \mathfrak{g}$-module. We will describe this structure more explicitly. First, we describe the $D_{V^\vee}$-module structure. Let θ be a vector field on $V^\vee$, and $\xi \in \mathfrak{g}$. It is enough to describe $\theta(f^{-1})$ and $(f^{-1})\xi$. Let us write $Z^\vee(\xi) = -\sum_{ij} \xi_{ij} a_i \otimes a_j^*$ as before.

Lemma 2.4. *We have*

$$\theta(f^{-1}) = -\frac{\sum_i \theta(a_i) \otimes a_i^*}{f^2} \in \mathcal{O}_U f^{-1},$$

and

$$(f^{-1})\xi = -\frac{\sum_{ij} \xi_{ij} a_i \otimes a_j^*}{f^2} \in \mathcal{O}_U f^{-1}.$$

[1] The $D_{V^\vee}$-module structure on $\mathcal{R}$ is given as follows: ∂_{a_i} acts on $\mathcal{O}_{V^\vee} \otimes S$ as $\partial_{a_i} \otimes 1 + 1 \otimes a_i^*$.

Proof. Let $v \in V^\vee$, regarded as a section of $\mathcal{L}$. Then v^{-1} is a rational section of ω_X, and $\omega = 1 \otimes v^{-1}$ is a rational section of $\mathcal{O}_{V^\vee} \boxtimes \omega_X$, obtained by pullback of a rational section of ω_X. Note that $g = (1 \otimes v)/f \in \mathcal{O}_U$, and we can write $f^{-1} = g(1 \otimes v^{-1})$. By definition, for a vector field θ on $V^\vee$, $\theta(\omega) = 0$, and for $\xi \in \mathfrak{g}$, $\omega\xi = -1 \otimes \mathrm{Lie}_\xi v^{-1}$, where $\mathrm{Lie}_\xi : \omega_X \to \omega_X$ is the Lie derivative (see [**6**, VI, §3] for the definition of right D-module structures on ω_X). Therefore

$$\theta(f^{-1}) = \theta(g)\omega, \quad (f^{-1})\xi = (g\xi)\omega - g(1 \otimes \mathrm{Lie}_\xi v^{-1}).$$

Note that

$$\begin{aligned}\theta(g) = \theta(\frac{1}{\sum a_i \otimes \frac{a_i^*}{v}}) &= -\frac{\sum \theta(a_i) \otimes \frac{a_i^*}{v}}{(\sum a_i \otimes \frac{a_i^*}{v})^2} \\ &= -g^2 \sum \theta(a_i) \otimes \frac{a_i^*}{v} = -g\frac{\sum \theta(a_i) \otimes a_i^*}{f}.\end{aligned}$$

Therefore, the first equation holds. On the other hand

$$(g)\xi = (\frac{1 \otimes v}{f})\xi = -\frac{1 \otimes Z^\vee(\xi)(v)}{f} - \frac{(1 \otimes v)\sum \xi_{ij} a_i \otimes a_j^*}{f^2}.$$

To prove the second, we need to understand $\mathrm{Lie}_\xi v^{-1}$. We consider a more general situation.

Let X be a Fano variety. Assume that $\mathcal{L} = \omega_X^{-1}$ is very ample, and $X \to \mathbb{P}(V)$ be the closed embedding where $V = \Gamma(X, \mathcal{L})^\vee$. Then $\mathfrak{g} = \Gamma(X, T_X)$ is a Lie algebra and $\mathcal{L}$ is naturally $\mathfrak{g}$-linearized. Therefore, $V^\vee = \Gamma(X, \mathcal{L})$ is a natural $\mathfrak{g}$-module with action $Z^\vee : \mathfrak{g} \to \mathrm{End}(V^\vee)$. As $Z^\vee(\xi) = -\sum_{ij} \xi_{ij} a_i \otimes a_j^*$, we have $Z^\vee(\xi)(v) = -\sum_{ij} \xi_{ij} a_i(v) a_j^*$ for $v \in V^\vee$. On the other hand, recall that $\mathfrak{g}$ acts on ω_X by Lie derivatives. Note that for $v \in V^\vee$, v^{-1} can be regarded as a rational section of ω_X.

Lemma 2.5. *Let $\xi \in \mathfrak{g}, 0 \neq v \in V^\vee$. Then*

$$\mathrm{Lie}_\xi v^{-1} = -\frac{Z^\vee(\xi)(v)}{v} v^{-1}.$$

Proof. Consider the 1-parameter subgroup $g_t = exp(t\xi)$. Then

$$\frac{d}{dt} g_t^*(v^{-1}) = -(v \circ g_t)^{-2} \frac{d}{dt}(v \circ g_t).$$

Now set $t = 0$. q.e.d.

Now Lemma 2.4 follows. q.e.d.

Note that explicitly, the $D_{V^\vee} \times \mathfrak{g}$-module structure on $\mathcal{O}_U f^{-1}$ can be described as follows. Let $\theta = \partial_{a_i}$ be a vector field on $V^\vee$ and $\xi = -\sum \xi_{ij} a_i \otimes a_j^* \in \mathfrak{g}$, $m = \frac{1}{f^{l+1}}(a \otimes b) \in \mathcal{O}_U f^{-1}$, where $a \in \mathcal{O}_{V^\vee}$ and $b \in S$ is homogeneous of degree k, then

$$\partial_{a_i}(m) = \frac{\partial_{a_i}(a) \otimes b}{f^{l+1}} + (-1)^{l+1}(l+1)\frac{a \otimes b a_i^*}{f^{l+2}},$$

$$(m)\xi = \frac{1}{f^{l+1}}(a \otimes Z^\vee(\xi)(b)) - \frac{l+1}{f^{l+2}}(\sum_{ij} \xi_{ij} a a_i \otimes b a_j^*).$$

We extend this to a $\hat{\mathfrak{g}}$-module by requiring that e acts by zero on $\mathcal{O}_U f^{-1}$.

Now, we have the following technical lemma. Recall that $\beta(e) = 1$.

Lemma 2.6. *The map $\phi : \mathcal{R} \otimes k_\beta \to \mathcal{O}_U f^{-1}$ given by*

$$\phi(a \otimes b) = \frac{(-1)^l l!}{f^{l+1}} a \otimes b$$

is a $D_{V^\vee} \times \hat{\mathfrak{g}}$-module homomorphism. In addition, it induces an isomorphism

$$\tau = (\mathcal{R} \otimes \beta) \otimes_{\hat{\mathfrak{g}}} k \simeq (\mathcal{O}_U f^{-1}) \otimes_{\hat{\mathfrak{g}}} k = (\mathcal{O}_U f^{-1}) \otimes_{\mathfrak{g}} k.$$

Proof. A direct calculation shows that ϕ is a $D_{V^\vee} \times \hat{\mathfrak{g}}$-module homomorphism. Namely, we know that ∂_{a_i} acts on $\mathcal{R}$ by $\partial_{a_i} \otimes 1 + 1 \otimes a_i^*$. Therefore,

$$\begin{aligned}\phi(\partial_{a_i}(a \otimes b)) &= \phi(\partial_{a_i}(a) \otimes b + a \otimes b a_i^*) \\ &= \frac{(-1)^l l!}{f^{l+1}}(\partial_{a_i}(a) \otimes b) + \frac{(-1)^{l+1}(l+1)!}{f^{l+2}}(a \otimes b a_i^*),\end{aligned}$$

which is the same as $\partial_{a_i}\phi(a \otimes b)$. The $\mathfrak{g}$-equivariance can be checked similarly.

Clearly ϕ is surjective, with the kernel spanned by $(l+1)a \otimes b + f(a \otimes b)$ for b homogeneous of degree l. But $(a \otimes b)\alpha(e) = (l+1)a \otimes b + f(a \otimes b)$. The lemma is proved. q.e.d.

To apply this lemma, recall the definition of $\pi_+^\vee$ for $\pi^\vee : U \to V^\vee$ a smooth morphism of algebraic varieties. As $\pi^\vee$ is an affine morphism,

$$\pi_+^\vee \mathcal{N} = \Omega^\bullet_{U/V^\vee} \otimes \mathcal{N}[\dim X].$$

In particular,

$$\begin{aligned}H^0 \pi_+^\vee \mathcal{N} = \operatorname{coker}((&\mathcal{O}_{V^\vee} \boxtimes \Omega_X^{\dim X - 1} \otimes D_X \otimes_{\mathfrak{g}} k)|_U \\ &\to (\mathcal{O}_{V^\vee} \boxtimes \omega_X \otimes D_X \otimes_{\mathfrak{g}} k)|_U).\end{aligned}$$

As $\operatorname{coker}(\Omega_X^{\dim X - 1} \otimes D_X \to \omega_X \otimes D_X) = \omega_X$ as right D_X-modules, $H^0\pi_+^\vee \mathcal{N}$ is exactly $(\mathcal{O}_{V^\vee} \boxtimes \omega_X)|_U \otimes_{\mathfrak{g}} k \simeq \tau$. This completes the proof of Theorem 2.1.

We continue to let X be a general smooth projective G-variety, and let $\beta(e) = 1$. We further assume that $k = \mathbb{C}$ and $\beta(\mathfrak{g}) = 0$, and consider some consequences of Theorem 2.1. By taking the solution sheaves on both sides in Corollary 2.2, we get an injective map

$$H_n(X - V(f_a)) \simeq \operatorname{Hom}(H^0\pi_+^\vee \mathcal{O}_U, \mathcal{O}_{V^\vee, a}) \to \operatorname{Hom}(\tau, \mathcal{O}_{V^\vee, a}), \tag{2.7}$$

where the first isomorphism follows from the same argument as in Corollary 2.3 and the Poincare duality. This gives an explicit lower bound for the solution rank of τ at any point a. For applications, we need to give a more geometric and explicit description of this map.

Note that we can interpret $1/f$ as a family (parametrized by $V^\vee$) of meromorphic top forms on X, whose fiber over $a \in V^\vee$ has poles along $V(f_a)$. We denote this family of top forms on X by Ω_a. These forms can also be given as follows.

Consider the principal $\mathbb{G}_m$-bundle $\pi^\vee : \mathring{\mathbb{L}}^\vee \to X$ (with right action). Then there is a natural one-to-one correspondence between sections of $\mathcal{L}$ and $\mathbb{G}_m$-equivariant morphism $f : \mathring{\mathbb{L}}^\vee \to k$, i.e. $f(m \cdot h^{-1}) = hf(m)$. We shall write f_a the function that represents the section a. Let $w = (w_1, .., w_n)$ be local coordinates on X, and z_w be the coordinate induced on the fibers of $\mathbb{L}^\vee$. Put $\omega = dz_w \wedge dw_1 \wedge \cdots \wedge dw_n$. Then it can be shown that ω defines a global non-vanishing form on $\mathbb{L}^\vee$. (See [**21**, Prop. 6.1].) Let x_0 be the vector field generated by $1 \in k = Lie(\mathbb{G}_m)$. Then $\Omega := i_{x_0}\omega$ is a G-invariant $\mathbb{G}_m$-horizontal form of degree $\dim X$ on $\mathring{\mathbb{L}}^\vee$. Moreover, since

$$\Omega_a := \frac{\Omega}{f_a}$$

is $G \times \mathbb{G}_m$-invariant, it defines a family of meromorphic top form on X with pole along $V(f_a)$ [**21**, Thm. 6.3]. Then the isomorphism in Lemma 2.6 sends the generator "1" of τ to Ω_a. Consider the "cycle-to-period" map defined in [**21**]

$$H_n(X - V(f_a)) \to \mathrm{Hom}(\tau, \mathcal{O}_{V^\vee,a}), \quad \gamma \mapsto \int_\gamma \Omega_a.$$

Corollary 2.7. *The cycle-to-period map* $H_n(X - V(f_a)) \to \mathrm{Hom}(\tau, \mathcal{O}_{V^\vee,a})$,

$$\gamma \mapsto \langle \gamma, \frac{\Omega}{f_a} \rangle = \int_\gamma \frac{\Omega}{f_a},$$

is injective.

The rest of the section will not be used in the sequel. We note that the argument of Corollary 2.3 has the following interesting topological consequence, which answers a question S. Bloch communicated to us. Let $X \subset \mathbb{P}^N$ be an n-dimensional smooth projective variety. Let $V(f) \to V^\vee = \Gamma(X, \mathcal{L})$ be the universal family of hyperplane sections of X.

Corollary 2.8. *Let $a \in V^\vee$. Then for a' close to a, the map $H_n(X - V(f_a)) \to H_n(X - V(f_{a'}))$ induced by parallel transport is injective.*

Proof. As argued in Corollary 2.3, $H_n(X - V(f_a))$ can be identified with the stalk of the classical solutions of some regular holonomic system on $V^\vee$. Since any analytic solution to a regular holonomic system at a extends to some neighborhood of a, the map between stalks of the classical solution sheaf of this regular holonomic system given by analytic continuation is injective. q.e.d.

Our result sheds new light on the well-studied toric case, i.e. the original GKZ A-hypergeometric differential equations. We assume that X is a toric variety, with the action of the torus $G = T$. Then $\hat{G} = T \times \mathbb{G}_m$. Then Theorem 2.1 takes a particular easy form in the following situation.

Corollary 2.9. [**12**] *If $\beta = \beta_0$, and X is smooth toric variety, $G = T$ is the algebraic torus of X, and Y_a is the anticanonical divisor of X given by the union of G-invariant toric divisors in X, then a is a rank 1 point.*

Proof. Note that in this case $D_{X,\beta}|_{X-Y_a} \simeq \mathcal{O}_{X-Y_a}$. Therefore, $\mathrm{Hom}(\tau, \mathcal{O}_{V^\vee,a}) \simeq H_n(T^n)$, which is one-dimensional. q.e.d.

3. !-fibers of τ

In the following three sections, we consider τ when $\beta(e)$ is not necessarily 1. Here we will give a formula of the !-fibers of τ at $a \in V^\vee$. For $a \in V^\vee$, let $i_a : \{a\} \to V^\vee$ be the inclusion and for simplicity, let us write

$$\tau_a^! = i_a^! \tau.$$

This is a complex of vector spaces and our goal is to give an expression of this complex.

By (2.2) we have

$$\tau_a^! = k_a \otimes^L_{\mathcal{O}_{V^\vee}} ((\mathcal{R} \otimes \beta) \otimes_{\hat{\mathfrak{g}}} k)[-\dim V],$$

where $k_a = \mathcal{O}_{V^\vee}/\mathfrak{m}_a$ is the residual field at a, and $\mathfrak{m}_a$ is the maximal ideal of $\mathcal{O}_{V^\vee}$ corresponding to a.

The advantage of this expression of $\tau_a^!$ is that we can first calculate $k_a \otimes^L_{\mathcal{O}_{V^\vee}} \mathcal{R}$ as a (complex of) right $\hat{\mathfrak{g}}$-modules, and then taking the Lie algebra coinvariants. Namely, we have the Koszul resolution of k_a, which gives the complex that calculates $\tau_a^!$

$$\tau_a^! = (\bigwedge V \otimes \mathcal{O}_{V^\vee} \otimes S) \otimes_{\hat{\mathfrak{g}}} (-\beta), \tag{3.1}$$

where $V \otimes \mathcal{O}_{V^\vee} \to \mathcal{O}_{V^\vee}$ is given by $v \otimes 1 \mapsto v - v(a)$. In general, this complex is difficult to compute. However, when $a = 0$, this is more tractable, as we shall see.

First, for a general point $a \in V^\vee$ we can express the degree r-term as

$$H^r \tau_a^! \simeq H_0(\hat{\mathfrak{g}}, S \otimes \beta), \tag{3.2}$$

where the action of $\hat{\mathfrak{g}}$ on S will be the sum of two actions (induced by the actions α_1 and α_2 of $\mathfrak{g}$ on $\mathcal{O}_{V^\vee} \otimes S$, as described in (2.4) and (2.5)). Concretely, the first action is via $Z^\vee : \hat{\mathfrak{g}} \to \mathrm{End} V^\vee \to \mathrm{End} S$, and the second is via the $\xi(b) = -\sum \xi_{ij} a_i(a) b a_j^*$ for $b \in S$. If $a \neq 0$, S is not a finite dimensional $\hat{\mathfrak{g}}$-module and this Lie algebra coinvariant is difficult to compute. On the other hand, if $a = 0$, the second action vanishes and S decomposes as finite dimensional representations of $\hat{\mathfrak{g}}$.

Lemma 3.1. *Assume that $\beta(e) \notin \mathbb{Z}_{\leq 0}$. Then $H^r \tau_0^! = 0$.*

Proof. The homothety $\mathbb{G}_m$ acts on S by nonnegative weights. Therefore, if $\beta(e) \notin \mathbb{Z}_{\leq 0}$, the coinvariant of $S \otimes \beta$ with respect to this $\mathbb{G}_m$ is zero. q.e.d.

From now on, we assume that $\beta(e) \notin \mathbb{Z}_{\leq 0}$.

Let us calculate $H^{r-1}\tau_0^!$. We have

$$\begin{array}{ccccc}
(V \wedge V) \otimes (\mathcal{R} \otimes \beta) & \xrightarrow{m_2} & V \otimes (\mathcal{R} \otimes \beta) & \xrightarrow{m_1} & \mathcal{R} \otimes \beta \\
\downarrow & & \downarrow & & \downarrow \\
(V \wedge V) \otimes (\mathcal{R} \otimes \beta) \otimes_{\hat{\mathfrak{g}}} k & \xrightarrow{d_2} & V \otimes (\mathcal{R} \otimes \beta) \otimes_{\hat{\mathfrak{g}}} k & \xrightarrow{d_1} & (\mathcal{R} \otimes \beta) \otimes_{\hat{\mathfrak{g}}} k.
\end{array}$$

Then

$$H^{r-1}\tau_0^! = m_1^{-1}((\mathcal{R} \otimes \beta)\hat{\mathfrak{g}})/(\operatorname{Im} m_2 + V \otimes (\mathcal{R} \otimes \beta)\hat{\mathfrak{g}})$$

As the Koszul complex is acyclic away from degree zero, we can rewrite the above as

$$H^{r-1}\tau_0^! = (\mathcal{R} \otimes \beta)\hat{\mathfrak{g}} \cap \operatorname{Im} m_1/(\operatorname{Im} m_1)\hat{\mathfrak{g}}.$$

Consider

$$\begin{aligned}
0 &\to (\mathcal{R} \otimes \beta)\hat{\mathfrak{g}} \cap \operatorname{Im} m_1/(\operatorname{Im} m_1)\hat{\mathfrak{g}} \to (\mathcal{R} \otimes \beta)\hat{\mathfrak{g}}/(\operatorname{Im} m_1)\hat{\mathfrak{g}} \\
&\to (\mathcal{R} \otimes \beta)\hat{\mathfrak{g}}/(\mathcal{R} \otimes \beta)\hat{\mathfrak{g}} \cap \operatorname{Im} m_1 \to 0.
\end{aligned}$$

Note that $0 = H^r\tau_0^!$ implies that $(\mathcal{R}\otimes\beta)\hat{\mathfrak{g}} + \operatorname{Im} m_1 = \mathcal{R}\otimes\beta$. Therefore,

$$(\mathcal{R} \otimes \beta)\hat{\mathfrak{g}}/(\mathcal{R} \otimes \beta)\hat{\mathfrak{g}} \cap \operatorname{Im} m_1 = \mathcal{R} \otimes \beta/\operatorname{Im} m_1.$$

We therefore can write

$$H^{r-1}\tau_0^! = \ker((\mathcal{R} \otimes \beta)\hat{\mathfrak{g}}/(\operatorname{Im} m_1)\hat{\mathfrak{g}} \to \mathcal{R} \otimes \beta/\operatorname{Im} m_1).$$

Therefore, there is a surjective map

$$H_1(\hat{\mathfrak{g}}, S \otimes \beta) \to H^{r-1}\tau_0^!,$$

where $\hat{\mathfrak{g}}$ acts on S via Z. (So S are direct sums of finite dimensional representations of $\hat{\mathfrak{g}}$.)

Lemma 3.2. *For $\beta(e) \notin \mathbb{Z}_{\leq 0}$, we have $H_1(\hat{\mathfrak{g}}, S \otimes \beta) = 0$. Therefore, $H^{r-1}\tau_0^! = 0$.*

Proof. Consider the $\hat{\mathfrak{g}}$ coinvariants functor as the composition of $\mathfrak{g}$ coinvariants functor, and the $\mathbb{C}$ coinvariants functor. The E_2 terms of the Grothendieck spectral sequence contributing to $H_1(\hat{\mathfrak{g}}, S \otimes \beta)$ are $H_1(\mathbb{C}, H_0(\mathfrak{g}, S \otimes \beta))$ and $H_0(\mathbb{C}, H_1(\mathfrak{g}, S \otimes \beta))$. As $S \otimes \beta$ breaks as direct sums according to weights as a $\mathfrak{g}$-module, and $\mathbb{C}$ acts on each given weight piece as the weight plus $\beta(e)$, it is clear that under the above assumption on $\beta(e)$, both $H_1(\mathbb{C}, H_0(\mathfrak{g}, S\otimes\beta))$ and $H_0(\mathbb{C}, H_1(\mathfrak{g}, S\otimes\beta))$ are zero. q.e.d.

4. The geometry

Let X be a smooth projective variety and $\mathcal{L}$ a very ample line bundle which gives $X \to \mathbb{P}(V)$, where $V^\vee = \Gamma(X, \mathcal{L})$. Let $\hat{\imath} : \hat{X} \to V$ be the closed embedding of the cone of X into V. Let $\mathbb{L}$ be the totally space of $\mathcal{L}^\vee$. Then

$$i_{\mathbb{L}} : \mathbb{L} \to X \times V$$

is a rank one subbundle of the trivial vector bundle over X with fiber V. The following diagram is commutative

$$\begin{array}{ccc} \mathbb{L} & \xrightarrow{i_{\mathbb{L}}} & X \times V \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \hat{X} & \xrightarrow{\hat{\imath}} & V \end{array}$$

and the left vertical arrow realizes $\mathbb{L}$ as the blow-up of $\hat{X}$ at the origin. We denote the open immersion

$$j_{\mathring{\mathbb{L}}} : \mathring{\mathbb{L}} = \mathbb{L} - X \to \mathbb{L},$$

where X is regarded as the zero section of $\mathbb{L}$.

Let $\mathbb{L}^\vee$ be the dual of $\mathbb{L}$, i.e., the total space of $\mathcal{L}$, and $j_{\mathring{\mathbb{L}}^\vee} : \mathring{\mathbb{L}}^\vee \to \mathbb{L}^\vee$ be the open subset away from the zero section. The dual of $i_{\mathbb{L}}$ is the evaluation map

$$ev : X \times V^\vee \to \mathbb{L}^\vee$$

which sends (x, a) to $a(x) \in \mathbb{L}^\vee$.

Let $i_{\mathbb{L}^\perp} : \mathbb{L}^\perp \to X \times V^\vee$ be the orthogonal complement of $\mathbb{L}$ in $X \times V^\vee$, i.e. the kernel of ev. The projection

$$\mathbb{L}^\perp \overset{i_{\mathbb{L}^\perp}}{\to} X \times V^\vee \overset{\pi^\vee}{\to} V^\vee$$

realizes $\mathbb{L}^\perp$ as the universal family of hyperplane sections of X. We still denote this projection by $\pi^\vee$. Let $j_U : U = X \times V^\vee - \mathbb{L}^\perp \to X \times V^\vee$ be the complement. For $a \in V^\vee$, the fiber U_a of $U \to V^\vee$ over a is $X - V(f_a)$, where f_a is the section of $\mathcal{L}$ given by a and $V(f_a)$ is its divisor. Note that the following diagram is Cartesian.

$$\begin{array}{ccc} U & \xrightarrow{j_U} & X \times V^\vee \\ {\scriptstyle ev}\downarrow & & \downarrow{\scriptstyle ev} \\ \mathring{\mathbb{L}}^\vee & \xrightarrow{j_{\mathring{\mathbb{L}}^\vee}} & \mathbb{L}^\vee. \end{array} \tag{4.1}$$

5. A formula for τ: CY case

We will complete the proof of Theorem 1.9 in this section.

Let $i_0 : \{0\} \to V$ be the inclusion of the origin, and $j_0 : \mathring{V} \to V$ be the open embedding of the complement. Let $\mathring{X} = \hat{X} - \{0\}$. The open inclusion $\mathring{X} \to \hat{X}$ is still denoted by j_0 and the closed inclusion $\mathring{X} \to \mathring{V}$ is denoted by $\mathring{i}$. By specializing (A.4), we have the following important sequence for $\hat{\tau} = \mathcal{F}our(\tau)$

$$0 \to i_{0,+}H^{-1}i_0^{+}\hat{\tau} \to H^0 j_{0,!}(\hat{\tau}|_{\mathring{V}}) \to \hat{\tau} \to i_{0,+}H^0 i_0^{+}\hat{\tau} \to 0. \tag{5.1}$$

First we make a simplification of this sequence.

Lemma 5.1. *For $\beta(e) \notin \mathbb{Z}_{\leq 0}$, $i_{0,+}H^0 i_0^{+}\hat{\tau} = 0$.*

Proof. Assume that $H^0 i_0^{+}\hat{\tau} = k^{\ell}$, so that $i_{0,+}H^0 i_0^{+}\hat{\tau} = \delta_0^{\ell}$. I.e. there is a surjective map of D-modules $\hat{\tau} \to \delta_0^{\ell}$ on V. Taking the Fourier transform, we therefore have a surjective map $\tau \to \mathcal{O}_{V^\vee}^{\ell}$. Taking the right exact functor $H^r i_0^!$, i.e., the rth cohomology of the !-fibers at $0 \in V^\vee$, we have a surjective map $H^r \tau_0^! \to k^{\ell}$. By Lemma 3.1, $\ell = 0$. q.e.d.

As a result, under our assumption

$$0 \to i_{0,+}H^{-1}i_0^{+}\hat{\tau} \to H^0 j_{0,!}(\hat{\tau}|_{\mathring{V}}) \to \hat{\tau} \to 0. \tag{5.2}$$

Let $d = \dim_k H^{-1}i_0^{+}\hat{\tau}$. Then $i_{0,+}H^{-1}i_0^{+}\hat{\tau} = \delta_0^{d}$. Taking the Fourier transform of this sequence, we therefore obtain

$$0 \to \mathcal{O}_{V^\vee}^{d} \to \mathcal{F}our(H^0 j_{0,!}(\hat{\tau}|_{\mathring{V}})) \to \tau \to 0. \tag{5.3}$$

We next understand $\mathcal{F}our(H^0 j_{0,!}(\hat{\tau}|_{\mathring{V}}))$. Clearly, $\hat{\tau}$ is set-theoretically supported on $\hat{X}$. Note that the Fourier transform of $Z(\xi) + \beta(\xi)$ is $Z^\vee(\xi) + \beta'(\xi)$, where

$$\beta'(\xi) = \mathrm{tr} Z(\xi) - \beta(\xi).$$

We have the following lemma.

Lemma 5.2. *For $\mathcal{L} = \omega_X^{-1}$, we have $\hat{\tau}|_{\mathring{X}} = D_{\mathring{X},\beta''}$, where $D_{\mathring{X},\beta''}$ is the D-module on $\mathring{X}$ as introduced in Lemma* A.6*, and β'' is defined as in equation* (1.1)*: i.e. $\beta''|_{\mathfrak{g}} := \beta'|_{\mathfrak{g}}$, $\beta''(e) := \beta'(e) - (r-1)$.*

Proof. Recall that $\hat{\tau}$ is defined as

$$\hat{\tau} = D_V / D_V I(\hat{X}) + D_V(Z^\vee(\xi) + \beta'(\xi), \xi \in \hat{\mathfrak{g}}).$$

Let $\mathcal{R}' := D_V / D_V I(\hat{X})$. Then similar to (2.2), $\hat{\tau} = (\mathcal{R}' \otimes k_{\beta'}) \otimes_{\hat{\mathfrak{g}}} k$. Consider the closed embedding $\mathring{X} \to \mathring{V}$. Then as explained in §A, there is a $D_{\mathring{V}} \times D_{\mathring{X}}$-bimodule, $D_{\mathring{V} \leftarrow \mathring{X}} = D_{\mathring{V}} / D_{\mathring{V}} \mathcal{I}(\mathring{X}) \otimes \omega_{\mathring{X}/\mathring{V}}$. The relative canonical sheaf $\omega_{\mathring{X}/\mathring{V}}$ is trivial as an $\mathcal{O}$ module, however, it carries a nontrivial right $\hat{\mathfrak{g}}$-action through its right $D_{\mathring{X}}$-module structure. It is clear that this right $\hat{\mathfrak{g}}$-action is such that $\mathfrak{g}$ acts trivially, and that e acts by the weight $-r+1$. This shows that $\mathring{i}_+ D_{\mathring{X},\beta''} = \hat{\tau}$, and therefore proves the lemma. q.e.d.

Note that $\mathring{\mathbb{L}} \simeq \mathring{X}$ and therefore $\hat{\tau}|_{\mathring{X}} \simeq D_{\mathring{X},\beta''}$ can be regarded as a D-module on $\mathring{\mathbb{L}}$, which is naturally $(\mathbb{G}_m, \beta''(e))$-equivariant. Then

$$H^0 j_{0,!}(\hat{\tau}|_{\mathring{V}}) = H^0 \pi_!(i_{\mathbb{L},!} j_{\mathring{\mathbb{L}},!} D_{\mathring{X},\beta''}). \tag{5.4}$$

According to Lemma A.14,

$$\mathcal{F}our(H^0\pi_!(i_{\mathbb{L},!} j_{\mathring{\mathbb{L}},!} D_{\mathring{X},\beta''})) = H^0 \pi_!^{\vee} \mathcal{F}our_X(i_{\mathbb{L},!} j_{\mathring{\mathbb{L}},!} D_{\mathring{X},\beta''}). \tag{5.5}$$

By (A.7),

$$\mathcal{F}our_X(i_{\mathbb{L},!} j_{\mathring{\mathbb{L}},!} D_{\mathring{X},\beta''}) = ev^! \mathcal{F}our_X(j_{\mathring{\mathbb{L}},!} D_{\mathring{X},\beta''})[1-r]. \tag{5.6}$$

Lemma 5.3. *There is a canonical isomorphism*

$$\mathcal{F}our_X(j_{\mathring{\mathbb{L}},!} D_{\mathring{X},\beta''}) \simeq j_{\mathring{\mathbb{L}}^\vee,+} D_{\mathring{\mathbb{L}}^\vee,-\beta''}.$$

Proof. Instead of the original formula, we can prove $\mathcal{F}our_X(j_{\mathring{\mathbb{L}}^\vee,+} D_{\mathring{\mathbb{L}}^\vee,-\beta''}) \simeq j_{\mathring{\mathbb{L}},!} D_{\mathring{X},\beta''}$. First note that the $+$-restriction of $\mathcal{F}our_X(j_{\mathring{\mathbb{L}}^\vee,+} D_{\mathring{\mathbb{L}}^\vee,-\beta''})$ along $X \to \mathbb{L}$ is zero. In fact, we have the following more general fact. We keep the notations $\mathbb{L}, \mathbb{L}^\vee, j_{\mathring{\mathbb{L}}}$ etc. We consider the $\mathbb{G}_m$-action on $\mathbb{L}$ by homotheties.

Lemma 5.4. *Let M be a $\mathbb{G}_m$-monodromic holonomic D-module on $\mathring{\mathbb{L}}$ (see §A for the terminology). Then the $+$-fiber of $\mathcal{F}our_X(j_{\mathring{\mathbb{L}},!} M)$ along $X \to \mathbb{L}^\vee$ is zero.*

Proof. One can check this pointwise on X and by the base change of Fourier transform (Lemma A.14), one can assume X is a point. Then it follows from Example A.12. q.e.d.

Therefore, by (A.4), it is enough to show $\mathcal{F}our_X(j_{\mathring{\mathbb{L}}^\vee,+} D_{\mathring{\mathbb{L}}^\vee,-\beta''})|_{\mathring{\mathbb{L}}} = D_{\mathring{\mathbb{L}},\beta''}$. By definition, we can write

$$\mathcal{F}our_X(j_{\mathring{\mathbb{L}}^\vee,+} D_{\mathring{\mathbb{L}}^\vee,-\beta''})|_{\mathring{\mathbb{L}}} = p_{\mathbb{L},+}(e^x \otimes D_{\mathring{\mathbb{L}}\times_X \mathring{\mathbb{L}}^\vee \to \mathring{\mathbb{L}}^\vee} \otimes_{\mathfrak{g}} (k_{\beta''}))[1].$$

In other words, $\mathcal{F}our_X(j_{\mathring{\mathbb{L}}^\vee,+} D_{\mathring{\mathbb{L}}^\vee,-\beta''})|_{\mathring{\mathbb{L}}}$ is calculated as the cokernel of the map

$$e^x \otimes D_{\mathring{\mathbb{L}}\times_X \mathring{\mathbb{L}}^\vee \to \mathring{\mathbb{L}}^\vee} \otimes_{\mathfrak{g}} (k_{\beta''}) \xrightarrow{\nabla} \Omega_{\mathring{\mathbb{L}}^\vee/X} \otimes_{\mathcal{O}_{\mathring{\mathbb{L}}^\vee}} e^x \otimes D_{\mathring{\mathbb{L}}\times_X \mathring{\mathbb{L}}^\vee \to \mathring{\mathbb{L}}^\vee} \otimes_{\mathfrak{g}} (k_{\beta''}).$$

Note that $M = D_{\mathring{\mathbb{L}}\times_X \mathring{\mathbb{L}}^\vee \to \mathring{\mathbb{L}}^\vee} \otimes_{\mathfrak{g}} (k_{\beta''})$ is a cyclic $D_{\mathring{\mathbb{L}}\times_X \mathring{\mathbb{L}}^\vee}$-module, with a canonical generator "1". For a local section $D \in D_{\mathring{\mathbb{L}}\times_X \mathring{\mathbb{L}}^\vee}$, let $[D] = D$"1" denote the corresponding local section of M. Note that $\Omega_{\mathring{\mathbb{L}}/X}$ and e^x are canonically trivialized as $\mathcal{O}$-modules. Indeed, by definition, the underlying $\mathcal{O}$-module of e^x is the structure sheaf. On the other hand, if locally on X, we choose s a section of $\mathcal{L}$, regarded as a coordinate function on $\mathbb{L}$, and t the dual coordinate on $\mathbb{L}^\vee$. Then the 1-form dt/t

is independent of the choice and defines the trivialization of $\Omega_{\mathring{\mathbb{L}}^\vee/X}$. Therefore, the underlying $\mathcal{O}$-modules of both terms in this complex are M. Then the D-module structure is given as follows: for $D \in D_{\mathring{\mathbb{L}}\times_X\mathring{\mathbb{L}}^\vee}$,

$$t\partial_t([D]) = [t\partial_t D] + [tsD], \quad s\partial_s([D]) = [s\partial_s D] + [tsD].$$

As a result, $\mathcal{F}our_X(j_{\mathring{\mathbb{L}}^\vee,+}D_{\mathring{\mathbb{L}}^\vee,\beta})|_{\mathring{\mathbb{L}}} = M/(t\partial_t + ts)M$.

More explicitly, as $\mathcal{O}$-modules, $e^x := m^! e^x$ is canonically trivialized, as was said above. Let $f \in \Gamma(\mathcal{O}_{V\times V^\vee})$, then unravelling the definitions, we have the following action of ∂_t on the element $f \otimes m^{-1}(1) \in e^x$:

$$\begin{aligned} \partial_t(f\otimes m^{-1}(1)) &= \partial_t f \otimes m^{-1}(1) + f\partial_t(st)\otimes m^{-1}(\partial_{st}1) \\ &= (\partial_t + s)f\otimes m^{-1}(1) \end{aligned} \tag{5.7}$$

Note that $\partial_{st} = 1$ in e^x. Therefore, for $1\otimes 1\otimes [D] \in e^x \otimes M$, we have

$$\begin{aligned} \nabla(1\otimes 1\otimes [D]) &= dt\otimes \partial_t(1\otimes [D]) = dt\otimes s\otimes [D] + dt\otimes 1\otimes [\partial_t D] \\ &= \frac{dt}{t}\otimes 1\otimes [tsD + t\partial_t D] \end{aligned} \tag{5.8}$$

So one gets the above identity for the Fourier transform.

To proceed, we first consider $N = D_{\mathring{\mathbb{L}}\times_X\mathring{\mathbb{L}}^\vee\to\mathring{\mathbb{L}}^\vee}/(t\partial_t+ts)D_{\mathring{\mathbb{L}}\times_X\mathring{\mathbb{L}}^\vee\to\mathring{\mathbb{L}}^\vee}$, which is a D-module on $\mathring{\mathbb{L}}$. We define a D-module homomorphism $D_{\mathring{\mathbb{L}}} \to N$, $\quad D \mapsto D$“1”, which we claim is an isomorphism. Indeed, we can assume that X is affine and the line bundle $\mathbb{L} \to X$ is trivial. Then it is a direct calculation.

Finally, note that both N and $D_{\mathring{\mathbb{L}}}$ are right $\hat{\mathfrak{g}}$-modules. The $\hat{\mathfrak{g}}$-module structure on N comes from $\hat{\mathfrak{g}} \to D_{\mathring{\mathbb{L}}^\vee}$ acting on $D_{\mathring{\mathbb{L}}\times_X\mathring{\mathbb{L}}^\vee\to\mathring{\mathbb{L}}^\vee}$ from the right, and the $\hat{\mathfrak{g}}$-module structure on $D_{\mathring{\mathbb{L}}}$ comes from $\hat{\mathfrak{g}} \to D_{\mathring{\mathbb{L}}}$ acting itself from the right. Under the above isomorphism, $D_{\mathring{\mathbb{L}}} \otimes k_{-\beta''} = N \otimes k_{\beta''}$. Now Lemma 5.3 follows. q.e.d.

Lemma 5.5. *Assume that $\beta(e) \notin \mathbb{Z}_{\le 0}$. We have $d = \dim H^{r-1}\tau_0^! = 0$.*

Proof. The second statement follows from Lemma 3.2. We need to establish the first equality. For simplicity, let us denote $\mathcal{N} := ev^!(D_{\mathring{\mathbb{L}}^\vee,-\beta})[1-r]$. This is a plain D-module on U.

Taking $i_0^!$ of (5.2), it is enough to show that

$$H^r i_0^! H^0 \pi_+^\vee \mathcal{N} = H^{r-1} i_0^! H^0 \pi_+^\vee \mathcal{N} = 0.$$

Consider the distinguished triangle

$$i_0^! H^{\le -1}\pi_+^\vee\mathcal{N} \to i_0^!\pi_+^\vee\mathcal{N} \to i_0^! H^0\pi_+^\vee\mathcal{N} \to .$$

The long exact sequence associated to this triangle is

$$\begin{aligned} H^{r-1}i_0^!\pi_+^\vee\mathcal{N} &\to H^{r-1}i_0^!H^0\pi_+^\vee\mathcal{N} \to H^r i_0^! H^{\le -1}\pi_+^\vee\mathcal{N} \\ &\to H^r i_0^!\pi_+^\vee\mathcal{N} \to H^r i_0^! H^0\pi_+^\vee\mathcal{N} \to 0. \end{aligned}$$

Note that U does not intersect with $X \times \{0\} \subset X \times V^\vee$. Therefore, $i_0^! \pi_+^\vee \mathcal{N} = 0$. This implies that $H^r i_0^! H^0 \pi_+^\vee \mathcal{N} = 0$, and $H^{r-1} i_0^! H^0 \pi_+^\vee \mathcal{N} = H^r i_0^! H^{\leq -1} \pi_+^\vee \mathcal{N}$. But $H^{\leq -1} \pi_+^\vee \mathcal{N}$ sits in cohomological degree ≤ -1 and $i_0^!$ has cohomological amplitude r, $H^r i_0^! H^{\leq -1} \pi_+^\vee \mathcal{N} = 0$. q.e.d.

We can now complete the proof of Theorem 1.9.

Proof. Combining (5.4)–(5.6) and Lemma 5.3, we can rewrite (5.2) as

$$0 \to \mathcal{O}_{V^\vee}^d \to H^0 \pi_+^\vee ev^! (D_{\mathring{\mathbb{L}}^\vee, -\beta''})[1-r] \to \tau \to 0. \tag{5.9}$$

Theorem 1.9 follows immediately from Lemma 5.5 and the sequence (5.9). q.e.d.

Remark 5.1. Note that explicitly,

$$ev^! (D_{\mathring{\mathbb{L}}^\vee, -\beta''})[1-r] = D_U / D_U T_{U/\mathring{\mathbb{L}}^\vee} + D_U(\xi - \beta''(\xi), \xi \in \hat{\mathfrak{g}}),$$

where $T_{U/\mathring{\mathbb{L}}^\vee}$ is the relative tangent sheaf, and $\hat{\mathfrak{g}}$ acts on $X \times V^\vee$ diagonally. In the special case $\beta(e) = 1$, it reduces to $\mathcal{N} = (\mathcal{O}_{V^\vee} \boxtimes D_{X,\beta})|_U$ as in Theorem 2.1.

6. General type hyperplane sections

Let X be a projective G-variety, $\mathcal{L}$ a very ample G-linearized invertible sheaf over X, and

$$X \to \mathbb{P}(V)$$

the associated G-equivariant embedding, where $V = \Gamma(X, \mathcal{L})^\vee$. Put $W = \Gamma(X, \mathcal{L} \otimes \omega_X)^\vee$, $r = \dim V$, and $s = \dim W$.

For simplicity, we assume that $\mathcal{L} \otimes \omega_X$ is base point free. (That $W \neq 0$ actually suffices for the following results.) Thus, we have a morphism $X \to \mathbb{P}(V) \times \mathbb{P}(W)$. Let

$$\mathcal{I} \subset k[V \times W]$$

be the bihomogeneous ideal defining the image, and let $\mathcal{I}_d$ be the subspace of $\mathcal{I}$ consisting of the $\deg_W = d$ elements.

Let $\mathbb{G}_m^2$ be the multiplicative group acting on $V \times W$ by homotheties. Let $\hat{G} = G \times \mathbb{G}_m^2$, whose Lie algebra is $\hat{\mathfrak{g}} = \mathfrak{g} \oplus k e^V \oplus k e^W$, where e^V, e^W act respectively on V, W by their identities. We denote by $Z^V : \hat{G} \to \mathrm{GL}(V)$ and $Z^W : \hat{G} \to \mathrm{GL}(W)$ the corresponding group representations, and $Z^V : \hat{\mathfrak{g}} \to \mathrm{End}(V)$, $Z^W : \hat{\mathfrak{g}} \to \mathrm{End}(W)$ the corresponding Lie algebra representations. In particular, $Z^V(e^V), Z^W(e^W)$ are the respective Euler vector fields on V, W. As before, we denote the Fourier transform by $\widehat{\ }: D_{V^\vee \times W^\vee} \to D_{V \times W}$.

Let $\hat{\imath} : \hat{X} \subset V$ be the cone of X, defined by the ideal $I(\hat{X})$. Let $\beta : \hat{\mathfrak{g}} \to k$ be a Lie algebra homomorphism. We extend the definition of a *tautological system* given in §1 as follows [**21**].

Definition 6.1. Let $\tau_{VW} = \tau_{VW}(G, X, \mathcal{L}, \beta)$ be the cyclic D-module on $V^\vee \times W^\vee$ given by

$$D_{V^\vee \times W^\vee}/D_{V^\vee \times W^\vee}\mathcal{J} + D_{V^\vee \times W^\vee}J^W + D_{V^\vee \times W^\vee}(Z^V(\xi) + Z^W(\xi) + \beta(\xi), \xi \in \hat{\mathfrak{g}}),$$

where

$$\mathcal{J} = \widehat{\mathcal{I}}, \qquad J^W = \widehat{\mathrm{Sym}\,^2 W^\vee}.$$

Note that when $\beta(e^W) = 0$, we have $\tau_{VW} = \tau \boxtimes \mathcal{O}_{W^\vee}$ where $\tau = \tau(G, X, \mathcal{L}, \beta)$ is as defined in §1. (See first paragraph of §7.)

To apply Definition 6.1 to the geometric problem at hand, we first prove

Proposition 6.1. *Let* Π *be the sheaf generated by the period integrals* Π_γ *of the universal family of hyperplane sections for* $\mathcal{L}$*. Then we have an injective map* $\Pi \to {}^{cl}\,\mathrm{Sol}(\tau_{VW})$*, with* $\beta(\mathfrak{g}) = 0$*,* $\beta(e^V) = 1$ *and* $\beta(e^W) = -1$*.*

Proof. By construction [**21**], $\Pi_\gamma = \int_\gamma \frac{f_b \Omega}{f_a}$, where $b \in W^\vee$, $a \in V^\vee$, and Ω is a G-invariant $\mathbb{G}_m^2$-horizontal form of degree $\dim X$ on $\mathring{\mathbb{L}}^\vee \oplus \mathring{\mathbb{K}}^\vee$ [**9**]. Here $\mathbb{L}^\vee, \mathbb{K}^\vee$ are the respective total spaces of $\mathcal{L}$, ω_X. Note that the $\frac{f_b \Omega}{f_a}$ define a family of meromorphic forms on X. Observe that $\mathcal{I}_0$ is nothing but $I(\hat{X}) \subset k[V]$, the defining ideal of X in $\mathbb{P}(V)$. Thus by [**21**, Theorem 8.9], Π_γ is annihilated by the Fourier transform $\widehat{\mathcal{I}_0}$. Since Π_γ is linear along the component $W^\vee$, the period integral is automatically annihilated by $\widehat{\mathcal{I}_d}$ for any $d > 1$. Likewise, $J^W \Pi_\gamma = 0$. As shown in [**21**, §8], for a given homogeneous function $p \in \mathcal{I}_1$, we have $\hat{p}\frac{f_b \Omega}{f_a} = (-1)^l p \frac{f_b \Omega}{f_a}$ where $l = \deg_V p$. But since $p \in \mathcal{I}$, this form vanishes when it is restricted to X. It follows that $\widehat{\mathcal{I}_1}\Pi_\gamma = 0$. Finally, by [**21**, Theorem 8.9] again

$$(Z^V(\xi) + Z^W(\xi) + \beta(\xi))\Pi_\gamma = 0, \qquad \xi \in \mathfrak{g} \oplus k e^V,$$

where $\beta(\mathfrak{g}) = 0$ and $\beta(e^V) = 1$. But since Π_γ is linear along $W^\vee$, this condition is equivalent to

$$(Z^V(\xi) + Z^W(\xi) + \beta(\xi))\Pi_\gamma = 0, \qquad \xi \in \hat{\mathfrak{g}} \equiv \mathfrak{g} \oplus k e^V \oplus k e^W,$$

with $\beta(e^W) = -1$. Therefore, the period integrals Π_γ are analytic solutions to the differential systems associated to $\tau_{VW}(G, X, \mathcal{L}, \beta)$, as desired. q.e.d.

Returning to the general case of $\tau_{VW} \equiv \tau_{VW}(G, X, \mathcal{L}, \beta)$, we proceed to analyze it in a way parallel to §2. We shall follow most of the notations introduced there, but with a general line bundle $\mathcal{L}$ now playing the role of ω_X^{-1} there. We will spell out the changes that need to be made to incorporate new structures associated to $W^\vee$ and $\hat{\mathfrak{g}} = \mathfrak{g} \oplus ke^V \oplus ke^W$. Put

$$\mathcal{N} := (\mathcal{O}_{V^\vee \times W^\vee} \boxtimes D_{X,\beta})|_{U \times W^\vee}.$$

The following is a generalization of Theorem 2.1.

Theorem 6.2. *Assume that $\beta(e^V) = 1$ and $\beta(e^W) = -1$. Then there is a canonical isomorphism $\tau_{VW} \simeq H^0(\pi^\vee \times \mathrm{id}_{W^\vee})_+\mathcal{N}$.*

For simplicity, we assume that $\beta(\mathfrak{g}) = 0$. The key step of the proof is finding an appropriate analogue of Lemma 2.6, which we now formulate. Put

$$\mathcal{R}^V = D_{V^\vee}/D_{V^\vee}\mathcal{I}_0, \qquad \mathcal{R}^W = D_{W^\vee}/D_{W^\vee}J^W.$$

Then $\mathcal{R}^V, \mathcal{R}^W$ have *right* $\hat{\mathfrak{g}}$-module structures by right multiplications via Z^V, Z^W respectively. Put

$$\mathcal{R}^{VW} = R/R\widehat{\mathcal{I}}_1, \quad R := \mathcal{R}^V \boxtimes \mathcal{R}^W.$$

For the same reason as $\widehat{\mathcal{I}}_1$ also affords an action of G, $R/R\widehat{\mathcal{I}}_1$ has a *right* $\hat{\mathfrak{g}}$-module structures by right multiplications. (Note that $\widehat{\mathcal{I}}_1$ a priori lives in a bigger space, whereas we used the same notation to denote its image in the quotient R.) By definition we have

$$\tau_{VW} = (\mathcal{R}^{VW} \otimes k_\beta) \otimes_{\hat{\mathfrak{g}}} k.$$

Fix bases $a_1, .., a_r$ of V, and $b_1, .., b_s$ of W respectively. As in §2, we have as $\mathcal{O}_{V^\vee}$-modules

$$\mathcal{R}^V \simeq \mathcal{O}_{V^\vee} \otimes S^V,$$

where $S^V = \mathcal{O}_V/\mathcal{I}_0$, which is $\mathbb{Z}_{\geq 0}$-graded. The $D_{V^\vee}$-structure on $\mathcal{R}^V$ is then given by $\partial_{a_i} \mapsto \partial_{a_i} \otimes 1 + 1 \otimes a_i^*$. We can also convert the right $\hat{\mathfrak{g}}$-action on $\mathcal{R}^V$ to a left action α as before. Similarly, we have as $\mathcal{O}_{W^\vee}$-modules

$$\mathcal{R}^W \simeq \mathcal{O}_{W^\vee} \otimes S^W,$$

where $S^W = \mathcal{O}_W/\mathcal{O}_W \mathrm{Sym}\,^2 W^\vee$, which is $\mathbb{Z}/2\mathbb{Z}$-graded. The $D_{W^\vee}$-structure on $\mathcal{R}^W$ is then given by $\partial_{b_i} \mapsto \partial_{b_i} \otimes 1 + 1 \otimes b_i^*$. Put

$$f^V = \sum a_i \otimes a_i^*, \qquad f^W = \sum b_i \otimes b_i^*,$$

which are the universal sections of the line bundles $\mathcal{O}_{V^\vee} \boxtimes \mathcal{L}$ and $\mathcal{O}_{W^\vee} \boxtimes (\mathcal{L} \otimes \omega_X)$ respectively. By pulling them back to $V^\vee \times W^\vee \times X$, we shall view f^V, f^W as sections of $\mathcal{O}_{V^\vee \times W^\vee} \boxtimes \mathcal{L}$ and $\mathcal{O}_{V^\vee \times W^\vee} \boxtimes (\mathcal{L} \otimes \omega_X)$ respectively.

Recall that $U := V^\vee \times X - \mathbb{L}^\perp$ where $\mathbb{L}^\vee$ is the total space of $\mathcal{L}$, and let $\mathring{\mathbb{L}}^\vee$ be the complement of the zero section. As in the proof of Proposition 6.1, for given $b \in W^\vee$, $a \in V^\vee$, we can regard $f_b^W \Omega / f_a^V$ as a meromorphic form on X with pole along $V(f_a)$. As in §2, we have

$$\omega_{U \times W^\vee / V^\vee \times W^\vee} = (\mathcal{O}_{V^\vee \times W^\vee} \boxtimes \omega_X)|_{U \times W^\vee}$$

as $D_{V^\vee \times W^\vee} \times \hat{\mathfrak{g}}$-modules.

Lemma 6.3. *Define* $\phi : (\mathcal{R}^V \boxtimes \mathcal{R}^W) \otimes k_{\beta_0} \to (\mathcal{O}_{V^\vee \times W^\vee} \boxtimes \omega_X)|_{U \times W^\vee}$ *by*

$$(a \otimes p) \boxtimes (b \otimes q) \mapsto \frac{(-1)^l l! (f^W)^{\frac{1+(-1)^m}{2}}}{(f^V)^{l+1}} (ab) \boxtimes (pq\Omega),$$

where $l = \deg p$ *and* $m = \deg q \in \mathbb{Z}/2\mathbb{Z}$*. Then* ϕ *is a* $D_{V^\vee \times W^\vee} \times \hat{\mathfrak{g}}$*-module homomorphism, and it induces an isomorphisms of* $D_{V^\vee \times W^\vee}$*-modules*

$$\tau_{VW} \to \omega_{U \times W^\vee / V^\vee \times W^\vee} \otimes_{\hat{\mathfrak{g}}} k.$$

Proof. It is a verbatim argument as in Lemma 2.6 and eqn. (2.6). q.e.d.

To complete our proof of Theorem 6.2, we observe that the proof of Theorem 2.1 carries over with just two changes: $V^\vee \times W^\vee$ and $\pi^\vee \times \mathrm{id}_{W^\vee}$ to replace $V^\vee$ and $\pi^\vee$ respectively.

As a consequence, for $\beta(e^V) = 1$ and $\beta(e^W) = -1$ we have

Corollary 6.4. *Let* $k = \mathbb{C}$*, and* $(a, b) \in V^\vee \times W^\vee$*. Then the solution rank of* τ_{VW} *at* (a, b) *is given by* $\dim H^0_c(U_a, \mathcal{F}|_{U_a})$*, where* $\mathcal{F} = \mathrm{Sol}(D_{X,\beta})$*.*

Proof. This follows from a verbatim argument as in Corollary 2.3. q.e.d.

7. A formula for τ: general type case

We now return to the tautological system $\tau_{VW} = \tau_{VW}(G, X, \mathcal{L}, \beta)$ introduced in Definition 6.1. We continue to use the notations introduced in §6. Let $\beta : \hat{\mathfrak{g}} \equiv \mathfrak{g} \oplus ke^V \oplus ke^W \to k$ be a Lie algebra homomorphism. If $\beta(e^W) \neq 0, -1$, then $\hat{\tau}_{VW}$ are zero. To see this, let $b_1^*, ..., b_s^*$ denote a dual basis of $W^\vee$. Then in $\hat{\tau}_{VW}$, we have $b_i^* b_j^* \equiv 0$, hence

$$0 \equiv b_j^* (\sum_i -\partial_{b_i^*} b_i^* + \beta(e^W)) = (1 + \beta(e^W)) b_j^*$$

implying that $b_j^* \equiv 0$ for all j. But this implies that $\beta(e^W) \equiv 0$, hence $\hat{\tau}_{VW} \equiv 0$. Now consider the case $\beta(e^W) = 0$. Then $b_j^* \equiv 0$ in $\hat{\tau}_{VW}$ as before. It follows that $\hat{\tau}_{VW}$ is supported on $V \times \{0\}$, and its inverse Fourier transform becomes

$$\begin{aligned} \tau_{VW} = D_{V^\vee \times W^\vee} / D_{V^\vee \times W^\vee} J^V(\hat{X}) + D_{V^\vee \times W^\vee} (Z^V(\xi) \\ + \beta(\xi), \xi \in \hat{\mathfrak{g}}) + D_{V^\vee \times W^\vee} \widehat{W^\vee}, \end{aligned}$$

where $J^V(\hat{X})$ is the Fourier transform of the ideal of X in $\mathbb{P}(V)$. This yields $\tau_{VW} = \tau \boxtimes \mathcal{O}_{W^\vee}$, hence reducing τ_{VW} to a the special case of $\tau = \tau(G, X, \mathcal{L}, \beta)$ introduced in §1.

From now on, we assume that

$$\beta(e^W) = -1, \quad \beta(e^V) \notin \mathbb{Z}_{\leq 0}.$$

In this section, we prove the following general type analogue of Theorem 1.9.

First we define a Lie algebra homomorphism β'': $\mathfrak{g} \oplus ke^V \to k$ as

(7.1) $\beta''(\xi) = \mathrm{tr}_W Z^W(\xi) + \mathrm{tr}_V Z^V(\xi) - \beta(\xi), \xi \in \mathfrak{g},\ \beta''(e^V) = 1 - \beta(e^V).$

Theorem 7.1. *For $\beta(e^V) \notin \mathbb{Z}_{\leq 0}$ and $\beta(e^W) = -1$, there is a canonical isomorphism*

$$\tau_{VW} \simeq p^! H^0 \pi_+^\vee ev^! (D_{\mathring{\mathbb{L}}^\vee, -\beta''})[1 - r - s],$$

where $p : V^\vee \times W^\vee \to V^\vee$ is the projection, $r = \dim V^\vee$, $s = \dim W^\vee$, and β'' is defined as in equation (1.1).

As in §6, the Fourier transform $\hat{\tau}_{VW}$ is a D-module on $V \times W$. Consider the open embedding

$$j : V \times \mathring{W} \hookrightarrow V \times W$$

and closed embedding

$$i : V \to V \times W.$$

Then we have the following distinguished triangle (A.2)

$$i_+ i^! \hat{\tau}_{VW} \to \hat{\tau}_{VW} \to j_+ j^! \hat{\tau}_{VW} \to . \tag{7.2}$$

Since $b_i^* b_j^* \equiv 0$ in $\hat{\tau}_{VW}$ for any i, j, it follows that on $\mathring{W}$, we have $b_i^* \equiv 0$ for any i. Hence $\beta(e^W) \equiv \sum_{i=1}^s \partial_{b_i^*} b_i^* \equiv 0$ in $\hat{\tau}_{VW}$. But since $\beta(e^W) = -1$, we have $j^! \hat{\tau}_{VW} = 0$, hence

$$\hat{\tau}_{VW} \simeq i_+ H^0 i^! \hat{\tau}_{VW}. \tag{7.3}$$

Our main observation here is that we can compute the D-module $H^0 i^! \hat{\tau}_{VW}$ in a way that is parallel to our computation in the CY case of $\hat{\tau}$ in §5. To proceed, first we have the following analogue of Lemma 5.2 for general types.

Lemma 7.2.

$$(H^0 i^! \hat{\tau}_{VW})|_{\mathring{V}} \simeq \mathring{i}_+ D_{\mathring{X}, \beta''}. \tag{7.4}$$

Proof. $H^0 i^! \tau_{\hat{V}W}$ consists of elements of $\tau_{\hat{V}W}$ annihilated by all b_i^*. One finds that they are precisely the elements that can be written in the form $\sum s_j \otimes b_j^*$, where $s_j \in D_V$.

On the other hand, we have

$$\mathring{i}_+ D_{\mathring{X}, \beta''} = (D_{\mathring{V}} / D_{\mathring{V}} I(X) \otimes_{\mathcal{O}_{\mathring{V}}} \omega_{\mathring{X}/\mathring{V}}) \otimes_{D_{\mathring{X}}} D_{\mathring{X}, \beta''}, \tag{7.5}$$

where $\omega_{\mathring{X}/\mathring{V}}$ as a left $\mathcal{O}_{\mathring{V}}$-module is generated by global sections $b_1^*, ..., b_s^*$ (under the canonical identification $\omega \to \omega \wedge \frac{dt}{t}$) and relations among these generators as a left $\mathcal{O}_{\mathring{V}}$-module are precisely given by $I^{VW} = \oplus_{d>0} \mathcal{I}_d$ (see section 6 for notations). Note that $\hat{\mathfrak{g}}$ acts on $D_{\mathring{V}} \otimes_{\mathcal{O}_{\mathring{V}}} \mathcal{O}_{\mathring{V}} \langle b_1^*, ..., b_s^* \rangle$ from the right via tensor product. The action descends to an action on

$$
\begin{aligned}
&D_{\mathring{V}}/D_{\mathring{V}}I(X)\otimes_{\mathcal{O}_{\mathring{V}}}\mathcal{O}_{\mathring{V}}\langle b_1^*,...,b_s^*\rangle/I^{VW}\\
&\qquad= D_{\mathring{V}}/D_{\mathring{V}}I(X)\otimes_{\mathcal{O}_{\mathring{V}}}\omega_{\mathring{X}/\mathring{V}}.
\end{aligned}\tag{7.6}
$$

One checks that this action restricted on $\mathfrak{g}$ coincides with the right $\mathfrak{g}$ action on $\omega_{\mathring{X}/\mathring{V}}$ through its right $D_{\mathring{X}}$-module structure given by negative Lie derivatives, while for e^V, $\omega_{\mathring{X}/\mathring{V}}$ again introduces an extra weight of $-r+1$. Thus we have

$$
\begin{aligned}
&(D_{\mathring{V}}/D_{\mathring{V}}I(X)\otimes_{\mathcal{O}_{\mathring{V}}}\omega_{\mathring{X}/\mathring{V}})\otimes_{D_{\mathring{X}}}D_{\mathring{X},\beta''}\\
&\qquad\simeq(D_{\mathring{V}}/D_{\mathring{V}}I(X)\otimes_{\mathcal{O}_{\mathring{V}}}\mathcal{O}_{\mathring{V}}\langle b_1^*,...,b_s^*\rangle/I^{VW}\otimes k_{\beta'})\otimes_{\hat{\mathfrak{g}}}k,
\end{aligned}\tag{7.7}
$$

where $\hat{\mathfrak{g}}$ acts on $(D_{\mathring{V}}/D_{\mathring{V}}I(X)\otimes_{\mathcal{O}_{\mathring{V}}}\mathcal{O}_{\mathring{V}}\langle b_1^*,...,b_s^*\rangle)/I^{VW}$ explicitly as explained, and the Lie algebra homomorphism β': $\mathfrak{g}\oplus ke^V\to k$ is defined as $\beta'|_g=\beta''$, $\beta'(e^V)=\beta''(e^V)+r-1$.

Furthermore, from definition and the explanation in the beginning of the proof,

$$
(H^0i^!\hat{\tau}_{VW})|_{\mathring{V}}=(D_{\mathring{V}}/D_{\mathring{V}}I(X)\otimes_{\mathcal{O}_{\mathring{V}}}\mathcal{O}_{\mathring{V}}\langle b_1^*,...,b_s^*\rangle/I^{VW}\otimes k_{\beta'})\otimes_{\hat{\mathfrak{g}}}k.\tag{7.8}
$$

Combining (7.5), (7.7), and (7.8), the lemma is proved. q.e.d.

Next, by specializing (A.4), we have the following analogue of the sequence (5.1):

$$
\begin{aligned}
&0\to i_{0,+}H^{-1}i_0^+(H^0i^!\hat{\tau}_{VW})\to H^0j_{0,!}(H^0i^!\hat{\tau}_{VW})|_{\mathring{V}}\\
&\qquad\to H^0i^!\hat{\tau}_{VW}\to i_{0,+}H^0i_0^+(H^0i^!\hat{\tau}_{VW})\to 0.
\end{aligned}\tag{7.9}
$$

With $\hat{\tau}$ in the CY case now replaced by $H^0i^!\hat{\tau}_{VW}$, Lemmas 5.1 and 5.5 carry over readily to the general type case, with the following changes. The !-fiber of τ at $a\in V^\vee$ in the CY case is replaced by the !-fiber of τ_{VW} at $(a,b)\in V^\vee\times W^\vee$ in the general type case. The latter is now given in a parallel way by the Lie algebra homology of $\hat{\mathfrak{g}}$ with coefficients in $S^{VW}\otimes\beta$, where

$$
S^{VW}:=S^V\otimes S^W/S^V\otimes S^W\mathcal{I}_1.
$$

Here the $\hat{\mathfrak{g}}$-action on $S^V=\mathcal{O}_V/I(\hat{X})$ is given verbatim as in §3.2 as the sum of two actions α_1,α_2 (see before Lemma 3.1). The $\hat{\mathfrak{g}}$-action on $S^W=\mathcal{O}_W/\mathcal{O}_W\mathrm{Sym}\,^2W^\vee$ is given by $Z^{W^\vee}:\hat{\mathfrak{g}}\to\mathrm{End}W^\vee\to\mathrm{End}S^W$, where e^W acts trivially ($\beta(e^W)=-1$). This shows that the first and last terms of (7.9) are both zero, hence

$$
H^0j_{0,!}(H^0i^!\hat{\tau}_{VW})|_{\mathring{V}}\simeq H^0i^!\hat{\tau}_{VW}.
$$

Together with Lemma 7.2, this implies that

$$
\begin{aligned}
\mathcal{F}our(H^0i^!\hat{\tau}_{VW})&\simeq\mathcal{F}our(H^0j_{0,!}(H^0i^!\hat{\tau}_{VW})|_{\mathring{V}})\\
&\simeq H^0\mathcal{F}our(j_{0,!}\mathring{i}_+D_{\mathring{X},\beta''}).
\end{aligned}\tag{7.10}
$$

Next, to compute the right hand side, observe that (5.4)–(5.6) and Lemma 5.3 hold for an arbitrary very ample line bundle $\mathcal{L}$. This yields

$$\mathcal{F}our(j_{0,!}\mathring{i}_+ D_{\mathring{X},\beta''}) \simeq \pi_+^\vee ev^! D_{\mathring{\mathbb{L}}^\vee,-\beta''}[1-r]\,. \tag{7.11}$$

Finally, since p is dual to the inclusion i, and combining (7.3) and (7.10)–(7.11), it follows that

$$\begin{aligned}\tau_{VW} &\simeq \mathcal{F}our(i_+ H^0 i^! \tau_{\hat{V}W}) \simeq p^! \mathcal{F}our(H^0 i^! \tau_{\hat{V}W})\,[-s] \\ &\simeq p^! H^0 \pi_+^\vee ev^! D_{\mathring{\mathbb{L}}^\vee,-\beta''}[1-r-s]\end{aligned}$$

This completes the proof of Theorem 7.1.

8. Projective homogeneous spaces

To apply our results, we need to understand the D-module $ev^!(D_{\mathring{\mathbb{L}}^\vee,-\beta''})[1-r]$ in various situations. In this section, we assume that G is semisimple and X is a projective homogeneous G-variety, i.e. X is a partial flag variety, and $\beta(e)=1$. Theorem 2.1 takes a particularly easy form in this case. We first have

Corollary 8.1. *If $\beta(\mathfrak{g})=0$ and X is a homogeneous G-variety, then $\tau \simeq H^0\pi_+^\vee \mathcal{O}_U$.*

Proof. Recall Corollary 2.2. Then if X is homogeneous, $\mathfrak{g}\otimes\mathcal{O}_X \to T_X$ is surjective. Therefore $D_{X,0} = D_X/D_X T_X = \mathcal{O}_X$. q.e.d.

Note that this corollary implies that a tautological system in this case, which is a priori defined as a D-module by generators and relations, is of geometric origin, i.e. itself is a Gauss–Manin connection.

Corollary 8.2. *Conjecture 1.2 holds.*

For general types, Theorem 6.2 also specializes in an analogous way, and we get the following analogues of both Corollaries 8.1 and 8.2.

Corollary 8.3. *If $\beta(\mathfrak{g})=0$ and X is a homogeneous G-variety, then $\tau \simeq H^0(\pi^\vee \times \mathrm{id}_{W^\vee})_+\mathcal{O}_{U\times W^\vee}$.*

Corollary 8.4. *Let X be an n-dimensional projective homogeneous space of a semisimple group G. Assume $\beta(e^V)=1$ and $\beta(e^W)=-1$. Then the solution rank of $\tau_{VW}=\tau_{VW}(G,X,\mathcal{L},\beta)$ at $(a,b)\in V^\vee\times W^\vee$ is given by $\dim H_n(X-Y_a)$.*

We can also describe a rank 1 point for a general homogeneous variety X in the case of $\mathcal{L}=\omega_X^{-1}$, using the projected Richardson stratification of X studied in [**22**][**23**][**11**][**17**].

We follow the notations in [**17**]. Let G be a reductive algebraic group over an algebraically closed field k of characteristics zero, B a Borel subgroup and $P\supset B$ a parabolic subgroup in G. Put $B^+=B$ and let B^- be the opposite Borel subgroup. Let $Q(W,W_P)$ be the set of equivalence classes of P-Bruhat intervals [**17**, §2]. Each equivalence class is uniquely specified by a pair (u, w) of elements in the Weyl group. For $(u, w) \in Q(W, W_P)$, put $X_u^w := (B^- uB/B) \cap (B^+ wB/B)$, an open Richardson variety in G/B.

Proposition 8.5. [**17**, §3][**23**, §7] *There is a stratification of $X = G/P$ of the form $X = \coprod_{(u,w)\in Q(W,W_P)} \mathring{\Pi}_u^w$, where each stratum $\mathring{\Pi}_u^w$ is the isomorphic image of X_u^w under the natural projection $G/B \to G/P$.*

The next result and proof are communicated to us by T. Lam.

Proposition 8.6. *Let $\Pi_1, .., \Pi_s$ be the closures of the codimension 1 strata in X. Then $\cup_i \Pi_i$ is an anticanonical divisor in X, and its complement in X has one-dimensional middle cohomology.*

Proof. The first assertion follows from Lemma 5.4 [**17**]. Since $X - \cup_i \Pi_i$ is the largest stratum, it is isomorphic to an open Richardson variety X_u^w in G/B. It is well-known that (see for example [**24**])

$$H_c^N(X_u^w) = \mathrm{Hom}(M_u, M_w), \tag{8.1}$$

where $N = \dim X_u^w$ and M_w denotes the Verma module of the Lie algebra of G of highest weight $-w(\rho) - \rho$. By combining Theorems 1–4 [**3**], or by the Kazhdan–Lusztig conjecture, one has

$$\dim \mathrm{Hom}(M_u, M_w) = 1. \tag{8.2}$$

q.e.d.

Therefore, by Theorem 1.4 we have

Corollary 8.7. *Let $a \in \Gamma(X, \omega_X^{-1})$ be the defining section of the anticanonical divisor $\cup_i \Pi_i$. Then a is a rank 1 point of X.*

Remark 8.1. This section is torus invariant. Due to a theorem of Kostant that later was generalized by Luna: If a point f_a in $V^\vee$ is fixed by a reductive subgroup H, then Gv is closed if and only if $C_G(H)v$ is closed. If H is the maximal torus, then $C_G(H) = H$, so the orbit is closed, therefore this section is GIT semistable w.r.t. the action of G on $V^\vee$.

Example 8.8. Consider the Grassmannian $X = G(d, n)$. According to [**18**], $\cup_i \Pi_i$ is defined by the section $a = x_{1,2,..,d} x_{2,3,...,d+1} \cdots x_{n,1,..,d-1}$, where the $x_{i_1,..,i_d}$ are the Plücker coordinates of X. This generalizes a construction in [**5**] for $d = 2$.

9. Rank 1 points of 1-step flags

Notation. If m is an $p \times q$ matrix, and $J \subset (1, 2, .., p)$ is an ordered index set, then m_J denotes the submatrix of m given by the rows labelled by J, and we also call m_J the J-block of m. We denote by x_J, $J \subset$

$(1,2,..,n)$, the Plücker coordinates of the d-plane Grassmannian $F(d,n)$. Let M be the space of rank d matrices of size $n \times d$. Then GL_d acts freely and properly on M by right multiplication and $M/GL_d \simeq F(d,n)$. Under this identification, we denote the projection map of the Stiefel bundle $M \to X$ by $m \mapsto [m] := m \cdot GL_d$. Then x_J can be viewed as the function $x_J : M \to \mathbb{C}$, $m \mapsto \det(m_J)$. Given a section f of any line bundle on X, we denote by $X(f)$ the complement of $f = 0$ in X, and by $M(f)$ the preimage of $X(f)$ under $M \to X$.

Proposition 9.1. *The 1-step flag variety $X = F(d,n)$ admits a rank 1 point $f \in \Gamma(X, \omega_X^{-1})$ such that $(x_J)^k | f$ for some $J \subset (1,2,..,n)$ with $|J| = d$ and $k = \min(d, n-d)$. If $n = 2d$, then $f = (x_{1,..,d})^d (x_{d+1,..,n})^d$ is a rank 1 point.*

Proof. (a) Consider the case $n \geq l + d \geq 2d$. We have

$$\begin{aligned} &X_1 := F(d, n-l) \hookrightarrow X, \ E \mapsto E \oplus 0_l, \\ &X_2 := F(d, l) \hookrightarrow X, \ E \mapsto 0_{n-l} \oplus E. \end{aligned} \tag{9.1}$$

Here we view $\mathbb{C}^n = \mathbb{C}^{n-l} \oplus \mathbb{C}^l$. Let f_1 be a given rank 1 point of X_1 such that $(x_{1,..,d})^{k_1} | f_1$, $k_1 = \min(d, n-l-d)$, and f_2 a rank 1 point of X_2 such that $(x_{n-d+1,..n})^{k_2} | f_2$, $k_2 = \min(d, l-d)$. (In case $l = d$, $X_2 = \mathrm{pt}$, we simply take $f_2 = (x_{n-d+1,..,n})^d$; in case $n - l = d$, $X_1 = \mathrm{pt}$, we take $f_1 = (x_{1,..,d})^d$.) We can view f_1, f_2 as sections of $\mathcal{O}_X(n-l)$ and $\mathcal{O}_X(l)$ respectively on $X = F(d,n)$. Then the restriction of f_1 to X_1 under (9.1) becomes a section of $\mathcal{O}_{X_1}(n-l)$. Likewise the restriction of f_2 to X_2 becomes a section of $\mathcal{O}_{X_2}(l)$. We claim that $f = f_1 f_2 \in \Gamma(X, \omega_X^{-1})$ is a rank 1 point of X. We will first construct an explicit isomorphism

$$X_1(f_1) \times X_2(f_2) \times GL_d \to X(f).$$

Let M_1, M_2, M be the Stiefel bundles over X_1, X_2, X respectively. Since $x_J | f_2$, $J = (n-d+1, ..n)$, each $m_2' \in M_2(f_2)$ has a nonsingular J-block D. Define

$$M_1(f_1) \times M_2(f_2) \to M(f), \ m_1', m_2' \mapsto m = \begin{bmatrix} m_1' D \\ m_2' \end{bmatrix}.$$

This is well-defined since

$$f(m) = f_1(m_1' D) f_2(m_2') = (\det D)^{n-l} f_1(m_1') f_2(m_2').$$

The map is a bijection with inverse $m = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \mapsto m_1 (m_2)_J^{-1}, m_2$. Now let $h \in GL_d$ act on $M_1(f_1) \times M_2(f_2)$ by the formula $(m_1', m_2' h^{-1})$. Then the map is equivariant. It follows that we have an isomorphism

$$M_1(f_1) \times X_2(f_2) \to X(f).$$

Finally, since $x_{1,..,d} | f_1$ each $m_1' \in M_1(f_1)$ has a nonsingular top $d \times d$ block. It follows that the principal bundle $GL_d - M_1(f_1) \to X_1(f_1)$ is trivial. In fact, it has a (unique) section of the form $[m_1] \mapsto m_1'$ where m_1' is the unique representative in $[m_1]$ whose top $d \times d$ block is the identity matrix I_d. This proves that

$$X(f) \simeq X_1(f_1) \times X_2(f_2) \times GL_d.$$

Since the $X_i(f_i)$ are affine varieties, all de Rham cohomology of degree $> \dim X_i$, vanishes. Since f_1, f_2 are rank 1 points of X_1, X_2 respectively, we have $H^{\dim X_i}(X_i(f_i)) = \mathbb{C}$ by Theorem 1.4. It follows that

$$H^{\dim X}(X(f)) \simeq H^{\dim X_1}(X_1(f_1)) \otimes H^{\dim X_2}(X_2(f_2)) \otimes H^{d^2}(GL_d) \simeq \mathbb{C}.$$

So f is a rank 1 point of X such that $(x_{1,..,d})^{k_1}(x_{n-d+1,..n})^{k_2}|f$.

(b) To complete the proof of the proposition, we proceed by induction. For $X = F(1,2) = \mathbb{P}^1$, paragraph (a) with $n = 2$ and $l = d = 1$ shows that x_1x_2 is a rank 1 point of X, and the proposition holds. Assume that it holds for up to $F(d, n-1)$, and consider the case $X = F(d,n)$. For $n < 2d$ we have $F(d,n) \simeq F(n-d,n)$, in which case paragraph (a) with l, d playing the role of $d, n-d$, yields a rank 1 point f of $F(n-d,n)$ with $(x_J)^{n-d}|f$ and $|J| = n-d$. This in turn yields a rank 1 point of $F(d,n)$ divisible by $(x_{J^c})^{n-d}$ where $J^c = (1,..,n) - J$. For $n = 2d$, paragraph (a) with $n = l + d = 2d$ shows that $(x_{1,..,d})^d(x_{d+1,..,n})^d$ is a rank 1 point of X. For $n > 2d$, paragraph (a) with $l = d$ and our inductive hypothesis shows that X has a rank 1 point $f = f_1 \cdot (x_{n-d+1,..,n})^d$, where f_1 is a rank 1 point of $F(d, n-d)$. This completes the proof. q.e.d.

Corollary 9.2. *Let $n = l_1 + \cdots + l_s$ be a partition of n with $l_p \geq d$. Let f_p be a rank 1 point of $F(d,l_p) \hookrightarrow F(d,n)$, viewed as a degree l_p polynomial in the Plücker coordinates x_J of $X = F(d,n)$ with $J \subset (l_1 + \cdots + l_{p-1} + 1,.., l_1 + \cdots + l_p)$ and $|J| = d$, such that $(x_{l_1+\cdots+l_{p-1}+1,..,l_1+\cdots+l_{p-1}+d})|f_p$. Then $f = f_1 \cdots f_s$ is a rank 1 point of X. In fact, we have an isomorphism*

$$X(f) \simeq X_1(f_1) \times \cdots \times X_s(f_s) \times (GL_d)^{s-1},$$

where $X_p := F(d, l_p)$.

Proof. Start with $l = l_2 + \cdots + l_s$. Then paragraph (a) in the preceding proof gives

$$X(f) \simeq X_1(f_1) \times X_2'(f_2 \cdots f_s) \times GL_d,$$

where $X_2' := F(d, n - l_1)$. Now the result follows by induction on s. q.e.d.

10. Rank 1 points of r-step flags

Throughout this section, let $X = F(d_1,..,d_r,n)$ be the r-step flag variety with $r \geq 2$. We will give a recursive procedure that produces a rank 1 point of X, by assembling rank 1 points of lower step flag varieties. We begin with some notations and terminology.

Let $\mathcal{O}_i(1)$ be the standard hyperplane bundle on $F(d_i, n)$. The space of its sections is an irreducible $G = SL_n$ module of highest weight λ_{d_i}, the d_ith fundamental weight of G. We shall denote by λ_{d_i} the pullback of $\mathcal{O}_i(1)$ via the composition map

$$X \hookrightarrow F(d_1, n) \times \cdots \times F(d_r, n) \twoheadrightarrow F(d_i, n),$$

where the first is the incidence embedding and the second is the ith projection. Then $\mathrm{Pic}(X)$ is the free abelian group generated by $\lambda_{d_1}, ..., \lambda_{d_r}$. We also have (see [**20**])

(10.1)

$$-K_X = \omega_X^{-1} = (n-d_{r-1})\lambda_{d_r} + (d_r - d_{r-2})\lambda_{d_{r-1}} + \cdots + (d_3 - d_1)\lambda_{d_2} + d_2\lambda_{d_1}.$$

By the Borel–Weil theorem, the restriction map

$$\Gamma(F(d_1, n), \mathcal{O}_1(k_1)) \otimes \cdots \otimes \Gamma(F(d_r, n), \mathcal{O}_r(k_r)) \to \Gamma(X, \sum_i k_i \lambda_{d_i})$$

is a G-equivariant surjective map for any $k_1, .., k_r \in \mathbb{Z}$ (and both spaces are zero unless $k_i \geq 0$ for all i). Thus any homogeneous polynomial in the Plücker coordinates x_{J_i} with $|J_i| = d_i$, of multi-degree $(k_1, .., k_r) \in \mathbb{Z}^r_{\geq 0}$, can be viewed as a section of the line bundle $\sum_i k_i \lambda_{d_i}$ on X. Conversely, any section of this line bundle on X can be expressed as such a polynomial (not necessarily unique).

Let $k < n - d_r$ and consider the embeddings

$$X_1 := F(d_1, .., d_r, n-k) \hookrightarrow X, \;\; E^\bullet \mapsto E^\bullet \oplus 0_k,$$

$$X_2 := F(d_1 - k, .., d_r - k, n-k) \hookrightarrow X, \;\; E^\bullet \mapsto E^\bullet \oplus \mathbb{C}^k. \tag{10.2}$$

Here we view $\mathbb{C}^n = \mathbb{C}^{n-k} \oplus \mathbb{C}^k$, and X_1, X_2 are viewed as spaces consisting of r-step flags in the factor $\mathbb{C}^{n-k}$. For each Plücker coordinate $x_{J'}$ on X_1 with $J' \subset (1, 2, ..., n-k)$, is the restriction of $x_{J'}$, regarded as a Plücker coordinate on X. Likewise, any homogeneous polynomial f_1 in the $x_{J'}$, can be viewed as the restriction of a section $\bar{f}_1$ on X involving only the same Plücker coordinates. We shall often impose certain divisibility conditions (called the hyperplane property – see below) on $\bar{f}_1$, but will state them in terms of f_1. Similarly each Plücker coordinate $x_{J'}$ on X_2 is the restriction of $x_{J' \cup (n-k+1, .., n)}$ on X; any given homogeneous polynomial f_2 in the $x_{J'}$, is the restriction of a section $\tilde{f}_2$ on X involving only the $x_{J' \cup (n-k+1, .., n)}$. Again, divisibility conditions imposed on $\tilde{f}_2$ will be stated in terms of f_2.

As in the case of 1-step flags, we can view $X = M/H$, where

$$H := GL_{d_r} \times \cdots \times GL_{d_1}$$

and M is the space of r-tuple of matrices $m = (m_r, .., m_1)$, m_i a $d_{i+1} \times d_i$ matrix of rank d_i ($d_{r+1} \equiv n$), where $h = (h_r, .., h_1) \in H$ acts on M by the formula

$$m \cdot h^{-1} := (m_r h_r^{-1}, h_r m_{r-1} h_{r-1}^{-1}, .., h_2 m_1 h_1^{-1}). \tag{10.3}$$

Under the identification $X = M/H$, we denote the projection map $M \to X$ by $m \mapsto [m] := m \cdot H$, and call M *the Stiefel bundle* over X. We can view a Plücker coordinate x_J, $|J| = d_i$, on X as the function $x_J : M \to \mathbb{C}$, $x_J(m) = \det(m_r \cdots m_i)_J$. In particular, f_1 is a section on X_1 and $\bar{f}_1$ a section on X restricting to it as described above, then for $J = (1, .., n-k)$ we have

$$\bar{f}_1(m_r, .., m_1) = f_1((m_r)_J, m_{r-1}, .., m_1)$$

whenever $\mathrm{rk}\,(m_r)_J = d_r$. Let $m = (m_r, .., m_1) \in M$ where the m_i have the form

$$m_i = \begin{bmatrix} m_i' & * \\ O & I_k \end{bmatrix},$$

where I_k is the $k \times k$ identity matrix and O a zero block. Then $x_{J' \cup (n-k+1,..,n)}(m) = \det(m_r' \cdots m_i')_{J'}$ for any $J' \subset (1, .., n-k)$ with $|J'| = d_i - k$. So, if f_2 is a section on X_2 and $\tilde{f}_2$ a section on X restricting to it as described above, then

$$\tilde{f}_2(m_r, .., m_1) = f_2(m_r', ..., m_1').$$

Let f be a nonzero section of a line bundle on X, and let $X(f)$ be the complement of $f = 0$, and $M(f)$ the preimage of $X(f)$ under $M \to X$.

Definition 10.1. (Hyperplane property) We say that f has the hyperplane property if for some $J_i \subset (1, 2, .., n)$ with $|J_i| = d_i$, $i = 1, .., r$, we have $(x_{J_1} \cdots x_{J_r})|f$. In other words, the hypersurface $f = 0$ contains the union of hyperplanes $x_{J_i} = 0$.

Note that if f has the hyperplane property, we can always find a suitable permutation matrix $g \in GL_n$ such that the g-translate of f has the hyperplane property where $J_r = (n - d_r + 1, .., n)$. In the construction that follows, we will often arrange our section f so that this occurs. Next, we have the following elementary lemma.

Lemma 10.1. *Assume f has the hyperplane property $(x_{J_1} \cdots x_{J_r})|f$. Then the principal H-bundle $M(f) \to X(f)$, has a unique section $m = (m_r, .., m_1)$, where the $m_r, .., m_1$ are matrix valued functions on $X(f)$ such that*

$$(m_r \cdots m_i)_{J_i} = I_{d_i}.$$

Definition 10.2. (Special section) We call the section given in Lemma 10.1, the special section of $M(f)$ (which depends on the index sets $J_1, .., J_r$).

We now describe our recursive procedure that produces a rank 1 point of X with the hyperplane property.

Case 1. Assume $d_{r-1} + d_r < n$. Consider (cf. (10.2))

$$X_1 := F(d_1, .., d_{r-1}, d_r) \hookrightarrow F(d_1, .., d_{r-1}, n), \ E^\bullet \mapsto 0_{n-d_r} \oplus E^\bullet,$$
$$X_2 := F(d_r, n).$$

Let M_1, M_2, M be the Stiefel bundles over X_1, X_2, X respectively. Let f_1, f_2 be rank 1 points of X_1, X_2 respectively with the hyperplane properties

$$(x_{J_1} \cdots x_{J_{r-1}})|f_1,\ (x_{J_r})^k|f_2, \tag{10.4}$$

for some J_i with $|J_i| = d_i$, $i = 1, ..., r$, and $J_1 = (1, .., d_1)$, $J_r = (n - d_r + 1, .., n)$, $k = \min(d_r, n - d_r) > d_{r-1}$. Such an f_2 exists by Proposition 9.1. Put

$$f = \bar{f}_1 \cdot \bar{f}_2 \cdot (x_{J_r})^{-d_{r-1}}. \tag{10.5}$$

Then we have

$$(x_{J_1} \cdots x_{J_r})|f. \tag{10.6}$$

It follows easily from (10.1) that f is a section of ω_X^{-1}.

Lemma 10.2. *We have an $H = GL_{d_r} \times \cdots \times GL_{d_1}$ equivariant isomorphism*

$$\begin{aligned}
&M_1(f_1) \times M_2(f_2) \to M(f)\\
&(m'_{r-1}, .., m'_1), m'_r \mapsto m = (m'_r, D^{-1}m'_{r-1}, m'_{r-2}, .., m'_1),
\end{aligned}$$

where D is the J_r-block of m'_r. Therefore the map descends to an isomorphism $X_1(f_1) \times X_2(f_2) \to X(f)$.

Proof. For $m'_r \in M_2(f_2)$, its J_r-block D is a nonsingular matrix in GL_{d_2} since $(x_{J_r})^k|f_2$. Suppose $f_1(m'_{r-1}, .., m'_1)f_2(m'_r) \neq 0$. Then

$$f(m) = f_1((m'_r D^{-1} m'_{r-1})_{J_r}, m'_{r-2}, .., m'_1) f_2(m'_r)(\det(m'_r)_{J_r})^{-d_{r-1}}.$$

Since $(m'_r)_{J_r} = D$, it follows that $(m'_r D^{-1} m'_{r-1})_{J_r} = m'_{r-1}$ and we have

$$f(m) = f_1(m'_{r-1}, .., m'_1) f_2(m'_r)(\det D)^{-d_{r-1}} \neq 0.$$

So, the map is well-defined. Now, $h = (h_r, .., h_1) \in H$ acts on $M(f)$ by (10.3), and on $M_1(f_1) \times M_2(f_2)$ by the formula

$$m_r h_r^{-1},\ (m_{r-1}h_{r-1}^{-1}, h_{r-1}m_{r-2}h_{r-2}^{-1}, .., h_2 m_1 h_1^{-1}).$$

Therefore our map is H-equivariant. Moreover, the map

$$M(f) \to M_1(f_1) \times M_2(f_2),\ (m_r, .., m_1) \mapsto ((m_r)_{J_r} m_{r-1}, m_{r-2}, .., m_1), m_r$$

is well-defined and is the inverse of the map above. q.e.d.

The lemma and Theorem 1.4 imply

Proposition 10.3. *For $d_{r-1} + d_r < n$, if any s-step flag variety for $s < r$ admits a rank 1 point with the hyperplane property, then $X = F(d_1, .., d_r, n)$ admits one as well.*

By Proposition 9.1, for $d_1+d_2<n$ it follows that $F(d_1,d_2,n)$ admits a rank 1 point with the hyperplane property. This also implies that for $d_1+d_2>n$, then $F(d_1,d_2,n)\simeq F(n-d_2,n-d_1,n)$ admits one as well.

Case 2. Assume $d_{r-1}+d_r=n$ and $r=2$. Consider the following section of ω_X^{-1}:

$$f=(x_{1,..,d_1})^{d_2}(x_{d_1+1,..,n})^{d_2}.$$

Then by Lemma 10.1, the special section of $M(f)\to X(f)$ has the form

$$m=(m_2,m_1)=(\begin{bmatrix}A_2\\ I_{d_2}\end{bmatrix},m_1)\text{ such that }A_2m_1=I_{d_1}.$$

Since $m_1(o)$ has rank d_1 at each point $o\in X(f)$, the second equation shows that the function

$$m_1:X(f)\to M_1,\ o\mapsto m_1(o)$$

is onto. Here M_1 be the Stiefel bundle over $F(d_1,d_2)$. Moreover, the level set of this function at each point is an affine space of dimension $d_1d_2-d_1^2$. It follows that $X(f)$ is homotopy equivalent to M_1. Finally, the principal GL_{d_1}-bundle $M_1\to F(d_1,d_2)$ is over a simply connected base. Thus by the Serre spectral sequence, the highest degree nonzero cohomology group of M_1 is one-dimensional at degree $2d_1d_2-d_1^2=\dim X$. By Theorem 1.4, we have

Proposition 10.4. *For $d_1+d_2=n$, $X=F(d_1,d_2,n)$ admits the rank 1 point $f=(x_{1,..,d_1})^{d_2}(x_{d_1+1,..,n})^{d_2}$.*

Remark 10.1. The propositions in Cases 1–2 ($r=2$) now imply that any 2-step flag variety $F(d_1,d_2,n)$ admits a rank 1 point with the hyperplane property.

Case 3. Assume $d_{r-1}+d_r=n$ and $r\geq 3$. Consider (cf. (10.2))

$$X_1:=F(d_1,..,d_{r-2},d_{r-1})\hookrightarrow F(d_1,..,d_{r-2},n),\ E^\bullet\mapsto E^\bullet\oplus 0_{n-d_{r-1}},$$
$$X_2:=F(d_{r-1},d_r,n).$$

Let M_1,M_2,M be the Stiefel bundles over X_1,X_2,X respectively. Let f_1,f_2 be rank 1 points of X_1,X_2 respectively with the hyperplane properties

$$(x_{J_1}\cdots x_{J_{r-2}})|f_1,\ f_2=(x_{J_{r-1}})^{d_r}(x_{J_r})^{d_r},\tag{10.7}$$

for some J_i with $|J_i|=d_i$, $i=1,...,r$, and $J_1=(1,..,d_{r-1})$, $J_r=(n-d_r+1,..,n)$. Note that f_2 is given by Proposition 10.4. Put

$$f=\bar f_1\cdot\bar f_2\cdot(x_{J_{r-1}})^{-d_{r-2}}\in\Gamma(X,\omega^{-1}).\tag{10.8}$$

Then we have

$$(x_{J_1}\cdots x_{J_{r-1}}(x_{J_r})^{d_r})|f.\tag{10.9}$$

Since $x_{J_{r-1}}|f_2$, the $J_{r-1}=(1,..,d_{r-1})$-block D of $m_r'm_{r-1}'$ for $(m_r',m_{r-1}')\in M_2(f_2)$ is nonsingular.

Lemma 10.5. *We have an $H = GL_{d_r} \times \cdots \times GL_{d_1}$ equivariant isomorphism*

$$M_1(f_1) \times M_2(f_2) \to M(f)$$
$$(m'_{r-2}, .., m'_1), (m'_r, m'_{r-1}) \mapsto m = (m'_r, m'_{r-1}, D^{-1}m'_{r-2}, m'_{r-3}, .., m'_1),$$

where D is the $J_{r-1} = (1, .., d_{r-1})$-block of $m'_r m'_{r-1}$. Therefore the map descends to an isomorphism $X_1(f_1) \times X_2(f_2) \to X(f)$.

The proof is closely analogous to the lemma in Case 1, and will be omitted. The lemma and Theorem 1.4 imply

Proposition 10.6. *For $d_{r-1} + d_r = n$, if any s-step flag variety for $s < r$ admits a rank 1 point with the hyperplane property, then $X = F(d_1, .., d_r, n)$ admits one such f that satisfies $(x_{J_r})^{d_r}|f$ where $J_r = (n - d_r + 1, .., n)$.*

Case 4. Assume $d_1 + d_2 = n$. Then $X \simeq F(n - d_r, .., n - d_2, n - d_1, n)$, which belongs in Case 3, and the analogue of Proposition 10.6 is

Proposition 10.7. *For $d_1 + d_2 = n$, if any s-step flag variety for $s < r$ admits a rank 1 point with the hyperplane property, then $X = F(d_1, .., d_r, n)$ admits one such f that satisfies $(x_{J_1})^{d_1}|f$ where $J_1 = (1, .., d_1)$.*

Case 5. Assume $d_{r-1} + d_r > n$. There exists a unique a with $r > a > 1$ such that $d_a + d_{a+1} > n \geq d_{a-1} + d_a$. Assume $n > 2d_a$ first. We will consider $n = 2d_a$ and $n = d_{a-1} + d_a$ in Cases 6-7 below separately. Consider

$$X_1 := F(d_1, .., d_a, n - d_a) \hookrightarrow F(d_1, .., d_a, n),\ (E_1^i) \mapsto (E_1^i \oplus 0_{d_a}),$$
$$X_2 := F(d_{a+1} - d_a, .., d_r - d_a, n - d_a) \hookrightarrow F(d_{a+1}, , .., d_r, n),$$
$$(E_2^j) \mapsto (E_2^j \oplus \mathbb{C}^{d_a}).$$

Here we view $\mathbb{C}^n = \mathbb{C}^{n-d_a} \oplus \mathbb{C}^{d_a}$. Let M_1, M_2, M be the Stiefel bundles over X_1, X_2, X respectively. Let f_1, f_2 be rank 1 points of X_1, X_2 respectively with the hyperplane properties

$$(x_{J_1} \cdots x_{J_a})|f_1,\ (x_{J'_{a+1}} \cdots x_{J'_r})|f_2 \tag{10.10}$$

for some $J_i \subset (1, 2, .., n - d_a)$ with $|J_i| = d_i$ $(i = 1, ..., a)$ and $J_a = (n-2d_a+1, .., n-d_a)$, and for some $J'_i \subset (1, 2, .., n-d_a)$ with $|J'_i| = d_i - d_a$ $(i = a+1, .., r)$ and $J'_r = (n-d_r+1, .., n-d_a)$. Put $J := (n-d_a+1, .., n)$, $J_i := J'_i \cup J$, $i = a + 1, .., n$, and

$$f := \bar{f}_1 \cdot \tilde{f}_2 \cdot (x_J)^{d_{a+1}+d_a-n}. \tag{10.11}$$

Then f has the hyperplane property

$$(x_{J_1} \cdots \widehat{x_{J_a}} \cdots x_{J_r} x_J)|f. \tag{10.12}$$

Lemma 10.8. *The special section $m = (m_r, .., m_1)$ (cf. Lemma 10.1) of $M(f) \to X(f)$ has the following form:*

$$
\begin{aligned}
m_i &= \begin{bmatrix} m_i' & A_i \\ O & I_{d_a} \end{bmatrix},\ i = a+1, .., r \\
m_a &= \begin{bmatrix} A_a \\ I_{d_a} \end{bmatrix} \\
m_r \cdots m_a &= \begin{bmatrix} m_a' D \\ I_{d_a} \end{bmatrix} \\
m_{a-1} &= D^{-1} m_{a-1}' \\
m_i &= m_i',\ i = 1, .., a-2
\end{aligned} \tag{10.13}
$$

where D is a GL_{d_a}-valued function, $A_a, .., A_r$ are matrix valued functions, and $(m_a', .., m_1')$, $(m_r', .., m_{a+1}')$ are matrix valued functions taking values in the special sections of the $M_1(f_1) \to X_1(f_1)$, $M_2(f_2) \to X_2(f_2)$ respectively.

Proof. For $o \in X(f)$, we will write $m_i \equiv m_i(o)$, $m_i' \equiv m_i'(o)$, $D \equiv D(o)$, etc. Then $m = m(o) \in M(f)$ means that

$$0 \neq f(m) = \bar{f}_1(m_r \cdots m_a, m_{a-1}, .., m_1)\tilde{f}_2(m_r, .., m_{a+1}) \det(m_r)_{J_r}.$$

(a) Since $x_{J_r'}|f_2$, we have $x_{J_r}|\tilde{f}_2$, and so our m_r has the correct form, i.e. $(m_r)_{J_r} = I_{d_r}$ (hence $\det(m_r)_{J_r} = 1$), and $(m_r')_{J_r'} = I_{d_r - d_a}$. Since $(x_{J_{a+1}'} \cdots x_{J_r'})|f_2$, we have $(x_{J_{a+1}} \cdots x_{J_r})|\tilde{f}_2$, hence $(m_r \cdots m_i)_{J_i} = I_{d_i}$. By induction on i, it is easy to see that our $m_r, .., m_i$ above have the correct form, so that

$$m_r \cdots m_i = \begin{bmatrix} m_r' \cdots m_i' & * \\ O & I_{d_a} \end{bmatrix} \tag{10.14}$$

and that $(m_r' \cdots m_i')_{J_i'} = I_{d_i - d_a}$ for $i = a+1.., r$. This shows that $(m_r', .., m_{a+1}')$ actually lies in the special section of $M_2(f_2) \to X_2(f_2)$, as asserted.

(b) Since $x_J|f$, we have $(m_r \cdots m_a)_J = I_{d_a}$. From (10.14), it follows that $(m_a)_J = I_{d_a}$. Since $x_{J_a}|f_1$, hence $x_{J_a}|f$, it follows that $(m_r \cdots m_a)_{J_a}$ is a nonsingular matrix $D \in GL_{d_a}$. Thus m_a has the correct form as asserted, and $(m_a')_{J_a} = I_{d_a}$. This also shows that $(m_r \cdots m_a)_{1,2,..,n-d_a} = m_a' D$ has rank d_a, hence

$$0 \neq \bar{f}_1(m_r \cdots m_a, m_{a-1}, .., m_1) = f_1(m_a' D, m_{a-1}, .., m_1).$$

Since f_1 is GL_{d_a}-equivariant, this is equivalent to

$$0 \neq f_1(m_a', D^{-1} m_{a-1}, m_{a-2}, .., m_1).$$

This implies that

$$(m'_a, m'_{a-1}, .., m'_1) = (m'_a, D^{-1}m_{a-1}, m_{a-2}, .., m_1)$$

lies in the special section of $M_1(f_1) \to X_1(f_1)$, as asserted.

This completes the proof. q.e.d.

We now use the special section $m : X(f) \to M(f)$ described in the preceding lemma to define a map

$$\begin{aligned} &X(f) \to X_1(f_1) \times X_2(f_2) \times GL_{d_a} \\ (10.15) \qquad &o \mapsto [m'_a(o), .., m'_1(o)], [m'_r(o), .., m'_{a+1}(o)], D(o). \end{aligned}$$

We will prove that this is an isomorphism. We will need the following elementary lemma.

Lemma 10.9. *Let m'_2 be an $(n-d_1) \times (d-d_1)$ matrix, and A_1, A_2 be $(d-d_1) \times d_1$ and $(n-d_1) \times d_1$ matrices. Put*

$$m_2 = \begin{bmatrix} m'_2 & A_2 \\ O & I_{d_1} \end{bmatrix}, \ m_1 = \begin{bmatrix} A_1 \\ I_{d_1} \end{bmatrix}$$

and assume that $J' \subset (1, .., n-d_1)$, $|J'| = d_2 - a_1$, and that the $J = J' \cup (n-d_1+1, .., n)$-block of m_2 is I_d (which is equivalent to that $(A_2)_{J'} = O$ and $(m'_2)_{J'} = I_{d-d_1}$). Then A_1, A_2 can be uniquely expressed as polynomial functions in terms of m'_2 and m_2m_1.

Lemma 10.10. *The map* (10.15) *is an isomorphism.*

Proof. We will explicitly construct the inverse of (10.15). It is enough to show that given a point $m' := ((m'_a, .., m'_1), (m'_r, .., m'_{a+1}), D)$ in the special section of the bundle $M_1(f_1) \times M_2(f_2) \times GL_{d_a} \to X_1(f_1) \times X_2(f_2) \times GL_{d_a}$, the relations (10.13) *uniquely* determine a point $m = (m_r, .., m_1) \in M$, expressible polynomially in terms of m'. In fact, it is enough to show that the $A_a, .., A_r$ can be so-expressed. Note that the relations (10.13) ensures that m lies in the special section of the bundle $M(f) \to X(f)$.

By (10.13), we have for $i = 1, .., a+1$,

$$m_r \cdots m_i = \begin{bmatrix} m'_r \cdots m'_i & m'_r \cdots m'_{i+1}A_i + \cdots + m'_rA_{r-1} + A_r \\ O & I_{d_a} \end{bmatrix}.$$

Since $m_a = \begin{bmatrix} A_a \\ I_{d_a} \end{bmatrix}$, Lemma 10.9 implies that A_a and $m'_r \cdots m'_{a+2}A_{a+1} + \cdots + m'_rA_{r-1} + A_r$ can be uniquely expressed polynomially in terms of m'. It follows that the right hand block of $m_r \cdots m_{a+1}$:

$$\begin{aligned} (m_r \cdots m_{a+1})_R &= \begin{bmatrix} m'_r \cdots m'_{a+2}A_{a+1} + \cdots + m'_rA_{r-1} + A_r \\ I_{d_a} \end{bmatrix} \\ &= \begin{bmatrix} m'_r \cdots m'_{a+2} & m'_r \cdots m'_{a+1}A_{a+2} + \cdots + m'_rA_{r-1} + A_r \\ O & I_{d_a} \end{bmatrix} \begin{bmatrix} A_{a+1} \\ I_{d_a} \end{bmatrix} \\ &= m_r \cdots m_{a+2} \begin{bmatrix} A_{a+1} \\ I_{d_a} \end{bmatrix} \end{aligned}$$

can be so-expressed. By Lemma 10.9 again, the right hand block of $m_r \cdots m_{a+2}$ and A_{a+1} can also be so-expressed. Continuing this way, we see that $A_a, .., A_r$ all can be so-expressed. This completes the proof. q.e.d.

The lemma and Theorem 1.4 imply

Proposition 10.11. *For $d_a + d_{a+1} > n > 2d_a$ with $r > a > 1$, if any s-step flag variety for $s < r$ admits a rank 1 point, then $X = F(d_1, .., d_r, n)$ admits one as well.*

Case 6. Assume $n = 2d_a$ with $r > a > 1$. Consider

$$\begin{aligned} X_1 = F(d_1, .., d_a) &\equiv F(d_1, .., d_a, d_a) \hookrightarrow F(d_1, .., d_a, n), \\ (E^\bullet) &\mapsto (0_{n-d_a} \oplus E^\bullet) \\ X_2 = F(d_{a+1} - d_a, .., d_r - d_a, n - d_a) &\hookrightarrow F(d_{a+1}, .., d_r, n), \\ (E^\bullet) &\mapsto (E^\bullet \oplus \mathbb{C}^{d_a}). \end{aligned}$$

Here we view $\mathbb{C}^n = \mathbb{C}^{n-d_a} \oplus \mathbb{C}^{d_a}$. Let M_1, M_2, M be the Stiefel bundles over X_1, X_2, X respectively. Let f_1, f_2 be rank 1 points of X_1, X_2 respectively with the hyperplane properties

$$(x_{J_1} \cdots x_{J_a})|f_1,\ (x_{J'_{a+1}} \cdots x_{J'_r})|f_2 \tag{10.16}$$

for some $J_i \subset (1, 2, .., n - d_a)$ with $|J_i| = d_i$ $(i = 1, ..., a - 1)$ and $J_{a-1} = (d_a - d_{a-1} + 1, .., d_a)$, and for some $J'_i \subset (1, 2, .., n - d_a)$ with $|J'_i| = d_i - d_a$ $(i = a + 1, .., r)$ and $J'_r = (n - d_r + 1, .., n - d_a)$. Put $J := (n - d_a + 1, .., n)$, $J_i := J'_i \cup J$, $i = a + 1, .., n$, and

$$f := \bar{f}_1 \cdot \tilde{f}_2 \cdot (x_J)^{d_{a+1} - d_{a-1} - 1}(x_{1,..,d_a}). \tag{10.17}$$

Then f has the hyperplane property

$$(x_{J_1} \cdots \widehat{x_{J_a}} \cdots x_{J_r} x_J)|f. \tag{10.18}$$

Lemma 10.12. *The special section $m = (m_r, .., m_1)$ (cf. Lemma 10.1) of $M(f) \to X(f)$ has the following form:*

$$\begin{aligned} m_i &= \begin{bmatrix} m'_i & A_i \\ O & I_{d_a} \end{bmatrix},\ i = a + 1, .., r \\ m_a &= \begin{bmatrix} A_a \\ I_{d_a} \end{bmatrix} \\ m_r \cdots m_a &= \begin{bmatrix} D \\ I_{d_a} \end{bmatrix} \\ m_{a-1} &= D^{-1} m'_{a-1} \\ m_i &= m'_i,\ i = 1, .., a - 2, \end{aligned} \tag{10.19}$$

where D is a GL_{d_a}-valued function, $A_a, .., A_r$ are matrix valued functions, and $(m'_{a-1}, .., m'_1)$, $(m'_r, .., m'_{a+1})$ are matrix valued functions tak-

ing values in the special sections of the $M_1(f_1) \to X_1(f_1)$, $M_2(f_2) \to X_2(f_2)$ respectively.

The proof is a degenerate version of the lemma in Case 5 (with m'_a missing but with $(1, .., d_a)$ play the role of J_a), and will be omitted. The lemma and Theorem 1.4 imply

Proposition 10.13. *For $n = 2d_a$ with $r > a > 1$, if any s-step flag variety for $s < r$ admits a rank 1 point f with the hyperplane property, then $X = F(d_1, .., d_r, n)$ admits one as well.*

Case 7. Assume $n = d_{a-1} + d_a$ with $r > a > 1$. If $a = 2$ then it is Case 4, so we can assume $a \geq 3$ (and $r \geq 4$). Consider

$$X_1 := F(d_1, .., d_{a-2}, d_{a-1}) \hookrightarrow F(d_1, .., d_{a-2}, n),\ E^\bullet \mapsto E^\bullet \oplus 0_{n-d_{a-1}},$$
$$X_2 := F(d_{a-1}, .., d_r, n).$$

Here we view $\mathbb{C}^n = \mathbb{C}^{d_{a-1}} \oplus \mathbb{C}^{n-d_{a-1}}$. Let M_1, M_2, M be the Stiefel bundles over X_1, X_2, X respectively. Let f_1, f_2 be rank 1 points of X_1, X_2 respectively with the hyperplane properties

$$(x_{J_1} \cdots x_{J_{a-2}})|f_1,\ ((x_{J_{a-1}})^{d_{a-1}} x_{J_a} \cdots x_{J_r})|f_2, \tag{10.20}$$

for some $J_i \subset (1, 2, .., d_{a-1})$ with $|J_i| = d_i$ $(i = 1, ..., a-2)$, and for some $J_i \subset (1, 2, .., n)$ with $|J_i| = d_i$ $(i = a-1, .., r)$ and $J_{a-1} = (1, .., d_{a-1})$. Note that such an f_2 exists by Proposition 10.7 in Case 4, if any s-step flag variety for $s < r$ admits a rank 1 point with the hyperplane property.

Put

$$f = \bar{f}_1 \cdot \bar{f}_2 \cdot (x_{J_{a-1}})^{-d_{a-2}} \in \Gamma(X, \omega_X^{-1}). \tag{10.21}$$

Then f has the hyperplane property

$$(x_{J_1} \cdots x_{J_r})|f. \tag{10.22}$$

Lemma 10.14. *We have an $H = GL_{d_r} \times \cdots \times GL_{d_1}$ equivariant isomorphism*

$$\begin{aligned} &M_1(f_1) \times M_2(f_2) \to M(f) \\ &(m'_{a-2}, .., m'_1), (m'_r, .., m'_{a-1}) \\ &\quad \mapsto m = (m'_r, .., m'_{a-1}, D^{-1} m'_{a-2}, m'_{a-3}, .., m'_1), \end{aligned}$$

where D is the J_{a-1}-block of $m'_r \cdots m'_{a-1}$. Hence the map descends to an isomorphism

$$X_1(f_1) \times X_2(f_2) \to X(f).$$

The proof is almost identical to the lemmas in Cases 1 and 3, and will be omitted. The lemma and Theorem 1.4 imply

Proposition 10.15. *For $d_{a-1} + d_a = n$ with $r > a > 1$, if any s-step flag variety for $s < r$ admits a rank 1 point with the hyperplane property, then $X = F(d_1, .., d_r, n)$ admits one as well.*

Now combining the propositions in all Cases 1–7 yields a complete recursive procedure for constructing a rank 1 point with the hyperplane property for any r-step flag variety, proving Corollary 1.6.

Example 10.16. Consider $X = F(1,2,3,5)$, which belongs in Case 3. Let $X_1 = F(1,2)$ and take $f_1 = x_1x_2$. Let $X_2 = F(2,3,5)$, which belongs in Case 2, and we can take $f_2 = (x_{12})^3(x_{345})^3$ as a rank 1 point of X_2, by Proposition 10.4. Therefore,

$$f = x_1x_2(x_{12})^3(x_{345})^3(x_{12})^{-1}$$

is rank 1 point of X according to the construction in Case 3.

Example 10.17. Consider the flag variety of SL_5, $X = F(1,2,3,4,5)$, which belongs in Case 7 with $a = 3$. Let $X_1 = F(1,2)$ and take $f_1 = x_1x_2$. Let $X_2 = F(2,3,4,5) \simeq F(1,2,3,5)$, which is the preceding example. Applying this isomorphism to the rank 1 point there, we get $f_2 = x_{2345}x_{1345}(x_{345})^2(x_{12})^3$ as a rank 1 point of X_2. Therefore,

$$f = x_1x_2x_{2345}x_{1345}(x_{345})^2(x_{12})^3(x_{12})^{-1}$$

is a rank 1 point of X according to the construction in Case 7.

Appendix A. Theory of D-modules

We recall the theory of *algebraic* D-modules. A standard reference is [**6**].

Let X be an algebraic variety over k of characteristics zero. Let $\mathrm{Hol}(D_X)$ be the category of holonomic (left) D-modules on X. Its bounded derived category is denoted by $D^b_h(X)$.

Let $f : X \to Y$ be a morphism, there are the following pairs of adjoint (derived) functors (following the notation of Borel's book)

$$f^+ : D^b_h(Y) \rightleftharpoons D^b_h(X) : f_+, \qquad f_! : D^b_h(X) \rightleftharpoons D^b_h(Y) : f^!.$$

Recall the definition of f_+ in the following cases (assuming X and Y are smooth): in the case, there is an $f^{-1}D_Y \times D_X$-bimodule $D_{Y\leftarrow X}$ on X, and

$$f_+(M) = Rf_*(D_{Y\leftarrow X} \otimes^L M).$$

Without mentioning the exact definition of this bimodule $D_{Y\leftarrow X}$, we concentrate on the following special cases. Let $d_{X,Y} = \dim X - \dim Y$.

(i) $f : X \to Y$ is smooth. Then f_+ (up to shift) is the usual construction of the Gauss–Manin connection. I.e.

$$f_+(M) = Rf_*(M \otimes \Omega^\bullet_{X/Y}[d_{X,Y}]).$$

In particular, $H^i f_+\mathcal{O}_X$ is the D-module on Y formed by the $(i+d_{X,Y})$th relative De Rham cohomology. In particular, if f is an open embedding, then $f_+(M) = Rf_*M$ as quasi-coherent sheaves on Y. Observe that under the this normalization of the cohomological degrees, $H^0 f_+\mathcal{O}_X$ is the usual "middle dimension" cohomology of the family $f : X \to Y$.

Example A.1. A particular example: $j : \mathbb{G}_m = \mathrm{Spec}k[x, x^{-1}] \to \mathbb{A}^1 = \mathrm{Spec}k[x]$ the open embedding. Then $j_+\mathcal{O}_{\mathbb{G}_m}$ as a D-module on $\mathbb{A}^1$ is isomorphic to $k[x, \partial_x]/(x\partial_x + 1)$.

(ii) $f : X \to Y$ is a closed embedding given by the ideal $\mathcal{I}$. Then

$$f_+(M) = f_*(D_Y/D_Y\mathcal{I} \otimes \omega_{X/Y} \otimes M),$$

where $\omega_{X/Y}$ is the relative canonical sheaf $\omega_{X/Y} = \omega_X \otimes (\omega_Y^{-1}|_X)$.

Example A.2. A particular example: let Y be $\mathbb{A}^n = \mathrm{Spec}k[x_1, \dots, x_n]$ and $i : X \to Y$ be the inclusion of the vector space given by $x_1 = \cdots = x_r = 0$. Then $x_{r+1}, \dots, x_n$ form a coordinate system on X. Let $M = \mathcal{O}_X = D_X/D_X(\partial_{r+1}, \dots, \partial_n)$. Then

$$i_+M = D_Y/D_Y(x_1, \dots, x_r, \partial_{r+1}, \dots, \partial_n)$$

called the delta sheaf supported on X, denoted by δ_X.

Observe that there is the following exact sequence of $D_{\mathbb{A}^1}$-modules

$$0 \to \mathcal{O}_{\mathbb{A}^1} \to j_+\mathcal{O}_{\mathbb{G}_m} \to \delta_{\{0\}} \to 0. \tag{A.1}$$

Next, we recall the definition of $f^!$. There is a $D_X \times f^{-1}D_Y$-bimodule $D_{X\to Y}$ on X, and by definition

$$f^!(M) = D_{X\to Y} \otimes^L_{f^{-1}D_Y} f^{-1}M[d_{X,Y}].$$

As quasi-coherent $\mathcal{O}_X$-modules,

$$f^!(M) = Lf^*M[d_{X,Y}].$$

Again, let us mention the following special cases.

(i) $f : X \to Y$ is smooth. In this case, $f^![-d_{X,Y}]$ is exact, and as quasi-coherent sheaves, $f^![-d_{X,Y}](M) = f^*M$. In particular, if f is an open embedding, then $f^!M = M|_X$.

(ii) $f : X \to Y$ is a closed embedding, given by the ideal sheaf $\mathcal{I}$. In this case

$$H^0f^!(M) \otimes \omega_{X/Y} = \{m \in M \mid xm = 0 \text{ for any } x \in \mathcal{I}\}.$$

The following distinguished triangle generalizes (A.1): Let $i : X \to Y$ be a closed embedding and $j : U \to Y$ be the complement:

$$i_+i^!M \to M \to j_+j^!M \to . \tag{A.2}$$

Indeed, in the case $Y = \mathbb{A}^1$ and $X = \mathbb{G}_m$, $M = \mathcal{O}_{\mathbb{A}^1}$, we recover (A.1).

The following theorem (Kashiwara's lemma) is of fundamental importance,

Theorem A.3. *Let $i : X \to Y$ be a closed embedding.*

(i) If M is a D_Y-module, set-theoretically supported on X. Then $H^ii^!M = 0$ for $i > 0$.

(ii) Let $D_Y-\mathrm{Mod}_X$ be the category of D_Y-modules, set-theoretically supported on X, and $D_X-\mathrm{Mod}$ be the category of D_X-modules. Then there is an equivalence of categories

$$i_+ : D_X-\mathrm{Mod} \rightleftharpoons D_Y-\mathrm{Mod}_X : H^0 i^!.$$

In the sequel, we will make use of the following notation: let $i : X \to Y$ be a locally closed embedding. If M is a D-module on Y, set-theoretically supported on $\bar{X}$, then $H^0 i^! M$ will be denoted by $M|_X$.

This finishes the discussion of the functors $f_+, f^!$. Then $f_!$ is defined to be the left adjoint of $f^!$ and f^+ is defined to be the left adjoint of f_+. Recall that there is the duality functor $\mathbb{D}_X : D^b_h(X) \to D^b_h(X)$. We can also express $f^+ = \mathbb{D}_X f^! \mathbb{D}_Y$ and $f_! = \mathbb{D}_Y f_+ \mathbb{D}_X$. It is known that

(i) If $f : X \to Y$ is a closed embedding (or more generally if f is proper), $f_! = f_+$.

(ii) If $f : X \to Y$ is an open embedding, $f^! = f^+$.

Remark A.1. The definitions of $f_+, f^!$ do not require the holonomicity, and therefore they are defined on the whole category of (not necessarily holonomic) D-modules. However, as functors on the whole category of D-modules, they do not admit adjoint functors and therefore $f_!, f^+$ are not defined in general.

Example A.4. Let $j : \mathbb{G}_m \to \mathbb{A}^1$ as before. One can show that $j_! \mathcal{O}_{\mathbb{G}_m} \simeq k[x, \partial]/x\partial$.

The dual version of (A.1) is

$$0 \to \delta_{\{0\}} \to j_! \mathcal{O}_{\mathbb{G}_m} \to \mathcal{O}_{\mathbb{A}^1} \to 0, \tag{A.3}$$

and the dual version of (A.2) is

$$j_! j^! M \to M \to i_+ i^+ M \to . \tag{A.4}$$

Now let $k = \mathbb{C}$. Let $D^b_{rh}(X)$ be the bounded derived category of holonomic D-modules with regular singularities, and let $D^b_c(X^{an})$ be the bounded derived category of constructible sheaves on X^{an} (we denote X equipped with the classical topology by X^{an}). Then Riemann–Hilbert correspondence is an equivalence

$$\begin{aligned}&\mathrm{RH} : D^b_{rh}(X) \simeq D^b_c(X),\\ &\mathrm{RH}(M) = \omega_{X^{an}} \otimes^L M^{an} = \Omega^\bullet_{X^{an}} \otimes M^{an}[\dim X],\end{aligned}$$

where $\omega_{X^{an}}$ is the canonical sheaf on X^{an}, regarded as a right D-module via Lie derivative, and the derived tensor product is over $D_{X^{an}}$. This correspondence is compatible with the six operation functors. In particular,

$$\mathrm{RH}\, f_+ \simeq f_* \,\mathrm{RH}, \quad \mathrm{RH}\, f_! \simeq f_! \,\mathrm{RH}, \quad \mathrm{RH}\, f^! \simeq f^! \,\mathrm{RH}, \quad \mathrm{RH}\, f^+ \simeq f^* \,\mathrm{RH}\,.$$

If M is a plain D-module, then $\mathrm{RH}(M)$ is a perverse sheaf on X^{an}. While the above equivalence is covariant, sometimes one also consider the contravariant version

$$\mathrm{Sol} : D^b_{rh}(X) \simeq D^b_c(X)^{op}, \qquad \mathrm{Sol}(M) = R\mathrm{Hom}_{D_{X^{an}}}(M^{an}, \mathcal{O}_{X^{an}}).$$

The relation between Sol and RH is $\mathrm{RH} = \mathrm{Sol}\,\mathbb{D}_X[\dim X]$.

Remark A.2. Let M be a D-module on X. In the paper we also talk about the solution sheaf of M, by which we mean the classical (non-derived) solutions of M, and is defined as

$${}^{cl}\,\mathrm{Sol}(M) = \mathrm{Hom}_{D_{X^{an}}}(M^{an}, \mathcal{O}_{X^{an}}).$$

This is a plain sheaf on X^{an}.

Next, we discuss background materials on equivariant D-modules, most which can be found in [**6**][**15**]. Let G be a connected algebraic group and $\mathfrak{g} = \mathrm{Lie}G$. Let us regard $\mathfrak{g}$ as right invariant vector fields on G, and for a Lie algebra homomorphism $\chi : \mathfrak{g} \to k$, we define a character D-module on G by

$$\mathcal{L}_\chi = D_G/D_G(\xi + \chi(\xi), \xi \in \mathfrak{g}). \tag{A.5}$$

This is a rank one local system on G. In particular, it is holonomic. It is called a character sheaf because if we denote by $\mathrm{mult} : G \times G \to G$ the multiplication map of G, then there is a canonical isomorphism $\mathrm{mult}^!\,\mathcal{L}_\chi \simeq \mathcal{L}_\chi \boxtimes \mathcal{L}_\chi[\dim G]$ satisfying the cocycle condition under the further !-pullback to $G \times G \times G$.

Let Z be a G-variety and $\mathrm{act} : G \times Z \to Z$ be the action map. A (G, χ)-equivariant, or a G-monodromic against χ, D-module on Z is a D-module on Z together with an isomorphism

$$\theta : \mathrm{act}^!\, M \simeq \mathcal{L}_\chi \boxtimes M[\dim G]$$

satisfying the usual cocycle condition under the further !-pullback to $G \times G \times Z$.

The following lemma is well-known, which can be proved as in [**6**, Theorem 12.11]. See also [**15**, §II.5].

Lemma A.5. *Assume that there are only finitely many orbits under the action of G on Z, then any (G, χ)-equivariant D-module is holonomic. In addition, if $\mathcal{L}_\chi$ is regular singular, then any (G, χ)-equivariant D-module is regular singular.*

We will need the following lemma. Let $U\mathfrak{g}$ be the universal enveloping algebra of $\mathfrak{g}$. Then χ defines a one-dimensional $U\mathfrak{g}$-module, denoted by k_χ. Note that if Z is a G-variety, we have the corresponding infinitesimal action $da : \mathfrak{g} \to T_Z$, which extends to $U\mathfrak{g} \to D_Z$.

Lemma A.6. *The D-module*

$$D_{Z,\chi} = D_Z/D_Z(da(\xi) + \chi(\xi), \xi \in \mathfrak{g}) = (D_Z \otimes k_\chi) \otimes_{U\mathfrak{g}} k$$

is a natural (G,χ)-equivariant D-module on Z.

More generally, note that D_Z is naturally G-equivariant as $\mathcal{O}$-modules, i.e., there is an isomorphism of $\mathcal{O}$-modules $\theta : \operatorname{act}^* D_Z \simeq p_Z^* D_Z$ satisfying the cocycle condition. Let $I \subset D_Z$ be a G-invariant left ideal, then

$$D_Z/I + D_Z(da(\xi) + \chi(\xi), \xi \in \mathfrak{g})$$

is (G,χ)-equivariant.

Note that in the above lemma, we do not need to assume that G acts on Z with finitely many orbits. See [**15**, §II.3].

Note that if $i : H \to G$ is a connected closed subgroup, $i^! \mathcal{L}_\chi[-\dim H] = D_H/D_H(\xi + \chi(\xi), \xi \in \mathfrak{h}) = \mathcal{L}_{\chi|_\mathfrak{h}}$. We have the following simple observation.

Lemma A.7. *Let $Z = G/H$ be a homogeneous G-variety. Let $\chi : \mathfrak{g} \to k$ be a Lie algebra homomorphism and $\mathcal{L}_\chi$ be the rank character D-module on G as in* (A.5)*. Then if $\mathcal{L}_{\chi|\mathfrak{h}} \neq \mathcal{O}_{H^\circ}$, where H° is the neutral connected component of H, there is no D-module on Z, equivariant with respect to G against χ.*

Proof. Let M be a non-zero (G,χ)-equivariant D-modules on Z. Let $i : H^\circ \to G$ be the inclusion, and $i_e : eH \to Z$ be the inclusion of the identity coset. Consider the diagram

$$\begin{array}{ccc} H^\circ \times eH & \longrightarrow & eH \\ {\scriptstyle i\times i_e}\big\downarrow & & \big\downarrow{\scriptstyle i_e} \\ G \times Z & \longrightarrow & Z. \end{array}$$

Then $i^! \mathcal{L}_\chi \otimes i_e^! M = (i \times i_e)^! \operatorname{act}^! M = \mathcal{O}_H \otimes i_e^! M[\dim Z]$. Therefore, $\mathcal{L}_{\chi|_\mathfrak{h}} = \mathcal{O}_H$. q.e.d.

Example A.8. Let $\lambda \in k^\times$, and let $\mathcal{L}_\lambda$ be the D-module on $\mathbb{G}_m$ given by $x\partial + \lambda$. I.e. $\mathcal{L}_\lambda$ is the local system on $\mathbb{G}_m$ with monodromy $\exp(-2\pi\sqrt{-1}\lambda)$ (via the Riemann–Hilbert correspondence if $k = \mathbb{C}$). This is a character D-module on G with $\chi(x\partial) = \lambda$. If $\lambda \in \mathbb{Z}$, then $\mathcal{L}_\lambda \simeq \mathcal{O}_{\mathbb{G}_m}$. Let $j : \mathbb{G}_m \to \mathbb{A}^1$ be the open embedding. Then both $j_+\mathcal{L}_\lambda$ and $j_!\mathcal{L}_\lambda$ are (G,λ)-equivariant D-modules on $\mathbb{G}_m$. If λ is not an integer, then $j_!\mathcal{L}_\lambda \simeq j_+\mathcal{L}_\lambda$. In this case, this D-module is irreducible on $\mathbb{A}^1$.

Our last topic is the Fourier transform. Let "e^x" be the character D-module on $\mathbb{A}^1$ defined by $\partial - 1$. Let V be a vector space and $V^\vee$ be its dual. We have the natural pairing

$$m : V \times V^\vee \to \mathbb{A}^1.$$

The pullback of e^x along m is still denoted by e^x, regarded as a plain D-module on $V \times V^\vee$. Let $p_V, p_{V^\vee}$ be the projections of $V \times V^\vee$ to the two factors. The Fourier transform is defined as

$$\mathcal{F}our(M) = p_{V^\vee,+}(p_V^!(M) \otimes e^x).$$

Fourier transform $\mathcal{F}our$ is an exact functor, and can be described in the following simple way. Let M be a D-module on V, and therefore is identified with a module over the Weyl algebra $k[a_1, \dots, a_n, \partial_{a_1}, \dots, \partial_{a_n}]$. Then $\mathcal{F}our(M)$ as a vector space is identified with M, and the D-module structure is given by $a_i^* m = \partial_{a_i} m$ and $\partial_{a_i^*} = -a_i m$. In other words, if we denote the ring homomorphism

$$\widehat{\ }: D_V \to D_{V^\vee}, \quad \widehat{a_i} = -\partial_{a_i^*}, \ \widehat{\partial_{a_i}} = a_i^*, \tag{A.6}$$

then $\mathcal{F}our(M) = D_{V^\vee} \otimes_{D_V} M$. See [**7**, p. 85].

Example A.9. Let $W \subset V$ be a vector subspace, and $W^\perp$ be the orthogonal complement of W in $V^\vee$. Then $\mathcal{F}our(\delta_W) = \delta_{W^\perp}$.

Example A.10. More generally, let $i : W \subset V$ be a vector subspace, and $0 \to W^\perp \to V^\vee \xrightarrow{p} W^\vee \to 0$ be the dual sequence. Let M be a D-module on W. Then

$$\mathcal{F}our(i_+ M) = p^! \mathcal{F}our(M)[\dim W - \dim V].$$

Example A.11. Let $V = \mathbb{A}^1$ and we identify $V^\vee = \mathbb{A}^1$ via the natural multiplication $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$. Then under the Fourier transform, the exact sequence (A.1) becomes (A.3).

Example A.12. Recall the character D-module $\mathcal{L}_\lambda$ on $\mathbb{G}_m$. Let $j : \mathbb{G}_m \to \mathbb{A}^1$ be the open immersion. Then

$$\mathcal{F}our(j_+ \mathcal{L}_\lambda) = j_! \mathcal{L}_{-\lambda+1}.$$

Fourier transform preserves holonomicity. If M is holonomic, the we can also write

$$\mathcal{F}our(M) = p_{V^\vee,!}(p_V^*(M) \otimes e^x).$$

However, Fourier transform does not necessarily preserves the regular singularity. For example, the Fourier transform of the delta sheaf on $\mathbb{A}^1$ supported at $1 \in \mathbb{A}^1(k)$ is e^x. However, under certain circumstance, one can show that $\mathcal{F}our(M)$ is regular singular. Let $\mathbb{G}_m$ act on V via homotheties, i.e. $\mathrm{mult} : \mathbb{G}_m \times V \to V, \mathrm{mult}(a, v) = av$. Let $\lambda : \mathrm{Lie}\mathbb{G}_m \to k$ be a map. Recall the notion of $(\mathbb{G}_m, \lambda)$-equivariant D-modules. We say a holonomic D-module on V to be $\mathbb{G}_m$ monodromic if each of its irreducible constitutes is $(\mathbb{G}_m, \lambda)$-equivariant for some λ. Observe that e^x is not $\mathbb{G}_m$-monodromic.

Let $D^b_{rh,m}(V)$ be the full subcategory of $D^b_{rh}(V)$ whose cohomology sheaves are regular holonomic and $\mathbb{G}_m$-monodromic.

Lemma A.13. *The Fourier transform restricts to an equivalence*

$$\mathcal{F}our : D^b_{rh,m}(V) \simeq D^b_{rh,m}(V^\vee).$$

Proof. [**7**, Theorems 7.4, 7.24]. q.e.d.

Fourier transform can be generalized to family versions. Let X be a base variety, and $\mathbb{V}$ a vector bundle over X, $\mathbb{V}^\vee$ the dual bundle, so there is

$$m : \mathbb{V} \times_X \mathbb{V}^\vee \to \mathbb{A}^1.$$

Then one can define

$$\mathcal{F}our_X(M) = p_{\mathbb{V}^\vee,+}(p^!_{\mathbb{V}}(M) \otimes e^x).$$

Note that the family version of Example A.10 still holds. More precisely, let $i : \mathbb{W} \subset \mathbb{V}$ be a subbundle on $p : \mathbb{V}^\vee \to \mathbb{W}^\vee$ be the dual map. Then

$$\mathcal{F}our_X(i_+M) = p^!\mathcal{F}our_X(M)[\operatorname{rk}\mathbb{W} - \operatorname{rk}\mathbb{V}]. \tag{A.7}$$

Note that family version of Fourier transforms commute with base change. Namely, if $f : Y \to X$ is a map. Then $\mathcal{F}our_Y(f^!M) = f^!\mathcal{F}our_X(M)$.

Let us consider the family version of Example A.12. So we assume that $\mathbb{V} = \mathbb{L}$ is a line bundle, on which $\mathbb{G}_m$ acts by homotheties. Let $\mathring{\mathbb{L}} = \mathbb{L} - X$, where X is regarded as the zero section of $\mathbb{L}$. Let $\mathbb{L}^\vee$ be the dual vector bundle of $\mathbb{L}$ and $\mathring{\mathbb{L}}^\vee$ is defined similarly. Let M be a $(\mathbb{G}_m, \lambda)$-equivariant D-module on $\mathring{\mathbb{L}}$.

The following lemma is useful.

Lemma A.14. *Let X be proper and $\mathbb{V} = X \times V$ be the trivial bundle over X. Let $\pi : X \times V \to V$ and $\pi^\vee : X \times V^\vee \to V^\vee$ be the projections. Then*

$$\mathcal{F}our \circ \pi_! \simeq \pi^\vee_! \circ \mathcal{F}our_X.$$

Proof. This follows from the base change theorem for D-modules (cf. [**6**, VI, §8]). Namely, as X is proper, $\pi_+ = \pi_!$, etc. We have the following commutative diagrams with both squares Cartesian:

$$\begin{tikzcd}
 & X \times V \times V^\vee \arrow[dl, "p_V"'] \arrow[d, "\pi^{V,V^\vee}"] \arrow[dr, "p_{V^\vee}"] & \\
X \times V \arrow[d, "\pi"'] & V \times V^\vee \arrow[dl, "p_V"'] \arrow[dr, "p_{V^\vee}"] & X \times V^\vee \arrow[d, "\pi^\vee"] \\
V & & V^\vee.
\end{tikzcd}$$

Then

$$\begin{aligned}
\mathcal{F}our(\pi_!(M)) &= p_{V^\vee,+}(p^!_V \pi^\vee_+(M) \otimes e^x) \\
&= p_{V^\vee,+}(\pi^{V,V^\vee}_+ p^!_V(M) \otimes e^x) \\
&= p_{V^\vee,+}\pi^{V,V^\vee}_+(p^!_V(M) \otimes e^x) \\
&= \pi^\vee_+ \mathcal{F}our_X(M).
\end{aligned}$$

q.e.d.

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