J. DIFFERENTIAL GEOMETRY
95 (2013) 337-354

NONCONSTANT CR MORPHISMS BETWEEN COMPACT STRONGLY PSEUDOCONVEX CR MANIFOLDS AND ÉTALE COVERING BETWEEN RESOLUTIONS OF ISOLATED SINGULARITIES

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Dedicated to Professor Yum-Tong Siu on the occasion of his 70th Birthday

Abstract

Strongly pseudoconvex CR manifolds are boundaries of Stein varieties with isolated normal singularities. We prove that any non-constant CR morphism between two $(2n-1)$-dimensional strongly pseudoconvex CR manifolds lying in an n-dimensional Stein variety with isolated singularities are necessarily a CR biholomorphism. As a corollary, we prove that any nonconstant self map of $(2n-1)$-dimensional strongly pseudoconvex CR manifold is a CR automorphism. We also prove that a finite étale covering map between two resolutions of isolated normal singularities must be an isomorphism.

1. Introduction

Rigidity phenomena in complex geometry have received a lot of attention historically. Borel and Narasimhan [**BN67**] obtained some general results on rigidity of morphisms with domain space carrying no non-constant pseudoconvex function which is bounded above and target space being covered by an analytic subset of a bounded domain in $\mathbb{C}^n$. In 1975, Kobayashi and Ochiai [**KO75**] proved that there are only finitely many surjective morphisms between two fixed projective manifolds of general type. In 1981, Kalka, Shiffman, and Wong [**KSW81**] developed a general theory to study the finiteness and rigidity theorems for holomorphic mappings. Let $\mathrm{Hol}_k(X,Y)$ denote the complex space of holomorphic maps of rank $\geqslant k$ from the compact complex space X into the complex manifold Y. They proved that if Y satisfies certain convexity or cohomological conditions, then for suitable k, $\mathrm{Hol}_k(X,Y)$ is either discrete or finite. They also showed that if the tangent bundle of Y satisfies the k-pseudoconvexity condition, then $\mathrm{Hol}_k(X,Y)$ is discrete. As a corollary, they asserted that if Y is a compact Hermitian manifold with negative holomorphic sectional curvature, then the set of

Received 11/14/2012.

surjective holomorphic maps from X onto Y is finite. On the other hand, if Y is a compact Kähler manifold with $c_1(Y)$ represented by a negative semidefinite form and either $c_n(Y) \neq 0$ or $c_1^n(Y) \neq 0$, Kalka, Shiffman, and Wong showed that $\mathrm{Hol}_n(X, Y)$ is discrete. Recently, Hwang, Kebekus, and Peternell [**HKP06**] proved the following beautiful result. Let Y be a projective n-dimensional manifold which is not uniruled. If either $\pi_1(Y)$ is finite or $c_n(Y) \neq 0$, then for each connected normal compact complex variety X, the space of surjective morphism from X to Y is discrete.

In 1977, Wong [**Won77**] proved an important result in complex geometry—that any strongly pseudoconvex domain with smooth boundary in $\mathbb{C}^n$ with noncompact automorphism group must be biholomorphically equivalent to the unit ball. In 1978, S. T. Yau [**Yau78**] proved the Calabi Conjecture. As a consequence, he [**Yau77**] proved that $3c_2 \geqslant c_1^2$ for any Kähler surface with ample canonical bundle and the equality holds if and only if the surface is covered by the ball in $\mathbb{C}^2$. Using this results, he [**Yau77**] proved the Severi Conjecture, which states that every complex surface which is homotopic to the complex projective plane $\mathbb{CP}^2$ is biholomorphic to $\mathbb{CP}^2$.

As a consequence of his strong rigidity theorem [**Mos83**] Mostow showed that two compact quotients of the ball of complex dimension $\geqslant 2$ with isomorphic fundamental groups are either biholomorphic or conjugate biholomorphic. S. T. Yau conjectured that his phenomenon of strong rigidity should hold also for compact Kähler manifolds of complex dimension $\geqslant 2$ with same homotopy type and negative sectional curvature. In his famous paper [**Siu80**], Siu proved that Yau's conjecture is true when the curvature tensor of one of the two compact Kähler manifolds is strongly negative with no curvature assumption on the other manifold.

CR manifolds are abstract models of boundaries of complex manifolds. Strongly pseudoconvex CR manifolds have rich geometric and analytic structures. The harmonic theory for the $\overline{\partial}_b$ complex on compact strongly pseudoconvex CR manifolds was developed by Kohn [**Koh65**]. Using this theory, Boutet de Monvel [**Bou03**] proved that if X is a compact strongly pseudoconvex CR manifold of real dimension $2n-1$, $n \geqslant 3$, then there exist C^∞ functions $f_1, \dots, f_N$ on X such that each $\overline{\partial}_b f_j = 0$ and $f = (f_1, \dots, f_N)$ defines an embedding of X in $\mathbb{C}^N$. Thus, any compact strongly pseudoconvex CR manifold of dimension $\geqslant 5$ can be CR embedded in some complex Euclidean space. On the other hand, 3-dimensional strongly pseudoconvex compact orientable CR-manifolds are not necessarily embeddable. Throughout this paper, our strongly pseudoconvex CR manifolds are always assumed to be compact orientable and embedded in some $\mathbb{C}^N$. By a beautiful theorem of Harvey and Lawson ([**HL75**],[**HL**]), these CR manifolds are the boundaries of subvarieties with only isolated normal singularities. Rigidity problems of CR-immersions into spheres and hyperquadrics were studied by Ebenfelt, Huang, and Zaitsev [**EHZ04**] [**EHZ05**]. Let X be a strongly pseudoconvex CR manifold of dimension $2n-1$. For $p \in X$, let f: $(X, p) \longrightarrow S^{2n+2d-1}$ be a local CR immersion of X near p into unit sphere $S^{2n+2d-1}$ in $\mathbb{C}^{n+d}$. The beautiful result of [**EHZ04**] states that if $d < \frac{n}{2} - 1$, then f is rigid in the sense that any other immersion of (X, p) into $S^{2n+2d-1}$ is of the form $\phi \circ f$, where ϕ is biholomorphic automorphism of the unit ball $B \subseteq \mathbb{C}^{n+d}$. As a striking corollary, they show that if X and X' are two strongly pseudoconvex CR manifolds of dimension $2n-1$ in $S^{2n+2d-1}$ with $d < \frac{n}{2} - 1$ and if X and X' are locally CR equivalent at some points $p \in X$ and $p' \in X'$, then there exists a unitary linear transformation which maps X to X'.

In a remarkable paper [**Pin74**], Pinchuk showed that a proper holomorphic mapping between strongly pseudoconvex domains in $\mathbb{C}^n$ is locally biholomorphic. In fact, he proved that proper holomorphic self-maps of strongly pseudoconvex domains are necessarily biholomorphic. It was proved in [**BC82**] and [**DF82**] that proper holomorphic maps extend smoothly to the boundaries and hence induce CR morphisms between the boundaries.

In our previous paper [**Yau11**], we took another point of view. In view of Fornaess's theory on strongly pseudoconvex domains [**For76**], we investigate the rigidity property of CR morphisms between strongly pseudoconvex CR manifolds by means of the singularities theory. In this paper, we continue this investigation and prove the rigidity of CR morphisms between CR manifolds lying in the same variety.

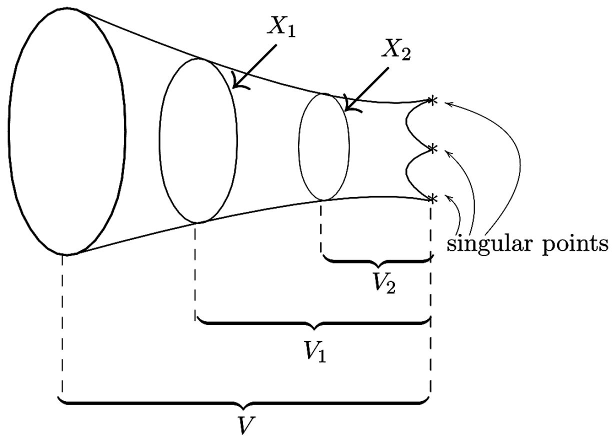
Theorem A. *Let X_1 and X_2 be two $(2n-1)$-dimensional compact strongly pseudoconvex CR manifolds lying in a Stein variety V of dimension n in $\mathbb{C}^N$. Let $V_1 \subseteq V$, $V_2 \subseteq V$ such that $\partial V_1 = X_1$ and $\partial V_2 = X_2$. Assume that the singular set S of V is nonempty and is equal to the singular set of V_i, $i = 1, 2$. Then nontrivial CR morphisms from X_1 to X_2 are necessarily CR biholomorphisms.*

As a corollary of Theorem A, we prove the following theorem, which is a conjecture of dimension 3 stated in [**Yau11**].

Theorem B. *Let X be a compact strongly pseudoconvex embeddable CR manifold of dimension ≥ 3 lying in $\mathbb{C}^N$. Then any nonconstant CR morphism from X to itself must be a CR automorphism.*

Remark. *A special case of Theorem B was proved in* [**Yau11**] *under the assumption that X bounds a complex variety with only isolated normal nonquotient singularities and* $\dim X = 3$. *In Theorem B here, X is allowed to be boundary of a complex submanifold in $\mathbb{C}^N$.*

The proofs of Theorem A can be reduced to the proof of the following theorem, which is of independent interest.



Theorem C. *Let V be a normal Stein space whose singular set is nonempty and finite. Let $f_1 : \widetilde{V}_1 \to V$ and $f_2 : \widetilde{V}_2 \to V$ be two resolutions of V. If $\widetilde{\Phi} : \widetilde{V}_1 \to \widetilde{V}_2$ is a finite étale covering map, $\widetilde{\Phi}$ must be an isomorphism.*

The difficulty of Theorem C is that $\widetilde{\Phi}$ may not commute with the resolution maps. For $n = 2$, Theorem C was proved in [**Yau11**].

In this paper, we derive a multiplicity formula in étale coverings between resolutions (see Theorem D below) and apply it to obtain Theorem C.

Theorem D. *Let V be an algebraic variety or a complex space with finitely many normal isolated singularities $x_i, i = 1, \dots, m$, on V. Positive numbers $cv_V(x_i)$ for each x_i can be defined (see §5) such that $\sum_{i=1}^{m} cv_V(x_i)$ is multiplicative in étale covering maps between resolutions. That is, if W is another algebraic variety or complex space with normal isolated singularities $y_j, j = 1, \dots, s$, $p_1 : \widetilde{W} \to W$ and $p_2 : \widetilde{V} \to V$ are resolutions of W and V, respectively, and $\Phi : \widetilde{W} \to \widetilde{V}$ is an étale covering map, then we have $\sum_{j=1}^{s} cv_W(y_j) = d \sum_{i=1}^{m} cv_V(x_i)$ where d is the degree of Φ.*

We call the number $cv_V(x)$ for a point $x \in V$ *canonical volume*, and it is defined in §5.

Remark. *Theorem C and Theorem D are true both in the sense of strong resolution of singularities of Hironaka or in the sense of weak resolution of singularities. Recall that $\pi : M \to V$ is a strong resolution of singularities of V if*

(1) π is a proper birational morphism and M is smooth,

(2) $M \setminus$ exceptional set *is dense in M and π restricts to $M \setminus$* exceptional set *is biholomorphic to $V \setminus$* singular set of V.

If we only require (1) above, then $\pi : M \to V$ is called a weak resolution of singularities.

Note that even if V is a smooth projective variety, an étale endomorphism $\Phi : V \to V$ is not necessarily an isomorphism. For example, let V be an abelian variety, $\hat{n} : V \to V$ is the morphism sending any point $x \in V$ to its n times, nx, then $\hat{n}$ is an étale covering map which is not an isomorphism if $n > 1$. Therefore, we need some restraints on V to force the degree to be 1.

A notion of volume for an isolated singular point of normal variety is defined in [**BFF12**]. The volume is multiplicative in étale covering maps. By Theorem A of [**BFF12**], if K_V is $\mathbb{Q}$-Cartier and V is not log canonical, the volume of the singularity is nonzero and we can determine the degree of an étale morphism. Our definition of canonical volume is like another multiplicative number between resolutions, and it determines the degree of étale coverings. Our method of proving the multiplicativity of canonical volume by taking the étale cover corresponding to a subgroup of the fundamental group is like the proof in the discussion of nearly étale maps in [**NZ09**].

In §2, we shall recall all the notations and theorems that we need later. In §3, we shall study the CR morphisms between two CR manifolds which may not lie on the same variety. In particular, we make some improvement of Theorem 1.1 in [**Yau11**]. Theorem A and Theorem B are proved in §4. In §5, we prove Theorem C and Theorem D.

Acknowledgments. The first author would like to express his deepest gratitude to professor János Kollár for the generous support and enlightening discussions.

The second author gratefully acknowledge the partial financial support from N.S.F. and Tsinghua University while the paper was written. The second and third authors are supported by the start-up fund from Tsinghua University.

2. Preliminaries

Definition 2.1. *Let X be a connected orientable manifold of real dimension $2n-1$. A CR structure on X is an $(n-1)$-dimensional subbundle S of the complexified tangent bundle $CT(X)$ such that*

(1) $S \cap \overline{S} = \{0\}$,
(2) *If L, L' are local sections of S, then so is $[L, L']$.*

A manifold with a CR structure is called a CR manifold. There is a unique subbundle $\mathcal{H}$ of the tangent bundle $T(X)$ such that $\mathbb{C}\mathcal{H} = S \oplus \overline{S}$. Furthermore, there is a unique homomorphism $J \colon \mathcal{H} \longrightarrow \mathcal{H}$ such that $J^2 = -1$ and $S = \{u - iu \colon u \in \mathcal{H}\}$. The pair $(\mathcal{H}, J)$ is called the real expression of the CR structure.

Definition 2.2. *Let $L_1, \ldots, L_{n-1}$ be a local frame of S. Then $\overline{L}_1, \ldots, \overline{L}_{n-1}$ is a local frame of $\overline{S}$ and one may choose a local section N of $T(X)$ which is purely imaginary such that $L_1, \ldots, L_{n-1}, \overline{L}_1, \ldots, \overline{L}_{n-1}, N$ is a local frame of $\mathbb{C}T(X)$. The matrix (c_{ij}) defined by*

$$[L_i, \overline{L}_j] = \Sigma a_{ij}^k L_k + \Sigma b_{ij}^k \overline{L}_k + \sqrt{-1} c_{ij} N$$

is Hermitian and is called the Levi form of S.

Proposition 2.1. *The number of nonzero eigenvalues and the absolute value of the signature of the Levi form (c_{ij}) at each point are independent of the choice of $L_1, \ldots, L_{n-1}, N$.*

Definition 2.3. *The CR manifold X is called strongly pseudoconvex if the Levi form is definite at each point of X.*

Theorem 2.1 (Boutet De Monvel [**Bou03**])**.** *If X is a compact strongly pseudoconvex CR manifold of dimension $(2n-1)$ and $n \geqslant 3$, then X is CR embeddable in $\mathbb{C}^N$.*

Although there are non-embeddable compact 3-dimensional CR manifolds, in this paper all CR manifolds are assumed to be embeddable in complex Euclidean space. The following beautiful theorem of Harvey and Lawson connects the theory of strongly pseudoconvex CR manifolds on the one hand and the theory of normal isolated singularities on the other hand.

Theorem 2.2 (Harvey–Lawson [**HL75**],[**HL**])**.** *For any compact connected embeddable strongly pseudoconvex CR manifold X, there is a unique complex variety V in $\mathbb{C}^N$ for some N such that the boundary of V is X and V has only normal isolated singularities.*

3. CR morphisms between CR manifolds lying on different varieties

The following proposition was the starting point of our investigation. It can be found in [**Yau11**]. It was proved by using the results of Fornaess [**For76**] and Prill [**Pri67**].

Proposition 3.1. *Let X_1 and X_2 be two compact strongly pseudoconvex CR manifolds of dimension $2n-1 \geqslant 3$ which bound complex varieties V_1 and V_2 in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$, respectively. Suppose the singular set S_i of V_i, $i = 1, 2$, is either an empty set or a set consisting of only isolated normal singularities. If $\Phi\colon X_1 \longrightarrow X_2$ is a non-constant CR morphism, then Φ is surjective and Φ can be extended to a proper surjective holomorphic map from V_1 to V_2 such that $\Phi(S_1) \subseteq S_2$, $\Phi^{-1}(X_2) = X_1$ and $\Phi\colon V_1 \setminus \Phi^{-1}(S_2) \longrightarrow V_2 \setminus S_2$ is a covering map. Moreover, if S_2 does not have quotient singularity, then $\Phi^{-1}(S_2) = S_1$.*

As an application of Proposition 3.1, we have the following theorem.

Theorem 3.1. *Let X_1 and X_2 be two compact strongly pseudoconvex CR manifolds of dimension $2n-1 \geqslant 5$ which bound complex variety V_1 and V_2 with only isolated normal singularities in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$ respectively. Let S_1 and S_2 be the singular sets of V_1 and V_2, respectively, and S_2 is nonempty. Suppose $2n - N_2 - 1 \geqslant 1$. Then any nonconstant CR morphism from X_1 to X_2 is a covering map. If $|S_1|$ is not divisible by $|S_2|$ or $|S_1| \leqslant 2|S_2| - 1$, then any nonconstant CR morphism from X_1 to X_2 is necessarily a CR biholomorphism.*

Proof. Let $\Phi\colon X_2 \longrightarrow X_2$ be a nonconstant CR morphism. Proposition 3.1 says the Φ can be extended to a proper surjective holomorphic map from V_1 to V_2 such that $\Phi(S_1) \subseteq S_2$ and $\Phi\colon V_1 \setminus \Phi^{-1}(S_2) \to V_2 \setminus S_2$ is a covering map of degree d. For any $q \in S_2$, we know that the punctured neighborhood of q in V_2 is $(2n - N_2 - 1)$-connected in view of a theorem of Hamm [**Ham81**]. Since $2n - N_2 - 1 \geqslant 1$ by assumption, the punctured neighborhood of q is simply connected. We claim that $\Phi^{-1}(q) \subseteq S_1$. If $\Phi^{-1}(q)$ is not contained in S_1, then there exists a smooth point q' of V_1 in $\Phi^{-1}(q)$. Recall that $\Phi^{-1}(q)$ is a finite set. We can find an open neighborhood U of q' which is biholomorphic to a domain in $\mathbb{C}^n$ such that $\Phi|_U$ from U to the germ of (V_2, q) is a branch covering with ramification locus $\{q'\}$. Since the punctured neighborhood of q in V_2 is simply connected, this implies $\Phi|_U$ is injective and hence $\Phi|_U$ is a biholomorphism. This leads to a contradiction because q is a singular point. We have shown that $\Phi^{-1}(q) = \{q'_1, \ldots, q'_d\} \subseteq S_1$. There are exactly d points in $\Phi^{-1}(q)$ because the punctured neighborhood of q is simply connected. If $|S_1| \leqslant 2|S_2| - 1$, by the pigeonhole principle, there exists $q \in S_2$ such that $\Phi^{-1}(q) = \{q'\}$. Since the punctured neighborhood of q is simply connected, we conclude that the degree of the covering map $\Phi\colon V_1 \setminus S_1 \longrightarrow V_2 \setminus S_2$ is 1. On the other hand, the above argument shows that $\Phi\colon V_1 \longrightarrow V_2$ is a covering map. In particular, $|S_1|$ is divisible by $|S_2|$. q.e.d.

4. Proof of Theorem A and Theorem B

We first recall a beautiful theory developed by Nash [**Nas95**]. In a 1968 preprint which was published as [**Nas95**], Nash introduced a beautiful theory on arc spaces and jet schemes for algebraic and analytic varieties. In what follows, we shall use Ishii and Kollár's treatment on Nash arc families of singularities [**IK03**].

Definition 4.1 (Nash)**.** *Let V be a complex variety with singular locus $S \subseteq V$. The space of arcs $J_\infty(V)$ of V parametries arcs α: $\operatorname{Spec}\mathbb{C}[[t]] \longrightarrow V$ such that $\alpha(0) \in S$. Decompose $J_\infty(V)$ into its irreducible components*

$$J_\infty(V) = \bigcup_{i \in I} C_i$$

where the C_i's are the components with an arc α such that $\alpha(t) \notin S$ for generic t. Denote C_i^0 the open subset of C_i consisting of arcs $\alpha\colon \operatorname{Spec}\mathbb{C}[[t]] \longrightarrow X$ such that $\alpha(t) \notin S$ for generic t.

Definition 4.2 (Nash)**.** *Let $\pi\colon M \longrightarrow V$ be a resolution of singularities of V, and let $A_1, \dots, A_r$ be the irreducible components of the exceptional set on M. For any arc $\alpha \in C_i^0$, we have $\alpha\colon \operatorname{Spec}\mathbb{C}[[t]] \longrightarrow V$ such that $\alpha(t) \notin S$ for generic t. Then α can be uniquely lifted to an arc $\widetilde{\alpha}\colon \mathbb{C}[[t]] \longrightarrow M$. This $\widetilde{\alpha}$ is called the lifting of α. There is a natural map*

$$\varphi\colon \bigcup_i C_i^0 \longrightarrow \bigcup_\ell A_\ell$$

given by $\varphi(\alpha) = \widetilde{\alpha}(0)$.

Theorem 4.1 (Nash [**Nas95**, p.35]; see also [**IK03**, p. 607])**.** *Let V be a complex variety and let $\pi\colon M \longrightarrow V$ be a resolution of singularities. Let $\{C_i\colon i \in I\}$ be the components of the space of arcs through singular locus of V, and let z_i denote the generic point of C_i. Then we have the following:*

(i) *$\varphi(z_i)$ is the generic point of an exceptional component $A_{\ell_i} \subseteq M$ for some ℓ_i.*

(ii) *For every $i \in I$, A_{ℓ_i} is an essential component on M, i.e., a component which appears on every possible resolution.*

(iii) *The resulting Nash map*

$$\mathcal{N}\colon \{\textit{irreducible components of the space of arcs through } \operatorname{sing} V\} \longrightarrow \{\textit{essential components on } M\}$$

given by $C_i \longrightarrow A_{\ell_i}$ is injective. In particular, there are only finitely many irreducible components of the space of arcs through $\operatorname{sing} V$.

Open problem (Nash). Is the Nash map bijective?

In the beautiful paper of Ishii and Kollár, they proved that the Nash map is bijective for toric singularities in any dimension. They also showed that the Nash map is not bijective in general.

Definition 4.3. *An essential component on M which is in the image of the Nash map is called Nash essential component.*

Definition 4.4. *An isolated singularity $(V,0)$ is called a Nash terminal singularity if for any resolution $\pi : (M,E) \to (V,0)$ the n-form $\omega \in \Gamma(M\backslash E, \Omega^n)$ which is holomorphic on $M\backslash E$ must vanish along all the Nash essential component E_i of E. And an isolated singularity $(V,0)$ is called a Gorenstein singularity if there is some neighborhood U of 0 in V and a holomorphic n-form ω on $U-\{0\}$ such that ω has no zeroes on $U-\{0\}$.*

We are now ready to prove Theorem A.

Proof of Theorem A. Let $\Phi\colon X_1 \longrightarrow X_2$ be a nonconstant CR morphism. In view of Proposition 3.1, Φ can be extended to a proper holomorphic map from V_1 to V_2 such that $\Phi\colon V_1 \setminus \Phi^{-1}(S) \longrightarrow V_2 \setminus S$ is a covering map of degree d and $\Phi(S) = S$. Let $S = \{q_1, \dots, q_m\}$. Then $\Phi^{-1}(S) = \{q_1, \dots, q_m, p_1, \dots, p_k\}$. We shall prove that $\Phi^{-1}(S) = S$. Let $\pi\colon M \longrightarrow V_2$ be a resolution of singularities of V_2 such that the exceptional sets

$$E_1 = \pi^{-1}(q_1) = \bigcup_{i=1}^{\ell_1} A_i^1, \dots, E_m = \pi^{-1}(q_m) = \bigcup_{i=1}^{\ell_m} A_i^m$$

are normal crossing divisors.

Consider the fiber product $V_1 \underset{V_2}{\times} M$ of the maps $\Phi\colon V_1 \longrightarrow V_2$ and $\pi\colon M \longrightarrow V_2$. Let $\tau\colon \widetilde{M} \longrightarrow V_1 \underset{V_2}{\times} M$ be the normalization map. Then we have the following commutative diagram (Figure 1) where π_1 and π_2 are natural projections.

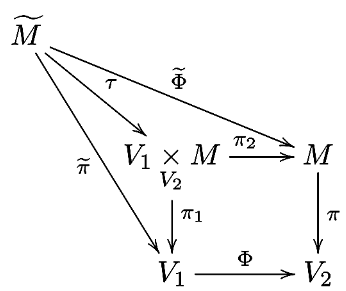


Figure 1

Notice that $\pi_1\colon V_1 \underset{V_2}{\times} M \longrightarrow V_1$ is a biholomorphism outside $\pi_1^{-1}(\Phi^{-1}(S))$ and $\pi_2\colon V_1 \underset{V_2}{\times} M \to M$ is a covering map outside $\bigcup_{i=1}^{m} E_i$. Thus,

$$\widetilde{\Phi} := \pi_2 \circ \tau\colon \widetilde{M} \longrightarrow M$$

is a d-fold branch covering. For each $A_i^j \subseteq E_j$, and any point $q_i^j \in A_i^j$ which is a smooth point in $\bigcup_{i=1}^{m} E_i$, we choose a germ of a curve Γ_i^j at the point q_i^j which intersects with $\bigcup_{i=1}^{m} E_i$ only at q_i^j and the intersection of A_i^j and Γ_i^j is transversal at q_i^j. Let $\Gamma = \bigcup \Gamma_i^j$, $1 \leqslant j \leqslant m$, $1 \leqslant i \leqslant \ell_j$. Notice that $\widetilde{\pi} := \pi_1 \circ \tau$ is a proper map which is a biholomorphism outside $\widetilde{E} := \widetilde{\pi}^{-1}(\Phi^{-1}(S)) = \widetilde{\Phi}^{-1}(E)$ where $E = E_1 \cup \cdots \cup E_m$. Observe that $\widetilde{E}$ has exactly $m + k$ connected components $\widetilde{E} = \widetilde{E}_1 \cup \cdots \cup \widetilde{E}_{m+k}$.

Clearly,

$$\widetilde{\Phi}_*(\widetilde{E}) = \sum_{i,j} d_i^j A_i^j, \text{ where } d_i^j \leqslant d.$$

By the projection formula (cf. p. 34 of [**Ful02**] or p. 426 of [**Har01**]),

$$\begin{aligned}\sum_{j=1}^{m}\sum_{i=1}^{\ell_j} d_i^j &= \Gamma \cdot \widetilde{\Phi}_*(\widetilde{E}) \\ &= \widetilde{\Phi}^*(\Gamma)\cdot \widetilde{E} \\ &\geqslant (\ell_1 + \cdots + \ell_m)d.\end{aligned}$$

The last inequality comes from the fact that $\widetilde{\Phi}^*(\Gamma_i^j)$ has d distinct branches because $\widetilde{\Phi}\colon \widetilde{M} \setminus \widetilde{\Phi}^{-1}(E) \longrightarrow M \setminus E$ is a d-fold covering space. Since $d_i^j \leqslant d$, we conclude that $d_i^j = d$ for all i, j. It follows that the branch locus of $\widetilde{\Phi}$ is contained in the singular locus of $\bigcup_{i=1}^{m} E_i$ which is of dimension $n-2$. As $\widetilde{M}$ is normal and M is smooth, $\widetilde{\Phi}\colon \widetilde{M} \longrightarrow M$ is a covering map by purity of branch locus. In particular, $\widetilde{M}$ is smooth.

Now we are ready to prove that $\Phi^{-1}(S) = S$, i.e., there are no $p_1, \ldots, p_k$ points in $\Phi^{-1}(S)$. Observe that $\widetilde{\pi}^{-1}(p_i)$, $1 \leqslant i \leqslant k$, and E_j, $1 \leqslant j \leqslant m$, are maximal compact connected analytic subsets in $\widetilde{M}$ and M, respectively. Since $\widetilde{\Phi}$ is a covering map, there is a neighborhood $\widetilde{U}_i$ of $\widetilde{\pi}^{-1}(p_i)$ which maps biholomorphically to a neighborhood U_j of E_j for some j via $\widetilde{\Phi}$. As $\widetilde{\pi}\colon \widetilde{M} \longrightarrow V_1$ is a point modification in a neighborhood of p_i, there is a neighborhood D_i of p_i such that

$$\widetilde{\pi}\colon \widetilde{\pi}^{-1}(D_i) \setminus \widetilde{\pi}^{-1}(p_i) \longrightarrow D_i \setminus \{p_i\}$$

is a biholomorphism. Similarly, there is a neighborhood O_j of q_j such that

$$\pi\colon \pi^{-1}(O_j) \setminus E_j \longrightarrow O_j \setminus \{q_j\}$$

is a biholomorphism. Therefore,

$$\pi \circ \widetilde{\Phi} \circ \widetilde{\pi}^{-1}\colon D_i \setminus \{p_i\} \longrightarrow O_j \setminus \{q_j\}$$

is a biholomorphism. Observe that p_i is a smooth point of D_i and q_j is an isolated normal singularity. It follows that $\pi \circ \widetilde{\Phi} \circ \widetilde{\pi}^{-1}$ extends to a biholomorphism from D_i to O_j. In particular, q_j is not a singular point. This contradiction shows that $\Phi^{-1}(S) = S$ and hence $\widetilde{\pi}\colon \widetilde{M} \longrightarrow V_1$ is also a resolution of singularities of V_1.

If $\Phi(q_i) = q_j$, then (V, q_i) is isomorphic to (V, q_j) as germs of singularities. This is because the resolution of (V, q_j) is a resolution of (V, q_i).

Now we first prove Theorem A under the assumption that S does not have Nash terminal singularity or S only has Gorenstein singularities.

Let A_1 be an irreducible component of E and $\widetilde{A}_1$ be an irreducible component of $\widetilde{\Phi}^{-1}(A_1)$. Then $\widetilde{\Phi}|_{\widetilde{A}_1}: \widetilde{A}_1 \longrightarrow A_1$ is a covering map. Suppose that the degree of $\widetilde{\Phi}|_{\widetilde{A}_1}$ is d. Observe that by projection formula, we have

$$\begin{aligned}
\widetilde{A}_1 \cdot \widetilde{A}_1 &= (\widetilde{\Phi}^*_{\widetilde{A}_1} A_1) \cdot (\widetilde{\Phi}|^*_{\widetilde{A}_1} A_1) \\
&= A_1 \cdot (\widetilde{\Phi}|_{\widetilde{A}_1 *} \widetilde{\Phi}|^*_{\widetilde{A}_1} A_1) \\
&= A_1 \cdot \widetilde{\Phi}|_{\widetilde{A}_1 *} \widetilde{A}_1 \\
&= A_1 \cdot (dA_1) = d(A_1 \cdot A_1). \qquad (4.1)
\end{aligned}$$

For any meromorphic n-form $\omega \in \Gamma(M \setminus E, \Omega^n)$ which is holomorphic on $M \setminus E$, we denote (ω) the divisor of ω along the exceptional set E. Observe that if ω_1 and ω_2 are two such meromorphic forms, then $(\omega_1 + \omega_2) \leq \min((\omega_1), (\omega_2))$. Since $\Gamma(M \setminus E, \Omega^n)/\Gamma(M, \Omega^n)$ is finite dimensional, there exists a meromorphic form $\omega_0 \in \Gamma(M \setminus E, \Omega^n)$ such that (ω_0) is the minimal one among all divisors along the exceptional set E of meromorphic n-forms which are holomorphic on $M \setminus E$. We shall denote this unique canonical divisor (ω_0) along the exceptional set by K_M. In view of the above theory developed by Nash, it is clear that the covering map $\widetilde{\Phi}: \widetilde{M} \longrightarrow M$ in Figure 1 sends Nash essential components in $\widetilde{M}$ to Nash essential components in M. Observe that there is a unique canonical divisor $K_{\widetilde{M}}$ (respectively K_M) of $\widetilde{M}$ (respectively M) with support in the exceptional set of $\widetilde{M}$ (respectively M). They are the divisor (ω_1) in $\widetilde{M}$ and the divisor (ω_0) in M. Since we are going to do the local analysis of the canonical divisors in the neighborhood of the exceptional sets, without loss of generality we may assume that $\widetilde{M}$ and M are resolution of singularities of V_0 where V_0 is a connected open set containing the singular locus of V. We claim that the coefficients of $K_{\widetilde{M}}$ and K_M along the same Nash essential component are exactly the same. This can be seen as follows: By Hironaka, there exists another resolution $\widetilde{\widetilde{M}}$ of V_0 in such a way that $\widetilde{\widetilde{M}}$ can be obtained by successive monoidal transforms on $\widetilde{M}$ or M, respectively. Since monoidal transform does not change the coefficient along the Nash essential component of the canonical divisor supported in exceptional set, the coefficients of $K_{\widetilde{\widetilde{M}}}$ and $K_{\widetilde{M}}$ along the same Nash essential component are the same. This proves our claim.

On the other hand, $\widetilde{\Phi}: \widetilde{M} \longrightarrow M$ is a d-fold covering. The pullback $\widetilde{\Phi}^*(\omega_0)$ is still a holomorphic n-form outside the exceptional set. Its divisor $(\widetilde{\Phi}^*(\omega_0))$ along the exceptional set is d-fold of the divisor (w_0) along the exceptional set.

Case 1. V has no Nash terminal singularity.

Then there exists a Nash essential component A_i^M of M such that the coefficient k_i of $K_M = (\omega_0)$ along this component is nonpositive. Denote the same Nash essential component in $\tilde{M}$ by $A_i^{\tilde{M}}$. Then the coefficient $\tilde{k_i}$ of $K_{\tilde{M}}$ along $A_i^{\tilde{M}}$ is exactly k_i. By the costruction of $K_{\tilde{M}}$, we have that k_i is less than or equal to the coefficient of $(\Phi^*(\omega_0))$ along $A_i^{\tilde{M}}$ which is exactly dk_i. If $k_i < 0$, then we have a contradiction unless $d = 1$.

Now we want to show that if the coefficient of $K_{\widetilde{M}}$ and K_M along the same Nash essential component are zero, then $d = 1$. By formula (4.1), we have $A_i^{\widetilde{M}} \cdot A_i^{\widetilde{M}} = d(A_i^M \cdot A_i^M)$. Since $K_{\widetilde{M}}|_{\widetilde{A_i^M}}$ and $K_M|_{A_i^M}$ are trivial, by adjunction formula, we have

$$\begin{array}{l} c(K_{A_i^{\widetilde{M}}}) = c(N_{A_i^{\widetilde{M}}}) = (A_i^{\widetilde{M}})^2 = d(A_i^M)^2 \\ \quad \| \\ c(K_{A_i^M}) = c(N_{A_i^M}) = (A_i^M)^2 \end{array}$$

where c denotes the Chern class. It follows that $d = 1$.

Case 2. V has only Gorenstein singularities.

Then there exists a nowhere vanishing holomorphic n-form ω_0 on $M \setminus E$. The divisor (ω_0) is supported on E and equal to K_M defined above along all Nash essential components . Observe that $\tilde{\Phi}^*(\omega_0)$ is a nowhere vanishing holomorphic n-form on $\tilde{M} \setminus \tilde{E}$. Its divisor $(\tilde{\Phi}^*(\omega_0))$ is supported on $\tilde{E}$ and is equal to $K_{\tilde{M}}$ defined above along all Nash essential components. It follows that if K_M along any Nash essential component has nonzero coeffient, then $d = 1$. The rest of the proof is the same as case 1.

This completes the proof of Theorem A when S has no Nash terminal singularity or S has only Gorenstein singularities.

For the general S, we can simply apply Theorem C, which will be proved in §5, to conclude that $\widetilde{\Phi}$ is biholomorphic. q.e.d.

Proof of Theorem B. By Theorem 2.2, there is a unique complex variety V in $\mathbb{C}^N$ for some N such that the boundary of V is X. If the singular set of V is nonempty, Theorem B follows immediately from Theorem A. On the other hand, if V is smooth, then Theorem B follows from Proposition 1.4 of Yau [**Yau11**], which was proved using Pinchuk's argument in [**Pin74**]. q.e.d.

Remark. *The proof of Theorem A also proves Theorem C in the case that V does not admit Nash terminal singularity.*

5. Proof of Theorem C and Theorem D

Locally, we shall sssume that $x \in V$ is a germ of complex analytic space with only one isolated singularity x. By Hironaka's paper[**Hir63**], it's biholomorphic equivalent to a germ of a complex algebraic singularity. Since there are only finitely many isolated singularities, after the equivalence and the resolution theorems of algebraic varieties over field of characteristic 0, we can construct a resolution $\pi : \widetilde{V} \to V$ of V such that $\widetilde{V}$ is smooth and π is a bimeromorphic proper morphism.

The key point in the proof of Theorem C of [**Yau11**] for the surface case is applying the minimal resolution. But in higher-dimensional cases, there is no minimal resolution in general. The above proof of Theorem C in §4 works if V does not admit Nash terminal singularity. Fortunately, by [**BCHM10**], there is a unique partial resolution $f : V^{can} \to V$ called the relative canonical model of V such that V^{can} has canonical singularity and the canonical divisor $K_{V^{can}}$ is f-ample. For surface, the relative canonical model is obtained by contracting all (-2)-rational curves in the minimal resolution of V. In general, the relative canonical model is isomorphic to

$$Proj \bigoplus_{m\geq 0} g_*\mathcal{O}(mK_Z)$$

where $g : Z \to V$ is any resolution of V.

Now we give the definition of the number $cv_V(x)$.

Definition 5.1. *Locally, let $(x \in V)$ be a germ such that x is the only isolated singularity. Take the relative canonical model $f : V^{can} \to V$ of V and denote E to be the exceptional set. Define $cv_V(x) = (K_{V^{can}})^{dimE} \cdot E$, called the* canonical volume *of x.*

If V^{can} is not isomorphic to V, we have E being nonempty and $cv_V(x) = (K_{V^{can}})^{dimE} \cdot E > 0$ by the f-ampleness. If V^{can} is isomorphic to V and x is a singular point, we set $cv_V(x) = (K_{V^{can}})^{dimE} \cdot E = (K_{V^{can}})^0 \cdot x = 1$. Finally, if x is a smooth point, we set $cv_V(x) = 0$.

In general, if V has finitely many isolated normal singularities $x_i, i = 1, \ldots, m$, then we consider the sum of canonical volume

$$\sum_{i=1}^{m} cv_V(x_i) = \sum_{i=1}^{m} (K_{V^{can}})^{dimE_{x_i}} \cdot E_{x_i}$$

where $f : V^{can} \to V$ is the relative canonical model of V and E_{x_i} is the exceptional set over x_i. From the definition, we see that $\sum_{i=1}^{m} cv_V(x_i) > 0$ for nonempty isolated normal singularities $x_i, i = 1, \ldots, m$, on V.

We can begin the proof of Theorem D.

Proof. Suppose we have resolutions $p_1 : \widetilde{W} \to W$ and $p_2 : \widetilde{V} \to V$, and we consider relative canonical models $p_1' : W^{can} \to W$ and $p_2' : V^{can} \to V$. We start from the following claim:

Claim: If $\Phi : W^{can} \to V^{can}$ is an étale covering map, then $\sum_{j=1}^{s} cv_W(y_j) = d\sum_{i=1}^{m} cv_V(x_i)$ where d is the degree of Φ.

Let E_1 be the exceptional set of p_1', and let E_2 be the exceptional set of p_2'. $E_1 = \sum_{j=1}^{s} E_{y_j}$ where E_{y_j} is the exceptional set over y_j, and similarly $E_2 = \sum_{i=1}^{m} E_{x_i}$ where E_{x_i} is the exceptional set over x_i. Since the canonical divisors $K_{W^{can}}$ and $K_{V^{can}}$ are p_1'-ample and p_2'-ample, respectively, if E_1 is not empty, we have $\sum_{j=1}^{s} cv_W(y_j) = \sum_{j=1}^{s} (K_{W^{can}})^{dimE_{y_j}} \cdot E_{y_j} > 0$. Similarly, $\sum_{i=1}^{m} cv_V(x_i) = \sum_{i=1}^{m} (K_{V^{can}})^{dimE_{x_i}} \cdot E_{x_i} > 0$ if E_2 is not empty.

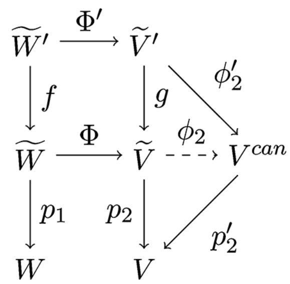
By Φ being an étale covering map, we have the pullback $\Phi^* K_{V^{can}} = K_{W^{can}}$. Also, $\Phi^* E_2 = E_1$ since E_1 and E_2 are the only proper sets in W^{can} and in V^{can}, respectively, if we shrink V and W. Therefore, from

$$\begin{aligned}\sum_{j=1}^{s} (K_{W^{can}})^{dimE_{y_j}} \cdot E_{y_j} &= \sum_{i=1}^{m} (\Phi^* K_{V^{can}})^{dimE_{x_i}} \cdot \Phi^* E_{x_i} \\ &= d\sum_{i=1}^{m} (K_{V^{can}})^{dimE_{x_i}} \cdot E_{x_i}\end{aligned}$$

where d is the degree of Φ, we have $\sum_{j=1}^{s} cv_W(y_j) = d\sum_{i=1}^{m} cv_V(x_i)$.

If $V^{can} \cong V$, then E_2 has dimension 0 and $E_2 = \sum_{i=1}^{m} x_i$. Since étale morphisms are locally isomorphisms, Φ sends singular points to singular points, and we have that $E_1 = \sum_{j=1}^{s} y_j$ is also 0-dimensional and $W^{can} \cong W$. The intersection $(K_{V^{can}})^0 \cdot x_i$ is just 1 by definition. By counting the singular points, we have that $\sum_{j=1}^{s} cv_W(y_j) = s = dm = d\sum_{i=1}^{m} cv_V(x_i) > 0$. The claim is proved.

Now, for two resolutions $p_1 : \widetilde{W} \to W$ and $p_2 : \widetilde{V} \to V$, we have birational map $\phi_2 : \widetilde{V} \dashrightarrow V^{can}$ over V (see the diagram below).



Take a common resolution $\widetilde{V}'$ of $\widetilde{V}$ and V^{can} with birational morphisms $g : \widetilde{V}' \to \widetilde{V}$ and $\phi_2' : \widetilde{V}' \to V^{can}$, and $\Phi' : \widetilde{W}' \to \widetilde{V}'$ is the base-change of $\Phi : \widetilde{W} \to \widetilde{V}$. Then Φ' is also an étale covering map with the same degree of Φ. After replacing Φ and ϕ_2 by Φ' and ϕ_2', respectively, we can assume $\phi_2 : \widetilde{V} \to V^{can}$ is a birational morphism.

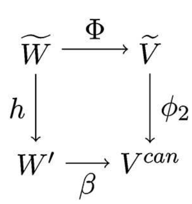
By the property of the canonical model, V^{can} and W^{can} have canonical singularities. We use a theorem in [**Tak03**]:

Theorem. ([**Tak03**], Theorem 1.1) *Let V be a normal analytic space, and let $f : \widetilde{V} \to V$ be a resolution of singularities. Then the induced homomorphism $f_* : \pi_1(\widetilde{V}) \to \pi_1(V)$ is an isomorphism if (V, Δ) is Kawamata log-terminal (klt) for some divisor Δ.*

Definition 5.2. *$(X, \triangle)$ is a pair where X is a normal variety and $\triangle$ is an effective $\mathbb{Q}$-divisor such that $K_X + \triangle$ is $\mathbb{Q}$-Cartier, and let $\triangle = \sum d_i \triangle_i$ be the prime decomposition. We say that $(X, \triangle)$ is (1) Kawamata log-terminal (klt) iff $d_i < 1$ for all i and there exists a projective birational morphism $\mu : Y \to X$ from a smooth variety Y with a normal crossing divisor E_i such that $K_Y \equiv \mu^*(K_X + \triangle) + \sum e_i E_i$ holds with $e_i > -1$; (2) canonical iff there exists a projective birational morphism $\mu : Y \to X$ from a smooth variety Y with a normal crossing divisor E_i such that $K_Y \equiv \mu^*(K_X + \triangle) + \sum e_i E_i$ holds with $e_i \geq 0$ for all i.*

Ramark. In [**Kol93**], it is proved that f induces isomorphism between algebraic fundamental groups when V is log terminal.

We see that $\pi_1(V^{can}) \cong \pi_1(\widetilde{V})$ by the above theorem since V^{can} has canonical singularities; we have that $(V^{can}, 0)$ is *klt*. Now we take the étale cover $\beta : W' \to V^{can}$ which gives the subgroup $\beta_* \pi_1(W') \subseteq \pi_1(V^{can})$ isomorphic to the subgroup $\Phi_* \pi_1(\widetilde{W}) \subseteq \pi_1(\widetilde{V})$.



We see that β is an étale covering map with the same degree as Φ. Note that W' has canonical singularities since étale morphisms are locally isomorphisms. Because $\phi_{2*}\Phi_*\pi_1(\widetilde{W}) = \beta_*\pi_1(W')$, there is a morphism $h : \widetilde{W} \to W'$ coming from the morphism $\phi_2 : \widetilde{V} \to V^{can}$ extending to the étale covers $\widetilde{W}$ and W' of $\widetilde{V}$ and V^{can}, respectively, and h is birational since ϕ_2 is. In fact, h is a resolution morphism from $\widetilde{W}$ to W'. We want to construct a morphism $q : W' \to W$ such that $p_1 = q \circ h$. Let z_i be a coordinate function defined on W. Since h is proper with connected fiber, $p_1^*(z_i)$ is a function on $\widetilde{W}$ which descends to W' as a continuous function $h_*p_1^*(z_i)$ which is holomorphic outside codimension 1 subvariety of W'. $h_*p_1^*(z_i)$ is actually holomorphic on the smooth part of W' because it is a continuous function on W'. Recall that the singular set of W' consists of isolated normal singularities. So $h_*p_1^*(z_i)$ is actually holomorphic on W'. This gives a morphism $q : W' \to W$ such that $p_1 = q \circ h$. As p_1 and q are birational, q is also birational. If $E \subseteq W'$ is an exceptional curve over W, by the projection formula, we have

$$K_{W'} \cdot E = \beta^* K_{V^{can}} \cdot E = K_{V^{can}} \cdot \beta_* E > 0$$

since β_*E is a sum of exceptional curves in V^{can} over V and $K_{V^{can}}$ is relative ample over V. So $K_{W'}$ is relatively ample over W. Then, by the uniqueness of relative canonical model, we have W' isomorphic to W^{can}. Replacing W' by W^{can}, we have an étale covering map $\beta : W^{can} \to V^{can}$ and this is the claim above, which gives $\sum_{j=1}^{s} cv_W(y_j) = d \sum_{i=1}^{m} cv_V(x_i)$ where $d = \deg\beta = \deg\Phi$.

q.e.d.

We now can prove Theorem C.

Proof. We take $W = V$ in Theorem C. Since the isolated singular points are nonempty, we have $\sum_{i=1}^{m} cv_V(x_i) > 0$. The equation $\sum_{i=1}^{m} cv_V(x_i) = d \sum_{i=1}^{m} cv_V(x_i)$, where d is the degree of Φ, gives $d = 1$; hence, Φ is an isomorphism. q.e.d.

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