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THE HARMONIC FIELD OF A RIEMANNIAN MANIFOLD

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Abstract

Sullivan's construction of minimal models for topological spaces is refined for the case of a simply connected closed Riemannian manifold, $(M, <, >)$, to define a unique finitely generated field extension, $\mathbf{k}$ of $\mathbf{Q}$, baptized the *harmonic field* of $(M, <, >)$, and a morphism, $m : (\Lambda V, d) \to A_{DR}(M)$, from a Sullivan model defined over $\mathbf{k}$. The Sullivan model and the morphism are determined up to isomorphism, and the natural extension of $H(m)$ to $H(\Lambda V, d) \otimes_{\mathbf{k}} \mathbf{R}$ is an isomorphism; in particular, $(\Lambda V, d)$ is isomorphic to a rational Sullivan model for M tensored with $\mathbf{k}$. Examples are constructed to show that every finitely generated extension field of $\mathbf{Q}$ occurs as a harmonic field of such a Riemannian manifold.

1. Introduction

Let $A_{DR}(M)$ denote the commutative differential graded algebra of differential forms on a simply connected closed smooth manifold, M, equipped with the exterior derivative. Then a classical theorem of de Rham provides an isomorphism, $H(M; \mathbf{Q}) \otimes \mathbf{R} \xrightarrow{\cong} H(A_{DR}(M))$. We denote the image of $H(M; \mathbf{Q})$ by $H_{\mathbf{Q}}(A_{DR}(M))$. An open question is whether this data alone is sufficient to determine the rational homotopy type of M.

Recall that a *Sullivan algebra* defined over a ground field $\mathbf{k}$ of characteristic zero is, in particular, a commutative differential graded algebra, (C, d), in which $C^0 = \mathbf{k}$, C is a free graded commutative algebra ΛV, and in which d satisfies an additional "nilpotence" condition (which automatically holds if $C^1 = 0$). The Sullivan algebra is *minimal* if $Im d \subset C^+ \cdot C^+$. (For the theory of Sullivan algebras and models see [**2**].)

Using the simplicial set of singular simplices (or alternatively, where one exists, a triangulation), Sullivan [**4**] and [**2**], Sec.12, constructs for every simply connected space a minimal Sullivan algebra defined over $\mathbf{Q}$, which determines its rational homotopy type. Thus the question is

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whether the Sullivan algebra, (C_M, d), for M can be constructed just from $A_{DR}(M)$ together with the de Rham isomorphism.

What is known ([**4**] and [**2**], Sec.12), is that there is a a morphism $\mu : (C_M, d) \otimes \mathbf{R} \to A_{DR}(M)$ such that $H(\mu)$ is an isomorphism of the form $H_{\mathbf{Q}}(\mu) \otimes \mathbf{R} : H(C(M,d)) \otimes \mathbf{R} \xrightarrow{\cong} H(M;\mathbf{Q}) \otimes \mathbf{R}$. Moreover, in [**4**] Sullivan provides a specific procedure, using Hodge theory for a Riemannian metric on M, to construct a minimal Sullivan algebra, (E,d), defined over $\mathbf{R}$, and an isomorphism $(E.d) \xrightarrow{\cong} (C_M, d) \otimes \mathbf{R}$.

Our purpose here is to show that a refinement of this construction determines a unique finitely generated extension field $\mathbf{k}$ of $\mathbf{Q}$ (to be called the *harmonic field* of the Riemannian manifold), together with a Sullivan algebra, (D,d), defined over $\mathbf{k}$, and a morphism $m : (D,d) \to A_{DR}(M)$ inducing an isomorphism $H(D,d) \xrightarrow{\cong} H_{\mathbf{Q}}(A_{DR}(M)) \otimes \mathbf{k}$. The Sullivan algebra, (D,d), to be called the harmonic Sullivan model, is finite dimensional in each degree, but may not be minimal. However, it and the quasi-isomorphism m are determined up to isomorphism by the Riemannian metric. This contrasts with the classical theory of Sullivan models, where the quasi-isomorphisms are only determined up to homotopy.

This result is in some sense analogous to that of an algebraic manifold defined over a subfield $\mathbf{k}$ of $\mathbf{C}$, where the de Rham cohomology has the form $H \otimes_{\mathbf{k}} \mathbf{C}$ and H is computed via the Kahler differentials.

Our construction exhibits (D,d) as a tensor product $(\Lambda V, d) \otimes \Lambda(U, dU)$ in which $(\Lambda V, d)$ is a minimal Sullivan algebra and $d : U \xrightarrow{\cong} d(U)$. In particular (cf. Theorem 1), $(\Lambda V, d) \cong (C_M, d) \otimes \mathbf{k}$ and so is independent of the metric. On the other hand, the harmonic field (and the Sullivan model, (D,d)) are new invariants, depending only on the image of the metric in the moduli space of metrics modulo the obvious action by the group of diffeomorphisms.

In Sec. 4 we construct a manifold P which is a principal S^3 fibre bundle over $S^2 \times S^2$ and show that its harmonic field is $\mathbf{Q}$ for all Riemannian metrics. However (Theorem 3), the possible harmonic fields for $M = S^3 \times P$ are precisely all the extension fields of the form $\mathbf{Q}(\lambda)$, where λ is an arbitrary real number. It follows (Corollary to Theorem 3) that any field of the form $\mathbf{Q}(\lambda_1, \dots \lambda_r)$ is the harmonic field for a suitable metric on M^r. In particular, it follows that every finitely generated extension of $\mathbf{Q}$ occurs as the harmonic field of a rationally elliptic ([**2**], Sec. 32) manifold. Moreover, if M is a product of odd spheres, then the harmonic field is $\mathbf{Q}$ for all Riemannian metrics, and the harmonic field of a Riemannian symmetric space is $\mathbf{Q}$.

This article leaves open a number of questions. As the examples in Sec. 4 indicate, the harmonic field of a Riemannian manifold depends on a combination of the analytic nature of the Hodge decomposition and the structure of the minimal rational Sullivan model. It would be interesting to know more generally how knowledge of the rational model limits the possibilities for the harmonic fields. For example, what can be said if the manifold is formal in the sense of rational homotopy, or at least in the case it is Kahler? Is there a bound, depending only on the rational homotopy type of M, on the number of generators of a harmonic field? Then, even absent knowledge of the rational homotopy type, there are questions: Are there manifolds for which every finitely generated extension of $\mathbf{Q}$ can occur? Are there manifolds for which $\mathbf{Q}$ can never be a harmonic field? In another direction, do the eigenvalues of the Laplacian give information about the harmonic field?

Additionally, the results of this paper introduce new questions about homotopy periods. The minimal Sullivan model, (C_M, d), determines via Sullivan's theory a natural isomorphism from the vector space $C_M^+/C_M^+ \cdot C_M^+$ to the vector space $Hom(\pi_*(M), \mathbf{Q})$. The isomorphism $C_M^+ \otimes \mathbf{R}/C_M^+ \otimes \mathbf{R} \cdot C_M^+ \otimes \mathbf{R}$ with $Hom(\pi_*(M), \mathbf{R})$ is then automatically compatible with the underlying rational vector spaces. On the other hand, let $\mathbf{k}$ be the harmonic field for M. Then the harmonic Sullivan model for M is the tensor product of a minimal Sullivan algebra (B, d) and a contractible one, and there is an inclusion (B, d) into $(C_M, d) \otimes \mathbf{R}$ that induces an isomorphism $H(B, d) \cong H(C_M, d) \otimes \mathbf{k}$. This inclusion also induces an isomorphism

$$B^+/B^+ \cdot B^+ \otimes_{\mathbf{k}} \mathbf{R} \cong C_M^+ \otimes \mathbf{R}/C_M^+ \otimes \mathbf{R} \cdot C_M^+ \otimes \mathbf{R}.$$

For each p, the image of $B^+/B^+ \cdot B^+$ in degree p will determine a finitely generated extension field, $\mathbf{l}_p$ of $\mathbf{k}$, which may be thought of as the "harmonic homotopy periods" of M. The question of whether these can be non-trivial remains one for future investigation. This is again analogous to the situation described above of algebraic varieties defined over a subfield $\mathbf{k}$ of the complex numbers, whose cohomology with coefficients in $\mathbf{k}$ may be computed via Kahler differentials, but who may have homotopy periods not in $\mathbf{k}$.

The reader will note that this article does not provide an answer to the original question. There was some hope that Sullivan's idea of using a Riemannian metric would produce the rational model. As it turns out, that idea does come within a finitely generated field extension of succeeding, but does not in general produce the rational model. Since even the addition of a Riemannian metric to the data seems insufficient, it seems unlikely that the answer is positive. On the other hand, there does not seem to be an obvious strategy to establish this.

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2. The harmonic Sullivan model of a harmonic decomposition

Throughout this article, and without further reference, all fields will be assumed to have characteristic zero. In particular, in this section, $\mathbf{K}$ will denote a fixed such field, and unless specified otherwise all vector spaces will be defined over $\mathbf{K}$.

Recall that a graded vector space has *finite type* if it is finite dimensional in each degree, and that a commutative differential graded algebra (cdga), (A, d), defined over $\mathbf{K}$, is *1-connected* if A is concentrated in non-negative degrees, $H^0(A, d) = \mathbf{K}$, and $H^1(A, d) = 0$. A (minimal) Sullivan model for (A, d) is a quasi-isomorphism from a (minimal) Sullivan algebra, $(\Lambda V, d)$ to (A, d), and we sometimes abuse langauage and call $(\Lambda V, d)$ a Sullivan model for (A, d). The minimal Sullivan models for (A, d) are determined up to isomorphism.

Definition 1. Let (A, d) be a 1-connected cdga defined over $\mathbf{K}$. A *rational homology structure* for (A, d) is a graded algebra, $H_{\mathbf{Q}}(A, d)$, defined over $\mathbf{Q}$, together with an isomorphism $H_{\mathbf{Q}}(A, d) \otimes \mathbf{K} \xrightarrow{\cong} H(A, d)$. For any subfield, $\mathbf{k}$, of $\mathbf{K}$ we write $H_{\mathbf{k}}(A, d) = H_{\mathbf{Q}}(A, d) \otimes \mathbf{k}$. Then a ***k**-(minimal) Sullivan model* for (A, d) is a morphism $\mu : (C, d) \to (A, d)$ from a (minimal) Sullivan algebra defined over $\mathbf{k}$ such that $H(\mu) : H(C, d) \xrightarrow{\cong} H_{\mathbf{k}}(A, d)$.

The uniqueness of $\mathbf{k}$-minimal Sullivan models for (A, d) follows from [**1**] and Theorem 6.8 in [**3**]:

Theorem 1. *Let (A, d) be a 1-connected cdga with homology of finite type, and equipped with a rational homology structure. If (C, d) and (C', d) are **k**-minimal Sullivan models for (A, d), then the isomorphism $H(C, d) \cong H(C', d)$ determined by the homology isomorphisms with $H_{\mathbf{k}}(A, d)$ can be realized by an isomorphism $(C, d) \cong (C', d)$.*

Proof. Both $(C, d) \otimes_{\mathbf{k}} \mathbf{K}$ and $(C', d) \otimes_{\mathbf{k}} \mathbf{K}$ are minimal Sullivan models for (A, d), and so there is an isomorphism $\varphi : (C, d) \otimes_{\mathbf{k}} \mathbf{K} \xrightarrow{\cong} (C', d) \otimes_{\mathbf{k}} \mathbf{K}$ for which $H(\varphi)$ restricts to an isomorphism $H(C, d) \xrightarrow{\cong} H(C', d)$. Now [**1**] and Theorem 6.8 in [**3**] imply that $(C, d) \cong (C', d)$. q.e.d.

Recall next that if (C, d) is a simply connected minimal Sullivan algebra with homology of finite type, then C itself has finite type and $C^1 = 0$. Moreover, we may always write $C = \Lambda W$ with

$$W = W(0) \oplus W(1), \tag{1}$$

where $d(W(0)) = 0$, $d : W^p(1)) \to \Lambda W^{<p}$, and $H(d) : W^p(1) \xrightarrow{\cong} \ker[H^{p+1}(\Lambda W^{<p}, d) \to H^{p+1}(\Lambda W^{<p}, d)]$. In this case we say that $(\Lambda W, d)$ has *normal form*. Note that the integers dim $W^p(1)$ depend only on the isomorphism class of the Sullivan algebra.

Definition 2. Suppose $(\Lambda W, d)$ is a minimal Sullivan model in normal form for a 1-connected cdga (A, d) with homology of finite type. Then the *critical degrees* for (A, d) are the integers p for which $W^p(1) \neq 0$. The set of critical degrees will be denoted $crit(A, d)$.

In what follows we shall need to consider (possibly finite) increasing sequences of vector spaces and cdga's defined over an increasing sequence of subfields of $\mathbf{K}$. Thus we make

Definition 3. Let $\mathbf{l}_* = \{\mathbf{l}_p\}_{p\geq 2}$ be an increasing sequence of subfields of $\mathbf{K}$. An *$\mathbf{l}_*$-graded vector space (resp. an $\mathbf{l}_*$-cdga, an $\mathbf{l}_*$-Sullivan algebra)* is a sequence, S_p, of $\mathbf{l}_p$-graded vector spaces, cdga's, or Sullivan algebras, together with inclusions $S_p \otimes_{\mathbf{l}_p} \mathbf{l}_{p+1} \subset S_{p+1}$. A *morphism*, φ, in any of these categories is a sequence of morphisms φ_p such that φ_{p+1} extends $\varphi_p \otimes_{\mathbf{l}_p} \mathbf{l}_{p+1}$.

Finally, suppose S is a graded vector space over a subfield $\mathbf{l}$ of a field $\mathbf{K}$.

Definition 4. If $T \subset S \otimes_{\mathbf{l}} \mathbf{K}$ is an $\mathbf{l}$-subspace then the *extension subfield* for T is the minimum intermediate field $\mathbf{l} \subset \mathbf{L} \subset \mathbf{K}$ such that $T \subset S \otimes_{\mathbf{l}} \mathbf{L}$. The $\mathbf{L}$-subspace $\mathbf{L} \cdot T$ is the *extension subspace* of T.

Remark: Suppose $\{t_i\}$ is a basis of T and $\{s_j\}$ is a basis of S. Each $t_i = \sum_j \lambda_{ij} s_j$, and $\mathbf{L}$ is the extension field generated by the λ_{ij}. In particular, if dim T is finite then $\mathbf{L}$ is a finitely generated field extension of $\mathbf{l}$. In this case dim $T \geq dim_{\mathbf{L}}(\mathbf{L} \cdot T)$ and equality holds if and only if the obvious map $T \otimes_{\mathbf{l}} \mathbf{L} \to \mathbf{L} \cdot T$ is an isomorphism.

We are now ready to define harmonic decompositions, which generalize the classical Hodge decompositions to general 1-connected cdga's, and to construct the corresponding harmonic Sullivan models.

Definition 5. A *harmonic decomposition* of a 1-connected cdga, (A, d), is a vector space direct sum $A = H \oplus X \oplus Y$, in which $d : X \xrightarrow{\cong} Y$ and $d(H) = 0$. The projections on H, X, and Y will be denoted respectively by ρ_H, ρ_X, and ρ_Y.

For our construction, we fix a 1-connected cdga, (A, d), with homology of finite type, defined over $\mathbf{K}$, and equipped with a rational homology structure $H_{\mathbf{Q}}(A, d) \otimes \mathbf{K} \xrightarrow{\cong} H(A, d)$ and a harmonic decomposition $A = H \oplus X \oplus Y$. Identify the graded spaces $H(A, d)$ and H via the obvious isomorphism and identify any map into $H(A, d)$ with a map into H. Then for any subfield $\mathbf{k} \subset \mathbf{K}$ the rational homology structure determines a linear isomorphism $H_{\mathbf{k}}(A, d) \otimes_{\mathbf{k}} \mathbf{K} \xrightarrow{\cong} H$, and we denote the image of $H_{\mathbf{k}}(A, d)$ by $H_{\mathbf{k}}$. Finally, let $\mu : (\Lambda(W(0) \oplus W(1)), d) \to (A, d)$ be a minimal Sullivan model for (A, d) in normal form.

Now let q be the least positive integer in which (A, d) has non trivial homology. We shall construct by induction:

(i) an increasing sequence $\mathbf{k}_* = \{\mathbf{k}_p\}$ of finitely generated extension fields of $\mathbf{Q}$ such that $\mathbf{k}_p = \mathbf{Q}$ for $p \leq q$;
(ii) a $\mathbf{k}_*$-Sullivan algebra, $\{(D_p, d)\}$ with each D_p of finite type; and
(iii) $\mathbf{k}_p$-linear morphisms $m_p : (D_p, d) \to (A, d)$ compatible with the inclusions and such that

$$\text{Im } H^{\leq p+2}(m_p) \subset H_{\mathbf{k}_p}(A, d) \tag{2}$$

and

(3) $H^i(m_p)$ is an isomorphism for $i \leq p$ and is injective for $i = p+1$;

all with certain other properties as specified below. In particular, each (D_p, d) will have the form

$$(D_p, d) = (\Lambda(V_p(0) \oplus V_p(1)), d) \otimes_{\mathbf{k}_p} \Lambda(U_p, d(U_p)),$$

in which $V_p(0) \oplus V_p(1)$ is concentrated in degrees $\leq p$,

$$(B_p, d) = (\Lambda(V_p(0) \oplus V_p(1)), d)$$

is a minimal Sullivan algebra in normal form, and

$$d : U_p \xrightarrow{\cong} dU_p.$$

Thus the inclusion $(B_p, d) \subset (D_p, d)$ will be a quasi-isomorphism.

Then we shall set $\mathbf{k} = \cup_p \mathbf{k}_p$ and $m : (D, d) \to (A, d) = \cup_p m_p : (D_p, d) \to (A, d)$: these will be, respectively, the harmonic field and the harmonic Sullivan model of (A, d).

We proceed by constructing $\mathbf{k}_*$ and $\mathbf{k}_*$-graded vector spaces $\{\widehat{V^p_{p-1}}(0)\}$, $\{\widehat{V^p_{p-1}}(1)\}$, and $\{\widehat{U^{p-1}_{p-1}}\}$ and set

$$V_p(0) = \mathbf{k}_p \otimes ((V_{p-1}(0) \oplus \widehat{V^p_{p-1}}(0)), \qquad V_p(1) = \mathbf{k}_p \otimes ((V_{p-1}(1) \oplus \widehat{V^p_{p-1}}(1)),$$

and $U_p = \mathbf{k}_p \otimes ((U_{p-1} \oplus \widehat{U^{p-1}_{p-1}})$; here the tensor products are over $\mathbf{k}_{p-1}$.

To begin, set $\mathbf{k}_p = \mathbf{Q}$ if $p \leq q$, and set $D_{q-1} = \mathbf{Q}$. Set $V_{\leq q}(1) = 0$, $V_{<q}(0) = 0$, $V_q(0) = H^q_{\mathbf{Q}}$, and $U_{\leq q} = 0$. Thus $(D_q, d) = (\Lambda V^q_q(0), 0)$. Finally, define m_q to be the morphism extending the identity $V^q(0) = H^q_{\mathbf{Q}}$.

Now assume by induction that the constructions of the fields $\mathbf{k}_p$ and the morphisms $m_p : (D_p, d) \to (A, d)$ have been carried out for $p \leq r$, and that (2) and (3) hold for $p \leq r$.

For the inductive step, first observe from (2) and (3) that we may choose $\widehat{V^{r+1}_r}(0) \subset H^{r+1}_{\mathbf{k}_r}(A, d)$ so that

$$H^{r+1}_{\mathbf{k}_r}(A, d) = \text{Im } H^{r+1}(m_r) \oplus \widehat{V^{r+1}_r}(0). \tag{4}$$

Let $\alpha : \widehat{V^{r+1}_r}(0) \to H^{r+1}_{\mathbf{k}_r}$ be the corresponding inclusion.

Next, choose $d : \widehat{V_r^{r+1}}(1) \to B_r^{r+2}$ to be a $\mathbf{k}_r$-linear map such that $d \circ d = 0$ and

$$H(d) : \widehat{V_r^{r+1}}(1) \xrightarrow{\cong} \ker\, [H(m_r) : H^{r+2}(B_r, d) \to H^{r+2}_{\mathbf{k}_r}(A, d)]. \tag{5}$$

Let $\beta : \widehat{V_r^{r+1}}(1) \to X$ be the unique $\mathbf{k}_r$-linear map such that

$$d \circ \beta = m_r \circ d : \widehat{V_r^{r+1}(1)} \to A. \tag{6}$$

Let $(\Lambda W, d)$ be a minimal model for (A, d). Then it follows from (2) and (3) that $\mathbf{K} \otimes_{\mathbf{k}_r} (B_r, d) \cong (\Lambda W^{\leq r}, d)$. In particular,

$$\widehat{V_r^{r+1}}(1) = 0, \text{ if } r+1 \notin crit(A, d). \tag{7}$$

Finally, let $\widehat{U_r^r} \subset \rho_Y \circ m_r(D_r^{r+1})$ be a $\mathbf{k}_r$ subspace such that

$$\rho_Y \circ m_r(D_r^{r+1}) = m_r \circ d(D_r^r) \oplus d \circ \gamma(\widehat{U_r^r}), \tag{8}$$

where γ is the inclusion.

Set d to be 0 in $\widehat{V_r^{r+1}}(0)$. Then m_r, α, β, and γ define a $\mathbf{k}_r$-linear morphism:

$$\widehat{m_r} : (\widehat{D_r}, d) = D_r \otimes_{\mathbf{k}_r} \Lambda(\widehat{V_r^{r+1}}(0) \oplus \widehat{V_r^{r+1}}(1)) \otimes_{\mathbf{k}_r} \Lambda(\widehat{U_r^r}, d\widehat{U_r^r}) \to (A, d).$$

We now come to the key step in which the extension from $\mathbf{k}_r$ to $\mathbf{k}_{r+1}$ is defined. Regard $H(\widehat{m_r})$ as a $\mathbf{k}_r$-linear map into H. As described above its image, which is a $\mathbf{k}_r$-vectorspace in H, determines an extension field $\mathbf{k}_{r+1}$. When $r+1 \notin crit(A, d)$ we define $\mathbf{k}_{r+1}$ to be the extension field of

$$\text{Im } H^{\leq r+3}(\widehat{m_r}). \tag{9}$$

When $r+1 \in crit(A, d)$ we need a more technical definition: in this case we define $\mathbf{k}_{r+1}$ to be the extension field of

$$\text{Im } H^{\leq r+3}(\widehat{m_r}) \oplus \rho_H \circ m_r(D_r^{r+1}). \tag{10}$$

Finally, extend $\widehat{m_r}$ to the $\mathbf{k}_{r+1}$-linear morphism,

$$m_{r+1} : (D_{r+1}, d) = \mathbf{k}_{r+1} \otimes_{\mathbf{k}_r} (\widehat{D_r}, d) \to (A, d)$$

This completes the construction of m_{r+1} and (D_{r+1}, d).

By construction, $\text{Im } H^{\leq r+3}(m_{r+1}) = \mathbf{k}_{r+1} \cdot \text{Im } H^{\leq r+3}(\widehat{m_r})$, and so (2) follows. On the other hand,

$$H^{\leq r+2}(\widehat{m_r}) : H^{\leq r+2}(D_r \oplus \widehat{V_r^{r+1}}(1)) \oplus \widehat{V_r^{r+1}}(0) \to H_{\mathbf{k}_r},$$

and it follows from the choice of $\mathbf{k}_{r+1}$ that

$$H^{\leq r+2}(m_{r+1}) = \mathbf{k}_{r+1} \otimes_{\mathbf{k}_r} [H^{\leq r+2}(\widehat{m_r}) : H^{\leq r+2}(\widehat{D_r}) \to H_{\mathbf{k}_r}];$$

this establishes (3).

Finally, it is clear from the construction that D_{r+1} has finite type and so, given the Remark above, $\mathbf{k}_{r+1}$ is a finitely generated extension of $\mathbf{k}_r$.

This closes the induction. Note that

(11) $$m_p : V_p(1) \oplus U_p \to X \text{ and } m_p : V_p(0) \to H_{\mathbf{k}_p}.$$

q.e.d.

Definition 6. If $\mathbf{k}_*$ and $\{\widehat{V_p^{p+1}}(0)\}$, $\{\widehat{V_p^{p+1}}(1)\}$, and $\{\widehat{U_p^p}\}$ satisfy (4)-(10) then $\mathbf{k} = \cup_p \mathbf{k}_p$ and $m = \cup_p m_p : (D,d) = \cup_p(D_p,d) \to (A,d)$ will be called, respectively, a *harmonic field* and a *harmonic Sullivan model* for (A,d).

Given a harmonic Sullivan model as above we shall write

(12)
$$V(0) = \cup V_p(0), \quad V(1) = \cup V_p(1), \quad V = V(0) \oplus V(1), \quad \text{and} \quad U = \cup U_p(0).$$

Proposition 1. *Let $m : (D,d) \to (A,d)$ be a harmonic Sullivan model. Then*

(i) *The linear maps in (11) are injections.*
(ii) *$m : (B,d) = (\Lambda V, d) \to (A,d)$ is a $\boldsymbol{k}$-minimal Sullivan model in normal form for (A,d), and (D,d) is a mininimal Sullivan algebra if and only if $U = 0$.*
(iii) *D has finite type.*
(iv) *If $H(A,d)$ is finite dimensional, then $\boldsymbol{k}$ is a finitely generated extension of $\boldsymbol{Q}$.*

Proof. Assertions (i) and (iii) are immediate from the construction, and assertion (ii) follows from (2) and (3). Finally, if $H^{>n}(A,d) = 0$, then it follows from (9) and (10) that $\mathbf{k}_r = \mathbf{k}_n$ for $r \geq n$, and so $\mathbf{k} = \mathbf{k}_n$. This gives (iv). q.e.d.

Definition 7. The morphism $m : (B,d) \to (A,d)$ will be called the *minimal component* of a harmonic Sullivan model for (A,d).

Lemma 1. *Let $m : (D,d) \to (A,d)$ be a harmonic Sullivan model. If $p \in crit(A,d)$ then*

(i) *$\rho_Y \circ m_p(D_p^p) = m_p \circ d(D_p^{p-1})$.*
(ii) *If $\Phi \in D_p^p$ and $m_p(\Phi)$ is a cycle, then there is a cycle $\Psi \in D_p^p$ such that $m_p(\Phi) = m_p(\Psi)$.*

Proof. (i) Evidently, $m_p \circ d(D_p^{p-1}) \subset \rho_Y \circ m_p(D_p^p)$. On the other hand, $D_p^p = \mathbf{k}_p \otimes_{\mathbf{k}_{p-1}} D_{p-1}^p \oplus V_p^p \oplus d(U_p^{p-1})$, and ρ_Y vanishes on $m_p(V_p)$. Thus (i) follows from (8) with $p = r+1$.

(ii) Write $m_p(\Phi) = \Phi_{p-1} + \Phi_0 + \Phi_1 + \Phi_2$ with the summands respectively in $D_{p-1}^p, V^p(0), V^p(1)$, and dU^{p-1}. Then $0 = \rho_Y \circ d \circ m_p(\Phi) = \rho_Y(d(\Phi_{p-1} + \Phi_1))$. In view of (11) and (5) it follows that $\Phi_1 = 0$, while trivially $\rho_H(\Phi_2) = 0$. Thus $\rho_H \circ m_p(\Phi) = \rho_H(\Phi_{p-1} + \Phi_0) \in H_{\mathbf{k}_p}$, by (10). Now by (2) and (3) there is a cycle $\Psi_0 \in D_p$ and an element $x \in X$ such that $m_p(\Phi) = \rho_H \circ m_p(\Psi_0) + dx$. Thus $dx \in \rho_Y \circ m_p(D_p^p)$, and it follows from (i) that $dx = m_p(d(\Psi_1))$. Set $\Psi = \Psi_0 \oplus d(\Psi_1)$. q.e.d.

The uniqueness of harmonic fields and harmonic Sullivan models is established in

Theorem 2. *Suppose* $m : (D, d) \to (A, d)$ *and* $m' : (D', d) \to (A, d)$ *are harmonic Sullivan models with harmonic fields* $\boldsymbol{k}$ *and* $\boldsymbol{k}'$. *Then* $\boldsymbol{k} = \boldsymbol{k}'$, *and there is an isomorphism* $\varphi : (D', d) \xrightarrow{\cong} (D, d)$ *such that* $m \circ \varphi = m'$. *In particular,* $(B', d) \cong (B, d)$.

Proof. We shall show that $\{\mathbf{k}'_r\} = \{\mathbf{k}_r\}$ and construct an isomorphism $\{\varphi_r\} : \{(D'_r, d)\} \xrightarrow{\cong} \{(D_r, d)\}$ such that $m_r \circ \varphi_r = m'_r$, both by induction. To begin, when $r = q$ we simply set $\varphi_q = id$.

Now assume for some r that for $p \leq r$, $\mathbf{k}_p = \mathbf{k}'_p$ and the φ_p have been constructed. Then Im $m_r =$ Im m'_r and it follows from (8) that both $\widehat{(U^r_r)'}$ and $\widehat{U^r_r}$ are complementary $\mathbf{k}_r$-subspaces of $m_r \circ d(D^r_r)$ in $\rho_Y \circ m_r(D^{r+1}_r)$. Thus there are bases, $\{u'_i\}$ of $\widehat{(U^r_r)'}$ and $\{u_i\}$ of $\widehat{U^r_r}$, and elements $\Phi_i \in D^r_r$ such that $d \circ \widehat{m'_r}(u'_i) = d \circ \widehat{m_r}(u_i) + d \circ m_r(\Phi_i)$. Extend φ_r to the isomorphism

$$\psi_U : (D'_r, d) \otimes \Lambda(\widehat{(U^r_r)'}, d\widehat{(U^r_r)'}) \xrightarrow{\cong} (D_r, d) \otimes \Lambda(\widehat{U_r}, d\widehat{U_r})$$

that sends u'_i to $u_i + \Phi_i$. (Here the tensor products are over $\mathbf{k}_r$.) Evidently the restrictions of $\widehat{m_r}$ and $\widehat{m'_r}$ satisfy $\widehat{m_r} \circ \psi_U = \widehat{m'_r}$.

Similarly we have Im $H(m_r) =$ Im $H(m'_r)$, and it follows from (4) that there are bases $\{v'_i\}$ of $\widehat{(V^{r+1}_{r+1})'}(0)$ and $\{v_i\}$ of $\widehat{V^{r+1}_{r+1}}(0)$ and cycles $\Phi_i \in A$ representing classes in Im $H^{r+1}(m_r)$ such that

$$\widehat{m'_r}(v'_i) = \widehat{m_r}(v_i) + \Phi_i. \tag{13}$$

Now write $\Phi_i = m_r(\Psi_i) + dx_i$ with Ψ_i a cycle in D^{r+1}_r and $x_i \in X$. Then (10) gives $\rho_Y(m_r(\Psi_i + dx_i)) = 0$. Thus $dx_i \in \rho_Y(m_r(\Psi_i))$, and it follows from (8) that $dx_i = d(\widehat{m_r}(\Omega_i)$ for some $\Omega_i \in D^r_r \oplus \widehat{U^r_r}$. Extend ψ_U to the isomorphism

$$\begin{aligned}\psi_0 : (D'_r, d) \otimes \Lambda(\widehat{(V^{r+1}_r)'}, 0) \otimes \Lambda(\widehat{(U^r_r)'}, d\widehat{(U^r_r)'}) \\ \xrightarrow{\cong} (D_r, d) \otimes \Lambda(\widehat{V^{r+1}_r}, 0) \otimes \Lambda(\widehat{U_r}, d\widehat{U_r})\end{aligned}$$

that sends v'_i to $v_i + \Psi_i + \Omega_i$. (Here again the tensor products are over $\mathbf{k}_r$.) As above we have that the restrictions of $\widehat{m_r}$ and $\widehat{m'_r}$ satisfy $\widehat{m_r} \circ \psi_0 = \widehat{m'_r}$.

If $r + 1 \notin crit(A, d)$, then $\psi_0 : (\widehat{D'_{r+1}}, d) \xrightarrow{\cong} (\widehat{D_{r+1}}, d)$ and it follows from (9) that $\mathbf{k}'_{r+1} = \mathbf{k}_{r+1}$. In this case we set $\varphi_{r+1} = \mathbf{k}_{r+1} \otimes_{\mathbf{k}_r} \psi_0$. Now suppose $r+1 \in crit(A, d)$. Use (6) to conclude that there are bases $\{w'_i\}$

of $\widehat{(V_{r+1}^{r+1})'}(1)$ and $\{w_i\}$ of $\widehat{V_{r+1}^{r+1}}(1)$ and elements $\Phi_i \in D_r^{r+1}$ such that

$$\varphi_r(dw_i^{'}) = d(w_i + \Phi_i). \tag{14}$$

It follows that $\widehat{m_r^{'}}(w_i^{'}) - m_{r+1}(w_i + \Phi_i)$ is a cycle and so $\widehat{m_r^{'}}(w_i^{'}) - m_{r+1}(w_i + \Phi_i) = \rho_H(m_{r+1}(\Phi_i)) + dx_i$. Now it follows from (10) that $\rho_H(m(\Phi_i)) \in H_{\mathbf{k}_{r+1}}$. It then follows from (4) that $\rho_H(m(\Phi_i)) = m_{r+1}(\Psi_i)+dt_i$, where Ψ_i is a cycle in $(\mathbf{k}_{r+1}\otimes_{\mathbf{k}_r} D_r^{r+1})\oplus V_{r+1}^{r+1}(0)$. Finally, $dt_i + dx_i = \rho_Y(\widehat{m_r^{'}}(w_i^{'}) - m_{r+1}(w_i + \Phi_i - \Psi_i) = \rho_Y \circ m_{r+1}(\Phi_i) - \Psi_i) \in m_{r+1}(D_{r+1}^{r+1})$. Now Lemma 1(i) asserts that $dt_i + dx_i = m_{r+1} \circ d(\Omega_i)$, where $\Omega_i \in D_{r+1}^r$. Thus $\widehat{m_r^{'}}(w_i^{'}) = m_{r+1}(w_i + \Phi_i - \Psi_i - d(\Omega_i))$.

But $D_{r+1}^r = D_r^r \oplus U_r^r$, so that $\Phi_i - \Psi_i - d(\Omega_i)) \in (D_r, d)\otimes\Lambda(\widehat{V_r^{r+1}}, 0)\otimes\Lambda(\widehat{U_r}, d\widehat{U_r})$. Thus ψ_0 extends to the isomorphism, $\psi_1 : \mathbf{k}_{r+1}\otimes_{\mathbf{k}_r} \widehat{D_{r+1}^{'}} \xrightarrow{\cong} D_{r+1}$ that sends $w_i^{'}$ to $w_i + \Phi_i - \Psi_i - d(\Omega_i)$.

By construction, $m_{r+1}\circ\psi_i = \widehat{m_{r'}} : \widehat{D_{r+1}^{'}} \to A$. Thus $\operatorname{Im} H^{\le r+3}(\widehat{m_{r'}}) \subset ImH^{\le r+3}(m_{r+1}) \subset H_{\mathbf{k}_{r+1}}^{\le r+3}(A, d)$. Moreover, by our induction hypothesis, $m_r^{'}((D_r^{r+1})^{'}) = m_r(D_r^{r+1})$. Thus it follows from (10) that $\rho_H(m_r^{'}((D^{'})_r^{r+1}) \subset H_{\mathbf{k}_{r+1}}(A, d)$ as well. Thus (10) (applied to $(D^{'}, d)$) yields $\mathbf{k}_{r+1}^{'} \subset \mathbf{k}_{r+1}$. Reversing the roles of $D^{'}$ and D yields the reverse inclusion and establishes that $\mathbf{k}_{r+1}^{'} \subset \mathbf{k}_{r+1}$.

We may therefore set $\varphi_{r+1} = \psi_1$, thereby closing the induction. q.e.d.

There is a well-defined notion of homotopy between morphisms from a Sullivan algebra, [**4**] and [**2**].

Corollary 1. If $(B^{'}, d)$ and (B, d) are the minimal components of harmonic Sullivan models for $(A^{'}, d)$ and (A, d) then φ induces an isomorphism $\varphi_B : (B^{'}, d) \xrightarrow{\cong} (B, d)$ such that $m \circ \varphi_B \sim m^{'} : (B^{'}, d) \to (A, d)$.

Remark: If (A, d) has a classical Sullivan model of the form $\mu : (C, d)\otimes\mathbf{K} \to (A, d)$ in which C is defined over $\mathbf{Q}$ and $H(\mu) : H(C, d) \xrightarrow{\cong} H_{\mathbf{Q}}(A, d)$, then it follows from Theorem 1 that there is an isomorphism $\xi : (C, d)\otimes\mathbf{k} \xrightarrow{\cong} (B, d)$. It is far from clear, however, that ξ can be chosen so that $m \circ \xi \sim \mu$.

Now suppose $(A(1), d)$ and $(A(2), d)$ are 1-connected cdga's with homology of finite type and equipped with rational homology structures and harmonic decompositions. Assume further that there is a quasi-isomorphism $\varphi : (A(1), d) \to (A(2), d)$ that maps $H_{\mathbf{Q}}(1), X(1)$, and $Y(1)$, respectively, to $H_{\mathbf{Q}}(2), X(2)$, and $Y(2)$.

Proposition 2. *With the hypotheses and notation above,* $(A(1), d)$ *and* $(A(2), d)$ *have the same harmonic fields, and their harmonic Sullivan models* $(D(1), d)$ *and* $(D(2), d)$ *are isomorphic.*

Proof. Because φ is a quasi-isomorphisn it restricts to an isomorphism from $H(1)$ to $H(2)$. Write $X(1) = X^{'}(1) \oplus \ker\, \varphi \cap X(1)$. Then

$$Y(1) = d(X^{'}(1)) \oplus \ker\, \varphi \cap Y(1). \tag{15}$$

In fact, it is obvious that $Y(1) \supset d(X^{'}(1)) \oplus \ker\, \varphi \cap Y(1)$. On the other hand, if $dx \in Y(1)$ then $\varphi(dx) = \varphi dx_1$ for some $x_1 \in X^{'}(1)$. Thus $\varphi(x - x_1)$ is a cycle in $A(2)$. Since φ preserves the harmonic decompositions, $\varphi(x - x_1) = 0$ and (15) is established.

Now extend φ to a surjective quasi-isomorphism, $\psi : (A(1), d) \otimes \Lambda(T, dT) \to (A(2), d)$, where $d : T \overset{\cong}{\to} dT$. Then $H(\ker\, \psi) = 0$, and using (15) we may extend $X(1)$ and $Y(1)$ to a harmonic decomposition $H(1) \oplus X \oplus Y$ of $(A(1), d) \otimes \Lambda(T, dT)$. Let $\mathbf{k}(1)$ be the harmonic field for $(A(1), d)$ and let $m(1) : (D(1), d) \to (A(1), d)$ be a harmonic Sullivan model. Regard $m(1)$ as a morphism into $(A(1), d) \otimes \Lambda(T, dT)$, and note that (4)–(10) are obviously satisfied. Thus $\mathbf{k}(1)$ is the harmonic field for $(A(1), d) \otimes \Lambda(T, dT)$ and $(D(1), d)$ is its harmonic Sullivan model.

Next, let $\mathbf{k}(2)$ be the harmonic field for $(A(2), d)$ and let $m(2) : (D(2), d) \to (A(2), d)$ be a harmonic Sullivan model. We shall construct a harmonic Sullivan model $m : (D(2), d) \to (A(1), d) \otimes \Lambda(T, dT)$ with harmonic field $\mathbf{k}(2)$ and such that $\psi \circ m = m(2)$. The proposition will then follow from Theorem 2.

As earlier, let q be the first positive integer such that $H^q(A(2), d) \neq 0$, and let $m : V_q^q(0) \to H(1)$ be the isomorphism such that $\varphi \circ m = m(2) : D_q(2) \to A(2)$. Suppose next that $m : (D_r(2) \to (A(1), d) \otimes \Lambda(T, dT)$ has been constructed. It is straightforward to verify that the linear maps $\alpha, \beta,$ and γ constructed earlier lift appropriately to define m in $D_{r+1}(2)$. q.e.d.

3. The harmonic Sullivan model of a Riemannian manifold

Let M be a simply connected closed Riemannian manifold. The metric determines an inner product on $A_{DR}(M)$ and a harmonic decompoaition of mutually orthogonal subspaces, $A_{DR}(M) = H \oplus X \oplus Y$, in which H and X consist, respectively, of the harmonic and coclosed forms, while the de Rham theorem endows $A_{DR}(M)$ with a rational homology structure. Note that while the sign of the adjoint of d depends on the choice of an orientation, the harmonic decomposition does not.

Definition 8. The harmonic field $\mathbf{k}$, the harmonic Sullivan model, $m : (D, d) \to A_{DR}(M)$ and the minimal component (B, d) of the harmonic Sullivan model, all as determined by the data above, will be called respectively the *harmonic field* and the *harmonic Sullivan model* and the *minimal component* of the harmonic Sullivan model of the Riemannian manifold.

From Theorem 1 and Proposition 1 we have:

Proposition 3. *Let M be a simply connected closed Riemannian manifold with harmonic field $\boldsymbol{k}$, minimal component (B, d) of a harmonic Sullivan model, and rational minimal Sullivan model (C_M, d). Then*

(i) $\boldsymbol{k} \otimes (C_M, d) \cong (B, d)$ *and so, given $\boldsymbol{k}$, is otherwise independent of the choice of metric.*
(ii) *The Sullivan field, $\boldsymbol{k}$, is a finitely generated field extension of $\boldsymbol{Q}$.*

Proposition 4. (i) *The harmonic field of the Riemannian product of two closed simply connected Riemannian manifolds M_1 and M_2 is the field generated by the harmonic fields of the two factors, and the harmonic Sullivan model of the product is the tensor product of the harmonic Sulivan models of the factors.*
(ii) *The harmonic field of a product of odd spheres with any Riemannian metric is $\boldsymbol{Q}$.*
(iii) *The harmonic field of a Riemannian symmetric space is $\boldsymbol{Q}$.*

Proof. (i) Consider the inclusion of $A_{DR}(M) \otimes A_{DR}(N)$ in $A_{DR}(M \times N)$, and let H_i, X_i, Y_i denote, respectively, the spaces of harmonic forms, coclosed forms, and closed forms on M_i. Then $H_1 \otimes H_2$ is the space of harmonic forms on the product, while $X_1 \otimes H_2$ and $H_1 \otimes X_2$ are coclosed forms on the product and $Y_1 \otimes H_2$ and $H_1 \otimes Y_2$ are closed forms on the product. Thus the tensor product of the harmonic Sullivan models satisfies the defining conditions for the harmonic Sullivan model of the product, and a straightforward step by step check establishes the assertion about the harmonic fields.

(ii) In this case there are no critical degrees. Thus $V(1) = 0$, and we may choose m to map $V(0)$ into $H_{\mathbf{Q}}$. It follows that $H(m)$ maps $H(D, d)$ into $H_{\mathbf{Q}}(A_{DR(M)})$, and so $\mathbf{k} = \mathbf{Q}$.

(iii) In this case $H_{\mathbf{Q}}$ is a subalgebra of $A_{DR}(M)$ and the inclusion $\mathbf{R} \otimes H_{\mathbf{Q}}$ in $A_{DR}(M)$ is a quasi-isomorphism. Thus by Proposition 2 a harmonic Sullivan model for $H_{\mathbf{Q}}$ (with $X = Y = 0$) is a harmonic Sullivan model for M. q.e.d.

4. Examples of harmonic fields

We construct a manifold P that is a principal S^3 fibre bundle over $S^2 \times S^2$, and establish

Theorem 3. *For any Riemannian metric on P, the harmonic field is $\boldsymbol{Q}$. However, the possible harmonic fields for $M = S^3 \times P$ are all the fields of the form $\boldsymbol{Q}(\lambda)$, where λ is any real number.*

From Proposition 4(i) we have then the

Corollary 1. Any field of the form $\mathbf{Q}(\lambda_1, \dots \lambda_r)$ is the harmonic field for some Riemannian metric on M^r.

Proof of theorem 3. We adopt the convention that for principal bundles the structure group acts on the right, and consider the classical principal S^1-fibre bundles $\rho_k : S^{2k+1} \to \mathbf{C}P^k$. In particular the S^1 action on S^7 extends to an action by S^3 and division by S^3 gives the Hopf fibre bundle $\zeta : S^7 \to S^4$. Thus ζ factors over ρ_3 to define a fibre bundle $\rho : \mathbf{C}P^3 \to S^4$ with fibre S^2.

Let $\psi : S^2 \times S^2 \to \mathbf{C}P^3$ be a map which pulls back the fundamental cohomology class α in $H^2(\mathbf{C}P^3; \mathbf{Z})$ to the sum of the fundamental cohomology classes of the two 2-spheres. Use $\rho \circ \psi$ to pull the fibration, $S^7 \to S^4$, back to a principal S^3 fibre bundle, $\pi : P \to S^2 \times S^2$, and let $\varphi : P \to S^7$ be the pull-back map. As with S^7, the S^3 action on P restricts to an S^1 action. Set $N = P/S^1$; then π factors over $\rho_P : P \to N$ to give a fibre bundle $N \to S^2 \times S^2$ with fibre S^2. Finally, note that φ factors to yield a map $\chi : N \to \mathbf{C}P^3$, and that $\zeta \circ \chi = \psi \circ \rho_P$.

Lemma 2. *The rational Sullivan minimal model for P has the form $\Lambda(y, y_L, y_R, a_L, a_R)$ in which a_L, a_R are cycles of degree 2 corresponding to the left and right 2-spheres, and $dy_L = a_L^2$, $dy_R = a_R^2$, and $dy = 2a_L a_R$.*

Proof. The rational Sullivan model for the Hopf bundle has the form $\Lambda(y, v, w)$, where v is a cycle mapping to a representative of the fundamental class, β, of S^4, and $dw = v^2$ and $dy = v$. It follows from [**2**, Sec. 15] that the model for P has the form $\Lambda(y, y_L, y_R, a_L, a_R)$ in which a_L, a_R are cycles of degree 2 corresponding to the left and right 2-spheres, $dy_L = a_L^2$, $dy_R = a_R^2$, and dy is any cycle mapping to a representative of $H(\rho \circ \psi)(\beta)$. Now it is a standard fact that $H(\rho)(\beta) = \alpha^2$. Since $H(\psi)(\alpha)$ is the sum of the fundamental classes of the 2-spheres, and since a_L^2 and a_R^2 are boundaries, we may take $dy = 2a_L a_R$. q.e.d.

It follows from Lemma 2 that the elements $1, a_L, a_R, ya_L, ya_R$, and $ya_L a_R$ are cycles representing a basis of the rational cohomology of P. In particular, $H^3(P; \mathbf{Q}) = 0$. Thus for any metric on P the harmonic Sullivan model (D, d) begins with a map $m : a_L, a_R \to \theta_L, \theta_R$, where θ_L, θ_R are harmonic forms representing the fundamental classes of the left and right 2-spheres. In particular, $D_2^3 = 0$ and so it follows from (8) that $\widehat{U_2^2} = 0$. Since three is the only critical degree, it follows that $U = 0$ and hence that the harmonic Sullivan model for P is the extension of m to a morphism that sends y, y_L, y_R to the coclosed forms $\Omega, \Omega_L, \Omega_R$ satisfying $d\Omega = 2\theta_L \theta_R$, $d\Omega_L = \theta_L^2$, and $d\Omega_R = \theta_R^2$.

On the other hand, if $\mu : \Lambda(y, y_L, y_R, a_L, a_R) \to A_{DR}(P)$ is the rational model morphism of Sullivan then, because $H^3(P, \mathbf{Q}) = 0$, it follows that the images under μ and m of y, y_L, y_R differ by elements in Imd. It follows that μ and m are homotopic as morphisms, and hence that $H(\mu) = H(m)$. Thus the harmonic field is indeed $\mathbf{Q}$.

We now turn attention to $M = S^3 \times P$, equipped with an arbitary Riemannian metric. Here the rational Sullivan model may be chosen of the form $\nu : \Lambda(z, y, y_L, y_R, a_L, a_R) \to A_{DR}(M)$ in which ν maps the cycle z to the harmonic form Φ which represents the fundamental class of S^3. The same argument as above then shows that the harmonic Sullivan model has the form

$$m : \Lambda(z, y, y_L, y_R, a_L, a_R) \to A_{DR}(M),$$

where $m(z) = \Phi$ and m sends the other variables either to harmonic forms θ_L, θ_R on M or else to coclosed forms on M, all satisfying the same conditions as above. However in this case we will have

$$m(y) = \nu(y) + d\beta + \lambda\Phi$$

where $\lambda \in \mathbf{R}$. Then $m - \nu$ maps the cycles ya_L, ya_R and ya_La_R to cycles that differ from Imd by $\lambda\Phi\theta_L, \lambda\Phi\theta_R$ and $\lambda\Phi\theta_L\theta_R$. It follows that the harmonic field for the metric is $\mathbf{Q}(\lambda)$.

It remains to show that any field of the form $\mathbf{Q}(\lambda)$ is the harmonic field for a suitable Riemannian metric on M. Recall that any Riemannian metric $<,>$ defines a $C^\infty(M)$ bilinear map,

$$<,>: A^k_{DR}(M) \times A^k_{DR}(M) \to C^\infty(M).$$

We will denote the corresponding inner product on $A^k_{DR}(M)$ by

$$(\Phi, \Psi) = \int_M < \Phi, \Psi > vol_M.$$

We begin by observing that each S^{2k+1} admits a 1-form, ω_k, invariant under the action of S^1, and whose restriction to each fibre is a nowhere vanishing 1-form, and such that $d\omega_k \in A^2_{DR}(CP^k)$ and represents a generator of $H^2(CP^k; \mathbf{Z})$. Moreover, $\omega_k(d\omega_k)^k$ represents a fundamental class for S^{2k+1}.

Now assign to N an arbitrary Riemannian metric and set $A_{DR}(\varphi)(\omega_3) = \omega$. Then ω restricts to a nowhere vanishing 1-form on each fibre of ρ_P. Thus the cotangent bundle, T^*_P, is the direct sum of the canonical horizontal subbundle, H_P, and the trivial line bundle determined by ω. The differential, $d\rho_P$ dualizes to isomorphisms $T^*_{\rho_P(x)} \cong (H_P)_x$, and so the metric in N determines a metric in H_P. Extend this to a metric in T^*_P by setting $< \omega, \omega > = 1$ and $< \omega, H_P > = 0$.

Next, assign S^3 the standard metric and note that this is invariant under the left and right actions of S^3. Moreover, ω_1 is invariant under the left action of S^3 , $d\omega_1$ is a horizontal 2-form on S^3 also invariant under the left action of S^3, and the horizontal subbundle is orthogonal to the line bundle determined by ω_1. Multiplying the metric by an appropriate constant, we arrange as well that $< \omega_1, \omega_1 > = 1$. In particular,

$< d\omega_1, d\omega_1 >$ is an S^3-invariant function; i.e., it is a non-zero constant. Set

$$< d\omega_1, d\omega_1 >= c.$$

Finally we define our metric in $M = S^3 \times P$ to restrict to the metrics for each of S^3 and P and to satisfy the following:

(i) The horizontal bundle for P (resp., the horizontal bundle for S^3) is orthogonal to the cotangent bundle for S^3 (resp., to the cotangent bundle for P), and

(ii) $< \omega_1, \omega >= (\sigma/c)(< d\omega_1, d\omega_1 > + < d\omega, d\omega >)$,

where σ is a positive real number to be determined, but satisfying $(\sigma/c)(< d\omega_1, d\omega_1 > + < d\omega, d\omega) < 1$.

Lemma 3. *The harmonic form representing the fundamental class of S^3 is given by*

$$\Phi = \omega_1 d\omega_1 - \sigma d(\omega_1\omega)$$

Proof. As we recalled above, $\omega_1(d\omega_1)$ represents the fundamental class of S^3. Thus $\Phi = \omega_1 d\omega_1 - d\gamma$, where $d\gamma$ is the unique closed form such that Φ is orthogonal to all closed forms. Elementary linear algebra then implies that $d\gamma$ is characterized by the equation

$$(\omega_1 d\omega_1, d\gamma) = (d\gamma, d\gamma).$$

But a straightforward check from the definitions above shows that $< \omega_1 d\omega_1, \sigma d(\omega_1\omega) > = < \sigma d(\omega_1\omega), \sigma d(\omega_1\omega) >$. q.e.d.

Now let A_I denote the subdifferential algebra of $A_{DR}(S^3 \times P)$ of differential forms invariant under the right action of S^1 on P and invariant under the actions on S^3 of S^3 from the left and S^1 from the right. The metric on $M = S^3 \times P$ is invariant under the combined action, and so integrating over the action shows that the harmonic decomposition of $A_{DR}(S^3 \times P)$ restricts to one in A_I. By Proposition 2 a harmonic Sullivan model for A_I is the harmonic Sullivan model for $(S^3 \times P, <, >)$. On the other hand, a straightforward check shows that

$$A_I = \mathbf{R}(1, \omega_1, d\omega_1) \otimes \Lambda\omega \otimes A_{DR}(N).$$

It follows in particular that every 2-cycle in A_I is a linear combination of $d\omega_1$ and a closed 2-form in $A_{DR}(N)$. Since a harmonic cycle must be orthogonal to $d\omega_1$, the harmonic cycles θ_L, θ_R, representing the left and right fundamental classes of the two 2-spheres, must belong to $A_{DR}(N)$.

Next, observe that $A_{DR}(\psi)(d\omega_3) = d\omega \in A^2_{DR}(N)$. Thus $d\omega = \theta_L + \theta_R + d\Gamma$ for some $\Gamma \varepsilon A^1_{DR}(N)$. It follows that

$$d\omega^2 = 2\theta_L\theta_R + d\theta$$

for some 3-form, θ, on N.

In particular, if $m : \Lambda(z, y, y_L, y_R, a_L, a_R) \to A_{DR}(M)$ is the Sullivan-Riemann model constructed in the first part of the proof we have that

$m(y) - \omega d\omega + \theta = d\Psi + \tau\Phi$, for some constant τ. Then by the first part of the proof, the Sullivan-Riemann field for the metric is $\mathbf{Q}(\tau)$.

On the other hand, since $m(y)$ is coclosed and Φ is harmonic, we conclude that

$$(\omega d\omega - \theta + \tau\Phi, \Phi) = 0. \tag{16}$$

But because θ is a 3-form on N we deduce from Lemma 3 that $< \theta, \Phi >= 0$. Denote $< d\omega_1, d\omega_1 > + < d\omega, d\omega >$ simply by f. Then a simple computation from (16) gives

$$\sigma^2 \int (< d\omega, d\omega > f/c - \tau f) = -\int c\tau.$$

Now in any field extension, $\mathbf{Q}(\lambda)$, we can find an arbitrarily small (possibly positive or negative) rational multiple, τ, of λ such that the equation above has a real solution for σ. Moreover for τ sufficiently small we will have σ also arbitrarily small and so satisfying the requirement in the definition of the metric. Thus $\mathbf{Q}(\lambda)$ is the harmonic field for the metric determined by σ. q.e.d.

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