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A LOWER BOUND FOR THE NUMBER OF NEGATIVE EIGENVALUES OF SCHRÖDINGER OPERATORS

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Abstract

We prove a lower bound for the number of negative eigenvalues for a Schödinger operator on a Riemannian manifold via the integral of the potential.

1. Introduction

Let (M, g) be a compact Riemannian manifold without boundary. Consider the following eigenvalue problem on M:

$$-\Delta u - Vu = \lambda u, \tag{1}$$

where Δ is the Laplace-Beltrami operator on M and $V \in L^\infty(M)$ is a given potential. It is well-known, that the operator $-\Delta - V$ has a discrete spectrum. Denote by $\{\lambda_k(V)\}_{k=1}^\infty$ the sequence of all its eigenvalues arranged in increasing order, where the eigenvalues are counted with multiplicity.

Denote by $\mathcal{N}(V)$ the number of negative eigenvalues of (1), that is,

$$\mathcal{N}(V) = \operatorname{card}\{k : \lambda_k(V) < 0\}.$$

It is well-known that $\mathcal{N}(V)$ is finite. Upper bounds of $\mathcal{N}(V)$ have received enough attention in the literature, and for that we refer the reader to [**2**], [**5**], [**12**], [**11**], [**15**] and references therein.

However, a little is known about lower estimates. Our main result is the following theorem. We denote by μ the Riemannian measure on M.

Theorem 1.1. *Set* $\dim M = n$. *For any* $V \in L^\infty(M)$ *the following inequality is true:*

$$\mathcal{N}(V) \geq \frac{C}{\mu(M)^{n/2-1}} \left(\int_M V d\mu \right)_+^{n/2}, \tag{2}$$

where $C > 0$ *is a constant that in the case* $n = 2$ *depends only on the genus of* M *and in the case* $n > 2$ *depends only on the conformal class of* M.

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In the case $V \geq 0$ the estimate (2) was proved in [**6**, Theorems 5.4 and Example 5.12]. Our main contribution is the proof of (2) for signed potentials V (as it was conjectured in [**6**]), with the same constant C as in [**6**]. In fact, we reduce the case of a signed V to the case of non-negative V by considering a certain variational problem for V and by showing that the solution of this problem is non-negative. The latter method originates from [**14**].

In the case $n = 2$, inequality (2) takes the form

$$\mathcal{N}(V) \geq C \int_M V d\mu. \tag{3}$$

For example, the estimate (3) can be used in the following situation. Let M be a two-dimensional manifold embedded in $\mathbb{R}^3$ and the potential V be of the form $V = \alpha K + \beta H$ where K is the Gauss curvature, H is the mean curvature, and α, β are real constants (see [**8**], [**4**]). In this case (3) yields

$$\mathcal{N}(V) \geq C\left(K_{total} + H_{total}\right),$$

where K_{total} is the total Gauss curvature and H_{total} is the total mean curvature. We expect in the future many other applications of (2)-(3).

2. A variational problem

Fix positive integers k, N and consider the following optimization problem: find $V \in L^\infty(M)$ such that

$$\int_M V d\mu \to \max \text{ under restrictions } \lambda_k(V) \geq 0 \text{ and } \|V\|_{L^\infty} \leq N. \tag{4}$$

Clearly, the functional $V \mapsto \int_M V d\mu$ is weakly continuous in $L^\infty(M)$. Since the class of potentials V satisfying the restrictions in (4) is bounded in $L^\infty(M)$, it is weakly precompact in $L^\infty(M)$. In fact, we prove in the next lemma that this class is weakly compact, which will imply the existence of the solution of (4).

Lemma 2.1. *The class*

$$C_{k,N} = \{V \in L^\infty(M) : \lambda_k(V) \geq 0 \text{ and } \|V\|_{L^\infty} \leq N\}$$

is weakly compact in $L^\infty(M)$. Consequently, the problem (4) *has a solution $V \in L^\infty(M)$.*

Proof. It was already mentioned that the class $C_{k,N}$ is weakly precompact in $L^\infty(M)$. It remains to prove that it is weakly closed, that is, for any sequence $\{V_i\} \subset C_{k,N}$ that converges weakly in L^∞, the limit V is also in $C_{k.N}$. The condition $\|V\|_{L^\infty} \leq N$ is trivially satisfied by the limit potential, so all we need is to prove that $\lambda_k(V) \geq 0$. Let us use the minmax principle in the following form:

$$\lambda_k(V) = \inf_{\substack{E\subset W^{1,2}(M)\\ \dim E=k}} \sup_{u\in E\setminus\{0\}} \frac{\int_M |\nabla u|^2 d\mu - \int_M Vu^2 d\mu}{\int_M u^2 d\mu},$$

where E is a subspace of $W^{1,2}(M)$ of dimension k. The condition $\lambda_k(V) \geq 0$ is equivalent then to the following:

$$\begin{aligned} &\forall E\subset W^{1,2}(M) \text{ with } \dim E = k \quad \forall\varepsilon>0 \quad \exists u\in E\setminus\{0\} \\ &\text{such that } \int_M |\nabla u|^2 d\mu - \int_M Vu^2 d\mu \geq -\varepsilon\int_M u^2 d\mu. \end{aligned} \tag{5}$$

Fix a subspace $E\subset W^{1,2}(M)$ of dimension k and some $\varepsilon>0$. Since $\lambda_k(V_i)\geq 0$, we obtain that there exists $u_i\in E\setminus\{0\}$ such that

$$\int_M |\nabla u_i|^2 d\mu - \int_M V_i u_i^2 d\mu \geq -\varepsilon \int_M u_i^2 d\mu. \tag{6}$$

Without loss of generality we can assume that $\|u_i\|_{W^{1,2}(M)} = 1$. Then the sequence $\{u_i\}$ lies on the unit sphere in the finite-dimensional space E. Hence, it has a convergent (in $W^{1,2}(M)$-norm) subsequence. We can assume that the whole sequence $\{u_i\}$ converges in E to some $u\in E$ with $\|u\|_{W^{1,2}(M)}=1$. It remains to verify that u satisfies the inequality (5). By construction we have

$$\int_M |\nabla u_i|^2 d\mu \to \int_M |\nabla u|^2 d\mu \quad \text{and} \quad \int_M u_i^2 d\mu \to \int_M u^2 d\mu.$$

Next we have

$$\begin{aligned} \left|\int_M V_i u_i^2 d\mu - \int_M Vu^2 d\mu\right| &\leq \left|\int_M \left(V_i u_i^2 - V_i u^2\right) d\mu\right| \\ &+ \left|\int_M \left(V_i u^2 - Vu^2\right) d\mu\right| \\ &\leq N\|u_i-u\|_{L^2}^2 + \left|\int_M (V_i - V)\, u^2 d\mu\right|. \end{aligned}$$

By construction we have $\|u_i-u\|_{L^2}\to 0$ as $i\to\infty$. Since $u^2\in L^1(M)$, the L^∞ weak convergence $V_i \rightharpoonup V$ implies that

$$\int_M (V_i - V)\, u^2 d\mu \to 0 \ \text{ as } i\to\infty.$$

Hence, the inequality (5) follows from (6). q.e.d.

Lemma 2.2. *If N is large enough (depending on k and M) then any solution V of (4) satisfies $\lambda_k(V)=0$.*

Proof. Assume that $\lambda_k(V)>0$ and bring this to a contradiction. Consider the family of potentials

$$V_t = (1-t)V + tN \ \text{ where } t\in[0,1].$$

Since $V_t \geq V$, we have by a well-known property of eigenvalues that $\lambda_k(V_t) \leq \lambda_k(V)$. By continuity we have, for small enough t, that $\lambda_k(V_t) > 0$. Clearly, we have also $|V_t| \leq N$. Hence, V_t satisfies the restriction of the problem (4), at least for small t. If $\mu\{V < N\} > 0$ then we have for all $t > 0$

$$\int_M V_t d\mu > \int_M V d\mu,$$

which contradicts the maximality of V. Hence, we should have $V = N$ a.e.. However, if $N > \lambda_k(-\Delta)$ then $\lambda_k(-\Delta - N) < 0$ and $V \equiv N$ cannot be a solution of (4). This contradiction finishes the proof. q.e.d.

3. Proof of Theorem 1.1

The main part of the proof of Theorem 1.1 is contained in the following lemma.

Lemma 3.1. *Let $V_{\max}$ be a maximizer of the variational problem (4). Then $V_{\max}$ satisfies the inequality*

$$V_{\max} \geq 0 \text{ a.e. } on \ M.$$

3.1. Proof of Theorem 1.1 assuming Lemma 3.1. Choose N large enough, say

$$N > \sup_M |V|.$$

Set $k = \mathcal{N}(V) + 1$ so that $\lambda_k(V) \geq 0$. For the maximizer $V_{\max}$ of (4) we have

$$\int_M V\, d\mu \leq \int_M V_{\max}\, d\mu.$$

On the other hand, since $V_{\max} \geq 0$, we have by [**6**]

$$\mathcal{N}(V_{\max}) \geq \frac{C}{\mu(M)^{n/2-1}} \left(\int_M V_{\max}\, d\mu\right)^{n/2}.$$

Also, we have

$$\lambda_k(V_{\max}) \geq 0,$$

which implies

$$\mathcal{N}(V_{\max}) \leq k - 1 = \mathcal{N}(V).$$

Hence, we obtain

$$\begin{aligned}\mathcal{N}(V) &\geq \mathcal{N}(V_{\max}) \geq \frac{C}{\mu(M)^{n/2-1}} \left(\int_M V_{\max}\, d\mu\right)^{n/2} \\ &\geq \frac{C}{\mu(M)^{n/2-1}} \left(\int_M V\, d\mu\right)_+^{n/2},\end{aligned}$$

which was to be proved.

3.2. Some auxiliary results. Before we can prove Lemma 3.1, we need some auxiliary lemmas. The following lemma can be found in [**9**].

Lemma 3.2. *Let $V(t,x)$ be a function on $\mathbb{R}\times M$ such that, for any $t\in\mathbb{R}$, $V(t,\cdot)\in L^{\infty}(M)$ and $\partial_t V(t,\cdot)\in L^{\infty}(M)$. For any $t\in\mathbb{R}$, consider the Schrödinger operator $L_t=-\Delta-V(t,\cdot)$ on M and denote by $\{\lambda_k(t)\}_{k=1}^{\infty}$ the sequence of the eigenvalues of L_t counted with multiplicities and arranged in increasing order. Let λ be an eigenvalue of L_0 with multiplicity m; moreover, let*

$$\lambda=\lambda_{k+1}(0)=...=\lambda_{k+m}(0).$$

Let U_λ be the eigenspace of L_0 that corresponds to the eigenvalue λ and $\{u_1,...,u_m\}$ be an orthonormal basis in U_λ. Set for all $i,j=1,...,m$

$$Q_{ij}=\int_M \left.\frac{\partial V}{\partial t}\right|_{t=0} u_i u_j d\mu.$$

and denote by $\{\alpha_i\}_{i=1}^{m}$ the sequence of the eigenvalues of the matrix $\{Q\}_{i,j=1}^{m}$ counted with multiplicities and arranged in increasing order. Then we have the following asymptotic, for any $i=1,...,m$,

$$\lambda_{k+i}(t)=\lambda_{k+i}(0)-t\alpha_i+o(t)\ \text{as}\ t\to 0.$$

Given a connected open subset Ω of M with smooth boundary, the Dirichlet problem

$$\begin{cases}\Delta u=0 \ \text{ in } \Omega\\ u|_{\partial\Omega}=f\end{cases}$$

has for any $f\in C(\partial\Omega)$ a unique solution that can be represented in the form

$$u(y)=\int_{\partial\Omega} Q(x,y)\,f(x)\,d\sigma(x)$$

for any $y\in\Omega$, where $Q(x,y)$ is the Poisson kernel of this problem and σ is the surface measure on $\partial\Omega$. For any $y\in\Omega$, the function $q(x)=Q(x,y)$ on $\partial\Omega$ will be called the Poisson kernel at the source y. Note that $q(x)$ is continuous, positive and

$$\int_{\partial\Omega} q\,d\sigma=1.$$

Lemma 3.3. *Let Ω be a connected open subset of M with smooth boundary and x_0 be a point in Ω. Then, for any constant $N\geq 1$ there exists $\varepsilon=\varepsilon(\Omega,N,x_0)>0$ such that for any measurable set $E\subset\Omega$ with*

$$\mu(E)\leq\varepsilon$$

and for any positive solution $v\in C^2(\Omega)$ of the inequality

$$\Delta v+Wv\geq 0 \ \text{in}\ \Omega, \tag{7}$$

where

$$W = \begin{cases} N & in \;\; E, \\ -\frac{1}{N} & in \; \Omega \setminus E, \end{cases} \tag{8}$$

the following inequality holds

$$v(x_0) < \int_{\partial\Omega} v\, q d\sigma, \tag{9}$$

where q is the Poisson kernel of the Laplace operator at the source x_0.

Proof. For any $\delta > 0$ denote by A_δ the set of points in Ω at the distance $\leq \delta$ from $\partial\Omega$ (see Fig. 1) and consider the potential V_δ in Ω defined by

$$V_\delta = \begin{cases} N & \text{in } A_\delta, \\ -\frac{1}{N} & \text{in } \Omega \setminus A_\delta. \end{cases} \tag{10}$$

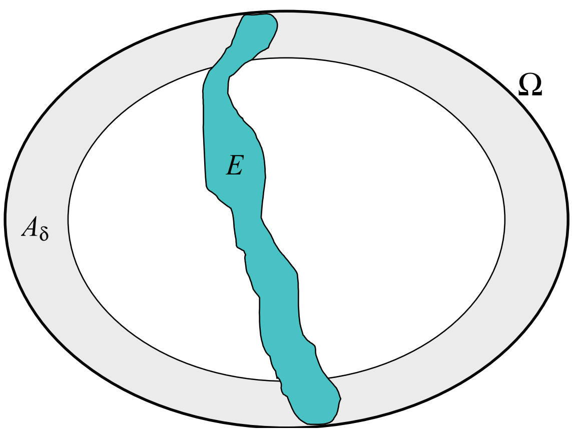

Figure 1

Since $\left\| V_\delta^+ \right\|_{L^p(\Omega)}$ can be made sufficiently small by the choice of $\delta > 0$, the following boundary value problem has a unique positive solution:

$$\begin{cases} \Delta w + V_\delta w = 0 \text{ in } \Omega \\ w = f \text{ on } \partial\Omega, \end{cases} \tag{11}$$

for any positive continuous function f on $\partial\Omega$. Denote by $q_\delta(x)$, $x \in \partial\Omega$, the Poisson kernel of (11) at the source x_0. Letting $\delta \to 0$, we obtain that the solution of (11) converges to that of

$$\begin{cases} \Delta w - \frac{1}{N} w = 0 \text{ in } \Omega \\ w = f \text{ on } \partial\Omega. \end{cases} \tag{12}$$

Denoting by q_0 the Poisson kernel of (12) at the source x_0, we obtain that $q_\delta \searrow q_0$ on $\partial\Omega$ as $\delta \searrow 0$ and, moreover, the convergence is uniform.

Let q be the Poisson kernel of the Laplace operator Δ in Ω, as in the statement of the theorem. Since any solution of (12) is strictly subharmonic in Ω, we obtain that $q_0 < q$ on $\partial\Omega$. In particular, there is a constant $\eta > 0$ depending only on Ω, N, x_0 such that

$$q_0 < (1-\eta)\, q \text{ on } \partial\Omega.$$

Since the convergence $q_\delta \to q$ is uniform on $\partial\Omega$, we obtain that, for small enough δ (depending on Ω, N, x_0),

$$q_\delta < (1-\eta/2)\, q \text{ on } \partial\Omega.$$

Fix such δ. Consequently, we obtain for the solution w of (11) that

$$w(x_0) < (1-\eta/2) \int_{\partial\Omega} fq d\sigma. \tag{13}$$

Note that the function W from (8) can be increased without violating (7). Define a new potential W_δ by

$$W_\delta = \begin{cases} N & \text{in } A_\delta \cup E, \\ -\frac{1}{N} & \text{in } \Omega \setminus A_\delta \setminus E. \end{cases} \tag{14}$$

Observe that, for any $p > 1$

$$\left\| W_\delta^+ \right\|_{L^p(\Omega)}^p \le N^p \left(\mu(A_\delta) + \varepsilon \right),$$

so that by the choice of ε and further reducing δ this norm can be made arbitrarily small. By a well-known fact (see [**13**]), if $\| W_\delta^+ \|_{L^p(\Omega)}$ is sufficiently small, then the operator $-\Delta - W_\delta$ in Ω with the Dirichlet boundary condition on $\partial\Omega$ is positive definite, provided $p = n/2$ for $n > 2$ and $p > 1$ for $n = 2$.

So, we can assume that the operator $-\Delta - W_\delta$ is positive definite. In particular, the following boundary value problem

$$\begin{cases} \Delta u + W_\delta u = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = v \end{cases} \tag{15}$$

has a unique positive solution u. Comparing this with (7) and using the maximum principle for the operator $\Delta + W_\delta$, we obtain $u \ge v$ in Ω. Since $u = v$ on $\partial\Omega$, the required inequality (9) will follow if we prove that

$$u(x_0) < \int_{\partial\Omega} uq d\sigma. \tag{16}$$

Set $\Omega_\delta = \Omega \setminus A_\delta$ and prove that

$$\sup_{\Omega_\delta} u \le C \int_{\partial\Omega} u d\sigma, \tag{17}$$

for some constant C that depends on Ω, N, δ, n. By choosing ε and δ sufficiently small, the norm $\|W_\delta\|_{L^p}$ can be made arbitrarily small for any p. Hence, function u satisfies the Harnack inequality

$$\sup_{\Omega_\delta} u \le C \int_{\Omega_\delta} u d\mu \tag{18}$$

where C depends on Ω, N, δ (see [**1**], [**7**]). Let h be the solution of the following boundary value problem

$$\begin{cases} -\Delta h - W_\delta h = 1_{\Omega_\delta} \text{ in } \Omega \\ h = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega_\delta = \Omega \setminus A_\delta$. Since $\|W_\delta\|_{L^q}$ is bounded for any q, we obtain by the known a priori estimates, that

$$\|h\|_{W^{2,p}(\Omega)} \le C \|1_{\Omega_\delta}\|_{L^p(\Omega)},$$

where $p > 1$ is arbitrary and C depends on Ω, N, δ, p (see [**10**]). Choose $p > n$ so that by the Sobolev embedding

$$\|h\|_{C^1(\Omega)} \le C \|h\|_{W^{2,p}(\Omega)}.$$

Since $\|1_{\Omega_\delta}\|_{L^p(\Omega)}$ is uniformly bounded, we obtain by combining the above estimates that

$$\|h\|_{C^1(\Omega)} \le C,$$

with a constant C depending on Ω, N, δ, n.

Multiplying the equation $-\Delta h - W_\delta h = 1_{\Omega_\delta}$ by u and integrating over Ω, we obtain

$$\int_{\Omega_\delta} u d\mu = \int_{\partial\Omega} \frac{\partial h}{\partial \nu} u \, d\sigma \le C \int_{\partial\Omega} u d\sigma$$

which together with (18) implies (17).

Let w be the solution (11) with the boundary condition $f = u$, that is,

$$\begin{cases} \Delta w + V_\delta w = 0 \text{ in } \Omega \\ w = u \text{ on } \partial\Omega. \end{cases}$$

Let us consider the difference

$$\varphi = u - w.$$

Clearly, we have in Ω

$$\Delta\varphi + V_\delta \varphi = (\Delta u + V_\delta u) - (\Delta w + V_\delta w) = (V_\delta - W_\delta) u$$

and $\varphi = 0$ on $\partial\Omega$. Denoting by G_{V_δ} the Green function of the operator $-\Delta - V_\delta$ in Ω with the Dirichlet boundary condition, we obtain

$$\varphi(x_0) = \int_\Omega G_{V_\delta}(x_0, y) (W_\delta - V_\delta) u(y) d\mu(y).$$

Since we are looking for an upper bound for $\varphi(x_0)$, we can restrict the integration to the domain $\{V_\delta \le W_\delta\}$. By (14) and (10) we have

$$\{V_\delta \le W_\delta\} = (\Omega \setminus A_\delta) \cap (A_\delta \cup E) = E \setminus A_\delta =: E'$$

and, moreover, on E' we have

$$W_\delta - V_\delta = N + \frac{1}{N} < 2N,$$

whence it follows that

$$\varphi(x_0) \le 2N \int_{E'} G_{V_\delta}(x_0, y)\, u(y)\, d\mu(y).$$

Using (17) to estimate here $u(y)$, we obtain

$$\varphi(x_0) \le 2NC \left(\int_{E'} G_{V_\delta}(x_0, y)\, d\mu(y) \right) \int_{\partial\Omega} u d\sigma.$$

Since $\mu(E') \le \varepsilon$ and the Green function $G_{V_\delta}(x_0, \cdot)$ is integrable, we see that $\int_{E'} G_{V_\delta}(x_0, \cdot)\, d\mu$ can be made arbitrarily small by choosing $\varepsilon > 0$ small enough. Choose ε so small that

$$2NC \int_{E'} G_{V_\delta}(x_0, y)\, d\mu(y) < \eta/2 \inf_{\partial\Omega} q,$$

which implies that

$$\varphi(x_0) < \eta/2 \int_{\partial\Omega} uqd\sigma.$$

Since by (13)

$$w(x_0) < (1 - \eta/2) \int_{\partial\Omega} uqd\sigma,$$

we obtain

$$u(x_0) = \varphi(x_0) + w(x_0) < \int_{\partial\Omega} uqd\sigma,$$

which was to be proved. q.e.d.

Let $V_{\max}$ be a solution of the problem (4). Denote by U the eigenspace of $-\Delta - V_{\max}$ associated with the eigenvalue $\lambda_k(V_{\max}) = 0$ assuming that N is sufficiently large.

Lemma 3.4. *Fix some $c > 0$ and consider the set*

$$F = \{V_{max} \le -c\}.$$

Then, for any Lebesgue point $x \in F$, then there exists a non-negative function $q \in L^\infty(M)$ such that

1) $\int_M q\, d\mu = 1$;
2) *for any $u \in U \setminus \{0\}$ we have*

$$u^2(x) < \int_M u^2 q\, d\mu. \tag{19}$$

Proof. Set $V = V_{\max}$. Any function $u \in U$ satisfies $\Delta u + Vu = 0$, which implies by a simple calculation that the function $v = u^2$ satisfies

$$\Delta v + 2Vv \ge 0.$$

Next, we apply Lemma 3.3 with $W = 2V$ where instead of parameter N there we will use $N' = \max(2N, \frac{1}{2c})$. Choose r so small that

$$\mu\left(F \cap B\left(x,r\right)\right) > \left(1-\varepsilon\right)\mu\left(B\left(x,r\right)\right),$$

where $\varepsilon = \varepsilon\left(N'\right)$ is given in Lemma 3.3. Since $W \leq 2N \leq N'$ in $B\left(x,r\right)$ and

$$\begin{aligned}\mu\left(\left\{W > -\frac{1}{N'}\right\} \cap B\left(x,r\right)\right) &\leq \mu\left(\left\{W > -2c\right\} \cap B\left(x,r\right)\right)\\ &= \mu\left(\left\{V > -c\right\} \cap B\left(x,r\right)\right)\\ &< \varepsilon\mu\left(B\left(x,r\right)\right),\end{aligned}$$

all the hypotheses of Lemma 3.3 in $\Omega = B\left(x,r\right)$ are satisfied. Let q be the function from Lemma 3.3. Extending q by setting $q = 0$ outside $B\left(x,r\right)$ we obtain (19). q.e.d.

3.3. Proof of main Lemma 3.1. We can now prove Lemma 3.1, that is, that $V_{\max} \geq 0$. Consider again the set

$$F = \left\{V_{max} \leq -c\right\},$$

where $c > 0$. We want to show that, for any $c > 0$,

$$\mu(F) = 0,$$

which will imply the claim. Assume the contrary, that is $\mu(F) > 0$ for some $c > 0$. Denote by F_L the set of Lebesgue points of F. For any $x \in F_L$ denote by q_x the function q that is given by Lemma 3.4. For $x \notin F_L$ set $q_x = \delta_x$. Then $x \mapsto q_x$ is a Markov kernel and, for all $x \in M$ and $u \in U$

$$u^2\left(x\right) \leq \int_M u^2 q_x d\mu. \tag{20}$$

Denote by $\mathcal{M}$ the set of all probability measures on M. Define on $\mathcal{M}$ a partial order: $\nu_1 \preceq \nu_2$ if and only if

$$\int_M u^2 d\nu_1 \leq \int_M u^2 d\nu_2 \text{ for all } u \in U \setminus \{0\}. \tag{21}$$

Define $\nu_0 \in \mathcal{M}$ by

$$d\nu_0 = \frac{1}{\mu\left(F_L\right)}\mathbf{1}_{F_L}d\mu$$

and measure $\nu_1 \in \mathcal{M}$ by

$$\nu_1 = \int_M q_x d\nu_0\left(x\right).$$

Since $\nu_0(F_L) > 0$, we obtain for any $u \in U \setminus \{0\}$ that

$$\begin{aligned}\int_M u^2 d\nu_1 &= \int_M \left(\int_M u^2 q_x d\mu\right) d\nu_0(x) \\ &\geq \int_{F_L} \left(\int_M u^2 q_x d\mu\right) d\nu_0(x) + \int_{M\setminus F_L} \left(\int_M u^2 q_x d\mu\right) d\nu_0(x) \\ &> \int_{F_L} u^2(x)\, d\nu_0(x) + \int_{M\setminus F_L} u^2(x)\, d\nu_0(x) \\ &= \int_M u^2 d\nu_0. \qquad (22)\end{aligned}$$

In particular, we have $\nu_0 \preceq \nu_1$. Consider the following subset of $\mathcal{M}$:

$$\mathcal{M}_1 = \{\nu \in \mathcal{M} : \nu \succeq \nu_1\}.$$

Let us prove that $\mathcal{M}_1$ has a maximal element. By Zorn's Lemma, it suffices to show that any chain (=totally ordered subset) $\mathcal{C}$ of $\mathcal{M}_1$ has an upper bound in $\mathcal{M}_1$. It follows from $\dim U < \infty$ that there exists an increasing sequence $\{\nu_i\}_{i=1}^{\infty}$ of elements of $\mathcal{C}$ such that, for all $u \in U$,

$$\lim_{i\to\infty} \int_M u^2 d\nu_i \to \sup_{\{\nu\in\mathcal{C}\}} \int_M u^2 d\nu.$$

The sequence $\{\nu_i\}_{i=1}^{\infty}$ of probability measures is w^*-compact. Without loss of generality we can assume that this sequence is w^*-convergent. It follows that the measure

$$\nu_{\mathcal{C}} = w^*\text{-}\lim \nu_i \in \mathcal{M}_1$$

is an upper bound for $\mathcal{C}$.

By Zorn's Lemma, there exists a maximal element ν in $\mathcal{M}_1$. Note that the measure ν can be alternatively constructed by using a standard balayage procedure (see e.g. [**3**, Proposition 2.1, p. 250]). Consider first the measure ν' defined by $\nu' = \int_M q_x d\nu(x)$. It follows from (20) that for any $u \in U$

$$\begin{aligned}\int_M u^2 d\nu' &= \int_M \left(\int_M u^2 q_x d\mu\right) d\nu \\ &\geq \int_M u^2 d\nu,\end{aligned}$$

that is, $\nu' \succeq \nu$, in particular, $\nu' \in \mathcal{M}_1$. Since ν is a maximal element in $\mathcal{M}_1$, it follows that $\nu' = \nu$, which implies the identity

$$\int_M u^2 d\nu = \int_M \left(\int_M u^2 q_x d\mu\right) d\nu. \qquad (23)$$

Now we can prove that $\nu(F_L) = 0$. Assuming from the contrary that $\nu(F_L) > 0$, we obtain, for any $u \in U \setminus \{0\}$.

$$\begin{aligned}
\int_M u^2 d\nu &= \int_M \left(\int_M u^2 q_x d\mu\right) d\nu(x) \\
&\geq \int_{F_L} \left(\int_M u^2 q_x d\mu\right) d\nu(x) + \int_{M\setminus F_L} \left(\int_M u^2 q_x d\mu\right) d\nu(x) \\
&> \int_{F_L} u^2(x)\, d\nu(x) + \int_{M\setminus F_L} u^2(x)\, d\nu(x) \\
&= \int_M u^2 d\nu, \qquad (24)
\end{aligned}$$

which is a contradiction. Finally, it follows from (22) and $\nu \in \mathcal{M}_1$ that, for any $u \in U \setminus \{0\}$,

$$\int_M u^2 d\nu_0 < \int_M u^2 d\nu.$$

Measure ν can be approximated in w^*-sense by measures with bounded densities sitting in $M \setminus F_L$. Therefore, there exists a non-negative function $\varphi \in L^\infty(M)$ that vanishes on F_L and such that

$$\int_M \varphi d\mu = 1$$

and, for any $u \in U \setminus \{0\}$,

$$\int_M u^2 \varphi_0 d\mu < \int_M u^2 \varphi d\mu \qquad (25)$$

where $\varphi_0 = \frac{1}{\mu(F_L)} \mathbf{1}_{F_L}$. Consider now the potential

$$V_t = V_{max} + t\varphi_0 - t\varphi.$$

We have for all t

$$\int_M V_t d\mu = \int_M V_{\max} d\mu$$

and for $t \to 0$

$$\lambda_k(V_t) = \lambda_k(V_{\max}) - t\alpha + o(t),$$

where α is the minimal eigenvalue of the quadratic form

$$Q(u,u) = \int_M u^2 (\varphi_0 - \varphi)\, d\mu,$$

which by (25) is negative definite. Therefore, $\alpha < 0$, which together with $\lambda_k(V_{\max}) = 0$ implies that, for all small enough $t > 0$

$$\lambda_k(V_t) > 0.$$

Finally, let us show that $|V_t| \leq N$ a.e. Indeed, on F we have

$$V_t \leq -c + t\varphi_0 < N$$

for small enough $t > 0$, and on $M \setminus F_L$ we have

$$V_t \leq V_{\max} - t\varphi \leq V_{\max} \leq N.$$

Therefore, $V \leq N$ a.e. for small enough $t > 0$. Similarly, we have on F_L

$$V_t \geq V_{\max} + t\varphi_0 \geq V_{\max} \geq -N$$

and on $M \setminus F$

$$V_t \geq -c - t\varphi \geq -N$$

for small enough $t > 0$, which implies that $|V_t| \leq N$ a.e. for small enough $t > 0$.

Hence, we obtain that V_t is a solution to our optimization problem (4), but it satisfies $\lambda_k(V_t) > 0$, which contradicts the optimality of V_t by Lemma 2.2.

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