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ANTI-HOLOMORPHIC MULTIPLICATION AND A REAL ALGEBRAIC MODULAR VARIETY

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Abstract

An anti-holomorphic multiplication by the integers $\mathcal{O}_d$ of a quadratic imaginary number field, on a principally polarized complex abelian variety $A_{\mathbb{C}}$ is an action of $\mathcal{O}_d$ on $A_{\mathbb{C}}$ such that the purely imaginary elements act in an anti-holomorphic manner. The coarse moduli space $X_{\mathbb{R}}$ of such A (with appropriate level structure) is shown to consist of finitely many isomorphic connected components, each of which is an arithmetic quotient of the quaternionic Siegel space, that is, the symmetric space for the complex symplectic group. The moduli space $X_{\mathbb{R}}$ is also identified as the fixed point set of a certain anti-holomorphic involution τ on the complex points $X_{\mathbb{C}}$ of the Siegel moduli space of all principally polarized abelian varieties (with appropriate level structure). The Siegel moduli space $X_{\mathbb{C}}$ admits a certain rational structure for which the involution τ is rationally defined. So the space $X_{\mathbb{R}}$ admits the structure of a rationally defined, real algebraic variety.

1. Introduction

1.1

Let $\mathfrak{h}_n = \mathbf{Sp}(2n,\mathbb{R})/\mathbf{U}(n)$ be the Siegel upper half space of rank n and let $\Gamma = \mathbf{Sp}(2n,\mathbb{Z})$. The quotient $\Gamma\backslash\mathfrak{h}_n$ has three remarkable properties:

(a) It has the structure of a quasi-projective complex algebraic variety.

(b) It is a coarse moduli space for principally polarized abelian varieties.

(c) It has a natural compactification (the Baily-Borel Satake compactification) which admits a model defined over the rational numbers.

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Among the many missing ingredients in the theory of automorphic forms for groups of non-Hermitian type are analogues of these three facts. The associated locally symmetric spaces do not appear to have an algebraic structure; they do not appear to be associated with a natural class of elliptic curves or abelian varieties, and although they have many compactifications, there does not appear to be a canonical or "best" one. In the early 1970's G. Shimura, J. Millson and M. Kuga [10] asked whether it might be possible to address these shortcomings by realizing a locally symmetric space W for a group of non-Hermitian type as a subspace of a locally symmetric space X for a group of Hermitian type; perhaps interpreting W as a moduli space of a class of real abelian varieties. These ideas were partially investigated by A. Adler [1], H. Jaffee [10], S. Kudla [12], K.-Y. Shih [22], and G. Shimura [24, 26, 27]. In [24], Shimura showed that results of this type cannot be expected in general. He found a moduli space $X_{\mathbb{C}}$ (for a certain class of abelian varieties) which had a model defined over $\mathbb{R}$, such that the locus $X_{\mathbb{R}}$ of real points did not represent a moduli space for the corresponding real abelian varieties.

1.2

We wish to revisit this question for quotients

$$W = \Gamma \backslash Y_n \tag{1.2.1}$$

of the symmetric space $Y_n = \mathbf{Sp}(2n, \mathbb{C})/\mathbf{U}(n, \mathbb{H})$ (the "quaternionic Siegel space", cf. §10) by the principal congruence subgroup

$$\Gamma = \mathbf{Sp}(2n, \mathcal{O}_d)[M]$$

of $\mathbf{Sp}(2n, \mathcal{O}_d)$ of level M. Here, $d < 0$ is a square-free integer and $\mathcal{O}_d$ is the ring of integers in the quadratic imaginary number field $\mathbb{Q}(\sqrt{d})$. If $M \geq 3$ then the space W is a smooth manifold of (real) dimension $2n^2 + n$. It does not have an (obvious) algebraic structure. In the case $n = 1$, W is an arithmetic quotient of the real hyperbolic 3-space, Y_1.

In this paper we show, for appropriate level M, that a certain disjoint union $X_{\mathbb{R}}$ of finitely many copies of W admits analogs to all three of the above statements. That is:

(a) The smooth manifold $X_{\mathbb{R}}$ is the set of fixed points of an anti-holomorphic involution τ of a quasi-projective complex algebraic variety $X_{\mathbb{C}}$.

(b) The space $X_{\mathbb{R}}$ may be naturally identified with the (coarse) moduli space of n-dimensional abelian varieties with level M structure and with anti-holomorphic multiplication (see below) by $\mathcal{O}_d$.

(c) The complex variety $X_{\mathbb{C}}$ and the involution τ admit a model that is defined over the rationals $\mathbb{Q}$.

1.3

The algebraic variety $X_{\mathbb{C}}$ is just the Siegel moduli space $\Gamma(M)\backslash\mathfrak{h}_{2n}$ of principally polarized abelian varieties with level M structure. The involution τ extends to an anti-holomorphic involution of the Baily-Borel Satake compactification $\overline{X}$ of $X_{\mathbb{C}}$ and hence defines a real structure on $\overline{X}$. In §9 we make use of a result of Shimura [25] to prove an analogue of statement (c) above by showing that $\overline{X}$ admits a rational structure that is compatible with this real structure.

1.4

In this paper we introduce the concept of anti-holomorphic multiplication of the ring of integers $\mathcal{O}_d$ on a principally polarized abelian variety A: it is an action of $\mathcal{O}_d$ on A by real endomorphisms which are compatible with the polarization, such that the purely imaginary elements of $\mathcal{O}_d$ act in an anti-holomorphic manner; see §7.3 for the precise definition. If such an action exists then the (complex) dimension of A is even (so elliptic curves do not admit anti-holomorphic multiplication). The definition of anti-holomorphic multiplication extends in an obvious manner to more general CM fields, cf. §11.1. This appears to be a very interesting structure which merits further study.

1.5

The key technical tool in this paper, which appears to be a missing ingredient in the earlier work on this question, is Proposition 7.7, an analog of the lemma of Comessatti and Silhol ([29]). It describes "normal forms" for the period matrix of an abelian variety with anti-holomorphic multiplication. This in turn relies on a structure theorem (Proposition 6.4) for symplectic modules over a Dedekind ring.

1.6

The results in this paper, the parallel results for $\mathbf{GL}(n, \mathbb{R})$ in [8], the paper [14], and recent results of [2] and [31] suggest that there are rich, largely unexplored phenomena involving real structures and moduli space interpretations of arithmetic quotients of non-Hermitian symmetric spaces. In this paper we have chosen particular arithmetic groups for which the results are (relatively) easy to state and prove, and for which the associated Shimura variety is defined over the rational numbers. Although it is possible to establish similar results for many other arithmetic groups, we do not know to what extent these results may be generalized to arbitrary arithmetic groups. (See also §11.)

1.7

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2. Statement of results

2.1

Throughout this paper we fix a square-free integer $d < 0$ and denote by $\mathcal{O}_d$ the ring of integers in the quadratic imaginary number field $\mathbb{Q}(\sqrt{d})$. Let Q_0 be the "standard" symplectic form, whose matrix is $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. For any ring R we use any of the standard notations $\mathbf{Sp}(2n, R)$, $\mathbf{Sp}(R^{2n}, Q_0)$, or $\mathbf{Sp}(R^{2n}, J)$ to denote the symplectic group consisting of all $g \in \mathbf{GL}(2n, R)$ such that ${}^t g J g = J$, or equivalently, $Q_0(gx, gy) = Q_0(x, y)$ for all $x, y \in R^{2n}$. It consists of matrices

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

such that

$$g^{-1} = \begin{pmatrix} {}^t D & -{}^t C \\ -{}^t B & {}^t A \end{pmatrix}.$$

Let $K_s \subset \mathbf{Sp}(2n,\mathbb{C})$ be the maximal compact subgroup that is fixed under the Cartan involution $\theta_s(g) = {}^t(\bar{g})^{-1}$. It is isomorphic to the unitary group $\mathbf{U}(n,\mathbb{H})$ over the quaternions. Let $Y_n = \mathbf{Sp}(2n,\mathbb{C})/K_s$ be the associated symmetric space. It is not compact and not Hermitian; for $n = 1$ it is the real hyperbolic 3-space. In §10 (which is not needed for the main results in this paper) we describe Y_n as a certain quaternionic Siegel space, on which $\mathbf{Sp}(2n,\mathbb{C})$ acts by fractional linear transformations.

Let $G = \mathbf{Sp}(4n,\mathbb{R})$ and let $\tau : G \to G$ be the involution $\tau(g) = NgN^{-1}$ of (5.8.2). Then τ commutes with the Cartan involution $\theta : G \to G$ of (5.8.2) so it passes to an (anti-holomorphic) involution (also denoted τ) on the (usual) Siegel space $\mathfrak{h}_{2n} \cong G/G^\theta$, which is given by (5.13.1), $\tau(Z) = \mathfrak{b}\overline{Z}\,{}^t\mathfrak{b}^{-1}$. In §5.8 we describe an injective homomorphism

$$\phi = \Psi \circ \psi : \mathbf{Sp}(2n,\mathbb{C}) \to \mathbf{Sp}(4n,\mathbb{R})$$

such that $\phi\theta_s = \theta\phi$ and whose image is exactly the set of fixed points G^τ of τ. It has the property that

$$\phi^{-1}(\mathbf{Sp}(4n,\mathbb{Z})) = \mathbf{Sp}(2n,\mathcal{O}_d),$$

an arithmetic group that we will denote by Λ_0.

Lemma 2.2. *The mapping ϕ passes to a closed embedding $\phi : Y_n \to \mathfrak{h}_{2n}$ whose image is the set $\mathfrak{h}_{2n}^\tau$ of points fixed by the anti-holomorphic involution τ.*

The proof appears in §5.15. Let $\Gamma \subset \mathbf{Sp}(4n,\mathbb{Z})$ be a torsion-free arithmetic subgroup which is preserved by the involution τ. If $d \equiv 1 \pmod 4$ then assume also that Γ is contained in the principal congruence subgroup $\Gamma(2)$ of level 2. Set $X = \Gamma\backslash\mathfrak{h}_{2n}$ and let $\pi : \mathfrak{h}_{2n} \to X$ be the projection. Let $\Lambda = \phi^{-1}(\Gamma) \subset \mathbf{Sp}(2n,\mathbb{C})$ so that $\phi(\Lambda) = \Gamma^\tau$ is the τ-invariants in Γ. Set $W = \Lambda\backslash Y_n$. Then ϕ also passes to a closed embedding $\phi : W \to X$ whose image is $\pi(\phi(Y_n))$. If $h \in G$ we denote by ${}^h\phi(W) = \pi(h\phi(Y_n))$. Set

$$\widetilde{\Gamma} = \left\{\gamma \in \mathbf{Sp}(4n,\mathbb{Z}) \mid \tau(\gamma)\gamma^{-1} \in \Gamma\right\}.$$

Let $\langle\tau\rangle = \{1,\tau\}$ be the group generated by τ and let $H^1(\langle\tau\rangle,\Gamma)$ be the (nonabelian) cohomology of Γ.

Theorem 2.3. *There is a canonical isomorphism*

$$H^1(\langle\tau\rangle,\Gamma) \cong \Gamma\backslash\widetilde{\Gamma}/\phi(\Lambda_0). \tag{2.3.1}$$

The involution τ passes to an anti-holomorphic involution $\tau : X \to X$ and hence defines a real structure on X. The set of real points $X_{\mathbb{R}} = X^{\tau}$ is the disjoint union

$$X_{\mathbb{R}} = \coprod_{h} {}^{h}\phi(W) \tag{2.3.2}$$

of finitely many disjoint translations of $\phi(W)$, indexed by

$$h \in \Gamma\backslash\widetilde{\Gamma}/\phi(\Lambda_0) \cong H^1(\langle\tau\rangle, \Gamma).$$

The proof is in §3.10. For the next two results we fix an integer $M \geq 3$. If $d \equiv 1 \pmod 4$ then we assume also that M is even. Denote by $\Gamma(M)$ the principal congruence subgroup of level M in $\mathbf{Sp}(4n, \mathbb{Z})$. In the preceding theorem, take $\Gamma = \Gamma_M = \Gamma(M) \cap \tau(\Gamma(M))$. Then Equation (5.10.1) says that the arithmetic group $\Lambda = \phi^{-1}(\Gamma)$ is the principal congruence subgroup $\mathbf{Sp}(2n, \mathcal{O}_d)(M)$ of level M in the symplectic group over $\mathcal{O}_d$.

Let $\overline{X}$ denote the Baily-Borel compactification of $X = \Gamma_M\backslash\mathfrak{h}_{2n}$. It carries the structure of a complex projective algebraic variety. In §9.9 we prove the following.

Theorem 2.4. *The locus $X_{\mathbb{R}}$ is the set of real points of a quasi-projective algebraic variety which has a model defined over the rational numbers. That is, there exists a holomorphic embedding in projective space $\overline{X} \to \mathbb{P}^m$ such that the image of $\overline{X}$ is defined over the rational numbers $\mathbb{Q}$, and such that the involution $\tau : X \to X$ is the restriction of an anti-holomorphic involution $\tau : \mathbb{P}^m \to \mathbb{P}^m$, also defined over $\mathbb{Q}$, which preserves $\overline{X}$.*

If A is an abelian variety with a principal polarization and a level M structure, an anti-holomorphic multiplication by $\mathcal{O}_d$ on A is a homomorphism $\mathcal{O}_d \to \mathrm{End}_{\mathbb{R}}(A)$ which is compatible with the polarization and level structures, such that $\sqrt{d}$ acts as an anti-holomorphic mapping, cf. §7.3. In §8.6 we prove the following:

Theorem 2.5. *The real algebraic variety $X_{\mathbb{R}}$ may be canonically identified with the coarse moduli space of abelian varieties with principal polarization, level M structure, and anti-holomorphic multiplication by the ring $\mathcal{O}_d$.*

In summary, this coarse moduli space $X_{\mathbb{R}}$ (of abelian varieties with anti-holomorphic multiplication) may be realized as the locus of real points of an algebraic variety defined over $\mathbb{Q}$. It consists of finitely many isomorphic connected components, each of which is diffeomorphic to the arithmetic quotient (or locally symmetric space)

$$W = \mathbf{Sp}(2n, \mathcal{O}_d)(M)\backslash\mathbf{Sp}(2n, \mathbb{C})/\mathbf{U}(n, \mathbb{H}).$$

3. Nonabelian cohomology

3.1

Let H be a group and let $\tau : H \to H$ be an involution. Let $\langle\tau\rangle$ be the group $\{1, \tau\}$ and let $H^1(\langle\tau\rangle, H)$ be the first nonabelian cohomology set. For any $\gamma \in H$ let $f_\gamma : \langle\tau\rangle \to H$ be the mapping $f_\gamma(1) = 1$ and $f_\gamma(\tau) = \gamma$. Then f_γ is a 1-cocycle iff $\gamma\tau(\gamma) = 1$, in which case its cohomology class is denoted $[f_\gamma]$. Two cocycles f_γ and $f_{\gamma'}$ are cohomologous iff there exists $h \in H$ so that $\gamma' = \tau(h)\gamma h^{-1}$.

Let G be a reductive algebraic group defined over $\mathbb{R}$, let θ be a Cartan involution with $K = G^\theta$ the maximal compact subgroup of θ-fixed points, and let $E = G/K$ be the resulting symmetric space. Suppose $\tau : G \to G$ is an involution which commutes with θ. Denote by G^τ, K^τ, and E^τ the corresponding fixed point sets in G, K, and E respectively. For notational simplicity we will often write $\hat{g}$ for $\tau(g)$. The coboundary $\delta : E^\tau \to H^1(\langle\tau\rangle, K)$ may be defined as follows. If $g \in G$ and if $gK \in E^\tau$ then $\tau(gK) = \tau(g)K = gK$ so there exists $k \in K$ so that

$$\hat{g} = gk. \tag{3.1.1}$$

Applying τ to this equation gives $g = gk\hat{k}$, hence k defines a 1-cocycle $f_k = \delta(gK)$.

Proposition 3.2. *The cohomology sequence*

$$1 \longrightarrow K^\tau \longrightarrow G^\tau \longrightarrow E^\tau \xrightarrow{\delta} H^1(\langle\tau\rangle, K) \xrightarrow{i} H^1(\langle\tau\rangle, G)$$

is exact. Moreover:

1. *The mapping δ is trivial.*
2. *The mapping i is a bijection.*
3. *The inclusion $G^\tau \subset G$ induces a diffeomorphism*

$$G^\tau/K^\tau \cong E^\tau.$$

3.3 Proof

Exactness of the cohomology sequence is standard ([21] §5.4). In the paragraph below we will show that δ is a locally constant mapping. Since τ acts by isometries, E^τ is connected and in fact the unique geodesic between any two points $x, x' \in E^\tau$ is fixed under τ. It follows that δ takes E^τ to a single cohomology class which, taking $g = 1$ in Equation (3.1.1), is necessarily trivial. Hence $\ker(i)$ is trivial. It follows by "twisting" ([21] §5.3) that the mapping i is injective. Part (3) also follows: clearly the mapping $G^\tau/K^\tau \to E^\tau$ is well-defined and injective; and part (1) guarantees that it is also surjective.

Now we will show that δ is locally constant. First observe that if $k \in K$ is sufficiently close to the identity and if f_k is a 1-cocycle, then it is also a coboundary. For in this case we may write $k = \exp(\dot{k})$ where $\dot{k} \in \mathfrak{k} = \mathrm{Lie}(K)$. Let $\tau' : \mathfrak{k} \to \mathfrak{k}$ be the differential of τ. From $k\tau(k) = 1$ we obtain $\tau'(\dot{k}) = -\dot{k}$. Then the element $a = \exp(-\frac{1}{2}\dot{k})$ satisfies $\tau(a)a^{-1} = a^{-2} = k$ which shows that the cohomology class defined by k is trivial.

Now suppose that $gK, g_0K \in E^\tau$. Set $\tau(g) = gk$ and $\tau(g_0) = g_0k_0$ and let $u = k_0^{-1}k$. Since $\hat{k} = k^{-1}$ and $\hat{k}_0 = k_0^{-1}$ we find that $uk_0^{-1}\hat{u}k_0 = 1$. This means that $u \in K$ defines a 1-cocycle in $H^1(\langle\mu\rangle, K)$ where $\mu : K \to K$ is the involution $\mu(v) = k_0^{-1}\hat{v}k_0$. By the preceding paragraph, if g, g_0 are sufficiently close then this cocycle gives the trivial cohomology class so there exists $a \in K$ such that

$$u = \mu(a)a^{-1} = k_0^{-1}\hat{a}k_0a^{-1}$$

or $k = \hat{a}k_0a^{-1}$. This says that the cocycles defined by k and by k_0 are cohomologous, which completes the proof that δ is locally constant.

Finally it remains to be shown that i is surjective. In fact there is a splitting $j : H^1(\langle\tau\rangle, G) \to H^1(\langle\tau\rangle, K)$. Let $G = KP$ be the Cartan decomposition of G that is determined by θ. Then $P = \exp(\mathfrak{p})$ where $\mathfrak{p}$ is the -1-eigenspace of θ on $\mathfrak{g} = \mathrm{Lie}(G)$, so that $\theta(p) = p^{-1} \in P$ for all $p \in P$. Let $g = kp \in G$ and suppose that f_g is a 1-cocycle. Then $\hat{k}\hat{p}kp = 1$ or

$$(\hat{k}k)(k^{-1}\hat{p}k) = p^{-1} \in P.$$

It follows from the Cartan decomposition that

$$\hat{k}k = 1 \quad \text{and} \quad k^{-1}\hat{p}kp = 1 \tag{3.3.1}$$

so we may define $j(f_g) = f_k$. We claim that f_k represents the same cohomology class as f_g in $H^1(\langle\tau\rangle, G)$, from which it will follow that j is well-defined and that i is surjective. Equation (3.3.1) says that f_p is a 1-cocycle for the involution μ of G defined by $\mu(x) = k^{-1}\hat{x}k$. If $p = \exp \dot{p}$, set $a = \exp(-\dot{p}/2)$. If $\mu' : \mathfrak{p} \to \mathfrak{p}$ denotes the differential of μ then $\mu'(\dot{p}) = -\dot{p}$ so $\mu(a) = a^{-1}$ and $p = \mu(a)a^{-1}$. It follows that $g = kp = kk^{-1}\hat{a}ka^{-1} = \hat{a}ka^{-1}$ which says that f_g and f_k are cohomologous. q.e.d.

3.4

For the remainder of this section we assume $\mathbf{G}$ is a reductive algebraic group defined over $\mathbb{Q}$, that θ is a Cartan involution of $\mathbf{G}$, and that τ is an involution of $\mathbf{G}$ that commutes with θ. We often write $\widehat{g}$ for $\tau(g)$. Let $G = \mathbf{G}(\mathbb{R})$ denote the group of real points, $K = G^\theta$ the corresponding maximal compact subgroup, and $E = G/K$ the associated symmetric space. Fix an arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ and let $\pi : E \to X = \Gamma\backslash E$ be the projection.

To every 1-cocycle f_γ in $H^1(\langle\tau\rangle, \Gamma)$ we associate the "γ-twisted" involutions $\tau\gamma : E \to E$ by $x \mapsto \tau(\gamma x)$ and $\tau\gamma : \Gamma \to \Gamma$ by $\gamma' \mapsto \tau(\gamma\gamma'\gamma^{-1})$. Let

$$E^{\tau\gamma} = \{x \in E \,|\, \tau(x) = \gamma x\} \tag{3.4.1}$$

be the fixed point set in E of the involution $\tau\gamma$ and let $\Gamma^{\tau\gamma}$ be the fixed group in Γ of the involution $\tau\gamma$. Set $X(\tau\gamma) = \pi(E^{\tau\gamma})$. Recall the following theorem of Rohlfs ([17], [18], [19], [8]):

Theorem 3.5. *Suppose Γ is torsion-free. Then the association $f_\gamma \mapsto X^{\tau\gamma}$ determines a one to one correspondence between the cohomology set $H^1(\langle\tau\rangle, \Gamma)$ and the connected components of the fixed point set X^τ.*

3.6 Proof

The twisted involution $\tau\gamma : E \to E$ acts by isometries so ([11] I §13.5) the fixed point set $E^{\tau\gamma}$ is nonempty. If $x, x' \in E^{\tau\gamma}$ then the unique geodesic joining them is also fixed by $\tau\gamma$, so $E^{\tau\gamma}$ is connected. Its image in X is a connected subset $X(\tau\gamma)$ of X^τ which depends only on the cohomology class of f_γ. It is easy to check that f_γ and $f_{\gamma'}$ are cohomologous iff $X(\tau\gamma) \cap X(\tau\gamma') \neq \phi$. q.e.d.

3.7

In general the cohomology set $H^1(\langle\tau\rangle, \Gamma)$ may be difficult to compute, the connected component $X(\tau\gamma)$ may be difficult to describe, and distinct connected components may fail to be isomorphic. We will introduce additional hypotheses which will allow us to address these three issues. Let $\widetilde{\Gamma} \subset G(\mathbb{Q})$ be a τ stable arithmetic group that contains Γ. Let $\theta : G \to G$ be the Cartan involution corresponding to K. Consider the following hypotheses:

(1) G is Zariski connected and the fixed subgroup $G^\tau = \mathbf{G}^\tau(\mathbb{R})$ is Zariski connected.

(2) $H^1(\langle\tau\rangle, K)$ is trivial.

(3) τ acts trivially on $\Gamma\backslash\widetilde{\Gamma}$.

(4) $H^1(\langle\tau\rangle, \Gamma) \to H^1(\langle\tau\rangle, \widetilde{\Gamma})$ is trivial.

(5) Γ is torsion-free.

Lemma 3.8. *Suppose $\Gamma \subset \widetilde{\Gamma} \subset \mathbf{G}(\mathbb{Q})$ are τ-stable arithmetic subgroups. Suppose the Cartan involution θ commutes with τ. Then the following statements hold:*

(a) *Under hypothesis* (1) *above, G^τ is reductive, θ restricts to a Cartan involution of G^τ, and K^τ is a maximal compact subgroup of G^τ.*

(b) *Under hypothesis* (2) *above, the mapping $G^\tau/K^\tau \to E^\tau$ is an isomorphism. If $\gamma \in G$ and if f_γ is a* 1*-cocycle, then under hypothesis* (2)*, its class in $H^1(\langle\tau\rangle, G)$ is trivial if and only if $E^{\tau\gamma}$ is nonempty.*

(c) *Under hypothesis* (3) *above, the association $\gamma \mapsto \widehat{\gamma}\gamma^{-1}$ defines a mapping $\widetilde{\Gamma} \to \Gamma$ which passes to an injection*

$$\Gamma\backslash\widetilde{\Gamma}/\widetilde{\Gamma}^\tau \to H^1(\langle\tau\rangle, \Gamma). \tag{3.8.1}$$

Under hypotheses (3) *and* (4) *this injection is a bijection.*

(d) *Under hypothesis* (4)*, for each cohomology class*

$$[f_\gamma] \in H^1(\langle\tau\rangle, \Gamma)$$

there exists $h \in \widetilde{\Gamma}$ such that $\gamma = \tau(h)h^{-1}$, in which case,

$$E^{\tau\gamma} = hE^\tau \quad \textit{and} \quad \Gamma^{\tau\gamma} = {}^h\Gamma^\tau = h\Gamma^\tau h^{-1}.$$

(e) *Under hypotheses* (4), (3), *and* (5) *the fixed point set* X^τ *is the disjoint union of isomorphic copies*

$$X^\tau = \coprod_{h\in\Gamma\backslash\widetilde{\Gamma}/\widetilde{\Gamma}^\tau} {}^h\Gamma^\tau\backslash hE^\tau$$

of the quotient $\Gamma^\tau\backslash E^\tau$.

In summary, if hypotheses (1) through (5) are satisfied, then the fixed point set X^τ consists of finitely many isomorphic copies of the arithmetic quotient $\Gamma^\tau\backslash G^\tau/K^\tau$ indexed by $H^1(\langle\tau\rangle,\Gamma) \cong \Gamma\backslash\widetilde{\Gamma}/\widetilde{\Gamma}^\tau$.

3.9 Proof

Part (a) is proven in [20] Chapt. 1 Thm. 4.2 and Cor. 4.5 (pages 15 and 17). Now consider part (b). Clearly $G^\tau/K^\tau \subset E^\tau$ so it suffices to show that G^τ acts transitively on E^τ. Let $x = gK \in E^\tau$. Then $\tau(g)K = gK$ so the element $k = g^{-1}\tau(g)$ lies in K. Moreover, f_k is a cocycle, so by hypothesis (2) there exists $u \in K$ with $k = u\tau(u)^{-1} = g^{-1}\tau(g)$. Then $gu \in G^\tau$ and $x = gK = guK$. To prove the second statement in part (b), let f_γ be a 1-cocycle and suppose there exists a point $gK \in E^{\tau\gamma}$. Then $\widehat{\gamma}\widehat{g}K = gK$ so there exists (a unique) $k \in K$ with $\gamma g = \widehat{g}k$, or $\gamma = \widehat{g}kg^{-1}$. Hence f_γ is cohomologous to f_k, which is trivial by (2). Part (c) follows from the long exact cohomology sequence for the groups $\Gamma \subset \widetilde{\Gamma}$ and "twisting", however it is also easy to verify directly. Let $\gamma \in \widetilde{\Gamma}$. By (3) there exists a unique $a \in \Gamma$ such that $\widehat{\gamma} = a\gamma$. Moreover, f_a is a 1-cocycle, so we have defined a mapping $\phi : \widetilde{\Gamma} \to H^1(\langle\tau\rangle,\Gamma)$. Suppose $\gamma' \in \widetilde{\Gamma}$ determines the same cohomology class $\phi(\gamma)$, that is, suppose $a' = \widehat{\gamma}'(\gamma')^{-1} = \widehat{b}ab^{-1}$ for some $b \in \Gamma$. Let $x = \gamma^{-1}b^{-1}\gamma'$. Then $x \in \widetilde{\Gamma}^\tau$ because $\widehat{x} = \widehat{\gamma}^{-1}\widehat{b}^{-1}\widehat{\gamma}' = \widehat{\gamma}^{-1}\widehat{b}^{-1}\widehat{b}ab^{-1}\gamma' = \gamma^{-1}a^{-1}ab^{-1}\gamma' = x$. Consequently $\gamma' \in \Gamma\gamma\widetilde{\Gamma}^\tau$, which verifies the injectivity statement. Hypothesis (4) immediately implies that ϕ is surjective, which proves (c). Part (d) is straightforward. Part (e) follows from Rohlfs' theorem and parts (a)-(d). q.e.d.

3.10 Proof of Theorem 2.3

This follows from Lemma 3.8 provided we can verify hypotheses (1) through (5) of §3.7. Of these, (1), (3) and (5) are obvious. Hypothesis (2) will be proven in Proposition 5.11 and hypothesis (4) will be proven in Proposition 6.10. q.e.d.

4. Remarks on involutions

4.1

Let V be a real vector space with a symplectic form S and a nondegenerate symmetric bilinear form R and let $\mathbf{Sp}(V,S)$ be the group of linear automorphisms of V that preserve S. Let V^* be the dual vector space. If $g \in \mathbf{GL}(V)$ define $g^* \in \mathbf{GL}(V^*)$ by $g^*(\lambda)(v) = \lambda(g^{-1}v)$ for any $\lambda \in V^*$. Let $N \in \mathbf{GL}(V)$ and suppose that $N^2 = dI$ for some real number d. Define automorphisms τ and θ of $\mathbf{GL}(V)$ by

$$\tau(g) = NgN^{-1} \text{ and } R(gu,v) = R(u, \theta(g)^{-1}v)$$

(for all $u, v \in V$). Then τ and θ are involutions, and in fact θ is a Cartan involution: its fixed point set is the orthogonal group $\mathbf{O}(V,R)$. Define $S^\sharp : V \to V^*$ by $S^\sharp(u)(v) = S(u,v)$. Let $S^\flat : V^* \to V$ be the mapping that is uniquely determined by the relation $S(S^\flat(\lambda), x) = \lambda(x)$ for any $\lambda \in V^*$ and $x \in V$. Then $S^\flat = \left(S^\sharp\right)^{-1}$. Define $R^\sharp$, $R^\flat$ similarly.

Lemma 4.2. *The following statements hold:*

1. *If $S(Nu,v) = S(u,Nv)$ (all $u,v \in V$) then τ preserves $\mathbf{Sp}(V,S)$.*

2. *If $R(Nu,v) = -R(u,Nv)$ (all $u,v \in V$) then $\tau\theta = \theta\tau$.*

3. *If $R^\flat S^\sharp R^\flat S^\sharp = cI$ is a multiple of the identity, then the involution θ preserves the symplectic group $\mathbf{Sp}(V,S)$ and its restriction to $\mathbf{Sp}(V,S)$ is a Cartan involution.*

4.3 Proof

Part (1) is straightforward. For part (2), compute

$$R(N\theta(g)N^{-1}u, v) = R(u, Ng^{-1}N^{-1}v) = R(\theta(NgN^{-1})u, v),$$

so $\tau\theta(g) = \theta\tau(g)$. For part (3) consider the following diagram:

$$\begin{tikzcd}
V \arrow[r, "S^\sharp"] \arrow[d, "g"'] & V^* \arrow[r, "R^\flat"] \arrow[d, "g^*"'] & V \arrow[r, "S^\sharp"] \arrow[d, "\theta(g)"] & V^* \arrow[r, "R^\flat"] \arrow[d, "\theta(g)^*"] & V \arrow[d, "g"] \\
V \arrow[r, "S^\sharp"'] & V^* \arrow[r, "R^\flat"'] & V \arrow[r, "S^\sharp"'] & V^* \arrow[r, "R^\flat"'] & V.
\end{tikzcd}$$

The second and fourth square commute for every $g \in \mathbf{GL}(V)$. The first square commutes iff $g \in \mathbf{Sp}(V, S)$ (which we assume). The outside rectangle commutes by hypothesis. It follows that the third square also commutes, but this is equivalent to the statement that $\theta(g) \in \mathbf{Sp}(V, S)$. Finally, it follows from [20] I Thm. 4.2 that the restriction of θ to $\mathbf{Sp}(V, S)$ is also a Cartan involution. q.e.d.

5. An involution on the symplectic group

5.1

In this section we construct an involution τ on $\mathbf{Sp}(4n, \mathbb{R})$ which preserves a certain maximal compact subgroup K and which passes to an involution τ on the Siegel space. This involution is first constructed in a coordinate-free manner, but with respect to a non-standard symplectic form (S_2), and is denoted τ_h, see Lemma 5.6. Then we change coordinates so as to convert S_2 to the usual symplectic form, and obtain the involution τ. The impatient reader may skip directly to the matrix descriptions (5.8.2) and (5.13.1), which could be used as an (unmotivated) definition of τ.

5.2 The number field

Throughout this paper we fix a square-free integer $d < 0$ and choose a square root, $\sqrt{d}$. Let $\mathcal{O}_d$ be the ring of integers in the quadratic imaginary number field $\mathbb{Q}(\sqrt{d})$, that is, $\mathcal{O}_d = \mathbb{Z} + \mathbb{Z}\omega$ where $\omega = \sqrt{d}$ if $d \not\equiv 1 \pmod 4$ and $\omega = (1 + \sqrt{d})/2$ if $d \equiv 1 \pmod 4$. Let $h : \mathbb{C}^r \to \mathbb{R}^{2r}$ be the vector space isomorphism

$$h(x_1 + \omega y_1, x_2 + \omega y_2, \dots, x_r + \omega y_r) = (x_1, y_1, x_2, y_2, \dots, x_r, y_r).$$

Then there is a unique homomorphism

$$\psi_r : M_{r\times r}(\mathbb{C}) \to M_{2r\times 2r}(\mathbb{R}). \tag{5.2.1}$$

such that $h(gz) = \psi_r(g)h(z)$ for all $z \in \mathbb{C}^r$. It takes the matrix

$$(a_{ij} + \omega b_{ij})$$

($1 \le i,j \le r$; $a_{ij}, b_{ij} \in \mathbb{C}$) to the matrix that consists of 2×2 blocks $\mathbf{z}_{ij} = \psi_1(a_{ij} + \omega b_{ij})$ where

$$\psi_1(a+\omega b) = \begin{pmatrix} a & db \\ b & a \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a & mb \\ b & a+b \end{pmatrix} \tag{5.2.2}$$

when $d \not\equiv 1 \pmod 4$ or $d = 4m+1$ respectively.

The complex linear mapping $\mathbb{C}^r \to \mathbb{C}^r$ given by multiplication by $\sqrt{d}$ therefore corresponds to a real linear mapping $N_h = \psi_r(\sqrt{d}I) : \mathbb{R}^{2r} \to \mathbb{R}^{2r}$. As in §4 define the involution $\tau_h : \mathbf{GL}(2r,\mathbb{R}) \to \mathbf{GL}(2r,\mathbb{R})$ by

$$\tau_h(g) = N_h g N_h^{-1}. \tag{5.2.3}$$

5.3

Take $r = 2n$. Fix a complex symplectic form $s : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}$. Using the isomorphism h we obtain a bilinear mapping $S : \mathbb{R}^{4n} \times \mathbb{R}^{4n} \to \mathbb{R}^2$ whose components we denote by S_1 and S_2, that is, $s(h^{-1}u, h^{-1}v) = S_1(u,v) + \omega S_2(u,v)$ for all $u, v \in \mathbb{R}^{2n}$. Then S_1 and S_2 are (real) symplectic forms on $\mathbb{R}^{4n}$ and we denote by $\mathbf{Sp}(\mathbb{R}^{4n}, S_i)$ the corresponding symplectic groups. Since $s(\sqrt{d}x, y) = s(x, \sqrt{d}y) = \sqrt{d}s(x,y)$ we have

$$\begin{pmatrix} S_1(N_h x, y) \\ S_2(N_h x, y) \end{pmatrix} = \begin{pmatrix} S_1(x, N_h y) \\ S_2(x, N_h y) \end{pmatrix} = \beta \begin{pmatrix} S_1(x,y) \\ S_2(x,y) \end{pmatrix} \tag{5.3.1}$$

for all $x, y \in \mathbb{R}^{4n}$, where β is the matrix for N_h for $r = 1$; see §5.7. Using (5.3.1) it is easy to see that:

Lemma 5.4. *The mapping ψ_{2n} restricts to an isomorphism*

$$\mathbf{Sp}(\mathbb{C}^{2n}, s) \to \mathbf{Sp}(\mathbb{R}^{4n}, S_1) \cap \mathbf{Sp}(\mathbb{R}^{4n}, S_2). \tag{5.4.1}$$

For $i = 1, 2$ the involution τ_h preserves the group $\mathbf{Sp}(\mathbb{R}^{4n}, S_i)$. The subgroup of $\mathbf{Sp}(\mathbb{R}^{4n}, S_i)$ that is fixed by this involution is exactly the intersection (5.4.1).

5.5 Choice of s and θ

Let us take s to be the standard symplectic form Q_0 on $\mathbb{C}^{2n}$ whose matrix is $J = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$; we write $\mathbf{Sp}(2n,\mathbb{C}) = \mathbf{Sp}(\mathbb{C}^{2n}, Q_0)$. We obtain symplectic forms S_1 and S_2 on $\mathbb{R}^{4n}$ by §5.3. Take $G = \mathbf{Sp}(\mathbb{R}^{4n}, S_2)$.

Let H be the positive definite Hermitian form on $\mathbb{C}^{2n}$ given by

$$H(z,w) = z \cdot \overline{w}.$$

Use the isomorphism h to convert H into a mapping $\mathbb{R}^{4n} \times \mathbb{R}^{4n} \to \mathbb{C}$ and let R_h and S_h be the real and imaginary parts of this bilinear mapping, that is, $H(h^{-1}x, h^{-1}y) = R_h(x,y) + iS_h(x,y)$. By §4 the positive definite form R_h determines a Cartan involution θ_h on $\mathbf{GL}(4n, \mathbb{R})$.

Lemma 5.6. *The involutions θ_h and τ_h commute. Both τ_h and θ_h preserve the symplectic group $\mathbf{Sp}(\mathbb{R}^{4n}, S_i)$ (for $i = 1, 2$).*

Consequently θ_h restricts to a Cartan involution θ_h on

$$G_h = \mathbf{Sp}(\mathbb{R}^{4n}, S_2)$$

(resp. θ_s on $G_s = \mathbf{Sp}(2n, \mathbb{C})$) ([20] I, Thm. 4.2). The fixed point set $K_h = G_h^{\theta_h}$ (resp. $K_s = G_s^{\theta_s}$) is a maximal compact subgroup of G_h (resp. of G_s). The involution τ_h passes to an involution τ_h of the symmetric space $D_h = G_h/K_h$.

The proof of Lemma 5.6 consists of verifying the conditions (1), (2), and (3) of Lemma 4.2 (for $S = S_i$), which amount to several calculations with matrices (cf. §5.7). q.e.d.

5.7 Matrix descriptions

If e is a $k \times k$ matrix let

$$\mathrm{Diag}^n(e) = \mathrm{Diag}(e, e, \ldots, e)$$

be the $nk \times nk$ matrix with n identical diagonal blocks, each consisting of e. Let $\mathrm{Sp}(e)$ be the $2k \times 2k$ matrix $\begin{pmatrix} 0 & e \\ -e & 0 \end{pmatrix}$. We shall use the following 2×2 matrices.

symbol	$d \not\equiv 1 \pmod 4$	$d = 4m+1$
β	$\begin{pmatrix} 0 & d \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 2m \\ 2 & 1 \end{pmatrix}$
μ	$\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}$
ν	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$
r	$\begin{pmatrix} 1 & 0 \\ 0 & -d \end{pmatrix}$	$\begin{pmatrix} 1 & 1/2 \\ 1/2 & -m \end{pmatrix}$
r'	$\begin{pmatrix} -d & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -m & -1/2 \\ -1/2 & 1 \end{pmatrix}$

Set $\mathfrak{b} = \mathrm{Diag}^n(\beta)$, $\mathfrak{u} = \mathrm{Diag}^n(\mu)$, $\mathfrak{v} = \mathrm{Diag}^n(\nu)$, $\mathfrak{r} = \mathrm{Diag}^n(r)$, and $\mathfrak{r}' = \mathrm{Diag}^n(r')$. The bilinear forms R_h, S_1, and S_2 have matrices $R_h = \mathrm{Diag}(\mathfrak{r}, \mathfrak{r})$, $S_1 = \mathrm{Sp}(\mathfrak{u})$, and $S_2 = \mathrm{Sp}(\mathfrak{v})$. The matrix for N_h is $N_h = \mathrm{Diag}(\mathfrak{b}, \mathfrak{b})$. The Cartan involution on $\mathbf{GL}(4n, \mathbb{R})$ is $\theta_h(g) = R_h^{-1}\, {}^t g^{-1} R_h$. To prove Lemma 5.6 it is necessary to verify condition (3) of Lemma 4.2, which amounts to checking that $R_h^{-1} S_i R_h^{-1} S_i = \mathrm{const} \cdot I_{4n}$ for $i = 1, 2$, a task which may be safely assigned to the undergraduate assistant.

5.8

The symplectic form S_2 is integrally equivalent to the standard symplectic form Q_0 whose matrix is

$$J = J_{2n} = \begin{pmatrix} 0 & I_{2n} \\ -I_{2n} & 0 \end{pmatrix}.$$

which is easier to compute with. An isomorphism

$$\Psi : \mathbf{Sp}(\mathbb{R}^{4n}, S_2) \to \mathbf{Sp}(4n, \mathbb{R}) = \mathbf{Sp}(\mathbb{R}^{4n}, Q_0)$$

is given by $\Psi(g) = TgT^{-1}$ where

$$T = \begin{pmatrix} I_{2n} & 0 \\ 0 & \mathrm{Diag}^n(\nu) \end{pmatrix}$$

so that

$$Q_0(Tx, Ty) = S_2(x, y) \text{ for all } x, y \in \mathbb{R}^{4n}. \tag{5.8.1}$$

Using the isomorphism Ψ the involutions τ_h, θ_h become the following involutions τ, θ on $\mathbf{Sp}(4n, \mathbb{R}) = \mathbf{Sp}(\mathbb{R}^{4n}, Q_0)$:

$$\tau(g) = NgN^{-1} \text{ and } \theta(g) = R^{-1}\, {}^t g^{-1} R \tag{5.8.2}$$

where

$$N = T^{-1} N_h T = \begin{pmatrix} \mathfrak{b} & 0 \\ 0 & {}^t\mathfrak{b} \end{pmatrix} \text{ and } R = T^{-1} R_h T = \begin{pmatrix} \mathfrak{r} & 0 \\ 0 & \mathfrak{r}' \end{pmatrix}. \tag{5.8.3}$$

In particular, it follows from (5.3.1) that

$$Q_0(Nx, y) = Q_0(x, Ny) \text{ for all } x, y \in \mathbb{R}^{4n}. \tag{5.8.4}$$

The involution τ preserves the maximal compact subgroup K that is fixed by the Cartan involution θ on $\mathbf{Sp}(4n,\mathbb{R})$. The induced mappings on the symmetric space can also be explicitly described. The symmetric space for $\mathbf{Sp}(4n,\mathbb{R})$ may be identified with the Siegel space

$$\mathfrak{h}_{2n} = \left\{Z \in M_{2n\times 2n}(\mathbb{C}) \;\middle|\; {}^tZ = Z, \mathrm{Im}(Z) > 0\right\} \tag{5.8.5}$$

on which $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathbf{Sp}(4n,\mathbb{R})$ acts by fractional linear transformations, $g \cdot Z = (AZ+B)(CZ+D)^{-1}$. The maximal compact subgroup K is the stabilizer of the following basepoint

$$x_1 = \frac{i}{\sqrt{-d}}\mathrm{Diag}^n\begin{pmatrix} -d & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad x_1 = \frac{i}{\sqrt{-d}}\mathrm{Diag}^n\begin{pmatrix} -2m & -1 \\ -1 & 2 \end{pmatrix}$$

if $d \not\equiv 1 \pmod 4$ or if $d = 4m+1$ respectively. The symmetric space for $\mathbf{Sp}(\mathbb{R}^{4n}, S_2)$ may be identified with

$$\mathfrak{h}_{2n}\mathfrak{v} = \{Z\mathfrak{v} \in M_{2n\times 2n}(\mathbb{C}) |\; Z \in \mathfrak{h}_{2n}\} \tag{5.8.6}$$

on which $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathbf{Sp}(\mathbb{R}^{4n}, S_2)$ acts by fractional linear transformations, $g \cdot Z\mathfrak{v} = (AZ\mathfrak{v}+B)(CZ\mathfrak{v}+D)^{-1}$. The mapping Ψ passes to a mapping which we also denote by $\Psi : \mathfrak{h}_{2n}\mathfrak{v} \to \mathfrak{h}_{2n}$ and which is given by $\Psi(W) = W\mathfrak{v}^{-1}$. Then $\Psi(gW) = \Psi(g)\Psi(W)$ for all $g \in \mathbf{Sp}(\mathbb{R}^{4n}, S_2)$ and all $W \in \mathfrak{h}_{2n}\mathfrak{v}$. The mapping $\psi : Y_n \to \mathfrak{h}_{2n}\mathfrak{v}$ will be described in §10. In summary we have a commutative diagram, the last line of which provides the names of the involutions associated with a given column:

$$\begin{array}{ccccc}
K_s & \longrightarrow & K_h & \longrightarrow & K \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{Sp}(2n,\mathbb{C}) & \xrightarrow{\psi} & \mathbf{Sp}(\mathbb{R}^{4n}, S_2) & \xrightarrow{\Psi} & \mathbf{Sp}(\mathbb{R}^{4n}, Q_0) \\
\downarrow & & \downarrow & & \downarrow \\
Y_n & \xrightarrow{\psi} & \mathfrak{h}_{2n}\mathfrak{v} & \xrightarrow{\Psi} & \mathfrak{h}_{2n}. \\
\theta_s & & \theta_h, \tau_h & & \theta, \tau
\end{array}$$

Definition 5.9. Define $\phi : \mathbf{Sp}(2n,\mathbb{C}) \to \mathbf{Sp}(4n,\mathbb{R})$ and $\phi : Y_n \to \mathfrak{h}_{2n}$ to be $\phi = \Psi \circ \psi$ in the above diagram.

5.10 Remark

Let M be a nonzero integer and let $\mathbf{GL}(4n,\mathbb{Z})[M]$ be the principal congruence subgroup of level M. It follows from (5.2.2) that

$$\psi^{-1}(\mathbf{GL}(4n,\mathbb{Z})[M]) = \mathbf{Sp}(2n,\mathcal{O}_d)[M]$$

is the principal congruence subgroup of level M. Since $\Psi \in \mathbf{GL}(4n,\mathbb{Z})$ it also follows that

$$\phi^{-1}(\mathbf{Sp}(4n,\mathbb{Z})[M]) = \mathbf{Sp}(2n,\mathcal{O}_d)[M]. \tag{5.10.1}$$

Proposition 5.11. *The nonabelian cohomology sets $H^1(\langle\tau\rangle, K)$ and $H^1(\langle\tau\rangle, \mathbf{Sp}(4n,\mathbb{R}))$ are both trivial.*

5.12 Proof

By Proposition 3.2 it suffices to show that $H^1(\langle\tau\rangle, K)$ is trivial. Let $\mathfrak{b}_0 = \mathrm{Diag}^n\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)$. Then $\mathfrak{b}_0^{-1} = {}^t\mathfrak{b}_0 = -\mathfrak{b}_0$. We claim there exists is an isomorphism $\Phi : (K,\tau) \cong (\mathbf{U}(2n), \tau')$ where $\tau'(u) = \mathfrak{b}_0\bar{u}\mathfrak{b}_0^{-1}$ for all $u \in \mathbf{U}(2n)$. Assuming the claim for the moment, let us prove that $H^1(\langle\tau'\rangle, \mathbf{U}(2n))$ is trivial. Let $\mu : \mathbf{GL}(2n,\mathbb{C}) \to \mathbf{GL}(2n,\mathbb{C})$ be the involution $\mu(A) = \mathfrak{b}_0\,{}^tA^{-1}\mathfrak{b}_0^{-1}$. Its restriction to $\mathbf{U}(2n)$ coincides with τ'. To prove $H^1(\langle\tau'\rangle, \mathbf{U}(2n))$ is trivial, by Proposition 3.2 it suffices to show that $i : H^1(\langle\mu\rangle, \mathbf{U}(2n)) \to H^1(\langle\mu\rangle, \mathbf{GL}(2n,\mathbb{C}))$ is the trivial mapping. So let $u \in \mathbf{U}(2n)$ and assume that $u\mu(u) = 1$. Then $u\mathfrak{b}_0\,{}^tu^{-1}\mathfrak{b}_0^{-1} = 1$ so $u\mathfrak{b}_0$ is antisymmetric. Regarding $u\mathfrak{b}_0$ as a bilinear form, it is nondegenerate, so it can be converted into the symplectic form $\mathfrak{b}_0$ by a change of basis. In other words, there exists $A \in \mathbf{GL}(2n,\mathbb{C})$ such that $Au\mathfrak{b}_0\,{}^tA = \mathfrak{b}_0$ or

$$u = A^{-1}\mathfrak{b}_0\,{}^tA^{-1}\mathfrak{b}_0^{-1} = A^{-1}\mu(A) \tag{5.12.1}$$

This equation says that the cocycle defined by u becomes trivial in $H^1(\langle\mu\rangle, \mathbf{GL}(2n,\mathbb{C}))$ as desired.

The isomorphism $\Phi : (K,\tau) \to (\mathbf{U}(2n), \tau')$ is obtained by changing the basepoint $x_1 \in \mathfrak{h}_{2n}$ (whose isotropy group is $K = G^\theta$) to the basepoint $iI_{2n} \in \mathfrak{h}_{2n}$ (whose isotropy group we denote by $K' = G^{\theta'} \cong \mathbf{U}(2n)$). Let

$$\mathfrak{a} = \begin{pmatrix} \mathrm{Diag}^n(\alpha) & 0 \\ 0 & \mathrm{Diag}^n(\alpha') \end{pmatrix} \in \mathbf{GSp}(4n,\mathbb{R}) \text{ and } N_0 = \begin{pmatrix} \mathfrak{b}_0 & 0 \\ 0 & {}^t\mathfrak{b}_0 \end{pmatrix}$$

where α, α' are defined as follows:

symbol	$d \not\equiv 1 \pmod 4$	$d \equiv 1 \pmod 4$
α	$\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{-d} \end{pmatrix}$	$\begin{pmatrix} 2 & 1 \\ 0 & \sqrt{-d} \end{pmatrix}$
α'	$\begin{pmatrix} \sqrt{-d} & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \sqrt{-d} & 0 \\ -1 & 2 \end{pmatrix}$

Define $\Phi : \mathbf{Sp}(4n, \mathbb{R}) \to \mathbf{Sp}(4n, \mathbb{R})$ by $\Phi(g) = \mathfrak{a} g \mathfrak{a}^{-1}$. The mapping Φ converts the commuting involutions θ, τ into commuting involutions $\theta'(g) = {}^t g^{-1}$ and $\tau'(g) = N_0 g N_0^{-1}$, so it takes (K, τ) to (K', τ'), (however the mapping Φ does not preserve the integral structure). The identification $\mathbf{U}(2n) \cong K' \subset \mathbf{Sp}(4n, \mathbb{R})$ is given by $A + iB \mapsto \left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right)$. Restricting the involution τ' to $\mathbf{U}(2n)$ gives $\tau'(u) = \mathfrak{b}_0 \bar{u} \mathfrak{b}_0^{-1}$. q.e.d.

5.13 The involution on Siegel space

There is a unique involution $\tau : \mathfrak{h}_{2n} \to \mathfrak{h}_{2n}$ so that $\tau(gZ) = \tau(g)\tau(Z)$ for all $g \in \mathbf{Sp}(4n, \mathbb{R})$ and $Z \in \mathfrak{h}_{2n}$; it is given by

$$\tau(Z) = \widehat{Z} = \mathfrak{b} \bar{Z} \, {}^t\mathfrak{b}^{-1} \tag{5.13.1}$$

where $\mathfrak{b} = \mathrm{Diag}^n(\beta)$. If $Z = (\mathbf{z}_{ij})$ is divided into 2×2 blocks $\mathbf{z}_{ij}$ then $Z \in \mathfrak{h}_{2n}^{\tau}$ is fixed under τ iff $\mathbf{z}_{ij} = \beta \bar{\mathbf{z}}_{ij} \, {}^t\beta$ (for $1 \le i, j \le n$), or

$$\mathbf{z}_{ij} = \begin{pmatrix} d\bar{z}_{ij} & w_{ij} \\ \bar{w}_{ij} & z_{ij} \end{pmatrix} \quad \text{if} \quad d \not\equiv 1 \pmod 4 \tag{5.13.2}$$

$$\mathbf{z}_{ij} = \sigma^{-1} \begin{pmatrix} d\bar{z}_{ij} & w_{ij} \\ \bar{w}_{ij} & z_{ij} \end{pmatrix} {}^t\sigma^{-1} \quad \text{if} \quad d \equiv 1 \pmod 4 \tag{5.13.3}$$

for some $z_{ij}, w_{ij} \in \mathbb{C}$, and where $\sigma = \left(\begin{smallmatrix} 2 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Conversely, if $Z \in \mathfrak{h}_{2n}$ is divided into 2×2 blocks $\mathbf{z}_{ij} = \left(\begin{smallmatrix} x_{ij} & w_{ij} \\ y_{ij} & z_{ij} \end{smallmatrix}\right)$ then $Z = \tau(Z)$ iff $x_{ij} = d\bar{z}_{ij}$ and $y_{ij} = \bar{w}_{ij}$. These are linear equations in the coordinates, so $\mathfrak{h}_{2n}^{\tau}$ is an open subset of a certain linear subspace of the space of symmetric $2n \times 2n$ matrices.

Proposition 5.14. *The embedding $\phi : \mathbf{Sp}(2n, \mathbb{C}) \to \mathbf{Sp}(4n, \mathbb{R})$ passes to an embedding $\phi : Y_n = \mathbf{Sp}(2n, \mathbb{C})/K_s \to \mathfrak{h}_{2n}$ whose image is the fixed point set $\mathfrak{h}_{2n}^{\tau}$. In particular, $\phi(Y_n)$ is a real algebraic submanifold of $\mathfrak{h}_{2n}$. If $g \in \mathbf{Sp}(4n, \mathbb{R})$ and if $g \ne \pm I$ then $\phi(Y_n) \cap \mathfrak{h}_{2n}^{g}$ is a proper real algebraic subvariety of $\phi(Y_n)$, where $\mathfrak{h}_{2n}^{g}$ denotes the points in $\mathfrak{h}_{2n}$ that are fixed by g.*

5.15 Proof

The first two statements follow from Proposition 5.11 and Lemma 3.8 part (b). If $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathbf{Sp}(4n, \mathbb{R})$ then the points $Z \in \mathfrak{h}_{2n}$ that are fixed by g satisfy

$$AZ + B = ZCZ + ZD \tag{5.15.1}$$

which is a system of linear and quadratic equations in the matrix entries for Z, so this fixed point set is a real algebraic subvariety of $\mathfrak{h}_{2n}$, as is its intersection with $\mathfrak{h}^{\tau}_{2n}$. We will now show that this intersection is a proper subvariety of $\mathfrak{h}^{\tau}_{2n}$ unless $g = \pm I$. We consider only the case $d \not\equiv 1 \pmod 4$; the case $d \equiv 1 \pmod 4$ is similar.

Let Y be the $2n \times 2n$ matrix consisting of 2×2 blocks along the diagonal $\mathbf{y}_i = \begin{pmatrix} -dy_i & 0 \\ 0 & y_i \end{pmatrix}$ (for $1 \leq i \leq n$) where $y_i > 0$. It follows from (5.13.2) that $itY \in \mathfrak{h}^{\tau}_{2n}$ for all $t > 0$. If itY is fixed under g then (5.15.1) gives $t^2 YCY = B$ and $YD = AY$ from which it follows that $B = C = 0$. Since g is symplectic we also obtain $D = {}^t\!A^{-1}$. So we are reduced to considering those matrices $A \in \mathbf{GL}(2n, \mathbb{R})$ such that $Z = AZ\,{}^t\!A$ for all $Z \in \mathfrak{h}^{\tau}_{2n}$.

We outline one of many possible ways to see this implies $A = \pm I$. By taking $Z = \mathrm{Diag}(\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n)$ to consist of 2×2 blocks $\mathbf{z}_i = \left(\begin{smallmatrix} d\bar{z}_i & r_i \\ r_i & z_i \end{smallmatrix}\right)$ along the diagonal, and by varying one block but fixing the others, we may conclude that $A = \mathrm{Diag}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ also consists of 2×2 blocks, and that $\mathbf{z}_i = \mathbf{a}_i \mathbf{z}_i\,{}^t\mathbf{a}_i$. Comparing real and imaginary parts of this equation gives $\mathbf{a}_i = \pm I$. It is then easy to see that the signs must all coincide. q.e.d.

5.16 γ-real points

If $\gamma \in \mathbf{Sp}(4n, \mathbb{Z})$ and $\gamma\hat{\gamma} = I$ (that is, if f_γ is a 1-cocycle), then a point $Z \in \mathfrak{h}_{2n}$ is said to be a γ *real point* if $\tau(Z) = \gamma Z$, the set of which was denoted $\mathfrak{h}^{\tau\gamma}_{2n}$ in (3.4.1). If $\Gamma \subset \mathbf{Sp}(4n, \mathbb{Z})$ is a torsion-free subgroup that is preserved by the involution τ then the set of τ fixed points in the quotient $\Gamma\backslash\mathfrak{h}_{2n}$ is precisely the image of the set

$$\mathfrak{h}^{\tau\Gamma}_{2n} = \bigcup_{\gamma \in Z^1(\langle\tau\rangle, \Gamma)} \mathfrak{h}^{\tau\gamma}_{2n}.$$

6. Symplectic $\mathcal{O}_d$ modules

6.1

The main result in this section (Proposition 6.4), which classifies symplectic $\mathcal{O}_d$ modules, will be used in the proof of both main theorems (Theorem 2.3 and Theorem 8.5) of this paper. First, it is used to prove Proposition 6.10, which verifies hypothesis (4) (vanishing of non-abelian cohomology) of Lemma 3.8, which in turn is used to prove Theorem 2.3. Proposition 6.4 is also used in the proof of the Comessatti lemma (Proposition 7.7), which in turn is used to prove Theorem 8.5. Throughout this section we fix a square-free integer $d < 0$ and let $\mathcal{O}_d$ denote the ring of integers in the quadratic imaginary number field $\mathbb{Q}(\sqrt{d})$.

6.2

Recall [3] (VII.10 Prop. 24) that a finitely generated module P over the Dedekind domain $\mathcal{O}_d$ is torsion-free iff it is projective. If such a module P has rank n, then there exist $v_1, v_2, \ldots, v_n \in P$ such that

$$P \cong \mathcal{O}_d v_1 \oplus \mathcal{O}_d v_2 \oplus \cdots \oplus \mathcal{O}_d v_{n-1} \oplus \mathcal{I} v_n \tag{6.2.1}$$

for some fractional ideal $\mathcal{I}$.

Now suppose $P_0 \subset P$ is a submodule. Then there exist submodules $P_1, P_2 \subset P$ such that $P = P_1 \oplus P_2$ and so that $P_1 \supset P_0$ and $P_1 \otimes \mathbb{Q} = P_0 \otimes \mathbb{Q}$. For, let $M = P/P_0$ and consider its torsion-free quotient M/M^{tor} where M^{tor} denotes the torsion submodule of M. The preceding paragraph implies the composition

$$P \to M \to M/M^{\text{tor}} \tag{6.2.2}$$

admits a splitting $M/M^{\text{tor}} \to P$ whose image we denote by P_2. Then $P \cong P_1 \oplus P_2$ where

$$P_1 = \{x \in P |\ rx \in P_0 \text{ for some } r \in \mathcal{O}_d\}$$

is the kernel of the composition (6.2.2).

6.3 Polarizations

Let $Q : \mathbb{R}^{2r} \times \mathbb{R}^{2r} \to \mathbb{R}$ be a symplectic form and let $L \subset \mathbb{R}^{2r}$ be a lattice. We say that Q is a principal polarization of L if Q takes integer values on L and if, for some basis of L (and hence for any basis of L), the matrix for Q has determinant 1. In this case there exists a *symplectic basis* of L, meaning an ordered basis such that the resulting matrix for Q is

$$J_r = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}.$$

Now suppose $r = 2n$ and suppose that L is also an $\mathcal{O}_d$ module. Let us write $b \cdot v$ for the action of $b \in \mathcal{O}_d$ on a vector $v \in L$. We say the action of $\mathcal{O}_d$ is compatible with the polarization Q if

$$Q(b \cdot u, v) = Q(u, b \cdot v) \tag{6.3.1}$$

for all $u, v \in L$ and $b \in \mathcal{O}_d$, cf. Equation (7.3.4). It follows that $Q(b \cdot u, u) = 0$ for all $b \in \mathcal{O}_d$ and all $u \in L$.

Proposition 6.4. *Suppose Q is a symplectic form on $\mathbb{R}^{4n}$ that principally polarizes a lattice $L \subset \mathbb{R}^{4n}$. Suppose L has an $\mathcal{O}_d$ structure that is compatible with the polarization Q. Then there exists*

$$u_1, \ldots, u_n, v_1, \ldots, v_n \in L$$

such that the following ordered collection is a symplectic basis for L:

$$\{u_1, \omega \cdot u_1, \ldots, u_n, \omega \cdot u_n, \omega \cdot v_1, v_1, \ldots, \omega \cdot v_n, v_n\}$$

if $d \not\equiv 1 \pmod 4$, *and*

$$\{u_1, \omega \cdot u_1, \ldots, u_n, \omega \cdot u_n, (\omega - 1) \cdot v_1, v_1, \ldots, (\omega - 1) \cdot v_n, v_n\}$$

if $d \equiv 1 \pmod 4$.

In particular, $L \cong L_1 \oplus L_2$ is the direct sum of the free Lagrangian submodules

$$L_1 = \mathcal{O}_d u_1 \oplus \cdots \oplus \mathcal{O}_d u_n \quad \text{and} \quad L_2 = \mathcal{O}_d v_1 \oplus \cdots \oplus \mathcal{O}_d v_n.$$

In either case, with respect to this basis, the matrix for the action of $\sqrt{d} \in \mathcal{O}_d$ is the matrix N of Equation (5.8.3) (cf. §7.3).

Proposition 6.4 will be proven by induction on n.

6.5 The case $n = 1$

By §6.2, $L = \mathcal{O}_d v \oplus \mathcal{I} w$ for some $v, w \in L$ and some fractional ideal $\mathcal{I}$. So a $\mathbb{Z}$ basis of L is given by

$$\{v, \omega \cdot v, aw, b\omega \cdot w\} \tag{6.5.1}$$

for some $a, b \in \mathbb{Q}$. The symplectic form Q vanishes on $\mathcal{O}_d v$ since $Q(v, \omega \cdot v) = Q(\omega \cdot v, v) = -Q(v, \omega \cdot v)$, and it similarly vanishes on $\mathcal{I} w$. So with respect to this basis, the matrix for Q is $\begin{pmatrix} 0 & P \\ -{}^tP & 0 \end{pmatrix}$ where P is some integer matrix. On the other hand, Q is a principal polarization of L, so $\det P = \pm 1$. Apply P^{-1} to the basis $\{aw, b\omega \cdot w\}$ of $\mathcal{I} w$ to obtain a new basis $\{x, y\}$ of $\mathcal{I} w$. Then the matrix of Q with respect to the basis $\{v, \omega \cdot v, x, y\}$ is J_2. Now let us determine the relationship between x, y, and $\omega \cdot y$. Set $x = a'y + b'\omega \cdot y$ for some $a', b' \in \mathbb{Q}$. Then

$$\begin{aligned}
1 &= Q(v, x) = a'Q(v, y) + b'Q(v, \omega \cdot y) = b'Q(\omega \cdot v, y) = b' \\
0 &= Q(\omega \cdot v, x) = a'Q(\omega \cdot v, y) + b'Q(\omega \cdot v, \omega \cdot y) = a' + Q(\omega^2 \cdot v, y) \\
&= \begin{cases} a' + dQ(v, y) = a' & \text{if } d \not\equiv 1 \pmod 4 \\ a' + Q((\omega + m) \cdot v, y) = a' + 1 & \text{if } d \equiv 1 \pmod 4. \end{cases}
\end{aligned}$$

Hence $x = \omega \cdot y$ if $d \not\equiv 1 \pmod 4$ and $x = (\omega - 1) \cdot y$ if $d \equiv 1 \pmod 4$, as desired. In either case, $\omega \cdot y \in \mathcal{I} w$ so $\mathcal{I} w = \mathcal{O}_d y$ is free.

6.6 The case $n > 1$

We will prove in Lemma 6.7 below (by a somewhat roundabout argument) that there exist elements $x, y \in L$ so that

$$Q(x, y) = 0 \text{ and } Q(x, \omega \cdot y) = 1. \tag{6.6.1}$$

It follows that $Q(\omega \cdot x, y) = 1$ and $Q(\omega \cdot x, \omega \cdot y) = 0$. Let P_0 be the $\mathcal{O}_d$ span of $\{x, y\}$. It has a $\mathbb{Z}$ basis $\{x, \omega \cdot x, \omega \cdot y, y\}$ with respect to which the matrix of $Q|P_0$ is J_2.

We claim that L splits as a direct sum, $L = P_0 \oplus L_2$ of $\mathcal{O}_d$ submodules (of $\mathbb{Z}$ rank 4 and rank 4n-4 respectively) such that the restriction $Q|L_2$ is a principal polarization. By induction, the lattice L_2 has a basis of the desired type, from which it follows that L does also.

The claim is proven as follows. Using §6.2 there exists a splitting $L = P_1 \oplus P_2$ by submodules P_1 and P_2 such that $P_1 \supset P_0$ and $P_1 \otimes \mathbb{Q} = P_0 \otimes \mathbb{Q}$. Let $\{u, v, z, w\}$ be a $\mathbb{Z}$ basis for P_1 and let Q_1 be the matrix of

Q with respect to this basis. If A denotes the matrix that transforms this basis of $P_1 \otimes \mathbb{Q}$ into the basis $\{x, \omega \cdot x, \omega \cdot y, y\}$, then $J_2 = AQ_1\, {}^tA$. Since these are matrices of integers, it follows that $\det(A) = \pm 1$ hence $P_0 = P_1$. The next step is to modify the complement P_2 to obtain a complement L_2 which is principally polarized.

By §6.2 we may write $P_2 = \mathcal{O}_d w_1 \oplus \cdots \oplus \mathcal{O}_d w_{2n-3} \oplus \mathcal{I} w_{2n-2}$ for some $w_i \in P_2$. For $1 \leq i \leq 2n-2$ set

$$w_i' = w_i - \lambda_i x - \mu_i \omega \cdot x - \gamma_i \omega \cdot y - \nu_i y.$$

Then there are unique choices of integers $\lambda_i, \mu_i, \gamma_i, \nu_i \in \mathbb{Z}$ so that each w_i' is Q orthogonal to P_0. Let L_2 be the $\mathcal{O}_d$ span of the vectors w_i' (for $1 \leq i \leq 2n-2$). Then $L = P_0 \oplus L_2$ and L_2 is orthogonal to P_0. With respect to any choice of $\mathbb{Z}$ basis for L_2 (and the above basis for P_0) the matrix for Q is

$$\begin{pmatrix} J_2 & 0 \\ 0 & Q_2 \end{pmatrix} \tag{6.6.2}$$

where Q_2 is some integer matrix. However Q is a principal polarization, so the determinant of the matrix (6.6.2) is 1, from which it follows that the determinant of Q_2 is also 1. Therefore, the restriction of Q to L_2 is a principal polarization, as desired. The rest of this section is dedicated to proving the existence of the elements x, y, which we now state precisely.

Lemma 6.7. *Fix $n \geq 2$. Suppose $L \subset \mathbb{R}^{4n}$ is a lattice that is principally polarized by the symplectic form Q, and suppose L admits a compatible action of $\mathcal{O}_d$. Then there exists $x, y \in L$ so that* (6.6.1) *holds.*

6.8 Proof

If $\{u_1, \ldots, u_r\}$ is a collection of vectors in L let

$$\langle u_1, \ldots, u_r \rangle$$

denote their vector space span in $\mathbb{R}^{2n}$ and let

$$\langle u_1, \ldots, u_r \rangle^{\perp}$$

be the Q-annihilator of this span. Since Q is integral on L, the intersection $L \cap \langle u_1, \ldots, u_r \rangle^{\perp}$ is a lattice in $\langle u_1, \ldots, u_r, \rangle^{\perp}$.

Step 1. *There exists a Lagrangian $\mathcal{O}_d$ submodule $L_0 \subset L$.*

Suppose by induction that a $\mathbb{Q}$ linearly independent collection of vectors

$$\{u_1, \omega \cdot u_1, u_2, \omega \cdot u_2, \dots, u_r, \omega \cdot u_r\} \subset L \tag{6.8.1}$$

has been found so that Q vanishes on their $\mathbb{Q}$ span U_r (with the case $r = 0$ being trivial). The Q-annihilator $U_r^\perp$ has dimension $2n - r$ and the intersection $L \cap U_r^\perp$ is a lattice in $U_r^\perp$. If $r < n$ then there exists a vector $u_{r+1} \in U_r^\perp \cap L$ which is not contained in U_r. We claim the collection $\{u_1, \omega \cdot u_1, \dots, u_r, \omega \cdot u_r, u_{r+1}, \omega \cdot u_{r+1}\}$ is linearly independent and that Q vanishes on its vector space span U_{r+1}.

Suppose that $\omega \cdot u_{r+1}$ is a linear combination of the other vectors in this collection, say,

$$\omega \cdot u_{r+1} = \sum_{i=1}^{r} (a_i u_i + b_i \omega \cdot u_i) + c u_{r+1}$$

for some rational numbers a_i, b_i, and c. Multiplying by ω and collecting terms gives

$$(d - c^2) u_{r+1} = \sum_{i=1}^{r} (a_i + c b_i) \omega \cdot u_i + (b_i d + c a_i) u_i)$$

if $d \not\equiv 1 \pmod 4$. But $d - c^2 < 0$ so this contradicts the linear independence of (6.8.1). The case of $d \equiv 1 \pmod 4$ is similar.

Step 2. *There exists a Lagrangian $\mathcal{O}_d$ submodule $L_1 \subset L$ and a submodule $L_2 \subset L$ so that $L = L_1 \oplus L_2$.*

This follows from §6.2 and in fact $L_1 \otimes \mathbb{Q} = L_0 \otimes \mathbb{Q}$.

Now set $L_1 = \mathcal{O}_d y_1 \oplus \mathcal{O}_d y_2 \oplus \cdots \oplus \mathcal{O}_d y_{n-1} \oplus \mathcal{I} y_n$ for some fractional ideal $\mathcal{I}$. Then there exist rational numbers $a, b \in \mathbb{Q}$ so that the collection $\{y_1, \omega \cdot y_1, \dots, y_{n-1}, \omega \cdot y_{n-1}, y_n, a y_n + b\omega \cdot y_n\}$ forms a $\mathbb{Z}$ basis of L_1.

Step 3. Choose any $\mathbb{Z}$ basis for L_2. Together with the preceding basis for L_1 this gives a basis for $L = L_1 \oplus L_2$ with respect to which the matrix of Q is

$$\begin{pmatrix} 0 & T \\ -{}^tT & * \end{pmatrix}$$

for some matrix T of integers. It follows that $\det(T) = \pm 1$. Applying T^{-1} to this basis gives a new $\mathbb{Z}$ basis for L_2 such that the matrix for Q is

$$\begin{pmatrix} 0 & I \\ -I & * \end{pmatrix}.$$

Denote this new basis by $\{z_1, x_1, z_2, x_2, \ldots, z_n, x_n\}$. Then

$$Q(y_1, x_1) = 0 \text{ and } Q(\omega \cdot y_1, x_1) = 1.$$

Therefore the elements $y = y_1$ and $x = -x_1$ satisfy (6.6.1). q.e.d.

6.9 Application to nonabelian cohomology

Proposition 6.4 may be used to construct an $\mathcal{O}_d$ module structure on certain lattices. Let τ be the involution of $\mathbf{Sp}(4n, \mathbb{Z}) = \mathbf{Sp}(\mathbb{Z}^{4n}, Q_0)$ defined in §5.7. Let $\Gamma \subset \mathbf{Sp}(4n, \mathbb{Z})$ be a torsion-free (arithmetic) subgroup that is preserved under τ. If $d \equiv 1 \pmod 4$ then suppose also that Γ is contained in the principal congruence subgroup $\Gamma(2)$ of level 2. Set

$$\widetilde{\Gamma} = \left\{ h \in \mathbf{Sp}(4n, \mathbb{Z}) \;\middle|\; \widehat{h}h^{-1} \in \Gamma \right\}.$$

Proposition 6.10. *The mapping*

$$H^1(\langle \tau \rangle, \Gamma) \to H^1(\langle \tau \rangle, \widetilde{\Gamma})$$

is trivial.

6.11 Proof

Let $\gamma \in \Gamma$ and suppose f_γ is a 1-cocycle, that is, $\gamma\widehat{\gamma} = \gamma N \gamma N^{-1} = I$, cf. Equation (5.8.3). It follows that $(N\gamma)^2 = N\gamma N\gamma = N^2 = dI$. Using Equation (5.8.4) we obtain

$$Q_0(N\gamma u, N\gamma v) = dQ_0(u, v)$$

and hence

$$Q_0(N\gamma u, v) = Q_0(u, N\gamma v)$$

for all $u, v \in \mathbb{R}^{4n}$. We use this to define a Q_0-polarized $\mathcal{O}_d$ module structure on the standard lattice $\mathbb{Z}^{4n}$ by letting $\sqrt{d}$ act through $N\gamma$, that is, define

$$(a + b\sqrt{d}) \cdot u = au + bN\gamma u$$

whenever $a, b \in \mathbb{Z}$ If $d \equiv 1 \pmod 4$ it is necessary to check that the action of $\omega = (1 + \sqrt{d})/2$ also preserves the lattice $\mathbb{Z}^{4n}$, however this follows from the fact that $\gamma \equiv I \pmod 2$ when $d \equiv 1 \pmod 4$.

So we may apply Proposition 6.4 to conclude that $\mathbb{Z}^{4n}$ admits a symplectic basis with respect to which the matrix of $\sqrt{d}$ is N. In other words, there exists $h \in \mathbf{Sp}(4n, \mathbb{Z})$ such that $N\gamma = hNh^{-1}$. Using the fact that $N = N^{-1}dI$ we conclude that

$$\gamma = N^{-1}hNh^{-1} = NhN^{-1}h^{-1} = \widehat{h}h^{-1}$$

from which it follows that $h \in \widetilde{\Gamma}$ and that the cocycle f_γ is a coboundary. q.e.d.

7. Anti-holomorphic multiplication

7.1

In this section we recall [13] some standard facts and notation concerning abelian varieties. Let $L \subset \mathbb{C}^r$ be a lattice (that is, a free abelian subgroup of rank $2r$ so that $L \otimes_{\mathbb{C}} \mathbb{R} \to \mathbb{C}^{2r}$ is an isomorphism of real vector spaces). Then $A = \mathbb{C}^r/L$ is a complex torus. If $\omega_1, \omega_2, \ldots, \omega_r$ is a basis for the space of holomorphic 1-forms on A, and if $v_1, v_2, \ldots, v_{2r}$ is a basis for L then the corresponding *period matrix* Ω is the matrix with entries $\Omega_{ij} = \int_{v_j} \omega_i$. If $v_i' = \sum_j A_{ij} v_j$ and if $\omega_i' = \sum_j B_{ij}\omega_j$ are new bases then the resulting period matrix is

(7.1.1) $$\Omega' = B\Omega\, {}^tA.$$

A real symplectic form Q on $\mathbb{C}^r$ is *compatible* with the complex structure if $Q(iu, iv) = Q(u, v)$ for all $u, v \in \mathbb{C}^r$, (not to be confused with Equation (7.3.4) below). A compatible form Q is *positive* if the symmetric form $R(u, v) = Q(iu, v)$ is positive definite. If Q is compatible and positive then it is the imaginary part of a unique positive definite Hermitian form $H = R+iQ$. Let $L \subset \mathbb{C}^r$ be a lattice and let $H = R+iQ$

be a positive definite Hermitian form on $\mathbb{C}^r$. Recall that Q is a *principal polarization* of L if L admits a basis such that the resulting matrix for Q is J_r (cf. §6.3). A principally polarized abelian variety is a pair $(A = \mathbb{C}^r/L, H = R + iQ)$ where H is a positive definite Hermitian form on $\mathbb{C}^r$ and where $L \subset \mathbb{C}^r$ is a lattice that is principally polarized by Q.

Each $Z \in \mathfrak{h}_r$ determines a principally polarized abelian variety

$$(A_Z, H_Z)$$

as follows. Let Q_0 be the standard symplectic form on $\mathbb{R}^{2r} = \mathbb{R}^r \oplus \mathbb{R}^r$ with matrix $J = J_r = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ (with respect to the standard basis of $\mathbb{R}^r \oplus \mathbb{R}^r$). Let $F_Z : \mathbb{R}^r \oplus \mathbb{R}^r \to \mathbb{C}^r$ be the real linear mapping with matrix (Z, I), that is,

$$F_Z \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = Zx + y.$$

Then

$$Q_Z = (F_Z)_*(Q_0) \tag{7.1.2}$$

is a compatible, positive symplectic form that principally polarizes the lattice

$$L_Z = F_Z(\mathbb{Z}^r \oplus \mathbb{Z}^r). \tag{7.1.3}$$

(In fact F_Z(standard basis) is a symplectic basis for L_Z.) The Hermitian form corresponding to Q_Z is

$$H_Z(u, v) = Q_Z(iu, v) + iQ_Z(u, v) = {}^t u(\mathrm{Im}(Z))^{-1}\bar{v}$$

for $u, v \in \mathbb{C}^r$. The pair $(A_Z = \mathbb{C}^r/L_Z, H_Z)$ is the desired principally polarized abelian variety. If $z_1, z_2, \ldots, z_r$ are the standard coordinates on $\mathbb{C}^r$ then, with respect to the above symplectic basis of L, the differential forms $dz_1, dz_2, \ldots, dz_r$ have period matrix $\Omega = (Z, I)$.

7.2

The principally polarized abelian varieties $(A_Z = \mathbb{C}^r/L_Z, H_Z)$ and $(A_\Omega = \mathbb{C}^r/L_\Omega, H_\Omega)$ are isomorphic iff there exists a complex linear mapping $\xi : \mathbb{C}^r \to \mathbb{C}^r$ such that $\xi(L_\Omega) = L_Z$ and $\xi_*(H_\Omega) = H_Z$. Set $h = {}^t(F_Z^{-1}\xi F_\Omega) = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. Then: $h \in \mathbf{Sp}(2r, \mathbb{Z})$, $\Omega = h \cdot Z$, and

$\xi(M) = {}^t(CZ+D)M$ for all $M \in \mathbb{C}^r$, which is to say that the following diagram commutes:

$$\begin{tikzcd} \binom{x}{y} \arrow[d] & \mathbb{R}^r \oplus \mathbb{R}^r \arrow[r, "F_{h\cdot Z}"'] \arrow[d] & \mathbb{C}^r \arrow[d, "\xi"] & M \arrow[d] \\ {}^t h\binom{x}{y} & \mathbb{R}^r \oplus \mathbb{R}^r \arrow[r, "F_Z"'] & \mathbb{C}^r & {}^t(CZ+D)M \end{tikzcd} \tag{7.2.1}$$

(since $h \cdot Z$ is symmetric). The relationship between the mapping F_Z and the involution τ is the following. Let N be as in (5.8.3). If $Z \in \mathfrak{h}_r$ and $\widehat{Z} = \mathfrak{b}\overline{Z}\,{}^t\mathfrak{b}^{-1}$ then this diagram commutes:

$$\begin{tikzcd} \binom{x}{y} \arrow[d] & \mathbb{R}^r \oplus \mathbb{R}^r \arrow[r, "F_Z"'] \arrow[d] & \mathbb{C}^r \arrow[d] & M \arrow[d] \\ {}^t N\binom{x}{y} & \mathbb{R}^r \oplus \mathbb{R}^r \arrow[r, "F_{\widehat{Z}}"'] & \mathbb{C}^r & \mathfrak{b}\overline{M}. \end{tikzcd} \tag{7.2.2}$$

7.3

A *real endomorphism* of a principally polarized abelian variety $(A = \mathbb{C}^r/L, H = R + iQ)$ is an $\mathbb{R}$-linear mapping $f : \mathbb{C}^r \to \mathbb{C}^r$ such that $f(L) \subset L$; two such being considered equivalent if they induce the same mapping $A \to A$. As in §5.2, fix a square-free integer $d < 0$ and let $\mathcal{O}_d$ denote the ring of integers in the number field $\mathbb{Q}(\sqrt{d})$. Recall (for example, from [23] Equation (5.5.12) or [27] or [7] §3.1.1) that a *complex multiplication* by the ring $\mathcal{O}_d$ on A is a ring homomorphism $\Phi : \mathcal{O}_d \to \mathrm{End}_{\mathbb{R}}(A)$ such that $\Phi(1) = I$, and for all $b \in \mathcal{O}_d$ and $u, v \in \mathbb{C}^r$,

$$\Phi(b) : \mathbb{C}^r \to \mathbb{C}^r \text{ is complex linear,} \tag{7.3.1}$$

$$Q(\Phi(b)u, v) = Q(u, \Phi(\bar{b})v). \tag{7.3.2}$$

(If $r = 1$ then (7.3.2) follows from (7.3.1) and the relation $Q(u,v) = Q(iu, iv)$.) In analogy with the above, let us say that an *anti-holomorphic multiplication* by the ring $\mathcal{O}_d$ is a ring homomorphism $\Psi : \mathcal{O}_d \to \mathrm{End}_{\mathbb{R}}(A)$ such that $\Psi(1) = I$ and so that the mapping $\kappa = \Psi(\sqrt{d}) : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ satisfies

$$\kappa(au) = \bar{a}\kappa(u) \tag{7.3.3}$$

$$Q(\kappa(u), v) = Q(u, \kappa(v)) \tag{7.3.4}$$

for all $a \in \mathbb{C}$ and $u, v \in \mathbb{C}^r$. (Consequently, $\Psi(1)$ is complex linear and $\kappa = \Psi(\sqrt{d})$ is anti-linear.) In this case, Ψ is determined by the mapping κ, and Equation (7.3.4) implies that

$$Q(\Psi(b)(u), v)) = Q(u, \Psi(b)(v)) \tag{7.3.5}$$

for all $b \in \mathcal{O}_d$. If such an anti-holomorphic multiplication exists then r is even (and in fact $\langle u, v\rangle = Q(\kappa(u), v) + \sqrt{d}Q(u, v)$ is a complex symplectic form on $\mathbb{C}^r$ with respect to the complex structure defined by $\kappa/\sqrt{-d})$. Equivalently, a choice of anti-holomorphic multiplication by $\mathcal{O}_d$, if one exists, is a choice of $\mathcal{O}_d$-module structure on L that satisfies (7.3.3) and (7.3.5) for all $u, v \in L$ and all $b \in \mathcal{O}_d$.

7.4

For the remainder of this section take $r = 2n$ and write Q_0 for the (standard) symplectic form on $\mathbb{R}^{4n}$ whose matrix is $J = J_{2n}$ with respect to the standard basis of $\mathbb{R}^{4n}$. The following lemma states that certain points (the γ-real points, for appropriately chosen γ) in the Siegel space correspond to abelian varieties with anti-holomorphic multiplication. We use the involution τ defined in (5.8.2) and (5.13.1) and the corresponding matrix N of (5.8.3); see also §5.16.

Lemma 7.5. *Let $\gamma \in \mathbf{Sp}(4n, \mathbb{Z})$ and suppose $\gamma\widehat{\gamma} = 1$. If $d \equiv 1 \pmod 4$ then assume also that $\gamma \equiv I \pmod 2$. Fix $Z \in \mathfrak{h}_{2n}^{\tau\gamma}$. Then the mapping*

$$\kappa_Z = F_Z \circ {}^t(N\gamma) \circ F_Z^{-1} : \mathbb{C}^{2n} \to \mathbb{C}^{2n} \tag{7.5.1}$$

defines an anti-holomorphic multiplication by $\mathcal{O}_d$ on the principally polarized abelian variety (A_Z, H_Z).

7.6 Proof

Set $\eta = {}^t(N\gamma)$. Then

$$\eta^2 = {}^t(N\gamma N\gamma) = {}^t(N\gamma dN^{-1}\gamma) = d\,{}^t(\widehat{\gamma}\gamma) = dI$$

so the same is true of κ_Z. Also, $\eta = {}^t\gamma \begin{pmatrix} {}^t\mathfrak{b} & 0 \\ 0 & \mathfrak{b}^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & dI \end{pmatrix}$. The first two factors are in $\mathbf{Sp}(4n, \mathbb{R})$ so $Q_0(\eta u, \eta v) = dQ_0(u, v)$ for all $u, v \in \mathbb{R}^{4n}$. Hence $Q_Z(\kappa_Z u, \kappa_Z v) = dQ_Z(u, v)$ for all $u, v \in \mathbb{C}^{2n}$ which implies

(7.3.4). The mapping κ_Z preserves the lattice $L_Z = F_Z(\mathbb{Z}^{2n} \oplus \mathbb{Z}^{2n})$ since η preserves the integer lattice. If $d \equiv 1 \pmod 4$ and $\gamma \equiv I \pmod 2$ then ${}^t(N\gamma) + I \equiv 0 \pmod 2$ (since β, $\mathfrak{b}$, and N are all $\equiv I \pmod 2$, see §5.7). This shows that $\frac{1}{2}(I+\eta)$ preserves the lattice $\mathbb{Z}^{2n} \oplus \mathbb{Z}^{2n}$, hence $\mathcal{O}_d$ preserves L_Z.

Finally we check that $\kappa_Z : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is anti-linear. Let $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. By (7.2.1) and (7.2.2) the following diagram commutes:

(7.6.1)
$$\begin{tikzcd}
\mathbb{R}^{2n} \oplus \mathbb{R}^{2n} \arrow[r, "F_Z"'] \arrow[d, "{}^tN"'] & \mathbb{C}^{2n} \arrow[d] & M \arrow[d] \\
\mathbb{R}^{2n} \oplus \mathbb{R}^{2n} \arrow[r, "F_{\widehat{Z}}"'] \arrow[d, "{}^t\gamma"'] & \mathbb{C}^{2n} \arrow[d] & \mathfrak{b}\overline{M} \arrow[d, "\xi"] \\
\mathbb{R}^{2n} \oplus \mathbb{R}^{2n} \arrow[r, "F_{\gamma^{-1}\cdot\widehat{Z}}"'] & \mathbb{C}^{2n} & {}^t(CZ+D)\mathfrak{b}\overline{M}.
\end{tikzcd}$$

But $Z = \gamma^{-1} \cdot \widehat{Z}$ so the bottom arrow is also F_Z. Then κ_Z is the composition along the right-hand vertical column and it is given by $M \mapsto {}^t(CZ+D)\mathfrak{b}\overline{M}$ which is anti-linear. q.e.d.

The following proposition is an analog of the lemma ([29], [4]) of Comessatti and Silhol.

Proposition 7.7. *Suppose $A = (\mathbb{C}^{2n}/L, H = R + iQ)$ is a principally polarized abelian variety with antiholomorphic multiplication $\kappa : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ by $\mathcal{O}_d$. Then there exists a basis for the holomorphic 1-forms on A and there exists a symplectic basis for L such that the resulting period matrix is $\Omega = (Z, I)$ for some $Z \in \mathfrak{h}_{2n}^{\tau}$ which is fixed under the involution τ.*

7.8 Proof

Throughout this section, in order to simplify notation, but at the risk of some confusion with the usual multiplication, we will write $b \cdot v$ rather than $\Psi(b)v$, for any $b \in \mathcal{O}_d$ and $v \in \mathbb{C}^{2n}$. First consider the case $d \not\equiv 1 \pmod 4$. By Proposition 6.4 and by interchanging the u's and v's, there exist $u_1, \ldots, u_n, v_1, \ldots, v_n \in L$ so that the ordered collection

$$\{\omega \cdot u_1, u_1, \ldots, \omega \cdot u_n, u_n;\ v_1, \omega \cdot v_1, \ldots, v_n, \omega \cdot v_n\}$$

is a symplectic basis for L.

The space of holomorphic 1-forms on A is 2n-dimensional, so for each i $(1 \leq i \leq n)$ there exist a unique holomorphic 1-form η_i such that the following holds for all j (with $1 \leq j \leq n$):

$$\int_{v_j} \eta_i = 0 \quad \text{and} \quad \int_{\omega \cdot v_j} \eta_i = \delta_{ij}.$$

Set $\eta_i' = \overline{\kappa^* \eta_i} = \kappa^* \bar{\eta}_i$. Then the collection $\{\eta_1', \eta_1, \ldots, \eta_n', \eta_n\}$ is an ordered basis for the holomorphic 1-forms on A. Let us compute the period matrix with respect to these bases. Calculate that

$$\int_{\omega \cdot v_j} \eta_i' = \int_{\kappa v_j} \kappa^* \bar{\eta}_i = d \int_{v_j} \bar{\eta}_i = 0$$

$$\int_{v_j} \eta_i' = \int_{\kappa v_j} \bar{\eta}_i = \delta_{ij}.$$

It follows that the second "half" of the period matrix is the identity. Now let $z_{ij} = \int_{u_j} \eta_i$ and $w_{ij} = \int_{u_j} \eta_i'$ for $1 \leq i, j \leq n$. Then by a similar calculation, the first half of the period matrix consists of 2×2 blocks,

$$\mathbf{z}_{ij} = \begin{pmatrix} \int_{\omega \cdot u_j} \eta_i' & \int_{u_j} \eta_i' \\ \int_{\omega \cdot u_j} \eta_i & \int_{u_j} \eta_i \end{pmatrix} = \begin{pmatrix} d\bar{z}_{ij} & w_{ij} \\ \overline{w}_{ij} & z_{ij} \end{pmatrix}$$

which implies by (5.13.2) that $Z \in \mathfrak{h}_{2n}^{\tau}$.

Now consider the case $d = 4m + 1$. In this case Proposition 6.4 guarantees the existence of vectors $u_1, \ldots, u_n, v_1, \ldots, v_n \in L$ so that the ordered collection

(7.8.1) $\{(\omega - 1) \cdot u_1, u_1, \ldots, (\omega - 1) \cdot u_n, u_n;\ v_1, \omega \cdot v_1, \ldots, v_n, \omega \cdot v_n\}$

is a symplectic basis for L. For each i $(1 \leq i \leq n)$ there exist a unique holomorphic 1-form η_i such that $\int_{v_j} \eta_i = 0$ and $\int_{\omega \cdot v_j} \eta_i = \delta_{ij}$. Set $\eta_i' = \kappa^*(\bar{\eta}_i)$. Then

$$\int_{v_j} \eta_i' = \int_{\kappa v_j} \bar{\eta}_i = \int_{(2\omega - 1) \cdot v_j} \bar{\eta}_i = 2$$

$$\int_{\omega \cdot v_j} \eta_i' = \int_{\kappa \omega \cdot v_j} \bar{\eta}_i = \int_{\omega \cdot v_j} \bar{\eta}_i = 1$$

since $\sqrt{d}\omega = 2m+\omega$. So the second half of the period matrix is $\mathrm{Diag}^n(\sigma)$ where $\sigma = \left(\begin{smallmatrix} 2 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Set $z_{ij} = \int_{u_j} \eta_i$ and $w_{ij} = \int_{u_j} \eta_i'$. Then a simple calculation gives

$$\begin{pmatrix} \int_{(\omega-1)\cdot u_j} \eta_i' & \int_{u_j} \eta_i' \\ \int_{(\omega-1)\cdot u_j} \eta_i & \int_{u_j} \eta_i \end{pmatrix} = \begin{pmatrix} d\bar{z}_{ij} & w_{ij} \\ \overline{w}_{ij} & z_{ij} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}.$$

So the period matrix is $\left(Z'\mathrm{Diag}^n({}^t\sigma^{-1}), \mathrm{Diag}^n(\sigma)\right)$ where Z' consists of 2×2 blocks $\mathbf{z}_{ij} = \left(\begin{smallmatrix} d\bar{z}_{ij} & w_{ij} \\ \overline{w}_{ij} & z_{ij} \end{smallmatrix}\right)$. By (7.1.1), changing the basis

$$\{\eta_1', \eta_1, \ldots, \eta_n', \eta_n\}$$

by the action of $\mathrm{Diag}^n(\sigma^{-1})$ will give a period matrix (Z, I) where

$$Z = \mathrm{Diag}^n(\sigma^{-1}) Z' \mathrm{Diag}^n({}^t\sigma^{-1}).$$

By (5.13.3) the point $Z \in \mathfrak{h}_{2n}^{\tau}$ is fixed under τ as claimed. q.e.d.

8. A coarse moduli space for abelian varieties with anti-holomorphic multiplication

8.1 Level structures

Let $(A = \mathbb{C}^{2n}/L, H = R+iQ)$ be a principally polarized abelian variety. A level M structure on A is a choice of basis $\{U_1, \ldots, U_{2n}, V_1, \ldots, V_{2n}\}$ for the M-torsion points of A that is symplectic, in the sense that there exists a symplectic basis

$$\{u_1, \ldots, u_{2n}, v_1, \ldots, v_{2n}\}$$

for L such that

$$U_i \equiv \frac{u_i}{M} \text{ and } V_i \equiv \frac{v_i}{M} \pmod{L}$$

(for $1 \le i \le 2n$). For a given level M structure, such a choice

$$\{u_1, \ldots, u_{2n}, v_1, \ldots, v_{2n}\}$$

determines a mapping

$$F : \mathbb{R}^{2n} \oplus \mathbb{R}^{2n} \to \mathbb{C}^{2n} \tag{8.1.1}$$

such that $F(\mathbb{Z}^{2n} \oplus \mathbb{Z}^{2n}) = L$, by $F(e_i) = u_i$ and $F(f_i) = v_i$ where $\{e_1, \ldots, e_{2n}, f_1, \ldots, f_{2n}\}$ is the standard basis of $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$. The choice $\{u_1, \ldots, u_{2n}, v_1, \ldots, v_{2n}\}$ (or equivalently, the mapping F) will be referred to as a *lift* of the level M structure. It is well-defined modulo the principal congruence subgroup $\Gamma(M)$, that is, if $F' : \mathbb{R}^{2n} \oplus \mathbb{R}^{2n} \to \mathbb{C}^{2n}$ is another lift of the level structure, then $F' \circ F^{-1} \in \Gamma(M)$.

Suppose (A, H, κ) is a principally polarized abelian variety with anti-holomorphic multiplication by $\mathcal{O}_d$ as in §7.3. A level M structure $\{U_1, \ldots, U_{2n}, V_1, \ldots, V_{2n}\}$ on A is *compatible* with κ if for some (and hence for any) lift F of the level structure, the following diagram commutes $\pmod{L}$:

$$\begin{array}{ccc} \frac{1}{M}\left(\mathbb{Z}^{2n} \oplus \mathbb{Z}^{2n}\right) & \xrightarrow[F]{} & \frac{1}{M}L \\ {}^tN \big\downarrow & & \big\downarrow \kappa \\ \frac{1}{M}\left(\mathbb{Z}^{2n} \oplus \mathbb{Z}^{2n}\right) & \xrightarrow[F]{} & \frac{1}{M}L \end{array} \tag{8.1.2}$$

where N is the matrix (5.8.3).

We will refer to the collection

$$\mathcal{A} = \left(A = \mathbb{C}^{2n}/L, H = R + iQ, \kappa, \{U_i, V_j\}\right)$$

as a principally polarized abelian variety with anti-holomorphic multiplication and level M structure. If

$$\mathcal{A}' = \left(A' = \mathbb{C}^{2n}/L', H' = R + iQ, \kappa', \{U_i', V_j'\}\right)$$

is another such, then an isomorphism $\mathcal{A} \cong \mathcal{A}'$ is a complex linear mapping $\psi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ such that $\psi(L) = L'$, $\psi_*(H) = H'$, $\psi_*(\kappa) = \kappa'$, and such that for some (and hence for any) lift

$$\{u_1, \ldots, u_{2n}, v_1, \ldots, v_{2n}\}$$

and

$$\{u_1', \ldots, u_{2n}', v_1', \ldots, v_{2n}'\}$$

of the level structures,

$$\psi\left(\frac{u_i}{M}\right) \equiv \frac{u_i'}{M} \quad \text{and} \quad \psi\left(\frac{v_j}{M}\right) \equiv \frac{v_j'}{M} \pmod{L}.$$

Define $V(d, M)$ to be the set of isomorphism classes of principally polarized abelian varieties with anti-holomorphic multiplication by $\mathcal{O}_d$ and level M structure.

8.2

If $Z \in \mathfrak{h}_{2n}$, then for any $M \geq 1$ we define the *standard level M structure* on the abelian variety (A_Z, H_Z) to be the basis

$$\{F_Z(e_i/M), F_Z(f_j/M)\} \pmod{L}$$

where

$$\{e_1, \ldots, e_{2n}, f_1, \ldots, f_{2n}\}$$

is the standard basis of $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$.

Lemma 8.3. *Let $\gamma \in \mathbf{Sp}(4n, \mathbb{Z})$ and let $Z \in \mathfrak{h}_{2n}^{\tau\gamma}$, that is, $\widehat{Z} = \gamma \cdot Z$. Let $M \geq 3$. Then the standard level M structure on the abelian variety (A_Z, H_Z) is compatible with the anti-holomorphic multiplication κ_Z iff $\gamma \in \Gamma_M = \Gamma(M) \cap \widehat{\Gamma}(M)$.*

8.4 Proof

It follows immediately from diagram (7.6.1) that $\gamma \in \Gamma(M)$ iff the standard level M structure on (A_Z, H_Z) is compatible with κ_Z. Since $\Gamma(M)$ is torsion-free, $\widehat{\gamma}\gamma = I$ which implies $\gamma \in \widehat{\Gamma}(M)$; hence $\gamma \in \Gamma_M$. q.e.d.

By Lemma 8.3, each point $Z \in \mathfrak{h}_{2n}^{\Gamma(M)}$ determines a principally polarized abelian variety

$$\mathcal{A}_Z = (A_Z, H_Z, \kappa_Z, \{F_Z(e_i/M), F_Z(f_j/M)\})$$

with anti-holomorphic multiplication and (compatible) level M structure.

Theorem 8.5. *Fix $M \geq 3$. If $d \equiv 1 \pmod 4$, assume also that M is even. Then the association $Z \mapsto \mathcal{A}_Z$ determines a one to one correspondence between the real points* (2.3.2) *$X_{\mathbb{R}}$ of $X = \Gamma_M \backslash \mathfrak{h}_{2n}$ and the set $V(d, M)$ of isomorphism classes of principally polarized abelian varieties with anti-holomorphic multiplication by $\mathcal{O}_d$ and* (*compatible*) *level M structure.*

8.6 Proof

A point $x \in X$ is real iff it is the image of a Γ_M-real point $Z \in \mathfrak{h}_{2n}^{\tau\Gamma(M)}$. If two Γ_M-real points Z, Ω determine isomorphic varieties, say $\psi : \mathcal{A}_\Omega \cong \mathcal{A}_Z$ then by (7.2.1) there exists $h \in \mathbf{Sp}(4n, \mathbb{Z})$ such that $\Omega = h \cdot Z$. Since the isomorphism ψ preserves the level M structures, it follows also from (7.2.1) that $h \in \Gamma(M)$. We claim that $h \in \Gamma_M$. Let $\widehat{Z} = \gamma_Z \cdot Z$ and $\widehat{\Omega} = \gamma_\Omega \cdot \Omega$, with $\gamma_Z, \gamma_\Omega \in \Gamma_M$. Putting diagram (7.6.1) for Z together with the analogous diagram for Ω and diagram (7.2.1), and using the fact that $\psi_*(\kappa_\Omega) = \kappa_Z$ gives a diagram

$$\begin{array}{ccc} \mathbb{R}^{2n} \oplus \mathbb{R}^{2n} & \xrightarrow[F_\Omega]{} & \mathbb{C}^{2n} \\ {}^t(\gamma_\Omega^{-1}\widehat{h}\gamma_Z)\Big\downarrow & & \Big\downarrow \kappa_Z \psi \kappa_\Omega^{-1} = \psi \\ \mathbb{R}^{2n} \oplus \mathbb{R}^{2n} & \xrightarrow[F_Z]{} & \mathbb{C}^{2n} \end{array}$$

from which it follows that ${}^t(\gamma_\Omega \widehat{h} \gamma_Z) \in \Gamma(M)$, hence $\widehat{h} \in \Gamma(M)$, hence $h \in \Gamma_M$.

So it remains to show that every principally polarized abelian variety with anti-holomorphic multiplication and level M structure, $\mathcal{A} = (A, H, \kappa, \{U_i, V_j\})$ is isomorphic to some $\mathcal{A}_Z$. By the Comessatti lemma (Proposition 7.7) there exists $Z' \in \mathfrak{h}_{2n}$, such that $\widehat{Z'} = Z'$, and there exists an isomorphism

$$\psi' : (A_{Z'}, H_{Z'}, \kappa_{Z'}) \cong (A, H, \kappa)$$

between the principally polarized abelian varieties with anti-holomorphic multiplication. However the isomorphism ψ' must be modified because it does not necessarily take the standard level M structure on $(A_{Z'}, H_{Z'}, \kappa_{Z'})$ to the given level M structure on (A, H, κ).

Choose a lift $\{u_1, \ldots, u_{2n}, v_1, \ldots, v_{2n}\}$ of the level M structure and let $F : \mathbb{R}^{2n} \oplus \mathbb{R}^{2n} \to \mathbb{C}^{2n}$ be the corresponding mapping (8.1.1). Define

$$\begin{aligned} &(8.6.1) \qquad {}^t g^{-1} = F^{-1} \circ \psi' \circ F_{Z'} \in \mathbf{Sp}(4n, \mathbb{Z}) \\ &(8.6.2) \qquad Z = g \cdot Z' \\ &(8.6.3) \qquad \gamma = \widehat{g} g^{-1} = N^{-1} g N g^{-1}. \end{aligned}$$

As in §7.2, if $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ define $\xi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ by $\xi(w) = {}^t(CZ + D)w$. Define $\psi = \psi' \circ \xi$. We will show that $\gamma \in \Gamma_M$, that $\widehat{Z} = \gamma \cdot Z$, and that ψ induces an isomorphism $\psi : \mathcal{A}_Z \to \mathcal{A}$ of principally polarized abelian varieties with anti-holomorphic multiplication and compatible level M structures.

In the following diagram, F is the mapping (8.1.1) associated to the lift of the level M structure. The bottom square commutes by the definition of g, while the top square commutes by (7.2.1):

$$\begin{array}{ccc} \mathbb{R}^{2n} \oplus \mathbb{R}^{2n} & \xrightarrow{F_Z} & \mathbb{C}^{2n} \\ {\scriptstyle {}^t g}\downarrow & & \downarrow{\scriptstyle \xi} \\ \mathbb{R}^{2n} \oplus \mathbb{R}^{2n} & \xrightarrow[F_{Z'}]{} & \mathbb{C}^{2n}. \\ {\scriptstyle {}^t g^{-1}}\downarrow & & \downarrow{\scriptstyle \psi'} \\ \mathbb{R}^{2n} \oplus \mathbb{R}^{2n} & \xrightarrow{F} & \mathbb{C}^{2n}. \end{array} \tag{8.6.4}$$

First let us verify that $\xi : (A_Z, H_Z, \kappa_Z) \to (A_{Z'}, H_{Z'}, \kappa_{Z'})$ is an isomorphism of principally polarized varieties with anti-holomorphic multiplication by $\mathcal{O}_d$. It follows from (8.6.4) that $\xi_*(L_Z) = L_{Z'}$ and $\xi_*(H_Z) = H_{Z'}$. We claim that $\xi_*(\kappa_Z) = \kappa_{Z'}$, that is, $\kappa_{Z'} = \xi\kappa_Z\xi^{-1}$. But this follows from direct calculation using $\xi = F_{Z'}\,{}^t g F_Z$, $\kappa_Z = F_Z\,{}^t(N\gamma)F_Z^{-1}$, $\kappa_{Z'} = F_{Z'}\,{}^t N F_{Z'}$ and (8.6.3) (and it is equivalent to the statement that the pushforward by ${}^t g$ of the involution ${}^t(N\gamma)$ on $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$ is the involution ${}^t N$). It follows that

$$\psi_*(\kappa_Z) = \kappa. \tag{8.6.5}$$

We claim that the standard level M structure on (A_Z, H_Z) is compatible with κ_Z. By construction, the mapping ψ takes the standard level M structure on (A_Z, H_Z) to the given level M structure on (A, H). By assumption, the diagram (8.1.2) commutes $(\bmod L)$. By (8.6.4), $F = \psi \circ F_Z$. Using (8.6.5) it follows that the diagram

$$\begin{array}{ccc} \frac{1}{M}\left(\mathbb{Z}^{2n} \oplus \mathbb{Z}^{2n}\right) & \xrightarrow[F_Z]{} & \frac{1}{M}L_Z \\ {\scriptstyle {}^t N}\downarrow & & \downarrow{\scriptstyle \kappa_Z} \\ \frac{1}{M}\left(\mathbb{Z}^{2n} \oplus \mathbb{Z}^{2n}\right) & \xrightarrow[F_Z]{} & \frac{1}{M}L_Z \end{array}$$

commutes $(\bmod L_Z)$, which proves the claim. It also follows from Lemma 8.3 that $\gamma \in \Gamma_M$.

In summary, we have shown that

$$(A_Z, H_Z, \kappa_Z, \{F_Z(e_i/M), F_Z(f_j/M)\})$$

is a principally polarized abelian variety with anti-holomorphic multiplication and (compatible) level M structure, and that the isomorphism ψ preserves both the anti-holomorphic multiplication and the level structures. q.e.d.

9. Rational structure

9.1

Let $g \mapsto \hat{g} = NgN^{-1}$ be the involution on $\mathbf{Sp}(4n, \mathbb{R})$ where N denotes the matrix (5.8.3), as in §5.7. The resulting anti-holomorphic involution on $\mathfrak{h}_{2n}$ is given by $Z \mapsto \hat{Z} = \mathfrak{b}\overline{Z}\,{}^t\mathfrak{b}^{-1}$. As in §8, fix a level $M \geq 1$ and let $\Gamma = \Gamma_M = \Gamma(M) \cap \widehat{\Gamma}(M)$. It is well-known ([5] §V Thm. 2.5) that the arithmetic quotient $\Gamma(M)\backslash\mathfrak{h}_{2n}$ admits a model that is defined over a certain cyclotomic field. For our purposes, however, we need a model that is defined over a subfield of the real numbers, and we need it for the slightly different arithmetic quotient $X = \Gamma_M\backslash\mathfrak{h}_{2n}$. For these facts we will use results of [25].

Theorem 9.2. *There exists a projective embedding $X \to \mathbb{CP}^r$ and there exists an anti-holomorphic involution τ on $\mathbb{CP}^r$ such that:*

- *The closure $\overline{X}$ is the Baily-Borel Satake compactification of X.*
- *As a projective algebraic variety, $\overline{X}$ is defined over $\mathbb{Q}$.*
- *The involution τ is rationally defined and preserves $\overline{X}$.*
- *The restriction $\tau|X$ coincides with the involution (induced by) τ of §5.13.*

In summary, the set $X_{\mathbb{R}}$ described in Theorem 2.3 forms the set of real points of a complex quasi-projective algebraic variety defined over $\mathbb{Q}$. The proof will occupy the rest of this section.

Proposition 9.3. *The complex vector space of (holomorphic) Γ_M-modular forms on $\mathfrak{h}_{2n}$ is spanned by modular forms with rational Fourier coefficients.*

9.4 Proof

Let $\mathbf{G} = \mathbf{GSp}(4n)$, let $\mathbb{A}$ be the adeles of $\mathbb{Q}$ and let $S \subset \mathbf{G}(\mathbb{A})^+$ be an open subgroup containing $\mathbb{Q}^\times\mathbf{G}(\mathbb{R})^+$ (where $+$ denotes the identity component). Let $\Gamma_S = S \cap \mathbf{G}(\mathbb{Q})$. Suppose that $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is an arithmetic group which is contained in Γ_S and that:

(1) $S/\mathbb{Q}^\times\mathbf{G}(\mathbb{R})^+$ is compact.

(2) $\Gamma \cdot \mathbb{Q}^\times = \Gamma_S$.

(3) The set

$$\Delta = \left\{ \begin{pmatrix} I_{2n} & 0 \\ 0 & tI_{2n} \end{pmatrix} \in \mathbf{G}(\mathbb{A}) \;\middle|\; t \in \prod_p \mathbb{Z}_p^\times \right\}$$

is contained in S.

Then [25] Thm. 3 (ii) states that the complex vector space of Γ-modular forms with weight k on $\mathfrak{h}_{2n}$ is spanned by those forms whose Fourier coefficients are in the finite abelian extension k_S of $\mathbb{Q}$ that is determined by the set S. To apply this to our setting, let $S_1(M)$ be the collection of elements $x \in \mathbf{G}(\mathbb{A})^+$ such that each p-component $x_p \in \mathbf{GL}(2n, \mathbb{Z}_p)$ and satisfies

$$x_p \equiv \begin{pmatrix} I_{2n} & 0 \\ 0 & a_p I_{2n} \end{pmatrix} \pmod{M \cdot \mathbb{Z}_p}$$

for some $a_p \in \mathbb{Z}_p^\times$. Define $S(M) = S_1(M) \cdot \mathbb{Q}^\times$, let $S'(M) = NS(M)N^{-1}$, and $S = S(M) \cap S'(M)$. Then hypothesis (1) is satisfied. It is easy to see that $S(M), S'(M)$ both contain Δ, hence hypothesis (3) is satisfied for the set S. In this case, $k_S = \mathbb{Q}$ and

$$\Gamma_S = S \cap \mathbf{G}(\mathbb{Q}) = (\Gamma(M) \cap \widehat{\Gamma}(M)) \cdot \mathbb{Q}^\times = \Gamma_M \cdot \mathbb{Q}^\times$$

which verifies hypothesis (2). q.e.d.

9.5

Let $I_- = \begin{pmatrix} I_{2n} & 0 \\ 0 & -I_{2n} \end{pmatrix}$. Its action by fractional linear transformations maps the Siegel lower half space $\mathfrak{h}_{2n}^-$ to the upper half space $\mathfrak{h}_{2n}$, that is, $I_- \cdot Z = -Z$. Hence, for any holomorphic mapping $f : \mathfrak{h}_{2n} \to \mathbb{C}$ we may define $f' : \mathfrak{h}_{2n} \to \mathbb{C}$ by

$$f'(Z) = f(I_- \cdot N \cdot Z) = f(-\mathfrak{b} Z\, {}^t\mathfrak{b}^{-1}).$$

Proposition 9.6. *If $f : \mathfrak{h}_{2n} \to \mathbb{C}$ is a holomorphic Γ-modular form of weight k, with rational Fourier coefficients, then f' is also a holomorphic Γ-modular form of weight k, and*

(9.6.1) $$f(\widehat{Z}) = \overline{f'(Z)}$$

for all $Z \in \mathfrak{h}_{2n}$.

9.7 Proof

Suppose that $f(\gamma \cdot Z) = j(\gamma, Z)^k f(Z)$ for all $\gamma \in \Gamma$ and all $Z \in \mathfrak{h}_{2n}$ where

$$j\left(\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), Z\right) = \det(CZ + D)$$

is the standard automorphy factor. Then $j(I_-N, Z) = \det(-{}^t\mathfrak{b})$ is independent of Z. Let $\gamma \in \Gamma$ and set

$$\gamma' = I_-N\gamma N^{-1} I_-^{-1} \in \Gamma.$$

Then

$$\begin{aligned} f'(\gamma \cdot Z) &= f(\gamma' \cdot I_-N \cdot Z) \\ &= j(\gamma', I_-N \cdot Z)^k f'(Z) \\ &= \det(-{}^t\mathfrak{b})^k j(\gamma, Z)^k \det(-{}^t\mathfrak{b})^{-k} f'(Z) \\ &= j(\gamma, Z)^k f'(Z). \end{aligned}$$

which shows that f' is Γ-modular of weight k. Next, with respect to the standard maximal parabolic subgroup P_0 (which normalizes the standard 0-dimensional boundary component), the modular form f has a Fourier expansion,

$$f(Z) = \sum_s a_s \exp\left(2\pi i \langle s, Z\rangle\right)$$

which is a sum over lattice points $s \in L^*$ where $L = \Gamma \cap Z(\mathcal{U}_0)$ is the intersection of Γ with the center of the unipotent radical $\mathcal{U}_0$ of P_0 and where $a_s \in \mathbb{Q}$. Then

$$\begin{aligned} f(\hat{Z}) &= \sum_s a_s \exp\left(2\pi i \langle s, \mathfrak{b}\overline{Z}\,{}^t\mathfrak{b}^{-1}\rangle\right) \\ &= \overline{\sum_s a_s \exp\left(2\pi i \langle s, -\mathfrak{b}Z\,{}^t\mathfrak{b}^{-1}\rangle\right)} \\ &= \overline{f(-\mathfrak{b}Z\,{}^t\mathfrak{b}^{-1})} = \overline{f'(Z)}. \end{aligned}$$

q.e.d.

9.8

The Baily-Borel compactification $\overline{X}$ of X is the obtained by embedding X holomorphically into $\mathbb{CP}^m$ using $m+1$ (Γ-)modular forms (say $f_0, f_1, \ldots, f_m$) of some sufficiently high weight k, with rational Fourier coefficients, and then taking the closure of the image. Define an embedding $\Phi : X \to \mathbb{CP}^{2m+1}$ by

$$\Phi(Z) = \left(f_0(Z) : f_1(Z) : \cdots : f_m(Z) : f_0'(Z) : f_1'(Z) : \cdots : f_m'(Z)\right).$$

Denote these homogeneous coordinate functions by $x_j = f_j(Z)$ and $y_j = f_j'(Z)$. Define an involution $\sigma : \mathbb{CP}^{2m+1} \to \mathbb{CP}^{2m+1}$ by $\sigma(x_j) = \bar{y}_j$ and $\sigma(y_j) = \bar{x}_j$. Then Equation (9.6.1) says that this involution is compatible with the embedding Φ, that is, for all $Z \in X$ we have:

$$\sigma\Phi(Z) = \Phi(\widehat{Z}).$$

Define $\Psi : \mathbb{CP}^{2m+1} \to \mathbb{CP}^{2m+1}$ by setting $\xi_j = x_j + y_j$ and $\eta_j = i(x_j - y_j)$ for $0 \le j \le m$. Let $Y = \Psi\Phi(X)$ and let $\overline{Y}$ denote its closure.

Proposition 9.9. *The composition $\Psi\Phi : X \to \mathbb{CP}^{2m+1}$ is a holomorphic embedding which induces an isomorphism of complex algebraic varieties $\overline{X} \to \overline{Y}$. The variety $\overline{Y}$ is defined over the rational numbers, and the real points of Y are precisely the image of those points $Z \in X$ such that $\hat{Z} = Z$.*

9.10 Proof

The image $\Psi\Phi(X)$ is an algebraic subvariety of projective space that is preserved by complex conjugation, so it is defined over $\mathbb{R}$. The real points are obtained by setting $\bar{\xi}_j = \xi_j$ and $\bar{\eta}_j = \eta_j$ which gives $\bar{x}_j = y_j$ and $\bar{y}_j = x_j$ hence $\Phi(Z) = \sigma\Phi(Z)$, or $Z = \hat{Z}$. The Fourier coefficients of ξ_j and η_j are in $\mathbb{Q}[i]$ so the image $\Psi\Phi(X)$ is defined over $\mathbb{Q}[i]$. Since it is also invariant under $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, it follows that $\Psi\Phi(X)$ is defined over $\mathbb{Q}$. q.e.d.

9.11 Remark

The embedding $(f_0 : f_1 : \cdots : f_m) : X \hookrightarrow \mathbb{CP}^m$ determines the usual rational structure on $\overline{X}$, and the resulting complex conjugation is that induced by $Z \mapsto -\overline{Z}$ for $Z \in \mathfrak{h}_{2n}$.

10. The symmetric space for $\mathbf{Sp}(2n, \mathbb{C})$

10.1

In this section we sketch a proof of the well-known (but difficult to reference) fact that the complex symplectic group acts transitively on the quaternionic Siegel space by fractional linear transformations, and that the stabilizer of each point is a maximal compact subgroup. These facts are not needed in the rest of the paper, however they may help to make this symmetric space look a little more familiar.

10.2 A quaternion algebra

Associated to the imaginary quadratic field $\mathbb{Q}(\sqrt{d})$ we consider the quaternion algebra $\mathbb{H}$ over $\mathbb{R}$ that is generated by $\underline{\mathbf{1}}, \underline{\mathbf{i}}, \underline{\mathbf{j}}, \underline{\mathbf{k}}$ with $\underline{\mathbf{i}}^2 = \underline{\mathbf{k}}^2 = d\underline{\mathbf{1}}$, $\underline{\mathbf{j}}^2 = -\underline{\mathbf{1}}$ and $\underline{\mathbf{i}}\underline{\mathbf{j}} = \underline{\mathbf{k}}$. If $w = r\underline{\mathbf{1}} + s\underline{\mathbf{i}} + x\underline{\mathbf{j}} + y\underline{\mathbf{k}}$ set $\overline{w} = r\underline{\mathbf{1}} - s\underline{\mathbf{i}} - x\underline{\mathbf{j}} - y\underline{\mathbf{k}}$ and $w^* = r\underline{\mathbf{1}} + s\underline{\mathbf{i}} - x\underline{\mathbf{j}} + y\underline{\mathbf{k}}$. If we embed $\theta : \mathbb{C} \to \mathbb{H}$ by $\sqrt{d} \mapsto \underline{\mathbf{i}}$ then $\theta(\bar{z}) = \overline{\theta(z)}$ and we may write $\mathbb{H} = \theta(\mathbb{C}) \oplus \underline{\mathbf{k}}\theta(\mathbb{C})$. Define the purely quaternionic part of such an element $w \in \mathbb{H}$ to be

$$\mathrm{Qu}(w) = \underline{\mathbf{k}}^{-1}(x\underline{\mathbf{j}} + y\underline{\mathbf{k}}) = y + \frac{x}{d}\underline{\mathbf{i}}.$$

The mapping θ extends to an injective algebra homomorphism $\theta : M_{n\times n}(\mathbb{C}) \to M_{n\times n}(\mathbb{H})$ by applying θ to each matrix entry. For notational convenience we shall often omit the use of the symbol θ.

If $A, B \in \mathbf{GL}(n, \mathbb{H})$ then ${}^t(AB)^* = ({}^tB^*)({}^tA^*)$. If $\mathrm{Qu}(A) = 0$ then $A\underline{\mathbf{k}} = \underline{\mathbf{k}}\overline{A}$. An element $A \in M_{n\times n}(\mathbb{H})$ is Hermitian if $A = {}^t\overline{A}$. In this case $\langle z, z\rangle_A = {}^t\overline{z}Az$ is real, for all $z \in \mathbb{H}^n$. The element A is positive definite (written $A > 0$) if $\langle z, z\rangle_A > 0$ for all nonzero z. The unitary group $\mathbf{U}(n, \mathbb{H})$ (sometimes denoted $\mathbf{Sp}(n)$) consists of those $A \in M_{n\times n}(\mathbb{H})$ such that $A^{-1} = {}^t\overline{A}$.

10.3

Define the quaternionic Siegel space

$$Y_n = \left\{ W \in M_{n\times n}(\mathbb{H}) \mid W^* = {}^tW, \mathrm{Qu}(W) > 0 \right\}.$$

Proposition 10.4. *The symplectic group* $\mathbf{Sp}(2n, \mathbb{C})$ *acts transitively on the quaternionic Siegel space* Y_n *by fractional linear transformations: if* $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *then*

$$g \cdot W = (AW + B)(CW + D)^{-1}$$

where we have identified $\theta(A)$ with A, etc. Moreover,

$$\mathrm{Qu}(g\cdot W) = {}^t\overline{(CW+D)}^{-1}\mathrm{Qu}(W)(CW+D)^{-1}. \tag{10.4.1}$$

The stabilizer of the basepoint $W_0 = \frac{1}{\sqrt{-d}}\underline{\mathbf{k}}I_n \in Y_n$ is the unitary group $\mathbf{U}(n,\mathbb{H})$ over the quaternions, which is embedded in $\mathbf{Sp}(2n,\mathbb{C})$ by

$$A + \frac{1}{\sqrt{-d}}B\underline{\mathbf{k}} \mapsto \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}.$$

10.5 Proof

Equation (10.4.1) may be verified by a (tedious) direct computation. It follows, for any $g \in \mathbf{Sp}(2n,\mathbb{C})$, that $W \in Y_n$ iff $g\cdot W \in Y_n$. The remaining statements may be verified by direct computation. q.e.d.

10.6

Define the homomorphism $\mu : \mathbb{H} \to M_{2\times 2}(\mathbb{C})$ as follows:

	$d \not\equiv 1 \pmod 4$	$d = 4m+1$
$\mu(\underline{\mathbf{i}})$	$\left(\begin{smallmatrix} 0 & d \\ 1 & 0 \end{smallmatrix}\right)$	$\sigma^{-1}\left(\begin{smallmatrix} 0 & d \\ 1 & 0 \end{smallmatrix}\right)\sigma$
$\mu(\underline{\mathbf{j}})$	$\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$	$\sigma^{-1}\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}\sigma$
$\mu(\underline{\mathbf{k}})$	$\begin{pmatrix} 0 & -d\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$	$\sigma^{-1}\begin{pmatrix} 0 & -d\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}\sigma$

where $\sigma = \left(\begin{smallmatrix} 2 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. This mapping extends to a homomorphism

$$\mu : M_{r\times r}(\mathbb{H}) \to M_{2r\times 2r}(\mathbb{C})$$

which replaces each matrix entry with the 2×2 block defined above. The following fact is immediate:

Lemma 10.7. *The composition*

$$M_{r\times r}(\mathbb{C}) \xrightarrow{\ \theta\ } M_{r\times r}(\mathbb{H}) \xrightarrow{\ \mu\ } M_{2r\times 2r}(\mathbb{C})$$

takes values in $M_{2r\times 2r}(\mathbb{R})$ and it coincides with the mapping ψ_r of (5.2.1). In particular, it restricts to the injective homomorphism ψ_n : $\mathbf{Sp}(2n,\mathbb{C}) \to \mathbf{Sp}(\mathbb{R}^{4n}, S_2)$ of §5.7.

Proposition 10.8. *The mapping μ takes the quaternionic Siegel space $Y_n \subset M_{n\times n}(\mathbb{H})$ diffeomorphically to the symmetric space*

$$\mathfrak{h}_{2n}\mathfrak{v} \subset M_{2n\times 2n}(\mathbb{C})$$

for $\mathbf{Sp}(\mathbb{R}^{4n}, S_2)$. *Its image is the fixed point set of the involution* τ_h. *Moreover, for each* $g \in \mathbf{Sp}(2n, \mathbb{C})$ *and* $W \in Y_n$ *we have*

$$\mu(g \cdot W) = \psi_n(g) \cdot \mu(W). \tag{10.8.1}$$

10.9 Proof

Since μ and θ are algebra homomorphisms, for

$$g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathbf{Sp}(2n, \mathbb{C})$$

we find,

$$\begin{aligned}\mu(g\cdot W) &= \mu\left((\theta(A)W + \theta(B))\,(\theta(C)W+\theta(D))^{-1}\right)\\ &= (\mu\theta(A)\mu(W) + \mu\theta(B))(\mu\theta(C)\mu(W)+\mu\theta(D))^{-1}\\ &= \psi(g)\cdot\mu(W)\end{aligned}$$

which verifies (10.8.1). A direct calculation shows that μ takes the base point $W_0 = \frac{1}{\sqrt{-d}}\mathbf{k}I_n \in Y_n$ to the following base point $x_2 \in \mathfrak{h}_{2n}\mathfrak{v}$,

$$x_2 = \frac{i}{\sqrt{-d}}\mathrm{Diag}^n\begin{pmatrix} 0 & -d \\ 1 & 0\end{pmatrix} \quad \text{or}$$

$$x_2 = \frac{i}{\sqrt{-d}}\mathrm{Diag}^n\sigma^{-1}\begin{pmatrix} 0 & -d \\ 1 & 0\end{pmatrix}\sigma = \frac{i}{\sqrt{-d}}\mathrm{Diag}^n\begin{pmatrix} -1 & -2m \\ 2 & 1\end{pmatrix}$$

depending on whether $d \not\equiv 1 \pmod 4$ or $d = 4m+1$ respectively. It follows from (5.8.6) that μ takes Y_n to $\mathfrak{h}_{2n}\mathfrak{v}$ and it further follows from Proposition 5.14 that its image is precisely the fixed point set under τ_h.

q.e.d.

11. Concluding remarks

11.1

The definition of anti-holomorphic multiplication given in §7 extends in an obvious manner to more general CM fields. Let F be a totally real, degree m extension of $\mathbb{Q}$ and let $E = F[\sqrt{d}]$ be a totally imaginary quadratic extension of F (with $d \in \mathcal{O}_F$). Let $(A = \mathbb{C}^{2n}/L, H = R + iQ)$ be a principally polarized abelian variety as in §7. Then an anti-holomorphic multiplication by the ring of integers $\mathcal{O}_E$ is a homomorphism $\Psi : \mathcal{O}_E \to \mathrm{End}_{\mathbb{R}}(A)$ such that $\Psi(\mathcal{O}_F) \subset \mathrm{End}_{\mathbb{C}}(A)$, such that $\kappa = \Psi(\sqrt{d}) : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is anti-linear ($\kappa(ax) = \bar{a}\kappa(x)$ for all $a \in \mathbb{C}$ and $x \in \mathbb{C}^r$), and such that $Q(\Psi(b)x, y) = Q(x, \Psi(b)y)$ for all $b \in \mathcal{O}_E$ and $y \in \mathbb{C}^{2n}$. One might then expect (1) that the moduli space of principally polarized abelian varieties with anti-holomorphic multiplication by $\mathcal{O}_E$ and appropriate level structure may be identified with the locus of real points in a corresponding Hilbert-Siegel modular variety, and (2) that it consists of finitely many copies of $\Gamma\backslash D$ where $\Gamma \subset \mathbf{Sp}(2n, \mathcal{O}_e)$ is an appropriate level subgroup and where $D = Y_n \times \cdots \times Y_n$ is a product of m copies of the symmetric space $Y_n = \mathbf{Sp}(2n, \mathbb{C})/\mathbf{U}(n, \mathbb{H})$.

11.2

One might ask whether the closure of $X_{\mathbb{R}}$ in the Baily-Borel Satake compactification $\overline{X}$ coincides with the locus of real points $(\overline{X})_{\mathbb{R}}$ of the Baily-Borel compactification. Although we do not know the answer to this question, in [9] we were able to show, in the case $n = 1$ (that is, when $X_{\mathbb{R}}$ is an arithmetic quotient of real hyperbolic 3-space), that the difference $(\overline{X})_{\mathbb{R}} - \overline{X}_{\mathbb{R}}$ consists at most of finitely many points.

11.3

In [8] we consider a different rational structure on the Siegel modular variety $X = \Gamma\backslash\mathfrak{h}_n$ and a different anti-holomorphic involution τ', such that the resulting locus of real points (let us call it $X'_{\mathbb{R}}$) may be naturally identified with the moduli space of real abelian varieties (with appropriate level structure); and we show that this moduli space consists of finitely many copies of the locally symmetric space $\Lambda\backslash\mathbf{GL}(n, \mathbb{R})/\mathbf{O}(n)$ (for appropriate principal congruence subgroups Γ and Λ). The involution τ' arises from an involution on $\mathbf{Sp}(2n, \mathbb{R})$ whose fixed point set is $\mathbf{GL}(n, \mathbb{R})$. In this paper also, the key technical tool is the lemma of Comessatti and Silhol. Although the outline of [8] is parallel to that of the present paper, the technical details are completely different and we do not yet know how to formulate or prove the most natural general statement along these lines. Interesting related results are described in [2].

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