J. DIFFERENTIAL GEOMETRY
102 (2016) 243-284

ON SPACELIKE ZOLL SURFACES WITH SYMMETRIES

Pierre Mounoud & Stefan Suhr

Abstract

Three explicit families of spacelike Zoll surface admitting a Killing field are provided. It allows to prove the existence of spacelike Zoll surfaces not smoothly conformal to a cover of de Sitter space as well as the existence of Lorentzian Möbius strips of non constant curvature all of whose spacelike geodesics are closed. Further the conformality problem for spacelike Zoll cylinders is studied.

1. Introduction

A spacelike Zoll surface is a Lorentzian surface all of whose spacelike geodesics are closed and simple and have the same length. The basic example of a spacelike Zoll surface is de Sitter space, the homogeneous space $\mathrm{SO}_0(2,1)/\mathrm{SO}_0(1,1)$, and its finite coverings since it is not simply connected. It can be understood as the Lorentzian analogue of the round sphere as it has constant positive curvature. In [**9**], the authors proved that a spacelike Zoll surface is diffeomorphic to a cylinder or a Möbius strip. This purely topological classification leaves open the finer questions of a classification up to isometry or conformality. Recall that the cylinder as well as the Möbius strip admit uncountable many non equivalent conformal Lorentzian structures.

The purpose of this article is twofold. First it provides three infinite dimensional families of examples of spacelike Zoll surfaces, in order to test answers to the questions that arise in the study of these surfaces. Second it tries to initiate a study of the conformal properties of spacelike Zoll surfaces since this is the main difference, besides the topological one, to the Riemannian case.

In the Riemannian case several explicit families of Zoll surfaces, i.e. surfaces all of whose geodesics are simple and closed, are known. The most famous family is certainly the Zoll spheres of revolution, i.e. with a Killing vector field, classified by Zoll and Darboux (see [**3**] chap. 4). The first work in this direction for spacelike Zoll surfaces has been done by Boucetta [**5**], who provided examples of spacelike Zoll cylinders of

Received 03/07/2014.

revolution, i.e. admitting a periodic spacelike Killing field. However, contrary to a Riemannian 2-sphere, Killing fields of Lorentzian cylinders are not periodic in general. Already, on de Sitter space, there exist three conjugacy classes of Killing fields: the elliptic, the parabolic and the hyperbolic one, each corresponding to a conjugacy class of 1-dimensional subgroups of $\mathrm{SO}_0(2,1)$ acting on $\mathrm{SO}_0(2,1)/\mathrm{SO}_0(1,1)$. Following this line of ideas this paper investigates spacelike Zoll surfaces admitting a non trivial Killing field.

For general spacelike Zoll cylinders the dynamics and the causal character of a Killing vector field coincides with that of a Killing vector field on de Sitter space, see Proposition 3.4. Thus there exist only three types: elliptic, parabolic and hyperbolic. Adding technical assumptions allows to prove the following (see Theorems 5.6, 6.1 and 7.5 for more precise statements):

Theorem. *There exists infinite dimensional families of explicit spacelike Zoll surfaces of elliptic, parabolic and hyperbolic types.*

These metrics are constructed as deformations of a covering of de Sitter space, the deformation preserving a chosen Killing field K. In the elliptic case, this construction is completely analogue to the one made for Riemannian Zoll surfaces of revolution in chapter 4 of [**3**] and the family obtained gives a complete classification. However, when K has lightlike orbits, new ingredients have to be added and the deformations are realized via atlases adapted to K (see Definitions 5.1, 7.1). These atlases are inspired by ideas used by Ch. Bavard and the first author in [**1**]. At the moment the authors are not aware of an example of a spacelike Zoll surface with a Killing vector field that does not belong to one of these families, but conjecture that such metrics exist.

Besides the classification problem for Zoll metrics there is the rigidity problem for Zoll projective planes proven by Green [**8**] and recently extended by Pries [**11**] to surfaces all of whose geodesics are closed. These notes present a new feature of spacelike Zoll surfaces, opposing the Riemannian case:

Theorem. *There exist Lorentzian Möbius strips of non constant curvature all of whose spacelike geodesics are closed.*

The examples constructed are covered by smooth spacelike Zoll metrics with non constant curvature and parabolic or hyperbolic Killing vector fields invariant by antipody, see Corollaries 5.10 and 7.9. So far it is not clear whether the geodesics of the Möbius strips obtained are simple and all have the same length i.e. if these metrics are spacelike Zoll. It is interesting however to note that none of the three families contain real-analytic metrics invariant by antipody.

Dropping the assumption of a Killing vector field two major results on Riemannian Zoll surfaces remain: one is the theorem by Green [**8**] and its recent extension in [**11**] mentioned before. The other one is the theorem of Guillemin [**7**] saying that the space of Zoll metrics on S^2 in the conformal class of the constant curvature metric g_0 is a manifold near g_0 and the tangent space at g_0 is precisely the space of odd functions on S^2. If Guillemin confined his study to the conformal deformations of the round sphere it is because of the uniformization theorem. Note that a uniformization theorem does not exist for Lorentzian surfaces and there exists an infinite number of non isometric conformal classes of Lorentzian cylinders. So naturally the question appears which conformal classes of Lorentzian cylinders are represented by spacelike Zoll metrics.

The conformal class of an orientable Lorentzian surface is simply given by its pair of lightlike foliations. Hence, for $0 \leq n \leq \infty$ two surfaces are C^n-conformal if there exists a C^n-diffeomorphism exchanging their lightlike foliations. It is then clear that two metrics may be C^n but not C^{n+1}-conformal for any $0 \leq n \leq \infty$. As often in Lorentzian geometry, the question of determining the conformal classes is quite subtle. Here the conformal class are determined by considering the map from the space of lightlike geodesics into itself obtained by reflection on the conformal boundary. This map is clearly a conformal invariant.

This paper contains essentially three results on the conformal class of a spacelike Zoll cylinder (C, g). Without further assumption, it is shown that a two-fold cover of (C, g) conformally embeds into de Sitter, see Proposition 2.6. Furthermore, combining the results of Theorem 4.2 and Theorem 5.11, we have:

Theorem. *Let (C, g) be a spacelike Zoll cylinder with a nontrivial Killing vector field. Then (C, g) is C^0-conformal to a cover of de Sitter space.*

Besides, there exists parabolic spacelike Zoll cylinders that are not C^2-conformal to any cover of de Sitter space.

The authors conjecture that there exist spacelike Zoll metrics with a Killing field which are not C^1-conformal to a cover of de Sitter. An extended classification of parabolic spacelike Zoll cylinders could yield such a result.

The paper is organized as follows: section 2 studies spacelike Zoll surfaces without assuming the presence of a Killing field; section 3 gives the description of spacelike Zoll cylinders admitting a Killing field; section 4 determines the C^0-conformal class of these metrics; sections 5, 6 and 7 are devoted to the construction of the families of examples, finally section 8, following an idea of Blaschke, explains how it is possible to blend the preceding constructions in order to find examples that do not admit any Killing fields.

Acknowledgement. The first author wishes to thank J.F. Bony for useful discussions and J. Lafontaine for pointing his attention towards the question of the existence of non constant curvature spacelike Zoll Möbius strips. The second author wishes to thank J.-L. Flores for explanations in connection with the causal boundary and the conformal boundary of spacetimes.

The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement 307062.

2. General spacelike Zoll surfaces

Proposition 2.1. *Let (C, g) be a pseudo-Riemannian cylinder all of whose spacelike geodesics are closed. Then (C, g) is globally hyperbolic and the universal cover of $(C, -g)$ is not globally hyperbolic.*

Lemma 2.2. *Let (C, g) be a Lorentzian cylinder with at least one non timelike or non spacelike loop. Then (C, g) is space- and time-orientable.*

Proof. By exchanging g with $-g$, if necessary, we can assume that the loop in the assumption is non timelike. Further since C is a surface we can assume that the loop is simple. Well-known arguments in Lorentzian geometry (cp. [**10**]) allow us to additionally assume that the loop is smooth and regular.

Let $\gamma\colon [0,1] \to C$ be a simple closed, smooth and regular non timelike loop in (C, g). Further let v_l and v_r be lightlike vectors at $\gamma(0)$ pointing to the same side of γ. If γ itself is a lightlike pregeodesic, the subsequent argument will apply to the lightlike direction not tangent to γ. Denote with η_l and η_r the geodesics with direction v_l and v_r. Lift both to $\mathbb{R}^2$. Then not both lightlike geodesics can be invariant (even up to a finite quotient) under the group of deck transformations. So w.lo.g. we can assume that the lift of γ lies on one side of $\widetilde{\eta}_l$ the lift of η_l. Now consider the strip bounded by $\widetilde{\eta}_l$ and its translate by the deck transformation α induced by the fundamental class of γ. If $\dot\gamma(1)$ does not lie in the same connected component of $\{w \in TC_{\gamma(0)} |\ g(w, w) \geq 0\} \setminus \{0\}$ as $\dot\gamma(0)$, then $\widetilde{\gamma}$ and $\alpha \circ \widetilde{\gamma}$ lie in the strip bounded by $\widetilde{\eta}_l$ and $\alpha \circ \widetilde{\eta}_l$. But then α will have a fixed point in that strip, which contradicts the assumption that α is a deck transformation.

Consequently (C, g) is space-orientable. Together with the orientability of C this implies the time-orientability of (C, g) as well. q.e.d.

Proof of Proposition 2.1. First we prove that (C, g) is globally hyperbolic. W.l.o.g. we can assume that (C, g) is spacelike Zoll (see [**9**, Theorem 6.1]). It is well known that global hyperbolicity is passed down to finite quotients. Let γ be any spacelike geodesic of (C, g). By assumption γ is an embedded closed hypersurface. We claim that γ is a

Cauchy hypersurface in (C, g). Let η be an inextendable causal curve in (C, g) that does not intersect γ. Choose any curve from a point on γ to a point on η and parallel transport the tangent vector $\dot\gamma$ along that curve. Denote the transported vector by v. Since η is causal the spacelike geodesic with direction v is transversal to η (w.l.o.g. we can assume η to be smooth.). This induces a smooth family of closed curves transversal to a given inextendable curve at one end and disjoint form it at the other end. This is of course impossible. Therefore η intersects γ, as γ disconnects C this intersection is reduced to a point and γ is a Cauchy hypersurface.

Next we show that the universal cover $(\widetilde{C}, -\widetilde{g})$ of $(C, -g)$ is not globally hyperbolic. Consider a deck transformation ϕ of $\widetilde{C} \to C$ and a point $p \in \widetilde{C}$. W.l.o.g. we can assume that $\phi(p) \in I^+(p)$, relative to the time-orientable metric $-\widetilde{g}$. Else consider ϕ^{-1} instead of ϕ. Now if $(\widetilde{C}, -\widetilde{g})$ is globally hyperbolic then the space of causal arcs between p and $\phi(p)$ is compact. By the limit curve lemma and the assumption that every future pointing $(-\widetilde{g})$-timelike geodesic from p intersects $\phi(p)$ we can conclude that both lightlike geodesics emanating from p intersect $\phi(p)$. Since they are curves belonging to different transversal foliations this is impossible. q.e.d.

Proposition 2.3. *Let (C, g) be a Lorentzian cylinder all of whose spacelike geodesics are closed. Then any pair of spacelike geodesics intersects at least twice. The number of intersections is even and constant throughout the set of spacelike geodesics.*

Proof. The first assertion follows from the second, since obviously there are intersecting spacelike geodesics and any pair of loops in the cylinder has to intersect at least twice if they intersect once.

Since (C, g) is spacelike Zoll we can assume with [**9**] that the geodesic flow on T^1C is a free S^1-action. Fix a simple and closed unit speed spacelike geodesic γ and consider the tangent curve $\dot\gamma$. Further let η_1, η_2 be two closed and simple unit speed spacelike geodesics geometrically different from γ. Then the tangent curves $\dot\eta_1$ and $\dot\eta_2$ are disjoint from $\dot\gamma$. Since $\dot\gamma$ is a loop in the 3-manifold T^1C we can connect $\dot\eta_1(1)$ and $\dot\eta_2(1)$ via a path $\mu\colon [0,1] \to T^1C$ not intersecting $\dot\gamma$. The geodesics with initial direction $\mu(s)$ form a (smooth) homotopy by spacelike geodesics geometrically different from γ with endpoints η_1 and η_2. Since any intersection between geometrically different geodesics is transversal, the number of intersection between γ and η_1 has to coincide with the number of intersection between γ and η_2. q.e.d.

Lemma 2.4. *Every globally hyperbolic* 2*-dimensional cylinder is globally conformally flat.*

This fact is actually folklore. For the sake of completeness and in the absence of a reference we give a proof. Note that the lemma shows that every globally hyperbolic cylinder is conformal to one connected component of the complement of at most two simply connected and disjoint non timelike curves. Conversely of course every such component is globally hyperbolic.

Proof. Denote the lightlike foliations of (C,g) by $\mathcal{F}_{1,2}$. Let γ be any smooth Cauchy hypersurface in a globally hyperbolic cylinder (C,g). Choose a diffeomorphism $\varphi\colon \gamma \to S^1$. Define two maps $\alpha, \beta\colon C \to S^1$ to be identical to φ on γ, α to be constant on the leafs of $\mathcal{F}_1$ and β to be constant on the leafs of $\mathcal{F}_2$. Since the lightlike foliations are transversal the differentials of α and β are linearly independent at every point. Lifting everything to the universal cover gives two coordinates x, y whose level sets are lightlike. Therefore the metric in these coordinates reads $f(x,y)dxdy$ with $f(x+2\pi, y+2\pi) = f(x,y)$. Consequently f descends to the quotient and the metric $\frac{1}{f}g$ is flat. q.e.d.

Remark 2.5. Next we want to fix a conformal embedding of de Sitter space into $(S^1 \times \mathbb{R}, d\varphi^2 - dt^2)$. De Sitter space is isometric to $(S^1 \times \mathbb{R}, \cosh^2(t)d\varphi^2 - dt^2)$. So in order to construct a conformal embedding into the flat cylinder we have to find a reparameterization $\psi\colon (0,b) \to \mathbb{R}$ such that $(\mathrm{id} \times \psi)^*(\cosh^2(t)d\varphi^2 - dt^2)$ is diagonal. This is equivalent to solving the ODE $(\psi')^2(s) = \cosh^2 \psi(s)$. Since ψ is supposed to be a diffeomorphism we can assume that $\psi' > 0$. Therefore we have to solve the equation $\psi'(s) = \cosh \psi(s)$. In fact we do not need the solution ψ explicitly. All we require is the value b, i.e. the length of the domain of ψ. This can be done by integration: We know that

$$\int_0^s \frac{\psi'(\sigma)}{\cosh \psi(\sigma)} d\sigma = s$$

for all $s \in (0,b)$. For $t = \psi(s)$ we then see that (wlog $\lim_{s \to 0} \psi(s) = -\infty$)

$$\int_{-\infty}^t \frac{1}{\cosh(\tau)} d\tau = \psi^{-1}(t).$$

The left hand side is equal to $2 \arctan e^t$ (which give the solution $\psi = \log(\tan \frac{s}{2})$) and therefore tends to π for $t \to \infty$. Thus de Sitter space is conformal to a flat cylinder with circumference 2π and height π.

Proposition 2.6. *Let (C^2, g) be a Lorentzian spacelike Zoll cylinder. Then for all $\varepsilon > 0$ there exists a smooth conformal embedding of (C,g) into $(S^1 \times \mathbb{R}, d\varphi^2 - dt^2)$ whose image is contained in $S^1 \times (-\varepsilon, \pi + \varepsilon)$. Especially up to a twofold covering (C,g) admits a conformal embedding into de Sitter space.*

Note that the conformal embedding is not surjective in general. Finite coverings of de Sitter serve as examples.

Proof. Let (C, g) be a spacelike Zoll surface and $F\colon C \to S^1 \times \mathbb{R}$ be a conformal embedding. Consider the image of a lightlike geodesic of (C, g) in $S^1 \times \mathbb{R}$. We can assume that $(0, 0)$ lies on the image and that the image is symmetric about $(0, 0)$. Denote with (a, a) and $(-a, -a)$ its future and past endpoint respectively on $\partial F(C)$ in $S^1 \times \mathbb{R}$. Since (C, g) is spacelike Zoll the fundamental class of every spacelike geodesic generates $\pi_1(C)$ and via the conformal embedding their images generate $\pi_1(S^1 \times \mathbb{R})$. Note that due to the continuity of the geodesic flow for every pair of neighborhoods of (a, a) and $(-a, -a)$ there exist spacelike geodesics passing through $(0, 0)$ and intersecting these neighborhoods. The geodesics have to close up after one round. Since the circumference of the circle is 2π, the value of $4a$ is bounded from above by 2π, i.e. $a \leq \pi/2$.

Let γ be a lightlike geodesic in (C, g). Denote with x and y the future and past endpoints of $F \circ \gamma$ in $S^1 \times \mathbb{R}$ respectively. Since (C, g) is globally hyperbolic the boundary of $F(C)$ is achronal (see e.g. [**6**] Theorem 3.29 and 4.16). Therefore we have $F(C) \subseteq S^1 \times \mathbb{R} \setminus (I^+(x) \cup I^-(y))$. The last set being compact shows that $\partial F(C)$ consists of two closed and simple disjoint non timelike curves $\gamma^{\pm}\colon S^1 \to S^1 \times \mathbb{R}$ with $t \circ \gamma^+ > t \circ \gamma^-$. Since neither γ^+ nor γ^- can be everywhere lightlike, we can approximate both curves up to a given error $\varepsilon > 0$ by smooth, closed and simple disjoint spacelike curves $\gamma_\varepsilon^{\pm}\colon S^1 \to S^1 \times \mathbb{R}$. Note that the precompact component of $S^1 \times \mathbb{R} \setminus (\gamma_\varepsilon^+(S^1) \cup \gamma_\varepsilon^-(S^1))$ is a globally hyperbolic spacetime which is ε-close to $F(C)$ whenever $\gamma_\varepsilon^{\pm}$ are ε-close to $\gamma^{\pm}$.

Now consider the cylinder $(S^1 \times \mathbb{R}, d\varphi^2 - dt^2)$ as the quotient of $(\mathbb{R}^2, dxdy)$ by the $\mathbb{Z}$-operation generated by $(x, y) \mapsto (x + \sqrt{2}\pi, y + \sqrt{2}\pi)$. Denote with $\widetilde{\gamma}_\varepsilon^{\pm}$ the lift of $\gamma_\varepsilon^{\pm}$ to $\mathbb{R}^2$. W.l.o.g. we can assume that $\widetilde{\gamma}_\varepsilon^{\pm}$ are parameterized as graphs over the x-axis, i.e. $\widetilde{\gamma}_\varepsilon^{\pm}(s) = (s, \theta_\varepsilon^{\pm}(s))$ for some maps $\theta_\varepsilon^{\pm}\colon \mathbb{R} \to \mathbb{R}$. Since $\gamma_\varepsilon^{\pm}$ are spacelike, closed and simply $\theta_\varepsilon^{\pm}$ are $\mathbb{Z}$-equivariant diffeomorphisms.

Set $\theta_\varepsilon(s) := \frac{1}{2}(\theta_\varepsilon^+(s) + \theta_\varepsilon^-(s))$. θ_ε is obviously a diffeomorphism of the reals. Define the diffeomorphism $\Theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $\Theta(x, y) := (x, \theta_\varepsilon^{-1}(y))$. It is conformal and maps the spacelike curve $s \mapsto (s, \theta_\varepsilon(s))$ to the diagonal $\triangle = \{(x, y) |\, x = y\}$. We know that

$$\sup_s |\theta_\varepsilon^+(s) - \theta_\varepsilon^-(s)| \leq \sqrt{2}(\pi + \varepsilon),$$

i.e. the curves $\Theta \circ \widetilde{\gamma}_\varepsilon^{\pm}$ have distance at most $\frac{\pi+\varepsilon}{2}$ from the diagonal. This implies the same maximal distance from the sets $\{t = 0\}$ in the quotient space. Since for the chosen conformal embedding of de Sitter we have $\theta^{\pm}(s) = s \pm \frac{\pi}{\sqrt{2}}$, the claim follows. q.e.d.

3. Killing fields on spacelike Zoll surfaces.

From now on, we will be interested in spacelike Zoll cylinders admitting a Killing field K, i.e. a vector field with complete flow consisting of isometries. We will prove in this section that the dynamics of K are always similar to that of a Killing field of a cover of de Sitter space. See Proposition 3.4 for the precise statement.

Proposition 3.1. *Let (S,g) be a connected Lorentzian surface all of whose spacelike geodesics are closed. Then any locally Killing vector field of (S,g), i.e. with antisymmetric covariant derivative, is complete and therefore Killing.*

Proof. Let K be a locally Killing vector field on S and let Φ_K be its local flow. For any $z \in S$, we define ω_z by $\omega_z = \sup\{t; \Phi_K^t(z) \text{ exists}\}$. Let γ be a spacelike geodesic and $\omega_\gamma = \inf_{z\in\gamma}\omega_z$. Let us assume that there exists $x \in \gamma$ such that $\omega_x > \omega_\gamma$ then $\Phi^K_{\omega_\gamma}(\gamma)$ is a spacelike geodesic that is not contained in any compact subset of S. This clearly contradicts the assumption that all spacelike geodesics are closed. Therefore we have $\omega_x = \omega_\gamma$ for any $x \in \gamma$. Since any pair of points in S can be joined by a broken spacelike geodesic, the function ω is constant and the flow of K is complete. q.e.d.

Proposition 3.2. *Let (C,g) be a spacelike Zoll Lorentzian cylinder admitting a non trivial Killing field K. Then every spacelike geodesic of g is at least twice tangent to K or contains at least two zeros of K. It follows that*

1) *K is periodic if and only if it is spacelike and if and only if it has a recurrent orbit;*
2) *K is vanishing if and only if K is somewhere timelike;*
3) *any geodesic perpendicular to K contains all its zeros.*

In particular, K has to be spacelike somewhere. Further K has only finitely many lightlike orbits.

Proof. Let γ be a spacelike geodesic of g. If there exists t_0 such that $\gamma \setminus \gamma(t_0)$ is transverse to K then by pushing γ along the flow of K gives a spacelike geodesic intersecting γ in at most one point. But, according to Proposition 2.3 this is impossible.

It is well known that if K vanishes then it is somewhere timelike. Reciprocally, if K is timelike at a point $x \in C$, consider the geodesic γ defined by $\gamma(0) = x$, $g(\gamma'(0), \gamma'(0)) = 1$ and $g(\gamma'(0), K_x) = 0$. As γ cannot be tangent to K and as it cannot be everywhere transverse to it, K vanishes somewhere along γ. Moreover, if m is a zero of K and $\hat\gamma$ a spacelike geodesic containing m, we choose $\hat\gamma$ different from γ. By Proposition 2.3, $\hat\gamma$ intersects γ. As γ and $\hat\gamma$ cannot be tangent and as they are both perpendicular to K, the intersection can be only at a zero of K. It follows that there exists t_0 such that $\Phi_K^{t_0}(\gamma) = \hat{\gamma}$ and therefore $m \in \gamma$ and any zero of K is on γ.

According to [**9**], a spacelike Zoll surface has no closed lightlike geodesic, therefore a periodic Killing field has to be spacelike. Reciprocally, if K is a spacelike Killing field and if γ is a spacelike geodesic that is not an orbit of K then Clairaut's first integral imposes the value of $g(K, K)$ at the points where K and γ are tangent. It follows that K is tangent to γ only at points where the restriction of the function α defined by $\alpha(x) = g_x(K_x, K_x)$ to γ reaches its maximum. But as γ is compact this function also has a minimum, a point x_0 realizing this minimum has to be a critical point of α, considered as a function on C. The orbit of K through x_0 is therefore a spacelike geodesic and so is closed.

If K has a recurrent orbit, there exists a spacelike geodesic γ intersecting this orbit transversally at a point x and $t_0 > 0$ such that $\Phi_K^{t_0}(x) \in \gamma$. Since a geodesic η emanating from $\Phi_K^{t_0}(x)$ is uniquely determined by $g(\dot{\eta}, \dot{\eta})$ and $g(K, \dot{\eta})$, we have $\Phi_K^{t_0}(\gamma) = \gamma$. Consequently, $\Phi_K^{t_0}$ is an isometry of γ seen as a Riemannian circle. Therefore the orbit of $\Phi_K^{t_0}$ is dense or finite. It cannot be dense as K would be everywhere transverse to γ. Hence, it is finite and K has a closed orbit.

The proposition follows then from the fact that a (complete) Killing field on a Lorentzian surface that has a closed leaf is periodic. Indeed, every geodesic emanating from a point x contained in a closed orbit of K is mapped to itself by $\Phi_K^{t_0}$ for some $t_0 > 0$. The isometry $\Phi_K^{t_0}$ has a fixed point x and its differential $\mathrm{d}\,\Phi_K^{t_0}(x)$ is an element of $\mathrm{SO}_0(1, 1)$ (if C is not assumed to be orientable we replace t_0 by $2t_0$) having an eigenvalue equal to 1 (associated to the eigenvector K_x) therefore $\mathrm{d}\,\Phi_K^{t_0} = \mathrm{Id}$ and therefore $\Phi_K^{t_0} = \mathrm{Id}$.

Assume that K has an infinite number of lightlike orbits. Since spacelike Zoll surfaces are globally hyperbolic, any given spacelike geodesic γ intersects all lightlike geodesics that contains lightlike orbits of K. If the complement of the lightlike orbits has only finitely many connected components, then an open subset of C is foliated by lightlike orbits of K. If not, the complement has infinitely many connected components. Since K is smooth and γ is compact, there exist an infinite number of these components on which K is transversal to γ. Choose any such connected component. If K is spacelike on it, then $g(K, K)$ has a maximum on the intersection of γ with that component. But then one orbit of K is a spacelike geodesic and therefore closed. This contradicts the first part of the proof. So on these infinitely many connected components K has to be timelike. We can therefore choose γ to be orthogonal to K. But it means that γ cuts only timelike or singular integral curves of K, therefore γ contains an infinite number of 0 of K. Which is impossible since γ is compact. Therefore we can assume that an open subset of C is foliated by lightlike orbits of K. It follows from the Lorentzian version of Wadsley's theorem, see [**9**, Theorem 2.3], that the set of lengths of spacelike geodesics of a spacelike Zoll metric is bounded. On the other hand, if S is a strip foliated by lightlike orbits of K then S is flat and isometric to $(I \times \mathbb{R}, dxdy)$ for some interval I. Thus, for any $T > 0$ there exists a spacelike geodesic segment contained in S whose length is greater than T. Hence, (C, g) does not contain any strip foliated by lightlike orbits of K. q.e.d.

Proposition 3.3. *Let (C, g) be a spacelike Zoll cylinder admitting a non trivial Killing field K. Let η be a lightlike geodesic of g that is transverse to K and α be the function on C defined by $\alpha(x) = g_x(K_x, K_x)$. Then the function α tends to $+\infty$ at both ends of η. Moreover, if K is not periodic then α vanishes once or twice on η and if there exists $x \in \eta$ such that $\alpha(x) < 0$ then it vanishes exactly twice.*

Proof. Let $\eta\colon (t_{\inf}, t_{\sup}) \to C$ be a lightlike geodesic that contains a point x such that $\alpha(x) > 0$. For any t, if $\alpha(\gamma(t)) > 0$ then $(\alpha \circ \gamma)'(t) \neq 0$ or $(\alpha \circ \gamma)''(t) > 0$. Indeed, if $(\alpha \circ \gamma)'(t) = 0$ then the orbit of K through $\gamma(t)$ is a geodesic and therefore, as g is spacelike Zoll, contains conjugate points. The curvature of g being constant along this geodesic, it has to be positive and therefore $(\alpha \circ \gamma)''(t) > 0$ (see Lemma 4.9 of [**3**]).

Clairaut's first integral tells us that along any spacelike geodesic γ the value of $g(\dot\gamma, K)$ is constant. We denote it by k_γ. Even if we impose $g(\dot\gamma, \dot\gamma) = 1$, it can be chosen as big as wanted by taking an initial speed at a point where $\alpha > 0$ sufficiently close to a lightlike direction. Moreover if K is tangent to γ and if $g(\dot\gamma, \dot\gamma) = 1$ then the value of α at this point is equal to k_γ^2. Any spacelike geodesic being somewhere tangent to K, the function α is unbounded on C.

We suppose first that K is periodic. The saturation of any lightlike geodesic η by K is equal to C. We have seen in the proof of Proposition 3.2 that α has critical points. But, we just saw they are all local minima. It means that α has a minimum that is realized on a unique orbit of K that we denote by γ_0. We choose $\eta(0)$ such that $\eta(0) \in \gamma_0$, i.e. such that it realizes the minimum of α. The restriction of $\alpha \circ \eta$ to $]t_{\inf}, 0[$ and $]0, t_{\sup}[$ are strictly monotonous and, as any spacelike geodesic has to be tangent to K on both side of γ_0, we see that $\alpha \circ \gamma \to +\infty$ when t goes to $t_{\inf}$ or $t_{\sup}$.

We can assume now that α vanishes somewhere (but maybe not K). Let η be a lightlike geodesic that is transverse to K. Let us first suppose that $\alpha(x) > 0$ for some $x \in \eta$. Let V be the connected component of $\alpha^{-1}(]0, \infty[)$ that contains x. The vector field K sends lightlike geodesics to lightlike geodesics and leaves V invariant. On V the vector field K is transverse to any lightlike geodesic, thus the flow of K defines an open equivalence relation on the lightlike geodesics of V. Hence, V is the saturation of $\eta \cap V$ by the flow of K. Let γ be a spacelike geodesic intersecting V. As the function α vanishes on ∂V the restriction of α

to $\gamma \cap V$ has a local maximum and therefore γ has to be tangent to K somewhere in V. Therefore α is unbounded on V. Any level set of $\alpha|_V$ intersects η since the saturation of $\eta \cap V$ under K is V. Since $\alpha|_\eta$ is monotonous, the restriction of α to $\eta \cap V$ goes to $+\infty$ at one end and to 0 at the other end.

Let us now see that this other end corresponds to an intersection between η and a lightlike orbit of K. The function α is strictly monotonous on $\eta \cap V$, therefore for any $c > 0$, $\alpha^{-1}(c) \cap V$ is equal to one orbit of K. Any spacelike geodesic is twice tangent to K, therefore the boundary of V contains at least two non trivial lightlike orbits of K. We choose small lightlike transversal τ_1 and τ_2 along each of them such that $\alpha(\tau_1 \cap V) = \alpha(\tau_2 \cap V)$. As the level sets of α on V are equal to orbits of K, there exists t_1 such that $\Phi_K^{t_1}(\tau_1) \cap \tau_2 \neq \emptyset$. If τ_1 and τ_2 are pieces of leaves from the same lightlike foliation then $\Phi_K^{t_1}(\tau_1) \cap V = \tau_2 \cap V$. But it would mean that τ_1 and τ_2 are transversals of the same lightlike orbit of K, contrary to our assumption. Hence we can assume that η and the geodesic containing τ_1 are leaves of the same foliation. Consequently there exists t_2 such that $\Phi_K^{t_2}(\tau_1) \cap V \subset \eta$ and therefore η intersects a lightlike orbit of K.

In order to see that α goes to $+\infty$ we just have to prove that it takes positive values again. Let x be a point of η such that $\alpha(x) = 0$. According to Proposition 3.2, the function α has to take non zero values again. Let us suppose that there exists a point $y \in \eta$ such that $\alpha(y) < 0$. We choose a parametrization of η starting from y and 3 unit spacelike geodesics γ_i starting also from y. The geodesic γ_0 is perpendicular to K and the initial speeds satisfy

$$|g(\dot\gamma_1(0), \dot\eta(0))| < |g(\dot\gamma_0(0), \dot\eta(0))| < |g(\dot\gamma_2(0), \dot\eta(0))|.$$

That is γ_1 is the closest to η. We remark that the roles of γ_1 and γ_2 are permuted if η is replaced by $\hat\eta$ the other lightlike geodesic emanating from y.

As above we choose two numbers $a < 0 < b$ such that γ_2 is transverse to K on $]a, b[$ and such that K is tangent to γ_2 at the points $\gamma_2(a)$ and $\gamma_2(b)$. Let U be the saturation by K of $\gamma_2(]a, b[)$. Let us see that each orbit of K cuts at most once $\gamma_2(]a, b[)$. If $y \in \gamma_2(]a, b[)$ and $\Phi_K^{t_0}(y) \in \gamma_2(]a, b[)$ then, using Clairaut's first integral and the fact that a flow always preserves the orientation, we see that $\Phi_K^{t_0}(\gamma_2) = \gamma_2$. As the set of tangency points between γ and K is preserved by the flow of K it follows that $\Phi_K^{t_0}(\gamma_2(]a, b[)) = \gamma_2(]a, b[)$. If $t_0 \neq 0$ then K has closed orbits contrarily to our assumption. The map $(s, t) \mapsto \Phi_K^s(\gamma_2(t))$ therefore defines coordinates on U such that metric reads $\alpha(t)ds^2 + 2dsdt + dt^2$ (in order to obtain a 2 we may have to change K by one of its multiples), with $\alpha(0) < 0$. The open set U contains lightlike orbits of K. Let $c \in]0, b[$ the smallest number such that $\alpha(c) = 0$. It corresponds to an orbit of K that goes to a zero of K (a separatrix). It implies that $\alpha'(c) > 0$ (otherwise $D_K K(\gamma_2(c)) = 0$) therefore $\alpha'(t) > 0$ for $t \geq c$. Doing the same for the biggest number $d \in]a, 0[$ such that $\alpha(d) = 0$, we see that U contains exactly 2 lightlike orbits of K that are separatrices of saddle points.

The intersection of γ_0 with U is asymptotic to these lines therefore $\eta \cap U$ cannot cut any of them and it has also to be asymptotic to them in both direction. It implies that $\hat{\eta}$ cuts the two lightlike orbits of K contained in U (see figure 1). Swapping the roles of η and $\hat{\eta}$ we see that η cuts the two lightlike orbits of K contained in the open set U' obtained by saturating a segment of γ_1.

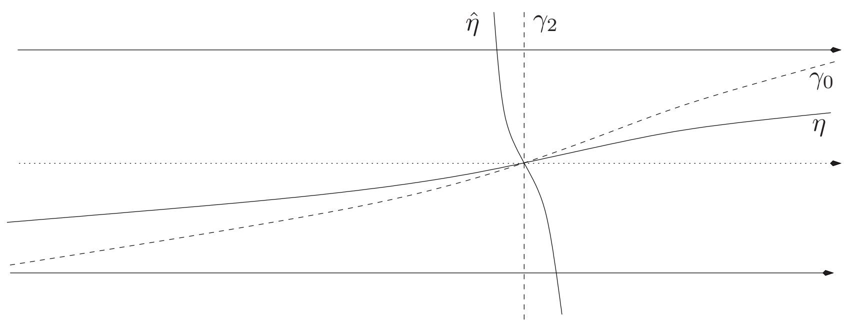

Figure 1. the positions of the curves γ_0, γ_2, η and $\hat{\eta}$ on U.

Thus there are points on η on both side of y where α takes positive values therefore α goes to infinity on both ends of η. q.e.d.

If (C, g) is a spacelike Zoll surface with a Killing field K then Proposition 3.3 says that in the coordinates obtained by K-saturation of a lightlike geodesic the metric reads $h(y)dx^2 + 2dxdy$ with h defined on an interval $I =]t^-, t^+[$ and $h \to +\infty$ when $y \to t^\pm$. We can actually precise this fact:

Proposition 3.4. *Let (C, g) be a spacelike Zoll cylinder admitting a non zero Killing field K.*

1) *When K is periodic, then (C, g) is the quotient by a horizontal translation, of a metric on $\mathbb{R}^2$ that reads $h(y)dx^2 + 2dxdy$ where h is a positive function that has a unique local minimum and satisfies $\lim_{y \to \pm\infty} h(y) = +\infty$.*
2) *When K does not vanish and is not periodic, then there exists a finite atlas $\{(U_i, \psi_i), i \in \mathbb{Z}/2k\mathbb{Z}\}$ such that $\psi_i(U_i) = \mathbb{R} \times I_i$ and $\psi_i^{-1*} g = h_i(y)dx^2 + 2dxdy$ where the h_i are non negative smooth functions, such that*
   - *$h_i(0) = 0$*
   - *the h_i are strictly monotonous on $]t_i^-, 0[$ and on $]0, t_i^+[$*

- $\lim_{y\to t_i^\pm} h_i(y) = +\infty$;
- $h_{2i}(t) = h_{2i+1}(t)$ *for any* $t > 0$ *and* $h_{2i}(t) = h_{2i-1}(t)$ *for any* $t < 0$.

3) *When* K *vanishes, then there exists a finite atlas* $\{(U_i, \psi_i), i \in \mathbb{Z}/4k\mathbb{Z}\}$ *of* C *minus the set of zeros of* K *such that* $\psi_i(U_i) = \mathbb{R}\times I_i$ *and* $(\psi_i^{-1})^*g = h_i(y)dx^2+2dxdy$ *where the* h_i *are smooth functions such that:*
   - *for any* $i \in A$, *there exists* $a_i < 0 < b_i$ *satisfying* $h_i(a_i) = h_i(b_i) = 0$,
   - h_i *is positive and strictly monotonous on* $]t_i^-, a_i[$ *and on* $]b_i, t_i^+[$,
   - h_i *is negative on* $]a_i, b_i[$,
   - $\lim_{y\to t_i^\pm} h_i(y) = +\infty$;
   - $h_{2i}(t) = h_{2i+1}(t - a_{2i} + a_{2i+1})$, *for any* $a_{2i} < t < b_{2i}$;
   - $h_{2i}(t) = h_{2i+3}(t - a_{2i} + a_{2i+3})$, *for any* $t < a_{2i}$;
   - $h_{2i-t}(t) = h_{2i}(t - b_{2i-1} + b_{2i})$, *for any* $t > b_{2i-1}$.

Proof. The first case is a direct consequence of Proposition 3.3 and the fact that, in this case, the saturation of any lightlike geodesic of the universal cover is the entire space. The interval I corresponds to the interval of definition of the geodesic.

Let us assume now that K is not periodic. Let η_1 be a lightlike geodesic such that $g(\dot\eta_1, K) = 1$. It follows from Proposition 3.3 that the map $(s,t) \mapsto \Phi_K^t(\eta_1(s))$ is a diffeomorphism onto its image, that we denote U_1. In these coordinates the metric reads $h_1 dt^2 + 2dsdt$. Let U_1^+ be a connected component of $\alpha^{-1}(]0,+\infty[) \cap U_1$. Let η_2 be another geodesic such that $g(\dot\eta_2, K) = 1$ and cutting η_1 at a point $p \in U_1^+$. We define U_2 and h_2 as above and U_2^+ as the connected component of $\alpha^{-1}(]0,+\infty[) \cap U_2$ that contains p. It is easily verified that η_2 cuts all the leaves of K contained in U_1^+ (cp. previous proof), therefore $U_1^+ = U_2^+$ and the functions h_1 and h_2 coincide on U_1^+. It implies that the derivative of h_1 on the boundary of U_1^+ in U_1 is equal to the derivative of h_2 on the boundary of U_2^+ in U_2. According to Proposition 3.3 it means that h_1 changes sign if and only if h_2 does. As any pair of points can be connected by a broken lightlike geodesic, it implies that if a function h_i takes negative values they all do.

The properties of the function h_i are also given by Proposition 3.3. The fact that the atlas is finite is equivalent to the fact that K has only a finite number of lightlike orbits and therefore follows from Proposition 3.2. The identities between the h_i's follow from the fact that the transition maps between the charts are isometries. q.e.d.

Let us remark that Proposition 3.4 actually says that there are only three possible dynamics for Killing fields of spacelike Zoll cylinders, the three dynamics that appear on de Sitter space. The study thus splits in three cases that we will call elliptic, parabolic and hyperbolic in reference to the constant curvature case. In order to be able to determine when such metrics are indeed spacelike Zoll, we have made assumptions on the h_i appearing in Proposition 3.4, see sections 5 and 7.

4. The Conformal Classes

In this section we prove the C^0-classification of the conformal classes of spacelike Zoll cylinders admitting a Killing vector field.

Definition 4.1. Let g, g' be Lorentzian metrics on a manifold M. A homeomorphism $\Phi\colon M \to M$ is called a conformal homeomorphism if it maps g-lightlike geodesics to g'-lightlike geodesics up to parameterization. If such a Φ exists (M, g) and (M, g') are called C^0-conformal.

Note that for surfaces C^0-conformality is equivalent to the property that the lightlike foliations are mapped onto each other. Denote with $[g]$ the conformal class of the pseudo-Riemannian metric g.

Theorem 4.2. *Let (C, g) be a spacelike Zoll cylinder with a non trivial Killing vector field K. Then (C, g) is C^0-conformal to the k-fold cover of de Sitter space, where $2k$ is the number of intersection points between any pair of distinct spacelike geodesics. Besides, if K is periodic then (C, g) is C^∞-conformal to the k-fold cover of de Sitter space.*

The proof will be given at the end of the section. In general the C^0-conformality cannot be improved to C^2-conformality as Theorem 5.11 shows.

Proposition 4.3. *Let (C, g) be a globally hyperbolic spacetime admitting a conformal embedding $F\colon (C, g) \to (S^1 \times \mathbb{R}, d\varphi^2 - dt^2)$. Assume that there exists a conformal homeomorphism $\Phi\colon (C, [g]) \to (C, [g])$ that leaves each lightlike foliation of (C, g) invariant. Then $F \circ \Phi \circ F^{-1}$ has a unique extension as a conformal homeomorphism of $(S^1 \times \mathbb{R}, d\varphi^2 - dt^2)$. If furthermore Φ is a C^n-diffeomorphism, so will be the extension.*

Proof. Since we are interested in the conformal structure only, we can assume from the very beginning that C is an open subset of $S^1 \times \mathbb{R}$ bounded by possible none, one or two closed and simple non timelike loops.

Consider $(S^1 \times \mathbb{R}, d\varphi^2 - dt^2)$ as the quotient of $(\mathbb{R}^2, dxdy)$ by the $\mathbb{Z}$-action generated by $(x, y) \mapsto (x + \sqrt{2}\pi, y + \sqrt{2}\pi)$. Lift everything to $\mathbb{R}^2$ and denote the lift of Φ with $\widetilde{\Phi}$. Since $\widetilde{\Phi}$ maps horizontal lines to horizontal lines and vertical lines to vertical lines, we see that $\widetilde{\Phi}(x, y) = (\widetilde{\Phi}_1(x), \widetilde{\Phi}_2(y))$. By the assumption that (C, g) is globally hyperbolic the intersection of any lightlike line in $(\mathbb{R}^2, dxdy)$ with $\widetilde{C}$ is an interval. This implies that the maps $\widetilde{\Phi}_1(x) = x \circ \widetilde{\Phi}$ and $\widetilde{\Phi}_2(y) = y \circ \widetilde{\Phi}$ are well defined. Since $x(\widetilde{C}) = y(\widetilde{C}) = \mathbb{R}$, we can define the extension of $\widetilde{\Phi}$ denoted by

$\widetilde{\Phi}_e$ to $\mathbb{R}^2$ by setting $\widetilde{\Phi}_e(x,y) := (\widetilde{\Phi}_1(x), \widetilde{\Phi}_2(y))$. This extension is unique if we impose the condition of conformality on the extension. Since $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$ are equivariant under the deck transformation group of $\mathbb{R}^2$ over $S^1 \times \mathbb{R}$ described above, $\widetilde{\Phi}_e$ descends to a conformal homeomorphism of $(S^1 \times \mathbb{R}, d\varphi^2 - dt^2)$. q.e.d.

Corollary 4.4. *If K is a smooth conformal vector field on a globally hyperbolic cylinder (C, g), then for every smooth conformal embedding $F\colon (C, g) \to (S^1 \times \mathbb{R}, d\varphi^2 - dt^2)$ there is a unique smooth extension $\overline{K}$ of F_*K to a smooth conformal vector field of $(S^1 \times \mathbb{R}, d\varphi^2 - dt^2)$.*

Proof. We have seen in the previous proof that the local flow of the lift of F_*K to the universal cover $(\mathbb{R}^2, dxdy)$ has the form $\Phi_t(x,y) = (\Phi_{1,t}(x), \Phi_{2,t}(y))$. This implies that the lift of F_*K has the form

$$(K_1(x), K_2(y))$$

for smooth functions K_1, K_2 on the lift of $F(C)$. Since the intersection of every horizontal and vertical line with the lift of $F(C)$ is non empty and connected, we can extend the functions K_1 and K_2 to $\mathbb{R}^2$ by setting $K_{(x,y)} = (K_1, K_2)$ where K_1 is the value of the x-part of K on the intersection of the vertical line through (x, y) with the lift of $F(C)$ and K_2 is the respective value on the intersection of the horizontal line with the lift of $F(C)$. Since the lift of F_*K is invariant under the group of deck transformations, it is now obvious that the constructed vector field induces a smooth conformal vector field on $(S^1 \times \mathbb{R}, d\varphi^2 - dt^2)$. q.e.d.

Corollary 4.5. *If (C, g) is spacelike Zoll and admits a nontrivial Killing vector field K, then the conformal boundary is piecewise smooth and spacelike. If K has no lightlike leaves then the boundary is spacelike and smooth.*

Proof. By Corollary 4.4 the Killing field K admits a unique conformal extension to $S^1 \times \mathbb{R}$ for every conformal embedding. Since the image of C is invariant under the flow of the extension, so is the conformal boundary. Therefore the conformal boundary consists of non timelike orbits of the extension since (C, g) is globally hyperbolic. By Proposition 3.2 K has only finitely many lightlike orbits. Therefore the conformal boundary contains only finitely many singularities of the extension, i.e. the common limit of lightlike orbits. The rest consists of spacelike or lightlike orbits.

We want to exclude the lightlike case. So assume that there is a lightlike orbit of K in the boundary of C. We denote it by I. Since the boundary is invariant under the flow of K the entire lightlike orbit of K is contained in the boundary. Let η be a lightlike geodesic asymptotic to a point in I. By Proposition 3.2 we know that $g(K, K) \to \infty$ as η approaches the boundary. Especially K will be spacelike near the boundary. We will now consider η only near I. Note that since

C is 2-dimensional $-g$ is Lorentzian again. Further $(C, -g)$ is time orientable by Lemma 2.2. Time orient $(C, -g)$ such that K is future pointing on η. Lift everything to the universal cover $(\widetilde{C}, -\widetilde{g})$. Now denote with $J^+(y)$ and $J^-(y)$ the causal future and past respectively of $y \in \widetilde{C}$ relative to $-\widetilde{g}$ with the lifted time orientation. Since the lifted boundary is lightlike as well we see, e.g. by considering the situation in a conformal embedding into $(\mathbb{R}^2, -dxdy)$, that for points x on $\widetilde{\eta}$ sufficiently close to the boundary the set $J^+(x) \cap J^-(\Phi^1_{\widetilde{K}}(x))$ is compact in $\widetilde{C}$, where $\Phi_{\widetilde{K}}$ denotes the flow of lifted Killing field $\widetilde{K}$. Recall that every Lorentzian metric on $\widetilde{C}$ is causal. It is well known that these two properties imply that the set of future pointing causal curves, modulo reparameterizations, from x to $\Phi^1_{\widetilde{K}}(x)$ is compact in the space of causal paths of $(\widetilde{C}, -\widetilde{g})$ (Proposition 8.7 in [**2**]). Therefore x and $\Phi^1_K(x)$ are connected by a maximal $-g$-timelike (i.e. g-spacelike) geodesic of g-length at least $\int_0^1 \sqrt{g(K,K)}dt = \sqrt{g(K,K)}(x)$. The right hand side diverges as $x \to \partial C$, thus showing that (C, g) contains arbitrarily long non selfintersecting spacelike geodesic arcs. This contradicts Wadsley's Theorem (cp. the last argument in the proof of Proposition 3.2). q.e.d.

Definition 4.6. (a) Let (C, g) a be globally hyperbolic cylinder and $F\colon (C, g) \to (S^1 \times \mathbb{R}, d\varphi^2 - dt^2)$ a conformal embedding. A ping-pong in $(\overline{F(C)}, d\varphi^2 - dt^2)$ is a piecewise smooth, closed and simple lightlike loop with vertices on the boundary.

(b) Let $k \in \mathbb{N}$. A globally hyperbolic cylinder (C, g) has the k-ping-pong-property (k-PPP) if every lightlike geodesic of (C, g) lies on a ping-pong and every ping-pong has exactly $2k$ vertices.

The definition of ping-pongs can be interpreted in a dynamical context. We describe this interpretation at the end of this section.

Remark 4.7. Ping-pongs can only exist in conformally compact globally hyperbolic cylinders. Further, ping-pongs are invariant under conformal homeomorphisms, i.e. the definition is independent of the conformal embedding F.

It is clear from the construction of the conformal class of de Sitter that the k-fold cover of de Sitter has the k-PPP.

The next Proposition is the first step in the proof of Theorem 4.2.

Proposition 4.8. *Let (C, g) be a spacelike Zoll cylinder admitting a non trivial Killing vector field. Then (C, g) has the k-PPP where $2k$ is the number of intersections of any pair of spacelike geodesics.*

Note that finite covers of de Sitter show that every $k \in \mathbb{N}$ appears.

Lemma 4.9. *If the conformal boundary of a globally hyperbolic cylinder in $S^1 \times \mathbb{R}$ has no lightlike parts, then every lightlike geodesic lies on* *at most one ping-pong. Further if every lightlike geodesic lies on a ping-pong, then the spacetime has the k-PPP for some $k \in \mathbb{N}$.*

Proof. If the conformal boundary has no lightlike parts the intersection of a lightlike line with it is unique. Therefore the vertices and sides of a ping-pong are uniquely determined by any side of it. Further if the conformal boundary has no lightlike parts, the intersection of a lightlike geodesic with the boundary varies continuously with the geodesic. Therefore the first selfintersection of a ping-pong varies continuously. Since the number of sides and vertices of a ping-pong is finite, it is constant throughout the set of lightlike geodesics. q.e.d.

Proof of Proposition 4.8. We will show that every lightlike geodesic is a side of a ping-pong by considering it as the limit of a sequence of spacelike geodesics.

Let $F\colon (C,[g]) \to (S^1 \times \mathbb{R}, [d\varphi^2 - dt^2])$ be a conformal embedding. We will not distinguish between (C,g) and its image under F. Reparameterize all spacelike geodesics of (C,g) as graphs over $S^1 \times \{0\}$, i.e. graphs of 1-Lipschitz functions on S^1.

Now let η be a lightlike geodesic of (C,g). Reparameterize η as a partial graph over $S^1 \times \{0\}$ and denote it with the same letter. Next consider a sequence of spacelike pregeodesics γ_n such that $\dot{\gamma}_n(0) \to \dot{\eta}(0)$. By the Theorem of Arzela-Ascoli a subsequence of γ_n converges uniformly to a $[d\varphi^2 - dt^2]$-non timelike curve $\gamma_\infty\colon S^1 \to S^1 \times \mathbb{R}$. By our assumptions η is a subarc of the limit curve.

Since γ_∞ is the limit of spacelike pregeodesics and (C,g) is spacelike Zoll, the limit curve has to be lightlike everywhere on the intersection with C. This follows from the fact that in C the curve γ_∞ is a non timelike pregeodesic as it is a limit of spacelike pregeodesics. If it is not lightlike, γ_∞ will be a spacelike pregeodesic and therefore nowhere lightlike, thus contradicting the initial assumption on the sequence.

Fix a closed and simple spacelike geodesic γ_0 of (C,g) not contained in the sequence $\{\gamma_n\}_{n\in\mathbb{N}}$. Since all γ_n's intersect γ_0 transversally in exactly $2k$ points, the limit curve intersects γ_0 in exactly $2k$ points as well. Note that the intersections cannot approach one another in the limit since on γ_0 the injectivity radius is bounded from below. Therefore γ_∞ contains exactly $2k$ lightlike pregeodesics of (C,g).

We claim that the limit curve has only vertices on the boundary of C in $S^1 \times \mathbb{R}$. Then γ_∞ will be a ping-pong with exactly $2k$ sides. If the $2k$ lightlike pregeodesics do not cover the entire limit curve, a piece of the boundary has to be part of the limit curve. Note that by Corollary 4.5 the conformal boundary of (C,g) consists of spacelike and constant orbits of the unique conformal extension of the g-Killing vector field K to $(S^1 \times \mathbb{R}, [d\varphi^2 - dt^2])$. Let $\gamma_\infty|_{[t_0,t_1]}$ be a subarc lying in a spacelike orbit of K and U a neighborhood with $K|_U$ spacelike. By restricting

U and $[t_0, t_1]$ we can assume that $g(K,K)|_{\gamma_n}$ has at most one critical point, a maximum, in U for all γ_n intersecting U. This follows from the classification of the Killing vector fields of spacelike Zoll cylinders in Proposition 3.4. In fact let t be a critical point of $g(K,K)|_{\gamma_n}$ that is not a maximum. Then γ_n is transversal to K. Thus the K-orbit through $\gamma_n(t)$ is itself a geodesic. If it is spacelike, it has to be closed and K is spacelike everywhere. In this case there is only one geodesic K-orbit and we can assume that it lies outside of U. In the other cases $K_{\gamma_n(t)}$ has to be a non spacelike and again we can assume that it is disjoint from U. Further note that the maxima of $g(K,K)|_{\gamma_n}$ are exactly the minima of

$$\operatorname{arcosh}\angle_{hyp}(K, \dot\gamma_n) = \frac{g(K, \dot\gamma_n)}{\sqrt{g(K,K)}\sqrt{g(\dot\gamma_n, \dot\gamma_n)}}.$$

Note that this definition makes sense without referring to g, since it coincides with the respective quotient in $(S^1 \times \mathbb{R}, d\varphi^2 - dt^2)$. Consequently, by Proposition 3.4, the quotient is monotonous in U except at its minima. From our assumptions we know that $\gamma_n|_{[t_0,t_1]}$ converges uniformly to $\gamma_\infty|_{[t_0,t_1]}$ a piece of a spacelike orbit of K. Use K and a curve orthogonal to K in U to introduce coordinates (w,z) on U such that $\partial_w = K$ and $\partial_z \perp K$ relative to $d\varphi^2 - dt^2$. Note that on the intersection with C the orthogonality also holds with respect to g. Choose constant $0 < C_1, C_2 < \infty$ such that the absolute value of the slope of lightlike vectors in these coordinates is bounded between C_1 and C_2. Reparameterize the γ_n and γ_∞ on the intersection with U as graphs over the w-axis. Let $w_0 := \gamma_\infty(t_0) < w_1 := \gamma_\infty(t_1)$. For $\varepsilon > 0$ choose N such that

$$\gamma_n|_{[w_0,w_1]} \subset U \cap \{|z| < \varepsilon C_2(w_1 - w_0)\}$$

for all $n \geq N$. Since the slope of $\dot\gamma_n$ is bounded by C_2 and $z(\gamma_n)$ has at most one critical point in U, the slope of $\dot\gamma_n$ is bounded by $C_2\varepsilon$ on a set $A \subseteq [w_0, w_1]$ of measure at least $(w_1 - w_0)(1 - 2C_2\varepsilon)$.

Now we can give a bound on $\frac{g(K,\dot\gamma_n)}{\sqrt{g(K,K)}\sqrt{g(\dot\gamma_n,\dot\gamma_n)}}$ on A. In fact we know that $d\varphi^2 - dt^2$ in the (w,z)-coordinates reads as $Edw^2 - Gdz^2$ for some positive smooth functions E, G on U. The upper bound on the slope of the lightlike directions is equivalent to saying $E - GC_2^2 \leq 0$, i.e. $\frac{G}{E} \geq \frac{1}{C_2^2}$. The lower bound on the slope is equivalent to saying $E - GC_1^2 \geq 0$, i.e. $\frac{G}{E} \leq \frac{1}{C_1^2}$. For $\dot\gamma_n = (1, \dot\gamma_{z,n})$ we then have

$$1 - \frac{G}{E}\dot\gamma_{z,n}^2 \geq 1 - \frac{\dot\gamma_{z,n}^2}{C_1^2} \geq 1 - \frac{\varepsilon^2 C_2^2}{C_1^2}$$

on A. As $E = g(K,K)$ we have

$$\frac{\sqrt{C_1^2 - C_2^2\varepsilon^2}}{C_1}\sqrt{g(K,K)} \leq \sqrt{g(\dot\gamma_n, \dot\gamma_n)}.$$

Thus we have

$$\begin{aligned} L^g(\gamma_n) &\geq \int_A \sqrt{g(\dot\gamma_n, \dot\gamma_n)}dt \\ &\geq (w_1 - w_0)(1 - 2C_2\varepsilon)\frac{\sqrt{C_1^2 - C_2^2\varepsilon^2}}{C_1} \inf_U \sqrt{g(K,K)}. \end{aligned}$$

Since we can choose U as small as we wish, $\inf_U \sqrt{g(K,K)}$ will diverge to ∞ by Proposition 3.4. Thus the g-length of the γ_n diverges as $n \to \infty$. This contradicts the corollary of Waldsley's theorem asserting that the geodesic flow on the unit tangent bundle of a spacelike Zoll manifold is periodic. q.e.d.

Proposition 4.10. *A globally hyperbolic cylinder (C, g) has the k-PPP and the conformal boundary contains no lightlike parts iff it is C^0-conformal to the k-fold cover of de Sitter space. Further if the conformal boundary is C^n-spacelike, then the conformal homeomorphism can be chosen to be a C^n-diffeomorphism.*

Assume that the globally hyperbolic cylinder (C, g) has the k-PPP and the conformal boundary contains no lightlike parts. Lift everything to the universal cover $(\mathbb{R}^2, dxdy)$ of $(S^1 \times \mathbb{R}, d\varphi^2 - dt^2)$ with the deck transformation group generated by $(x, y) \mapsto (x + \sqrt{2}\pi, y + \sqrt{2}\pi)$. Then the boundary of the universal cover $\widetilde{C}$ is the union of the graphs of two $\sqrt{2}\pi$-equivariant homeomorphisms $\theta^\pm\colon \mathbb{R} \to \mathbb{R}$, i.e. $\theta^\pm(x + \sqrt{2}\pi) = \theta^\pm(x) + \sqrt{2}\pi$, over the x-axis ($\theta^- < \theta^+$).

Lemma 4.11. *Assume that the conformal boundary of the globally hyperbolic cylinder (C, g) does not contain any lightlike parts. Then (C, g) has the k-PPP iff $((\theta^-)^{-1} \circ \theta^+)^k(x) = x + \sqrt{2}\pi$ for all $x \in \mathbb{R}$.*

Proof. Let $x \in \mathbb{R}$. Then $(x, \theta^+(x))$ is the future endpoint of a vertical lightlike $\widetilde{g}$-geodesic γ_x^+ of $(\widetilde{C}, \widetilde{g})$ in $\mathbb{R}^2$. The point $((\theta^-)^{-1} \circ \theta^+(x), \theta^+(x))$ is the past endpoint of the horizontal lightlike $\widetilde{g}$-geodesic $\gamma^-_{\theta^+(x)}$ of $(\widetilde{C}, \widetilde{g})$ in $\mathbb{R}^2$ whose future endpoint in $\mathbb{R}^2$ is $(x, \theta^+(x))$. Now we can consider the vertical lightlike geodesic of $(\widetilde{C}, \widetilde{g})$ whose past endpoint is $((\theta^-)^{-1} \circ \theta^+(x), \theta^+(x))$ and start the above construction over again. This defines inductively a series of wedges in $(\mathbb{R}^2, dxdy)$ with vertices in $\partial\widetilde{C}$ and sides in $\widetilde{C}$.

Now if (C, g) has the k-PPP take the lift of a ping-pong that contains a given lightlike $\widetilde{g}$-geodesic γ_x^+. The ping-pong in (C, g) returns to the same geodesic after k wedges in $(S^1 \times \mathbb{R}, d\varphi^2 - dt^2)$. By the first paragraph this implies that $((\theta^-)^{-1} \circ \theta^+)^k(x) = x + \sqrt{2}\pi$. Since any γ_x lies on the lift of a ping-pong, we see that the k-PPP implies the identity for $((\theta^-)^{-1} \circ \theta^+)^k$.

For the other direction we can restrict ourself to geodesic lifting to vertical lightlike geodesics since the claim for geodesics lifting to horizontal lightlike geodesics follows by considering the vertical lightlike geodesic with the same future endpoint as the given horizontal lightlike geodesic. If the identity $((\theta^-)^{-1}\circ\theta^+)^k(x) = x+\sqrt{2}\pi$ holds for all x then the projection to $S^1\times\mathbb{R}$ of the wedges constructed in the first paragraph will form a k-ping-pong, thus showing the lemma. q.e.d.

Proof of Proposition 4.10. The second assertion will readily follow from the construction in the first part. Further if (C, g) is C^0-conformal to the k-fold cover of de Sitter space, then the k-PPP is obvious for (C, g). The conformal boundary does not contain any lightlike parts either since this is invariant under conformal homeomorphisms.

Using the Lemma choose a $\sqrt{2}\pi$-equivariant homeomorphism $\psi\colon\mathbb{R}\to\mathbb{R}$ conjugating $(\theta^-)^{-1}\circ\theta^+$ to a translation by $\frac{\sqrt{2}\pi}{k}$. Applying $(\psi^{-1}\circ\theta^+\circ\psi)^{-1}$ to both sides we obtain

$$(\psi^{-1}\circ\theta^-\circ\psi)^{-1}(x)-(\psi^{-1}\circ\theta^+\circ\psi)^{-1}(x)=\frac{\sqrt{2}\pi}{k}.$$

Now we can isotope $\psi^{-1}\circ\theta^-\circ\psi$ and $\psi^{-1}\circ\theta^+\circ\psi$ simultaneously to translations. Note that for the k-fold cover of de Sitter the boundary is given by two translations whose difference is $\frac{\sqrt{2}\pi}{k}$. Thus the result of this isotopy is a conformal homeomorphism of (C, g) to the k-fold cover of de Sitter space.

Finally the conformal boundary is C^n-spacelike if, and only if the homeomorphisms $\theta^\pm$ are C^n-diffeomorphisms. Since $(\theta^-)^{-1}\circ\theta^+$ is periodic the conjugation ψ can be chosen to be C^n as well. This shows that in this case (C, g) is C^n-conformal to the k-fold cover of de Sitter space. q.e.d.

Proof of Theorem 4.2. The proof follows from Corollary 4.5, Proposition 4.8 and Proposition 4.10. In fact if (C, g) is spacelike Zoll then by Corollary 4.5 the conformal boundary does not contain any lightlike parts. Further by Proposition 4.8 (C, g) has the k-PPP for some $k\in\mathbb{N}$. By Proposition 4.10 these two properties imply that (C, g) is C^0-conformal to the k-fold cover of de Sitter space. If K is periodic then by Corollary 4.5 the boundary is smooth and by Proposition 4.10 the conformal homeomorphism is a smooth diffeomorphism. q.e.d.

As announced we give a dynamical interpretation of ping-pongs. Consider a globally hyperbolic conformally compact cylinder (C, g) whose conformal boundary has no lightlike parts. Then we know that every lightlike geodesic lies on a unique lightlike polygonal chain (with self intersections) with vertices on the conformal boundary. The map $(\theta^-)^{-1}\circ\theta^+$ can be defined as before and induces a homeomorphism of the future conformal boundary of (C, g). The difference is that i.g. the polygonal chains do not close, i.e. no power of this homeomorphism is the identity. In fact a periodic point corresponds to a closed chain. If the chain has no selfintersections the lightlike geodesic is a side of a ping-pong. It is conceivable that this map completely encodes the conformal class of (C, g) on every level of differentiability.

5. Parabolic case

In this section we will describe a family of parabolic spacelike Zoll surfaces, i.e. admitting an atlas similar to the one described at point 2 of Proposition 3.4. This family will allow us to construct several interesting examples, such as Moebius strip all of whose spacelike geodesics are closed with non constant curvature or spacelike Zoll cylinders not smoothly conformal to a cover of de Sitter space. We start with the following definition of a "parabolic atlas".

Definition 5.1. Let (C, g) be a Lorentzian cylinder with an atlas $\mathcal{A} = \{(U_i, \phi_i); i \in \mathbb{Z}/2k\mathbb{Z}\}$. We denote by $\Phi_{ij} = \phi_j \circ \phi_i^{-1}$ the transition functions of $\mathcal{A}$.
We will say that $\mathcal{A}$ is a parabolic atlas of (C, g) if:

1) for all $i \in \mathbb{Z}/2k\mathbb{Z}$, the image of ϕ_i is $\mathbb{R}^2$;
2) the transition functions are the following:

$$\begin{array}{rrccl} & \Phi_{2i,2i+1}: & H^+ & \to & H^+ \\ & & (x,y) & \mapsto & \left(-x + \frac{2}{y}, y\right), \\ \text{if } i \neq 0 & \Phi_{2i-1,2i}: & H^- & \to & H^- \\ & & (x,y) & \mapsto & \left(-x + \frac{2}{y}, y\right), \\ & \Phi_{2k-1,0}: & H^- & \to & H^- \\ & & (x,y) & \mapsto & \left(-x + \frac{2}{y} + \tau, y\right), \end{array}$$

where $H^+ = \{(x, y) \in \mathbb{R}^2; y > 0\}$, $H^- = \{(x, y) \in \mathbb{R}^2; y < 0\}$ and $\tau \in \mathbb{R}$;
3) for all $i \in \{1, \dots 2k\}$,

$$g_i = \phi_i^{-1*} g = y^2 dx^2 + 2dxdy + f_i(y)dy^2,$$

where f_i is a smooth function satisfying $1 - y^2 f_i(y) > 0$ for all $y \in \mathbb{R}$.

Remark 5.2. Note that a parabolic atlas induces an analytic structure on C. The Killing field K is according to the conditions analytic as well. In opposition the metric g need not be analytic, but the g-length of K is again an analytic function on C.

Clearly, Lorentzian cylinders admitting a parabolic atlas posses a Killing vector field K that is everywhere spacelike except on a finite number of lightlike orbit (K reads as ∂_x in any map ϕ_i). De Sitter

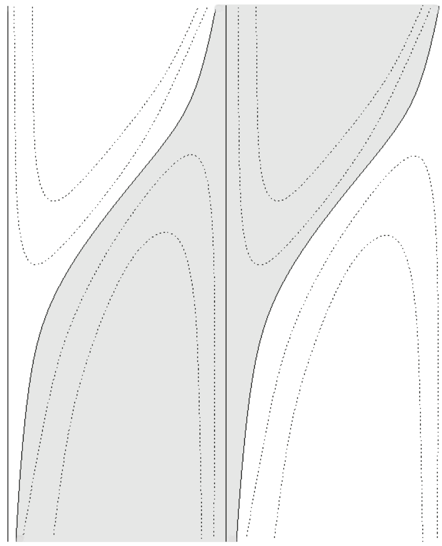

Figure 2. The manifold, the open set U_1 and the Killing field $(k = 2)$

space clearly admits such an atlas. It has the following parameters $k = 1$, $\tau = 0$ and $f_1 = f_2 = 0$. Let us note that if we modify only τ, the cylinder obtained still has constant curvature equal to 1 but is not isometric to de Sitter space (for example its spacelike geodesics are no more closed).

We remark also that for any $i \in \mathbb{Z}/2k\mathbb{Z}$ the restrictions of f_{2i} and f_{2i+1} to H^+ have to be equal as well as the restrictions of f_{2i-1} and f_{2i} to H^-. In particular if g is analytic then $f_1 = f_2 = \cdots = f_{2k}$.

Proposition 5.3. *Let (C, g) be a spacelike Zoll cylinder admitting a parabolic Killing field K, i.e. that is not periodic and does not vanish. Let σ be the curvature of g and $\alpha\colon C \to \mathbb{R}$ the function defined by $\alpha(p) = g_p(K_p, K_p)$. There exists $l > 0$ such that $(C, l \cdot g)$ admits a parabolic atlas if and only if for any $p \in C$, $\alpha(p) = 0$ implies $\sigma(p) > 0$ and $d\sigma(p) = 0$.*

Proof. According to Proposition 3.2, K has spacelike and lightlike orbits but no timelike ones. Let η be a lightlike geodesic of (C, g) transversal to K. According to Proposition 3.3, the function α vanishes somewhere on η and goes to $+\infty$ on both ends of η.

Let U be the saturation of η by K. There exists a lightlike geodesic vector field Y on U tangent to η such that $[Y, K] = 0$. It allows us to find coordinates on U such that g reads as $h(y)dx^2 + 2dxdy$ with $h \geq 0$ for $(x, y) \in \mathbb{R} \times I$, according to Proposition 3.4, h vanishes at only one point. The assumption on the curvature implies that $h(y) = 0$ implies $h''(y) > 0$. Choose the coordinates so that $h(0) = 0$ and denote by a

the function defined by $h(y) = y^2 e^{2a(y)}$. Rescaling g we assume that $a(0) = 0$. The hypothesis $\sigma'(0) = 0$ entails that $a'(0) = 0$.

Let $\gamma(t) = (x(t), y(t))$ be the unique curve satisfying:

$$\left\{ \begin{array}{r} g(\gamma'(t), \partial_x) = 1 \\ \gamma'(t).ye^{a(y)} = 1 \\ \gamma(0) = 0 \end{array} \right. \tag{1}$$

i.e.

$$\left\{ \begin{array}{r} x'y^2 e^{2a(y)} + y' = 1 \\ y'(t)e^{a(y)}(1 + a'(y)y) = 1 \\ \gamma(0) = 0 \end{array} \right.$$

Fact 5.4. *For all $y \in \mathbb{R}$, we have $1 + a'(y)y \neq 0$.*

Indeed, if we have $1 + a'(y)y = 0$ then the curve $t \mapsto (t, y)$ is a complete spacelike geodesic. As it is not closed, it contradicts the fact that the metric is spacelike Zoll.

Thanks to Fact 5.4, we can write

$$\begin{aligned} y' &= \frac{e^{-a(y)}}{1 + ya'(y)} \\ x' &= \frac{e^{-2a(y)}}{1 + ya'(y)} \frac{1 + ya'(y) - e^{-a(y)}}{y^2}. \end{aligned}$$

Therefore

$$\frac{\partial x}{\partial y} = e^{-a(y)} \frac{1}{y^2} \left(1 + ya'(y) - e^{-a(y)}\right)$$

as $a'(0) = 0$ we see that $\frac{\partial x}{\partial y}$ is well defined on $\mathbb{R}$ and smooth. It implies that γ is the graph of a function, in particular it cuts each horizontal line exactly once. The fact that h goes to infinity on both ends of η says that γ is defined on $\mathbb{R}$.

Hence, the map $\Phi : (u, v) \mapsto \gamma(v) + (u, 0)$ is a smooth diffeomorphism. Equation (1) exactly says that $\Phi^* g|_U$ has the desired form. Repeating this construction for any lightlike geodesic transverse to K gives us an atlas of (C, g) such that the metric has the right expression. The last things to check are the transition functions.

If η' is another lightlike geodesic and if we denote by V its saturation, then there are 3 possibilities: either $U = U'$, either $U \cap U' = \emptyset$ or $V := U \cap U'$ is an half plane of U ($\{v > 0\}$ or $\{v < 0\}$). The only case to deal with is the third one. In that case there exists a geodesic δ in V that is perpendicular to K and such that the orthogonal symmetry relatively to δ sends η on η' (see [**1**] for details). It is not difficult to check that this symmetry is the transition function we were looking for. It has the right expression up to a possible horizontal translation. However, it is not difficult to modify the atlas so that these translations are trivial except one.

Reciprocally, it is easily checked that a Lorentzian cylinder admitting a parabolic atlas satisfies the conditions on the curvature given in the statement. q.e.d.

Remark 5.5. If (C, g) is a spacelike Zoll cylinder admitting a parabolic atlas $\mathcal{A}$, then the parameter k of $\mathcal{A}$ is equal to the number k in Theorem 4.2.

At the moment we do not know if there exists a spacelike Zoll metric with a parabolic Killing field that does not admit a parabolic atlas.

We are able to describe all the spacelike Zoll surfaces admitting a parabolic atlas:

Theorem 5.6. *Let (C, g) be a Lorentzian cylinder admitting a parabolic atlas $\mathcal{A} = \{(U_i, \phi_i); i \in \mathbb{Z}/2k\mathbb{Z}\}$. The surface (C, g) is spacelike Zoll if and only if the parameter τ of $\mathcal{A}$ vanishes and there exist k smooth functions $\kappa_0, \dots \kappa_{k-1} \colon \mathbb{R} \to \mathbb{R}$ such that*

1) *for all $t \in \mathbb{R}$, for all $j \in \mathbb{Z}/k\mathbb{Z}$, $\kappa_j(t) \geq -1$;*
2) *all the functions κ_j have the same infinite Taylor expansion at 0 and satisfy $\kappa_j(0) = \kappa_j'(0) = 0$;*
3) *the function $\sum_j \kappa_j$ is odd;*
4) *for all $i \in \mathbb{Z}/2k\mathbb{Z}$ the function f_i such that*
$$g_i = \phi_i^{-1*} g = y^2 dx^2 + 2dxdy + f_i(y)dy^2,$$
satisfies
$$f_i(y) = \begin{cases} \dfrac{1-(1+\kappa_{\lfloor i/2 \rfloor})^2}{y^2} & \text{if } y > 0 \\[2ex] \dfrac{1-(1+\kappa_{\lceil i/2 \rceil})^2}{y^2} & \text{if } y < 0, \end{cases}$$
where $\lfloor . \rfloor$ (resp. $\lceil . \rceil$) is the lower (resp. upper) integral part.

Proof. Let $\mathcal{A}$ be a parabolic atlas of (C, g). We denote as above by g_i the expression of g in the coordinates (U_i, ϕ_i). We recall that there exist functions f_i such that the g_i's read as $y^2dx^2 + 2dxdy + f_i(y)dy^2$.

Lemma 5.7. *Let $\gamma_i : t \mapsto (x(t), y(t))$ be a unit spacelike geodesic of $(\mathbb{R}^2, g_i)$. There exists $c > 0$ such that γ_i is contained between the lines $y = c$ and $y = -c$ and is tangent exactly once to each of these lines. Moreover the geodesic segment between these points satisfies*
$$\frac{\partial x}{\partial y} = \frac{c\sqrt{1 - y^2 f_i(y)} - \sqrt{c^2 - y^2}}{y^2\sqrt{c^2 - y^2}}.$$

Proof. Let $\gamma_i : t \mapsto (x(t), y(t))$ be a unit spacelike geodesic of $(\mathbb{R}^2, g_i)$. It is well know that Killing vector fields induce first integrals for the geodesic field, more precisely the fact that the vector field ∂_x is Killing implies that $g_i(\partial_x,\gamma_i')$ is constant. Therefore, there exists $c\geq 0$ and $\epsilon_1\in\{\pm 1\}$ such that:

$$\begin{cases} y^2x'+y'=\epsilon_1 c \\ y^2x'^2+2x'y'+f_i(y)y'^2=1. \end{cases} \tag{2}$$

This system of equations can be solved if and only if $c^2-y^2\geq 0$ proving that $c\neq 0$ and $-c\leq y\leq c$. Solving it we find:

$$\begin{aligned} x' &= \frac{\epsilon_1 c(1-y^2f_i(y))+\epsilon\sqrt{(1-y^2f_i(y))(c^2-y^2)}}{y^2(1-y^2f_i(y))} \\ y' &= -\epsilon\sqrt{\frac{c^2-y^2}{1-y^2f_i(y)}}, \end{aligned}$$

where $\epsilon\in\{\pm 1\}$. It implies that

$$\frac{\partial x}{\partial y}=\frac{-\epsilon\epsilon_1 c\sqrt{1-y^2f_i(y)}-\sqrt{c^2-y^2}}{y^2\sqrt{c^2-y^2}}.$$

The number ϵ_1 determines the orientation of the geodesic and ϵ changes only when $y=\pm c$.

The fact that for any y_0 such that $0<|y_0|<c$ the integral

$$\int_0^{y_0}\frac{-c\sqrt{1-y^2f_i(y)}-\sqrt{c^2-y^2}}{y^2\sqrt{c^2-y^2}}dy$$

diverges implies that γ is tangent at most once to each line $y=\pm c$.

The fact that for any $y_0\in]0,c[$ and any $y_1\in]-c,0[$ the integrals

$$\int_{-c}^{c}\frac{c\sqrt{1-y^2f_i(y)}-\sqrt{c^2-y^2}}{y^2\sqrt{c^2-y^2}}dy$$

$$\int_{y_0}^{c}\frac{-c\sqrt{1-y^2f_i(y)}-\sqrt{c^2-y^2}}{y^2\sqrt{c^2-y^2}}dy$$

$$\int_{-c}^{y_1}\frac{-c\sqrt{1-y^2f_i(y)}-\sqrt{c^2-y^2}}{y^2\sqrt{c^2-y^2}}dy$$

converge implies that γ is tangent at least once to each line $y=\pm c$.

q.e.d.

Proposition 5.8. *Let γ be a unit spacelike geodesic of (C,g). The geodesic γ is closed if and only if*

$$\int_{-c}^{c}c\frac{\sum_{i\in\mathbb{Z}/2k\mathbb{Z}}(\sqrt{1-y^2f_i(y)}-1)}{y^2\sqrt{c^2-y^2}}dy=-\tau,$$

where τ is the term of translation appearing in $\Phi_{2k-1,0}$ and $c=|g(\gamma',K)|$.

Proof. Let γ be a geodesic of g and $p_0, \dots, p_l, \dots$ the points where γ is tangent to the Killing field K (such points exists according to Lemma 5.7). For each geodesic segment $[p_l, p_{l+1}]$, there exist an open set U_{i_l} containing it. Clearly, $i_{l+1} = i_l + 1$ and $i_0 = i_{2k}$. Without loss of generality we can suppose $i_l = l \mod 2k$.

For any $i \in \mathbb{Z}/2k\mathbb{Z}$, we set:

$$I_k(c) = \int_{-c}^{c} \frac{c\sqrt{1 - y^2 f_0(y)} - \sqrt{c^2 - y^2}}{y^2\sqrt{c^2 - y^2}} dy.$$

If p_0 has coordinates $(x_0, -c)$ on U_0 then the coordinates of p_1 on U_0 are

$$(x_0 + I_0(c), c)\,.$$

It follows that the coordinates of p_1 on U_1 are

$$\left(-\left[x_0 + I_0(c)\right] + \frac{2}{c}, c\right)$$

and the coordinates of p_2 on U_1 are (remark that the orientation of γ changed)

$$\left(-\left[x_0 + I_0(c)\right] + \frac{2}{c} - I_1(c), -c\right).$$

We can continue the same way, in order to obtain the coordinates of the points p_l on U_{l-1} and U_l. In particular, we see that the coordinates of p_{2k} on U_0 are

$$\left(x_0 + \sum_{l=0}^{2k-1} I_l(c) - \frac{4k}{c} + \tau, -c\right).$$

It implies that γ is closed if and only if $p_0 = p_{2k}$ if and only if

$$\sum_{l=0}^{2k-1} \int_{-c}^{c} \frac{c\sqrt{1 - y^2 f_l(y)} - \sqrt{c^2 - y^2}}{y^2\sqrt{c^2 - y^2}} dy = \frac{4k}{c} - \tau. \tag{3}$$

We consider now the metric g^0 of constant curvature 1 given by $g^0_i = y^2dx^2 + 2dxdy$ and γ^0 the g^0-spacelike geodesic starting horizontally from p_0. We denote by p^0_l the points of tangency of γ^0 with K. Doing the same computation as above we see that the coordinates of p^0_{2k} on U_0 are

$$\left(x_0 + 2k\int_{-c}^{c} \frac{c - \sqrt{c^2 - y^2}}{y^2\sqrt{c^2 - y^2}} dy - \frac{4k}{c} + \tau, -c\right).$$

Computing the integral above, we find that its value is $\frac{2}{c}$ and therefore the coordinates of p^0_{2k} are in fact

$$(x_0 + \tau, -c)$$

(reproving that all the spacelike geodesics of de Sitter space are closed). In order to finish the proof, we just replace $\frac{4k}{c}$ by $2k\int_{-c}^{c}\frac{c-\sqrt{c^2-y^2}}{y^2\sqrt{c^2-y^2}}dy$ in (3). q.e.d.

Lemma 5.9. *If $h:\mathbb{R}\to\mathbb{R}$ is a function such that the function*

$$H : c \mapsto \int_{-c}^{c} c\frac{h(y)}{y^2\sqrt{c^2-y^2}}dy$$

is constant, then h is odd and $H=0$.

Proof. We first remark that $H(c)$ only depends on the even part of h. Thus we will assume that h is even and prove that it has to vanish.

We define a function I on $\mathbb{R}^+$ by

$$I(a) = \int_0^a \frac{H(t)}{\sqrt{a^2-t^2}}dt.$$

We have

$$\begin{aligned} I(a) &= \int_0^a \frac{2t}{\sqrt{a^2-t^2}}\int_0^t \frac{h(s)}{s^2\sqrt{t^2-s^2}}ds\,dt \\ &= \int_0^a \frac{2h(s)}{s^2}\int_s^a \frac{t}{\sqrt{(a^2-t^2)(t^2-s^2)}}dt\,ds \\ &= \int_0^a \frac{2h(s)}{s^2}ds\int_0^{+\infty}\frac{dx}{1+x^2} = \pi\int_0^a\frac{h(s)}{s^2}ds \end{aligned}$$

with $x=\sqrt{\frac{t^2-s^2}{a^2-t^2}}$.

Moreover if H is constant, we see by direct computation that I is also constant. It follows from $I'=0$ that $h=0$. q.e.d.

For $i\in\{0,\dots,k-1\}$ we define the function κ_i by

$$\kappa_i(y) = \sqrt{1-y^2f_{2i}(y)}-1.$$

These functions clearly satisfy points 1 and 2 of the statement. It follows from Lemma 5.9 and Proposition 5.8 that the geodesics of (C,g) are all closed if and only if the function $c\mapsto\sum_{i\in\mathbb{Z}/2k\mathbb{Z}}\kappa_i$ is odd and $\tau=0$.

q.e.d.

Corollary 5.10. *There exist smooth Möbius strips all of whose spacelike geodesics are closed with non constant curvature and whose orientation cover admits a parabolic atlas. Moreover, if the orientation cover of a non constant curvature Möbius strip all of whose spacelike geodesics are closed admits a parabolic atlas then it is not analytic and it is C^0-conformal to a k-cover of de Sitter with $k>1$.*

Proof. Let (C,g) be a parabolic spacelike Zoll cylinder and $\mathcal{A}$ be a parabolic atlas of (C,g). If (C,g) is analytic (or if $k=1$) then the functions κ_i given by Theorem 5.6 have to be equal and therefore odd.

It follows that (C, g) cannot be the lift of a metric on the Möbius strip unless the κ_i vanish.

Let κ be a smooth function on $\mathbb{R}$ with support on $[1, 2]$ and values in $[-1, 1]$. We define now three functions κ_0, κ_1 and κ_2 by $\kappa_0(t) = \kappa(t)$, $\kappa_1(t) = -\kappa(-t)$ and $\kappa_3(t) = -\kappa(t) + \kappa(-t)$. These functions clearly satisfy points 1, 2 and 3 of the statement of Theorem 5.6. Therefore they induce a spacelike Zoll metric g on the cylinder (the parameters of the parabolic atlas being $k = 3$ and $\tau = 0$).

Let $\sigma : C \to C$ be the map sending U_i on U_{i+3} for any $i \in \mathbb{Z}/6\mathbb{Z}$ and that reads $(x, y) \mapsto (-x, -y)$ in coordinates. This map is clearly a smooth involution with no fixed points. Moreover, despite appearances (because the orientations of the frame (∂_x, ∂_y) are opposite on U_i and U_{i+3}), it does not preserve the orientation. Hence C/σ is a smooth Möbius strip. By a direct computation, we see that the metric g is invariant by σ and therefore defines a Lorentzian metric on the Möbius strip all of whose spacelike geodesics are closed. Further we can choose κ such that the curvature of g is non constant. q.e.d.

Theorem 5.11. *There exists a spacelike Zoll cylinder, admitting a parabolic atlas with parameter $k > 1$, that is not C^2-conformal to de Sitter and whose conformal boundary is not C^2.*

Proof. Let (C, g) be a time-oriented spacelike Zoll admitting a parabolic atlas and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be its lightlike foliations. We denote by $\mathcal{L}_1$ and $\mathcal{L}_2$ their spaces of leaves. As (C, g) is globally hyperbolic, we know that the $\mathcal{L}_i$ are diffeomorphic to circles. The time orientation and the orientation of (C, g) define an orientation on the $\mathcal{L}_i$.

We define the first reflexion map P (for Ping) from $\mathcal{L}_1$ to $\mathcal{L}_2$ that associates to a lightlike geodesic $\eta \in \mathcal{L}_1$ the lightlike geodesic $\bar{\eta} \in \mathcal{L}_2$ such that η and $\bar{\eta}$ intersects on the future conformal boundary of (C, g). It follows from the fact that the boundary contains no lightlike parts, see Corollary 4.5, that P is well defined and continuous. Actually, we do not need to have a conformal embedding in the flat cylinder to define the map P, $P(\eta)$ is the unique geodesic such that $\eta \cap P(\eta) = \emptyset$ and any $\hat{\eta} \in \mathcal{L}_1$ sufficiently closed to η and on one side intersects $P(\eta)$.

Clearly, any smooth parametrized transversal cutting at most once each leaf of $\mathcal{F}_i$ defines smooth local coordinates on $\mathcal{L}_i$. It follows from the definition of P in terms of reflexion on the future conformal boundary that P is smooth where the boundary is spacelike and smooth. Moreover, if it is smooth but lightlike at a point then the graph of P has a horizontal or vertical tangent at this point therefore this property can be read off of P. It follows from the other definition, that the regularity of P is a conformal invariant of (C, g). In particular, if (C, g) is conformal to a finite cover of de Sitter space then P has to be smooth. Corollary

4.5 implies that P is smooth except maybe at points of $\mathcal{L}_1$ corresponding to lightlike orbits of K. Let us look at P at a neighborhood of such a leaf.

Let $\eta_0 \in \mathcal{L}_1$ that is a lightlike orbit of K. For example η_0 is the curve contained in U_0 whose equation is $y=0$. The curve $y \mapsto (0,y)$ is transversal to $\mathcal{F}_1$ and cuts each element of $\mathcal{F}_1$ at most once, therefore it defines local coordinates on $\mathcal{L}_1$ around η_0. Possibly changing the time orientation, we can assume that the leaf $P(\eta_0)$ is the geodesic $\{y=0\}$ contained in U_1. We define as above coordinates on $\mathcal{L}_2$ around $P(\eta_0)$.

Lemma 5.12. *Let κ_0 and κ_1 be the functions given by Theorem* 5.6. *For $i = 0$ or 1, let h_i be the primitive vanishing at 0 of $s \mapsto \frac{\kappa_i(s)}{s^2}$ and let $\delta_i = \lim_{s\to-\infty} h_i(s)$. Denoting by $\overline{P}$ the expression of P in the coordinates defined above, we have for $y>0$*

$$\overline{P}(y) = \begin{cases} F^{-1}\left(\dfrac{2}{y} + \delta_1 - h_0(y)\right) & \text{if } y > 0 \\ G^{-1}\left(-\dfrac{2}{y} - \delta_0 + h_0(y)\right) & \text{if } y < 0 \end{cases}$$

where F is the map defined for $z<0$ by $F(z) = -\delta_1 + h_1(z) - \frac{2}{z}$ and G the map defined for $z>0$ by $G(z) = \delta_0 + \frac{2}{z} - h_0(z)$.

Proof. On U_i the metric reads $y^2dx^2 + 2dxdy + f_i(y)dy^2$ therefore any lightlike geodesic of U_i different from $\{y=0\}$ is transverse to ∂_x. The vector fields defined by $\frac{\kappa_{\lfloor i/2\rfloor}(y)}{y^2}\partial_x + \partial_y$ and $-\frac{2+\kappa_{\lfloor i/2\rfloor}(y)}{y^2}\partial_x + \partial_y$ for $y>0$ and by $\frac{\kappa_{\lceil i/2\rceil}(y)}{y^2}\partial_x + \partial_y$ and $-\frac{2+\kappa_{\lceil i/2\rceil}(y)}{y^2}\partial_x + \partial_y$ for $y<0$ are lightlike and smooth. Therefore any lightlike geodesic of $\{y>0\}$ is the graph of a function $y \mapsto \int \frac{\kappa_{\lfloor i/2\rfloor}(s)}{s^2}ds = h_i(y) + \text{cst}$ or $y \mapsto \int -\frac{2+\kappa_{\lfloor i/2\rfloor}(s)}{s^2}ds = \frac{2}{y} - h_{\lfloor i/2\rfloor}(y) + \text{cst}$ and any lightlike geodesic of $\{y<0\}$ is the graph of a function $y \mapsto \int \frac{\kappa_{\lceil i/2\rceil}(s)}{s^2}ds = h_{\lceil i/2\rceil}(y) + \text{cst}$ or $y \mapsto \int -\frac{2+\kappa_{\lceil i/2\rceil}(s)}{s^2}ds = \frac{2}{y} - h_{\lceil i/2\rceil}(y) + \text{cst}$.

Let η be a lightlike geodesic of $\mathcal{F}_1$ (the foliation that has $\{y=0\}$ as a leaf) intersecting H_0^+. It cuts $\{x=0\}$ at a point $(0,y_1)$. The image of $(0,y_1)$ by the transition function $\Phi_{0,1}$ is $(\frac{2}{y_1}, y_1)$, therefore $\eta \cap H_1^+$ is the graph of a map $h_0 + c_1$. As $(2/y_1, y_1) \in \eta$, we have $c_1 = \frac{2}{y_1} - h_0(y_1)$. It follows that $\eta \cap H_1^-$ is the graph of the map $h_1 + c_1$. This implies that η is asymptotic to the vertical line $\{x = \delta_1 + c_1\}$ when y goes to $-\infty$.

We now define the map F. Let z be a negative number and $\bar{\eta}$ be the geodesic of $\mathcal{F}_2$ that contains the point $(0,z)$ of H_1^-. Its intersection with H_1^- is the graph of the function $y \mapsto \frac{2}{y} - h_1(y) + h_1(z) - \frac{2}{z}$ therefore it is asymptotic to the vertical line $\{x = -\delta_1 + h_1(z) - \frac{2}{z}\}$ when y goes to $-\infty$. We set therefore $F(z) = -\delta_1 + h_1(z) - \frac{2}{z}$. By definition the function F is strictly increasing and therefore invertible.

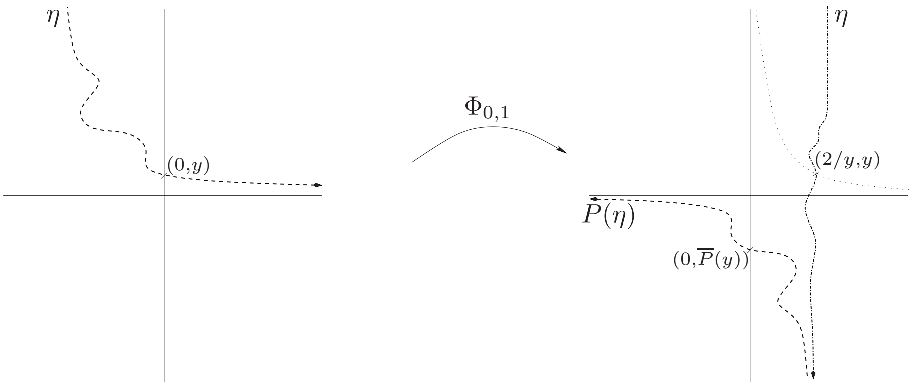

Figure 3. The map P for $y > 0$.

We have $\bar{\eta} = P(\eta)$ if and only if $\eta \cap H_1^-$ and $\bar{\eta} \cap H_1^-$ are asymptotic to the same vertical line. Indeed, if η is asymptotic to $\{x = a\}$ and $\bar{\eta}$ to $\{x = b\}$, then for $b < a$ the curves have to intersect and so $\bar{\eta} \neq P(\eta)$. If $a > b$ and $\bar{\eta} = P(\eta)$ then the leaf of $\mathcal{F}_1$ asymptotic to $\{x = \frac{a+b}{2}\}$ has to cut $\bar{\eta}$. But as $b < \frac{a+b}{2}$ they have to cut twice which is impossible since the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are transverse. It follows that $\bar{\eta} = P(\eta)$ if and only if $F(z) = \delta_1 + \frac{2}{y_1} - h_0(y_1)$ i.e. that $\overline{P}(y) = F^{-1}(\delta_1 + \frac{2}{y_1} - h_0(y_1))$ for any $y > 0$.

Let us see now what happens for a geodesic η that intersects H_0^-. We denote by $(0, y_2)$ the intersection of η with $\{x = 0\}$. Note that $y_2 < 0$. The curve $\eta \cap H_0^-$ is the graph of $y \mapsto \frac{2}{y} - h_0(y) + h_0(y_2) - \frac{2}{y_2}$. It is asymptotic to $\{x = -\delta_0 + h_0(y_2) - \frac{2}{y_2}\}$ when y goes to $-\infty$. Hence $P(\eta) \cap H_0$ is the graph of $y \mapsto h_0(y) - 2\delta_0 + h_0(y_2) - \frac{2}{y_2}$ and $P(\eta)$ cuts $y = 0$ at the point $(-2\delta_0 + h_0(y_2) - \frac{2}{y_2}, 0)$.

We now define the map F. Let z be a positive number and $\bar{\eta}$ be the geodesic of $\mathcal{F}_2$ that contains the point $(0, z)$ of H_1^+. Its intersection with H_0^+ contains the point $(2/z, z)$ therefore it is the graph of the function $y \mapsto h_0(y) - h_0(z) + \frac{2}{z}$. Thus it cuts the set $\{y = 0\}$ of U_0 at the point $(h_1(z) + \frac{2}{z}, 0)$. We define G by $G(z) = -h_0(z) + \frac{2}{z}$. Similarly to the previous case, we have $\bar{\eta} = P(\eta)$ if and only if $G(z) = -2\delta_0 + h_0(y_2) - \frac{2}{y_2}$ i.e. $\overline{P}(y) = G^{-1}(-2\delta_0 + h_0(y_2) - \frac{2}{y_2})$ for any $y < 0$. q.e.d.

We now choose some spacelike Zoll cylinder (C, g) admitting a parabolic atlas such that κ_0, κ_1 coincide with $y \mapsto y^2$ on a neighborhood of 0 but $\delta_0 \neq \delta_1$ (using Lemma 5.12's notations). Clearly, such a surface exits, but only for $k \geq 3$. Near 0 we thus have $h_0(s) = h_1(s) = s$ and

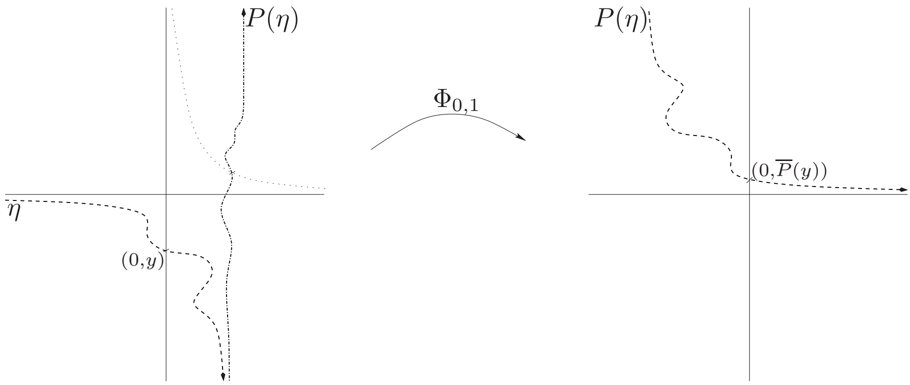

Figure 4. The map P for $y < 0$.

therefore for small z and large $y > 0$

$$F(z) = -\delta_1 + z - \frac{2}{z}, \qquad G(z) = \frac{2}{z} - z,$$
$$F^{-1}(y) = \frac{y + \delta_1 - \sqrt{(y+\delta_1)^2 + 8}}{2}, \qquad G^{-1}(y) = \frac{-y + \sqrt{y^2+8}}{2},$$

where we used that F^{-1} and G^{-1} tend to 0 when $y \to +\infty$. Hence, for small y,

$$\overline{P}(y) = \begin{cases} \frac{1}{2}\left(\frac{2}{y} + 2\delta_1 - y - \sqrt{\left[\frac{2}{y} + 2\delta_1 - y\right]^2 + 8}\right) & \text{if } y > 0 \\ \frac{1}{2}\left(\frac{2}{y} + 2\delta_0 - y + \sqrt{\left[\frac{2}{y} + 2\delta_0 - y\right]^2 + 8}\right) & \text{if } y < 0. \end{cases} \tag{4}$$

Consequently $\overline{P}$ is C^1 but not C^2. It means that the metric is not C^2-conformal to a finite cover of de Sitter space and that its conformal boundary is not C^2. q.e.d.

6. Elliptic case

Now we look at spacelike Zoll surfaces admitting a periodic Killing field. We will call them *elliptic* spacelike Zoll surfaces. This case is much simpler than the former one and very similar to the Riemannian one treated in [**3**]. Moreover, in this case we don't need to make any extra assumptions on the metric. Our result in this case is the following:

Theorem 6.1. *If (C, g) is an elliptic cylinder all of whose spacelike geodesics are closed then there exist a smooth function $f : \mathbb{R} \to \mathbb{R}$ such that $f(y)(y^2+1) - 1 < 0$ for all $y \in \mathbb{R}$ and numbers $l > 0$ and $\tau > 0$ such* *that $(C, l\,g)$ is isometric to the quotient of $\mathbb{R}^2$ endowed with the metric $(y^2+1)dx^2+2dxdy+f(y)dy^2$ by the translation $(x,y)\mapsto(x+\tau,y)$.*

Moreover such a metric has all its spacelike geodesics closed if and only if there exist p,q in $\mathbb{Z}^*$, and an odd function κ bounded below by $-\frac{p\tau}{2q\pi}$ such that

$$f(y)=\frac{1-\left(\kappa(y)+\frac{p\tau}{2q\pi}\right)^2}{y^2+1}.$$

In particular elliptic Möbius strips all of whose spacelike geodesics are closed have constant curvature.

Sketch of proof. It can be proven simply by following the scheme of proof of Section 5. The adaptation is straightforward. In particular, the metric has closed spacelike geodesics if and only if there exists integers p and q such that for any $c>0$

$$2q\int_{-c}^{c}\frac{\sqrt{c^2+1}\sqrt{1-f(y)(y^2+1)}}{(1+y^2)\sqrt{c^2-y^2}}dy=p\tau,$$

but the cylinder is spacelike Zoll if and only if $p=1$. Using the fact that

$$\int_{-c}^{c}\frac{\sqrt{c^2+1}}{(1+y^2)\sqrt{c^2-y^2}}dy=\pi$$

we find that the metric has closed spacelike geodesics if and only if

$$\int_{-c}^{c}\frac{\sqrt{c^2+1}\left(\sqrt{1-f(y)(y^2+1)}-\frac{p\tau}{2\pi q}\right)}{(1+y^2)\sqrt{c^2-y^2}}dy=0 \qquad \forall c>0 \tag{5}$$

Adapting Lemma 5.9, we see that $y\mapsto\sqrt{1-f(y)(y^2+1)}-\frac{p\tau}{2\pi q}$ is odd. For metrics g lifted from the Möbius strip this implies that $\sqrt{1-f(y)(y^2+1)}-\frac{p\tau}{2\pi q}$ vanishes and g has constant curvature. q.e.d.

7. The hyperbolic case

We are interested now in spacelike Zoll surfaces with a Killing field that vanishes somewhere. Again we start by describing a family of Lorentzian atlases.

Definition 7.1. Let (S,g) be a Lorentzian surface and let $\mathcal{A}=\{(U_i,\phi_i); i\in\mathbb{Z}/4k\mathbb{Z}\}$ be an atlas of it. We denote by $\Phi_{ij}=\phi_j\circ\phi_i^{-1}$ the transition functions of $\mathcal{A}$.
We will say that $\mathcal{A}$ is a hyperbolic atlas with parameter τ of (S,g) if:

1) for all $i\in\mathbb{Z}/4k\mathbb{Z}$, the image of ϕ_i is $\mathbb{R}^2$;

2) the transition functions are the following:

$$\begin{array}{lllll}
 & \Phi_{2i,2i+1}: & P_0 & \to & P_0 \\
 & & (x,y) & \mapsto & \left(-x+\log\left(\frac{y+1}{-y+1}\right), y\right), \\
\text{if } i \neq 0 & \Phi_{2i-2,2i+1}: & P_- & \to & P_- \\
 & & (x,y) & \mapsto & \left(-x+\log\left(\frac{y+1}{y-1}\right), y\right), \\
 & \Phi_{4k-2,1}: & P_- & \to & P_- \\
 & & (x,y) & \mapsto & \left(-x+\log\left(\frac{y+1}{y-1}\right)+\tau, y\right), \\
\text{if } i \neq 0 & \Phi_{2i-1,2i}: & P_+ & \to & P_+ \\
 & & (x,y) & \mapsto & \left(-x+\log\left(\frac{y+1}{y-1}\right), y\right), \\
 & \Phi_{4k-1,0}: & P_+ & \to & P_+ \\
 & & (x,y) & \mapsto & \left(-x+\log\left(\frac{y+1}{y-1}\right)-\tau, y\right),
\end{array}$$

where $P_+ = \{(x,y) \in \mathbb{R}^2; y > 1\}$, $P_0 = (x,y) \in \mathbb{R}^2; -1 < y < 1\}$ and $P_- = \{(x,y) \in \mathbb{R}^2; y < -1\}$ and $\tau \in \mathbb{R}$;

3) for all $i \in \{1, \dots 4k\}$,

$$g_i = \phi_i^{-1*} g = (y^2-1)\, dx^2 + 2dxdy + f_i(y)dy^2,$$

where f_i is a smooth function satisfying $1-(y^2-1)f_i(y) > 0$ for all $y \in \mathbb{R}$.

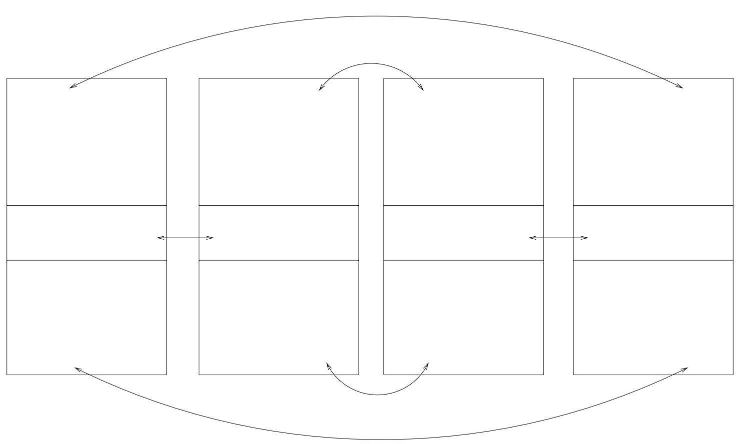

Figure 5. The gluing picture when $k = 1$.

Remark 7.2. 1) The function $\Phi_{i,j}$ are odd involutions.

2) If (S, g) admits a hyperbolic atlas then S is diffeomorphic to a cylinder with $2k$ points removed.

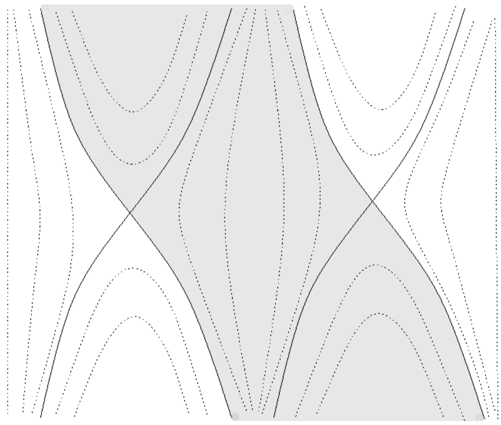

Figure 6. The manifold, the open set U_0 and the Killing field ($k = 1$).

3) De Sitter space (with 2 points removed) clearly admits such an atlas with the parameters $k = 1$, $\tau = 0$ and $f_1 = \cdots = f_4 = 0$.
4) Note that a hyperbolic atlas induces an analytic structure on C. The Killing field K is according to the conditions analytic as well. In opposition the metric g need not be analytic, but the g-length of K is again an analytic function on C.
5) The transition maps being isometries, the restriction of f_{2i} and f_{2i+1} coincide on P_0, f_{2i-2} and f_{2i+1} coincide on P_- and f_{2i-1} and f_{2i} coincide on P_+. It follows that it is sufficient to know the f_{2i} in order to know all the f_i. Further if $\mathcal{A}$ is analytic then $f_0 = f_1 = \cdots = f_{4k}$.

Proposition 7.3. *If (S, g) is a Lorentzian surface admitting a hyperbolic atlas then it can be isometrically embedded into a Lorentzian cylinder.*

Proof. Since we fill the holes one by one, we need to consider the case $k = 1$ only. In this case S is diffeomorphic to a cylinder with 2 points removed. Let us see how to fill one hole.

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function invariant by the flow of $u\partial_u - v\partial_v$ defined by $F(u, v) = f_0(uv - 1)$ if $u > 0$ and $F(u, v) = f_2(uv - 1)$ if $u < 0$. Let h be the Lorentzian metric on $\mathbb{R}^2$ given by $h = v^2du^2 + 2(1 + uv\,F)du\,dv + u^2F\,dv^2$.

Let $V_0 = \{(u, v) \in \mathbb{R}^2\,;\, u > 0\}$, $V_1 = \{(u, v) \in \mathbb{R}^2\,;\, v > 0 \text{ and } 2-uv > 0\}$, $V_2 = \{(u, v) \in \mathbb{R}^2\,;\, u < 0\}$ and $V_3 = \{(u, v) \in \mathbb{R}^2\,;\, v < 0 \text{ and } 2-uv >$

$0\}$. Let ψ_0, ψ_1, ψ_2 and ψ_3 the diffeomorphisms defined by

$$\begin{array}{rcl} \psi_0 : V_0 & \to & \mathbb{R}^2 \\ (u,v) & \mapsto & (\log(u), uv-1) \\ \psi_1 : V_1 & \to & \{(x,y)\,;\, y<1\} \\ (u,v) & \mapsto & \left(\log\left(\dfrac{v}{2-uv}\right), uv-1\right) \\ \psi_2 : V_2 & \to & \mathbb{R}^2 \\ (u,v) & \mapsto & (\log(-u)+\tau, uv-1) \\ \psi_3 : V_3 & \to & \{(x,y)\,;\, y<1\} \\ (u,v) & \mapsto & \left(\log\left(\dfrac{-v}{2-uv}\right)-\tau, uv-1\right) \end{array}$$

We let the reader check that $\psi_i^* g_i = h|_{V_i}$ and that $\psi_{i+1}\circ\psi_i^{-1} = \Phi_{i,i+1}$. Hence the first hole is filled, the second one can be filled the same way.
q.e.d.

The following proposition gives sufficient conditions to ensure that a spacelike Zoll cylinder admits a hyperbolic atlas. The conditions 2 and 3 are necessary in the sense that they are satisfied by metrics admitting a hyperbolic atlas. There exists spacelike Zoll metrics admitting a hyperbolic atlas that do not satisfy condition 1, see Theorem 7.5, but this condition is satisfied on a neighborhood of de Sitter space.

Proposition 7.4. *Let (C,g) be a spacelike Zoll cylinder with a Killing field K that is timelike somewhere (and therefore vanishing somewhere) and $\{p_0,\dots,p_{2k-1}\}$ be the set of zeros of K. Let η be a lightlike geodesic transverse to K.*

Then there exists $l>0$ such that $(C\smallsetminus\{p_0,\dots,p_{2k-1}\}, l\cdot g)$ admits a hyperbolic atlas if

1) *the curvature of g is positive at any point where K is timelike or lightlike,*
2) *K and α are analytic,*
3) *there exists $t_1\neq t_2$ such that $\alpha\circ\eta(t_1)=\alpha\circ\eta(t_2)=0$ and $(\alpha\circ\eta)'(t_1) = -(\alpha\circ\eta)'(t_2)$.*

Proof. Let $\{(U_i,\psi_i), i\in\mathbb{Z}/4k\mathbb{Z}\}$ be the atlas of $C\smallsetminus\{p_0,\dots,p_{2k-1}\}$ given by Proposition 3.4. In any of these charts g reads $h_i(y)dx^2+2dxdy$. Condition 3 of the statement above can be translated into $h_0'(a_0) = -h_0'(b_0)$, with a_0 and b_0 as defined in Proposition 3.4. It follows from the compatibility conditions between the h_i that for any $i\in\mathbb{Z}/4k\mathbb{Z}$, we have $h'(a_i)=h'(a_0)$ and $h'(b_i)=h'(b_0)$ and therefore $h'(a_i)=-h'(b_i)$ Let y_i be a critical point of h_i, we know that $h_i(y_i)<0$ and it follows from our assumption on the curvature that $h_i''(y_i)>0$. Therefore the function h_i has a unique minimum. Multiplying g and K by positive constants, if necessary, we may assume that the minimum of h_0 is -1 and $|h_0'(a_0)| = |h_0'(b_0)| = 2$.

The space of non constant orbits of K is an analytic non Hausdorff 1-dimensional manifold that we denote $\mathcal{L}$. The points where $\mathcal{L}$ is not separated correspond to the separatrix of the saddle points of K. The cardinality of this set is therefore $8k$. An atlas of $\mathcal{L}$ can be easily obtained from the atlas $\{(U_i, \psi_i)\}$. To each U_i corresponds a maximal connected Hausdorff submanifold V_i of $\mathcal{L}$. We endow $\mathcal{L}$ with an analytic vector field ∂_s whose restriction to V_0 is complete. This induces a coordinate s on each line $D_i = \{(x, y) \in U_i, x = 0\}$ such that $\alpha|_{D_i}(s)$ is analytic. It follows from the gluing picture of $\mathcal{L}$, that, up to an appropriate translation, the functions $\alpha|_{D_i}(s)$ all coincide. In particular it means that all the functions h_i have the same minimum and, using the fact that all h_i diverge to infinity at both ends, that s goes from $-\infty$ to $+\infty$ on each V_i, i.e. that ∂_s is complete.

Let λ_i be the function such that $h_i(y) = e^{\lambda_i(y)}(y - a_i)(y - b_i)$. Notice that $\lambda_i(a_i) = \lambda_i(b_i) = \ln\left(\frac{2}{|a_i - b_i|}\right)$. The map $\alpha|_{D_i}$ is a Morse function admitting a unique critical point. Therefore there exists a coordinate t on D_i (depending on i) such that $\alpha|_{D_i}(t) = t^2 - 1$ and $y(-1) = a_i$ and $y(1) = b_i$. Differentiating the equality $e^{\lambda_i(y(t))}(y(t) - a_i)(y(t) - b_i) = t^2 - 1$ we obtain $y'(-1)e^{\lambda_i(a_i)}(a_i - b_i) = -2$ i.e. $y'(-1) = 1$ and similarly $y'(1) = 1$. The metric g reads as $(t^2 - 1)dx^2 + 2\beta_i(t)dtdx$ in the coordinates (x, t). Let $t \mapsto c(t)$ be a solution of $c'(t)(t^2 - 1) = 1 - \beta_i(t)$. It follows from $y'(\pm 1) = 1$ that $\beta_i(\pm 1) = 1$. Consequently c', and therefore c, is smooth.

Using the solution $c(t)$ we define new coordinates (u, t) on U_i by $(u, t) \mapsto (c(t) + u, t)$. The metric in the coordinates (u, t) has the desired form. As in Proposition 5.3, we see that the transition functions are isometries preserving the second coordinate and sending the Killing field to its opposite. Consequently they have the desired expression up to a horizontal translation of length τ_i. However in this case, it is not obvious which translations can be assumed to be trivial.

In order to conclude we remark that the restrictions of the function α to arc length parametrized geodesics intersecting $\{p_0, \ldots, p_{2k-1}\}$ does not depend on the choice of these horizontal translations. Furthermore, it is proven in [**1**] that these functions completely determine the metric on a neighborhood of the zero. Thus it follows from Proposition 7.3 that $\sum_{k=0}^{3} \tau_{4j+k} = 0$ for any j. Now it is not difficult to modify the atlas in order to obtain a hyperbolic atlas. q.e.d.

Theorem 7.5. *Let (C, g) be a Lorentzian cylinder and $p_0, \ldots, p_{2k-1} \in C$ such that $(C \smallsetminus \{p_0, \ldots, p_{2k-1}\}, g)$ admits a hyperbolic atlas $\mathcal{A} = \{(U_i, \phi_i); i \in \mathbb{Z}/4k\mathbb{Z}\}$. If the parameter τ of $\mathcal{A}$ is 0 and if there exist $2k$ smooth functions $\kappa_0, \ldots \kappa_{2k-1}$ on $\mathbb{R}$ such that*

1) *for all* $t \in \mathbb{R}$ *and all* $j \in \mathbb{Z}/2k\mathbb{Z}$, $\kappa_j(t) \geq -1$*;*
2) *all the functions* κ_j *have the same infinite Taylor expansion at* -1 *and at* 1 *and satisfy* $\kappa_j(\pm 1) = 0$*;*
3) *for all* $i \in \{0, \ldots, 2k-1\}$ *the function* f_{2i} *such that*

$$g_{2i} = \phi_i^{-1*} g = (y^2 - 1)dx^2 + 2dxdy + f_{2i}(y)dy^2,$$

satisfies

$$f_{2i} = \frac{1 - (1 + \kappa_i)^2}{y^2 - 1};$$

4) *the restriction of the function* $\sum_j \kappa_j$ *to* $[-1, 1]$ *and the restrictions of the functions* $\sum_j \kappa_{2j}$ *and* $\sum_j \kappa_{2j+1}$ *to* $]-\infty, -1] \cup [1, +\infty[$ *are odd.*

then the cylinder (C, g) *is spacelike Zoll.*

Moreover, we have a reciprocal in the analytic case i.e. if (C, g) *is analytic and spacelike Zoll then the parameter* τ *is equal to* 0 *and* $\kappa_0 = \cdots = \kappa_{2k}$ *are odd functions.*

Proof. Let $\mathcal{A}$ be a hyperbolic atlas of

$$(C \setminus \{p_0, \ldots, p_{2k-1}\}, g).$$

We denote with K the associated Killing field and by g_i the expression of g in the coordinates (U_i, ϕ_i). We recall that there exist functions f_i such that the g_i's read as $(y^2 - 1)dx^2 + 2dxdy + f_i(y)dy^2$.

We first remark that on each U_i the foliation perpendicular to K does not depend on the functions f_i and so do the $\Phi_{i,j}$'s. Moreover, in the proof of Proposition 7.3 we saw that the transitions functions used to fill the holes are also independent of these functions. It means that the (unparameterized) spacelike geodesics orthogonal to K do not depend on the choice of the functions f_i but only on τ. In order to see when such a geodesic is close and simple we can assume that all the f_i are 0. The cylinder is then the quotient of the universal cover of de Sitter by the product of an elliptic element and the time τ of an hyperbolic flow. Consequently, the spacelike geodesics orthogonal to K are closed and simple if and only if $\tau = 0$. We assume now $\tau = 0$ and we study spacelike geodesics not perpendicular to K.

Lemma 7.6. *Let* $\gamma_i : t \mapsto (x(t), y(t))$ *be a unit spacelike geodesic of* $(\mathbb{R}^2, g_i)$ *that is not perpendicular to* ∂_x*. Then there exists* $c > 1$ *such that* γ_i *is contained between the lines* $y = c$ *and* $y = -c$*. Further* γ_i *is either tangent exactly once to each of these lines or it is asymptotic to the lines* $y = \pm 1$*. Moreover, in the first case the geodesic segment between the points of tangency satisfies:*

$$\frac{\partial x}{\partial y} = \frac{\sqrt{c^2 - 1}\sqrt{1 - (y^2 - 1)f_i(y)} - \sqrt{c^2 - y^2}}{(y^2 - 1)\sqrt{c^2 - y^2}}.$$

Furthermore, these two situations are exchanged by a transition map $\Phi_{2j,2j+1}$.

Proof. Let $\gamma_i : t \mapsto (x(t), y(t))$ be a unit spacelike geodesic of $(\mathbb{R}^2, g_i)$ that is not perpendicular to ∂_x.

Writing the first integrals of the geodesic flow, we have

$$\begin{cases} (y^2-1)x' + y' = \epsilon_1\sqrt{c^2-1} \\ (y^2-1)x'^2 + 2x'y' + f_i(y)y'^2 = 1, \end{cases} \tag{6}$$

with $c > 1$ and $\epsilon_1 = \pm 1$.

This system of equations can be solved if and only if $c^2 - y^2 \geq 0$ proving that $-c \leq y \leq c$. Solving it we find:

$$\begin{aligned} x' &= \frac{\epsilon_1\sqrt{c^2-1}(1-(y^2-1)f_i(y)) + \epsilon\sqrt{(1-y^2f_i(y))(c^2-y^2)}}{(y^2-1)(1-y^2f_i(y))} \\ y' &= -\epsilon\sqrt{\frac{c^2-y^2}{1-(y^2-1)f_i(y)}}, \end{aligned}$$

where $\epsilon = \pm 1$. It implies that

$$\frac{\partial x}{\partial y} = \frac{-\epsilon\epsilon_1\sqrt{c^2-1}\sqrt{1-(y^2-1)f_i(y)} - \sqrt{c^2-y^2}}{(y^2-1)\sqrt{c^2-y^2}}.$$

The number ϵ_1 determines the orientation of the geodesic and ϵ changes only when $y = \pm c$.

The fact that for any y_0 such that $1 < y_0 < c$ the integrals

$$\int_1^{y_0} \frac{-\sqrt{c^2-1}\sqrt{1-(y^2-1)f_i(y)} - \sqrt{c^2-y^2}}{(y^2-1)\sqrt{c^2-y^2}}dy,$$
$$\int_{-y_0}^{-1} \frac{-\sqrt{c^2-1}\sqrt{1-(y^2-1)f_i(y)} - \sqrt{c^2-y^2}}{(y^2-1)\sqrt{c^2-y^2}}dy$$

diverge, implies that γ is tangent at most once to each line $y = \pm c$.

If γ intersect $P_+ \cup P_-$, then the fact that for any $y_0 \in]1, c[$ and any $y_1 \in]-c, -1[$ the integrals

$$\int_{-c}^{c} \frac{\sqrt{c^2-1}\sqrt{1-(y^2-1)f_i(y)} - \sqrt{c^2-y^2}}{(y^2-1)\sqrt{c^2-y^2}}dy,$$
$$\int_{y_0}^{c} \frac{-\sqrt{c^2-1}\sqrt{1-(y^2-1)f_i(y)} - \sqrt{c^2-y^2}}{(y^2-1)\sqrt{c^2-y^2}}dy,$$
$$\int_{-c}^{y_1} \frac{-\sqrt{c^2-1}\sqrt{1-(y^2-1)f_i(y)} - \sqrt{c^2-y^2}}{(y^2-1)\sqrt{c^2-y^2}}dy$$

converge implies that γ is tangent at least once to each line $y = \pm c$. Between these points we have

$$\frac{\partial x}{\partial y} = \frac{\sqrt{c^2-1}\sqrt{1-(y^2-1)f_i(y)} - \sqrt{c^2-y^2}}{(y^2-1)\sqrt{c^2-y^2}}.$$

If $\gamma \subset P_0$ then $\frac{\partial x}{\partial y}$ has to be equal to

$$\frac{-\sqrt{c^2-1}\sqrt{1-(y^2-1)f_i(y)} - \sqrt{c^2-y^2}}{(y^2-1)\sqrt{c^2-y^2}}$$

and the geodesic is asymptotic to the lines $y = \pm 1$. q.e.d.

Proposition 7.7. *Let γ be a unit spacelike geodesic of (C, g) that is not perpendicular to K. Let i_0 be an* even *element of $\mathbb{Z}/4k\mathbb{Z}$ such that γ is tangent to K at some point of U_{i_0}. The geodesic γ is closed if and only if*

$$\int_{-c}^{c} \sqrt{c^2-1}\frac{\sum_{i\in\sigma_\gamma}(\sqrt{1-(y^2-1)f_i(y)}-1)}{(y^2-1)\sqrt{c^2-y^2}}dy = 0, \tag{7}$$

where $\sigma_\gamma = \left\{2i + \frac{1+(-1)^{i+1}}{2} + i_0 \in \mathbb{Z}/4k\mathbb{Z}\right\}$ *and* $\sqrt{c^2-1} = |g(\gamma', K)|$.

Proof. Similar to the parabolic case, the integral above expresses the shift between the geodesic γ and the geodesic of g^0 starting with the same initial speed at a point of tangency. The only difference is that when γ is cut along its points of tangency with K the segments obtained are contained in a U_i with $i \in \sigma_\gamma$. q.e.d.

Let g be a metric admitting a hyperbolic atlas with $\tau = 0$. Replacing $\sqrt{1-(y^2-1)f_{2i}(y)} - 1$ by κ_i in (7), we see that the spacelike geodesics having a point of tangency in U_0 are all closed if and only if

$$\int_{[-c,-1]\cup[1,c]} \frac{\sqrt{c^2-1}\sum_{i\in\sigma_\gamma}\kappa_{2i}}{(y^2-1)\sqrt{c^2-y^2}}dy + \int_{-1}^{1}\frac{\sqrt{c^2-1}\sum_{i\in\sigma_\gamma}\kappa_i}{(y^2-1)\sqrt{c^2-y^2}}dy = 0.$$

We also see that spacelike geodesics having a point of tangency in U_2 are all closed if and only if

$$\int_{[-c,-1]\cup[1,c]} \frac{\sqrt{c^2-1}\sum_{i\in\sigma_\gamma}\kappa_{2i+1}}{(y^2-1)\sqrt{c^2-y^2}}dy + \int_{-1}^{1}\frac{\sqrt{c^2-1}\sum_{i\in\sigma_\gamma}\kappa_i}{(y^2-1)\sqrt{c^2-y^2}}dy = 0.$$

This is the case, under the hypothesis of Theorem 7.5. The reciprocal is given by applying the following lemma to the function $\sum_{i\in\sigma_\gamma}\kappa_i(y)/(y^2-1)$.

Lemma 7.8. *Let $h : \mathbb{R} \to \mathbb{R}$ be a smooth function. If the function*

$$H : c \mapsto \int_0^c \sqrt{c^2-1}\frac{h(s)}{\sqrt{c^2-s^2}}ds$$

is constant on $]1,+\infty[$ *then it is equal to* 0. *If moreover* h *is analytic then* $h=0$.

Proof. We define the function $J(a)$ by:

$$J(a)=\int_1^a \frac{cH(c)}{\sqrt{a^2-c^2}\sqrt{c^2-1}}dc=\int_1^a \frac{c}{\sqrt{a^2-c^2}}\int_0^c \frac{h(s)}{\sqrt{c^2-s^2}}ds\, dc.$$

Repeating the computation in the proof of Lemma 5.9 we find

$$J(a)=\pi\int_0^a h(s)ds-2\int_0^1 h(s)\arctan\left(\frac{1-s^2}{a^2-1}\right)ds. \tag{8}$$

On the other hand if H is constant and equal to τ then $J(a)$ does not depend on a, in fact $J(a)=\tau\pi/2$. When a tends to 1 then (8) tends to 0 therefore $\tau=0$. Proving the first part.

It follows that $\int_0^c \frac{c\ h(s)}{\sqrt{c^2-s^2}}ds=0$ for any $c>1$. If h is analytic then this equality is in fact true for all $c>0$ and it follows from Lemma 5.9 that $h=0$. q.e.d.

Combining Lemma 7.6, Proposition 7.7 and Lemma 7.8 finishes the proof of Theorem 7.5. q.e.d.

It is not clear, whereas Lemma 7.8 can be extended to the smooth case. Indeed, it is possible to construct non zero C^n functions h such that the corresponding function J vanishes, even though this does not imply that H also vanishes. Adapting the construction of Corollary 5.10, we obtain:

Corollary 7.9. *There exist smooth spacelike Zoll Möbius strips with non constant curvature whose orientation covers admit hyperbolic atlases, but no analytic ones.*

Proof. Let $\mathcal{A}$ be a hyperbolic atlas with $k=2$ and $\tau=0$. Let $\sigma: C\to C$ be the involution such that $\sigma(U_{2i})=U_{2i+5}$ (and therefore $\sigma(U_{2i+1})=U_{2i-4}$) and $\phi_{2i+5}\circ\sigma\circ\phi_{2i}^{-1}(x,y)=-(x,y)=\phi_{2i}\circ\sigma\circ\phi_{2i+5}^{-1}(x,y)$. We let the reader check that σ is well defined. It has no fixed points and it is not orientation preserving therefore C/σ is a Möbius strip. Let κ be a smooth function with support in $[2,3]$ and values in $[-1,1]$. We define four functions by setting $\kappa_0=\kappa$, $\kappa_1(t)=-\kappa(t)+\kappa(-t)$, $\kappa_2(t)=-\kappa(-t)$ and $\kappa_3=0$. Let g be the spacelike Zoll metric provided by Theorem 7.5 with the functions κ_i. This metric is clearly invariant by σ and therefore induces a metric all of whose spacelike geodesics are closed on C/σ.
q.e.d.

8. Blaschke's examples

It is also possible to produce examples with no global Killing field. We just adapt Blaschke construction from [**4**, p. 162] to the Lorentzian case. We give only one of the possible constructions and let the reader imagine all the possible variations around it.

We start with de Sitter space seen as $\{(x,y,z) \in \mathbb{R}^3, -x^2+y^2+z^2 = 1\}$ endowed with the metric g^0 induced by $-dx^2 + dy^2 + dz^2$. Let K_1 be the elliptic Killing field given by $K_1(x,y,z) = (0,-z,y)$ and K_2 be the parabolic Killing field given by $K_2(x,y,z) = (y, x+z, -y)$. Let $V_1 = \{(x,y,z) \in \mathbb{R}^3; -x^2+y^2+z^2 = 1 \text{ and } g^0(K_1,K_1) \leq 2\}$. We see that $(x,y,z) \in V_1$ if and only if $|x| \leq 1$, therefore for any $(x,y,z) \in V_1$ we have $g^0(K_2,K_2) = (x+z)^2 \leq 9$. Hence, $V_2 = \{(x,y,z) \in \mathbb{R}^3; -x^2 + y^2+z^2 = 1 \text{ and } 16 \leq g^0(K_2,K_2) \leq 25\}$ and V_1 are disjoint.

Let κ_1 be an odd function with support in $[-1,1]$ bounded below by -1 and g^1 be the metric given by $g^1 = (v^2+1)du^2 + 2dudv + \frac{1-(\kappa_1(v)+1)^2}{v^2+1}dv^2$. According to Theorem 6.1, g^1 induces a spacelike Zoll metric on the quotient of $\mathbb{R}^2$ by the horizontal translation of length 2π (then $p = q = 1$). It can be seen as a perturbation of the de Sitter metric for which K_1 is still a Killing field. The support of this deformation being contained in V_1.

Let κ_2 be an odd function with support in $[-5,-4] \cup [4,5]$ bounded below by -1. Let g^2 be the metric on the cylinder given by a parabolic atlas such that $k = 1$, $\tau = 0$ and $g^2_0 = v^2du^2 + 2dudv + \frac{1-(\kappa_2(v)+1)^2}{v^2}dv^2$. According to Theorem 5.6, g^2 is spacelike Zoll. It can be seen as a perturbation of the de Sitter metric for which K_2 is still a Killing field, the support of this deformation being contained in V_2.

Let g be the metric on the cylinder that coincides with g^1 on V_1, with g^2 on V_2 and with g^0 elsewhere. Let γ be a spacelike geodesic of g. It follows from Proposition 2.3 that γ intersects V_1. If γ does not meet V_2 then it is clearly closed, otherwise it has to cross it. Let γ_1 be the g^1 geodesic that contains $\gamma \cap V^1$ and let $t_0 < t_1 < t_2 < t_3$ be such that $\gamma_1([t_1,t_2]) \subset V_1$, $\gamma_1(]t_0,t_1[) \cap V_1 = \gamma_1(]t_2,t_3[) \cap V_1 = \emptyset$ and $\gamma_1'(t_0)$ and $\gamma_1'(t_3)$ are proportional to K_1. The restrictions of γ_1 to $[t_0,t_1]$ and $[t_2,t_3]$ are geodesic segments of g^0. Let us see that these two segments are in fact on the same g^0-geodesic. We proved that g^1 is spacelike Zoll by comparing its geodesics to the g^0 one. The fact that for any $c > 0$ we have (compare to (5) in section 6):

$$\int_{-c}^{c} \frac{\sqrt{c^2+1}\,\kappa_1(v)}{(1+v^2)\sqrt{c^2-v^2}}dv = 0$$

says precisely that the g^0-geodesic that starts tangentially to K_1 from $\gamma_1(t_0)$ arrives tangentially to K_1 at the point $\gamma_1(t_3)$, proving our claim. This means that from the perspective of V_2 the perturbation on V_1 has no effect on the spacelike geodesics. In particular, in this case also γ is closed and therefore g is spacelike Zoll. Moreover, for $i = 1,2$, any global Killing field K of g has to be proportional to K_i on V_i, therefore if K is non trivial it has both lightlike leaves and periodic leaves. But such a behaviour contradicts Proposition 3.4, therefore $K = 0$.

References

[1] C. Bavard & P. Mounoud, *Extensions de tores lorentziens munis d'un champ de Killing*, in preparation.

[2] J.K. Beem, P.E. Ehrlich & K.L. Easley, *Global Lorentzian geometry. Second edition.* Monographs and Textbooks in Pure and Applied Mathematics, 202. Marcel Dekker, Inc., New York, 1996. xiv+635 pp. MR 1384756, Zbl 0846.53001.

[3] A.L. Besse, *Manifolds all of whose geodesics are closed.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 93. Springer-Verlag, Berlin-New York, 1978. MR 0641690, Zbl 0387.53010.

[4] W. Blaschke, *Vorlesungen über Differentialgeometrie,* Vol. I, Springer, Berlin, 1924, MR 0015247.

[5] M. Boucetta, *Métriques de Lorentz Zoll sur la surface de de Sitter,* Proyecciones 17 (1998), no. 1, 13–21, MR 1639716, Zbl 1082.53501.

[6] J.L. Flores, J. Herrera & M. Sánchez, *On the final definition of the causal boundary and its relation with the conformal boundary,* Adv. Theor. Math. Phys. 15 (2011), no. 4, 991–1057, MR 2929681, Zbl 1251.83008.

[7] V. Guillemin, *The Radon transform on Zoll surfaces,* Advances in Math. 22 (1976), no. 1, 85–119,MR 0426063, Zbl 0353.53027.

[8] L.W. Green, *Proof of Blaschke's sphere conjecture,* Bull. Amer. Math. Soc. 67 (1961) 156–158,MR 0124019, Zbl 0100.35804.

[9] P. Mounoud & S. Suhr, *Pseudo-Riemannian geodesic foliations by circles,* Math. Z. 274 (2013), no. 1-2, 225–238,MR 3054326, Zbl 1272.53016.

[10] R. Penrose, *Techniques of differential topology in relativity*, Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 7, Philadelphia: SIAM, 1972, MR 0469146, Zbl 0321.53001.

[11] C. Pries, *Geodesics closed on the projective plane,* Geom. Funct. Anal. 18 (2009), no. 5, 1774–1785, MR 2481742, Zbl 1160.53022.

Univ. Bordeaux, IMB
UMR 5251
F-33400 Talence, France
E-mail address: pierre.mounoud@math.u-bordeaux1.fr

Fachbereich Mathematik
Universität Hamburg
20146 Hamburg, Germany
E-mail address: suhr@dma.ens.fr

Current address:
DMA, ENS Paris and Université Paris Dauphine
75005 Paris, France