J. DIFFERENTIAL GEOMETRY
104 (2016) 239-273

PSEUDONORMS AND THEOREMS OF TORELLI TYPE

Chen-Yu Chi

Abstract

We show that for every positive integer n, there exists an explicit positive integer c_n that only depends on n such that if M and M' are canonically polarized complex projective manifolds of dimension n and if $H^0(M, mK_M)$ and $H^0(M', mK_{M'})$ are linearly isometric with respect to the pseudonorm $\langle\!\langle\ \rangle\!\rangle$ for some $m \geqslant c_n$, then M and M' are isomorphic. This generalizes a result of Royden for compact Riemann surfaces of genus greater than or equal to 2. The same approach is used to prove similar and weaker results for projective manifolds with nonnegative Kodaira dimension. We also introduce a kind of singular index for singularities of pairs that refines the traditional log canonical threshold.

1. Introduction

Global birational geometry investigates the classification of algebraic varieties of arbitrary dimensions. For a complex projective variety M, the m-pluricanonical spaces $H^0(M, mK_M)$ form a typical set of birational invariants of M and play an essential role in understanding its global geometry. If the Kodaira dimension $\kappa(M) \geqslant 0$, the linear systems $|mK_M|$ provide rational maps

$$\varphi_{|mK_M|} : M \dashrightarrow \mathbb{P}H^0(M, mK_M)^*,$$

the closure of whose images stablizes birationally to Proj $R(M, K_M)$ when m is sufficiently large and sufficiently divisible [**17**], which is the canonical model if M is of general type [**2**]. Therefore, given two complex projective manifolds of general type, a necessary and sufficient condition for them to be birational is that their canonical rings are isomorphic as graded **C**-algebras. The canonical ring of M is known to be finitely generated and it is its multiplicative structure that governs the birational geometry of M. This work studies the following natural question: Given two complex projective manifolds of general type, what kind of isomorphisms between their m-canonical spaces manage to give back a birational map between them? More specifically, we look for more analytic or geometric conditions on pluricanonical spaces instead of the multiplicative structure of canonical rings.

Received 3/4/2015.

A good solution to the above question will be of fundamental importance in classifying projective varieties of general type. In [**4**], S.-T. Yau has initiated a program of establishing a correspondence between birational equivalence classes of complex projective varieties and some special type of (possibly singular) pseudoconvex hypersurfaces in finite-dimensional complex vector spaces. This paper is actually the first step to realize Yau's program. We study a particular kind of norm-like functions defined on the pluricanonical spaces, called pseudonorms (to be defined below). The unit level sets of these functions will appear to be those pseudoconvex hypersurfaces in Yau's program. The next steps in realizing the program will be to study (affine) differential geometric invariants of these hypersurfaces and to give characterizations of the pseudonorms. We plan to treat these issues in separate works. After all these steps are achieved, classifying projective varieties birationally will be equivalent to classifying real pseudoconvex hypersurfaces of certain type in finite-dimensional vector spaces using (affine) differential geometric invariants. We expect our results to shed new light on the classification of projective algebraic manifolds, especially surfaces, of general type.

The origin of our approach dates back to the study of Teichmüller spaces by H. L. Royden. In 1971, he proved the following theorem:

Theorem 1.1. ([**18**]) *Let C and C' be compact Riemann surfaces of genus $g \geqslant 2$. If $H^0(C, 2K_C)$ is isometric to $H^0(C', 2K_{C'})$ with respect to the canonical norm $\| \ \|_2$ ($\langle\!\langle \ \rangle\!\rangle_2$ in Definition 1.1), then C is isomorphic to C'.*

For a complex projective manifold M of dimension n, we consider analogues of $\| \ \|_2$ that are pseudonorms $\langle\!\langle \ \rangle\!\rangle_m$ defined on its mth pluricanonical spaces $H^0(M, mK_M)$:

Definition 1.1. (1) For any Lebesgue measurable section s of $K_M^{\otimes m}$ we define a top form $\langle s \rangle_m$ as follows. On each coordinate chart $(U; \{z_j = x_j + iy_j\}_{j=1}^n)$, $s = f(z_1, \dots, z_n)(dz_1 \wedge \cdots \wedge dz_n)^{\otimes m}$ for some measurable function f. We define $\langle s \rangle_m$ by setting

$$\langle s \rangle_m|_U = |f(z_1, \dots, z_n)|^{\frac{2}{m}} dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n.$$

It is clear that $\langle s \rangle_m$ is a well-defined real measurable (n, n)-form that is nonnegative with respect to the canonical orientation associated to the complex structure on M. (2) The pseudonorm of s is defined to be

$$\langle\!\langle s \rangle\!\rangle_m = \int_M \langle s \rangle_m \in \mathbf{R}_{\geqslant 0} \cup \{\infty\}.$$

If M is compact, the restriction to holomorphic sections gives a map $\langle\!\langle \ \rangle\!\rangle_m : H^0(M, mK_M) \to \mathbf{R}_{\geqslant 0}$.

Note that $\langle\!\langle\ \rangle\!\rangle_m$ possesses all the properties of a norm except that it has different positive homogeneity. It defines a translation invariant distance structure on $H^0(M, mK_M)$ in the usual way, and we can speak of isometries with respect to it. A birational map $\psi : M \dashrightarrow M'$ uniquely determines an isomorphism $\psi^* : H^0(M', mK_{M'}) \to H^0(M, mK_M)$ for all m. It is a consequence of the change of variable formula for integration that ψ^* is a linear isometry with respect to $\langle\!\langle s \rangle\!\rangle_m$, i.e., the pseudonormed pluricanonical spaces $\big(H^0(M, mK_M), \langle\!\langle\ \rangle\!\rangle_m\big)$ form a set of birational invariants of M. Note that $\langle\!\langle\ \rangle\!\rangle_1$ is the square of the norm associated to a hermitian product on $H^0(M, K_M)$ (the Bergman pairing), and hence its unit level set is the round hermitian sphere, which carries no interesting birational geometric information. Therefore, when talking about pluricanonical spaces $H^0(M, mK_M)$ in the following, we always assume that $m \geqslant 2$.

Inspired by Royden's result, S.-T. Yau proposed to consider the following statement, which we denote as $(*)_m$ for any two projective manifolds of general type M and M' and each $m \in \mathbf{N}$:

> If $\iota : H^0(M', mK_{M'}) \to H^0(M, mK_M)$ is a linear isometry with respect to $\langle\!\langle\ \rangle\!\rangle_m$, then there exists a birational map $M \overset{\psi}{\dashrightarrow} M'$ and a unit complex number c such that $c\iota = \psi^*$.

Royden's result says that $(*)_2$ holds if M and M' are algebraic curves. One should note that there are no obvious implications between the $(*)_m$'s for different m, even for the case of curves. To study the statement $(*)_m$, we first note that if ψ maps M birationally to M', then we have the following commutative diagram of rational maps:

$$\begin{tikzcd} M \arrow[r, dashed, "\psi"] \arrow[d, dashed, "\phi_{|mK_M|}"'] & M' \arrow[d, dashed, "\phi_{|mK_{M'}|}"] \\ \mathbb{P}H^0(M, mK_M)^* \arrow[r, "\mathbb{P}(\psi^*)^*", "\sim"'] & \mathbb{P}H^0(M', mK_{M'})^*. \end{tikzcd}$$

Now suppose M and M' are both of general type, but are not assumed to be birational to each other. Suppose $\iota : H^0(M', mK_{M'}) \to H^0(M, mK_M)$ is an isomorphism. Consider the diagram below, where I is the isomorphism induced by ι. If m is sufficiently large the vertical pluricanonical maps map M and M' birationally to their images (cf. [**9**] and [**19**]). If I maps im $\phi_{|mK_M|}$ into the closure of im $\phi_{|mK_{M'}|}$, then there exists a rational map $M \dashrightarrow M'$ that factorizes the diagram

$$\begin{tikzcd} M \arrow[r, dotted, "?"] \arrow[d, dashed, "\phi_{|mK_M|}"'] & M' \arrow[d, dashed, "\phi_{|mK_{M'}|}"] \\ \mathbb{P}H^0(M, mK_M)^* \arrow[r, "I", "\sim"'] & \mathbb{P}H^0(M', mK_{M'})^*. \end{tikzcd} \tag{1.1}$$

For manifolds M and M' that are not necessarily of general type, one can consider a similar statement $(*')_m$:

If $\iota : H^0(M', mK_{M'}) \to H^0(M, mK_M)$ is a linear isometry with respect to $\langle\!\langle\ \rangle\!\rangle_m$, then the map

$$I = \mathbb{P}(\iota)^* : \mathbb{P}H^0(M, mK_M)^* \to \mathbb{P}H^0(M', mK_{M'})^*$$

identifies the closure of $\operatorname{im} \varphi_{|mK_M|}$ with that of $\operatorname{im} \varphi_{|mK_{M'}|}$.

The following are our main results, which can be viewed as theorems of Torelli type for birational or biregular equivalence.

Theorem 1.2. *For any pair of projective manifolds M and M' both of nonnegative Kodaira dimension, there exists a positive integer $C(M, M')$ depending on M and M' such that $(*')_{mC(M,M')}$ holds for positive integers $m > 2\max\{\dim M, \dim M'\} + 2$.*

The Cartier index of a projective variety with $\mathbf{Q}$-Cartier canonical divisor is defined to be the smallest positive integer j such that the jth multiple of the canonical divisor is Cartier. It is a consequence of [**2**] that every complex manifold of general type has a minimal model, which is $\mathbf{Q}$-factorial, and any two birational minimal models have the same Cartier index. We define the Cartier index of a complex manifold M of general type, denoted by j_M, to be that of any of its minimal model. The first part of the following result was conjectured by Yau and clearly follows from the above theorem and the assumption that K_M and $K_{M'}$ are big:

Theorem 1.3. *(1) For any pair of projective manifolds of general type M and M' of dimension n, there exists a positive integer $c(j_M, j_{M'})$ that only depends on the Cartier indices j_M and $j_{M'}$ of M and M' such that $(*)_{mc(j_M,j_{M'})}$ holds for positive integers $m > 2n + 1$.*

The form of $c(j_M, j_{M'})$ will be given in Section 4. Note that the statement "ψ^* and ι differ from each other by a constant factor" is obvious since the images of $\varphi_{|mK_M|}$ and $\varphi_{|mK_{M'}|}$ generate their respective ambient projective spaces.

For every $n \in \mathbf{N}$, it is known that there exists $c_n \in \mathbf{N}$ that depends only on n such that, for any canonically polarized manifold M of dimension n, mK_M is very ample if $m \geqslant c_n$. For example, c_n can be taken to be $\lceil \frac{1}{2}(n^3 + 2n^2 + 5n + 8) \rceil$ by [**1**] (see also [**5**]). We fix such a $c_n \geqslant 2$. Our technique coupled with effective very ampleness gives a uniform result that directly generalizes Theorem 1.1:

Theorem 1.4. *If M and M' are canonically polarized manifolds of dimension n with isometric m-pluricanonical spaces with respect to $\langle\!\langle\ \rangle\!\rangle_m$ for some $m \in \{ab | a \geqslant (2n+1), b \geqslant c_n\}$, then M and M' are isomorphic (and the induced isometry between their m-pluricanonical spaces equals to the given one up to the multiplication by a unit complex number).*

Now we explain the main idea of the proof of Theorem 1.2. Roughly speaking, we connect the original global problem with local geometry of singularities of pairs via pseudonorms. On the side of local geometry, for a complex manifold M, a point $x \in M$, and hypersurfaces $F \subset D$ in M, we introduce the *relative characteristic index* $\chi_{F,r}(M, D, x)$ to measure how singular D is in M at x with respect to F, which refines the traditional log canonical threshold $\mathrm{lct}_x(M, D)$. According to this singularity index, the points at which D is the most singular in M with respect to F form a nonempty locally analytically Zariski closed subset $C_{F,r}(M, D)$ of M, called the *relative characteristic indicatrix* of D in M with respect to F. Now for each m, let $F_m = \mathrm{Fix}|mK_M|$ and let s_{F_m} be a canonical section of $\mathcal{O}_M(F_m)$. For $\eta_0 \in H^0(M, mK_M)$, let D_{η_0} be the divisor defined by the vanishing of η_0. When D_{η_0} is sufficiently singular, for all $\eta \in H^0(M, mK_M)$, the function

$$t \longmapsto \langle\!\langle \eta_0 + t\eta \rangle\!\rangle_m - \langle\!\langle \eta_0 \rangle\!\rangle_m$$

has an asymptotic expansion as $t \longrightarrow 0$, whose leading coefficient vanishes exactly when $\eta \otimes s_{F_m}^{-1}$ vanishes along $C_{F_m,m}(M, D_{\eta_0})$. On the global side, we consider the diagram (1.1). If the images of vertical maps can be described in terms of $\langle\!\langle\ \rangle\!\rangle_m$, the bottom map I will identify one with another since ι preserves $\langle\!\langle\ \rangle\!\rangle_m$. By definition, for any generic $x \in M$, $\varphi_{|mK_M|}(x)$ is represented by the hyperplane $\{\eta | \eta(x) = 0\}$ in $H^0(M, mK_M)$. If there exists an m-canonical form η_0 for each generic x such that D_{η_0} is sufficiently singular and $C_{F_m,m}(M, D_{\eta_0}) = \{x\}$, then the condition "$\eta(x) = 0$" is equivalent to "the leading coefficient of the asymptotic expansion of $\langle\!\langle \eta_0 + t\eta \rangle\!\rangle_m - \langle\!\langle \eta_0 \rangle\!\rangle_m$ as $t \longrightarrow 0$ vanishes," a condition in terms of $\langle\!\langle\ \rangle\!\rangle_m$. Such η_0 always exists if m is sufficiently large and sufficiently divisible.

It should be mentioned that in the case of a compact Riemann surface C of genus at least 2 the same asymptotic result for $\|\ \|_2$ has been obtained by Royden [**18**]. He took a different viewpoint: instead of the image of the bicanonical map $\varphi_{|2K_C|}$, he considered its dual curve in $\mathbb{P}H^0(C, 2K_C)$. He found a characterization of the dual curve purely in terms of (the asymptotic behavior of) $\|\ \|_2$ on $H^0(C, 2K_C)$. Therefore, the projectivization of a linear isometry between bicanonical spaces of two such Riemann surfaces identifies the dual curves of the images of their bicanonical maps, which are isomorphic to themselves. In higher dimensions we have not found a practical way to generalize along the same line. Instead, we use the pluricanonical maps, which are more natural in birational geometry, and replace Royden's characterization of dual bicanonical curves by Lemma 3.1. In dimension 1, singularities of pairs are much easier to understand, because the multiplicity covers all information. In higher dimensions the situation is much more complicated, and that is why we need to introduce some new singularity indices. See Definition 2.1.

In Section 2 the local asymptotic expansion of certain types of integrals (Theorem 2.1) is introduced, which is the technical core of this paper and whose proof will be postponed to Section 5. Then we apply this local formula to justify the definition of several new singularity indices and to deduce the asymptotic expansion of pseudonorms (Theorem 2.3). In Section 3 we give a general result linking pseudonorms with the images of rational maps associated to sublinear systems of pluricanonical systems, and prove Theorem 1.4 as an application. Finally, we discuss Theorems 1.2 and 1.3 in Section 4.

Theorems 1.3 and 2.1 have been announced in [**4**] without proof. Assuming certain base point free conditions on pluricanonical systems, stronger versions of some special cases of Theorem 1.3 were proven there. A nice uniform result among them is the following theorem, whose proof will not be repeated here:

Theorem 1.5. ([**4**], *Theorem* 4.2) $(*)_m$ *holds for every* $m \geqslant 75$ *if* M *and* M' *are compact complex surfaces of general type.*

Acknowledgments. I am indebted to Professor Shing-Tung Yau for introducing this subject to me and sharing with me many of his bold and imaginative ideas. I am grateful to James McKernan, Mircea Mustaţă, and Chin-Lung Wang for their many valuable suggestions. I would also like to thank Michael Viscardi for reading through a draft of this work carefully and largely improving the writing. Finally, I would like to thank the referee for many helpful suggestions and enormous efforts to point out many errors and to make the expression largely improved.

2. Singularities of pairs and asymptotics of pseudonorms

2.1. A local asymptotic expansion. We will state the main local asymptotic result here, and will deduce from it the global asymptotic property of $\langle\!\langle\ \rangle\!\rangle$ in 2.3. We use the following notation:

$$n \in \mathbf{N}, 0 < p \leqslant 1/2, \Delta = \{(z_1, \dots, z_n) \in \mathbf{C}^n |\ |z_j| < 1, j = 1, \dots, n\},$$
$$g \in \mathcal{C}^\infty(\overline{\Delta}), \phi \in \mathcal{O}(\overline{\Delta}),$$
$$A = (a_1, \dots, a_n) \in \mathbf{R}^n_{\geqslant 0}, B = (b_1, \dots, b_n) \in \mathbf{R}^n,$$
$$l_j = \frac{b_j + 1}{a_j} \text{ if } a_j \neq 0 \text{ and } = \infty \text{ otherwise, } j = 1, \dots, n,$$
$$l(A, B) = \inf \{l_j |\ j = 1, \dots, n\}.$$

We assume that A and B are chosen so that

$$l(A, B) = l_1 = \cdots = l_{\mu(A,B)} < l_{\mu(A,B)+1} \leqslant \cdots \leqslant l_n$$

for some $1 \leqslant \mu(A,B) \leqslant n$. Notice that $l(A,B)$ and $\mu(A,B)$ depend only on the multi-indices A and B. Let

$$x_j = \mathrm{Re}\ z_j \quad \text{and} \quad y_j = \mathrm{Im}\ z_j,$$

$j = 1, \ldots, n$. We abbreviate $(x_1, y_1, \ldots, x_n, y_n)$, $(z_1, \ldots, z_n)$, $z_1^{a_1} \cdots z_n^{a_n}$, $|z_1|^{b_1} \cdots |z_n|^{b_n}$, and $dx_1 dy_1 \cdots dx_n dy_n$ as (X, Y), Z, Z^A, $|Z|^B$, and $dX\, dY$, respectively.

Theorem 2.1. *Suppose $A \in \mathbf{Z}_{\geqslant 0}$ and $B \in \mathbf{R}^n_{\geqslant 0}$. Let $l = l(A,B)$ and $\mu = \mu(A,B)$. For $t \in \mathbf{C}$, we let*

$$\Psi(t) = \int_{\overline{\Delta}} g(X,Y) \big| Z^A + t\phi(Z) \big|^{2p} \big| Z \big|^{2B} dX dY .$$

Then

$$\Psi(t) - \Psi(0) = \begin{cases} O\left(|t| \left(\ln \frac{1}{|t|}\right)^{\mu}\right) & \text{if } \ 2l + 2p \geqslant 1; \\[2ex] c(A,B,\phi)\ |t|^{2l+2p} \left(\ln \frac{1}{|t|}\right)^{\mu-1} & \\ +o\left(|t|^{2l+2p} \left(\ln \frac{1}{|t|}\right)^{\mu-1}\right) & \text{if } \ 2l + 2p < 1 \end{cases}$$

as $t \longrightarrow 0$, where $c(A,B,\phi)$ is a real number depending on ϕ. In the second case we have $c(A,B,\phi) \geqslant 0$, and if $g(0) \neq 0$ then

$$c(A,B,\phi) = 0 \iff \phi(0, \ldots, 0, z_{\mu+1}, \ldots, z_n) \equiv 0.$$

The proof consists of a long computation and will be given in Section 5.

2.2. The relative characteristic index and indicatrix. We introduce several quantities measuring how singular a divisor is at a point in the ambient manifold.

Fix a complex manifold M, effective divisors D and F on M such that $D - F$ is also an effective divisor, and $r > 0$. We first choose a common log resolution $\pi : \widetilde{M} \to M$ for (M, D) and (M, F) and write

$$\pi^* D = \sum_E d_E E,$$

$$\pi^* F = \sum_E f_E E,$$

and

$$K_{\widetilde{M}} = \pi^* K_M + \sum_E j_E E,$$

where E runs over all irreducible subvarieties of $\widetilde{M}$ of codimension 1. For every $x \in M$, we let

$$l_{(D,F),r;\pi}(x) := \inf_{\{E | x \in \pi(E)\}} \frac{j_E + \frac{f_E}{r} + 1}{d_E - f_E}$$

and

$$\mu_{(D,F),r;\pi}(x) := \max \left\{ q \;\middle|\; \begin{array}{c} \exists \text{ distinct irreducible divisors} \\ E_1, \ldots, E_q \text{ in } \widetilde{M} \text{ such that} \\ \frac{j_{E_k} + \frac{f_{E_k}}{r} + 1}{d_{E_k} - f_{E_k}} = l_{(D,F),r;\pi}(x) \\ \text{for } 1 \leqslant k \leqslant q \text{ and } x \in \pi\left(\bigcap_{k=1}^{q} E_k\right) \end{array} \right\}.$$

These quantities are essential in the proof of Theorems 2.1, 3.1, and 1.2. In most of the cases in which we are interested, r will be a positive integer $m > 1$, D will be the divisor associated to some section of mK_M or related line bundles, and F will be the fixed divisor of $|mK_M|$ or some of its sub-linear systems.

Lemma 2.1. *$l_{(D,F),r;\pi}(x)$ and $\mu_{(D,F),r;\pi}(x)$ are independent of the choice of π.*

Proof. First note that $l_{(D,F),r;\pi}(x) = \infty$ if and only if $x \notin \operatorname{supp}(D - F)$. Therefore, we only need to consider the case that $l_{(D,F),r;\pi}(x) < \infty$. Let λ be any positive integer such that $\lambda r \geqslant 2$ and $l_{(D,F),r;\pi}(x)/\lambda < 1$. Suppose $x \in M$ and the local defining functions of D and F are f_D and f_F, respectively. For any log resolution $\pi : \widetilde{M} \to M$, we first choose a coordinate neighborhood $(U, \{w_\alpha = u_\alpha + iv_\alpha\}_{\alpha=1}^n)$ of x with compact closure such that $\pi^{-1}(\overline{U})$ is disjoint from any prime divisor disjoint from $\pi^{-1}(x)$. We can cover $\pi^{-1}(\overline{U})$ by finitely many coordinate charts $(V, \{z_\beta = x_\beta + iy_\beta\}_{\beta=1}^n)$, each biholomorphic to the unit polydisc Δ, in any of which

$$(f_D \circ \pi)(z_1, \ldots, z_n),\ (f_E \circ \pi)(z_1, \ldots, z_n),$$

and

$$\pi^* dw_1 \wedge \cdots \wedge dw_n$$

are of the form

$$d(z_1, \ldots, z_n) z_1^{d_1} \cdots z_n^{d_n},\ f(z_1, \ldots, z_n) z_1^{f_1} \cdots z_n^{f_n},$$

and

$$j(z_1, \ldots, z_n) z_1^{j_1} \cdots z_n^{j_n} dz_1 \wedge \cdots \wedge dz_n,$$

respectively, where d, f, and j are nonvanishing holomorphic functions. Let $D = (d_1, \ldots, d_n)$, $F = (f_1, \ldots, f_n)$, and $J = (j_1, \ldots, j_n) \in \mathbf{Z}^n_{\geqslant 0}$. Finally, we choose smooth functions ρ_V on V with compact support such that the restriction of $\sum_V \rho_V$ to $\pi^{-1}(\overline{U})$ is identically 1 (a partial partition of unity). For any $t \in \mathbf{C}$ and any smooth function ρ on U with compact support, we consider the integral

$$I_{\lambda,\rho}(t) := \int_U \rho \left| f_D^\lambda + t f_E^\lambda \right|^{\frac{2}{\lambda r}} du_1 dv_1 \cdots du_n dv_n,$$

which is independent of the choice of π. By the change of variable formula for integration, it is equal to

$$\sum_V \int_V (\rho\circ\pi)(X,Y)\rho_V(X,Y)\big|d(Z)^\lambda Z^{\lambda D} + tf(Z)^\lambda Z^{\lambda F}\big|^{\frac{2}{\lambda r}}|Z|^{2J}dXdY.$$

Here we have adopted the convention of abbreviation in 2.1. Applying Theorem 2.1 by setting $g = (\rho\circ\pi)\rho|\frac{d}{f}|^{\frac{1}{r}}$, $\phi = \frac{f^\lambda}{d^\lambda}$, $p = \frac{1}{\lambda r}$, $A = \lambda(D-F)$, and $B = J + \frac{1}{r}F$ for each V, we see that $\big(l_{(D,F),r;\pi}(x), \mu_{(D,F),r;\pi}(x)\big)$ is characterized as the unique pair $(l,\mu)\in\mathbf{R}^2$ such that

$$\frac{I^{\lambda,\rho}(t)}{|t|^{\frac{2l+(2/r)}{\lambda}}\left(\ln\frac{1}{|t|}\right)^\mu}$$

tends to a nonzero limit as $t \longrightarrow 0$. For any two log resolutions there exist common $\lambda\in\mathbf{N}$ and $\rho\in\mathcal{C}_c^\infty(U)$ working for both. This completes the proof. q.e.d.

As a consequence we have justified the following definition:

Definition 2.1. Suppose $0 < r$. The log canonical threshold $l_{F,r}(M,D,x)$, the log canonical multiplicity $\mu_{F,r}(M,D,x)$, and the characteristic index $\chi_{F,r}(M,D,x)$ of (M,D) with respect to F of exponent r at $x\in M$ are defined to be $l_{(D,F),r;\pi}(x)$, $\mu_{(D,F),r;\pi}(x)$, and $\big(l_{(D,F),r;\pi}(x), \mu_{(D,F),r;\pi}(x)\big)$, respectively. If M is compact, we define the global characteristic index $\chi_{F,r}(M,D)$ and the characteristic indicatrix $C_{F,r}(M,D)$ of (M,D) with respect to F of exponent r by

$$\chi_{F,r}(M,D) := \sup_{y\in M}\chi_{F,r}(M,D,y)$$

with respect to the total order

$$(l_1,\mu_1) > (l_2,\mu_2) \Longleftrightarrow \begin{cases} l_1 = l_2 \text{ and } \mu_1 > \mu_2 \\ \text{or} \\ l_1 < l_2 \end{cases}$$

and

$$C_{F,r}(M,D) := \{x\in M\,|\chi_{F,r}(M,D,x) = \chi_{F,r}(M,D)\}\,.$$

It is clear that the first component of $\chi_{F,r}(M,D)$ is $\inf_{x\in M} l_{F,r}(M,D,x)$. We call it the global log canonical threshold of (M,D) with respect to F of exponent r and denote it by $l_{F,r}(M,D)$.

By Lemma 2.1 we can easily see that $l_{F,r}(M,D,\cdot)$ is lower semicontinuous and $\chi_{F,r}(M,D,\cdot)$ is upper semicontinuous (with respect to the above order) in the local analytically Zariski sense. Therefore, $C_{F,r}(M,D)$ is a nonempty locally analytically Zariski closed subset of M. $l_{F,r}(M,D,x)$ is related to the usual log canonical threshold lct in that we have the following inequality:

$$l_{F,r}(M,D,x) \geqslant l_{0,r}(M,D-F,x)(=\mathrm{lct}_x(M,D-F)). \tag{2.1}$$

If $x \notin \mathrm{supp}F$ (in particular, if $F = 0$),

$$l_{F,r}(M, D, x) = \mathrm{lct}_x(M, D - F). \tag{2.2}$$

In general, we have the following lemma.

Lemma 2.2. *Suppose* $x \in \mathrm{supp}F$. *Then*

$$l_{F,r}(M, D, x) = l_{0,r}(M, D-F, x) \Longrightarrow \mu_{F,r}(M, D, x) \leqslant \mu_{0,r}(M, D-F, x).$$

Proof. Recall the definition of $l_{(D,F),r;\pi}$ and $\mu_{(D,F),r;\pi}$ right before Lemma 2.1. Let $\mu = \mu_{0,r}(M, D - F, x)$. If $\mu_{F,r}(M, D, x) > \mu$, then there exist irreducible subvarieties $E_1, \dots, E_{\mu+1}$ of codimension 1 in the resolution $\widetilde{M}$ such that $x \in \pi\left(\cap_{k=1}^{\mu+1} E_k\right)$ and

$$\frac{j_{E_k} + \frac{f_{E_k}}{r} + 1}{d_{E_k} - f_{E_k}} = l_{F,r}(M, D, x) = l_{0,r}(M, D - F, x) \leqslant \frac{j_{E_k} + 1}{d_{E_k} - f_{E_k}},$$

$k = 1, \dots, \mu + 1$. This implies $f_{E_k} = 0$ and

$$l_{0,r}(M, D - F, x) = \frac{j_{E_k} + 1}{d_{E_k} - f_{E_k}},\ k = 1, \dots, \mu + 1,$$

and hence $\mu_{0,r}(M, D - F, x) \geqslant \mu + 1$, a contradiction. q.e.d.

Taking an admissible log resolution of $\mathrm{Bl}_x(M)$ to compute $l_{F,r}(M, D, x)$, we obtain that

$$l_{F,r}(M, D, x) \leqslant \frac{\dim M}{\mathrm{mult}_x(D - F)}. \tag{2.3}$$

Remark 2.1. (1) The "$F = 0$" version of both lct and μ have been used in [**4**] to prove base point free cases of our main theorems. They are generalized to the current version in order to prove our main theorems in full generality. The proof of their well-definedness given here is basically the one used in the author's thesis. During the Workshop on Higher Dimensional Algebraic Geometry 2010 in Taiwan I asked Mircea Mustaţă if one can show that $\mu_{0,r}(M, D, x)$ is well defined in the algebraic setting, and later he informed me of a technique using arc spaces. His method possibly also works for the general (F not necessarily 0) algebraic situation. (2) One can formulate Definition 2.1 in more general situations such as when D and F are analytic subsets of M with $\mathcal{I}_D \subset \mathcal{I}_F$. The generalized definition is justified by another analytic method and we prefer to discuss it in a separate article.

2.3. The asymptotic property of $\langle\!\langle\ \rangle\!\rangle_m$. Throughout 2.3 we assume M to be compact and fix η_0, $\eta \in H^0(M, mK_M)$ and an effective divisor F such that $D_{\eta_0} - F$ and $D_\eta - F$ are both effective, where D_{η_0} and D_η are the divisors defined by the vanishing of η_0 and η, respectively. We study the asymptotic behavior of the function $t \longmapsto \langle\!\langle \eta_0 + t\eta \rangle\!\rangle_m$ as $t \longrightarrow 0$.

We basically follow the local setup in the proof of Lemma 2.1. Let $\mathcal{U} = \big\{\big(U, (w_U^j)_{j=1}^n\big)\big\}$ be a finite open cover of coordinate charts on M. We choose a log resolution $\pi : \widetilde{M} \to M$ for (M, D_{η_0}) and a finite refinement $\mathcal{V} = \{(V, Z_V = X_V + iY_V)\}$ of $\pi^{-1}\mathcal{U} = \{\pi^{-1}U\}$ formed by charts in $\widetilde{M}$, where Z_V and (X_V, Y_V) abbreviate $(z_V^1, \dots, z_V^n)$ and $(x_V^1, y_V^1 \dots, x_V^n, y_V^n)$ respectively. Let $\tau : \mathcal{V} \to \mathcal{U}$ be such that $\pi(V) \subset \tau(V)$. Finally, we choose a partition of unity $\{\rho_V(X_V, Y_V)\}$ subordinate to $\mathcal{V}$. $\mathcal{V}$ and $\{\rho_V\}$ can be chosen so that

(i) the image of $Z_V : V \to \mathbf{C}^n$ is the unit polydisc Δ in $\mathbf{C}$;

(ii) if $U = \tau(V)$, then

$$\pi^*(dw_U^1 \wedge \cdots \wedge dw_U^n) = \big(j_V(Z_V)\big) Z_V^{J_V}\ dz_V^1 \wedge \cdots \wedge dz_V^n$$

for some nonvanishing $j_V \in \mathcal{O}(\overline{\Delta})$ and $J_V = (j_V^1, \dots, j_V^n) \in \mathbf{Z}_{\geqslant 0}^n$;

(iii) for s_F, a canonical section of $\mathcal{O}_M(F)$ defining F, $\pi^* s_F = f_V(Z_V) Z_V^{F_V}$ on V for some nonvanishing $f_V \in \mathcal{O}(\overline{\Delta})$ and $F_V = (f_V^1, \dots, f_V^n) \in \mathbf{Z}_{\geqslant 0}^n$;

(iv) following the notation in (ii), on V we have

$$\pi^*\eta_0 = c_V(Z_V)\big(j_V(Z_V)\big)^m Z_V^{A_V + F_V + mJ_V}(dz_V^1 \wedge \cdots \wedge dz_V^n)^{\otimes m}$$

and

$$\pi^*\eta = c_V(Z_V)\big(j_V(Z_V)\big)^m \phi_V(Z_V) Z_V^{F_V + mJ_V}(dz_V^1 \wedge \cdots \wedge dz_V^n)^{\otimes m},$$

where $\phi_V, c_V \in \mathcal{O}(\overline{\Delta})$, c_V is nonvanishing, and $A_V = (a_V^1, \dots, a_V^n) \in \mathbf{Z}_{\geqslant 0}^n$;

(v) for each V we have $l_V^1 = \cdots = l_V^{\mu_V} < l_V^{\mu_V + 1} \leqslant \cdots \leqslant l_V^n$, where $l_V^k = \frac{j_V^k + \frac{f_V^k}{m} + 1}{h_V^j - f_V^k}$, $k = 1, \dots, n$;

(vi) $\rho_V(0, 0) > 0$ for every V.

In the following proof of Theorem 2.2 we will have to consider two different kinds of pullbacks via $\pi : \widetilde{M} \to M$ of elements in $H^0(M, mK_M)$, and it is important not to mix them up. The first one is

$$\pi^* : H^0(M, mK_M) \to H^0(\widetilde{M}, mK_{\widetilde{M}}),$$

which acts on K_M as the usual pullback of differential forms via the map π. The second one is $\pi^{**} : H^0(M, mK_M) \to H^0\big(\widetilde{M}, \pi^*(mK_M)\big)$, the usual pullback map from the space of sections of a vector bundle to that of its pullback bundle via a map.

Theorem 2.2. *Set $r = m$ in Definition 2.1. Write*

$$\chi_{F,m}(M, D_{\eta_0}) = \big(l = l_{F,m}(M, D_{\eta_0}), \mu\big).$$

Then

$$\langle\!\langle \eta_0 + t\eta \rangle\!\rangle_m - \langle\!\langle \eta_0 \rangle\!\rangle_m = \begin{cases} O\left(|t| \left(\ln \frac{1}{|t|}\right)^{\mu}\right) & \text{if } l \geqslant \frac{m-2}{2m}; \\ c_{F,m}(\eta_0, \eta)\ |t|^{2l+\frac{2}{m}} \left(\ln \frac{1}{|t|}\right)^{\mu-1} \\ +o\left(|t|^{2l+\frac{2}{m}} \left(\ln \frac{1}{|t|}\right)^{\mu-1}\right) & \text{if } l < \frac{m-2}{2m} \end{cases}$$

as $t \to 0$, where $c_{F,m}(\eta_0, \eta)$ is a nonnegative number. Moreover, in the second case,

$$c_{F,m}(\eta_0, \eta) = 0 \iff \eta \otimes s_F^{-1} \textit{vanishes on } C_{F,m}(M, D_{\eta_0}).$$

Proof. In terms of $\mathcal{V}$ and g_V chosen above, we can write

$$\langle\!\langle \eta_0 + t\eta \rangle\!\rangle_m = \sum_{V \in \mathcal{V}} \int_{\Delta_0} \left(\rho_V |c_V|^{\frac{2}{m}} |j_V|^2\right) \left|Z_V^{A_V} + t\phi_V\right|^{\frac{2}{m}} \left|Z_V\right|^{\frac{2}{m}F_V + 2J_V} dX_V dY_V.$$

Following the notation of 2.1, for every $V \in \mathcal{V}$ we obtain a corresponding pair $(l_V, \mu_V) = \big(l(A_V, \frac{2}{m}F_V + 2J_V), \mu(A_V, \frac{2}{m}F_V + 2J_V)\big)$. It is clear that $(l, \mu) = \sup_V(l_V, \mu_V)$. Applying Theorem 2.1 by setting $g = \rho_V |c_V|^{\frac{2}{m}} |j_V|^2$, $\phi = \phi_V$, $p = \frac{1}{m}$, $A = A_V$, and $B = \frac{2}{m}F_V + 2J_V$ for each V yields the stated asymptotics. Comparing the coefficient of the leading term of each summand shows that

$$c_{F,m}(\eta_0, \eta) = \sum_{\{V | (l_V, \mu_V) = (l, \mu)\}} c(A_V, B_V, \phi_V).$$

Theorem 2.1 shows that $c_{F,m}(\eta_0, \eta) \geqslant 0$, and that $c_{F,m}(\eta_0, \eta) = 0$

$$\iff c(A_V, \frac{2}{m}F_V + 2J_V, \phi_V) = 0 \text{ for all } V \text{ such that } (l_V, \mu_V) = (l, \mu)$$
$$\iff \phi_V(0, \dots, 0, z_V^{\mu+1}, \dots, z_V^n) \equiv 0 \text{ for all } V \text{ such that } (l_V, \mu_V) = (l, \mu).$$

Consider those prime divisors E on $\widetilde{M}$ whose corresponding ratios (see Section 2.2) give $l_{F,m}(M, D_{\eta_0})$. The union S of intersections of any distinct μ of them is described by $z_V^1 = \cdots = z_V^{\mu} = 0$ in every V. Because of the conditions (ii) and (iii) satisfied by $\mathcal{V}$, the last equivalent condition above is the same as saying that $\pi^{**}\eta \otimes \pi^* s_F^{-1}$ vanishes on S. This is the same as saying that $\eta \otimes s_F^{-1}$ vanishes on $\pi(S) = C_{F,m}(M, D_{\eta_0})$. q.e.d.

Remark 2.2. Definition 1.1 can be generalized to hermitian line bundles. Let (L, h) be a pair of a holomorphic line bundle on M with a smooth hermitian h, and let η be a Lebesgue measurable section of

$K_M^{\otimes m} \otimes L$. Locally $\eta = f(w)(dw^1 \wedge \cdots \wedge dw^n)^{\otimes m}$, where f is a local section of L. We can define a Lebesgue measurable (n,n)-form by setting locally

$$\langle \eta \rangle_{m,h} = ||f(w)||_h^{\frac{2}{m}} du^1 \wedge dv^1 \cdots \wedge du^n \wedge dv^n.$$

Then we can define the pseudonorm of η by $\langle\!\langle \eta \rangle\!\rangle_{n,h} = \int_M \langle \eta \rangle_{n,h}$. This notion has already been used in [**3**]. Theorem 2.2 obviously generalizes to this situation. It will be interesting to see if one can obtain asymptotic results for line bundles with singular hermitian metrics.

3. The Image of Rational Maps $\varphi_{|V|}$

Again we assume M to be a compact complex manifold. In this section we consider linear subspaces $V \subset H^0(M, mK_M)$ and use Theorem 2.2 to study the image of the rational map $\varphi = \varphi_{|V|}$ associated to V. We will view $\mathbb{P}V^*$ as the set of linear subspaces of V of codimension 1 (which will be called hyperplanes in the following). The rational map $\varphi : M \dashrightarrow \mathbb{P}V^*$ is given by

$$x \longmapsto \left[\left\{\eta \in V \;\middle|\; \eta(x) = 0\right\}\right],$$

where $[\{\eta \in V \mid \eta(x) = 0\}]$ is the point corresponding to the hyperplane $\{\eta \in V \mid \eta(x) = 0\}$ in V. The domain of φ (as a set theoretical map) is $M \backslash \mathrm{Bs}|V|$. Throughout this section we assume that $\mathrm{Bs}(|V| - F) = \emptyset$, where $F = \mathrm{Fix}|V|$. Therefore,

$$\mathrm{Bs}|V| = \mathrm{supp} F.$$

Then φ can be extended to a regular map $\varphi_{|V|-F}$ by identifying $\mathbb{P}V^*$ with $\mathbb{P}(V \otimes s_F^{-1})^*$, where s_F is a canonical section of $\mathcal{O}_M(F)$. However, we prefer to distinguish between φ and the regular map extending it, especially in the proof of Lemma 3.1. Note that for a hyperplane H in V, $[H]$ lies in the image of φ if and only if $\mathrm{Bs}|H|$ is not contained in $\mathrm{Bs}|V|$. If this is the case, $\varphi(x) = [H]$ for all $x \in \mathrm{Bs}|H| \backslash \mathrm{Bs}|V|$.

Definition 3.1. We say that V concentrates singularities if for a generic $x \in M \backslash \mathrm{Bs}|V|$ there exists $\eta_x \in V$ such that

$$l_{F,m}(M, D_{\eta_x}) < \frac{m-2}{2m} \tag{3.1}$$

and

$$C_{F,m}(M, D_{\eta_x}) = \varphi_{|V|-F}^{-1}\big(\varphi_{|V|-F}(x)\big), \tag{3.2}$$

where D_{η_x} is the divisor associated to the section η_x.

Lemma 3.1. *Let M and M' be compact complex manifolds. Let $V \subset H^0(M, mK_M)$ and $V' \subset H^0(M', m'K_{M'})$ be linear subspaces such that* $\mathrm{Bs}(|V| - F) = \mathrm{Bs}(|V'| - F') = \emptyset$, *where* $F = \mathrm{Fix|V|}$ *and* $F' = \mathrm{Fix|V'|}$. *If V concentrates singularities and there exists a surjective linear map*

$$\iota : (V', \langle\!\langle\ \rangle\!\rangle_{m'}) \to (V, \langle\!\langle\ \rangle\!\rangle_m)$$

and a number $T > 0$ *such that*

$$\left|\langle\!\langle \iota\eta_0' \rangle\!\rangle_m - \langle\!\langle \iota\eta' \rangle\!\rangle_m\right| = T\left|\langle\!\langle \eta_0' \rangle\!\rangle_{m'} - \langle\!\langle \eta' \rangle\!\rangle_{m'}\right|$$

for any $\eta_0', \eta' \in V'$, *then*

$$I := \mathbb{P}(\iota)^* : \mathbb{P}V^* \to \mathbb{P}V'^*$$

maps the image of $\varphi_{|V|}$ *into that of (the regular extension of)* $\varphi_{|V'|}$.

Proof. For a generic point $x \in M \setminus \mathrm{supp}F$ we have

(1) $\mathrm{supp}F$ does not contain any connected component of $\varphi^{-1}_{|V|-F}\big(\varphi_{|V|-F}(x)\big)$;

(2) $\varphi^{-1}_{|V|-F}\big(\varphi_{|V|-F}(x)\big)$ is nonsingular.

Let s_F be a canonical section of $\mathcal{O}_M(F)$. Note that, for $\eta \in V$, the condition

$$\eta|_{C_{F,m}(M,D_{\eta_x})} \equiv 0 \tag{3.3}$$

is equivalent to the condition

$$\eta \otimes s_F^{-1}|_{C_{F,m}(M,D_{\eta_x})} \equiv 0. \tag{3.4}$$

Since V concentrates singularities, (3.1) holds. By Theorem 2.2, (3.4) is further equivalent to the condition

$$c_{F,m}(\eta_x, \eta) = 0. \tag{3.5}$$

For any $\eta \in V$, we have

$$\begin{aligned}&\langle\!\langle \eta_x + t\eta \rangle\!\rangle_m - \langle\!\langle \eta_x \rangle\!\rangle_m \\ &= c_{F,m}(\eta_x, \eta)\ |t|^{2l+\frac{2}{m}}\left(\ln\frac{1}{|t|}\right)^{\mu-1} + o\left(|t|^{2l+\frac{2}{m}}\left(\ln\frac{1}{|t|}\right)^{\mu-1}\right)\end{aligned}$$

as $t \longrightarrow 0$. Since ι is surjective, we can find $\eta_0' \in V'$ such that $\iota\eta_0' = \eta_x$. By substituting $\iota\eta'$ for η in the above equality, we see that for all $\eta' \in V'$,

$$\begin{aligned}&|\langle\!\langle \eta_0' + t\eta' \rangle\!\rangle_{m'} - \langle\!\langle \eta_0' \rangle\!\rangle_{m'}| \\ &= T\left|c_{F,m}(\eta_x, \iota\eta')\right|\ |t|^{2l+\frac{2}{m}}\left(\ln\frac{1}{|t|}\right)^{\mu-1} + o\left(|t|^{2l+\frac{2}{m}}\left(\ln\frac{1}{|t|}\right)^{\mu-1}\right)\end{aligned} \tag{3.6}$$

as $t \longrightarrow 0$. Since $x \notin \mathrm{Bs}|V|$, by (3.2), there exists some $\eta_1 \in V$ that violates condition (3.3), and hence violates (3.5). That is to say

$$c_{F,m}(\eta_x, \eta_1) \neq 0.$$

Choose any $\eta_1' \in V'$ such that $\iota\eta_1' = \eta_1$. (3.6) implies that

$$\begin{aligned}&|\langle\!\langle \eta_0' + t\eta_1' \rangle\!\rangle_{m'} - \langle\!\langle \eta_0' \rangle\!\rangle_{m'}| \\ &= T\left|c_{F,m}(\eta_x, \eta_1)\right|\ |t|^{2l+\frac{2}{m}}\left(\ln\frac{1}{|t|}\right)^{\mu-1} + o\left(|t|^{2l+\frac{2}{m}}\left(\ln\frac{1}{|t|}\right)^{\mu-1}\right)\end{aligned} \tag{3.7}$$

as $t \longrightarrow 0$ with the leading coefficient

$$T\,|c_{F,m}(\eta_x, \eta_1)| \neq 0. \tag{3.8}$$

We have

$$2l_{F',m'}(M', D_{\eta_0'}) + \frac{2}{m'} < 1, \tag{3.9}$$

otherwise, by Theorem 2.2, there is an integer μ' such that

$$\left|\langle\!\langle \eta_0' + t\eta_1' \rangle\!\rangle_{m'} - \langle\!\langle \eta_0' \rangle\!\rangle_{m'}\right| = O\left(|t|\left(\ln\frac{1}{|t|}\right)^{\mu'}\right)$$

as $t \longrightarrow 0$. This together with (3.7) and (3.8) imply that

$$2l + \frac{2}{m} \geq 1,$$

a contradiction to the choice of η_x (3.1). By (3.7), (3.9), and Theorem 2.2, we have

$$c_{F',m'}(\eta_0', \eta')| = T|c_{F,m}(\eta_x, \iota\eta')| \quad \text{for } \underline{\text{all}} \quad \eta' \in V'. \tag{3.10}$$

By the definition of $I : \mathbb{P}V^* \to \mathbb{P}V'^*$,

$$\begin{aligned}
&I\big(\varphi_{|V|}(x)\big) = I\left(\left[\left\{\eta \in V \,\middle|\, c_{F,m}(\eta_x, \eta) = 0\right\}\right]\right)\\
&= \left[\iota^{-1}\left\{\eta \in V \,\middle|\, c_{F,m}(\eta_x, \eta) = 0\right\}\right]\\
&= \left[\left\{\eta' \in V' \,\middle|\, c_{F',m'}(\eta_0', \eta') = 0\right\}\right] \quad \text{by (3.10)}\\
&= \left[\left\{\eta' \in V' \,\middle|\, \eta' \otimes s_{F'}^{-1} \text{ vanishes along } C_{F',m'}(M', D_{\eta_0'})\right\}\right],
\end{aligned}$$

where the last equality comes from the equivalence between conditions (3.4) and (3.5) for the situation of M' and V'. Note that

$$\left\{\eta' \in V' \,\middle|\, \eta' \otimes s_{F'}^{-1} \text{ vanishes along } C_{F',m'}(M', D_{\eta_0'})\right\}$$

is the image of

$$H := \left\{\eta'' \in V' \otimes s_{F'}^{-1} \,\middle|\, \eta'' \text{ vanishes along } C_{F',m'}(M', D_{\eta_0'})\right\}$$

under the isomorphism $\otimes s_{F'}$ between $V' \otimes s_{F'}^{-1} \subset H^0(M', m'K_{M'} - F')$ and $V' \subset H^0(M', m'K_{M'})$. We have the following commutative diagram of rational maps:

$$\begin{tikzcd}
 & \mathbb{P}V'^* \arrow[d, "\simeq"] \\
M \arrow[ur, dashed, "\varphi_{|V'|}"] \arrow[r, "\varphi_{|V'|-F'}"'] & \mathbb{P}(V' \otimes s_{F'}^{-1})^*.
\end{tikzcd}$$

$\mathrm{Bs}|H|\backslash\mathrm{Bs}(|V'|-F') = \mathrm{Bs}|H|$ contains $C_{F',m'}(M', D_{\eta'_0})$ by the definition of H. Therefore, $[H]$ lies in the image of $\varphi_{|V'|-F'}$. More precisely, being a hyperplane in V' whose elements vanish (as sections of $m'K_{M'}$) along $C_{F',m'}(M', D_{\eta'_0})$, $[H]$ is the image of all points in $C_{F',m'}(M', D_{\eta'_0})$. Therefore, $I\big(\varphi_{|V|}(x)\big)$ lies in the closure of the image of $\varphi_{|V'|}$. This completes the proof. q.e.d.

As an application we obtain the following theorem:

Theorem 1.4. *If M and M' are canonically polarized manifolds of dimension n with isometric m-pluricanonical spaces with respect to $\langle\!\langle\ \rangle\!\rangle_m$ for some $m \in \{ab | a \geqslant (2n+1), b \geqslant c_n\}$, then M and M' are isomorphic (and the induced isometry between their m-pluricanonical spaces equals the given one up to multiplication by a unit complex number).*

Proof. $\varphi_{|mK_M|}$ and $\varphi_{|mK_{M'}|}$ are embeddings if $m = ab$ for some $a \geqslant (2n+1)$ and $b \geqslant c_n \geqslant 2$. We will apply Lemma 3.1 by taking V and V' to be full pluricanonical spaces, $m = m'$, and $T = 1$. Note that in this case $F = F' = 0$. By symmetry, it suffices to show that $|mK_M|$ concentrates singularities. For a generic $x \in M$, we consider

$$W_x = \left\{ \eta \in H^0(M, mK_M) \,\middle|\, \mathrm{mult}_x \eta > \frac{2mn}{m-2} \right\}.$$

$\mathrm{Bs}|W_x| = \{x\}$, since for each $y \neq x$ in M there exists $\tilde{\eta} \in H^0(M, bK_M)$ such that $\tilde{\eta}(x) = 0$ but $\tilde{\eta}(y) \neq 0$, and a direct computation shows that $\tilde{\eta}^{\otimes a} \in W_x$. Bertini's theorem yields an $\eta_x \in W_x$ whose divisor D_{η_x} is smooth away from x. We have $l_{0,m}(M, D_{\eta_x}, x) < \frac{m-2}{2m}$ by (2.3) and $l_{0,m}(M, D_{\eta_x}, y) < \frac{m-2}{2m}$ by (2.1). Therefore, $C_{0,m}(M, D_{\eta_x}) = \{x\}$. q.e.d.

4. Birational Torelli Type Theorems

In this section we turn to birational geometry. The proof of Theorem 1.4 provides a good guideline for proving Theorem 1.2.

Theorem 1.2. *For any pair of projective manifolds M and M' both of nonnegative Kodaira dimension, there exists a positive integer $C(M, M')$ depending on M and M' such that $(*')_{mC(M,M')}$ holds for positive integers $m > 2\max\{\dim M, \dim M'\} + 2$.*

Proof. We first illustrate how to choose $C(M, M')$. For any positive integer r, we let F_r denote $\mathrm{Fix}|rK_M|$ and let φ_r denote $\varphi_{|rK_M - F_r|}$. We may choose a positive integer k such that the kth truncation $R(M, K_M)^{(k)}$ of $R(M, K_M)$ is generated by $H^0(M, kK_M)$ by [**2**]. This implies that $F_{mk} = mF_k$ for all positive integers m. By replacing M by a nonsingular birational model, we may further assume that $\mathrm{Bs}|kK_M - F_k| = \emptyset$ (and hence $\mathrm{Bs}|m(kK_M - F_k)| = \emptyset$ and φ_{mk} is a morphism for every positive integer m). We have the following commutative diagram of morphisms

$$\begin{tikzcd} & M \arrow[dl, "\varphi_{mk}"'] \arrow[dr, "\varphi_k"] & \\ \mathrm{im}\ \varphi_{mk} \arrow[rr, "\nu_m"] & & \mathrm{im}\ \varphi_k, \end{tikzcd}$$

where ν_m is the morphism induced by the map

$$\mathrm{Sym}^m H^0(M, kK_M - F_k) \longrightarrow H^0(M, m(kK_M - F_k))$$

and the mth Veronese embedding

$$\mathbb{P}H^0(M, kK_M - F_k)^* \longrightarrow \mathbb{P}\mathrm{Sym}^m H^0(M, kK_M - F_k)^*.$$

ν_m is not just a rational map but a morphism, because

$$\big|\mathrm{im}\left(\mathrm{Sym}^m H^0(M, kK_M - F_k) \to H^0\big(M, m(kK_M - F_k)\big)\right)\big|$$

is base point free. We may take k sufficiently large such that ν_m is an isomorphism and φ_k has connected fibers. (For example, see [**16**, Theorem 2.1.27].) Note that we may choose k such that all properties above hold for M' (possibly after replacement by a suitable nonsingular birational model) as well. Let k_0 be the smallest such positive integer k. We define $C(M, M')$ to be k_0. In the following, to avoid unnecessary notational complication, we will use the notation k_0 instead of $C(M, M')$.

By symmetry, the proof will be completed if we can apply Lemma 3.1 to the case $V = |mk_0K_M|$ and $V' = |mk_0K_{M'}|$ for every $m > 2\dim M + 2$. It suffices to verify that $|mk_0K_M|$ concentrates singularities when $m > 2\dim M + 2$. In the argument below m will be a fixed positive integer greater than $2\dim M + 2$.

Choose a Zariski open subset U in im φ_{mk_0} satisfying the following properties:

(i) U is nonsingular;
(ii) $\varphi_{mk_0}|_{\varphi_{mk_0}^{-1}(U)} : \varphi_{mk_0}^{-1}(U) \longrightarrow U$ is a submersion.

For $x \in \varphi_{mk_0}^{-1}(U) \setminus \mathrm{supp}F_{mk_0}$, we consider

$$W_x = \left\{ \eta \in H^0\big(M, m(k_0K_M - F_{k_0})\big) \;\middle|\; \begin{array}{c} \mathrm{mult}_y \eta > \frac{2mk_0 \dim M}{mk_0 - 2} \text{ for} \\ \\ \text{all } y \in \varphi_{mk_0}^{-1}\big(\varphi_{mk_0}(x)\big). \end{array} \right\}.$$

We claim that

$$\mathrm{Bs}|W_x| = \varphi_{mk_0}^{-1}\big(\varphi_{mk_0}(x)\big).$$

To see this, first note that

$$\mathrm{Bs}|W_x| \supseteq \varphi_{mk_0}^{-1}\big(\varphi_{mk_0}(x)\big)$$

by the definition of W_x. On the other hand, suppose that $x' \in M$ and $\varphi_{mk_0}(x') \neq \varphi_{mk_0}(x)$. $\varphi_{k_0}(x') \neq \varphi_{k_0}(x)$, since ν_m is an isomorphism. A hyperplane which contains $\varphi_{k_0}(x)$ and does not contain $\varphi_{k_0}(x')$ gives a section $\tilde{\eta}$ of $H^0(M, k_0K_M - F_{k_0})$ such that

$$\mathrm{mult}_x\tilde{\eta} > 0, \text{ and } \mathrm{mult}_{x'}\tilde{\eta} = 0.$$

It is easy to check that $\tilde{\eta}^{\otimes m} \in W_x$ if $m > 2\dim M + 2$ and $\tilde{\eta}^{\otimes m}(x') \neq 0$. Therefore $x' \notin \mathrm{Bs}|W_x|$.

Now we are ready to construct the desired section η_x (see Definition 3.1). By Bertini's theorem, there exists $\hat{\eta}_x \in W_x$ whose multiplicity at points away from $\varphi_{mk_0}^{-1}\big(\varphi_{mk_0}(x)\big)$ is lower than 1. We let $s_{F_{mk_0}}$ be a canonical section of $\mathcal{O}_M(F_{mk_0})$ and $\eta_x = \hat{\eta}_x \otimes s_{F_{mk_0}} \in H^0(M, mk_0K_M)$. Denote the divisors associated to η_x and $\hat{\eta}_x$ by D_{η_x} and $D_{\hat{\eta}_x}$, respectively. It remains to show that

$$l_{F_{mk_0},mk_0}(M, D_{\eta_x}) < \frac{mk_0 - 2}{2mk_0} \tag{4.1}$$

and

$$C_{F_{mk_0},mk_0}(M, D_{\eta_x}) = \varphi_{mk_0}^{-1}\big(\varphi_{mk_0}(x)\big). \tag{4.2}$$

Since $x \notin \mathrm{supp}F_{mk_0}$, we have

$$l_{F_{mk_0},mk_0}(M, D_{\eta_x}, x) = \mathrm{lct}_x(M, D_{\hat{\eta}_x}) \leqslant \frac{\dim M}{\mathrm{mult}_x D_{\hat{\eta}_x}} < \frac{mk_0 - 2}{2mk_0}$$

by (2.2) and the fact that $\hat{\eta}_x \in W_x$. Therefore, (4.1) holds. In particular,

$$l_{F_{mk_0},mk_0}(M, D_{\eta_x}, x) < 1. \tag{4.3}$$

In order to prove (4.2), we have to compare $l_{F_{mk_0},mk_0}(M, D_{\eta_x}, z)$ for all $z \in M$ first. There are two cases:

Case 1. $z \notin \varphi_{mk_0}^{-1}\big(\varphi_{mk_0}(x)\big)$.

By (2.1) and the fact that $\hat{\eta}_x$ has order lower than 1 away from $\varphi_{mk_0}^{-1}\big(\varphi_{mk_0}(x)\big)$, we have

$$l_{F_{mk_0},mk_0}(M, D_{\eta_x}, z) \geqslant \mathrm{lct}_x(M, D_{\hat{\eta}_x}) = 1. \tag{4.4}$$

Case 2. $z \in \varphi_{mk_0}^{-1}\big(\varphi_{mk_0}(x)\big)$.

In this case we claim that

$$\chi_{F_{mk_0},mk_0}(M, D_{\eta_x}, z) = \big(l_{F_{mk_0},mk_0}(M, D_{\eta_x}, z), \mu_{F_{mk_0},mk_0}(M, D_{\eta_x}, z)\big)$$

does not depend on z, or equivalently, $\chi_{F_{mk_0},mk_0}(M, D_{\eta_x}, \cdot)$ is constant on $\varphi_{mk_0}^{-1}\big(\varphi_{mk_0}(x)\big)$. It suffices to show that $\chi_{F_{mk_0},mk_0}(M, D_{\eta_x}, \cdot)$ is locally constant on $\varphi_{mk_0}^{-1}\big(\varphi_{mk_0}(x)\big)$ since $\varphi_{mk_0}^{-1}\big(\varphi_{mk_0}(x)\big)$ is connected. Let $n = \dim M$ and $p = \dim H^0\big(M, m(k_0K_M - F_{k_0})\big) - 1$. Recall that U is chosen so that properties (i) and (ii) above hold. Therefore, we may choose a coordinate chart

$$(P, \{t = (t_1, \ldots, t_p)\})$$

of $\mathbb{P}H^0\big(M, m(k_0K_M - F_{k_0})\big)^*$ centered at $\varphi_{mk_0}(z) = \varphi_{mk_0}(x)$ and a coordinate chart

$$(Q, \{\tau = (\tau_1, \dots, \tau_m)\})$$

of M centered at z with

$$\tau(Q) = \{(\tau_1, \dots, \tau_n) \in \mathbf{C}^n : |\tau_j| < 1, j = 1, \dots, n\}$$

such that

(1) $\varphi_{mk_0}(Q) \subseteq P$,
(2) $\mathrm{im}\varphi_{mk_0}$ is described locally by $t_{q+1} = \cdots = t_p = 0$, and
(3) the map φ_{mk_0} is described locally by

$$(\tau_1, \dots, \tau_q, \dots, \tau_n) \longmapsto (\tau_1, \dots, \tau_q, 0, \dots, 0).$$

$(\tau_{q+1}, \dots, \tau_n)$ is then a local coordinate system of $\varphi^{-1}_{mk_0}(\varphi_{mk_0}(x))$. For any $a = (a_{q+1}, \dots, a_n)$ with $|a_j| < 1$, $j = q+1, \dots, n$, we let Q_a be the subset of Q with

$$\tau(Q_a) = \{(\tau_1, \dots, \tau_q, a_{q+1}, \dots, a_n) \in \mathbf{C}^n : |\tau_j| < 1, j = 1, \dots, q\}$$

via the coordinate system τ. By the definition of φ_{mk_0}, there exists a hyperplane

$$H \subset \mathbb{P}H^0\big(M, m(k_0K_M - F_{k_0})\big)^*$$

such that $D_{\hat{\eta}_x} = \varphi^{-1}_{mk_0}(H)$. Choose a defining function $f(t_1, \dots, t_p)$ of H in the chart P. Then $f(\tau_1, \dots, \tau_q, 0, \dots, 0)$ is a defining function of $D_{\hat{\eta}_x}$ in the chart Q. This function does not involve coordinates along the direction of $\varphi^{-1}_{mk_0}(U)$. This means that $D_{\hat{\eta}_x} \cap Q$ is the total space of an isotrivial deformation of singularity with base $\varphi^{-1}_{mk_0}(\varphi_{mk_0}(x)) \cap Q$. Therefore, when using log resolutions of $(Q, D_{\hat{\eta}_x} \cap Q)$ to compute $\chi_{0,mk_0}(M, D_{\hat{\eta}_x}, z)$ for $z \in \varphi^{-1}_{mk_0}(\varphi_{mk_0}(x))$, we may take a log resolution of $(Q_z, D_{\hat{\eta}_x} \cap Q_z)$ and then take the product of it with $\varphi^{-1}_{mk_0}(\varphi_{mk_0}(x)) \cap Q$. Using this particular kind of log resolution we see that $\chi_{0,mk_0}(M, D_{\hat{\eta}_x}, \cdot)$ is locally constant, and hence constant along $\varphi^{-1}_{mk_0}(\varphi_{mk_0}(x))$. Next we show that

$$\chi_{F_{mk_0}, mk_0}(M, D_{\eta_x}, z) = \chi_{0,mk_0}(M, D_{\hat{\eta}_x}, z) \tag{4.5}$$

for $z \in \varphi^{-1}_{mk_0}(\varphi_{mk_0}(x))$. It is clear from the definition that this holds for z in the Zariski dense subset $\varphi^{-1}_{mk_0}(\varphi_{mk_0}(x)) - \mathrm{supp}F_{mk_0}$ of $\varphi^{-1}_{mk_0}(\varphi_{mk_0}(x))$. Now suppose that $z \in \varphi^{-1}_{mk_0}(\varphi_{mk_0}(x)) \cap \mathrm{supp}F_{mk_0}$. By the upper semi-continuity of $\chi_{F_{mk_0}, mk_0}(M, D_{\eta_x}, \cdot)$ and Lemma 2.2, we conclude that (4.5) holds, and hence $\chi_{F_{mk_0}, mk_0}(M, D_{\eta_x}, \cdot)$ is constant on $\varphi^{-1}_{mk_0}(\varphi_{mk_0}(x))$.

Finally, (4.2) holds by (4.1) and (4.4). This completes the proof. q.e.d.

Now we consider projective manifolds of general type. We first recall a fundamental fact. By [**9**] and [**19**], for each $n \in \mathbf{N}$, there exists $m_n \in \mathbf{N}$ such that $\varphi_{|mK_M|} : M \dashrightarrow \mathbb{P}H^0(M, mK_M)^*$ maps M birationally to its image for any n-dimensional projective manifold M of general type and any $m \geqslant m_n$. We fix such an m_n greater than $n+1$.

Theorem 1.3. *Let M and M' be n-dimensional projective manifolds of general type. For a sufficiently large and sufficiently divisible integer m, if $\iota : H^0(M', rK_{M'}) \to H^0(M, rK_M)$ is a linear isometry with respect to $\langle\!\langle\ \rangle\!\rangle_m$, then one can find a birational map $\psi : M \dashrightarrow M'$ and a unit complex number c such that $c\iota = \psi^*$. Moreover, every number $r = 2m(n+2)!j_{M,M'}m_n$ will do the job when $m > 2n+2$, where $j_{M,M'}$ is the least common multiple of the Cartier indices of M and M'.*

Since the proof is the same as that of Theorem 1.2, we just outline how it goes. We need to analyze the pair of numbers k and m in the proof of Theorem 1.2. The key point is to achieve $F_{mk} = mF_k$. Kollár's effective base point freeness theorem ([**15**], 1.1 Theorem) says that if a log pair (Y, Δ) is proper and klt of dimension n, L is a nef Cartier divisor on Y, and $a \in \mathbf{N}$ is such that $aL - (K_Y + \Delta)$ is nef and big, then $\mathrm{Bs}|2(n+2)!(a+n)L| = \emptyset$. Applying this in the case $\Delta = 0$, $a = 2$ and $L = jK_Y$, where Y is a minimal model of M (which exists by [**2**] and [**9**]) and $j = j_Y$, the Cartier index of Y, we have that $\mathrm{Bs}|2m(n+2)!jK_Y| = \emptyset$ if $m \geqslant n+2$. By passing to a birational model of M, we may assume that we have a morphism $\pi : M \to Y$. Consider the ramification formula $jK_M = \pi^*(jK_Y) + E$. E is effective since $(Y, 0)$ is a terminal pair. Then we have $F_{2m(n+2)!j} = 2m(n+2)!E$ if $m \geqslant n+2$ (by possibly passing to a nonsingular birational model of M). The other parts of the proof of 1.2 work as before. So we reach the same condition $m > 2n+2$. The factor m_n is necessary to make φ_{mK_M} and $\varphi_{mK_{M'}}$ map M and M' birationally to their images.

5. The Proof of Theorem 2.1

Finally, we come to the technical core of this paper. In this section we will give a proof of Theorem 2.1. The proof will be broken into a sequence of lemmas. We will deal with several integrals in the following. In most cases the integrand will be a function multiplied with a "volume element" in the classical sense. In some cases we will use exterior differential forms to compute the integrands after changing variables, but when integrating we always go back to the classical setting.

We need some basic estimates for the asymptotic order of some monomial integrals of specific forms. We adopt the same setting and the same rule of abbreviating notations as in 2.1. Recall that for $0 < p \leqslant 1/2$, $A \in \mathbf{Z}^n_{\geqslant 0}$, and $B \in \mathbf{R}^n_{\geqslant 0}$ the function

$$\Psi : \mathbf{C} \longrightarrow \mathbf{R}_{\geqslant 0}$$

is defined by setting

$$\Psi(t) = \int_{\overline{\Delta}} g(X,Y)\big|Z^A + t\phi(Z)\big|^{2p}\big|Z\big|^{2B} dX dY.$$

We will assume that $A \in \mathbf{Z}^n_{\geqslant 0}$ and $B \in \mathbf{R}^n_{\geqslant 0}$ in Lemmas 5.3, 5.4, and 5.5.

Definition 5.1. For $(\alpha, \beta) \in \mathbf{R} \times \mathbf{Z}$, we define $F_{\alpha,\beta} : \mathbf{C} \setminus \{0\} \longrightarrow \mathbf{R}$ by setting for $t \in \mathbf{C} \setminus \{0\}$

$$F_{\alpha,\beta}(t) = \begin{cases} |t|^\alpha \left(\ln \frac{1}{|t|}\right)^{\beta - 1} & \text{if } \alpha \neq 0, \\ \\ |t|^\alpha \left(\ln \frac{1}{|t|}\right)^{\beta} & \text{if } \alpha = 0. \end{cases}$$

The following lemma is obvious.

Lemma 5.1. *Let $(l_1, \mu_1), (l_2, \mu_2) \in \mathbf{R} \times \mathbf{Z}$. We have*

$$(l_1, \mu_1) > (l_2, \mu_2) \Longleftrightarrow F_{l_2,\mu_2}(t) = o\big(F_{l_1,\mu_1}(t)\big) \ \text{as } t \longrightarrow 0,$$

where $>$ is the order relation introduced in Definition 2.1.

Definition 5.2. For $A \in \mathbf{R}^n_{\geqslant 0}$, $B \in \mathbf{R}^n$, and $t \in \mathbf{C}$, we define

$$I_+(A,B,t) = \int\limits_{\overline{\Delta} \cap \{|Z|^A \leqslant |t|\}} \big|Z\big|^{2B} \, dX dY$$

(which could be ∞) and

$$I_-(A,B,t) = \int\limits_{\overline{\Delta} \cap \{|Z|^A \geqslant |t|\}} \big|Z\big|^{2B} \, dX dY.$$

We state a calculus lemma without proof.

Lemma 5.2. *Let $l = l(A,B)$ and $\mu = \mu(A,B)$ (see* 2.1*). We have*

$$I_-(A,B,t) = \begin{cases} O(1) & \text{if } \ l > 0, \\ \\ O\big(F_{2l,\mu}(t)\big) & \text{if } \ l \leqslant 0, \end{cases}$$

and

$$I_+(A,B,t) = O\big(F_{2l,\mu}(t)\big) \ \text{if } \ l > 0$$

as $t \longrightarrow 0$.

The proof of this lemma is elementary and will be omitted. For readers who are interested in its proof, we refer them to the changes of variables used in the proof of Lemma 5.5 below.

There are two types of inequalities that will be used many times below: for any $0 < p < 1$, we fix two constants $\delta(p) > 0$ and $C(p) > 0$ such that

$$\big||1 + w_1|^{2p} - |1 + w_2|^{2p}\big| \leqslant C(p)|w_1 - w_2| \tag{5.1}$$

for any two complex numbers w_1 and w_2 with $|w_j| \leqslant \delta(p)$; we also have the "triangle inequality"

$$\left||a+b|^{2p} - |a|^{2p}\right| \leqslant |b|^{2p}, \tag{5.2}$$

for all $a, b \in \mathbf{C}$.

Here we make a convention that will be valid throughout the rest of the paper. By our assumption (Section 2.1) g and ϕ both admit smooth extension to a neighborhood of $\overline{\Delta}$, so we may assume that they and their first-order partial derivatives are all bounded by a number $N > 0$, which will be fixed in the following. We will also have to apply a "mean value theorem"-type argument at several places. In these arguments, there will be some positive constants that can usually be chosen to depend only on N and $C(p)$ but whose precise values do not matter much for our purpose. If this is the case, we will denote them by $N_1, N_2, \dots$ without figuring out their precise dependence in N and $C(p)$.

From now on we assume that $A \in \mathbf{Z}^n_{\geqslant 0}$ and $B \in \mathbf{R}^n_{\geqslant 0}$.

Lemma 5.3. *Let $l = l(A,B)$. If $2l + 2p > 1$, then*

$$\Psi(t) - \Psi(0) = O(|t|)$$

as $t \longrightarrow 0$.

Proof. Fix δ such that $0 < \delta < \delta(p)$.

$$\Psi(t) - \Psi(0) = \int_{\overline{\Delta}} g(X,Y)\left(\left|Z^A + t\phi(Z)\right|^{2p} - \left|Z^A\right|^{2p}\right)\left|Z\right|^{2B} dX\,dY$$

$$= \int_{\overline{\Delta}\cap\{|Z|^A \leqslant \frac{N|t|}{\delta}\}} g(X,Y)\left(\left|Z^A + t\phi(Z)\right|^{2p} - \left|Z^A\right|^{2p}\right)\left|Z\right|^{2B} dX\,dY$$

$$+ \int_{\overline{\Delta}\cap\{|Z|^A \geqslant \frac{N|t|}{\delta}\}} g(X,Y)\left(\left|1 + \frac{t\phi(Z)}{Z^A}\right|^{2p} - 1\right)\left|Z\right|^{2pA+2B} dX\,dY.$$

By (5.2), the first term is bounded in absolute value by

$$N^{1+2p}\ |t|^{2p}\ I_+\left(A, B, \frac{N|t|}{\delta}\right),$$

which, as $t \longrightarrow 0$, is

$$O\left(|t|^{2l+2p}\left(\ln\frac{1}{|t|}\right)^{\mu-1}\right)$$

by Lemma 5.2, and hence is $O(1)$ as $t \longrightarrow 0$.

Note that

$$\left|\frac{t\phi(Z)}{Z^A}\right| \leqslant \frac{|t|N}{|Z|^A} \leqslant \delta < \delta(p).$$

By (5.1), the second term is bounded in absolute value by

$$C(p)N^2\ |t|\ I_-\left(A, B+\left(p-\frac{1}{2}\right)A, \frac{N|t|}{\delta}\right),$$

which is $O(|t|)$ as $t \longrightarrow 0$ by Lemma 5.2, since $2l+2p>1$ implies

$$l\big(A, B+(p-(1/2))A\big) = l+p-\frac{1}{2} > 0.$$

q.e.d.

To keep notation shorter, we adopt the following abbreviations for coordinates (recall the definition of μ in 2.1):

$$Z = (z, z')$$

with

$$z = (z_1, \dots, z_\mu) \text{ and } z' = (z_{\mu+1}, \dots, z_n),$$

and

$$(X, Y) = (x, y, x', y')$$

with

$$(x, y) = (x_1, y_1, \dots, x_\mu, y_\mu)$$

and

$$(x', y') = (x_{\mu+1}, y_{\mu+1}, \dots, x_n, y_n),$$

where

$$x_j = \mathrm{Re}\ z_j,\ \ y_j = \mathrm{Im}\ z_j,$$

$j = 1, \dots, n$. Let

$$\overline{\Delta}_0 = \{z : |z_j| \leqslant 1,\ j = 1, \dots, \mu\},$$

and

$$\overline{\Delta}' = \{z' : |z_j| \leqslant 1,\ j = \mu+1, \dots, n\}.$$

Lemma 5.4. *Let $l = l(A,B)$ and $\mu = \mu(A,B)$. If $2l+2p \leqslant 1$, then*

$$\Psi(t) - \Psi(0) - \int\limits_{\overline{\Delta}} g(0,0,x',y')\left(\left|Z^A + t\phi(0,z')\right|^{2p} - \left|Z^A\right|^{2p}\right)\left|Z\right|^{2B} dX\,dY$$

$$= \quad o\big(|t| F_{2l+2p-1,\mu}(t)\big) \quad \textit{as} \ \ t \longrightarrow 0.$$

Proof. Write

$$\Psi(t) - \Psi(0) - \int\limits_{\overline{\Delta}} g(0,0,x',y')\left(\left|Z^A + t\phi(0,z')\right|^{2p} - \left|Z^A\right|^{2p}\right)\left|Z\right|^{2B} dX\,dY$$

$$= \quad I_1 \quad + \quad I_2 \quad + \quad I_3 \quad + \quad I_4,$$

where

$$I_1 = \int\limits_{\overline{\Delta}\cap\{|Z|^A \leqslant \frac{N|t|}{\delta}\}} g(X,Y)\left(\left|Z^A + t\phi(Z)\right|^{2p} - \left|Z^A + t\phi(0,z')\right|^{2p}\right) \left|Z\right|^{2B} dX dY,$$

$$I_2 = \int\limits_{\overline{\Delta}\cap\{|Z|^A \leqslant \frac{N|t|}{\delta}\}} \left(g(X,Y) - g(0,0,x',y')\right)\left(\left|Z^A + t\phi(0,z')\right|^{2p} - \left|Z^A\right|^{2p}\right) \left|Z\right|^{2B} dX dY,$$

$$I_3 = \int\limits_{\overline{\Delta}\cap\{|Z|^A \geqslant \frac{N|t|}{\delta}\}} g(X,Y)\left(\left|1 + \frac{t\phi(Z)}{Z^A}\right|^{2p} - \left|1 + \frac{t\phi(0,z')}{Z^A}\right|^{2p}\right) \left|Z\right|^{2B+2pA} dX dY,$$

$$I_4 = \int\limits_{\overline{\Delta}\cap\{|Z|^A \geqslant \frac{N|t|}{\delta}\}} \left(g(X,Y) - g(0,0,x',y')\right)\left(\left|1 + \frac{t\phi(0,z')}{Z^A}\right|^{2p} - 1\right) \left|Z\right|^{2B+2pA} dX dY.$$

We let $e_r = (0,\dots,0,\overset{r-\text{th}}{1},0,\dots,0) \in \mathbf{R}^n$, $r = 1,\dots,n$.

Estimate of I_1:

By (5.2) and the mean value theorem (applied to the function $\phi(\cdot,z')$),

$$\left||Z^A + t\phi(Z)|^{2p} - |Z^A + t\phi(0,z')|^{2p}\right| \leqslant |t|^{2p}\left|\phi(Z) - \phi(0,z')\right|^{2p}$$

$$\leqslant N_1\ |t|^{2p}\left(\sum_{r=1}^{\mu} |z_r|\right)^{2p} \leqslant N_2\ |t|^{2p}\sum_{r=1}^{\mu} |z_r|^{2p}$$

for some N_1 and $N_2 > 0$. Therefore,

$$|I_1| \leqslant N\ N_2\ |t|^{2p}\sum_{r=1}^{\mu} I_+\left(A, B + pe_r, \frac{N|t|}{\delta}\right).$$

For $r = 1,\dots,\mu(=\mu(A,B))$, we have two possibilities:

(1) $\mu(A,B) = 1$. Then $l' := l(A, B + pe_1) > l(A,B) = l$.

(2) $\mu(A,B) > 1$. Then $l' := l(A, B + pe_r) = l(A,B)$ and

$$\mu' := \mu(A, B + pe_r) = \mu(A,B) - 1.$$

According to the order defined in Definition 2.1, we have in both cases

$$(l,\mu) = \big(l(A,B),\mu(A,B)\big) > \big(l(A,B+pe_r),\mu(A,B+pe_r)\big) = (l',\mu').$$

In particular, $l' \geqslant l > 0$. By Lemmas 5.1 and 5.2, we have

$$|t|^{2p} I_+ \left(A, B + pe_r, \tfrac{N|t|}{\delta}\right) = O\big(|t|^{2p} F_{2l',\mu'}(t)\big) = o\big(|t| F_{2l+2p-1,\mu}(t)\big)$$

as $t \longrightarrow 0$, $r = 1, \ldots, \mu$.
Therefore,

$$I_1 = o\big(|t| F_{2l+2p-1,\mu}(t)\big) \quad \text{as} \quad t \longrightarrow 0.$$

Estimate of I_2:

We have

$$\big|g(X,Y) - g(0,0,x',y')\big| \leqslant N_3 \sum_{r=1}^{\mu} |z_r| \tag{5.3}$$

on $\overline{\Delta}$ for some $N_3 > 0$. By (5.2),

$$\big||Z^A + t\phi(0,z')|^{2p} - |Z^A|^{2p}\big| \leqslant |t|^{2p} \, \big|\phi(0,z')\big|^{2p} \leqslant N^{2p} |t|^{2p}.$$

Therefore, by Lemma 5.2,

$$|I_2| \leqslant N^{2p} N_3 \; |t|^{2p} \sum_{r=1}^{\mu} I_+ \left(A, B + \frac{1}{2} e_r, \frac{N|t|}{\delta}\right).$$

Similarly, for $r = 1, \ldots, \mu(A,B)$, we have two possibilities:

(1) $\mu(A,B) = 1$. Then $l'' := l(A, B + (1/2)e_1) > l$.

(2) $\mu(A,B) > 1$. Then $l'' := l(A, B + (1/2)e_r) = l$ and

$$\mu'' := \mu(A, B + (1/2)e_r) = \mu - 1.$$

We have in both cases

$$(l,\mu) > (l'',\mu'').$$

In particular, $l'' \geqslant l > 0$. By Lemmas 5.1 and 5.2, we have

$$|t|^{2p} I_+ \left(A, B + \tfrac{1}{2} e_r, \tfrac{N|t|}{\delta}\right) = O\big(|t|^{2p} F_{2l'',\mu''}(t)\big) = o\big(|t| F_{2l+2p-1,\mu}(t)\big)$$

as $t \longrightarrow 0$, $r = 1, \ldots, \mu$. Therefore,

$$I_2 = o\big(|t| F_{2l+2p-1,\mu}(t)\big) \quad \text{as} \quad t \longrightarrow 0.$$

Estimate of I_3:

Note that the domain of integration indicates that

$$\left|\frac{t\phi(Z)}{Z^A}\right| \leqslant \delta \quad \text{and} \quad \left|\frac{t\phi(0,z')}{Z^A}\right| \leqslant \delta.$$

By (5.1) and the mean value theorem, there exists $N_4 > 0$ such that

$$\left| \left|1 + \frac{t\phi(Z)}{Z^A}\right|^{2p} - \left|1 + \frac{t\phi(0,z')}{Z^A}\right|^{2p} \right| \leqslant C(p) \left| \frac{t\phi(Z)}{Z^A} - \frac{t\phi(0,z')}{Z^A} \right|$$

$$\leqslant N_4 \frac{|t|}{|Z|^A} \sum_{r=1}^{\mu} |z_r|.$$

Therefore,

$$|I_3| \leqslant N\ N_4\ |t| \sum_{r=1}^{\mu} I_- \left(A, B + \left(p - \frac{1}{2}\right) A + \frac{1}{2} e_r, \frac{N|t|}{\delta} \right).$$

Two situations may occur:

(1) $l''' := l\big(A, B + (p - \frac{1}{2})A + \frac{1}{2}e_r\big) > 0$. Then, by Lemma 5.2,

$$|t| I_- \left(A, B + \left(p - \tfrac{1}{2}\right) A + \tfrac{1}{2} e_r, \tfrac{N|t|}{\delta} \right) = |t| O(1) = O(|t|),$$

which can be shown to be

$$o\big(|t| F_{2l+2p-1,\mu}(t)\big)$$

as $t \longrightarrow 0$ by definition.

(2) $l''' := l\big(A, B + (p - \frac{1}{2})A + \frac{1}{2}e_r\big) \leqslant 0$. Let

$$\mu''' := \mu(A, B + (p - (1/2))A + (1/2)e_r).$$

Then, by definition,

$$\mu''' = \mu(A, B + (1/2)e_r).$$

(More precisely, $\mu(A, B + kA) = \mu(A, B)$ for all $k \in \mathbf{R}$.) On the other hand,

$$l''' = l(A, B + (1/2)e_r) + p - (1/2) \geqslant l + p - (1/2),$$

and the last equality holds only if

$$\mu = \mu(A, B) > \mu(A, B + (1/2)e_r),$$

as in the previous two estimates. In summary, we always have

$$(l, \mu) > (l''', \mu''').$$

By Lemma 5.2,

$$|t| I_- \left(A, B + \left(p - \tfrac{1}{2}\right) A + \tfrac{1}{2} e_r, \tfrac{N|t|}{\delta} \right) = |t| O\big(F_{2l''',\mu'''}(t)\big)$$

$$= \quad o\big(|t| F_{2l+2p-1,\mu}(t)\big) \quad \text{as} \quad t \longrightarrow 0, \quad r = 1, \ldots, \mu.$$

Therefore,

$$I_3 = o\big(|t|F_{2l+2p-1,\mu}(t)\big) \ \text{ as } \ t \longrightarrow 0.$$

Estimate of I_4:

Again, the domain of integration indicates that $\left|\frac{t\phi(0,z')}{Z^A}\right| \leqslant \delta$, and hence

$$\left| \left|1+\tfrac{t\phi(0,z')}{Z^A}\right|^{2p} - 1\right| \leqslant C(p)\left|\tfrac{t\phi(0,z')}{Z^A}\right| \leqslant NC(p)\tfrac{|t|}{|Z|^A} \ .$$

By (5.3),

$$\big|g(X,Y)-g(0,0,x',y')\big| \leqslant N_3 \sum_{r=1}^{\mu} |z_r|.$$

Therefore,

$$|I_4| \leqslant NC(p)N_3 \ |t| \sum_{r=1}^{\mu} I_-\left(A, B+\left(p-\frac{1}{2}\right)A+\frac{1}{2}e_r, \frac{N|t|}{\delta}\right)$$

and the remained part is the same as the last part of the estimate of I_3. We get

$$I_4 = o\big(|t|F_{2l+2p-1,\mu}(t)\big) \ \text{ as } \ t \longrightarrow 0.$$

This completes the proof. q.e.d.

Lemma 5.5. *Let $l = l(A,B)$ and $\mu = \mu(A,B)$. We have*

$$\int_{\Delta} g(0,0,x',y')\left(\left|Z^A+t\phi(0,z')\right|^{2p} - \left|Z^A\right|^{2p}\right)\left|Z\right|^{2B} dXdY$$

$$= \begin{cases} O\left(|t|\left(\ln\frac{1}{|t|}\right)^{\mu}\right) & \text{if } \ 2l+2p=1, \\ c(A,B,\phi)\ |t|^{2l+2p}\left(\ln\frac{1}{|t|}\right)^{\mu-1} \\ +o\left(|t|^{2l+2p}\left(\ln\frac{1}{|t|}\right)^{\mu-1}\right) & \text{if } \ 2l+2p<1 \end{cases}$$

as $t \longrightarrow 0$. If $2l+2p<1$, then $c(A,B,\phi) \geqslant 0$ and

$$c(A,B,\phi) = 0 \Longleftrightarrow \phi(0,\dots,0,z_{\mu+1},\dots,z_n) \equiv 0.$$

Proof. The first part of the proof will be devoted to showing (5.5), below. For any multi-index $A \in \mathbf{Z}^n_{\geqslant 0}$, we write $A = (a,a')$, $a = (a_1,\dots,a_\mu)$, and $a' = (a_{\mu+1},\dots,a_n)$. By Fubini's theorem,

$$\int_{\Delta} g(0,0,x',y')\left(\left|Z^A+t\phi(0,z')\right|^{2p} - \left|Z^A\right|^{2p}\right)\left|Z\right|^{2B} dXdY$$

$$= \int_{\overline{\Delta}'} g(0,0,x',y') \left(\int_{\overline{\Delta}_0} \left(\left| 1 + \frac{t\phi(0,z')}{z^a z'^{a'}} \right|^{2p} - 1 \right) |z|^{2b+2pa} dxdy \right)$$
$$|z'|^{2b'+2pa'} dx'dy'.$$

We denote the inner integral by

$$I(z') = \int_{\overline{\Delta}_0} \left(\left| 1 + \frac{t\phi(0,z')}{z^a z'^{a'}} \right|^{2p} - 1 \right) |z|^{2b+2pa} dxdy.$$

Consider the following change of variables:

$$z_j = \left(|t|^{\frac{1}{\mu}} r_j \right)^{\frac{1}{a_j}} e^{\frac{i\theta_j}{a_j}}, \; j = 1, \ldots, \mu.$$

We have

$$\frac{dz_j \wedge d\overline{z}_j}{|z_j|^2} = -\frac{2i}{a_j^2} \frac{dr_j}{r_j} \wedge d\theta_j$$

and

$$dx_j \wedge dy_j = \frac{i}{2} dz_j \wedge d\overline{z}_j = \frac{1}{a_j^2} \left(|t|^{\frac{1}{\mu}} r_j \right)^{\frac{2}{a_j}} \frac{dr_j}{r_j} \wedge d\theta_j$$

for each j. Under this transformation the original region $\overline{\Delta}_0$ is transformed into

$$0 \leqslant r_j \leqslant |t|^{-\frac{1}{\mu}} \;\text{ and }\; 0 \leqslant \theta_j \leqslant 2a_j\pi,$$

$j = 1, \ldots, \mu$. We adopt the following abbreviations:

$$R = r_1 \cdots r_\mu,$$

$$\frac{dR}{R} = \frac{dr_1}{r_1} \cdots \frac{dr_\mu}{r_\mu},$$

and

$$d\Theta = d\theta_1 \cdots d\theta_\mu.$$

We have

$$
\begin{aligned}
I(z') &= \int_{\theta_j=0}^{2a_j\pi}\int_{r_j=0}^{|t|^{-\frac{1}{\mu}}}\left(\left|1+\frac{t\phi(0,z')}{|t|Re^{i(\theta_1+\cdots+\theta_\mu)}z'^{a'}}\right|^{2p}-1\right)\\
&\qquad |t|^{2l+2p}\frac{R^{2l+2p}}{a_1^2\cdots a_\mu^2}\frac{dR}{R}d\Theta\\
&= \frac{|t|^{2l+2p}}{a_1\cdots a_\mu}\int_{\theta_j=0}^{2\pi}\int_{r_j=0}^{|t|^{-\frac{1}{\mu}}}\left(\left|1+\frac{t\phi(0,z')}{|t|Re^{i(\theta_1+\cdots+\theta_\mu)}z'^{a'}}\right|^{2p}-1\right)\\
&\qquad R^{2l+2p}\frac{dR}{R}d\Theta\\
&= \frac{(2\pi)^{\mu-1}|t|^{2l+2p}}{a_1\cdots a_\mu}\int_{\theta=0}^{2\pi}\int_{r_j=0}^{|t|^{-\frac{1}{\mu}}}\left(\left|1+\frac{t\phi(0,z')}{|t|Re^{i\theta}z'^{a'}}\right|^{2p}-1\right)\\
&\qquad R^{2l+2p}\frac{dR}{R}d\theta.
\end{aligned}
$$

If $\mu=1$, then

$$
I(z')=\frac{|t|^{2l+2p}}{a_1}\int_0^{2\pi}\int_0^{|t|^{-1}}\left(\left|1+\frac{t\phi(0,z')}{|t|\rho e^{i\theta}z'^{a'}}\right|^{2p}-1\right)\rho^{2l+2p}\frac{d\rho}{\rho}d\theta.
$$

By setting $w=\frac{|t|}{t}\rho e^{i\theta}z'^{a'}=u+iv$, we obtain

$$
I(z')=\frac{|t|^{2l+2p}|z'|^{-(2l+2p)a'}}{a_1}\int\limits_{|w|\leqslant\frac{|z'|^{a'}}{|t|}}\left(\left|1+\frac{\phi(0,z')}{w}\right|^{2p}-1\right)|w|^{2l+2p-2}dudv.
$$

Now we consider the case $\mu>1$. Choose a basis $\{v_1,\ldots,v_{\mu-1}\}$ of $\mathbf{R}^{\mu-1}$ and let $v_\mu=-(v_1+\cdots+v_{\mu-1})$. We define linear functions

$$
L_j:\tau=(\tau_1,\ldots,\tau_{\mu-1})\in\mathbf{R}^{\mu-1}\longmapsto v_j\cdot\tau\in\mathbf{R},
$$

where $\cdot$ means the euclidean inner product on $\mathbf{R}^{\mu-1}$. Finally, we let

$$
D=\left|\det\begin{pmatrix}v_1\\ \cdot\\ \cdot\\ \cdot\\ v_{\mu-1}\end{pmatrix}\right|.
$$

Now we make another change of variables for $(r_1,\ldots,r_\mu)$ with new variables $(\tau_1,\ldots,\tau_{\mu-1},\rho)$ as follows:

$$
r_j=\rho^{\frac{1}{\mu}}e^{\frac{L_j(\tau)}{\mu}},
$$

$j=1,\ldots,\mu$. We have

$$
r_1\cdots r_\mu=\rho
$$

and

$$\frac{dr_j}{r_j} = \frac{1}{\mu}\left(\frac{d\rho}{\rho} + v_j \cdot d\tau\right)$$

for $j = 1, \ldots, \mu$. Therefore

$$\frac{dr_1}{r_1} \wedge \cdots \wedge \frac{dr_\mu}{r_\mu} = \det\left[\frac{1}{\mu}\begin{pmatrix} v_1 & 1 \\ & \cdots \\ v_{\mu-1} & 1 \\ v_\mu & 1 \end{pmatrix}\right] d\tau_1 \wedge \cdots \wedge d\tau_{\mu-1} \wedge \frac{d\rho}{\rho}$$

and

$$\frac{dr_1}{r_1} \cdots \frac{dr_\mu}{r_\mu} = \frac{D}{\mu^{\mu-1}} \frac{d\rho}{\rho}\ d\tau_1 \cdots d\tau_{\mu-1}.$$

Under this transformation, the condition

$$0 \leqslant r_j \leqslant |t|^{-\frac{1}{\mu}}$$

for $j = 1, \ldots, \mu$ becomes

$$\tau \in \mathbf{R}^{\mu-1} \text{ and } 0 \leqslant \rho \leqslant \frac{1}{|t|} e^{-G(\tau)},$$

where

$$G(\tau) = \max_{1 \leqslant j \leqslant \mu} L_j(\tau).$$

Let

$$V_0 = \frac{\mu D}{\mu - 1}$$

be the volume of the polytope generated by $v_1, \ldots, v_{\mu-1}$. As in the case $\mu = 1$, setting

$$w = \frac{t}{|t|}\rho e^{i\theta} z'^{a'} = u + iv,$$

we obtain

$$I(z') = \frac{(2\pi)^{\mu-1}(\mu-1)V_0 |t|^{2l+2p}|z'|^{-(2l+2p)a'}}{a_1 \cdots a_\mu \mu^\mu} J(z'),$$

where

$$\begin{aligned} J(z') &= \int_{\mathbf{R}^{\mu-1}} \left[\int_0^{2\pi}\int_0^{\frac{1}{|t|}e^{-G(\tau)}} \left(\left|1 + \frac{t\phi(0,z')}{|t|\rho e^{i\theta} z'^{a'}}\right|^{2p} - 1\right) \rho^{2l+2p} \frac{d\rho}{\rho} d\theta\right] d\tau \\ &= \int_{\mathbf{R}^{\mu-1}} \left[\int_{|w| \leqslant \frac{|z'|^{a'}}{|t|} e^{-G(\tau)}} \left(\left|1 + \frac{\phi(0,z')}{w}\right|^{2p} - 1\right) |w|^{2l+2p-2} du dv\right] d\tau. \end{aligned}$$

By Fubini's theorem,

$$J(z') = \int_{|w| \leqslant \frac{|z'|^{a'}}{|t|}} V\left(\ln\left(\frac{|z'|^{a'}}{|w||t|}\right)\right)\left(\left|1 + \frac{\phi(0,z')}{w}\right|^{2p} - 1\right) |w|^{2l+2p-2} du dv,$$

where

$$V(q) := \operatorname{Vol}\ \left\{\tau \in \mathbf{R}^{\mu-1}\,|G(\tau) \leqslant q\right\}$$

for any $q > 0$. $V(q) = V(1)q^{\mu-1}$ since G is positively homogeneous. Therefore,

$$\begin{aligned} J(z') &= V(1) \int\limits_{|w|\leqslant \frac{|z'|^{a'}}{|t|}} \left(\left| 1 + \frac{\phi(0,z')}{w} \right|^{2p} - 1 \right) \\ &\qquad |w|^{2l+2p-2} \left(\ln \left(\frac{|z'|^{a'}}{|w||t|} \right) \right)^{\mu-1} dudv \\ &= \sum_{\alpha+\beta+\gamma=\mu-1} \frac{V(1)(\mu-1)!}{\alpha!\beta!\gamma!} \left(\ln \frac{1}{|t|} \right)^{\alpha} \left(\ln |z'^{a'}| \right)^{\beta} K_\gamma \left(\frac{|z'|^{a'}}{|t|} \right), \end{aligned}$$

where

$$K_\gamma(s) = \int\limits_{|w|\leqslant s} \left(\left| 1 + \frac{\phi(0,z')}{w} \right|^{2p} - 1 \right) |w|^{2l+2p-2} \left(\ln \frac{1}{|w|} \right)^{\gamma} dudv. \tag{5.4}$$

A direct computation shows that $V_0 V(1) = \frac{\mu^\mu}{(\mu-1)^2}$. In summary, if we let $w = u + iv$, then

$$\begin{aligned} &\int\limits_{\Delta} g(0,0,x',y') \left(\left| Z^A + t\phi(0,z') \right|^{2p} - \left| Z^A \right|^{2p} \right) |Z|^{2B} \, dXdY \\ &= \sum_{\alpha+\beta+\gamma=\mu-1} \frac{C'_\mu}{\alpha!\beta!\gamma! a_1 \cdots a_\mu} |t|^{2l+2p} \left(\ln \frac{1}{|t|} \right)^{\alpha} \\ &\qquad \int\limits_{\Delta'} g(0,0,x',y') K_\gamma \left(\frac{|z'^{a'}|}{|t|} \right) |z'|^{2(b'-la')} \left(\ln |z'^{a'}| \right)^{\beta} dx'dy', \end{aligned} \tag{5.5}$$

where

$$C'_\mu = \begin{cases} 1 & \text{if } \mu = 1, \\ (\mu-2)!(2\pi)^{\mu-1} & \text{if } \mu > 1. \end{cases}$$

Now it remains to figure out the asymptotic behavior of the terms of the right-hand side of (5.5). We fix δ such that $0 < \delta < \min\{N, \delta(p)\}$ and decompose $\left\{ w : |w| \leqslant \frac{|z'|^{a'}}{|t|} \right\}$ (which is the domain of integration in forming

$$K_\gamma \left(\frac{|z'^{a'}|}{|t|} \right)$$

from (5.4)) into two parts:

$$\Omega_1(t) = \left\{ w : |w| \leqslant \frac{N}{\delta} \right\} \text{ and } \Omega_2(t) = \left\{ w : \frac{N}{\delta} \leqslant |w| \leqslant \frac{|z'|^{a'}}{|t|} \right\}.$$

On $\overline{\Delta}' \times \Omega_1(t)$, by (5.2), we have

$$\left| \left(\left| 1 + \frac{\phi(0,z')}{w} \right|^{2p} - 1 \right) |w|^{2l+2p-2} \left(\ln \frac{1}{|w|} \right)^{\gamma} |z'|^{2(b'-la')} \left(\ln |z'^{a'}| \right)^{\beta} \right|$$

$$\leqslant N^{2p} |w|^{2l-2} \left| \left(\ln \frac{1}{|w|} \right)^{\gamma} \right| \left| |z'|^{2(b'-la')} \left(\ln |z'^{a'}| \right)^{\beta} \right|. \tag{5.6}$$

The right-hand side of (5.6) is integrable since $l > 0$ and $b' - la' > -1$ (by the definition of $l = l(A, B)$). It is also independent of t. Therefore, the left-hand side of (5.6) contributes an $O(1)$ (as $t \longrightarrow 0$) to the integral factor of the right-hand side of (5.5).

On $\overline{\Delta}' \times \Omega_2(t)$, by (5.1), we have

$$\left| \left(\left| 1 + \frac{\phi(0,z')}{w} \right|^{2p} - 1 \right) |w|^{2l+2p-2} \left(\ln \frac{1}{|w|} \right)^{\gamma} |z'|^{2(b'-la')} \left(\ln |z'^{a'}| \right)^{\beta} \right|$$

$$\leqslant C(p) |w|^{2l+2p-3} \left| \left(\ln \frac{1}{|w|} \right)^{\gamma} \right| |z'|^{2(b'-la')} \left| \left(\ln |z'^{a'}| \right)^{\beta} \right|. \tag{5.7}$$

There are two situations:

(i) $2l + 2p = 1$.

$$\int_{\overline{\Delta}'} \int_{\Omega_2(t)} \left| \left(\left| 1 + \frac{\phi(0,z')}{w} \right|^{2p} - 1 \right) |w|^{-1} \left(\ln \frac{1}{|w|} \right)^{\gamma} \right|$$

$$\left| g(0,0,x',y') \left(\ln |z'^{a'}| \right)^{\beta} |z'|^{2(b'-la')} \right| du dv dx' dy' \tag{5.8}$$

$$\leqslant C(p) N \int_{\overline{\Delta}'} \int_{\Omega_2(t)} |w|^{-2} \left| \left(\ln \frac{1}{|w|} \right)^{\gamma} \right| du dv \ |z'|^{2(b'-la')} \left| \left(\ln |z'^{a'}| \right)^{\beta} \right| dx' dy'.$$

The inner integral can be evaluated explicitly by using polar coordinate system on $\Omega_2(t)$: write $w = re^{i\psi}$. Then the inner integral becomes

$$2\pi \int_{\frac{N}{\delta}}^{\frac{|z'|^{a'}}{|t|}} \left| \left(\ln \frac{1}{r} \right)^{\gamma} \right| \frac{dr}{r} = \frac{2\pi(-1)^{\gamma}}{\gamma+1} \left[\left(\ln \frac{|z'|^{a'}}{|t|} \right)^{\gamma+1} - \left(\ln \frac{N}{\delta} \right)^{\gamma+1} \right]. \tag{5.9}$$

which can be written as a sum each of whose terms is either a constant or a constant multiple of

$$\left(\ln(|z'|^{a'})\right)^{\kappa}\left(\ln\frac{1}{|t|}\right)^{\lambda}$$

for some nonnegative integers κ and λ with $\kappa+\lambda=\gamma+1$. Substituting (5.9) back into (5.8) and using that fact that $b'-la'>-1$, we see that the right-hand side of (5.8) contributes

$$O\left(|t|\left(\ln\frac{1}{|t|}\right)^{\mu}\right)\quad (as\ t\longrightarrow 0)$$

to the right-hand side of (5.5).

(ii) $2l+2p<1$.

$b'-la'>-1$ by the definition of $l=l(A,B)$, and hence in (5.7),

$$N|w|^{2l+2p-3}\left|\left(\ln\frac{1}{|w|}\right)^{\gamma}\right||z'|^{2(b'-la')}\left|\left(\ln|z'^{a'}|\right)^{\beta}\right|$$

is integrable on $\overline{\Delta}'\times\left\{w:|w|\geqslant\frac{N}{\delta}\right\}$. By Lebesgue's dominated convergence theorem, as $t\longrightarrow 0$,

$$\int_{\overline{\Delta}'}\int_{\Omega_1(t)}\left(\left|1+\frac{\phi(0,z')}{w}\right|^{2p}-1\right)|w|^{2l+2p-2}\left(\ln\frac{1}{|w|}\right)^{\gamma}$$

$$g(0,0,x',y')|z'|^{2(b'-la')}\left(\ln|z'^{a'}|\right)^{\beta}dudvdx'dy'$$

converges to

$$\int_{\overline{\Delta}'}\int_{\mathbf{C}}\left(\left|1+\frac{\phi(0,z')}{w}\right|^{2p}-1\right)|w|^{2l+2p-2}\left(\ln\frac{1}{|w|}\right)^{\gamma}$$

$$g(0,0,x',y')\left(\ln|z'^{a'}|\right)^{\beta}|z'|^{2(b'-la')}dudv\ dx'dy'.$$

Therefore, among the terms of the right-hand side of (5.5), the leading term is the one with $\gamma=\mu-1$. Consequently,

$$\int_{\overline{\Delta}_0}g(0,0,x',y')\left(\left|Z^A+t\phi(0,z')\right|^{2p}-\left|Z^A\right|^{2p}\right)|Z|^{2B}dXdY$$

$$=\quad c(A,B,\phi)|t|^{2l+2p}\left(\ln\frac{1}{|t|}\right)^{\mu-1}+o\left(|t|^{2l+2p}\left(\ln\frac{1}{|t|}\right)^{\mu-1}\right),$$

where $c(A,B,\phi)=$

$$C_\mu\int_{\overline{\Delta}'}\int_{\mathbf{C}}\left(\left|1+\frac{\phi(0,z')}{w}\right|^{2p}-1\right)|w|^{2l+2p-2}g(0,0,x',y')|z'|^{2(b'-la')}dudv\ dx'dy',$$

where

$$C_\mu = \begin{cases} \frac{1}{a_1} & \text{if } \mu = 1, \\ \frac{(2\pi)^{\mu-1}}{a_1 \cdots a_\mu (\mu-1)} & \text{if } \mu > 1. \end{cases}$$

Taking (when $\phi(0, z') \neq 0$)

$$\zeta = \frac{w}{\phi(0, z')} = \xi + i\eta,$$

we have

$$\begin{aligned} & \int_{\overline{\Delta}'} \int_{\mathbf{C}} \left(\left| 1 + \frac{\phi(0, z')}{w} \right|^{2p} - 1 \right) \\ & |w|^{2l+2p-2} g(0, 0, x', y') |z'|^{2(b'-la')} dudv \; dx'dy' \\ = \; & \int_{\overline{\Delta}'} \int_{\mathbf{C}} \left(\left| 1 + \frac{1}{\zeta} \right|^{2p} - 1 \right) \\ & |\zeta|^{2l+2p-2} |\phi(0, z')|^{2p} g(0, 0, x', y') |z'|^{2(b'-la')} d\xi d\eta \; dx'dy' \\ = \; & \int_{\mathbf{C}} \left(\left| 1 + \frac{1}{\zeta} \right|^{2p} - 1 \right) \\ & |\zeta|^{2l+2p} \frac{id\zeta d\overline{\zeta}}{2|\zeta|^2} \int_{\overline{\Delta}'} |\phi(0, z')|^{2p} g(0, 0, x', y') |z'|^{2(b'-la')} dx'dy'. \end{aligned}$$

It is an easy exercise to see that the first factor is positive. The second factor is obviously nonnegative, and is 0 if and only if

$$\phi(0, \ldots, 0, z_{\mu+1}, \ldots, z_n) \equiv 0.$$

This completes the proof. q.e.d.

Combining Lemmas 5.3, 5.4, and 5.5, we see that Theorem 2.1 holds.

Note added in proof: V. Markovic generalizes Royden's theorem for more general classes of Riemann surfaces than compact ones, whose approach simply applies Rudin's work on isometries between L_p spaces. Markovic's argument can be carried out verbatim to higher dimensions and provides a direct way of completely achieving Yau's proposal mentioned in the introduction. This was relayed to us by S. Antonakoudis after the current paper had been finished.

References

[1] U. Angehrna & Y.-T. Siu, *Effective freeness and point separation for adjoint bundles* Invent. Math. **122** (1995), no. 2, 291–308, MR 1358978, Zbl 0847.32035.

[2] C. Birkar, P. Cascini, C.D. Hacon & J. McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468, MR 2601039, Zbl 1210.14019.

[3] C.-Y. Chi, S.-S. Wang & C.-L. Wang, *Extensions of multiply twisted pluricanonical forms*, Pure Appl. Math. Q. **7** (2011), no. 4, Special Issue: In memory of Eckart Viehweg, 1129–1163, MR 2918157, Zbl 06107772.

[4] C.-Y. Chi and S.-T. Yau, *A geometric approach to problems in birational geometry*, Proc. Natl. Acad. Sci. USA **105** (2008), no. 48, 18696–18701, MR 2470870, Zbl 1205.14016.

[5] J.-P. Demailly, *Effective bounds for very ample line bundles* Invent. Math. **126** (1996), no. 1–3, 243–261, MR 1369417, Zbl 0862.14004.

[6] L. Ein, R. Lazarsfeld & M. Mustaţă, *Contact loci in arc spaces*, Compos. Math. **140** (2004), no. 5, 1229–1244, MR 2081163, Zbl 1060.14004.

[7] R. Friedman, *Algebraic surfaces and Holomorphic Vector Bundles*, New York: Springer (1998), MR 1600388, 0902.14029.

[8] F.P. Gardiner and N. Lakic, *Quasiconformal Teichmüller Theory*, Mathematical Surveys and Monographs, ISSN 0076-5376; vol. 76, MR 1730906, Zbl 0949.30002.

[9] C. Hacon & J. M^{c}Kernan, *Boundedness of pluricanonical maps of varieties of general type*, Invent. Math. **166** (2006), no. 1, 1–25, MR 2242631, Zbl 1121.14011.

[10] C. Hacon & J. M^{c}Kernan, *Existence of minimal models for varieties of log general type II*, J. Amer. Math. Soc. **23** (2010), no. 2, 469–490, MR 2601040, Zbl 1210.14021.

[11] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math. (2) **79** (1964), 109–203; ibid. (2) **79** (1964), 205–326, MR 0199184, Zbl 0122.38603.

[12] H. Hironaka, *Introduction to the theory of infinitely near singular points*, Memorias de Matematica del Instituto "Jorge Juan," No. 28 Consejo Superior de Investigaciones Cienti'ficas, Madrid (1974), MR 0399505, Zbl 1055.14013.

[13] Y. Kawamata, K. Matsuda & K. Matsuki, *Introduction to the minimal model problem*, Algebraic geometry–Sendai, Adv. Stud. Pure Math., **10** (1985), 283–360, MR 0946243, Zbl 0628.00007.

[14] J. Kollár, *Singularities of pairs*, Algebraic geometry—Santa Cruz 1995, 221–287, Proc. Sympos. Pure Math., **62**, Part 1, Amer. Math. Soc., Providence, RI, 1997, MR 1492525, Zbl 905.14002.

[15] J. Kollár, *Effective base point freeness*, Math. Ann. **296** (1993), no. 4, 595–605, MR 1233485, Zbl 0818.14002.

[16] R. Lazarsfeld, *Positivity in Algebraic Geometry. I*, Springer-Verlag, Berlin (2004), MR 2095471, Zbl 1093.14501.

[17] S. Mori, *Classification of higher-dimensional varieties*, Algebraic geometry—Bowdoin 1985, 269–331, Proc. Sympos. Pure Math., **46**, Part 1, Amer. Math. Soc., Providence, RI, 1987, MR 0927961, Zbl 656.14022.

[18] H.L. Royden, *Automorphisms and isometries of Teichmüller space* 1971 Advances in the Theory of Riemann Surfaces, Ann. of Math. Studies, **66**, 369–383, MR 0288254, Zbl 0218.32011.

[19] S. Takayama, *Pluricanonical systems on algebraic varieties of general type*, Invent. Math. **165** (2006), no. 3, 551–587, MR 2242627, Zbl 1108.14031.

Department of Mathematics
National Taiwan University
Taipei, Taiwan
E-mail address: chi@math.ntu.edu.tw