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ON THE DIMENSION DATUM OF A SUBGROUP AND ITS APPLICATION TO ISOSPECTRAL MANIFOLDS

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Abstract

The dimension datum of a subgroup of a compact Lie group is a piece of spectral information about that subgroup. We find some new invariants and phenomena of the dimension data and apply them to construct the first example of a pair of isospectral, simply connected closed Riemannian manifolds which are of different homotopy types. We also answer questions proposed by Langlands.

1. Introduction

Let G be a compact Lie group (not necessarily connected). We denote by $\widehat{G}$ the set of irreducible (continuous) representations of G up to equivalence, by $G^\natural$ the set of conjugacy classes in G, and by μ_G the normalized Haar measure on G.

Let H be a closed subgroup of G. It is known [**18**] that the following three objects associated to H contain the same information about H:

- the function $\mathscr{D}_H : \widehat{G} \to \mathbb{Z}$, $V \mapsto \dim V^H$, called the *dimension datum* of H;
- the equivalence class of the G-representation $L^2(G/H)$;
- the push-forward of μ_H by the composition $H \hookrightarrow G \to G^\natural$, as a measure on $G^\natural$.

It is natural to consider the following:

Question 1.1. To what extent is H (up to G-conjugacy) determined by its dimension datum $\mathscr{D}_H$?

One may consider the dimension datum to be spectral in nature, and paraphrase the question (following Bers and Kac [**9**]) as "can one hear the shape of a subgroup?" Ideas around this question have been used for the determination of the monodromy groups in arithmetic geometry and for some constructions in inverse spectral geometry (on the problem "can one hear the shape of a drum?"). In the theory of automorphic forms, Langlands [**14**] has suggested using the dimension datum as a key ingredient in his programme "Beyond Endoscopy." The idea is to use

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the dimension datum to identify the conjectural subgroup ${}^{\lambda}H_{\pi} \subset {}^{L}\mathscr{G}$ associated to an automorphic representation π of $\mathscr{G}(\mathbb{A})$, where ${}^{L}\mathscr{G}$ is the L-group of $\mathscr{G}$ in the form of [**2**, 2.4(2)]. Strictly speaking, for this one should formulate and consider the dimension datum problem for complex reductive groups. By a well-known principle, which we review in Section 8, this is equivalent to the problem we are considering here, by taking G to be a maximal compact subgroup of ${}^{L}\mathscr{G}$.

In particular, Langlands [**14**, 1.1] wrote that it will be important to establish the following.

Theorem 1.2. *If the function $\mathscr{D}_H$ is given, then there are only finitely many possibilities for the conjugacy class of H.*

(Langlands in fact concerns the finiteness of the possible G°-conjugacy classes of H, but that is the same as the finiteness of the possible G-conjugacy classes.) The first contribution in this paper is a proof of this expectation of Langlands. A different and independent proof will appear in another paper by the third author [**33**]. More finiteness results will be given in Section 3. Next, we will prove

Theorem 1.3. *The dimension datum $\mathscr{D}_H$ determines the cardinality $|H/H^{\circ}|$ of H/H°, the dimension of H, the rank of H, the G-conjugacy class of the maximal tori of H, and $\mathscr{D}_{H^{\circ}}$, where H° is the neutral component of H.*

The most striking result about dimension data is due to Larsen and Pink [**18**]. It implies the following, when combined with Theorem 1.3.

Theorem 1.4. (Larsen and Pink) *If H_1 and H_2 are semisimple subgroups of G such that $\mathscr{D}_{H_1} = \mathscr{D}_{H_2}$, then H_1° is isomorphic to H_2°.*

Here, a compact Lie group is called semisimple if its Lie algebra is semisimple. In view of this result, one may hope to extend it by dropping the semisimplicity hypothesis, or to prove that certain numerical invariants of H°, such as the order of the Weyl group, the Coxeter number (see the paragraph before Theorem 6.1 for the definition), the Betti numbers, or the semisimple rank, are determined by $\mathscr{D}_H$. However, all these hopes are falsified by Part (1) of the following theorem.

Theorem 1.5. *Let $n \geqslant 2$, $n_1 = \lfloor (n-1)/2 \rfloor$, $n_2 = \lfloor n/2 \rfloor$. Put $H_1 = \mathrm{U}(n)$, $H_{k+1} = \mathrm{Sp}(n_k) \times \mathrm{SO}(2n-2n_k)$, $k = 1, 2$. Embed H_1 into $\mathrm{U}(2n)$ by $\mathrm{st} \oplus \mathrm{st}^*$, where st is the standard representation of H_1. Embed H_{k+1} into $\mathrm{U}(2n)$ by $\mathrm{st}' \otimes \mathbf{1} \oplus \mathbf{1} \otimes \mathrm{st}''$, where st' and st'' are the standard representations of $\mathrm{Sp}(n_k)$ and $\mathrm{SO}(2n-2n_k)$ respectively. Then the images of H_1, H_2, and H_3 lie in $\mathrm{SU}(2n)$. Moreover, for any compact Lie group G containing $\mathrm{SU}(2n)$, we have:*

(1) *If n is odd, then $\mathscr{D}_{H_1} = \mathscr{D}_{H_2}$.*
(2) *If n is even, then $2\mathscr{D}_{H_1} = \mathscr{D}_{H_2} + \mathscr{D}_{H_3}$.*

Part (2) of this theorem can be used to show that a problem raised by Langlands (Question 5.3) has a negative answer in general. See Corollary 5.4.

Part (1) of the theorem gives the first examples of *connected* subgroups H_1 and H_2 such that $\mathscr{D}_{H_1} = \mathscr{D}_{H_2}$ but H_1 is not isomorphic to H_2. There are abundant examples of this sort in the literature (see Section 7), but with *finite* H_1, H_2. Larsen and Pink [**18**] also showed that one can have connected H_1, H_2 such that $\mathscr{D}_{H_1} = \mathscr{D}_{H_2}$ and H_1 is not G-conjugate to H_2. But the examples of Larsen and Pink are semisimple groups and hence H_1 is isomorphic to H_2 by Theorem 1.4.

We remark that although the hope of extending Theorem 1.4 without the semisimplicity hypothesis is shattered by Theorem 1.5, one may try to classify all counterexamples. This and the description of linear relations amongst $\mathscr{D}_H$ will be pursued by the third author [**32**]. On the other hand, although the Coxeter number is not determined by $\mathscr{D}_H$, we have

Theorem 1.6. *If H_1 and H_2 are closed subgroups of G such that $\mathscr{D}_{H_1} = \mathscr{D}_{H_2}$, then $|h_1 - h_2| \leqslant 1$, where h_i is the Coxeter number of H_i°, $i = 1, 2$.*

Application to isospectral manifolds. We now describe a geometric application of Theorem 1.5 (1). Recall that two closed Riemannian manifolds are called *isospectral* if the eigenvalues of their Laplacians, counting multiplicities, coincide. Many examples reveal that closed isospectral manifolds can be neither locally isometric nor homeomorphic (see [**8**] and the references therein). The relationship between isospectral manifolds and dimension data of connected subgroups was discovered by Sutton [**28**], who proved that if $\mathscr{D}_{H_1} = \mathscr{D}_{H_2}$, then the Riemannian manifolds G/H_1 and G/H_2 are isospectral. This generalized Sunada's and Pesce's method for producing isospectral manifolds ([**27**], [**23**]). Sutton [**28**] also showed that for non-conjugate H_1 and H_2 with $\mathscr{D}_{H_1} = \mathscr{D}_{H_2}$, G/H_1 and G/H_2 are not locally isometric. The simplest example of this kind constructed by Sutton, which is based on an example of Larsen and Pink, has dimension on the order of 10^{10}, and it is difficult to determine whether they are homeomorphic ([**28**, Remark 3.7]). Theorem 1.5 (1) can be used to construct similar examples of dimension as small as 26. Moreover, since the subgroups H_1 and H_2 we use have different homotopic invariants, it is easy to show that G/H_1 and G/H_2 have different homotopy types. This provides the first example of pairs of isospectral, simply connected, non-homeomorphic closed manifolds:

Theorem 1.7. *Let G, H_1, and H_2 be as in Theorem* 1.5 *(1). Assume that G is connected and simply connected. Then the compact homogeneous Riemannian manifolds G/H_1 and G/H_2 are isospectral, simply connected, and have different homotopy types.*

Sections 2–6 are devoted to the proofs of the above-mentioned results. Various examples are given in Section 7 to further illustrate the difference between $\sim$ and $\sim_{\mathrm{LP}}$, and between $\prec$ and $\prec_{\mathrm{LP}}$. These four relations are defined in the next paragraph.

Notation and conventions. By a subgroup of a compact Lie group, we always mean a closed subgroup. For two subgroups H and H' of G, we write (following Langlands)

- $H \sim H'$ if H is G-conjugate to H',
- $H \sim_{\mathrm{LP}} H'$ if $\mathscr{D}_H = \mathscr{D}_{H'}$,
- $H \prec H'$ if H is G-conjugate to a subgroup of H',
- $H \prec_{\mathrm{LP}} H'$ if $\mathscr{D}_H(V) \geqslant \mathscr{D}_{H'}(V)$ for all $V \in \widehat{G}$.

We often use the same notation for an equivalence class of representations, and a particular representation in that class, since there is no danger of confusion. Similarly, we often use the same notation for a G-conjugacy class of subgroups, and a particular subgroup in that class.

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2. A finiteness theorem

In this section, we will prove Theorem 1.2, which says that the fibers of the map

$$\mathscr{D} : \{\text{subgroups of } G\}/(G\text{-conjugacy}) \to \mathbb{Z}^{\widehat{G}}, \qquad H \mapsto \mathscr{D}_H$$

are finite, where $\mathbb{Z}^{\widehat{G}}$ is the set of all functions $\widehat{G} \to \mathbb{Z}$. The key is the following finiteness theorem of Mostow ([**20**]; see also [**12**, Theorem 4.23]).

Theorem 2.1. *Let G be a compact Lie group and let X be a compact G-manifold. Then the set $\{G_x : x \in X\}/(G\text{-conjugacy})$ is finite, where $G_x = \{g \in G : g.x = x\}$.*

We first prove two lemmas.

Lemma 2.2. *Let $n \in \mathbb{Z}^{\widehat{G}}$ be such that $0 \leqslant n(V) \leqslant \dim V$ for all $V \in \widehat{G}$. Then there exist a finite set $\{H_1, \ldots, H_m\}$ of subgroups of G and a finite subset S of $\widehat{G}$ such that*

(1) $\mathscr{D}_{H_j}(V) \geqslant n(V)$ *for all* $1 \leqslant j \leqslant m$ *and* $V \in \widehat{G}$, *and*
(2) *if* H *is a subgroup of* G *satisfying* $\mathscr{D}_H(V) \geqslant n(V)$ *for* $V \in S$, *then* $H \prec H_j$ *for some* j, *and hence* $\mathscr{D}_H(V) \geqslant n(V)$ *for all* $V \in \widehat{G}$.

Proof. The lemma is evident if G is finite. For the rest of this proof, we assume that G is infinite. Then $\widehat{G}$ is countably infinite. We enumerate $\widehat{G}$ as $\{V_i\}_{i\geqslant 1}$ and fix a G-invariant inner product on each V_i. Let X_i be the compact G-manifold consisting of n_i-tuples of orthonormal vectors in V_i, where $n_i = n(V_i)$. Then a subgroup H of G satisfies $\mathscr{D}_H(V_i) \geqslant n_i$ if and only if $X_i^H \neq \emptyset$.

We inductively construct a rooted tree $\mathscr{T}$ and a map $x \mapsto G(x)$ from the set of nodes of $\mathscr{T}$ to the set of subgroups of G as follows. For the root x_0 of $\mathscr{T}$ we set $G(x_0) = G$. Suppose that the nodes x together with $G(x)$ of level up to $k \geqslant 0$ have been constructed. Let x be a node of level k. If $X_i^{G(x)} \neq \emptyset$ for all $i \geqslant 1$, we do not attach any more nodes to x. Otherwise, let $i(x)$ be the smallest integer such that $X_{i(x)}^{G(x)} = \emptyset$. Then by the above theorem of Mostow, there exist finitely many proper subgroups $G_1, \ldots, G_r$ of $G(x)$ such that the stabilizer subgroup in $G(x)$ of every point in $X_{i(x)}$ is conjugate to some G_j. We create r new nodes $y_1, \ldots, y_r$ of level $k+1$, link them to x, and set $G(y_j) = G_j$. By doing so for each node x of level k, we obtain all the nodes of level $k+1$. This completes the inductive construction.

From the construction we see that $\mathscr{T}$, $i(x)$, and $G(x)$ satisfy the following properties.

(i) If x is a terminal node of $\mathscr{T}$, then $X_i^{G(x)} \neq \emptyset$ for all $i \geqslant 1$.
(ii) If x is a non-terminal node of $\mathscr{T}$ and H is a subgroup of $G(x)$ with $X_{i(x)}^H \neq \emptyset$, then there exists a node y of level one greater than that of x, which is adjacent to x, such that $H \prec G(y)$.
(iii) If $x_1, \ldots, x_s$ form a path in $\mathscr{T}$ with increasing levels, then $G(x_1) \supsetneq \cdots \supsetneq G(x_s)$.

Since there is no infinite descending chain of subgroups of G, from (iii) we see that there is no infinite path in $\mathscr{T}$. Note also that each node of $\mathscr{T}$ is adjacent to finitely many other nodes. By König's lemma (see [**19**, page 298]), $\mathscr{T}$ has only finitely many nodes. We claim that

$$\{H_1, \ldots, H_m\} = \{G(x) : x \text{ is a terminal node of } \mathscr{T}\}$$

and

$$N = \max\{i(x) : x \text{ is a non-terminal node of } \mathscr{T}\}$$

satisfy the requirement of the theorem.

By (i) above, we have $X_i^{H_j} \neq \emptyset$ for all $i \geqslant 1$ and $1 \leqslant j \leqslant m$. So (1) is satisfied. Let H be a subgroup of G such that $X_i^H \neq \emptyset$ for $1 \leqslant i \leqslant N$. We inductively construct a path $x_0, \ldots, x_k$ in $\mathscr{T}$ with increasing level, starting from the root x_0, ending at a terminal node x_k, and such that

$H \prec G(x_j)$ for each $j = 0, \ldots, k$, as follows. Note that $G(x_0) = G$ contains H. Suppose that $x_0, \ldots, x_j$ have been constructed. If x_j is a terminal node, then the path ends there. Otherwise, since $i(x_j) \leqslant N$, we have $X^H_{i(x_j)} \neq \emptyset$. We have $H \prec G(x_j)$ by the induction hypothesis. Then by (ii), there exists a node x_{j+1} of level $j+1$, which is adjacent to x_j, such that $H \prec G(x_{j+1})$. This completes the construction of the path. Now $G(x_k)$, which is equal to some H_j, contains a conjugate of H. This proves (2) and completes the proof of the lemma. q.e.d.

Lemma 2.3. *Let $H \subsetneq H'$ be subgroups of G. Then there exists $V \in \widehat{G}$ such that* $\dim V^H > \dim V^{H'}$.

Proof. Let U be a nontrivial irreducible subrepresentation of $W = \mathrm{Ind}_H^{H'}\mathbf{1}$, where $\mathbf{1}$ is the trivial representation of H. Then $U^{H'} = 0$. By Frobenius reciprocity, we have $U^H \neq 0$. Thus $U^H \supsetneq U^{H'}$. Let V be an irreducible constituent of $\mathrm{Ind}_{H'}^G U$. By Frobenius reciprocity again, the restriction of V to H' contains U. This implies that $V^H \supsetneq V^{H'}$. Hence $\dim V^H > \dim V^{H'}$. q.e.d.

Proof of Theorem 1.2. Let $n \in \mathbb{Z}^{\widehat{G}}$. We want to show that $\mathscr{D}^{-1}(n)$ is finite. We may and do assume $0 \leqslant n(V) \leqslant \dim V$ for all $V \in \widehat{G}$ (for otherwise $\mathscr{D}^{-1}(n)$ is empty). Therefore, by Lemma 2.2, there exists a finite set $\{H_1, \ldots, H_m\}$ of subgroups of G such that if a subgroup H of G satisfies $\mathscr{D}_H(V) \geqslant n(V)$ for all V, then $H \prec H_j$ for some $j \in \{1, \ldots, m\}$.

Let H be a subgroup of G such that $\mathscr{D}_H = n$; then some H_j contains a conjugate of H. We claim that H_j is indeed equal to this conjugate of H. For otherwise, by Lemma 2.3, there exists $V \in \widehat{G}$ such that $n(V) = \dim V^H > \dim V^{H_j} \geqslant n(V)$, a contradiction. This completes the proof. q.e.d.

3. More finiteness results: isolated points in $\mathrm{Im}(\mathscr{D})$

Notice that Lemma 2.2 has the following interesting consequence.

Proposition 3.1. *Given $n \in \mathbb{Z}^{\widehat{G}}$, there exists a finite subset S of $\widehat{G}$ such that if H is a subgroup of G and $\mathscr{D}_H(V) \geqslant n(V)$ for $V \in S$, then $\mathscr{D}_H(V) \geqslant n(V)$ for all $V \in \widehat{G}$.*

This leads to the following question concerning a subgroup H of G. A variant of this question was discussed by Larsen in [**17**] and the results in this section are heavily influenced by Larsen's ideas.

Question 3.2. Let $n = \mathscr{D}_H \in \mathbb{Z}^{\widehat{G}}$ be the dimension datum of a subgroup H. Is there a finite set $S \in \widehat{G}$ such that for any subgroup H', $\mathscr{D}_{H'}(V) = n(V)$ for $V \in S$ implies $\mathscr{D}_{H'}(V) = n(V)$ for all $V \in \widehat{G}$?

Equivalently,

Question 3.3. Is $\mathscr{D}_H$ an isolated point in the image $\mathrm{Im}(\mathscr{D})$ of $\mathscr{D}$? Here $\mathrm{Im}(\mathscr{D}) \subset \mathbb{Z}^{\widehat{G}}$ is given the induced topology from $\mathbb{Z}^{\widehat{G}}$ (with product topology while regarding each factor $\mathbb{Z}$ discrete).

If the answer is affirmative, one may say that (for fixed H and varying H') verifying $\mathscr{D}_H = \mathscr{D}_{H'}$ can be checked on a finite set of representations. For example, the question has an affirmative answer when $H = \{1\}$ (by letting $n(V) = \dim V$ in Proposition 3.1, or by considering a faithful representation of G), or more generally when H is finite (by the theorem below). Such results, in particular with a concrete set S, would be very useful for the application of dimension data to monodromy groups or automorphic forms [**11**].

Theorem 3.4. *Let H be a subgroup of G. Put $n = \mathscr{D}_H$ and*

$$\{K_1, \dots, K_r\} = \mathscr{D}^{-1}(n).$$

Then $\mathscr{D}_H$ is isolated in $\mathrm{Im}(\mathscr{D})$ if and only if $K_1, \dots, K_r$ are all semisimple.

Lemma 3.5. *Let J be a compact Lie group such that J° is a torus T. There exists a finite subgroup F of J such that F meets every component of J.*

Proof. Let $\pi_0 = \pi_0(J)$ and consider the extension $1 \to T \to J \to \pi_0 \to 1$ and the corresponding class E in $H^2(\pi_0, T)$. The group $H^2(\pi_0, T)$, being compact and killed by $|\pi_0|$, is finite, say of order N. Therefore, the class E, being in the kernel of $H^2(\pi_0, T) \xrightarrow{N} H^2(\pi_0, T)$, comes from $H^2(\pi_0, T_N)$, where T_N is the subgroup of N-torsion elements in T. This gives a group F which is an extension of π_0 by T_N, embedded in J, and meets every component of J. q.e.d.

Lemma 3.6. *If H is not semisimple, then $\mathscr{D}_H$ is not isolated in $\mathrm{Im}(\mathscr{D})$.*

Proof. Let D be the derived group of H° and apply the preceding lemma to $J = H/D$. Let $F_n = F.T_{2^n}$ and let K_n be the inverse image of F_n under $H \twoheadrightarrow J$. It is clear that $\{K_n\}_{n \geqslant 1}$ is an increasing sequence of subgroups, and $\bigcup_{n \geqslant 1} K_n$ is dense in K. It follows that for any $V \in \widehat{G}$, $\dim V^H = \dim V^{K_n}$ for all n sufficiently large.

If H is not semisimple, then the torus $T = (H/D)^\circ$ is of positive dimension, the subgroups K_n are proper subgroups of H, and $K_n \subsetneq K_{n+1}$ for all n sufficiently large. It follows that for any finite $S \subset \widehat{G}$, we have $\mathscr{D}_H|_S = \mathscr{D}_{K_n}|_S$ for all n sufficiently large. This shows that $\mathscr{D}_H$ is not isolated. q.e.d.

Proof of Theorem 3.4. The "only if" part is clear from the preceding lemma. Let us assume that $K_1, \dots, K_r$ are all semisimple. By [**17**, Theorem 1.3], for each $i = 1, \dots, r$, there exists a finite set U_i of proper subgroups of K_i such that every proper subgroup of K_i is contained in a K_i-conjugate of some element of U_i. For each $J \in U_i$, pick a representation $V_{i,J} \in \widehat{G}$ such that $\dim V_{i,J}^J > \dim V_{i,J}^{K_i} = n(V_{i,J})$, by Lemma 2.3.

Apply Lemma 2.2 to get finitely many subgroups $H_1, \dots, H_m$ and a finite set $S \subset \widehat{G}$. We may and do assume that $H_i = K_i$ for $i = 1, \dots, r$, and $\mathscr{D}_{H_j} \neq n$ for $j = r+1, \dots, m$. For each $j = r+1, \dots, m$, choose a representation $W_j \in \widehat{G}$ such that $\dim W_j^{H_j} \neq n(W_j)$. Notice that this actually implies $\dim W_j^{H_j} > n(W_j)$. Consider

$$S' = S \cup \{V_{i,J} : i = 1, \dots, r, J \in U_i\} \cup \{W_{r+1}, \dots, W_m\}.$$

We claim that $\mathscr{D}_{H'}|_{S'} = n|_{S'}$ implies $\mathscr{D}_{H'} = n$. Indeed, since $\mathscr{D}_{H'}(V) \geqslant n(V)$ for all $V \in S$, a conjugate of H' lies in a suitable H_j. We can not have $j > r$ since that would give $\dim W_j^{H'} \geqslant \dim W_j^{H_j} > n(W_j)$. Therefore, a conjugate of H' lies in a K_i, for some $1 \leqslant i \leqslant r$. If it is not equal to the whole K_i, it lies in a conjugate of J for some $J \in U_i$, and we have $\dim V_{i,J}^{H'} \geqslant \dim V_{i,J}^J > n(V_{i,J})$. Therefore, a conjugate of H is equal to one of $K_1, \dots, K_m$, and the theorem is proved. q.e.d.

Corollary 3.7. *If $\mathscr{D}_{H^\circ}$ is isolated in $\mathrm{Im}(\mathscr{D})$, then $\mathscr{D}_H$ is also isolated in $\mathrm{Im}(\mathscr{D})$.*

Proof. Let $\{K_1, \dots, K_r\} = \mathscr{D}^{-1}(\mathscr{D}_H)$. Then $\mathscr{D}_{K_i^\circ} = \mathscr{D}_{H^\circ}$ by Theorem 1.3 (the proof of this theorem, to be presented in the next section, doesn't use this corollary). Therefore, K_i°, and hence K_i also, is semisimple by the preceding theorem. By the theorem again, $\mathscr{D}_H$ is isolated in $\mathrm{Im}(\mathscr{D})$. q.e.d.

4. Determination of some invariants I

This section is devoted to the proof of Theorem 1.3. The determination of $\dim H$ for connected semisimple H and that of the G-conjugacy class of maximal tori of H for connected H have been proved in [**18**].

We first recall the measure-theoretic characterization of dimension data (due to [**18**]) mentioned in the introduction. Let $p : G \to G^\natural$ be the quotient map and let $\mathscr{M}$ be the space of (real-valued) measures on $G^\natural$, where $G^\natural$ is endowed with the quotient Borel structure. For a subgroup H of G, we view the normalized Haar measure μ_H on H as a measure on G, supported on H. Then the measure $\mu_H^\natural := p_*(\mu_H) \in \mathscr{M}$ depends only on the conjugacy class of H. From the Peter-Weyl theorem, we obtain

Lemma 4.1. *The linear map*

$$D : \mathscr{M} \to \mathbb{R}^{\widehat{G}}, \qquad D(\mu)(V) = \int_{G^\natural} \mathrm{Tr}(g|V)\, d\mu(g)$$

is injective. In particular, the function $\mathscr{D}_H = D(\mu_H^\natural)$ *determines the measure* $\mu_H^\natural$*, and vice versa.*

We assume for the moment that G and H are connected. Let T be a fixed maximal torus of G, and let $W = W(G,T)$ be the Weyl group. To analyze $\mathscr{D}(H)$ and $\mu_H^\natural$, we may assume without loss of generality that T contains a maximal torus T_H of H. Let $\Phi_H \subset X(T_H)$ be the root system of H, and let Φ_H^+ be a system of positive roots. Let

$$f_H = \prod_{\alpha\in\Phi_H^+}(1-[\alpha]),$$

which is an element in the group algebra $\mathbb{Q}[X(T_H)]$. We identify $\mathbb{Q}[X(T_H)]$ with a subset of the space of (complex-valued) functions on T_H in the natural way. Let

$$\Gamma = \{w \in W : w(T_H) = T_H\}.$$

For a function f on T_H, we set

$$\sigma(f) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\gamma(f).$$

Lemma 4.2. *We have* $\mu_H^\natural = p_*(f_H\mu_{T_H}) = p_*(\sigma(f_H)\mu_{T_H})$.

Proof. Let $W_H = W(H,T_H)$ be the Weyl group of H, and let

$$F_H = \frac{1}{|W_H|}\prod_{\alpha\in\Phi_H}(1-[\alpha]).$$

Then we have

$$\begin{aligned}
|W_H|F_H &= \left(\prod_{\alpha\in\Phi_H^+}(1-[\alpha])\right)\left(\prod_{\alpha\in\Phi_H^+}(1-[-\alpha])\right)\\
&= \left(\sum_{w\in W_H}\operatorname{sgn}(w)[\delta - w\delta]\right)\left(\sum_{w'\in W_H}\operatorname{sgn}(w')[-\delta + w'\delta]\right)\\
&= \sum_{w,w'\in W_H}\operatorname{sgn}(ww')[w'\delta - w\delta]\\
&= \sum_{w'\in W_H} w'\left(\sum_{w''\in W_H}\operatorname{sgn}(w'')[\delta - w''\delta]\right)\\
&= \sum_{w\in W_H} w\left(\prod_{\alpha\in\Phi_H^+}(1-[\alpha])\right) = \sum_{w\in W_H} w(f_H),
\end{aligned}$$

where $\delta = \frac{1}{2}\sum_{\alpha\in\Phi_H^+}\alpha$. From the Weyl integration formula we obtain

$$\begin{aligned}\mu_H^\natural = p_*(F_H\mu_{T_H}) &= \frac{1}{|W_H|}\sum_{w\in W_H} p_*(w(f_H)\mu_{T_H})\\ &= p_*(f_H\mu_{T_H}) = p_*(\sigma(f_H)\mu_{T_H}).\end{aligned}$$

This completes the proof. q.e.d.

Lemma 4.3. *For $f, f' \in L^1(T_H, \mu_{T_H})$, if $p_*(f\mu_{T_H}) = p_*(f'\mu_{T_H})$, then $\sigma(f) = \sigma(f')$.*

Proof. The argument is similar to that in [**18**, page 380]. Since

$$\begin{aligned}p_*\left(\sum_{w\in W} w_*(f\mu_{T_H})\right) &= |W|p_*(f\mu_{T_H})\\ &= |W|p_*(f'\mu_{T_H}) = p_*\left(\sum_{w\in W} w_*(f'\mu_{T_H})\right)\end{aligned}$$

and the restriction of p_* to the set of W-invariant measures on T is injective, we obtain

$$\sum_{w\in W} w_*(f\mu_{T_H}) = \sum_{w\in W} w_*(f'\mu_{T_H}).$$

By restricting the measures on both sides to T_H, we obtain

$$|\Gamma|\sigma(f)\mu_{T_H} = \sum_{\gamma\in\Gamma}\gamma_*(f\mu_{T_H}) = \sum_{\gamma\in\Gamma}\gamma_*(f'\mu_{T_H}) = |\Gamma|\sigma(f')\mu_{T_H}.$$

Thus $\sigma(f) = \sigma(f')$. q.e.d.

Proof of Theorem 1.3. Let H, H' be subgroups of G with $\mathscr{D}(H) = \mathscr{D}(H')$. By embedding G into a connected compact Lie group, say $\mathrm{U}(n)$, we may assume without loss of generality that G is connected. By Lemma 4.1, we have $\mu_H^\natural = \mu_{H'}^\natural$. Let B be a small ball (with respect to some bi-invariant Riemannian metric on G) centered at the identity of G such that $B\cap(H\smallsetminus H^\circ) = B\cap(H'\smallsetminus H'^\circ) = \emptyset$. Then B is invariant under the conjugation of G. Since μ_H is the sum of $\frac{1}{|H/H^\circ|}\mu_{H^\circ}$ and a measure supported on $H\smallsetminus H^\circ$, from Lemma 4.2 we obtain

$$\chi_{p(B)}\mu_H^\natural = \frac{\chi_{p(B)}\mu_{H^\circ}^\natural}{|H/H^\circ|} = \frac{p_*(\chi_B f_{H^\circ}\mu_{T_{H^\circ}})}{|H/H^\circ|}.$$

A similar identity holds for H'. So we have

$$\frac{p_*(\chi_B f_{H^\circ}\mu_{T_{H^\circ}})}{|H/H^\circ|} = \frac{p_*(\chi_B f_{H'^\circ}\mu_{T_{H'^\circ}})}{|H'/H'^\circ|}. \tag{4.1}$$

This implies that $\operatorname{supp}(\chi_B f_{H^\circ}\mu_{T_{H^\circ}}) = T_{H^\circ}\cap B$ and $\operatorname{supp}(\chi_B f_{H'^\circ}\mu_{T_{H'^\circ}}) = T_{H'^\circ}\cap B$ are conjugate. So T_{H° and $T_{H'^\circ}$ are conjugate.

We may assume without loss of generality that $T_{H^\circ} = T_{H'^\circ}$. By Lemma 4.3 and (4.1), the functions $\sigma(f_{H^\circ})/|H/H^\circ|$ and $\sigma(f_{H'^\circ})/|H'/H'^\circ|$ agree on $T_{H^\circ} \cap B$. Since they are analytic on T_{H°, they must agree on T_{H°. From Lemma 4.2 we obtain

$$\mu^\natural_{H^\circ}/|H/H^\circ| = \mu^\natural_{H'^\circ}/|H'/H'^\circ|.$$

By evaluating at $G^\natural$ on both sides, we get $|H/H^\circ| = |H'/H'^\circ|$. Hence $\mu^\natural_{H^\circ} = \mu^\natural_{H'^\circ}$. Now from Lemma 4.1 we obtain $\mathscr{D}(H^\circ) = \mathscr{D}(H'^\circ)$.

It remains to prove that $\dim H = \dim H'$. For $\alpha \in \Phi^+_{H^\circ}$, let $d\alpha$ be the complex-valued linear function on the Lie algebra $\mathrm{Lie}(T_{H^\circ})$ of T_{H° such that $\alpha(\exp(v)) = e^{d\alpha(v)}$ for all $v \in \mathrm{Lie}(T_{H^\circ})$. Then we have

$$f_{H^\circ}(\exp(v)) = \prod_{\alpha\in\Phi^+_{H^\circ}} (1 - e^{d\alpha(v)}) = \prod_{\alpha\in\Phi^+_{H^\circ}} \left(-\sum_{n=1}^{\infty} \frac{(d\alpha(v))^n}{n!} \right).$$

Thus the smallest order of nontrivial terms in the power series expansion of the analytic function $f_{H^\circ} \circ \exp$ on $\mathrm{Lie}(T_{H^\circ})$, and hence that of $\sigma(f_{H^\circ}) \circ \exp$, is equal to $|\Phi^+_{H^\circ}|$. A similar result holds for H'°. Since $\sigma(f_{H^\circ}) = \sigma(f_{H'^\circ})$, we have $|\Phi^+_{H^\circ}| = |\Phi^+_{H'^\circ}|$. Hence

$$\dim H = 2|\Phi^+_{H^\circ}| + \dim T_{H^\circ} = 2|\Phi^+_{H'^\circ}| + \dim T_{H'^\circ} = \dim H'.$$

This completes the proof. q.e.d.

5. Theorem 1.5 and its applications

We first recall the following determinant identities.

Lemma 5.1. *Consider the elements in* $\mathbb{Q}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ *defined by*

$$\tilde{a}_n = \det\left[t_j^{i-j}\right]_{i,j=1}^n, \qquad \tilde{b}_n = \det\left[t_j^{i-j} - t_j^{2n+1-i-j}\right]_{i,j=1}^n,$$

$$\tilde{c}_n = \det\left[t_j^{i-j} - t_j^{2n+2-i-j}\right]_{i,j=1}^n, \qquad \tilde{d}_n = \frac{1}{2}\det\left[t_j^{i-j} + t_j^{2n-i-j}\right]_{i,j=1}^n.$$

Then we have

$$\tilde{a}_n = \prod_{1\leqslant i<j\leqslant n} (1 - t_i t_j^{-1}),$$

$$\tilde{b}_n = \prod_{1\leqslant i<j\leqslant n} (1 - t_i t_j)(1 - t_i t_j^{-1}) \prod_{i=1}^n (1 - t_i),$$

$$\tilde{c}_n = \prod_{1\leqslant i<j\leqslant n} (1 - t_i t_j)(1 - t_i t_j^{-1}) \prod_{i=1}^n (1 - t_i^2),$$

$$\tilde{d}_n = \prod_{1\leqslant i<j\leqslant n} (1 - t_i t_j)(1 - t_i t_j^{-1}).$$

Proof. The identity for $\tilde{a}_n$ is a variant of the Vandermonde determinant. The other three identities are variants of (2.3)–(2.5) in [**13**, Lemma 2]. q.e.d.

Lemma 5.2. *Let $x_0 = 1$, and consider the polynomials*

$$a_n = \det\left[x_{|i-j|}\right]_{i,j=1}^n, \qquad b_n = \det\left[x_{|i-j|} - x_{2n+1-i-j}\right]_{i,j=1}^n,$$

$$c_n = \det\left[x_{|i-j|} - x_{2n+2-i-j}\right]_{i,j=1}^n, \quad d_n = \frac{1}{2}\det\left[x_{|i-j|} + x_{2n-i-j}\right]_{i,j=1}^n$$

in $\mathbb{Q}[x_1, x_2, \ldots]$. Then

$$a_{2n+1} = c_n d_{n+1}, \tag{5.1}$$

$$2a_{2n} = c_n d_n + c_{n-1} d_{n+1}. \tag{5.2}$$

Proof. This is proved in [**30**, Section 4.5.2]. We sketch another proof below.

We view x_i as real variables, denote the matrices in the definitions of a_n, c_n, and d_n by A_n, C_n, and D_n, respectively, and let D'_n be the matrix obtained from D_n by dividing the last row by 2. Let L_n be the linear transformation on $\mathbb{R}^n$ with matrix A_n relative to the standard ordered basis $\{e_1, \ldots, e_n\}$. To prove (5.1), consider the decomposition $\mathbb{R}^{2n+1} = V_1 \oplus V_2$, where

$$V_1 = \operatorname{span}\{e_1 - e_{2n+1}, e_2 - e_{2n}, \ldots, e_n - e_{n+2}\},$$
$$V_2 = \operatorname{span}\{e_1 + e_{2n+1}, e_2 + e_{2n}, \ldots, e_n + e_{n+2}, 2e_{n+1}\}.$$

It is easy to verify that L_{2n+1} preserves the decomposition, and with respect to the ordered basis of V_1 (resp. V_2) specified above, the matrix for $L_{2n+1}|_{V_1}$ (resp. $L_{2n+1}|_{V_2}$) is C_n (resp. D'_{n+1}). Thus (5.1) follows.

Let P_{2n} be the linear transformation on $\mathbb{R}^{2n}$ defined by $P_{2n}(e_i) = x_i e_{2n}$, $1 \leqslant i \leqslant 2n$. Then $2a_{2n} = \det(L_{2n} + P_{2n}) + \det(L_{2n} - P_{2n})$. Consider the subspaces of $\mathbb{R}^{2n}$ defined by

$$V^+ = \operatorname{span}\{e_1 - e_{2n-1}, e_2 - e_{2n-2}, \ldots, e_{n-1} - e_{n+1}\},$$
$$V^- = \operatorname{span}\{e_1 + e_{2n-1}, e_2 + e_{2n-2}, \ldots, e_{n-1} + e_{n+1}, 2e_n\}.$$

Then V^+ (resp. V^-) is invariant under $L_{2n} + P_{2n}$ (resp. $L_{2n} - P_{2n}$), and with respect to its ordered basis specified above, the matrix for $(L_{2n} + P_{2n})|_{V^+}$ (resp. $(L_{2n} - P_{2n})|_{V^-}$) is C_{n-1} (resp. D'_n). Moreover, the linear transformation on the quotient space $\mathbb{R}^{2n}/V^+$ (resp. $\mathbb{R}^{2n}/V^-$) induced by $L_{2n} + P_{2n}$ (resp. $L_{2n} - P_{2n}$) has matrix D'_{n+1} (resp. C_n) with respect to the ordered basis

$$\{2e_{2n} + V^+, e_1 + e_{2n-1} + V^+, \ldots, e_{n-1} + e_{n+1} + V^+, 2e_n + V^+\}$$
$$(\text{resp.} \quad \{-2e_{2n} + V^-, e_1 - e_{2n-1} + V^-, \ldots, e_{n-1} - e_{n+1} + V^-\}).$$

Thus $\det(L_{2n} + P_{2n}) = c_{n-1} d_{n+1}$, $\det(L_{2n} - P_{2n}) = c_n d_n$. This proves (5.2). q.e.d.

Proof of Theorem 1.5. It suffices to prove the theorem when $G = \mathrm{SU}(2n)$. Let

$$T = \{\mathrm{diag}(t_1, \ldots, t_{2n}) : t_i \in \mathrm{U}(1), t_1 \cdots t_{2n} = 1\},$$

and let

$$T_{H_1} = \{t = \mathrm{diag}(t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}) : t_i \in \mathrm{U}(1)\},$$

which is a maximal torus of H_1. Note that $\Gamma \simeq S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$, which acts on T_{H_1} in the natural way. Let $\epsilon_i \in X(T_{H_1})$ $(1 \leqslant i \leqslant n)$ be the character $\epsilon_i(t) = t_i$. We choose

$$\Phi^+_{H_1} = \{\epsilon_i - \epsilon_j : 1 \leqslant i < j \leqslant n\}.$$

Then

$$f_{H_1}(t) = \prod_{1\leqslant i<j\leqslant n} (1 - t_i t_j^{-1}) = \tilde{a}_n.$$

It is easy to see that H_{k+1} $(k = 1, 2)$ has a conjugate, which we still denote by H_{k+1}, such that it contains T_{H_1} as a maximal torus and has a system of positive roots

$$\begin{aligned}\Phi^+_{H_{k+1}} = &\{\epsilon_i \pm \epsilon_j : 1 \leqslant i < j \leqslant n_k\} \cup \{2\epsilon_i : 1 \leqslant i \leqslant n_k\} \\ &\cup \{\epsilon_i \pm \epsilon_j : n_k + 1 \leqslant i < j \leqslant n\}.\end{aligned}$$

Then

$$\begin{aligned}f_{H_{k+1}}(t) = &\prod_{1\leqslant i<j\leqslant n_k} (1 - t_i t_j)(1 - t_i t_j^{-1}) \prod_{i=1}^{n_k} (1 - t_i^2) \\ &\times \prod_{n_k+1\leqslant i<j\leqslant n} (1 - t_i t_j)(1 - t_i t_j^{-1}) = \tilde{c}_{n_k} \tilde{d}^{(n_k)}_{n-n_k},\end{aligned}$$

where $\tilde{d}^{(n_k)}_{n-n_k}$ is obtained from $\tilde{d}_{n-n_k}$ by replacing $t_1, \ldots, t_{n-n_k}$ with $t_{n_k+1}, \ldots, t_n$, respectively. By Lemmas 4.1 and 4.2, it suffices to prove that

$$\begin{cases} \sigma(\tilde{a}_n) = \sigma(\tilde{c}_{n_1} \tilde{d}^{(n_1)}_{n-n_1}) & n \text{ odd}, \\ 2\sigma(\tilde{a}_n) = \sigma(\tilde{c}_{n_1} \tilde{d}^{(n_1)}_{n-n_1}) + \sigma(\tilde{c}_{n_2} \tilde{d}^{(n_2)}_{n-n_2}) & n \text{ even}. \end{cases}$$

Consider the linear map $\nu : \mathbb{Q}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] \to \mathbb{Q}[x_1, x_2, \ldots]$ determined by $\nu(1) = 1$ and $\nu(t_{i_1}^{k_1} \cdots t_{i_r}^{k_r}) = x_{|k_1|} \cdots x_{|k_r|}$, where $k_j \neq 0$. It is easy to see that

(i) ν is Γ-invariant. In particular, $\nu \circ \sigma = \nu$.
(ii) The restriction of ν on $\mathbb{Q}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]^\Gamma$ is injective.
(iii) For $f \in \mathbb{Q}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, we denote by $\mathrm{supp}(f)$ the smallest subset J of $\{1, \ldots, n\}$ such that $f \in \mathbb{Q}[t_j^{\pm 1} : j \in J]$. Then for f_1, f_2 with $\mathrm{supp}(f_1) \cap \mathrm{supp}(f_2) = \emptyset$, we have $\nu(f_1 f_2) = \nu(f_1)\nu(f_2)$.

So it suffices to prove that

$$\begin{cases} \nu(\tilde{a}_n) = \nu(\tilde{c}_{n_1})\nu(\tilde{d}^{(n_1)}_{n-n_1}) & n \text{ odd,} \\ 2\nu(\tilde{a}_n) = \nu(\tilde{c}_{n_1})\nu(\tilde{d}^{(n_1)}_{n-n_1}) + \nu(\tilde{c}_{n_2})\nu(\tilde{d}^{(n_2)}_{n-n_2}) & n \text{ even.} \end{cases}$$

Since $\mathrm{supp}(t_j^{i-j}) \subset \{j\}$, monomials in different columns of the matrix $\left[t_j^{i-j}\right]_{i,j=1}^n$ have mutually disjoint supports. Thus

$$\nu(\tilde{a}_n) = \nu\left(\det\left[t_j^{i-j}\right]_{i,j=1}^n\right) = \det\left[\nu(t_j^{i-j})\right]_{i,j=1}^n = \det\left[x_{|i-j|}\right]_{i,j=1}^n = a_n.$$

Similarly, we have

$$\nu(\tilde{c}_{n_k}) = \det\left[\nu(t_j^{i-j} - t_j^{2n_k+2-i-j})\right]_{i,j=1}^{n_k} = c_{n_k},$$
$$\nu(\tilde{d}^{(n_k)}_{n-n_k}) = \frac{1}{2}\det\left[\nu(t_{n_k+j}^{i-j} + t_{n_k+j}^{2n-2n_k-i-j})\right]_{i,j=1}^{n-n_k} = d_{n-n_k}.$$

Taking Lemma 5.2 into account, this completes the proof. q.e.d.

Remark. The technique of computing in the ring $\mathbb{Q}[x_1, x_2, \ldots]$ in the above proof is due to Larsen and Pink. They also introduced the elements b_n, c_n, d_n (which are denoted as b'_n, c'_n, d'_n in [**18**]) in this ring. What is new here is that we introduce a new family of polynomials a_n, and obtain simple formulas of a_n, b_n, c_n, d_n as determinants (notice that we also have $\nu(\tilde{b}_n) = b_n$ by the same argument). For the sake of clarity, we make our proof self-contained without direct reference to [**18**].

Proof of Theorem 1.7. The isospectrality of G/H_1 and G/H_2 follows from [**28**, Theorem 2.3] and Theorem 1.5. Consider the homotopy exact sequence

$$\cdots \to \pi_2(G) \to \pi_2(G/H_1) \to \pi_1(H_1) \to \pi_1(G) \to \pi_1(G/H_1) \to \cdots .$$

Since $\pi_1(G)$ and $\pi_2(G)$ are trivial by our hypothesis on G ([**5**, Proposition V(7.5)]), we have $\pi_1(G/H_1) = 1$ and

$$\pi_2(G/H_1) \simeq \pi_1(H_1) \simeq \pi_1(\mathrm{U}(n)) \simeq \mathbb{Z}.$$

Similarly, we have $\pi_1(G/H_2) = 1$ and

$$\pi_2(G/H_2) \simeq \pi_1(H_2) \simeq \pi_1(\mathrm{Sp}(n_1) \times \mathrm{SO}(2n - 2n_1)) \simeq \mathbb{Z}/2\mathbb{Z}.$$

This shows that G/H_1 and G/H_2 are simply connected, and they have different homotopy types. q.e.d.

Next, we will show an implication of Theorem 1.5 (2) for another question raised by Langlands in [**14**, 1.1 and 1.6]. Denote by $\mathbb{R}[\widehat{G}]$ the free $\mathbb{R}$-module with basis $\widehat{G}$, so that its dual space $\mathrm{Hom}(\mathbb{R}[\widehat{G}], \mathbb{R})$ is $\mathbb{R}^{\widehat{G}}$.

Question 5.3. Let $\mathscr{L}$ be a set of subgroups of G. Can we find a collection $\{a_H\}_{H\in\mathscr{L}}$ of elements in $\mathbb{R}[\widehat{G}]$ with the following property?

$$\text{For all } H, H' \in \mathscr{L}, \qquad (a_H, \mathscr{D}_{H'}) = \begin{cases} 1 & \text{if } H' \sim_{\mathrm{LP}} H, \\ 0 & \text{if } H' \not\prec_{\mathrm{LP}} H, \end{cases}$$

where $(-,-)$ is the natural pairing between $\mathbb{R}[\widehat{G}]$ and $\mathbb{R}^{\widehat{G}}$.

Langlands proposed that the existence of $\{a_H\}_{H\in\mathscr{L}}$ may facilitate a way to deal with the dimension datum of ${}^\lambda H_\pi$ using the trace formula.

In [**14**, 1.2], Langlands started with the class $\mathscr{L}_1 = \{H \subset G : H \to G/G^\circ \text{ is surjective}\}$. He then analyzed the case $G = \mathrm{SU}(2)\times F$, where F is a finite group, in [**14**, 1.3] and decided that it is necessary to restrict to a smaller class ([**14**, 1.4]): $\mathscr{L}_2 = \{H \subset G : H \cap G^\circ = H^\circ \text{ and } H/H^\circ \simeq G/G^\circ\}$ so that there is a chance of an affirmative answer for Question 5.3. (Langlands expects this restricted class to be enough for his purpose in that $\mathscr{L}_2$ should contain all his conjectural groups ${}^\lambda H_\pi$'s; see also [**1**, Section 5].) Indeed, for $G = \mathrm{SU}(2) \times F$ one can show the existence of $\{a_H\}_{H\in\mathscr{L}}$ for $\mathscr{L} = \mathscr{L}_2$. However, Langlands suspected ([**14**, discussions following (14)]) that in general Question 5.3 cannot be solved exactly (for $\mathscr{L} = \mathscr{L}_2$). The following result confirms this.

Corollary 5.4. *Let $n \geqslant 2$ be even. Let H_1, H_2, H_3 be as in Theorem* 1.5*, and let G be any connected compact Lie group containing* $\mathrm{SU}(2n)$*. Then the answer to Question* 5.3 *is negative for any class $\mathscr{L}$ containing $\{H_1, H_2, H_3\}$.*

We first give a simple lemma which offers a basic obstruction to the existence of $\{a_H\}_{H\in\mathscr{L}}$.

Lemma 5.5. *If $\{a_H\}_{H\in\mathscr{L}}$ exists, then $\{\mathscr{D}_{H_1}, \ldots, \mathscr{D}_{H_n}\}$ is linearly independent for any $H_1, \ldots, H_n$ such that $\mathscr{D}_{H_i} \neq \mathscr{D}_{H_j}$ for $i \neq j$.*

Proof. Let $\sum_{i=1}^n c_i \mathscr{D}_{H_i} = 0$ be a non-trivial linear relation. We may and do assume that $c_i \neq 0$ for all $i = 1, \ldots, n$. Assume also that H_1 is a minimal element in the partially ordered set $(\{H_1, \ldots, H_n\}, \prec_{\mathrm{LP}})$. Then $(a_{H_1}, \mathscr{D}_{H_i}) = 0$ for $i = 2, \ldots, n$. The linear relation then implies $(a_{H_1}, \mathscr{D}_{H_1}) = 0$, a contradiction. q.e.d.

Proof of Corollary 5.4. By the lemma and Theorem 1.5 (2), it suffices to show that $\mathscr{D}_{H_1}, \mathscr{D}_{H_2}, \mathscr{D}_{H_3}$ are distinct. Since $\dim H_1 = \dim H_3 = n^2$ and $\dim H_2 = n^2 + 2$, Theorem 1.3 implies that $\mathscr{D}_{H_2}$ is different from $\mathscr{D}_{H_1}$ and $\mathscr{D}_{H_3}$. By Theorem 1.5 (2) again, we obtain $\mathscr{D}_{H_1} \neq \mathscr{D}_{H_3}$. q.e.d.

Example 5.6. For the polynomials defined in Lemma 5.2, one can check the equality (see [**32**])

$$\begin{aligned} 2b_1b_2^2c_1 + 4b_1^2c_2d_2 + 4b_1^2c_1d_3 + 4b_2c_1^2d_2 + 4b_2^2d_2 \\ - b_1c_1c_2d_2 - b_1c_1^2d_3 - 16b_1b_2c_1d_2 = 0 \end{aligned}$$

holds. This is an explicit form of the equation described in the last paragraph of Section 3 on page 393 of [**18**]. Similar to the proof of Theorem 1.5, from this we can construct rank-6 connected semisimple subgroups $H_1, \dots, H_8$ of $G = \mathrm{SU}(15)$ such that $\mathscr{D}_{H_1}, \dots, \mathscr{D}_{H_8}$ are distinct and linearly dependent. This again has an implication for Question 5.3 by the above lemma.

Question 5.7. Find all linear relations among $\{\mathscr{D}_H : H \in \mathscr{L}\}$.

In view of the preceding discussion, this is a natural and important question. For G connected and $\mathscr{L} = \mathscr{L}_2$ being the class of connected subgroups, this question has been solved by the third author [**32**].

6. Determination of some invariants II

For any compact connected Lie group H, we define a sequence of numbers $\epsilon_n(H)$ as follows. Take a maximal torus T of H and let Φ be the root system of (H, T). Let $D : T \to \mathbb{R}$ be the Weyl discriminant $D(t) = |W|^{-1} \prod_{\alpha \in \Phi}(1 - \alpha(t))$, where W is the Weyl group of (H, T). We then define for $n \geqslant 1$,

$$\epsilon_n(H) = \frac{1}{|T_n|} \sum_{t \in T_n} D(t),$$

where T_n is the subgroup of n-torsion points on T. It is clear that $\epsilon_n(H)$ depends only on the isomorphism class of H. In view of Weyl's integration formula, one may regard ϵ_n as an analogue of the ϵ_n-invariant defined for finite groups in Example 7.1.

Put $X = X^*(T)$ and

$$F = \frac{1}{|W|} \prod_{\alpha \in \Phi} (1 - [\alpha]) \in \mathbb{Q}[X],$$
$$f = \prod_{\alpha \in \Phi^+} (1 - [\alpha]) = \sum_{w \in W} \mathrm{sgn}(w)[(1 - w)\delta] \in \mathbb{Z}[X].$$

Here Φ^+ is a fixed system of positive roots in Φ and $\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$. We will denote X/nX by X_n, the image of F under $\mathbb{Q}[X] \to \mathbb{Q}[X_n]$ by F_n, and the image of f under $\mathbb{Z}[X] \to \mathbb{Z}[X_n]$ by f_n.

By *the constant term* of an element $g \in \mathbb{Q}[X]$ or $\mathbb{Q}[X_n]$, we mean the coefficient of $[0]$ in g.

Let $h = 1$ if H is a torus; otherwise let h be the largest among the Coxeter numbers (see [**3**, VI.1.11]) of the irreducible components of Φ. We call h the *Coxeter number of* H. Let $\tilde{H}$ be the simply connected cover of the derived group of H. We decompose $\tilde{H}$ as $\tilde{H}_h \times \tilde{H}'_h$, where $\tilde{H}_h$ (resp. $\tilde{H}'_h$) is the product of those simple factors of $\tilde{H}$ with Coxeter number $= h$ (resp. $< h$). Corresponding to this decomposition we have $\Phi = \Phi_h \times \Phi'_h$. Put $\delta^\vee = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha^\vee$. Let H_h, H'_h, Z_h, Z'_h be the image of $\tilde{H}_h, \tilde{H}'_h, Z(\tilde{H}_h), Z(\tilde{H}'_h)$ in H, respectively, where $Z(G)$ denotes the center of G.

Theorem 6.1. *We have*

(1) $\epsilon_n(H)$ *is a non-negative integer for all* $n \geqslant 1$, *and* $\epsilon_n(H) = 1$ *for* n *sufficiently large.*
(2) $\epsilon_n(H_1 \times H_2) = \epsilon_n(H_1) \times \epsilon_n(H_2)$ *for all* $n \geqslant 1$.
(3) *If* H_1 *and* H_2 *are connected subgroups of a compact Lie group* G *such that* $H_1 \sim_{\mathrm{LP}} H_2$, *then* $\epsilon_n(H_1) = \epsilon_n(H_2)$ *for all* $n \geqslant 1$.
(4) $\epsilon_n(H)$ *is the constant term of* F_n.
(5) $\epsilon_n(H)$ *is the constant term of* f_n.
(6) *Let* S *be the neutral component of the center* Z *of* H. *Put* $\bar{H} = H/S$. *Then* $\epsilon_n(H) = \epsilon_n(\bar{H})$ *for all* $n \geqslant 1$.
(7) $\epsilon_n(H) = \begin{cases} 1 & \text{if } n > h, \\ 0 & \text{if } n < h. \end{cases}$
Moreover, the average of $D(t)$ *over* $\{t \in T : t^n = z\}$ *is* 1 *for all* $n > h$, $z \in Z$.
(8) $\epsilon_h(H) = \epsilon_h(H/SH'_h)$.
(9) $\epsilon_h(H) = \begin{cases} |Z/(SZ'_h)| & \text{if } \exp(2\pi i \delta^\vee) \in SZ'_h, \\ & \text{where } Z \text{ and } S \text{ are defined in (6)}, \\ 0 & \text{otherwise}. \end{cases}$

Remark. The theorem implies that we can repack the information contained in the sequence $\{\epsilon_n(H)\}_{n\geqslant 1}$ into a pair (h', z), where

$$\begin{aligned} h' &:= \min\{n \geqslant 1 : \epsilon_n(H) > 0\} \\ &= \min\{\mathrm{ord}(x) : x \in H \text{ is regular of finite order}\}, \end{aligned}$$

and $z := \epsilon_{h'}(H)$. Then h' and z are numerical invariants which are determined by the dimension datum (relative to any $H \hookrightarrow G$). We have $h \leqslant h' \leqslant h + 1$ by the above theorem. Theorem 1.6 is an immediate consequence of this.

Proof of Properties (1)–(8). Properties (2), (3), and (4) are immediate from the definition of ϵ_n. Property (5) follows from (4) and the formula $|W| \cdot F = \sum_{w \in W} w(f)$ (see the proof of Lemma 4.2). From (5) we see that ϵ_n are integers. We have $\epsilon_n(H) \geqslant 0$ since $D(t) \geqslant 0$. Write $f = \sum_{x \in X} m_x[x]$. For large n, no non-zero x with $m_x \neq 0$ is divisible by n, so $\epsilon_n(H) = 1$ by (5). We can also derive $\lim_{n\to\infty} \epsilon_n(H) = 1$ by Weyl's integration formula. So we have (1).

The split exact sequence $0 \to \bar{X} \to X \to X^*(S) \to 0$ implies that we have an injection $\bar{X}_n \hookrightarrow X_n$. This gives (6) by using (4) or (5).

We now prove (7). By (6), it suffices to prove (7) when G is semisimple, a condition we will assume throughout the proof of (7). If $n < h$, every n-torsion point t of T is non-regular (see [**4**, Exercise IX.4.14(d)]), i.e. it satisfies $\alpha(t) = 0$ for some $\alpha \in \Phi$. So $\epsilon_n(H) = 0$.

Assume $n > h$. Let $P \supset X$ be the dual of the coroot lattice. We claim that $(1-w)\delta \notin nP$ for $w \neq 1$.

Recall that $(\delta, \alpha^\vee) = 1$ for all simple coroots α. It follows that $-(h-1) \leqslant (\delta, \alpha^\vee) \leqslant (h-1)$ for all coroots $\alpha^\vee$. Moreover, $(\delta, \alpha^\vee) = \pm(h-1)$ if and only if $\pm\alpha^\vee$ is the highest coroot in an irreducible component of Φ with Coxeter number h.

Observe that $(1-w)\delta \in nP$ if and only if $((1-w)\delta, \alpha^\vee) \in n\mathbb{Z}$ for all simple coroots $\alpha^\vee$. But

$$((1-w)\delta, \alpha^\vee) = (\delta, \alpha^\vee) - (w\delta, \alpha^\vee) = 1 - (\delta, w^{-1}\alpha^\vee) \in [-(h-2), h].$$

The above number is divisible by n only when it is zero. So $(1-w)\delta \in nP$ $\Leftrightarrow (1-w)\delta = 0 \Leftrightarrow w = 1$. This proves the claim.

It follows that the only non-zero term in f of the form $c_\alpha[\alpha]$ ($\alpha \in nX$) is $1 \cdot [0]$, and the same statement holds when f is replaced by $F =$ the average of $w(f)$ over $w \in W$. This implies $\epsilon_n(H) = 1$, and more generally the last statement in (7). We have completed the proof of (7).

To prove (8), let T'' (resp. T') be a maximal torus of H_h (resp. SH'_h). Then $T = T''T'$ is a maximal torus of H and $A := T'' \cap T' = Z_h \cap (SZ'_h)$. Notice that $D|_{T''}$ and $D|_{T'}$ are the D-function for H_h and SH'_h, respectively, and $D(t''t') = D(t'')D(t')$ for $t'' \in T''$, $t' \in T'$.

The average of D over T_n is the same as the average of D over $\{(t'', t') \in T'' \times T' : (t'')^n = (t')^{-n} \in A\}$. Fix $t'' \in T''$ such that $z := (t'')^n \in A$; then the average of $D(t''t')$ over $\{t' \in T' : (t')^{-n} = z\}$ is $D(t'')$ if $n = h$, by the last statement of (7).

Therefore, the average of D over T_h is the same as the average of D over the inverse image of $\bar{T}''_h$ in T'', where $\bar{T}'' = T''/A$ is a maximal torus of H/SH'_h. This proves (8). q.e.d.

It remains to analyze the case of $n = h$ and prove (9). Assume first that H is simple and $\Delta^\vee = \{\alpha_1^\vee, \ldots, \alpha_l^\vee\}$ is the set of simple coroots. Let $\beta^\vee = \sum_{i=1}^l n_i\alpha_i^\vee$ be the highest coroot. Let C be the alcove on the apartment $A = X \otimes_{\mathbb{Z}} \mathbb{R}$ defined by $\{x \in A : \alpha_0^\vee(x) \geqslant 0, \ldots, \alpha_l^\vee(x) \geqslant 0\}$, where $\alpha_0^\vee = 1 - \beta^\vee$. We put $n_0 = 1$ and give the i^{th} vertex of C weight n_i. Then $x = h^{-1}\delta$ is the (weighted) barycenter of C, and is characterized by $\alpha_i^\vee(x) = 1/h$ for $i = 0, \ldots, h$. For any $w \in W$, let $\tilde{w}$ be the affine map $x \in A \mapsto w(x - h^{-1}\delta) + h^{-1}\delta$, which is the only affine map fixing $h^{-1}\delta$ with tangential part w.

Lemma 6.2. *Let Q be the root lattice and P the dual of the coroot lattice. For $w \in W$, the following conditions are equivalent:*

(a) $\tilde{w} \in W \ltimes P$. (b) $\tilde{w}(C) = C$.
(c) $\tilde{w}(\Delta_0^\vee) = \Delta_0^\vee$, *where* $\Delta_0^\vee = \Delta^\vee \cup \{\alpha_0^\vee\}$.
(d) $w(\Delta^\vee \cup \{-\beta^\vee\}) = \Delta^\vee \cup \{-\beta^\vee\}$. (e) $(1-w)\delta \in hP$.

The set Ω consisting of those $w \in W$ satisfying these conditions is a subgroup of W, and $w \mapsto \tilde{w}(\alpha_0^\vee)$ is a bijection between Ω and $\{\alpha_i^\vee \in$

$\Delta_0^\vee : n_i = 1\}$. *There is an isomorphism* $\iota : \Omega \to P/Q$ *defined by any of the following equivalent ways:*

(A) $\iota(w) = (1 - w)h^{-1}\delta + Q$.
(B) $\iota(w)$ *is the image of* $\tilde{w}$ *under* $W \ltimes P \to (W \ltimes P)/(W \ltimes Q) = P/Q$.
(C) *If* $w \neq 1$ *and* $\tilde{w}(\alpha_0^\vee) = \alpha_i^\vee$, $\iota(w)$ *is the* i^{th} *fundamental weight* ω_i *modulo* Q. *For* $w = 1$, $\iota(w) = 0$.

Finally, $\operatorname{sgn}(w) = (-1)^{\langle \iota(w), 2\delta^\vee \rangle}$ *for all* $w \in \Omega$, *where* $\delta^\vee = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha^\vee$.

Proof. Assume (a). Then $\tilde{w}$ takes alcoves to alcoves ([**3**, VI.2.3]). Since $\tilde{w}$ fixes the interior point $h^{-1}\delta$ of C, we must have (b). The implications (b) $\Rightarrow$ (c) $\Rightarrow$ (d), and (e) $\Rightarrow$ (a) are obvious. Assume (d). By the proof of (7), $\langle (1-w)\delta, \alpha_i^\vee \rangle \in \{0, h\}$ for $i = 1, \ldots, l$. So we have (e). This completes the proof of the equivalence of (a)–(e).

It is clear from (d) or (e) that Ω is a subgroup. The bijection with $\{\alpha_i^\vee : n_i = 1\}$ is [**3**, VI.2.3, Prop. 6]. The equivalence of the three descriptions of ι follows from that of (a)–(e), and it is clear from (A) or (B) that ι is a homomorphism. Moreover, [**3**, VI.2.3, Corollary] shows that ι is a bijection.

To prove the identity $\operatorname{sgn}(w) = (-1)^{\langle \iota(w), 2\delta^\vee \rangle}$, assume $w(-\beta^\vee) = \alpha_i^\vee$ with $i \neq 0, n_i = 1$. Recall that $\operatorname{sgn}(w) = (-1)^{|S|}$, where $S = \{\alpha^\vee \in \Phi_+^\vee : w^{-1}.\alpha^\vee \in -\Phi_+^\vee\}$. Write $\alpha^\vee = \sum_{j=1}^{l} c_j \alpha_j^\vee \in \Phi_+^\vee$. Then $c_i = \langle \omega_i, \alpha^\vee \rangle$ is either 0 or 1 (since $n_i = 1$) and $\alpha^\vee \in S \Leftrightarrow c_i = 1$. Thus we have $\langle \omega_i, 2\delta^\vee \rangle = |S|$. q.e.d.

Let $\Omega_X \subset \Omega$ be the subgroup $\iota^{-1}(X/Q)$. Then the above lemma implies $(w-1)\delta \in hX \Leftrightarrow w \in \Omega_X$. Write $f = \sum_{x \in X} m_x[x]$ and put $f_h = \sum_{x \in hX} m_x[x] \in \mathbb{Z}[hX]$. Then the lemma gives

$$f_h = [0] + \sum_{1 \leqslant i \leqslant l, n_i = 1, \omega_i \in X} (-1)^{\langle \omega_i, 2\delta^\vee \rangle}[h\omega_i],$$

and the image of f_h under the map $\mathbb{Z}[hX] \simeq \mathbb{Z}[X] \to \mathbb{Z}[X/Q]$ is $\sum_{x \in X/Q} (-1)^{\langle x, 2\delta^\vee \rangle}[x]$.

Corollary 6.3. *Assume that* H *is simple and* H_{ad} *is its adjoint group. Let* c *be a regular element of order* h *in* H_{ad} *and* $\tilde{c} \in H$ *a lift of* c. *The following are equivalent:*

(a) $\epsilon_h(H) > 0$.
(b) $\epsilon_h(H) = |Z(H)|$.
(c) $\Omega_X \subset \ker(\operatorname{sgn} : W \to \{\pm 1\})$.
(d) *The order of* $\tilde{c}$ *is* h.
(e) $\exp(2\pi i \delta^\vee) = 1$ *in* H.
(f) $\delta^\vee \in X^\vee$.
(g) $X/Q \subset \ker\big(\langle -, \delta^\vee \rangle : P/Q \to \frac{1}{2}\mathbb{Z}/\mathbb{Z}\big)$.

Proof. Recall that $\Omega_X \simeq X/Q$ is dual to $Z(H)$. Therefore the equivalence of (a), (b), (c), (g) is clear from the preceding discussion. It is also clear that (g), (e), and (f) are mutually equivalent. Finally [**4**, Exercise IX.4.14(b)] says that (f) is equivalent to (d). q.e.d.

Corollary 6.4. *Let H be a simple adjoint group of rank l and let $h_1 = h$ or $h+1$. Let $c \in T$ be a regular element of order h_1. Then*

$$D(c) = \frac{1}{|W|}\prod_{i=1}^{l}\prod_{j=1}^{d_i-1}|1-\zeta^j|^2 = \begin{cases} h^l \cdot |P/Q| \cdot |W|^{-1}, & \text{if } h_1 = h,\\ (h+1)^l \cdot |W|^{-1}, & \text{if } h_1 = h+1,\end{cases}$$

where ζ is a primitive h_1-th root of unity, and $d_1, \ldots, d_l$ are the degrees of W.

Proof. The first equality follows from the definition of $D(t)$. By ([**10**, Cor. to Prop. 1], or [**4**, Exercise IX.4.14(c)] in case $h_1 = h$), there is a unique W-orbit of regular h_1-torsion points on T, which is represented by $c = \exp(2\pi i\delta^\vee/h_1)$. By applying Lemma 6.2 when $h_1 = h$ or the proof of (7) when $h_1 = h+1$ to the dual root system, we see that $\mathrm{Stab}_W(c)$ is isomorphic to the dual of P/Q when $h_1 = h$, and is trivial when $h_1 = h+1$. The second equality follows from these and $\epsilon_{h_1}(H) = 1$. q.e.d.

Corollary 6.5. *Assume that H is simple.*

(a) *If H has a regular conjugacy class of order h, it has a unique one. This happens exactly when* (a)–(g) *of Corollary* 6.3 *hold.*
(b) *The group H always has a unique regular conjugacy class of order $h+1$.*

Proof. We stated (a), which is [**4**, Exercise IX.4.14], for contrast. We now give the proof of (b), whose argument also gives another proof of (a).

Since $\epsilon_{h+1}(H) > 0$, a regular conjugacy class of order $h+1$ exists. Let $\tilde{c} \in T$ be H-regular of order $h+1$. Let $\pi : H \to H_{\mathrm{ad}}$ be the natural homomorphism from H to its adjoint group. Then $c = \pi(\tilde{c})$ is regular of order dividing $h+1$, and hence is of order exactly $h+1$ (by [**4**, Exercise 4.14d]). Observe that $\mathrm{Stab}_W(\tilde{c})$ is trivial since $\mathrm{Stab}_W(c)$ is trivial by the proof of the preceding corollary. This gives $\epsilon_{h+1}(H) \geqslant |W| \cdot D(\tilde{c})/(h+1)^l = |W| \cdot D(c)/(h+1)^l = 1$. Since $\epsilon_{h+1}(H) = 1$ by (7), H has no other regular conjugacy classes of order $h+1$. q.e.d.

Remark. (1) See [**4**, Exercise IX.4.13(d)] for an explicit form of $\exp(2\pi i\delta^\vee)$ for each simple group. (2) In the above proofs, all references to [**4**, Exercise IX.4.14] can be replaced by simple arguments using the theory of Kac coordinates ([**26**]). (3) One may ask for an explicit construction of an element in the class characterized by part (b) of the above corollary. This can be done as follows. Let $\alpha_1, \ldots, \alpha_l$ be the simple roots, $\omega_1^\vee, \ldots, \omega_l^\vee$ the fundamental coweights, and let $\beta = \sum_{i=1}^{l} m_i\alpha_i$ be the highest root. Make the conventions that $m_0 = 1$ and $\omega_0^\vee = 0$. Then $\{\omega_i^\vee : m_i = 1\}$ forms a set of representatives of $Q^\vee/P^\vee$. Therefore, there exists an i_0 such that $m_{i_0} = 1$ and $\delta^\vee \equiv \omega_{i_0}^\vee \pmod{P^\vee}$. Then $\exp(2\pi i(\delta^\vee + \omega_{i_0}^\vee)/(h+1))$ is regular of order $h+1$.

Proof of Property (9). We observe that by (6), it suffices to prove (9) when H is semisimple, a condition we now assume (if we want, we may also use (8) to reduce (9) to the case of $H = H_h$; this would make the proof below a bit simpler). Again let Q and P be the root lattice and the dual of the coroot lattice, respectively. Corresponding to the decomposition $\Phi = \Phi_h \times \Phi'_h$, we have a decomposition $W = W_h \times W'_h$ and similar decompositions of P, Q, and $\Omega \subset W$. Let $\Omega'_X \subset P/Q$ be the intersection of P_h/Q_h and X/Q. Let Ω_X be the subgroup of Ω corresponding to Ω'_X under the isomorphism $\iota : \Omega \to P/Q$. We observe

- $(1-w)\delta \in hX$ if and only if $w \in \Omega_X$.
- $|\Omega_X| = |\Omega'_X| = |Z/Z'_h|$.
- The conditions $\Omega_X \subset \ker\bigl(\mathrm{sgn} : W \to \{\pm 1\}\bigr)$, $\Omega'_X \subset \ker\bigl(\langle -, \delta^\vee\rangle : P/Q \to \frac{1}{2}\mathbb{Z}/\mathbb{Z}\bigr)$, $\delta^\vee \in X^\vee + \check{P}'_h$, and $\exp(2\pi i \delta^\vee) \in Z'_h$ are mutually equivalent, where $X^\vee = X^*(T)$, $\check{P}'_h$ is the dual of Q'_h.

Now (9) is clear from (5) and the above observations. q.e.d.

Example 6.6. Let $H_1 = \mathrm{U}(2)$, $H_2 = \mathrm{U}(1) \times \mathrm{SU}(2)$. Then $\epsilon_2(H_1) = 1$, $\epsilon_2(H_2) = 0$. Therefore, for any embeddings of H_1 and H_2 into a common target group G, we have $H_1 \not\sim_{\mathrm{LP}} H_2$. This generalizes to two families of similar examples: take (H_1, H_2) to be $(\mathrm{U}(n), \mathrm{U}(1) \times \mathrm{SU}(n-1))$, $n \geqslant 2$, or $(\mathrm{GSp}(m), U(1) \times \mathrm{Sp}(m))$, $m \geqslant 1$; then $H_1 \not\sim_{\mathrm{LP}} H_2$ in any G containing both H_1 and H_2.

Example 6.7. Let H_1 and H_2 be simple adjoint groups of type B_n and C_n, respectively, $n \geqslant 3$. Then H_1 and H_2 have the same dimension, rank, and ϵ_n-invariant for all $n \geqslant 1$. But Theorem 1.4 implies $H_1 \not\sim_{\mathrm{LP}} H_2$ inside any G. Therefore, the criterion in Example 7.1 doesn't generalize to compact connected groups.

Example 6.8. Let H_1 and H_3 be as in Corollary 5.4. By Theorem 6.1 (9) and [**4**, Exercise IX.4.13(d)], we have $\epsilon_n(H_1) = 1$ and $\epsilon_n(H_3) = 0$. This gives a direct proof of the distinctness of $\mathscr{D}_{H_1}$ and $\mathscr{D}_{H_3}$.

7. Miscellaneous examples

We will first review various examples in the literature of (G, H, H') with $H \sim_{\mathrm{LP}} H'$ and $H \not\sim H'$.

Example 7.1. When G is finite, the relation $\sim_{\mathrm{LP}}$ is called Gassmann-equivalence. There are abundant examples of Gassmann-equivalent but non-conjugate or non-isomorphic subgroups in the literature ([**24**], [**25**], [**29**]). In particular, it is known that two finite groups H_1, H_2 are Gassmann-equivalent inside certain G if and only if $\epsilon_n(H_1) = \epsilon_n(H_2)$ for all $n \geqslant 1$, where $\epsilon(H_i)$ is the number of n-torsion elements in H_i.

Example 7.2. Examples with H, H' finite and $G = \mathrm{U}(n)$ were studied in the representation theory of finite groups. The most remarkable kind comes from a *Brauer pair* (H, H'). This means that H and H' are non-isomorphic finite groups with identical character table in the following strong sense: there are bijections $i : \widehat{H} \to \widehat{H}'$ and $j : H^\natural \to (H')^\natural$ such that for any $\pi \in \widehat{H}$ of degree d, $\pi' := i(\pi)$ is also of degree d and the diagram

$$\begin{tikzcd} H^\natural \arrow[d, "j"'] \arrow[dr, "\pi^\natural"] & \\ & U(d)^\natural \\ (H')^\natural \arrow[ur, "(\pi')^\natural"'] & \end{tikzcd}$$

is commutative. It follows that for any faithful representation $\pi : H \hookrightarrow \mathrm{U}(n)$, there is a corresponding faithful representation $\pi' = i(\pi) : H' \hookrightarrow \mathrm{U}(n)$ such that $\pi(H) \sim_{\mathrm{LP}} \pi'(H')$. But obviously $\pi(H) \not\sim \pi'(H')$ since H and H' are not isomorphic. See [**6**] and [**21**].

Example 7.3. Many examples with finite $H \simeq H'$ arise in the work of Larsen ([**15**], [**16**]). He considered a finite group H, a connected G, and embeddings $\phi_1, \phi_2 : H \to G$ such that $\phi_1^\natural = \phi_2^\natural$ but ϕ_1 is not G-conjugate to ϕ_2. Then clearly $\phi_1(H) \sim_{\mathrm{LP}} \phi_2(H)$. But we have $\phi_1(H) \not\sim \phi_2(H)$ if all automorphisms of H are inner. Similar examples with H connected are given by S. Wang [**31**].

Next, we will give some examples of the relation $\prec_{\mathrm{LP}}$, to address the following question.

Question 7.4. Let H and H' be subgroups of G. It is clear that if there exist $K \sim_{\mathrm{LP}} H$, $K' \sim_{\mathrm{LP}} H'$, such that $K \prec K'$, then $H \prec_{\mathrm{LP}} H'$. Is the converse true?

This may seem reasonable at first. Indeed, an affirmative answer will make Langlands' formulas (1) and (2) in [**14**, 1.1] most consistent. Our examples will show that it is the contrary in general. They will also serve as illustrations to Lemma 2.2.

Example 7.5. Let $G = S_n$ be a symmetric group with $n \geqslant 2$, and for $0 \leqslant p \leqslant \lfloor n/2 \rfloor$, put $H_p = S_p \times S_{n-p}$, regarded as the subgroup of S_n stabilizing $\{1, \dots, p\}$. Then we have

$$G = H_0 \succ_{\mathrm{LP}} H_1 \succ_{\mathrm{LP}} H_2 \succ_{\mathrm{LP}} \cdots \succ_{\mathrm{LP}} H_{\lfloor n/2 \rfloor}.$$

Indeed, from the representation theory of S_n, one sees that there exist mutually inequivalent irreducible representations $\pi_0, \dots, \pi_{\lfloor n/2 \rfloor}$ such that $\mathrm{Ind}_{H_p}^G \mathbf{1} \simeq \pi_0 + \pi_1 + \cdots + \pi_p$. More precisely, π_p is indexed by the partition $(n-p, p)$ of n in Frobenius' parametrization of $\widehat{S}_n$ with partitions of n.

It is clear that $H_p \not\prec H_q$ for $p \neq q$ $(p, q \geqslant 1)$, since H_p meets the conjugacy class parametrized by the partition $(n-p, p)$, but H_q does not.

Claim. *Let H' be a subgroup of S_n. The following are equivalent:*
(1) *$H' \prec H_p$ for some $p \geqslant 1$,*
(2) *$H' \prec_{\mathrm{LP}} H_1$,*
(3) *$\dim V^{H'} \geqslant 2$, where $V = \mathrm{Ind}_{H_1}^{G} \mathbf{1} = \pi_0 + \pi_1$.*

Indeed, (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is clear. Moreover, $\dim V^{H'}$ is the number of H'-orbits in $\{1, \ldots, n\}$. From this it is clear that (3) $\Rightarrow$ (1).

Claim. *Let H' be a subgroup of S_n. Then $H' \sim_{\mathrm{LP}} H_p \Leftrightarrow H' \sim H_p$.*

It suffices to verify ($\Rightarrow$) when $p \geqslant 1$. Assume $H' \sim_{\mathrm{LP}} H_p$ with $p \geqslant 1$. Then $H' \prec H_q$ for some $q \geqslant 1$ by the preceding claim. By assumption, H' meets the conjugacy class indexed by the partition $(n-p, p)$. But H_p is the only one among $H_1, \ldots, H_{\lfloor n/2 \rfloor}$ that meets this class. Therefore, $q = p$ and this forces $H' \sim H_p$ by cardinality consideration.

Example 7.6. Let $G = \mathrm{U}(n)$ with $n \geqslant 2$, and for $0 \leqslant p \leqslant \lfloor n/2 \rfloor$, put $H_p = \mathrm{U}(p) \times \mathrm{U}(n-p)$, regarded as the subgroup of G stabilizing the decomposition $\mathbb{C}^n = \mathbb{C}^p \times \mathbb{C}^{n-p}$. Then we again have

$$G = H_0 \succ_{\mathrm{LP}} H_1 \succ_{\mathrm{LP}} H_2 \succ_{\mathrm{LP}} \cdots \succ_{\mathrm{LP}} H_{\lfloor n/2 \rfloor}.$$

Indeed, it is customary to parametrize $\widehat{G}$ as $\{\pi_\lambda\}_{\lambda \in \Lambda}$, where Λ is the set of decreasing n-tuples of integers. By a theorem of Helgason [**7**, Theorem 12.3.13], and the calculation in [**7**, pages 577–578], the G-representation $L^2(G/H_p)$ is multiplicity-free, and is the orthogonal direct sum of π_λ, over all $\lambda \in \Lambda$ of the form $\lambda = (\lambda_1, \ldots, \lambda_p, 0, \ldots, 0, -\lambda_p, \ldots, -\lambda_1)$. Therefore, if $p \leqslant q \leqslant \lfloor n/2 \rfloor$, then $\mathscr{D}_{H_p}(\pi_\lambda) > 0 \Rightarrow \mathscr{D}_{H_p}(\pi_\lambda) = 1 \Rightarrow \mathscr{D}_{H_q}(\pi_\lambda) = 1$. Thus $H_p \succ_{\mathrm{LP}} H_q$.

It is clear that $H_p \not\prec H_q$ for $p \neq q$ $(p, q \geqslant 1)$, since H_p doesn't stabilize a q-dimensional subspace in the standard representation of $\mathrm{U}(n)$.

Claim. *Let H' be a subgroup of* $\mathrm{U}(n)$. *The following are equivalent:*
(1) *$H' \prec H_p$ for some $p \geqslant 1$,*
(2) *$H' \prec_{\mathrm{LP}} H_1$,*
(3) *$\dim V^{H'} \geqslant 2$, where V is the adjoint representation of* $\mathrm{U}(n)$.

Indeed, it is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). By Schur's lemma, (3) implies that the standard representation of $\mathrm{U}(n)$ is reducible as an H'-representation, and hence $H' \prec H_p$ for some $p \geqslant 1$.

Claim. *Let H' be a subgroup of* $\mathrm{U}(n)$. *Then $H' \sim_{\mathrm{LP}} H_p \Leftrightarrow H' \sim H_p$.*

Again we only have to check ($\Rightarrow$) while assuming $H' \sim_{\mathrm{LP}} H_p$ with $p \geqslant 1$. Then H' is connected of rank n. It is well known that then H' is conjugate to $\mathrm{U}(r_1)\times\cdots\times\mathrm{U}(r_k)$ for some $r_1+\cdots+r_k=n$. By considering $\dim V^{H'}$, we see that $k=2$ and $H'\sim H_q$ for some q. But $\mathscr{D}_{H_q}=\mathscr{D}_{H_p}$ only when $q=p$. So $H'\sim H_p$.

The two preceding examples give nice instances of Lemma 2.2. Take $n=\mathscr{D}_{H_1}$. Then the conclusion of Lemma 2.2 is satisfied with the subgroups $\{H_1,\ldots,H_{\lfloor n/2\rfloor}\}$ and the set $S=\{\pi_1\}$ (resp. $S=\{V_1\}$) in the case of Example 7.5 (resp. Example 7.6), where $V_1=\pi_{(1,0,\ldots,0,-1)}$ is the non-trivial subrepresentation of the adjoint representation of $\mathrm{U}(n)$.

Finally, we remark that the relation $H\prec_{\mathrm{LP}}H'$ seems harder to handle than $H\sim_{\mathrm{LP}}H'$. The technique of [**18**] allows one to verify $H\sim_{\mathrm{LP}}H'$ by an algorithm. But we do not know any good characterization for $H\prec_{\mathrm{LP}}H'$.

8. Appendix: The category of compact Lie groups up to conjugation

Let $\mathscr{K}$ (resp. $\bar{\mathscr{K}}$) be the category defined as follows: the objects are compact Lie groups, and the set of morphisms from A to B is $\mathrm{Hom}(A,B)$ (resp. $\mathrm{Hom}(A,B)/B$), where $\mathrm{Hom}(A,B)$ is the set of smooth homomorphisms from A to B, and B acts on the right of $\mathrm{Hom}(A,B)$ by $f.b=$ the homomorphism $a\mapsto b^{-1}f(a)b$. The composition in $\mathscr{K}$ is the usual one, and the composition of $g.C\in\mathrm{Mor}_{\bar{\mathscr{K}}}(B,C)$ with $f.B\in\mathrm{Mor}_{\bar{\mathscr{K}}}(A,B)$ is defined to be $(g\circ f).C\in\mathrm{Mor}_{\bar{\mathscr{K}}}(A,C)$.

We verify easily that $\bar{\mathscr{K}}$ is indeed a category. We call it the category of compact Lie groups up to conjugation. This seems a suitable formalism for studying dimension datum problems. For example, the subgroups of G up to G-conjugation are exactly the subobjects (in $\bar{\mathscr{K}}$) of G up to isomorphism. Notice also that $\mathrm{Aut}_{\bar{\mathscr{K}}}(G)$ is the group of outer automorphisms of G.

Most applications of dimension data to arithmetic geometry and automorphic forms require us to use the language of reductive groups over a field isomorphic to $\mathbb{C}$ instead of that of compact Lie groups. The principle for translating between the two settings is well known, but is usually only given in the literature as the first statement in the theorem below. For the convenience of the reader, we will give a version which is most adequate for the present purpose.

A linear algebraic group G over $\mathbb{R}$ or $\mathbb{C}$ is called reductive if its neutral component is reductive. We will identify a reductive group G over $\mathbb{C}$ with $G(\mathbb{C})$ in what follows. We define the category $\mathscr{C}$ (resp. $\bar{\mathscr{C}}$) of complex reductive groups (resp. up to conjugation) in the same way we defined $\mathscr{K}$ (resp. $\bar{\mathscr{K}}$): the objects are reductive algebraic groups over $\mathbb{C}$, and the set of morphisms from A to B is $\mathrm{Hom}(A,B)$ (resp. $\mathrm{Hom}(A,B)/B$), where $\mathrm{Hom}(A,B)$ is the set of algebraic homomorphisms from A to B.

It is well known [**22**, page 246] that a compact Lie group A carries a unique real algebraic structure $\underline{A}$, which is reductive, and every continuous morphism between compact Lie groups is real algebraic. Therefore we can define the functors of complexification: $A \mapsto \underline{A} \otimes_{\mathbb{R}} \mathbb{C}$, from $\mathscr{K}$ to $\mathscr{C}$ (resp. from $\bar{\mathscr{K}}$ to $\bar{\mathscr{C}}$), in an obvious manner. Recall that $\underline{A} \otimes_{\mathbb{R}} \mathbb{C}$ is identified with $\underline{A}(\mathbb{C})$ by our convention.

Theorem 8.1. *The functor of complexification $\mathscr{K} \to \mathscr{C}$ is faithful and essentially surjective. The functor $\bar{\mathscr{K}} \to \bar{\mathscr{C}}$ is an equivalence of categories.*

Proof. The first statement is well known [**22**, Theorems 11 and 12, pages 246–247], and it shows that the functor $\bar{\mathscr{K}} \to \bar{\mathscr{C}}$ is essentially surjective. Therefore it suffices to show that $\bar{\mathscr{K}} \to \bar{\mathscr{C}}$ is *fully* faithful, i.e. the natural map $\mathrm{Hom}(A,B)/B \to \mathrm{Hom}(\underline{A}(\mathbb{C}), \underline{B}(\mathbb{C}))/\underline{B}(\mathbb{C})$ is bijective for any compact Lie groups A and B. Recall that the Cartan decomposition [**22**, Theorem 2, page 239] states that $B \times P \to \underline{B}(\mathbb{C})$, $(k,p) \mapsto kp$, is a bijection, where $P = \exp(\sqrt{-1}\,\mathrm{Lie}\,B)$.

We first verify that if $f_1, f_2 \in \mathrm{Hom}(A,B)$ are in the same $\underline{B}(\mathbb{C})$-orbit, then they are in the same B-orbit. Indeed, if $f_2(a) = b^{-1} f_1(a) b$, we can write $b = kp$ with $k \in B$, $p \in P$. Then we have $kpf_2(a) = f_1(a)kp$, and $(kf_2(a))(f_2(a)^{-1}pf_2(a)) = (f_1(a)k)p$. Since $f_2(a) \in B$ and B normalizes P, the uniqueness of the Cartan decomposition gives $kf_2(a) = f_1(a)k$, $f_2(a) = k^{-1}f_1(a)k$, which holds for all $a \in A$.

Next, we show that every $f \in \mathrm{Hom}(\underline{A}(\mathbb{C}), \underline{B}(\mathbb{C}))$ is $\underline{B}(\mathbb{C})$-conjugate to an element of $\mathrm{Hom}(A,B)$. The image $f(A)$, being compact, lies in bBb^{-1} for some $b \in \underline{B}(\mathbb{C})$. It follows that $f.b \in \mathrm{Hom}(A,B)$. q.e.d.

Corollary 8.2. *Let G be a compact Lie group. Let $\widehat{\underline{G}(\mathbb{C})}$ be the set of irreducible rational representations of $\underline{G}(\mathbb{C})$, up to equivalence. There is a canonical bijection $\widehat{G} \to \widehat{\underline{G}(\mathbb{C})}$. Identify $\widehat{G}$ with $\widehat{\underline{G}(\mathbb{C})}$ using this bijection. Then we have $\dim V^H = \dim V^{\underline{H}(\mathbb{C})}$ for any $V \in \widehat{G}$ and any subgroup H of G.*

Proof. Let $\pi : G \to U(n)$ correspond to an n-dimensional representation V of G. Notice that $\pi : G \to \mathrm{U}(n)$ is irreducible if and only if π does not factor through $\mathrm{U}(m) \times \mathrm{U}(n-m) \hookrightarrow \mathrm{U}(n)$ in the category $\bar{\mathscr{K}}$ for any $1 \leqslant m \leqslant n-1$, and for any subgroup H, $\dim V^H$ is the largest integer d such that $\pi|_H$ factors through the composition $\{e\} \times \mathrm{U}(n-d) \subset \mathrm{U}(d) \times \mathrm{U}(n-d) \hookrightarrow \mathrm{U}(n)$ in $\bar{\mathscr{K}}$. The corollary follows from these statements, their analogues in $\bar{\mathscr{G}}$, and the theorem. q.e.d.

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