J. DIFFERENTIAL GEOMETRY
95 (2013) 355-388

THE EXISTENCE OF EMBEDDED MINIMAL HYPERSURFACES

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Abstract

We give a shorter proof of the existence of nontrivial closed minimal hypersurfaces in closed smooth $(n+1)$-dimensional Riemannian manifolds, a theorem proved first by Pitts for $2 \leq n \leq 5$ and extended later by Schoen and Simon to any n.

0. Introduction

In this paper we give a proof of the following theorem, a natural generalization of the classical existence of nontrivial simple closed geodesics in closed 2-dimensional Riemannian manifolds.

Theorem 0.1. *Let M be an $(n+1)$-dimensional smooth closed Riemannian manifold. Then there is a nontrivial embedded minimal hypersurface $\Sigma \subset M$ without boundary with a singular set $\operatorname{Sing}\Sigma$ of Hausdorff dimension at most $n-7$.*

More precisely, Σ is a closed set of finite $\mathcal{H}^n$-measure and $\operatorname{Sing}\Sigma \subset \Sigma$ is the smallest closed set S such that $M \setminus S$ is a smooth embedded hypersurface ($\Sigma \setminus \operatorname{Sing}\Sigma$ is in fact analytic if M is analytic). In this paper *smooth* will always mean C^∞. In fact, the result remains true for any C^4 Riemannian manifold M, Σ then will be of class C^2 (see [**18**]). Moreover $\int_{\Sigma\setminus\operatorname{Sing}\Sigma} \omega = 0$ for any exact n-form on M. The case $2 \leq n \leq 5$ was proved by Pitts in his groundbreaking monograph [**16**], an outstanding contribution which triggered all the subsequent research in the topic. The general case was proved by Schoen and Simon in [**18**], building heavily upon the work of Pitts.

The monograph [**16**] can be ideally split into two parts. The first half of the book implements a complicated existence theory for suitable "weak generalizations" of global minimal submanifolds, which is a version of the classical min-max argument introduced by Birkhoff for $n = 1$ (see [**5**]). The second part contains the regularity theory needed to prove Theorem 0.1. The curvature estimates of [**19**] for stable minimal surfaces are a key ingredient of this part: the core contribution of

Received 2/5/2009.

[**18**] is the extension of these fundamental estimates to any dimension, which enabled the authors to complete Pitts' program for $n > 5$.

[**18**] gives also a quite readable account of parts of Pitts' regularity theory. To our knowledge, there is instead no contribution to clarify other portions of the monograph, at least in general dimension. Indeed, for $n = 2$, the unpublished PhD thesis of Smith (see [**21**]) gives a powerful variant of Pitts' approach. Building on ideas of Simon, the author proved the existence of minimal embedded 2-spheres in any M which is topologically a 3-sphere (further theorems in general Riemannian 3-manifolds have been claimed in [**17**]; [**6**] and [**10**] contain a complete proof of the Simon–Smith Theorem and of a statement in the direction of [**17**]). Smith's aproach relies heavily on the features of 2-dimensional surfaces in 3-manifolds, most notably on the celebrated paper [**13**], and therefore it is not feasible in higher dimensions.

This paper gives a much simpler proof of Theorem 0.1. Our contribution draws heavily on the existing literature and follows Pitts in many aspects. However, we introduce some new ideas which, in spite of their simplicity, allow us to shorten the proof dramatically. These contributions are contained in Sections 3 and 4 of the paper, but we prefer to give a complete account of the proof of Theorem 0.1, containing all the necessary technical details. We leave aside only those facts which are either (by now) classical results or for which we can give a precise reference.

0.1. Min-max surfaces. In what follows M will denote an $(n+1)$-dimensional smooth Riemannian manifold without boundary. First of all we need to generalize slightly the standard notion of a 1-parameter family of hypersurfaces, allowing for some singularities.

Definition 0.2. A family $\{\Gamma_t\}_{t\in[0,1]^k}$ of closed subsets of M with finite $\mathcal{H}^n$-measure is called a *generalized smooth family* if

- (s1) For each t there is a finite set $P_t \subset M$ such that Γ_t is a smooth hypersurface in $M \setminus P_t$;
- (s2) $\mathcal{H}^n(\Gamma_t)$ depends smoothly on t and $t \mapsto \Gamma_t$ is continuous in the Hausdorff sense;
- (s3) on any $U \subset\subset M \setminus P_{t_0}$, $\Gamma_t \stackrel{t\to t_0}{\longrightarrow} \Gamma_{t_0}$ smoothly in U.

$\{\Gamma_t\}_{t\in[0,1]}$ is a *sweepout* of M if there exists a family $\{\Omega_t\}_{t\in[0,1]}$ of open sets such that

- (sw1) $(\Gamma_t \setminus \partial\Omega_t) \subset P_t$ for any t;
- (sw2) $\Omega_0 = \emptyset$ and $\Omega_1 = M$;
- (sw3) $\mathrm{Vol}(\Omega_t \setminus \Omega_s) + \mathrm{Vol}(\Omega_s \setminus \Omega_t) \to 0$ as $t \to s$.

Remark 0.3. The convergence in (s3) means, as usual, that, if $U \subset\subset M \setminus P_{t_0}$, then there is $\delta > 0$ such that, for $|t - t_0| < \delta$, $\Gamma_t \cap U$ is the graph of a function g_t over $\Gamma_{t_0} \cap U$. Moreover, given $k \in \mathbb{N}$ and $\varepsilon > 0$, $\|g_t\|_{C^k} < \varepsilon$ provided δ is sufficiently small.

We introduce the singularities P_t for two important reasons. They allow for the change of topology which, for $n > 2$, is a fundamental tool of the regularity theory. Moreover, it is easy to exhibit sweepouts as in Definition 0.2, as is witnessed by the following proposition.

Proposition 0.4. *Let $f : M \to [0,1]$ be a smooth Morse function. Then $\{\{f = t\}\}_{t\in[0,1]}$ is a sweepout.*

The obvious proof is left to the reader. For any generalized family $\{\Gamma_t\}$, we set

$$\mathcal{F}(\{\Gamma_t\}) \;:=\; \max_{t\in[0,1]} \mathcal{H}^n(\Gamma_t). \tag{0.1}$$

A key property of sweepouts is an obvious consequence of the isoperimetric inequality.

Proposition 0.5. *There exists $C(M) > 0$ such that $\mathcal{F}(\{\Gamma_t\}) \geq C(M)$ for every sweepout.*

Proof. Let $\{\Omega_t\}$ be as in Definition 0.2. Then there is $t_0 \in [0,1]$ such that $\mathrm{Vol}(\Omega_{t_0}) = \mathrm{Vol}(M)/2$. We then conclude

$$\mathcal{H}^n(\Gamma_{t_0}) \;\geq\; c_0^{-1}(2^{-1}\mathrm{Vol}\,(M))^{\frac{n}{n+1}},$$

where c_0 is the isoperimetric constant of M. q.e.d.

For any family Λ of sweepouts, we define

$$m_0(\Lambda) \;:=\; \inf_{\Lambda} \mathcal{F} \;=\; \inf_{\{\Gamma_t\}\in\Lambda} \left[\max_{t\in[0,1]} \mathcal{H}^n(\Gamma_t)\right]. \tag{0.2}$$

By Proposition 0.5, $m_0(\Lambda) \geq C(M) > 0$. A sequence $\{\{\Gamma_t\}^k\} \subset \Lambda$ is *minimizing* if

$$\lim_{k\to\infty} \mathcal{F}(\{\Gamma_t\}^k) \;=\; m_0(\Lambda)\,.$$

A sequence of surfaces $\{\Gamma^k_{t_k}\}$ is a *min-max sequence* if $\{\{\Gamma_t\}^k\}$ is minimizing and $\mathcal{H}^n(\Gamma^k_{t_k}) \to m_0(\Lambda)$. The min-max construction is applied to families of sweepouts which are closed under a very natural notion of homotopy.

Definition 0.6. Two sweepouts $\{\Gamma^0_s\}$ and $\{\Gamma^1_s\}$ are homotopic if there is a generalized family $\{\Gamma_t\}_{t\in[0,1]^2}$ such that $\Gamma_{(0,s)} = \Gamma^0_s$ and $\Gamma_{(1,s)} = \Gamma^1_s$. A family Λ of sweepouts is called *homotopically closed* if it contains the homotopy class of each of its elements.

Ultimately, this paper gives a proof of the following Theorem, which, together with Proposition 0.4, implies Theorem 0.1 for $n \geq 2$ (recall that Morse functions exist on every smooth compact Riemannian manifold without boundary; see corollary 6.7 of [**14**]).

Theorem 0.7. *Let $n \geq 2$. For any homotopically closed family Λ of sweepouts there is a min-max sequence $\{\Gamma^k_{t_k}\}$ converging (in the sense of varifolds) to an embedded minimal hypersurface Σ as in Theorem* 0.1. *Multiplicity is allowed.*

The smoothness assumption on the metric g can be relaxed easily to C^4. The ingredients of the proof where this regularity is needed are: the regularity theory for the Plateau problem, the unique continuation for classical minimal surfaces, and the Schoen–Simon compactness theorem. C^4 suffices for all of them.

The paper is organized as follows: Section 1 contains some preliminaries, Section 2 gives an overview of the proof of Theorem 0.7, Section 3 contains the existence theory, and Sections 4 and 5 contain the regularity theory.

Acknowledgments. Both authors are grateful for the support by the DFG Sonderforschungsbereich / Transregio 71.

1. Preliminaries

1.1. Notation. Throughout this paper our notation will be consistent with the one introduced in section 2 of [**6**]. We summarize it in the following table.

$\mathrm{Inj}\,(M)$	the injectivity radius of M;
$B_\rho(x)$, $\overline{B}_\rho(x)$, $\partial B_\rho(x)$	the open and closed ball, the distance sphere in M;
$\mathrm{diam}(G)$	the diameter of $G \subset M$;
$d(G_1, G_2)$	$\inf_{x\in G_1, y\in G_2} d(x,y)$;
$\mathcal{B}_\rho$	the ball of radius ρ and centered in 0 in $\mathbb{R}^n$;
$\exp_x$	the exponential map in M at $x \in M$;
$\mathrm{An}(x,\tau,t)$	the open annulus $B_t(x) \setminus \overline{B}_\tau(x)$;
$\mathcal{AN}_r(x)$	the set $\{\mathrm{An}(x,\tau,t)$ with $0 < \tau < t < r\}$;
$\mathcal{X}(M)$, $\mathcal{X}_c(U)$	smooth vector fields, smooth vector fields compactly supported in U.

Remark 1.1. In [**6**] the authors erroneously define d as the Hausdorff distance. However, for the purposes of both this and that paper, the correct definition of d is the one given here, since in both cases the following fact plays a fundamental role: $d(A,B) > 0 \Longrightarrow A \cap B = \emptyset$. Note that, unlike the Hausdorff distance, d is not a distance on the space of compact sets.

1.2. Caccioppoli sets and Plateau's problem. We give here a brief account of the theory of Caccioppoli sets. A standard reference is [**12**]. Let $E \subset M$ be a measurable set and consider its indicator function $\mathbf{1}_E$

(taking the value 1 on E and 0 on $M \setminus E$). The perimeter of E is defined as

$$\mathrm{Per}\,(E) := \sup\left\{\int_M \mathbf{1}_E \,\mathrm{div}\,\omega \;:\; \omega \in \mathcal{X}(M),\ \|\omega\|_{C^0} \le 1\right\}.$$

A Caccioppoli set is a set E for which $\mathrm{Per}\,(E) < \infty$. In this case the distributional derivative $D\mathbf{1}_E$ is a Radon measure and $\mathrm{Per}\,E$ corresponds to its total variation. As usual, the perimeter of E in an open set U, denoted by $\mathrm{Per}\,(E, U)$, is the total variation of $D\mathbf{1}_E$ in the set U.

We follow De Giorgi and, given a Caccioppoli set $\Omega \subset M$ and an open set $U \subset M$, we consider the class

$$\mathcal{P}(U, \Omega) \;:=\; \{\Omega' \subset M \;:\; \Omega' \setminus U = \Omega \setminus U\}\,. \tag{1.1}$$

The theorem below states the fundamental existence and interior regularity theory for De Giorgi's solution of the Plateau problem, which summarizes results of De Giorgi, Almgren, Simons and Federer (see [**12**] for the case $M = \mathbb{R}^{n+1}$ and section 37 of [**20**] for the general case).

Theorem 1.2. *Let $U, \Omega \subset M$ be, respectively, an open and a Caccioppoli set. Then there exists a Caccioppoli set $\Xi \in \mathcal{P}(U, \Omega)$ minimizing the perimeter. Moreover, any such minimizer is, in U, an open set whose boundary is smooth outside of a singular set of Hausdorff dimension at most $n - 7$.*

1.3. Theory of varifolds. We recall here some basic facts from the theory of varifolds; see for instance chapters 4 and 8 of [**20**] for further information. Varifolds are a convenient way of generalizing surfaces to a category that has good compactness properties. An advantage of varifolds, over other generalizations (like currents), is that they do not allow for cancellation of mass. This last property is fundamental for the min-max construction. If U is an open subset of M, any finite nonnegative measure on the Grassmannian $G(U)$ of unoriented n-planes on U is said to be an *n-varifold in U*. The space of n-varifolds is denoted by $\mathcal{V}(U)$ and we endow it with the topology of the weak* convergence in the sense of measures. Therefore, a sequence $\{V^k\} \subset \mathcal{V}(U)$ converges to V if

$$\lim_{k\to\infty} \int \varphi(x, \pi)\, dV^k(x, \pi) \;=\; \int \varphi(x, \pi)\, dV(x, \pi)$$

for every $\varphi \in C_c(G(U))$. Here π denotes an n-plane of $T_x M$. If $U' \subset U$ and $V \in \mathcal{V}(U)$, then $V \llcorner U'$ is the restriction of the measure V to $G(U')$. Moreover, $\|V\|$ is the nonnegative measure on U defined by

$$\int_U \varphi(x)\, d\|V\|(x) \;=\; \int_{G(U)} \varphi(x)\, dV(x, \pi) \qquad \forall \varphi \in C_c(U)\,.$$

The support of $\|V\|$, denoted by supp $(\|V\|)$, is the smallest closed set outside which $\|V\|$ vanishes identically. The number $\|V\|(U)$ will be called the *mass of V in U*.

Recall also that an n-dimensional rectifiable set is the countable union of closed subsets of C^1 surfaces (modulo sets of $\mathcal{H}^n$-measure 0). If $R \subset U$ is an n-dimensional rectifiable set and $h : R \to \mathbb{R}_+$ is a Borel function, then the *varifold* V *induced by* R is defined by

$$\int_{G(U)} \varphi(x,\pi)\, dV(x,\pi) = \int_R h(x)\varphi(x, T_x R)\, d\mathcal{H}^n(x) \tag{1.2}$$

for all $\varphi \in C_c(G(U))$. Here $T_x R$ denotes the tangent plane to R in x. If h is integer-valued, then we say that V is an *integer rectifiable varifold*. If $\Sigma = \bigcup n_i \Sigma_i$, then by slight abuse of notation we use Σ for the varifold induced by Σ via (1.2).

If $\psi : U \to U'$ is a diffeomorphism and $V \in \mathcal{V}(U)$, $\psi_\sharp V \in \mathcal{V}(U')$ is the varifold defined by

$$\int \varphi(y,\sigma)\, d(\psi_\sharp V)(y,\sigma) \;=\; \int J\psi(x,\pi)\, \varphi(\psi(x), d\psi_x(\pi))\, dV(x,\pi)\,,$$

where $J\psi(x,\pi)$ denotes the Jacobian determinant (i.e., the area element) of the differential $d\psi_x$ restricted to the plane π; cf. equation (39.1) of [**20**]. Obviously, if V is induced by a C^1 surface Σ, V' is induced by $\psi(\Sigma)$.

Given $\chi \in \mathcal{X}_c(U)$, let ψ be the isotopy generated by χ, i.e., $\frac{\partial \psi}{\partial t} = \chi(\psi)$. The first and second variation of V with respect to χ are defined as

$$\begin{aligned} & [\delta V](\chi) \;=\; \frac{d}{dt}(\|\psi(t,\cdot)_\sharp V\|)(U)\Big|_{t=0} \\ \text{and} \qquad & [\delta^2 V](\chi) \;=\; \frac{d^2}{dt^2}(\|\psi(t,\cdot)_\sharp V\|)(U)\Big|_{t=0}\;; \end{aligned}$$

cf. sections 16 and 39 of [**20**]. V is said to be *stationary* (resp. *stable*) in U if $[\delta V](\chi) = 0$ (resp. $[\delta^2 V](\chi) \geq 0$) for every $\chi \in \mathcal{X}_c(U)$. If V is induced by a surface Σ with $\partial\Sigma \subset \partial U$, V is stationary (resp. stable) if and only if Σ is minimal (resp. stable).

Stationary varifolds in a Riemannian manifold satisfy the monotonicity formula, i.e., there exists a constant Λ (depending on the ambient manifold M) such that the function

$$f(\rho) \;:=\; e^{\Lambda\rho}\frac{\|V\|(B_\rho(x))}{\omega_n \rho^n} \tag{1.3}$$

is nondecreasing for every x (see theorem 17.6 of [**20**]; $\Lambda = 0$ if the metric of M is flat). This property allows us to define the *density* of a stationary varifold V at x, by

$$\theta(x,V) \;=\; \lim_{r\to 0}\frac{\|V\|(B_r(x))}{\omega_n r^n}.$$

1.4. Schoen–Simon curvature estimates. Consider an orientable $U \subset M$. We look here at closed sets $\Gamma \subset M$ of codimension 1 satisfying the following regularity assumption:

(SS) $\Gamma \cap U$ is a smooth embedded hypersurface outside a closed set S with $\mathcal{H}^{n-2}(S) = 0$.

Γ induces an integer rectifiable varifold V. Thus Γ is said to be minimal (resp. stable) in U with respect to the metric g of U if V is stationary (resp. stable). The following compactness theorem, a consequence of the Schoen–Simon curvature estimates (compare with theorem 2 of section 6 in [**18**]), is a fundamental tool in this note.

Theorem 1.3. *Let U be an orientable open subset of a manifold and $\{g^k\}$ and $\{\Gamma^k\}$, respectively, sequences of smooth metrics on U and of hypersurfaces $\{\Gamma^k\}$ satisfying (SS). Assume that the metrics g^k converge smoothly to a metric g, that each Γ^k is stable and minimal relative to the metric g^k, and that $\sup \mathcal{H}^n(\Gamma^k) < \infty$. Then there are a subsequence of $\{\Gamma^k\}$ (not relabeled), a stable stationary varifold V in U (relative to the metric g), and a closed set S of Hausdorff dimension at most $n-7$ such that*

(a) *V is a smooth embedded hypersurface in $U \setminus S$;*
(b) *$\Gamma^k \to V$ in the sense of varifolds in U;*
(c) *Γ^k converges smoothly to V on every $U' \subset\subset U \setminus S$.*

Remark 1.4. The precise meaning of (c) is as follows: fix an open $U'' \subset U'$ where the varifold V is an integer multiple N of a smooth oriented surface Σ. Choose a normal unit vector field on Σ (in the metric g) and corresponding normal coordinates in a tubular neighborhood. Then, for k sufficiently large, $\Gamma^k \cap U''$ consists of N disjoint smooth surfaces Γ^k_i which are graphs of functions $f^k_i \in C^\infty(\Sigma)$ in the chosen coordinates. Assuming, w.l.o.g., $f^k_1 \leq f^k_2 \leq \ldots \leq f^k_N$, each sequence $\{\Gamma^k_i\}_k$ converges to Σ in the sense of Remark 0.3.

Note the following obvious corollary of Theorem 1.3: if Γ is a stationary and stable surface satisfying (SS), then the Hausdorff dimension of $\mathrm{Sing}\,\Gamma$ is, in fact, at most $n-7$. Since we will deal very often with this type of surface, we will use the following notational convention.

Definition 1.5. Unless otherwise specified, a *hypersurface* $\Gamma \subset U$ is a closed set of codimension 1 such that $\overline{\Gamma} \setminus \Gamma \subset \partial U$ and $\mathrm{Sing}\,\Gamma$ has Hausdorff dimension at most $n-7$. The words "stable" and "minimal" are then used as explained at the beginning of this subsection. For instance, the surface Σ of Theorem 0.1 is a *minimal hypersurface.*

2. Proof of Theorem 0.7

2.1. Isotopies and stationarity. It is easy to see that not all min-max sequences converge to stationary varifolds (see [**6**]). In general, for any minimizing sequence $\{\{\Gamma_t\}^k\}$ there is at least one min-max sequence converging to a stationary varifold. For technical reasons, it is useful to consider minimizing sequences $\{\{\Gamma_t\}^k\}$ with the additional property that *any* corresponding min-max sequence converges to a stationary varifold. The existence of such a sequence, which roughly speaking follows from "pulling tight" the surfaces of a minimizing sequence, is an important conceptual step and goes back to Birkhoff in the case of geodesics and to the fundamental work of Pitts in the general case (see also [**7**] and [**8**] for other applications of these ideas). In order to state it, we need some terminology.

Definition 2.1. Given a smooth map $F : [0,1] \to \mathcal{X}(M)$, for any $t \in [0,1]$ we let $\Psi_t : [0,1] \times M \to M$ be the one-parameter family of diffeomorphisms generated by the vectorfield $F(t)$. If $\{\Gamma_t\}_{t\in[0,1]}$ is a sweepout, then $\{\Psi_t(s,\Gamma_t)\}_{(t,s)\in[0,1]^2}$ is a homotopy between $\{\Gamma_t\}$ and $\{\Psi_t(1,\Gamma_t)\}$. These will be called *homotopies induced by ambient isotopies.*

We recall that the weak* topology on the space $\mathcal{V}(M)$ (varifolds with bounded mass) is metrizable and we choose a metric $\mathcal{D}$ which induces it. Moreover, let $\mathcal{V}_s \subset \mathcal{V}(M)$ be the (closed) subset of stationary varifolds.

Proposition 2.2. *Let Λ be a family of sweepouts which is closed under homotopies induced by ambient isotopies. Then there exists a minimizing sequence $\{\{\Gamma_t\}^k\} \subset \Lambda$ such that, if $\{\Gamma^k_{t_k}\}$ is a min-max sequence, then $\mathcal{D}(\Gamma^k_{t_k}, \mathcal{V}_s) \to 0$.*

This proposition is proposition 4.1 of [**6**]. Though stated for the case $n = 2$, this assumption, in fact, is never used in the proof given in that paper. Therefore we do not include a proof here.

2.2. Almost mimimizing varifolds. It is well known that a stationary varifold can be far from regular. To overcome this issue, we introduce the notion of almost minimizing varifolds.

Definition 2.3. Let $\varepsilon > 0$ and $U \subset M$ be open. A boundary $\partial\Omega$ in M is called ε-*almost minimizing* (ε-a.m.) in U if there is NO 1-parameter family of boundaries $\{\partial\Omega_t\}$, $t \in [0,1]$, satisfying the following properties:

(2.1) (s1), (s2), (s3), (sw1), and (sw3) of Definition 0.2 hold;

(2.2) $\Omega_0 = \Omega$ and $\Omega_t \setminus U = \Omega \setminus U$ for every t;

(2.3) $\mathcal{H}^n(\partial\Omega_t) \leq \mathcal{H}^n(\partial\Omega) + \frac{\varepsilon}{8}$ for all $t \in [0,1]$;

(2.4) $\mathcal{H}^n(\partial\Omega_1) \leq \mathcal{H}^n(\partial\Omega) - \varepsilon$.

A sequence $\{\partial\Omega^k\}$ of hypersurfaces is called *almost minimizing in U* if each $\partial\Omega^k$ is ε_k-a.m. in U for some sequence $\varepsilon_k \to 0$.

Roughly speaking, $\partial\Omega$ is a.m. if any deformation which eventually brings down its area is forced to pass through some surface which has substantially larger area. A similar notion was introduced for the first time in the pioneering work of Pitts and a corresponding one is given in [**21**] using isotopies (see section 3.2 of [**6**]). Following in part section 5 of [**6**] (which uses a combinatorial argument inspired by a general one of [**2**] reported in [**16**]), we prove in Section 3 the following existence result.

Proposition 2.4. *Let Λ be a homotopically closed family of sweep-outs. There is a function $r : M \to \mathbb{R}_+$ and a min-max sequence $\Gamma^k = \Gamma^k_{t_k}$ such that*

(a) *$\{\Gamma^k\}$ is a.m. in every $An \in \mathcal{AN}_{r(x)}(x)$ with $x \in M$;*
(b) *Γ^k converges to a stationary varifold V as $k \to \infty$.*

In this part we introduce, however, a new ingredient. The proof of Proposition 2.4 has a variational nature: assuming the nonexistence of such a min-max sequence, we want to show that on an appropriate minimizing sequence $\{\{\Gamma_t\}^k\}$, the energy $\mathcal{F}(\{\Gamma_t\}^k)$ can be lowered by a fixed amount, contradicting its minimality. Note, however, that we have one-parameter families of surfaces, whereas the variational notion of Definition 2.3 focuses on a single surface. Pitts (who in turn has a stronger notion of almost minimality) avoids this difficulty by considering discretized families, and this, in our opinion, makes his proof quite hard. Instead, our notion of almost minimality allows us to stay in the smooth category: the key technical point is the "freezing" presented in Section 3.2 (compare with Lemma 3.1).

2.3. Replacements. We complete the program in Sections 4 and 5 showing that our notion of almost minimality is still sufficient to prove regularity. As a starting point, as in the theory of Pitts, we consider *replacements*.

Definition 2.5. Let $V \in \mathcal{V}(M)$ be a stationary varifold and $U \subset M$ be an open set. A stationary varifold $V' \in \mathcal{V}(M)$ is called a *replacement for V in U* if $V' = V$ on $M \setminus \bar{U}$, $\|V'\|(M) = \|V\|(M)$, and $V \llcorner U$ is a stable minimal hypersurface Γ.

We show in Section 4 that almost minimizing varifolds do posses replacements.

Proposition 2.6. *Let $\{\Gamma^j\}$, V, and r be as in Proposition 2.4. Fix $x \in M$ and consider an annulus $An \in \mathcal{AN}_{r(x)}(x)$. Then there is a varifold $\tilde{V}$, a sequence $\{\tilde{\Gamma}^j\}$, and a function $r' : M \to \mathbb{R}_+$ such that*

(a) *$\tilde{V}$ is a replacement for V in An and $\tilde{\Gamma}^j$ converges to $\tilde{V}$ in the sense of varifolds;*
(b) *$\tilde{\Gamma}^j$ is a.m. in every $An' \in \mathcal{AN}_{r'(y)}(y)$ with $y \in M$;*
(c) *$r'(x) = r(x)$.*

The strategy of the proof is the following. Fix an annulus An. We would like to substitute $\Gamma^j = \partial\Omega^j$ in An with the surface minimizing the area among all those which can be continuously deformed into Γ^j according to our homotopy class: we could appropriately call it a solution of the $(8j)^{-1}$ *homotopic Plateau problem*. As a matter of fact, we do not know any regularity for this problem. However, if we consider a corresponding minimizing sequence $\{\partial\Omega^{j,k}\}_k$, we will show that it converges, up to subsequences, to a varifold V^j which is regular in An. This regularity is triggered by the following observation: on any sufficiently small ball $B \subset An$, $V^j \llcorner B$ is the boundary of a Caccioppoli set Ω^j which solves the Plateau problem in the class $\mathcal{P}(\Omega^j, B)$ (in the sense of Theorem 1.2).

In fact, by standard blow-up methods of geometric measure theory, V^j is close to a cone in any sufficiently small ball $B = B_r(y)$. For k large, the same property holds for $\partial\Omega^{j,k}$. Modifying suitably an idea of [**21**], this property can be used to show that any (sufficiently regular) competitor $\tilde{\Omega} \in \mathcal{P}(\Omega^{j,k}, B)$ can be homotopized to $\Omega^{j,k}$ without passing through a surface of large energy. In other words, minimizing sequences of the homotopic Plateau problem are in fact minimizing for the usual Plateau problem at sufficiently small scales.

Having shown the regularity of V^j in An, we use the Schoen–Simon compactness theorem to show that V^j converges to a varifold $\tilde{V}$ which in An is a stable minimal hypersurface. A suitable diagonal sequence $\Gamma^{j,k(j)}$ gives the surfaces $\tilde{\Gamma}^j$.

2.4. Regularity of V. One would like to conclude that, if V' is a replacement for V in an annulus contained in a convex ball, then $V = V'$ (and hence V is regular in An). However, two stationary varifolds might coincide outside of a convex set and be different inside: the standard unique continuation property of classical minimal surfaces fails in the general case of stationary varifolds (see the appendix of [**6**] for an example). We need more information to conclude the regularity of V. Clearly, applying Proposition 2.6 three times, we conclude:

Proposition 2.7. *Let V and r be as in Proposition* 2.4*. Fix $x \in M$ and $An \in \mathcal{AN}_{r(x)}(x)$. Then:*

(a) *V has a replacement V' in An such that*
(b) *V' has a replacement V'' in any*

$$An' \in \mathcal{AN}_{r(x)}(x) \cup \bigcup_{y \neq x} \mathcal{AN}_{r'(y)}(y)$$

such that

(c) *V'' has a replacement V''' in any $An'' \in \mathcal{AN}_{r''(y)}(y)$ with $y \in M$.*

r' and r'' are positive functions (which might depend on V' and V'').

In fact, the process could be iterated infinitely many times. However, it turns out that three iterations are sufficient to prove regularity, as stated in the following proposition. Its proof is given in Section 5, where we basically follow [**18**] (see also [**6**]).

Proposition 2.8. *Let V be as in Proposition* 2.7*. Then V is induced by a minimal hypersurface Σ (in the sense of Definition* 1.5*).*

3. The existence of almost mimimizing varifolds

In this section we prove Proposition 2.4. At various steps in the regularity theory we will have to construct comparison surfaces which are deformations of a given surface. However, each initial surface will be just a member of a one-parameter family and in order to exploit our variational properties we must in fact construct "comparison families." If we consider a family as a moving surface, it becomes clear that difficulties arise when we try to embed the deformation of a single "time-slice" into the dynamics of the family itself. The main new point of this section is therefore the following technical lemma, which allows to use the "static" variational principle of Definition 2.3 to construct a "dynamic" competitor.

Lemma 3.1. *Let $U \subset\subset U' \subset M$ be two open sets and $\{\partial\Xi_t\}_{t\in[0,1]}$ a sweepout. Given an $\varepsilon > 0$ and a $t_0 \in [0,1]$, assume $\{\partial\Omega_s\}_{s\in[0,1]}$ is a one-parameter family of surfaces satisfying* (2.1)*,* (2.2)*,* (2.3)*, and* (2.4)*, with $\Omega = \Xi_{t_0}$. Then there is $\eta > 0$, such that the following holds for every a, b, a', b' with $t_0 - \eta \le a < a' < b' < b \le t_0 + \eta$. There is a* competitor sweepout $\{\partial\Xi'_t\}_{t\in[0,1]}$ *with the following properties:*

(a) $\Xi_t = \Xi'_t$ *for* $t \in [0,a] \cup [b,1]$ *and* $\Xi_t \setminus U' = \Xi'_t \setminus U'$ *for* $t \in (a,b)$*;*
(b) $\mathcal{H}^n(\partial\Xi'_t) \le \mathcal{H}^n(\partial\Xi_t) + \frac{\varepsilon}{4}$ *for every* t*;*
(c) $\mathcal{H}^n(\partial\Xi'_t) \le \mathcal{H}^n(\partial\Xi_t) - \frac{\varepsilon}{2}$ *for* $t \in (a', b')$*.*

Moreover, $\{\partial\Xi'_t\}$ *is homotopic to* $\{\partial\Xi_t\}$*.*

Bulding on Lemma 3.1, Proposition 2.4 can be proved using a clever combinatorial argument due to Pitts and Almgren. Indeed, for this part our proof follows literally the exposition of section 5 of [**6**]. This section is therefore split into two parts. In the first one we use the Almgren–Pitts combinatorial argument to show Proposition 2.4 from Lemma 3.1, which will be proved in the second one.

3.1. Almost minimizing varifolds.

Before coming to the proof, we introduce some further notation.

Definition 3.2. Given a pair of open sets (U^1, U^2) we call a hypersurface $\partial\Omega$ ε-a.m. in (U^1, U^2) if it is ε-a.m. in at least one of the two open sets. We denote by $\mathcal{CO}$ the set of pairs (U^1, U^2) of open sets with

$$\mathrm{d}\,(U^1, U^2) \ge 4 \min\{\mathrm{diam}(U^1), \mathrm{diam}(U^2)\}.$$

The following trivial lemma will be of great importance.

Lemma 3.3. *If* (U^1, U^2) *and* (V^1, V^2) *are such that*

$$\mathrm{d}\,(U^1, U^2) \geq 2\min\{diam(U^1), diam(U^2)\}\,,$$
$$\mathrm{d}\,(V^1, V^2) \geq 2\min\{diam(V^1), diam(V^2)\}\,,$$

then there are indices $i, j \in \{1, 2\}$ *with* $\mathrm{d}\,(U^i, V^j) > 0$.

We are now ready to state the Almgren–Pitts combinatorial lemma: Proposition 2.4 is indeed a corollary of it.

Proposition 3.4 (Almgren–Pitts combinatorial lemma)**.** *Let* Λ *be a homotopically closed family of sweepouts. There is a min-max sequence* $\{\Gamma^N\} = \{\partial\Omega^{k(N)}_{t_{k(N)}}\}$ *such that*

- Γ^N *converges to a stationary varifold;*
- *For any* $(U^1, U^2) \in \mathcal{CO}$, Γ^N *is* $1/N$*-a.m. in* (U^1, U^2)*, for* N *large enough.*

Proof of Proposition 2.4. We show that a subsequence of the $\{\Gamma^k\}$ in Proposition 3.4 satisfies the requirements of Proposition 2.4. For this, fix $k \in \mathbb{N}$ and $r > 0$ such that $\mathrm{Inj}\,(M) > 9r > 0$. Then $(B_r(x), M\setminus\overline{B}_{9r}(x)) \in \mathcal{CO}$ for all $x \in M$. Therefore we have that Γ^k is (for k large enough) $1/k$-almost minimizing in $B_r(x)$ or $M \setminus \overline{B}_{9r}(x)$. Therefore, having fixed $r > 0$,

(a) either $\{\Gamma^k\}$ is (for k large) $1/k$-a.m. in $B_r(y)$ for every $y \in M$;
(b) or there are a (not relabeled) subsequence $\{\Gamma^k\}$ and a sequence $\{x^k_r\} \subset M$ such that Γ^k is $1/k$-a.m. in $M \setminus \overline{B}_{9r}(x^k_r)$.

If for some $r > 0$ (a) holds, we clearly have a sequence as in Proposition 2.4. Otherwise, there are a subsequence of $\{\Gamma^k\}$, not relabeled, and a collection of points $\{x^k_j\}_{k,j\in\mathbb{N}} \subset M$ such that

- for any fixed j, Γ^k is $1/k$-a.m. in $M \setminus \overline{B}_{1/j}(x^k_j)$ for k large enough;
- $x^k_j \to x_j$ for $k \to \infty$ and $x_j \to x$ for $j \to \infty$.

We conclude that, for any J, there is K_J such that Γ^k is $1/k$-a.m. in $M \setminus \overline{B}_{1/J}(x)$ for all $k \geq K_J$. Therefore, if $y \in M \setminus \{x\}$, we choose $r(y)$ such that $B_{r(y)} \subset\subset M\setminus\{x\}$, whereas $r(x)$ is chosen arbitrarily. It follows that $An \subset\subset M\setminus\{x\}$, for any $An \in \mathcal{AN}_{r(z)}(z)$ with $z \in M$. Hence, $\{\Gamma^k\}$ is $1/k$-a.m. in An, provided k is large enough, which completes the proof of the proposition. q.e.d.

Proof of Proposition 3.4. We start by picking a minimizing sequence $\{\{\Gamma_t\}^k\}$ satisfying the requirements of Proposition 2.2 and such that $\mathcal{F}(\{\Gamma_t\}^k) < m_0 + \frac{1}{8k}$. We then assert the following claim, which clearly implies the proposition.

Claim. For N large enough, there exists $t_N \in [0, 1]$ such that $\Gamma^N := \Gamma^N_{t_N}$ is $\frac{1}{N}$-a.m. in all $(U^1, U^2) \in \mathcal{CO}$ and $\mathcal{H}^n(\Gamma^N) \geq m_0 - \frac{1}{N}$.

Define

$$K_N := \left\{ t \in [0,1] : \mathcal{H}^n(\Gamma_t^N) \geq m_0 - \frac{1}{N} \right\}.$$

Assume the claim is false. Then there is a sequence $\{N_k\}$ such that the assertion of the claim is violated for every $t \in K_{N_k}$. By a slight abuse of notation, we do not relabel the corresponding subsequence and from now on we drop the super- and subscripts N.

Thus, for every $t \in K$ we get a pair $(U_{1,t}, U_{2,t}) \in \mathcal{CO}$ and two families $\{\partial\Omega_{i,t,\tau}\}_{\tau\in[0,1]}^{i\in\{1,2\}}$ such that

(i) $\partial\Omega_{i,t,\tau} \cap (U_{i,t})^c = \partial\Omega_t \cap (U_{i,t})^c$;
(ii) $\partial\Omega_{i,t,0} = \partial\Omega_t$;
(iii) $\mathcal{H}^n(\partial\Omega_{i,t,\tau}) \leq \mathcal{H}^n(\partial\Omega_t) + \frac{1}{8N}$;
(iv) $\mathcal{H}^n(\partial\Omega_{i,t,1}) \leq \mathcal{H}^n(\partial\Omega_t) - \frac{1}{N}$.

For every $t \in K$ and every $i \in \{1,2\}$, we choose $U'_{i,t}$ such that $U_{i,t} \subset\subset U'_{i,t}$ and

$$\mathrm{d}\,(U'_{1,t}, U'_{2,t}) \geq 2\min\{\mathrm{diam}(U'_{1,t}), \mathrm{diam}(U'_{2,t})\}.$$

Then we apply Lemma 3.1 with $\Xi_t = \Omega_t$, $U = U_{i,t}$, $U' = U'_{i,t}$ and $\Omega_\tau = \Omega_{i,t,\tau}$. Let $\eta_{i,t}$ be the corresponding constant η given by Lemma 3.1 and let $\eta_t = \min\{\eta_{1,t}, \eta_{2,t}\}$.

Next, cover K with intervals $I_i = (t_i - \eta_i, t_i + \eta_i)$ in such a way that

- $t_i + \eta_i < t_{i+2} - \eta_{i+2}$ for every i;
- $t_i \in K$ and $\eta_i < \eta_{t_i}$.

Step 1: Refinement of the covering. We are now going to refine the covering I_i to a covering J_l such that:

- $J_l \subset I_i$ for some $i(l)$;
- there is a choice of a U_l such that $U'_l \in \{U'_{1,t_{i(l)}}, U'_{2,t_{i(l)}}\}$ and

$$\mathrm{d}\,(U'_i, U'_j) > 0 \qquad \text{if } \overline{J}_i \cap \overline{J}_j \neq \emptyset; \tag{3.1}$$

- each point $t \in [0,1]$ is contained in at most two of the intervals J_l.

The choice of our refinement is in fact quite obvious. We start by choosing $J_1 = I_1$. Using Lemma 3.3, we choose indices r, s such that $\mathrm{dist}(U'_{r,t_1}, U'_{s,t_2}) > 0$. For simplicity we can assume $r = s = 1$. We then set $U'_1 = U'_{1,t_1}$. Next, we consider two indices ρ, σ such that $\mathrm{d}\,(U'_{\rho,t_2}, U'_{\sigma,t_3}) > 0$. If $\rho = 1$, we then set $J_2 = I_2$ and $U'_2 = U'_{1,t_2}$. Otherwise, we cover I_2 with two open intervals J_2 and J_3, with the property that $\overline{J}_2$ is disjoint from $\overline{I}_3$ and $\overline{J}_3$ is disjoint from $\overline{I}_1$. We then choose $U'_2 = U'_{1,t_2}$ and $U'_3 = U'_{2,t_2}$. From this we are ready to proceed inductively. Note therefore that, in our refinement of the covering, each interval I_j with $j \geq 2$ gets either "split into two halves" or remains the same (compare with Figure 1, left).

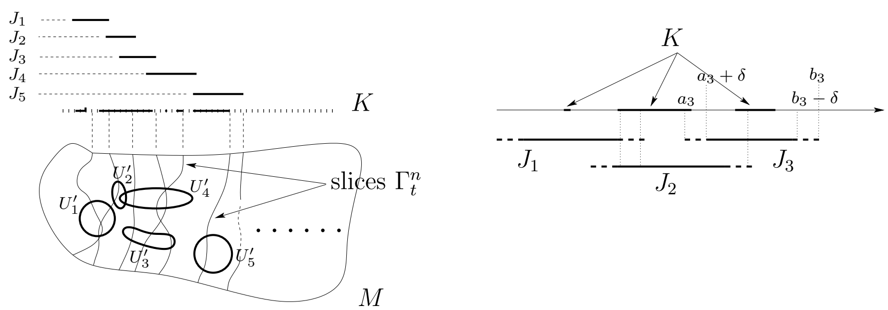

Figure 1. The left picture shows the refinement of the covering. We split I_2 into $J_2 \cup J_3$ because $U'_4 = U'_{1,t_3}$ intersects $U'_2 = U'_{1,t_2}$. The refined covering has the property that $U'_i \cap U'_{i+1} = \emptyset$. In the right picture the segments $(a_k, b_k) = J_k$ and $(a_k+\delta, b_k-\delta)$. Any point $\tau \in K$ belongs to at least one $(a_i+\delta, b_i-\delta)$ and to at most one $J_j \setminus (a_j+\delta, b_j-\delta)$.

Next, fixing the notation $(a_i, b_i) = J_i$, we choose $\delta > 0$ with the following property:

(C) Each $t \in K$ is contained in at least one segment $(a_i+\delta, b_i-\delta)$ (compare with Figure 1, right).

Step 2: Conclusion. We now apply Lemma 3.1 to conclude the existence of a family $\{\partial\Omega_{i,t}\}$ with the following properties:

- $\Omega_{i,t} = \Omega_t$ if $t \notin (a_i, b_i)$ and $\Omega_{i,t} \setminus U'_i = \Omega_t \setminus U'_i$ if $t \in (a_i, b_i)$;
- $\mathcal{H}^n(\partial\Omega_{i,t}) \le \mathcal{H}^n(\partial\Omega_t) + \frac{1}{4N}$ for every t;
- $\mathcal{H}^n(\partial\Omega_{i,t}) \le \mathcal{H}^n(\partial\Omega_t) - \frac{1}{2N}$ if $t \in (a_i+\delta, b_i-\delta)$.

Note that, if $t \in (a_i, b_i) \cap (a_j, b_j)$, then $j = i+1$ and in fact $t \notin (a_k, b_k)$ for $k \ne i, i+1$. Moreover, $\mathrm{dist}(U'_i, U'_{i+1}) > 0$. Thus, we can define a new sweepout $\{\partial\Omega'_t\}_{t\in[0,1]}$

- $\Omega'_t = \Omega_t$ if $t \notin \cup J_i$;
- $\Omega'_t = \Omega_{i,t}$ if t is contained in a single J_i;
- $\Omega'_t = \big[\Omega_t \setminus (U'_i \cup U'_{i+1})\big] \cup \big[\Omega_{i,t} \cap U'_i\big] \cup \big[\Omega_{i+1,t} \cap U'_{i+1}\big]$ if $t \in J_i \cap J_{i+1}$.

In fact, it is as well easy to check that $\{\partial\Omega'_t\}_{t\in[0,1]}$ is homotopic to $\{\partial\Omega_t\}$ and hence belongs to Λ.

Next, we want to compute $\mathcal{F}(\{\partial\Omega'_t\})$. If $t \notin K$, then t is contained in at most two J_i's, and hence $\partial\Omega'_t$ can gain at most $2 \cdot \frac{1}{4N}$ in area:

$$t \notin K \quad \Rightarrow \quad \mathcal{H}^n(\partial\Omega'_t) \le \mathcal{H}^n(\partial\Omega_t) + \frac{1}{2N} \le m_0(\Lambda) - \frac{1}{2N}\,. \tag{3.2}$$

If $t \in K$, then t is contained in at least one segment $(a_i+\delta, b_i-\delta) \subset J_i$ and in at most a second segment J_l. Thus, the area of $\partial\Omega'_t$ looses at least $\frac{1}{2N}$ in U'_i and gains at most $\frac{1}{4N}$ in U'_l. Therefore we conclude

$$t \in K \quad \Rightarrow \quad \mathcal{H}^n(\partial\Omega'_t) \;\leq\; \mathcal{H}^n(\partial\Omega_t) - \frac{1}{4N} \;\leq\; m_0(\Lambda) - \frac{1}{8N}\,. \tag{3.3}$$

Hence $\mathcal{F}(\{\partial\Omega'_t\}) \leq m_0(\Lambda) - (8N)^{-1}$, which is a contradiction to $m_0(\Lambda) = \inf_\Lambda \mathcal{F}$. q.e.d.

3.2. Proof of Lemma 3.1. The proof consists of two steps.

Step 1: Freezing. First of all, we choose open sets A and B such that

- $U \subset\subset A \subset\subset B \subset\subset U'$;
- $\partial\Xi_{t_0} \cap C$ is a smooth surface, where $C = B \setminus \overline{A}$.

This choice is clearly possible since there are only finitely many singularities of $\partial\Xi_{t_0}$. Next, we fix two smooth functions φ_A and φ_B such that

- $\varphi_A + \varphi_B = 1$;
- $\varphi_A \in C^\infty_c(B)$, $\varphi_B \in C^\infty_c(M \setminus \overline{A})$.

Now, we fix normal coordinates $(z,\sigma) \in \partial\Xi_{t_0} \cap C \times (-\delta,\delta)$ in a regular δ-neighborhood of $C \cap \partial\Xi_{t_0}$. Because of the convergence of Ξ_t to Ξ_{t_0}, we can fix $\eta > 0$ and an open $C' \subset C$, such that the following holds for every $t \in (t_0 - \eta, t_0 + \eta)$:

- $\partial\Xi_t \cap C$ is the graph of a function g_t over $\partial\Xi_{t_0} \cap C$;
- $\Xi_t \cap C \setminus C' = \Xi_{t_0} \cap C \setminus C'$;
- $\Xi_t \cap C' = \{(z,\sigma) : \sigma < g_t(z)\} \cap C'$,

(compare with Figure 2). Obviously, $g_{t_0} \equiv 0$. We next introduce the functions

$$g_{t,s,\tau} \;:=\; \varphi_B g_t + \varphi_A((1-s)g_t + s g_\tau) \tag{3.4}$$

for $t, \tau \in (t_0 - \eta, t_0 + \eta)$, $s \in [0,1]$. Since g_t converges smoothly to g_{t_0} as $t \to t_0$, by choosing η arbitrarily small, we can make $\sup_{s,\tau} \|g_{t,s,\tau} - g_t\|_{C^1}$ arbitrarily small. Next, if we express the area of the graph of a function g over $\partial\Xi_{t_0} \cap C$ as an integral functional of g, this functional depends obviously only on g and its first derivatives. Thus, if $\Gamma_{t,s,\tau}$ is the graph of $g_{t,s,\tau}$, then we can choose η so small that

$$\max_{s,\tau} \mathcal{H}^n(\Gamma_{t,s,\tau}) \;\leq\; \mathcal{H}(\partial\Xi_t \cap C) + \frac{\varepsilon}{16}\,. \tag{3.5}$$

Now, given $t_0 - \eta < a < a' < b' < b < t_0 + \eta$, we choose $a'' \in (a, a')$ and $b'' \in (b', b)$ and fix the following:

- a smooth function $\psi : [a,b] \to [0,1]$ which is identically equal to 0 in a neighborhood of a and b and equal to 1 on $[a'', b'']$;
- a smooth function $\gamma : [a,b] \to [t_0 - \eta, t_0 + \eta]$ which is equal to the identity in a neighborhood of a and b and indentically t_0 in $[a'', b'']$.

Next, define the family of open sets $\{\Delta_t\}$ as follows:

- $\Delta_t = \Xi_t$ for $t \notin [a,b]$;
- $\Delta_t \setminus \overline{B} = \Xi_t \setminus \overline{B}$ for all t;
- $\Delta_t \cap A = \Xi_{\gamma(t)} \cap A$ for $t \in [a,b]$;
- $\Delta_t \cap C \setminus C' = \Xi_{t_0} \cap C \setminus C'$ for $t \in [a,b]$;
- $\Delta_t \cap C' = \{(z,\sigma) : \sigma < g_{t,\psi(t),\gamma(t)}(z)\}$ for $t \in [a,b]$.

Note that $\{\partial\Delta_t\}$ is in fact a sweepout homotopic to $\partial\Xi_t$. In addition:

- $\Delta_t = \Xi_t$ if $t \notin [a,b]$, and Δ_t and Ξ_t coincide outside of B (and hence outside of U') for *every* t;
- $\Delta_t \cap A = \Xi_{\gamma(t)} \cap A$ for $t \in [a,b]$ (and hence $\Delta_t \cap U = \Xi_{\gamma(t)} \cap U$).

Therefore, $\Delta_t \cap U = \Xi_{t_0} \cap U$ for $t \in [a'', b'']$, i.e., $\Delta_t \cap U$ is *frozen* in the interval $[a'', b'']$. Moreover, because of (3.5),

$$\mathcal{H}^n(\partial\Delta_t \cap C) \leq \mathcal{H}^n(\partial\Xi_t \cap C) + \frac{\varepsilon}{16} \qquad \text{for } t \in [a,b]. \tag{3.6}$$

Step 2: Dynamic competitor. Next, fix a smooth function $\chi : [a'', b''] \to [0,1]$ which is identically 0 in a neighborhood of a'' and b'' and which is identically 1 on $[a', b']$. We set

- $\Xi'_t = \Delta_t$ for $t \notin [a'', b'']$;
- $\Xi'_t \setminus A = \Delta_t \setminus A$ for $t \in [a'', b'']$;
- $\Xi'_t \cap A = \Omega_{\chi(t)} \cap A$ for $t \in [a'', b'']$.

The new family $\{\partial\Xi'_t\}$ is also a sweepout, obviously homotopic to $\{\partial\Delta_t\}$ and hence homotopic to $\{\partial\Xi_t\}$. We next estimate $\mathcal{H}^n(\partial\Xi'_t)$. For $t \notin [a,b]$, $\Xi'_t \equiv \Xi_t$ and hence

$$\mathcal{H}^n(\partial\Xi'_t) \; = \; \mathcal{H}^n(\partial\Xi_t) \qquad \text{for } t \notin [a,b]. \tag{3.7}$$

For $t \in [a,b]$, we anyhow have $\Xi'_t = \Xi_t$ on $M \setminus B$ and $\Xi'_t = \Delta_t$ on C. This shows the property (a) of the lemma. Moreover, for $t \in [a,b]$ we have

$$\begin{aligned} \mathcal{H}^n(\partial\Xi'_t) - \mathcal{H}^n(\partial\Xi_t) \quad &\leq \quad [\mathcal{H}^n(\partial\Delta_t \cap C) - \mathcal{H}^n(\partial\Xi_t \cap C)] \\ &\qquad + [\mathcal{H}^n(\partial\Xi'_t \cap A) - \mathcal{H}^n(\partial\Xi_t \cap A)] \\ &\overset{(3.6)}{\leq} \quad \frac{\varepsilon}{16} + [\mathcal{H}^n(\partial\Xi'_t \cap A) - \mathcal{H}^n(\partial\Xi_t \cap A)]. \end{aligned} \tag{3.8}$$

To conclude, we have to estimate the part in A in the time interval $[a,b]$. We have to consider several cases separately.

(i) Let $t \in [a, a''] \cup [b'', b]$. Then $\Xi'_t \cap A = \Delta_t \cap A = \Xi_{\gamma(t)} \cap A$. However, $\gamma(t), t \in (t_0 - \eta, t_0 + \eta)$, and, having chosen η sufficiently small, we can assume

$$|\mathcal{H}^n(\partial\Xi_s \cap A) - \mathcal{H}^n(\partial\Xi_\sigma \cap A)| \; \leq \; \frac{\varepsilon}{16} \tag{3.9}$$

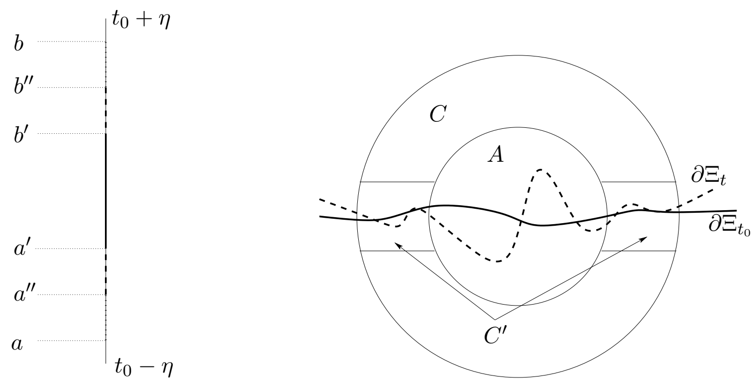

Figure 2. The left picture shows the intervals involved in the construction. If we focus on the smaller set A, then the sets Ξ'_t coincide with Δ_t and evolve from Ξ_a to Ξ_{t_0} (resp. Ξ_{t_0} to Ξ_b) in $[a, a'']$ (resp. $[b'', b]$); they then evolve from Ξ_{t_0} to Ω_1 (resp. Ω_1 to Ξ_{t_0}) in $[a'', a']$ (resp. $[b', b'']$). In the right picture, the sets in the region C. Indeed, the evolution takes place in the region C' where we patch smoothly Ξ_{t_0} with $\Xi_{\gamma(t)}$ into the sets Δ_t.

for every $\sigma, s \in (t_0-\eta, t_0+\eta)$. (Note: this choice of η is independent of a and b!) Thus, using (3.8), we get

$$\mathcal{H}^n(\partial\Xi'_t) \;\leq\; \mathcal{H}^n(\partial\Xi_t) + \frac{\varepsilon}{8}. \tag{3.10}$$

(ii) Let $t \in [a'', a'] \cup [b'', b']$. Then $\partial\Xi'_t \cap A = \partial\Omega_{\chi(t)} \cap A$. Therefore we can write, using (3.8),

$$\begin{aligned}\mathcal{H}^n(\partial\Xi'_t) - \mathcal{H}^n(\partial\Xi_t) \quad &\leq \quad \frac{\varepsilon}{16} + [\mathcal{H}^n(\partial\Xi_{t_0} \cap A) - \mathcal{H}^n(\partial\Xi_t \cap A)] \\ &\qquad + [\mathcal{H}^n(\partial\Omega_{\chi(t)} \cap A) - \mathcal{H}^n(\partial\Xi_{t_0} \cap A)] \\ &\overset{(3.9),(2.3)}{\leq} \quad \frac{\varepsilon}{16} + \frac{\varepsilon}{16} + \frac{\varepsilon}{8} \;=\; \frac{\varepsilon}{4}. \end{aligned} \tag{3.11}$$

(iii) Let $t \in [a', b']$. Then we have $\Xi'_t \cap A = \Omega_1 \cap A$. Thus, again using (3.8),

$$\begin{aligned}\mathcal{H}^n(\partial\Xi'_t) - \mathcal{H}^n(\partial\Xi_t) \quad &\leq \quad \frac{\varepsilon}{16} + [\mathcal{H}^n(\partial\Omega_1 \cap A) - \mathcal{H}^n(\partial\Xi_{t_0} \cap A)] \\ &\qquad + [\mathcal{H}^n(\partial\Xi_{t_0} \cap A) - \mathcal{H}^n(\partial\Xi_t \cap A)] \\ &\overset{(2.4),(3.9)}{\leq} \quad \frac{\varepsilon}{16} - \varepsilon + \frac{\varepsilon}{16} \;<\; -\frac{\varepsilon}{2}. \end{aligned} \tag{3.12}$$

Gathering the estimates (3.7), (3.10), (3.11), and (3.12), we finally obtain the properties (b) and (c) of the lemma. This finishes the proof.

4. The existence of replacements

In this section we fix $An \in \mathcal{AN}_{r(x)}(x)$ and we prove the conclusion of Proposition 2.6.

4.1. Setting. For every j, consider the class $\mathcal{H}(\Omega^j, An)$ of sets Ξ such that there is a family $\{\Omega_t\}$ satisfying $\Omega_0 = \Omega^j$, $\Omega_1 = \Xi$, (2.1), (2.2), and (2.3) for $\varepsilon = \frac{1}{j}$ and $U = An$. Consider next a sequence $\Gamma^{j,k} = \partial\Omega^{j,k}$ which is minimizing for the perimeter in the class $\mathcal{H}(\Omega^j, An)$: this is the minimizing sequence for the $(8j)^{-1}$-*homotopic Plateau problem* mentioned in Subsection 2.3. Up to subsequences, we can assume that

- $\Omega^{j,k}$ converges to a Caccioppoli set $\tilde{\Omega}^j$;
- $\Gamma^{j,k}$ converges to a varifold V^j;
- V^j (and a suitable diagonal sequence $\tilde{\Gamma}^j = \Gamma^{j,k(j)}$) converges to a varifold $\tilde{V}$.

The proof of Proposition 2.6 will then be broken into three steps. In the first one we show

Lemma 4.1. *For every j and every $y \in An$ there is a ball $B = B_\rho(y) \subset An$ and a $k_0 \in \mathbb{N}$ with the following property. Every open set Ξ such that*

- *$\partial\Xi$ is smooth except for a finite set,*
- *$\Xi \setminus B = \Omega^{j,k} \setminus B$,*
- *and $\mathcal{H}^n(\partial\Xi) < \mathcal{H}^n(\partial\Omega^{j,k})$*

belongs to $\mathcal{H}(\Omega^j, An)$ if $k \geq k_0$.

In the second step we use Lemma 4.1 and Theorem 1.2 to show:

Lemma 4.2. *$\partial\tilde{\Omega}^j \cap An$ is a stable minimal hypersurface in An and $V^j \llcorner An = \partial\tilde{\Omega}^j \llcorner An$.*

Recall that in this section we use the convention of Definition 1.5. In the third step we use Lemma 4.2 to conclude that the sequence $\tilde{\Gamma}^j$ and the varifold $\tilde{V}$ meet the requirements of Proposition 2.6.

4.2. Proof of Lemma 4.1. The proof of the lemma is achieved by exhibiting a suitable homotopy between $\Omega^{j,k}$ and Ξ. The key idea is:

- First deform $\Omega^{j,k}$ to the set $\tilde{\Omega}$ which is the union of $\Omega^{j,k} \setminus B$ and the cone with vertex y and base $\Omega^{j,k} \cap \partial B$;
- Then deform $\tilde{\Omega}$ to Ξ.

The surfaces of the homotopizing family do not gain too much in area, provided $B = B_\rho(y)$ is sufficiently small and k sufficiently large: in this case the area of the surface $\Gamma^{j,k} \cap B$ will, in fact, be close to the area of the cone. This "blow down–blow up" procedure is an idea which we borrow from [**21**] (see section 7 of [**6**]).

Proof of Lemma 4.1. We fix $y \in An$ and $j \in \mathbb{N}$. Let $B = B_\rho(y)$ with $B_{2\rho}(y) \subset An$ and consider an open set Ξ as in the statement of the Lemma. The choice of the radius of the ball $B_\rho(y)$ and of the constant k_0 (which are both independent of the set Ξ) will be determined at the very end of the proof.

Step 1: Stretching $\Gamma^{j,k} \cap \partial B_r(y)$. First of all, we choose $r \in (\rho, 2\rho)$ such that, for every k,

$$\text{(4.1)} \qquad \Gamma^{j,k} \text{ is regular in a neighborhood of } \partial B_r(y) \text{ and intersects it transversally.}$$

In fact, since each $\Gamma^{j,k}$ has finitely many singularities, Sard's lemma implies that (4.1) is satisfied by a.e. r. We assume moreover that 2ρ is smaller than the injectivity radius. For each $z \in \overline{B}_r(y)$ we consider the closed geodesic arc $[y, z] \subset \overline{B}_r(y)$ joining y and z. As usual, (y, z) denotes $[y, z] \setminus \{y, z\}$. We let K be the open cone consisting

$$K = \bigcup_{z \in \partial B \cap \Omega^{j,k}} (y, z)\,. \tag{4.2}$$

We now show that $\Omega^{j,k}$ can be homotopized through a family $\tilde{\Omega}_t$ to a $\tilde{\Omega}_1$ in such a way that

- $\max_t \mathcal{H}^n(\partial\tilde{\Omega}_t) - \mathcal{H}^n(\partial\Omega^{j,k})$ can be made arbitrarily small;
- $\tilde{\Omega}_1$ coincides with K in a neighborhood of $\partial B_r(y)$.

First of all, consider a smooth function $\varphi : [0, 2\rho] \to [0, 2\rho]$, with

- $|\varphi(s) - s| \leq \varepsilon$ and $0 \leq \varphi' \leq 2$;
- $\varphi(s) = s$ if $|s - r| > \varepsilon$ and $\varphi \equiv r$ in a neighborhood of r.

Set $\Phi(t, s) := (1 - t)s + t\varphi(s)$. Moreover, for every $\lambda \in [0, 1]$ and every $z \in \overline{B}_r(y)$ let $\tau_\lambda(z)$ be the point $w \in [y, z]$ with $\operatorname{dist}(y, w) = \lambda \operatorname{dist}(y, z)$. For $1 < \lambda < 2$, we can still define $\tau_\lambda(z)$ to be the corresponding point on the geodesic that is the extension of $[y, z]$. (Note that by the choice of ρ this is well defined.) We are now ready to define $\tilde{\Omega}_t$ (compare with Figure 3, left):

- $\tilde{\Omega}_t \setminus An(y, r - \varepsilon, r + \varepsilon) = \Omega^{j,k} \setminus An(y, r - \varepsilon, r + \varepsilon)$;
- $\tilde{\Omega}_t \cap \partial B_s(y) = \tau_{s/\Phi(t,s)}(\Omega^{j,k} \cap \partial B_{\Phi(t,s)})$ for every $s \in (r - \varepsilon, r + \varepsilon)$.

Thanks to (4.1), for ε sufficiently small $\tilde{\Omega}_t$ has the desired properties. Moreover, since Ξ coincides with $\Omega^{j,k}$ on $M \setminus B_\rho(y)$, the same argument can be applied to Ξ. This shows that

$$\text{(4.3)} \qquad \text{w.l.o.g. we can assume } K = \Xi = \Omega^{k,j} \text{ in a neighborhood of } \partial B_r(y).$$

Step 2: The homotopy We then consider the following family of open sets $\{\Omega_t\}_{t\in[0,1]}$ (compare with Figure 3, right):

- $\Omega_t \setminus \overline{B}_r(y) = \Omega^{j,k} \setminus \overline{B}_r(y)$ for every t;

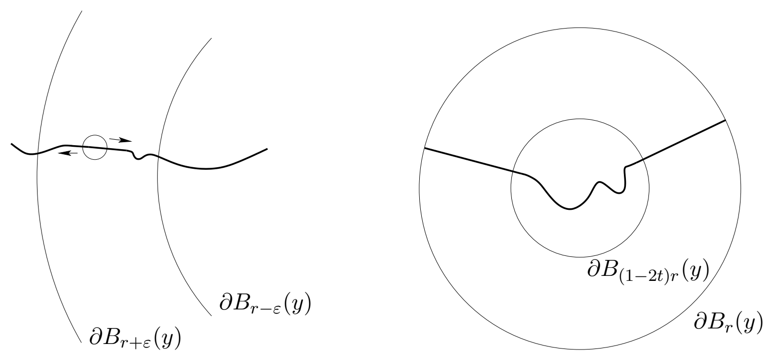

Figure 3. The left picture illustrates the stretching of $\Gamma^{j,k}$ into a cone-like surface in a neighborhood of $\partial B_r(y)$. The right picture shows a slice $\Omega_t \cap B_r(y)$ for $t \in (0, 1/2)$.

- $\Omega_t \cap An(y, |1-2t|r, r) = K \cap An(y, |1-2t|r, r)$ for every t;
- $\Omega_t \cap \overline{B}_{(1-2t)r}(y) = \tau_{1-2t}(\Omega^{k,j} \cap \overline{B}_r(y))$ for $t \in [0, \frac{1}{2}]$;
- $\Omega_t \cap \overline{B}_{(2t-1)r}(y) = \tau_{2t-1}(\Xi \cap \overline{B}_r(y))$ for $t \in [\frac{1}{2}, 1]$.

Because of (4.3), this family satisfies (s1)–(s3), (sw1), and (sw3). It remains to check,

$$\max_t \mathcal{H}^n(\partial\Omega_t) \;\le\; \mathcal{H}^n(\partial\Omega^{j,k}) + \frac{1}{8j} \qquad \forall k \ge k_0 \tag{4.4}$$

for a suitable choice of ρ, r and k_0.

First of all, we observe that, by the smoothness of M, there are constants μ and ρ_0, depending only on the metric, such that the following holds for every $r < 2\rho < 2\rho_0$ and $\lambda \in [0,1]$:

$$\mathcal{H}^n(K) \;\le\; \mu r \mathcal{H}^{n-1}(\partial\Omega^{j,k} \cap \partial B_r(y)) \tag{4.5}$$

$$\mathcal{H}^n([\partial(\tau_\lambda(\Omega^{j,k} \cap \overline{B}_r(y)))] \cap B_{\lambda r}(y)) \;\le\; \mu \mathcal{H}^n(\partial\Omega^{j,k} \cap B_r(y)) \tag{4.6}$$

$$\mathcal{H}^n([\partial(\tau_\lambda(\Xi \cap \overline{B}_r(y)))] \cap B_{\lambda r}(y)) \;\le\; \mu \mathcal{H}^n(\partial\Xi \cap B_r(y)) \tag{4.7}$$

$$\int_0^{2\rho} \mathcal{H}^{n-1}(\partial\Omega^{j,k} \cap \partial B_\tau(y))\, d\tau \;\le\; \mu \mathcal{H}^n(\partial\Omega^{j,k} \cap B_{2\rho}(y))\,. \tag{4.8}$$

In fact, for ρ small, μ will be close to 1. (4.5), (4.6) and (4.7) give the obvious estimate

$$\begin{aligned} \max_t \mathcal{H}^n(\partial\Omega_t) - \mathcal{H}^n(\partial\Omega^{j,k}) \;\;&\le\;\; \mu \mathcal{H}^n(\partial\Omega^{j,k} \cap B_{2\rho}(y)) \\ &\quad + \mu r \mathcal{H}^{n-1}(\partial\Omega^{j,k} \cap \partial B_r(y))\,. \end{aligned} \tag{4.9}$$

Moreover, by (4.8) we can find $r \in (\rho, 2\rho)$ which, in addition to (4.9), satisfies

$$\mathcal{H}^{n-1}(\partial\Omega^{j,k} \cap \partial B_r(y)) \;\le\; \frac{2\mu}{\rho} \mathcal{H}^n(\partial\Omega^{j,k} \cap B_{2\rho}(y))\,. \tag{4.10}$$

Hence, we conclude

$$\max_t \mathcal{H}^n(\partial\Omega_t) \;\leq\; \mathcal{H}^n(\partial\Omega^{j,k}) + (\mu+4\mu^2)\mathcal{H}^n(\partial\Omega^{j,k}\cap B_{2\rho}(y))\,. \tag{4.11}$$

Next, by the convergence of $\Gamma^{j,k} = \partial\Omega^{j,k}$ to the stationary varifold V^j, we can choose k_0 such that

$$\mathcal{H}^n(\partial\Omega^{j,k}\cap B_{2\rho}(y)) \;\leq\; 2\|V^j\|(B_{4\rho}(y)) \qquad \text{for } k\geq k_0. \tag{4.12}$$

Finally, by the monotonicity formula,

$$\|V^j\|(B_{4\rho}(y)) \leq C_M\|V^j\|(M)\rho^n\,. \tag{4.13}$$

We are hence ready to specify the choice of the various parameters:

- We first determine the constants μ and $\rho_0 < \mathrm{Inj}\,(M)$ (which depend only on M) which guarantee (4.5), (4.6), (4.7), and (4.8);
- We subsequently choose $\rho < \rho_0$ so small that
$$2(\mu+4\mu^2)C_M\|V^j\|(M)\rho^n < (8j)^{-1},$$
and k_0 so that (4.12) holds.

At this point ρ and k are fixed, and, choosing $r\in(\rho,2\rho)$ satisfying (4.1) and (4.10), we construct $\{\partial\Omega_t\}$ as above, concluding the proof of the lemma.

4.3. Proof of Lemma 4.2. Fix $j\in\mathbb{N}$ and $y\in An$, and let $B = B_\rho(y)\subset An$ be the ball given by Lemma 4.1. We claim that $\tilde\Omega^j$ minimizes the perimeter in the class $\mathcal{P}(\tilde\Omega^j, B_{\rho/2}(y))$. Assume, by contradiction, that Ξ is a Caccioppoli set with $\Xi\setminus B_{\rho/2}(y) = \tilde\Omega^j\setminus B_{\rho/2}(y)$ and

$$\mathrm{Per}\,(\Xi) \;<\; \mathrm{Per}\,(\tilde\Omega^j) - \eta\,. \tag{4.14}$$

Note that, since $\mathbf{1}_{\Omega^{j,k}}\to\mathbf{1}_{\tilde\Omega^j}$ strongly in L^1, up to extraction of a subsequence we can assume the existence of $\tau\in(\rho/2,\rho)$ such that

$$\lim_{k\to\infty}\|\mathbf{1}_{\tilde\Omega^j}-\mathbf{1}_{\Omega^{j,k}}\|_{L^1(\partial B_\tau(y))} \;=\; 0\,. \tag{4.15}$$

We also recall that, by the semicontinuity of the perimeter,

$$\mathrm{Per}\,(\tilde\Omega^j) \;\leq\; \liminf_{k\to\infty}\mathcal{H}^n(\partial\Omega^{j,k})\,. \tag{4.16}$$

Define therefore the set $\Xi^{j,k}$ by setting

$$\Xi^{j,k} \;=\; (\Xi\cap B_\tau(y))\cup(\Omega^{j,k}\setminus B_\tau(y))\,.$$

(4.14), (4.15) and (4.16) imply

$$\limsup_{k\to\infty}[\mathrm{Per}\,(\Xi^{j,k}) - \mathcal{H}^n(\partial\Omega^{j,k})] \;\leq\; -\eta\,. \tag{4.17}$$

Fix next k and recall the following standard way of approximating $\Xi^{j,k}$ with a smooth set. We first fix a compactly supported convolution kernel φ, then consider the function $g_\varepsilon := \mathbf{1}_{\Xi^{j,k}}*\varphi_\varepsilon$, and finally look at a smooth level set $\Delta_\varepsilon := \{g_\varepsilon > t\}$ for some $t\in(\frac14,\frac34)$. Then $\mathcal{H}^n(\partial\Delta_\varepsilon)$ converges to Per $(\Xi^{j,k})$ as $\varepsilon \to 0$ (see [**12**] in the euclidean case and [**15**] for the general one).

Clearly, Δ_ε does not coincide anymore with $\Omega^{j,k}$ outside $B_\rho(y)$. Thus, fix $(a,b) \subset (\tau,\rho)$ with the property that $\Sigma := \Omega^{j,k} \cap \overline{B}_b(y) \setminus B_a(y)$ is smooth. Fix a regular tubular neighborhood T of Σ and corresponding normal coordinates (ξ,σ) on it. Since $\Xi^{j,k} \setminus B_\tau(y) = \Omega^{j,k} \setminus B_\tau(y)$, for ε sufficiently small $\partial\Delta_\varepsilon \cap \overline{B}_b(y) \setminus B_a(y) \subset T$ and $T \cap \Delta_\varepsilon$ is the set $\{\sigma < f_\varepsilon(\xi)\}$ for some smooth function f_ε. Moreover, as $\varepsilon \to 0$, $f_\varepsilon \to 0$ smoothly.

Therefore, a patching argument entirely analogous to the one of the freezing construction (see Subsection 3.2) allows us to modify $\Xi^{j,k}$ to a set $\Delta^{j,k}$ with the following properties:

- $\partial\Delta^{j,k}$ is smooth outside of a finite set;
- $\Delta^{j,k} \setminus B = \Omega^{j,k} \setminus B$;
- $\limsup_k(\mathcal{H}^n(\partial\Delta^{j,k}) - \mathcal{H}^n(\partial\Omega^{j,k})) \leq -\eta < 0$.

For k large enough, Lemma 4.1 implies that $\Xi^{j,k} \in \mathcal{H}(\Omega^j, An)$, which would contradict the minimality of the sequence $\Omega^{j,k}$.

Next, in order to show that the varifold V^j is induced by $\partial\tilde{\Omega}^j$, it suffices to show that in fact $\mathcal{H}^n(\partial\Omega^{j,k})$ converges to $\mathcal{H}^n(\partial\tilde{\Omega}^j)$ (since we have not been able to find a precise reference for this well-known fact, we give a proof in the appendix; compare with Proposition A.1). On the other hand, if this is not the case, then we have

$$\mathcal{H}^n(\partial\tilde{\Omega}^j \cap B_{\rho/2}(y)) \ < \ \limsup_{k\to\infty} \mathcal{H}^n(\partial\Omega^{j,k} \cap B_{\rho/2}(y))$$

for some $y \in An$ and some ρ to which we can apply the conclusion of Lemma 4.1. We can then use $\tilde{\Omega}^j$ in place of Ξ in the argument of the previous step to contradict, once again, the minimality of the sequence $\{\Omega^{j,k}\}_k$. The stationarity and stability of the surface $\partial\tilde{\Omega}^j$ is, finally, an obvious consequence of the variational principle.

4.4. Proof of proposition 2.6. Consider the varifolds V^j and the diagonal sequence $\tilde{\Gamma}^j = \Gamma^{j,k(j)}$ of Section 4.1. Observe that $\tilde{\Gamma}^j$ is obtained from Γ^j through a suitable homotopy which leaves everything fixed outside An. Consider $An(x,\varepsilon,r(x)-\varepsilon)$ containing An. It follows from the a.m. property of $\{\Gamma^j\}$ that $\{\tilde{\Gamma}^j\}$ is also a.m. in $An(x,\varepsilon,r(x)-\varepsilon)$.

Note next that if a sequence is a.m. in an open set U and U' is a second open set contained in U, then the sequence is a.m. in U' as well. This trivial observation and the discussion above implies that $\tilde{\Gamma}^j$ is a.m. in any $An \in \mathcal{AN}_{r(x)}(x)$.

Fix now an annulus $An' = An(x,\varepsilon,r(x)-\varepsilon) \supset\supset An$. Then $M = An' \cup (M \setminus An)$. For any $y \in M \setminus An$ (and $y \neq x$), consider $r'(y) := \min\{r(y), \mathrm{dist}(y,An)\}$. If $An'' \in \mathcal{AN}_{r'(y)}(y)$, then $\Gamma^j \cap An'' = \tilde{\Gamma}^j \cap An''$, and hence $\{\tilde{\Gamma}^j\}$ is a.m. in An''. If $y \in An'$, then we can set $r'(y) =$

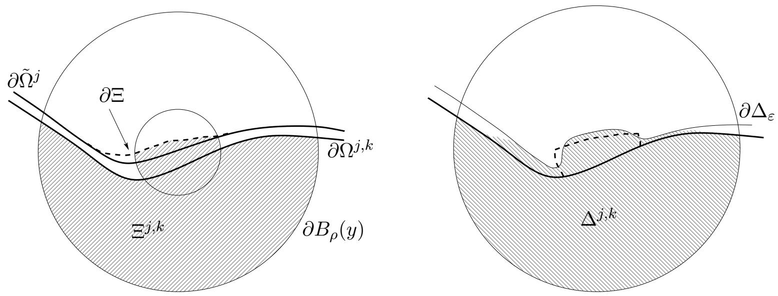

Figure 4. On the left, the set $\tilde{\Omega}^j$, the competitor Ξ, one set of the sequence $\{\Omega^{j,k}\}_k$, and the corresponding $\Xi^{j,k}$. On the right, the smoothing Δ_ε of $\Xi^{j,k}$ and the final set $\Delta^{j,k}$ (a competitor for $\Omega^{j,k}$).

$\min\{r(y), \mathrm{dist}(y, \partial An')\}$. If $An'' \in \mathcal{AN}_{r'(y)}(y)$, then $An'' \subset An'$, and, since $\{\tilde{\Gamma}^j\}$ is a.m. in An' by the argument above, $\{\tilde{\Gamma}^j\}$ is a.m. in An''.

We next show that $\tilde{V}$ is a replacement for V in An. By Theorem 1.3, $\tilde{V}$ is a stable minimal hypersurface in An. It remains to show that $\tilde{V}$ is stationary. $\tilde{V}$ is obviously stationary in $M \setminus An$, because it coincides with V there. Next, let $An' \supset\supset An$. Since $\{An', M \setminus An\}$ is a covering of M, we can subordinate a partition of unity $\{\varphi_1, \varphi_2\}$ to it. By the linearity of the first variation, we get $[\delta\tilde{V}](\chi) = [\delta\tilde{V}](\varphi_1\chi) + [\delta\tilde{V}](\varphi_2\chi) = [\delta\tilde{V}](\varphi_1\chi)$. Therefore it suffices to show that $\tilde{V}$ is stationary in An'. Assume, by contradiction, that there is $\chi \in \mathcal{X}_c(An')$ such that $[\delta\tilde{V}](\chi) \le -C < 0$ and denote by ψ the isotopy defined by $\frac{\partial\psi(x,t)}{\partial t} = \chi(\psi(x,t))$. We set

$$\tilde{V}(t) := \psi(t)_\sharp \tilde{V} \qquad \Sigma^j(t) = \psi(t, \tilde{\Gamma}^j). \tag{4.18}$$

By continuity of the first variation there is $\varepsilon > 0$ such that $\delta\tilde{V}(t)(\chi) \le -C/2$ for all $t \le \varepsilon$. Moreover, since $\Sigma^j(t) \to \tilde{V}(t)$ in the sense of varifolds, there is J such that

$$[\delta\Sigma^j(t)](\chi) \le -\frac{C}{4} \quad \text{for } j > J \text{ and } t \le \varepsilon. \tag{4.19}$$

Integrating (4.19), we conclude $\mathcal{H}^n(\Sigma^j(t)) \le \mathcal{H}^n(\tilde{\Gamma}^j) - Ct/8$ for every $t \in [0, \varepsilon]$ and $j \ge J$. This contradicts the a.m. property of $\tilde{\Gamma}^j$ in An', for j large enough.

Finally, observe that $\mathcal{H}^n(\tilde{\Gamma}^j) \le \mathcal{H}^n(\Gamma^j)$ by construction and

$$\liminf_n (\mathcal{H}^n(\tilde{\Gamma}^j) - \mathcal{H}^n(\Gamma^j)) \ge 0,$$

because otherwise we would contradict the a.m. property of $\{\Gamma^j\}$ in An. We thus conclude that $\|V\|(M) = \|\tilde{V}\|(M)$. q.e.d.

5. The regularity of varifolds with replacements

In this section we prove Proposition 2.8. We recall that we adopt the convention of Definition 1.5. We first list several technical facts from geometric measure theory.

5.1. Maximum principle. The first one is just a version of the classical maximum principle.

Theorem 5.1. *(i) Let V be a stationary varifold in a ball $\mathcal{B}_r(0) \subset \mathrm{R}^{n+1}$. If $\mathrm{supp}\,(V) \subset \{z_{n+1} \geq 0\}$ and $\mathrm{supp}\,(V) \cap \{z_{n+1} = 0\} \neq \emptyset$, then $\mathcal{B}_r(0) \cap \{z_{n+1} = 0\} \subset \mathrm{supp}\,(V)$.*

(ii) Let W be a stationary varifold in an open set $U \subset M$ and K be a smooth strictly convex closed set. If $x \in \mathrm{supp}\,(V) \cap \partial K$, then $\mathrm{supp}\,(V) \cap B_r(x) \setminus K \neq \emptyset$ for every positive r.

For (ii) we refer, for instance, to appendix B of [**6**], whereas (i) is a very special case of the general result of [**22**].

5.2. Tangent cones. The second device is a fundamental tool of geometric measure theory. Consider a stationary varifold $V \in \mathcal{V}(U)$ with $U \subset M$ and fix a point $x \in \mathrm{supp}\,(V) \cap U$. For any $r < \mathrm{Inj}\,(M)$ consider the rescaled exponential map $T^x_r : \mathcal{B}_1 \ni z \mapsto \exp_x(rz) \in B_r(x)$, where $\exp_x$ denotes the exponential map with base point x. We then denote by $V_{x,r}$ the varifold $(T^x_r)^{-1}_\sharp V \in \mathcal{V}(\mathcal{B}_1)$. Then, as a consequence of the monotonicity formula, one concludes that for any sequence $\{V_{x,r_n}\}$ there exists a subsequence converging to a stationary varifold V^* (stationary for the euclidean metric!), which in addition is a cone (see corollary 42.6 of [**20**]). Any such cone is called *tangent cone to V in x*. For varifolds with the replacement property, the following is a fundamental step towards the regularity (first proved by Pitts for $n \leq 5$ in [**16**]).

Lemma 5.2. *Let V be a stationary varifold in an open set $U \subset M$ having a replacement in any annulus $An \in \mathcal{AN}_{r(x)}(x)$ for some positive function r. Then*

- *V is integer rectifiable;*
- *$\theta(x, V) \geq 1$ for any $x \in U$;*
- *any tangent cone C to V at x is a minimal hypersurface for general n and (a multiple of) a hyperplane for $n \leq 6$.*

Proof. First of all, by the monotonicity formula there is a constant C_M such that

$$\frac{\|V\|(B_\sigma(x))}{\sigma^n} \leq C_M \frac{\|V\|(B_\rho(x))}{\rho^n} \tag{5.1}$$

for all $x \in M$ and all $0 < \sigma \leq \rho < \mathrm{Inj}\,(M)$. Fix $x \in \mathrm{supp}\,(\|V\|)$ and $0 < r < \min\{r(x)/2, \mathrm{Inj}\,(M)/4\}$. Next, we replace V with V' in the annulus $An(x, r, 2r)$. We observe that $\|V'\| \not\equiv 0$ on $An(x, r, 2r)$;

otherwise there would be $\rho \leq r$ and ε such that supp $(\|V'\|) \cap \partial B_\rho(x) \neq \emptyset$ and supp $(\|V'\|) \cap \mathcal{AN}(x, \rho, \rho+\varepsilon) = \emptyset$. By the choice of ρ, this would contradict Theorem 5.1(ii).

Thus we have found that $V' \llcorner An(x, r, 2r)$ is a non-empty stable minimal hypersurface and hence there is $y \in An(x, r, 2r)$ with $\theta(y, V') \geq 1$. By (5.1),

$$\begin{aligned} \frac{\|V\|(B_{4r}(x))}{(4r)^n} &= \frac{\|V'\|(B_{4r}(x))}{(4r)^n} \geq \frac{\|V'\|(B_{2r}(y))}{(4r)^n} \\ &\geq \frac{\omega_n}{2^n C_M}\, \theta(y, V') \geq \frac{\omega_n}{2^n C_M}. \end{aligned} \tag{5.2}$$

Hence, $\theta(x, V)$ is uniformly bounded away from 0 on supp $(\|V\|)$ and Allard's rectifiability theorem (see theorem 42.4 of [**20**]) gives that V is rectifiable.

Let C denote a tangent cone to V at x and let $\rho_k \to 0$ a sequence with $V^x_{\rho_k} \to C$. Note that C is stationary. We replace V by V'_k in $An(x, \lambda\rho_k, (1-\lambda)\rho_k)$, where $\lambda \in (0, 1/4)$ and set $W'_k = (T^x_{\rho_k})_\sharp V'_k$. Up to subsequences we have $W'_k \to C'$ for some stationary varifold C'. By the definition of a replacement we obtain

$$C' = C \quad \text{in } \mathcal{B}_\lambda \cup An(0, 1-\lambda, 1), \tag{5.3}$$

$$\|C'\|(\mathcal{B}_\rho) = \|C\|(\mathcal{B}_\rho) \quad \text{for } \rho \in (0, \lambda) \cup (1-\lambda, 1). \tag{5.4}$$

Moreover, since C is a cone,

$$\frac{\|C'\|(\mathcal{B}_\sigma)}{\sigma^n} = \frac{\|C'\|(\mathcal{B}_\rho)}{\rho^n} \quad \text{for all } \rho, \sigma \in (0, \lambda) \cup (1-\lambda, 1). \tag{5.5}$$

By the monotonicity formula for stationary varifolds in euclidean spaces, (5.5) implies that C' as well is a cone (see for instance 17.5 of [**20**]). Moreover, by the Compactness theorem 1.3, $C' \llcorner An(0, \lambda, 1-\lambda)$ is a stable embedded minimal hypersurface. Since C and C' are integer rectifiable, the conical structure of C implies that supp (C) and supp (C') are closed cones (in the usual meaning for sets) and the densities $\theta(\cdot, C)$ and $\theta(\cdot, C')$ are 0-homogeneous functions (see theorem 19.3 of [**20**]). Thus (5.3) implies $C = C'$ and hence that C is a stable minimal hypersurface in $An(0, \lambda, 1-\lambda)$. Since λ is arbitrary, C is a stable minimal hypersurface in the punctured ball. Thus, if $n \leq 6$, by Simons' theorem (see Theorem B.2 in [**20**]) C is in fact a multiple of a hyperplane. If instead $n \geq 7$, since $\{0\}$ has dimension $0 \leq n-7$, C is a minimal hypersurface in the whole ball $\mathcal{B}_1$ (recall Definition 1.5). q.e.d.

5.3. Unique continuation and two technical lemmas on varifolds. To conclude the proof we need yet three auxiliary results. All of them are justified in Appendix A. The first one is a consequence of the classical unique continuation for minimal surfaces.

Theorem 5.3. *Let U be a smooth open subset of M and let $\Sigma_1, \Sigma_2 \subset U$ be two connected* smooth *embedded minimal hypersurfaces with $\partial\Sigma_i \subset \partial U$. If Σ_1 coincides with Σ_2 in some open subset of U, then $\Sigma_1 = \Sigma_2$.*

The other two are elementary lemmas for stationary varifolds.

Lemma 5.4. *Let $r < Inj(M)$ and let V be a stationary varifold. Then*

$$\operatorname{supp}(V) \cap \overline{B}_r(x) \;=\; \overline{\bigcup_{0<s<r} \operatorname{supp}(V \llcorner B_s(x)) \cap \partial B_s(x)}. \tag{5.6}$$

Lemma 5.5. *Let $\Gamma \subset U$ be a relatively closed set of dimension n and S a closed set of dimension at most $n-2$ such that $\Gamma \setminus S$ is a smooth embedded hypersurface. Assume Γ induces a varifold V which is stationary in U. If Δ is a connected component of $\Gamma \setminus S$, then Δ induces a stationary varifold.*

5.4. Proof of Proposition 2.8. The proof consists of five steps.

Step 1: Set up. Let $x \in M$ and $\rho \le \min\{r(x)/2, Inj(M)/2\}$. Then we choose a replacement V' for V in $An(x,\rho,2\rho)$ coinciding with a stable minimal embedded hypersurface Γ'. Next, choose $s \in (0,\rho)$ and $t \in (\rho, 2\rho)$ such that $\partial B_t(x)$ intersects Γ' transversally. Then we pick a second replacement V'' of V' in $An(x,s,t)$, coinciding with a stable minimal embedded hypersurface Γ'' in the annulus $An(x,s,t)$. Now we fix a point $y \in \partial B_t(x) \cap \Gamma'$ that is a regular point of Γ' and a radius $r > 0$ sufficiently small such that $\Gamma' \cap B_r(y)$ is topologically an n-dimensional ball in M and $\gamma = \Gamma' \cap \partial B_t(x) \cap B_r(y)$ is a smooth $(n-1)$-dimensional surface. This can be done due to our regularity assumption on y. Then we choose a diffeomorphism $\zeta : B_r(y) \to \mathcal{B}_1$ such that

$$\zeta(\partial B_t(x)) \subset \{z_1 = 0\} \quad \text{and} \quad \zeta(\Gamma'') \subset \{z_1 > 0\},$$

where $z_1, \ldots, z_{n+1}$ are orthonormal coordinates in $\mathcal{B}_1$. Finally, suppose

$$\begin{aligned} \zeta(\gamma) &= \{(0, z_2, \ldots, z_n, g'((0, z_2, \ldots, z_n))\}, \\ \zeta(\Gamma') \cap \{z_1 \le 0\} &= \{(z_1, \ldots, z_n, g'((z_1, \ldots, z_n))\} \end{aligned}$$

for some smooth function g'. Note that

- any kind of estimates (like curvature estimates or area bound or monotonicity) for a minimal surface $\Gamma \subset B_r(y)$ translates into similar estimates for the surface $\zeta(\Gamma)$;
- varifolds in $B_r(y)$ are pushed forward to varifolds in $\mathcal{B}_1$ and there is a natural correspondence between tangent cones to V in ξ and tangent cones to $\zeta_\sharp V$ in $\zeta(\xi)$.

We will use the same notation for the objects in $B_r(y)$ and their images under ζ.

Step 2: Tangent cones. We next claim that any tangent cone to V'' at any point $w \in \gamma$ is a unique flat space. Note that all these w

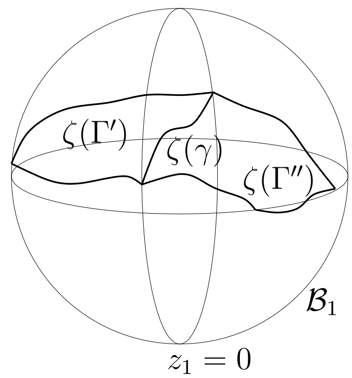

Figure 5. The surfaces Γ', Γ'', and γ in the coordinates z.

are regular points of Γ'. Therefore by our transversality assumption every tangent cone C at w coincides in $\{z_1 < 0\}$ with the half space $T_w\Gamma' \cap \{z_1 < 0\}$. We wish to show that C coincides with $T_w\Gamma'$. By the constancy theorem (see theorem 41.1 in [**20**]), it suffices to show $\operatorname{supp}(C) \subset T_w\Gamma'$.

Note first that if $z \in T_w\Gamma' \cap \{z_1 = 0\}$ is a regular point for C, then by Theorem 5.3, C coincides with $T_w\Gamma'$ in a neighborhood of z. Therefore, if $z \in \operatorname{supp}(C) \cap \{z_1 = 0\}$, either z is a singular point, or $C = T_w\Gamma'$ in a neighborhood of z. Assume now by contradiction that $p \in \operatorname{supp}(C) \setminus T_w\Gamma'$. Since, by Lemma 5.2 and the fact that Γ'' has replacements due to Proposition 2.7, $\operatorname{Sing} C$ has dimension at most $n-7$, and we can assume that p is a regular point of C. Consider next a sequence N^j of smooth open neighborhoods of $\operatorname{Sing} C$ such that $T_w\Gamma' \setminus \overline{N}^j$ is connected and $N^j \to \operatorname{Sing} C$. Let Δ^j be the connected component of $C \setminus \overline{N}^j$ containing p. Then Δ^j is a smooth minimal surface with $\partial\Delta^j \subset \partial N^j$. We conclude that Δ^j cannot touch $\{z_1 = 0\}$: it would touch it in a regular point of $\operatorname{supp}(C) \cap \{z_1 = 0\}$ and hence it would coincide with $T_w\Gamma' \setminus \overline{N}^j$, which is impossible because it contains p. If we let $\Delta = \cup \Delta^j$, then Δ is a connected component of the regular part of C, which does not intersect $\{z_1 = 0\}$. Let W be the varifold induced by Δ: by Lemma 5.5 W is stationary. Since C is a cone, W is also a cone. Thus $\operatorname{supp}(W) \ni 0$. On the other hand, $\operatorname{supp}(W) \subset \{z_1 \geq 0\}$. Thus, by Theorem 5.1(i), $\{z_1 = 0\} \subset \operatorname{supp}(W)$. But this would imply that $\{z_1 = 0\} \cap T_w\Gamma'$ is in the singular set of C: this is a contradiction because the dimension of $\{z_1 = 0\} \cap T_w\Gamma'$ is $n-1$.

Step 3: Graphicality. In this step we show that the surfaces Γ' and Γ'' can be "glued" together at $\partial B_t(x)$; that is,

$$\Gamma'' \subset \Gamma' \text{ in } B_t(x) \setminus B_{t-\varepsilon}(x) \text{ for some } \varepsilon > 0. \tag{5.7}$$

For this we fix $z \in \gamma$ and, using the notation of Step 2, consider the (exterior) unit normal $\tau(z)$ to the graph of g'. Let $T^z_r \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the dilation of the $(n+1)$-space given by

$$T^z_r(\bar{z}) = \frac{\bar{z} - z}{r}.$$

By Step 2 we know that any tangent cone to V'' at z is given by the tangent space $T_z\Gamma'$, and therefore the rescaled surfaces $\Gamma_r = T^z_r(\Gamma'')$ converge to the half space $H = \{v \colon \tau(z) \cdot v = 0, v_1 > 0\}$. We claim that this implies that we have

$$\lim_{\bar{z} \to z, \bar{z} \in \Gamma''} \frac{|(\bar{z} - z) \cdot \tau(z)|}{|\bar{z} - z|} = 0 \tag{5.8}$$

uniformly on compact subsets of γ. We argue by contradiction and assume the claim is wrong. Then there is a sequence $\{z_j\} \subset \Gamma''$ with $z_j \to z$ and $|(z_j - z) \cdot \tau(z)| \geq k|z_j - z|$ for some $k > 0$. We can assume that z_j is a regular point of Γ'' for all $j \in \mathbb{N}$. We set $r_j = |z_j - z|$; then there is a positive constant $\bar{k}$ such that $\mathcal{B}_{2\bar{k}r_j}(z_j) \cap H = \emptyset$. This implies that $\mathrm{dist}(H, \mathcal{B}_{\bar{k}r_j}(z_j)) \geq \bar{k}r_j$. By the minimality of Γ'' we can apply the monotonicity formula and find

$$\|V''\|(\mathcal{B}_{\bar{k}r_j}(z_j)) \geq C\bar{k}^n r_j^n$$

for some positive constant C depending on the diffeomorphism ζ. In other words, there is a considerable amount of the varifold that is far from the half space H. But this contradicts the fact that the corresponding full space is the only tangent cone. We also point out that this convergence is uniform on compact subsets of γ.

Now we denote by ν the smooth normal field to Γ'' with $\nu \cdot (0, \ldots, 0, 1) \geq 0$. Let Σ be the space $\{(0, \alpha_1, \ldots, \alpha_n) : \alpha_i \in \mathbb{R}\}$. Then we assume that $z_j \to z$, set $r_j = \mathrm{dist}(z_j, \Sigma)$ and define the rescaled hypersurfaces $\Gamma_j = T^{z_j}_{r_j}(\Gamma'' \cap \mathcal{B}_{r_j}(z_j))$. Then all the Γ_j are smooth stable minimal surfaces in $\mathcal{B}_1$, and thus we can apply Theorem 1.3 to extract a subsequence that converges to a stable minimal hypersurface in the ball $\mathcal{B}_{1/2}$. But by (5.8) we know that this limit surface is simply $T_z\Gamma' \cap \mathcal{B}_{1/2}$. Since the convergence is in the C^1 topology, we have

$$\lim_{\bar{z} \to z, \bar{z} \in \Gamma''} \nu(\bar{z}) = \tau(z).$$

Again this convergence is uniform in compact subsets of γ.

For any $z \in \gamma$ Theorem 1.3 gives us a radius $\sigma > 0$ and a function $g'' \in C^2(\{z_1 \geq 0\})$ with

$$\begin{aligned}
\Gamma'' \cap B_\sigma(z) &= \{(z_1, \ldots, z_n, g''(z_1, \ldots, z_n)) : z_1 > 0\} \\
g''(0, z_2, \ldots, z_n) &= g'(0, z_2, \ldots, z_n) \\
Dg''(0, z_2, \ldots, z_n) &= Dg'(0, z_2, \ldots, z_n).
\end{aligned}$$

Using elliptic regularity theory (see [**11**]), we conclude that g' and g'' are the restriction of a smooth function g giving a minimal surface Δ. Now using Theorem 5.3, we conclude that $\Delta \subset \Gamma'$, and hence that Γ'' is a subset of Γ' in a neighborhood of z. Since this is vaild for every $z \in \gamma$, we conclude (5.7).

Step 4: Regularity in the annuli. In this step we show that V is a minimal hypersurface in the punctured ball $B_\rho(x) \setminus \{x\}$. First of all, we prove

$$\Gamma' \cap An(x,\rho,t) = \Gamma'' \cap An(x,\rho,t). \tag{5.9}$$

Assume for instance that $p \in \Gamma'' \setminus \Gamma'$. Without loss of generality we can assume that p is a regular point. Then let Δ be the connected component of $\Gamma'' \setminus (\operatorname{Sing}\Gamma'' \cup \operatorname{Sing}\Gamma')$ containing p. Δ is necessarily contained in $\overline{B}_{t-\varepsilon}(x)$; otherwise, by (5.7) and Theorem 5.3 Δ would coincide with a connected component of $\Gamma' \setminus (\operatorname{Sing}\Gamma'' \cup \operatorname{Sing}\Gamma')$, contradicting $p \in \Gamma''\setminus\Gamma'$. But then Δ induces, by Lemma 5.5, a stationary varifold V, with $\operatorname{supp}(V) \subset \overline{B}_{t-\varepsilon}(x)$. So, for some $s \le t-\varepsilon$, we have $\partial B_s(x) \cap \operatorname{supp}(V) \neq \emptyset$ and $\operatorname{supp}(V) \subset \overline{B}_s(x)$, contradicting Theorem 5.1(ii). This proves $\Gamma'' \subset \Gamma'$. Precisely the same argument can be used to prove $\Gamma' \subset \Gamma''$.

Thus we conclude that $\Gamma' \cup \Gamma''$ is in fact a minimal hypersurface in $An(x,s,2\rho)$. Since s is arbitrary, this means that Γ' is in fact contained in a larger minimal hypersurface $\Gamma \subset B_{2\rho}(x) \setminus \{x\}$ and that, moreover, $\Gamma'' \subset \Gamma$ for any second replacement V'', whatever is the choice of s (t being instead fixed).

Now fix such a V'' and note that $V'' \llcorner B_s(x) = V \llcorner B_s(x)$. Note, moreover, that by Theorem 5.1(ii) we necessarily conclude

$$\operatorname{supp}(V \llcorner B_s(x)) \cap \partial B_s(x) \ \subset\ \overline{\Gamma''} \ \subset\ \Gamma\,.$$

Thus, using Lemma 5.4, we conclude $\operatorname{supp}(V) \subset \Gamma$, which hence proves the desired regularity of V.

Step 5: Conclusion. The only thing left to analyize are the centers of the balls $B_\rho(x)$ of the previous steps. Clearly, if $n \ge 7$, we are done because by the compactness of M we only have to add possibly a finite set of points, that is, a 0-dimensional set, to the singular set. In other words, the centers of the balls can be absorbed in the singular set.

If, on the other hand, $n \le 6$, we need to show that x is a regular point. If $x \notin \operatorname{supp}(\|V\|)$, we are done, so we assume $x \in \operatorname{supp}(\|V\|)$. By Lemma 5.2 we know that every tangent cone is a multiple $\theta(x,V)$ of a plane (note that $n \le 6$). Consider the rescaled exponential maps of Section 5.2 and note that the rescaled varifolds V_r coincide with $(T^x_r)^{-1}(\Gamma) = \Gamma_r$. Using Theorem 1.3 we get the C^1-convergence of subsequences in $\mathcal{B}_1 \setminus \mathcal{B}_{1/2}$ and hence the integrality of $\theta(x,V) = N$.

Fix geodesic coordinates in a ball $B_\rho(x)$. Thus, given any small positive constant c_0, if $K \in \mathbb{N}$ is sufficiently large, there is a hyperplane π_K

such that, on $An(x, 2^{-K-2}, 2^{-K})$, the varifold V is the union of $m(K)$ disjoint graphs of Lipschitz functions over the plane π_K, all with Lipschitz constants smaller than c_0, counted with multiplicity $j_1(K), \ldots, j_m(K)$, with $j_1 + \ldots + j_m = N$. We do not know a priori that there is a *unique* tangent cone to V at x. However, if K is sufficiently large, it follows that the tilt between two consecutive planes π_K and π_{K+1} is small. Hence $j_i(K) = j_i(K+1)$ and the corresponding Lipschitz graphs do join, forming m disjoint smooth minimal surfaces in the annulus $An(x, 2^{-K-3}, 2^{-K})$, topologically equivalent to n-dimensional annuli. Repeating the process inductively, we find that $V \llcorner B_\rho(x) \setminus \{x\}$ is in fact the union of m smooth disjoint minimal hypersurfaces $\Gamma^1, \ldots, \Gamma^m$ (counted with multiplicities $j_1 + \ldots + j_m = N$), which are all, topologically, punctured n-dimensional balls.

Since $n \geq 2$, by Lemma 5.5, each Γ^i induces a stationary varifold. Every tangent cone to Γ^i at x is a hyperplane, and, moreover, the density of Γ^i (as a varifold) is everywhere equal to 1. We can therefore apply Allard's regularity Theorem (see [**1**]) to conclude that each Γ^i is regular. On the other hand, the Γ^i are disjoint in $B_r(x) \setminus \{x\}$ and they contain x. Therefore, if $m > 1$, we contradict the classical maximum principle. We conclude that $m = 1$ and hence that x is a regular point for V.

Appendix A. Proofs of the technical lemmas

A.1. Varifolds and Caccioppoli set limits.

Proposition A.1. *Let $\{\Omega^k\}$ be a sequence of Caccioppoli sets and U an open subset of M. Assume that*

(i) $D\mathbf{1}_{\Omega^k} \to D\mathbf{1}_\Omega$ *in the sense of measures in U;*
(ii) $\mathrm{Per}\,(\Omega^k, U) \to \mathrm{Per}\,(\Omega, U)$

for some Caccioppoli set Ω and denote by V^k and V the varifolds induced by $\partial^*\Omega^k$ and $\partial^*\Omega$. Then $V^k \to V$ in the sense of varifolds.

Proof. First, we note that by the rectifiability of the boundaries we can write

$$V^k = \mathcal{H}^n \llcorner \partial^*\Omega^k \otimes \delta_{T_x \partial^*\Omega^k} \ , \ V = \mathcal{H}^n \llcorner \partial^*\Omega \otimes \delta_{T_x \partial^*\Omega} \,, \tag{A.1}$$

where $\partial^*\Omega, \partial^*\Omega^k$ are the reduced boundaries and $T_x\partial^*\Omega$ is the approximate tangent plane to Ω in x (see chapter 3 of [**12**] for the relevant definitions). With the notation $\mu \otimes \alpha_x$ we understand, as usual, the measure ν on a product space $X \times Y$ given by

$$\nu(E) = \int\int \mathbf{1}_E(x, y)\, d\alpha_x(y)\, d\mu(x)\,,$$

where μ is a Radon measure on X and $x \mapsto \alpha_x$ is a weak* μ-measurable map from X into $\mathcal{M}(Y)$ (the space of Radon measures on Y).

By (ii) we have $\|V^k\| \to \|V\|$ and hence there is $W \in \mathcal{V}(U)$ such that (up to subsequences) $V^k \to W$. In addition, $\|V\| = \|W\|$. By the disintegration theorem (see theorem 2.28 in [**3**]) we can write $W = \mathcal{H}^n \llcorner \partial^*\Omega \otimes \alpha_x$. The proposition is proved, once we have proved

(Cl) $\alpha_{x_0} = \delta_{T_{x_0}\partial^*\Omega}$ for $\mathcal{H}^n$-a.e. $x_0 \in \partial^*\Omega$.

To prove this, we reduce the situation to the case where Ω is a half space by a classical blow-up analysis. Having fixed a point x_0, a radius r, and the rescaled exponential maps $T_r^{x_0} : \mathcal{B}_1 \to B_r(x_0)$ as in Subsection 5.2, we define

- $V_r^k := (T_r^{x_0})^{-1}_\sharp V^k$ and $V_r := (T_r^{x_0})^{-1}_\sharp V$;
- $\Omega_r^k := (T_r^{x_0})^{-1}(\Omega^k)$ and $\Omega_r := (T_r^{x_0})^{-1}(\Omega)$.

Clearly, V_r^k and Ω_r^k are related by the same formulas as in (A.1). Next, let G be the set of radii r such that

$$\mathcal{H}^n(\partial^*\Omega^k \cap \partial B_r(x_0)) = \mathcal{H}^n(\partial^*\Omega \cap \partial B_r(x_0)) = 0$$

for every k and observe that the complement of G is a countable set. Denote by H the set $\{x_1 < 0\}$. Then, after a suitable choice of orthonormal coordinates in $\mathcal{B}_1$, we have

(a) $D\mathbf{1}_{\Omega_r^k} \to D\mathbf{1}_{\Omega_r}$ and $\mathrm{Per}\,(\Omega_r^k, \mathcal{B}_1) \to \mathrm{Per}\,(\Omega_r, \mathcal{B}_1)$ for $k \to \infty$ and $r \in G$;
(b) $D\mathbf{1}_{\Omega_r} \to D\mathbf{1}_H$ and $\mathrm{Per}\,(\Omega_r, \mathcal{B}_1) \to \mathrm{Per}\,(H, \mathcal{B}_1)$ for $r \to 0, r \in G$;
(c) $T_0\partial^* H = T_{x_0}\partial^*\Omega$;
(d) $V_r^k \to V_r$ for $k \to \infty$ and $r \in G$.

(The assumption $r \in G$ is essential: see proposition 1.62 of [**3**] or proposition 2.7 of [**9**]).

Next, for $\mathcal{H}^n$-a.e. $x_0 \in \partial^*\Omega$ we have in addition

(e) $V_r \to \mathcal{H}^n \llcorner \partial^* H \otimes \alpha_{x_0}$

(in fact, if $\mathcal{D} \subset C(\mathbb{P}^n\mathbb{R})$ is a dense set, the claim holds for every x_0 which is a point of approximate continuity for all the functions $x \mapsto \int \varphi(y) d\alpha_x(y)$ with $\varphi \in \mathcal{D}$).

By a diagonal argument we get sets $\tilde\Omega^k = \Omega^k_{r(k)}$ such that

(f) $D\mathbf{1}_{\tilde\Omega^k} \to D\mathbf{1}_H$ and $\mathrm{Per}\,(\tilde\Omega^k, \mathcal{B}_1) \to \mathrm{Per}\,(H, \mathcal{B}_1)$;
(g) $\mathcal{H}^n \llcorner \partial^*\tilde\Omega^k \otimes \delta_{T_x\partial^*\tilde\Omega^k} \to \mathcal{H}^n \llcorner \partial^* H \otimes \alpha_{x_0}$.

Let $e_1 = (1, 0, \ldots 0)$ and ν be the exterior unit normal to $\partial^*\Omega^k$. Then (f) implies

$$\lim_{k\to\infty} \int_{\partial^*\tilde\Omega_k} \|\nu - e_1\|^2 = \lim_{k\to\infty} \left(2\mathcal{H}^n(\partial^*\tilde\Omega_k) - 2\int_{\partial^*\tilde\Omega_k} \langle \nu, e_1 \rangle \right) = 0\,.$$

This obviously gives $\mathcal{H}^n \llcorner \partial^*\tilde\Omega^k \otimes \delta_{T_x\partial^*\tilde\Omega^k} \to \mathcal{H}^n \llcorner \partial^* H \otimes \delta_{T_0\partial^* H}$, which together with (c) and (g) gives $\alpha_{x_0} = \delta_{T_0\partial^* H} = \delta_{T_{x_0}\partial^*\Omega}$, which is indeed the claim (Cl). q.e.d.

A.2. Proof of Theorem 5.3. Let $W \subset U$ be the maximal open set on which Σ_1 and Σ_2 coincide. If $W \neq U$, then there is a point $p \in \overline{W} \cap U$. In a ball $B_\rho(p)$, Σ_2 is the graph of a smooth function w over Σ_1 (as usual, we use normal coordinates in a regular neighborhood of Σ_1). By a straightfoward computation, w satisfies a differential inequality of the form $|A^{ij} D^2_{ij} w| \leq C(|Dw| + |w|)$ where A is a smooth function with values in symmetric matrices, satisfying the usual ellipticity condition $A^{ij}\xi_i\xi_j \geq \lambda|\xi^2|$, where $\lambda > 0$. Let $x \in W$ be such that $\mathrm{dist}(x, p) < \varepsilon$. Then w vanishes at infinite order in x and hence, according to the classical result of Aronszajn (see [**4**]), $w \equiv 0$ on a ball $B_r(x)$ where r depends on λ, A, C and $\mathrm{dist}(x, \partial B_\rho(p))$, but not on ε. Hence, by choosing $\varepsilon < r$ we contradict the maximality of W.

A.3. Proof of Lemma 5.4. Let T be the set of points $y \in \mathrm{supp}\,(V)$ such that the approximate tangent plane to V in y is transversal to the sphere $\partial B_{|y-x|}(x)$. The claim follows from the density of T in $\mathrm{supp}\,(V)$. The (quite short) proof of this statement can be found for instance in appendix B of [**6**] (compare with lemma B.2 therein).

A.4. Proof of Lemma 5.5. Set $\Gamma_r := \Gamma \setminus S$ and denote by H the mean curvature of Γ_r and by ν the unit normal to Γ_r. Obviously $H = 0$. Let V' be the varifold induced by Δ. We claim that

$$[\delta V'](\chi) \;=\; \int_\Delta \mathrm{div}_\Delta\, \chi \;=\; -\int_\Delta H\chi \cdot \nu \tag{A.2}$$

for any vector field $\chi \in \mathcal{X}_c(U)$.

The first identity is the classical computation of the first variation (see lemma 9.6 of [**20**]). To prove the second identity, fix a vector field χ and a constant $\varepsilon > 0$. W.l.o.g. we assume $S \subset \Gamma$. By the definition of the Hausdorff measure, there exists a covering of S with balls $B_{r_i}(x_i)$ centered on $x_i \in S$ such that $r_i < \varepsilon$ and $\sum_i r_i^{n-1} \leq \varepsilon$. By the compactness of $S \cap \mathrm{supp}\,(\chi)$ we can find a finite covering $\{B_{r_i}(x_i)\}_{i\in\{1,\dots,N\}}$. Fix smooth cutoff functions φ_i with

- $\varphi_i = 1$ on $M \setminus B_{2r_i}(x_i)$ and $\varphi_i = 0$ on $B_{r_i}(x_i)$;
- $0 \leq \varphi_i \leq 1$, $|\nabla\varphi_i| \leq C r_i^{-1}$.

(Note that C is in fact only a geometric constant.) Then $\chi_\varepsilon := \chi\Pi\varphi_i$ is compactly supported in $U \setminus S$. Thus,

$$\int_\Delta \mathrm{div}_\Delta\, \chi_\varepsilon \;=\; -\int_\Delta H\chi_\varepsilon \cdot \nu\,. \tag{A.3}$$

The RHS of (A.3) obviously converges to the RHS of (A.2) as $\varepsilon \to 0$. As for the left-hand side, we estimate

$$\begin{aligned} \int_\Delta |\mathrm{div}_\Delta(\chi - \chi_\varepsilon)| &\leq \sum_i \int_{B_{r_i}(x_i)\cap\Delta} (\|\nabla\chi\|_{C^0} + \|\chi\|_{C^0}\|\nabla\varphi_i\|_{C^0}) \\ &\leq \sum_i \|V\|(B_{r_i}(x_i))\|\chi\|_{C^1}(1 + Cr_i^{-1}) \qquad \text{(A.4)} \\ &\leq C\|\chi\|_{C^1}\sum_i (r_i^n + Cr_i^{n-1}) \;<\; C\varepsilon \end{aligned}$$

where the second inequality in the last line follows from the monotonicity formula. We thus conclude that the LHS of (A.3) converges to the LHS of (A.2).

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