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CURVATURES OF EMBEDDED MINIMAL DISKS BLOW UP ON SUBSETS OF C^1 CURVES

BRIAN WHITE

Abstract

Using results of Colding-Minicozzi and an extension due to Meeks, we prove that a sequence of properly embedded minimal disks in a 3-ball must have a subsequence whose curvature blow-up set lies in a union of disjoint C^1 curves.

1. Introduction

Let D_n be a sequence of minimal disks that are properly embedded in an open subset U of $\mathbf{R}^3$ or more generally of a 3-dimensional Riemannian manifold. By passing to a subsequence, we may assume that there is a relatively closed subset K of U such that the curvatures of the D_n blow up at each point of K (i.e., such that for each $p \in K$, there are points $p_n \in D_n$ converging to p such that curvature of D_n at p_n tends to infinity as $n \to \infty$) and such that $D_n \setminus K$ converges smoothly on compact subsets of $U \setminus K$ to a minimal lamination L of $U \setminus K$. It is natural to ask what kinds of singular sets K and laminations L can arise in this way. In this paper, we prove:

Theorem 1. *Every point of K contains a neighborhood W such that $K \cap W$ is (after a rotation of $\mathbf{R}^3$) contained in the graph of a C^1 function from $\mathbf{R}$ to $\mathbf{R}^2$.*

This extends previous results of Colding-Minicozzi and of Meeks. In particular, if one replaces "C^1" by "Lipschitz" in Theorem 1, then the result is implicit in the work of Colding and Minicozzi. (See [CM04c, Section I.1] and [CM04c, Theorem 0.1] for a very similar result.) Thus if K is a curve, it must be a Lipschitz curve. Meeks later showed that if K is a Lipschitz curve, then it must be a $C^{1,1}$ curve [Mee04].

Meeks and Weber [MW07] showed that every $C^{1,1}$ curve arises as such a blow-up set K. Hoffman and White [HW11] showed that every closed subset of a line arises as such a blow-up set. (Kleene [Kle12] gave another proof of the Hoffman-White result. Special cases had been proved earlier by Colding-Minicozzi [CM04a], Brian Dean [Dea06], and Siddique Kahn [Kha09].)

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The following questions remain open:

1) Can C^1 in Theorem 1 be replaced by $C^{1,1}$? The Meeks-Weber examples show that one cannot prove more regularity than $C^{1,1}$.
2) If C^1 can be replaced by $C^{1,1}$, does every closed subset of a $C^{1,1}$ curve arise as the blow-up set K of some sequence D_n? If C^1 cannot be replaced by $C^{1,1}$, does every closed subset of a C^1 curve arise as such a K?

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2. Results

We begin with some definitions. For simplicity, we work in $\mathbf{R}^3$, although the results generalize easily to arbitrary smooth Riemannian 3-manifolds; see the remark at the end of the paper. A **configuration** is a triple (U, K, L) where U is an open ball in $\mathbf{R}^3$, an open halfspace in $\mathbf{R}^3$ or all of $\mathbf{R}^3$, where K is a relatively closed subset of U, and where L is a minimal lamination of $U \setminus K$. Here K should be thought of as a singular set: the configurations (U, K, L) we are most interested in arise as limits of smooth, properly embedded minimal surfaces, in which case K will be the set of points where the curvature blows up.

We define the **curvature** of a configuration (U, K, L) at a point $p \in L$ to be the norm of the second fundamental form at p of the leaf that contains p. We define the curvature of the configuration (U, K, L) to be ∞ at each point of K.

A plane P (i.e., a two-dimensional linear subspace of $\mathbf{R}^3$) is said to be **tangent** to (U, K, L) at a point p if and only if

1) $p \in L$ and P is the tangent plane at p to the leaf of the lamination that contains p, or
2) $p \in K$.

Thus each point in L has a unique tangent plane, whereas each point in K has (by definition) *every* plane as a tangent plane.

If (U, K, L) is a configuration, the **lift** of (U, K, L) is

$$\Phi(U, K, L) = \{(x, P) : x \in K \cup L \text{ and } P \text{ is a tangent plane to } (U, K, L) \text{ at } x\}.$$

Note that the lift is a relatively closed subset of the Grassmann bundle $U \times G$, where G is the set of all 2-dimensional linear subspaces of $\mathbf{R}^3$. Note also that a configuration is determined by its lift: if $\Phi(U, K, L) = \Phi(U, K', L')$ then $K = K$ and $L = L'$.

Theorem 2. *Let (U_n, K_n, L_n) be a sequence of configurations such that U_n converges to a nonempty open set U. Suppose also that the lifts $\Phi(U_n, K_n, L_n)$ converge in the Gromov-Hausdorff sense to a relatively* *closed subset V of $U \times G$. Then V is the lift $\Phi(U,K,L)$ of a configuration (U,K,L). Furthermore,*

1) *For each point $q \in K$, the curvatures of the (U_n,K_n,L_n) blow up at q, meaning that there is a sequence $q_n \in K_n \cup L_n$ such that q_n converges to q and such that the curvature of (U_n,K_n,L_n) at q_n tends to ∞ as $n \to \infty$.*
2) *For each compact subset C of $U \setminus K$, the curvatures of the (U_n,K_n,L_n) are uniformly bounded on C as $n \to \infty$.*
3) *The laminations L_n converge to the lamination L on compact subsets of U.*

Here (and throughout the paper) convergence of open sets U_n to open set U means convergence of $\mathbf{R}^3 \setminus U_n$ to $\mathbf{R}^3 \setminus U$ in the Gromov-Hausdorff topology. In particular, if U_n and U are balls, convergence of U_n to U means that the centers and radii of the U_n converge to the center and radius of U.

Proof. Let K be the set of points q in U such that

$$\{q\} \times G \subset V.$$

First we prove that (1) holds. For suppose it fails at a point $q \in K$. By passing to a subsequence, we may assume (for some ball W centered at q) that the curvatures of the (U_n,K_n,L_n) are uniformly bounded on W. In other words, W is disjoint from each K_n and the curvatures of the lamination $L_n \cap W$ are uniformly bounded. By replacing W by a smaller ball, we can then ensure that the tangent planes to L_n at any two points of $L_n \cap W$ make an angle of at most $\pi/20$ (for example) with each other. It follows that if (x,P) and (x',P') are points of V with $x, x' \in W$, then the angle between P and P' is at most $\pi/20$. But this contradicts the fact that $\{q\} \times G \subset V$, thus proving (1).

Next we prove that (2) holds. Suppose that $q \in U \setminus K$. Then

$$(*) \qquad \{P \in G : (q,P) \in V\}$$

is a closed subset of G but is not equal to G. Thus there is a closed set $\Sigma \subset G$ with nonempty interior such that Σ is disjoint from the set (*). In other words,

$$(\{q\} \times \Sigma) \cap V = \emptyset.$$

By the Gromov-Hausdorff convergence $\Phi(U_n,K_n,L_n) \to V$, it follows that there is an open ball W centered at q and compactly contained in U such that

$$(\overline{W} \times \Sigma) \cap \Phi(U_n,K_n,L_n) = \emptyset$$

for all sufficiently large n, say $n \geq N$. It follows immediately that

(i) $K_n \cap W = \emptyset$ for $n \geq N$, and
(ii) the Gauss map of $L_n \cap W$ omits Σ for $n \geq N$.

By a theorem of Osserman [Oss60], (i) and (ii) imply that the curvatures of the L_n are uniformly bounded (for $n \geq N$) on compact subsets of W. This together with (i) implies that the curvatures of the (U_n, K_n, L_n) are uniformly bounded on compact subsets of W. This proves (2).

It remains only to prove (3). Note that the curvature bounds in (2) imply that every subsequence of the L_n has a further subsequence that converges on compact subsets of $U \setminus K$ to a lamination L of $U \setminus K$. But clearly L is determined by V.[1] Thus the limit L is independent of the subsequence, which means that the original sequence L_n converges to L on compact subsets of $U \setminus K$. q.e.d.

We say that configurations (U_n, K_n, L_n) **converge** to configuration (U, K, L) provided U_n converges to U and $\Phi(U_n, K_n, L_n)$ converges in the Gromov-Hausdorff topology to $\Phi(U, K, L)$. From Theorem 2 together with compactness of the space of closed sets under Gromov-Hausdorff convergence, we deduce

Corollary 3 (Compactness of configurations)**.** *Suppose* (U_n, K_n, L_n) *is a sequence of configurations such that* U_n *converges to a nonempty open set* U*. Then a subsequence of the* (U_n, K_n, L_n) *converges to a configuration* (U, K, L)*.*

A **configuration of disks** is a configuration $(U, \emptyset, L)$ in which each leaf of L is a properly embedded minimal disk in U. We let $\mathcal{D}$ be the set of all configurations of disks. We let $\overline{\mathcal{D}}$ be the set of all configurations that are limits of configurations of disks. Note that $\overline{\mathcal{D}}$ is closed under sequential convergence.

Theorem 4. *Suppose that* $(U, K, L) \in \overline{\mathcal{D}}$*. Then* U *is covered by open balls* $\mathbf{B}$ *with the following properties:*

1) *For each point* $p \in K \cap \mathbf{B}$*, there is a leaf* L_p *of* $L \cap \mathbf{B}$ *such that* $L_p \cup \{p\}$ *is a minimal graph over a planar region and is properly embedded in* $\mathbf{B}$*.*
2) *If* $q_n \in K \cap \mathbf{B}$ *converges to* $q \in K \cap \mathbf{B}$*, then* $L_{q_n} \cup \{q_n\}$ *converges smoothly to* $L_q \cup \{q\}$*.*
3) *The singular set* $K \cap B$ *is contained in a* C^1 *embedded curve* Γ *such that at each point* q *of* $K \cap B$*, the curve* Γ *is orthogonal to* $L_q \cup \{q\}$ *at* q*.*

(See Remark 7 for the generalization to arbitrary Riemannian 3-manifolds.)

Proof. Assertion (1) is due to Colding and Minicozzi [CM04b, Theorem 5.8]. Assertion (2) follows immediately from Assertion (1). To prove Assertion (3), we use the following theorem due to Colding-Minicozzi and Meeks:

[1]In fact, $V \cap ((U \setminus K) \times G)$ is the lift of $(U \setminus K, \emptyset, L)$, so the latter may be recovered from the former using the projection map from $U \times G$ to U.

Theorem 5. *If $(\mathbf{R}^3, K, L) \in \overline{\mathcal{D}}$ and if K is nonempty, then K is a line and the lamination L is the foliation consisting of all planes perpendicular to L.*

(According to [CM04c, Theorem 0.1], L is a foliation consisting of parallel planes and K is a Lipschitz curve transverse to those planes. According to [Mee04], the Lipschitz curve must be a straight line perpendicular to those planes.)

We also use the following proposition, which is a restatement of the C^1 case of Whitney's Extension Theorem[Whi34, Theorem I]:

Proposition 6. *Let K be a relatively closed subset of an open subset $\mathbf{B}$ of $\mathbf{R}^n$. Suppose $\mathcal{V}$ is a continuous line field on K, i.e., a continuous function that assigns to each $p \in K$ a line $\mathcal{V}(p)$ in $\mathbf{R}^n$. Suppose also that if $p_i, q_i \in K$ with $p_i \neq q_i$ converge to $p \in K$, then $\overleftrightarrow{p_i q_i}$ converges to $\mathcal{V}(p)$.*

Then each point $p \in K$ has a neighborhood W such that $K \cap W$ is contained in the graph Γ of a C^1 function from $\mathcal{V}(p)$ to $(\mathcal{V}(p))^\perp$ such that at each point $q \in W \cap K$, $\mathcal{V}(q)$ is tangent to Γ at q.

We will apply Proposition 6 with $\mathcal{V}(p) = (\operatorname{Tan}_p L_p)^\perp$. By Assertion (2) of Theorem 4, $\mathcal{V}(p)$ depends continuously on $p \in K$. Let $p_j, q_j \in K \cap \mathbf{B}$ with $p_j \neq q_j$ converge to $q \in K \cap \mathbf{B}$. It suffices to prove that $\overleftrightarrow{p_j q_j}$ converges to $\mathcal{V}(q)$.

Let $\phi_n : \mathbf{R}^3 \to \mathbf{R}^3$ be translation by $-q_n$ followed by dilation by $1/|p_n - q_n|$:

$$\phi_n(x) = \frac{x - q_n}{|p_n - q_n|}.$$

By passing to a subsequence, we may assume that $\phi_n(p_n)$ converges to a point p^* with $|p^*| = 1$. Thus

$$\overleftrightarrow{p_n q_n} = \overleftrightarrow{\phi_n(p_n)\phi_n(q_n)} = \overleftrightarrow{\phi_n(p_n)O} \to \overleftrightarrow{p^* O}.$$

Thus it suffices to prove that $\overleftrightarrow{p^* 0}$ is equal to $\mathcal{V}(q)$.

Note that $\phi_n(U_n) \to \mathbf{R}^3$. Now consider the configurations

$$(\phi_n(U), \phi_n(K), \phi_n(L)).$$

By passing to a further subsequence, we may assume that these configurations converge to a configuration $(\mathbf{R}^3, K', L') \in \overline{\mathcal{D}}$. Note that K' is nonempty since 0 and p^* are in K'. Thus by Theorem 5, K' is a line and L' consists of all planes perpendicular to K'. Since K' contains 0 and p^*, in fact K' is the line through 0 and p^*.

Now by Assertion (2) of the theorem, the leaves $\phi_n(L_{q_n} \cup \{q_n\})$ converge smoothly to $\operatorname{Tan}_q L_q$. Thus $\operatorname{Tan}_q L_q$ is one of the leaves of L', which means that $\operatorname{Tan}_q L_q$ is perpendicular to K'. In other words, K' is the line $\mathcal{V}(q)$. q.e.d.

Remark 7. The definitions and theorems in this paper generalize to arbitrary smooth Riemannian 3-manifolds. In particular, Theorem 4 remains true if U is an open geodesic ball of radius r in a 3-dimensional Riemannian manifold, provided all the geodesic balls of radius $\leq r$ centered at points in U are mean convex. (This guarantees that if D is a minimal disk properly embedded in U, then the intersection of D with any geodesic ball in U is a union of disks.) The proof is almost identical to the proof in the Euclidean case.

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Mathematics Department
Stanford University
Stanford, CA 94305
E-mail address: white@math.stanford.edu