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BERNSTEIN THEOREM AND REGULARITY FOR A CLASS OF MONGE-AMPÈRE EQUATIONS

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Abstract

In this paper we first introduce a transform for convex functions and use it to prove a Bernstein theorem for a Monge-Ampère equation in half space. We then prove the optimal global regularity for a class of Monge-Ampère type equations arising in a number of geometric problems such as Poincaré metrics, hyperbolic affine spheres, and Minkowski type problems.

1. Introduction

In this paper we introduce a transform for convex functions, prove a Bernstein theorem for a Monge-Ampère equation in half space, and establish the global regularity of solutions to a class of Monge-Ampère equations by new techniques. A well-known transform for convex functions is the Legendre transform, given by

$$\begin{aligned} y &= Du(x), \\ u^*(y) &= x \cdot y - u(x). \end{aligned} \tag{1.1}$$

The Legendre transform is very useful in the study of Monge-Ampère type equations and optimal transportation, and in the theory of convex bodies.

In this paper we introduce the following new transform for convex functions:

$$\begin{aligned} y &= x/u(x), \\ u^{\#}(y) &= 1/u(x), \end{aligned} \tag{1.2}$$

where u is defined in a domain $\Omega \subset \mathbb{R}^n$. This transform is closely related to the concept of polar set in the theory of convex bodies; a geometric interpretation of it will be given in Section 2. See in particular Lemma 2.1 for its relation to the Legendre transform (1.1). Some properties of the transform are summarized in the following theorem.

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Theorem 1.1. *Let u be a convex function. Then*
(a) $u^{\#}$ is convex if $u^*/u > 0$, and concave if $u^*/u < 0$.
(b) $(u^*)^{\#} = (u^{\#})^*$.
(c) Denote $Q[u] = |u|^{n+2} \det D^2 u$, and $w = (u^{\#})^*$. Then

$$Q[u] = 1/Q[w] \tag{1.3}$$

if $u^{\#}$ is convex, and $Q[u] = 1/|Q[w]|$ if $u^{\#}$ is concave.

The transform $u \to u^{\#}$ has some other properties that should be useful in studying Monge-Ampère type equations. For example, if u is a convex function satisfying $u(0) = |Du(0)| = 0$ and $|Du(x)| \to \infty$ as $|x| \to \infty$, then the transform (1.2) sends $x = 0$ to $y = \infty$ and $x = \infty$ to $y = 0$. This property is shared by the Kelvin transform

$$\begin{aligned} y &= \psi(x) := x/|x|^2, \\ v(y) &= J^{(n-2)/2n} u(x), \end{aligned} \tag{1.4}$$

where J is the Jacobian of the mapping ψ. Another property shared by the transforms (1.2) and (1.4) is the invariance of certain quantities, that is, $Q[u]$ for (1.2) and $|u|^{-\frac{n+2}{n-2}} \Delta u$ for (1.4). Moreover, $Q[u]$ is closely related to the Blaschke-Santalo inequality, just like $|u|^{-\frac{n+2}{n-2}} \Delta u$ is related to the Sobolev inequality.

The Kelvin transform is very useful in the study of semilinear elliptic equations involving critical Sobolev exponents. It is natural to use the transform (1.2) to study the equation

$$\det D^2 u = \frac{\eta(x)}{|u|^{n+2}} \quad \text{in } \Omega, \tag{1.5}$$

where Ω is a convex domain in $\mathbb{R}^n$, and $\eta \in C(\overline{\Omega})$ is a positive function. Equation (1.5) arises in several geometric problems such as the Hilbert metric (Poincaré metric) in convex domains [**24**], affine spheres [**6**, **7**], the p-Minkowski problem [**25**], and the Minkowski problem in centro-affine geometry [**10**]. A parabolic version of equation (1.5) has been used in image processing. In Section 2 we will see that equation (1.5) is related to the Euler equation of the well-known Blaschke-Santalo inequality [**26**, **30**].

A special case of (1.5) is when $\eta \equiv 1$, which is the equation for affine hyperbolic spheres. By the rotation of coordinates,

$$\begin{aligned} y_1 &= -x_{n+1}, \\ y_k &= x_k, \qquad k = 2, \dots, n, \\ y_{n+1} &= x_1, \end{aligned} \tag{1.6}$$

equation (1.5) (with $\eta \equiv 1$) can be rewritten as

$$\det D^2 u = \Big(\frac{u_{x_1}}{x_1}\Big)^{n+2} \quad \text{in } \mathbb{R}^{n,+} := \mathbb{R}^n \cap \{x_1 > 0\}. \tag{1.7}$$

Equation (1.7) and the more general equation (4.2) also arise in optimal transportation; namely u is the potential function of the optimal transportation from $\mathbb{R}^{n,+}$ to $\mathbb{R}^{n,+}$.

Equation (1.7) is invariant under both transforms (1.1) and (1.2). As an application of the transform (1.2), we prove the following Bernstein property.

Theorem 1.2. *Let u be a smooth convex solution to (1.7) in $\mathbb{R}^{n,+}$. If $u(0,\tilde{x}) = |\tilde{x}|^2/2$, where $\tilde{x} = (x_2, \dots, x_n)$, then*

$$\begin{aligned} \textit{either} \qquad & u(x) = |x|^2/2, \\ \textit{or} \qquad & u(x) = |\tilde{x}|^2/2. \end{aligned} \tag{1.8}$$

Bernstein theorems usually concern the classification of entire solutions and have been a key issue in PDEs. But the Bernstein theorem in half-space is also important. For example, an interesting problem is the Bernstein theorem for the equation

$$\begin{aligned} \det D^2 u &= 1 && \text{in } \mathbb{R}^{n,+}, \\ u &= \tfrac{1}{2}|\tilde{x}|^2 && \text{on } \{x_1 = 0\} \end{aligned} \tag{1.9}$$

(namely, a smooth convex solution to (1.9) must be $u(x) = \frac{1}{2}|x|^2$ up to a linear function). If the Bernstein property for (1.9) is true, then one can recover the global $C^{2,\alpha}$ regularity of convex solutions to the Dirichlet problem of the Monge-Ampère equation [**4**, **18**, **31**], and it can also be used to establish the global $W^{2,p}$ estimate, extending the interior estimate of Caffarelli [**2**].

As another application of the transform (1.2), in this paper we also prove the regularity up to boundary for the graphs of solutions to the Dirichlet problem

$$\begin{aligned} \det D^2 u = \eta(x)/|u|^{n+2} \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.10}$$

where Ω is a bounded, uniformly convex domain in $\mathbb{R}^n$, and η is a positive, sufficiently smooth function. There are a number of deep works on boundary regularity for related problems such as (1.12), and (1.14) below [**7**, **12**, **19**, **32**, **33**]. Let $u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ be a solution to (1.10). If $\eta \equiv 1$, then $(-u)^{-1} \sum u_{x_i x_j} dx_i dx_j$ is a Hilbert metric in Ω [**24**], and u^*, the Legendre transform of u, defines an affine hyperbolic sphere [**7**]. The existence and uniqueness of smooth solutions $u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ to (1.10) was proved in [**7**]. Due to the singularity on the right hand side of (1.10), the gradient Du necessarily blows up at the boundary. One cannot expect the regularity of the solution u up to the boundary. But we want to know if its graph $\mathcal{M}_u$, as a hypersurface in $\mathbb{R}^{n+1}$, is smooth up to the boundary. In this paper we prove

Theorem 1.3. *Suppose Ω is a bounded, uniformly convex domain in $\mathbb{R}^n$ with C^∞ boundary. Suppose $\eta > 0$ and $\eta \in C^\infty(\overline{\Omega})$. Let u be a convex solution to (1.10). Then the graph $\mathcal{M}_u$ is $C^{n+2,\alpha}$ smooth up to its boundary, for any $\alpha \in (0,1)$.*

In §5.5 we will introduce a compatibility condition, which implies that the $C^{n+2,\alpha}$ regularity is probably optimal, even if $\partial\Omega$ and η are C^∞ smooth. A similar phenomenon occurs for Fefferman's equation (1.12) below. We point out that the global $C^{1,1}$ estimate in Theorem 1.3 (in the two-dimensional case) was first observed by Loewner and Nirenberg [**24**], and the $C^{2,\alpha}$ regularity was obtained by Lin and Wang [**22**] by constructing proper sub- and super-solutions (see Remark 4.1 for details). In this paper we will present a different proof for the $C^{2,\alpha}$ regularity. Here our main interest is the optimal regularity, stated in Theorem 1.3. It is also a question raised by S.T. Yau in his lecture at the Chinese Academy of Sciences in 2006.

Due to the singularity on the right hand side of (1.7), our proofs of Theorems 1.2 and 1.3 involve new techniques. Our proof of Theorem 1.2 and the $C^{2,\alpha}$ regularity in Theorem 1.3 is based on the transform (1.2) and the method of moving planes. The method of moving planes and the Kelvin transform have been used in semilinear elliptic equations to prove the rotational symmetry of solutions and Liouville theorems. But for Monge-Ampère type equations it is new to use it to establish the regularity of solutions, and this should be of interest itself. We expect more applications of the transform (1.2) in Monge-Ampère type equations.

For the $C^{k,\alpha}$ ($k \geq 3$) regularity of (1.7), we introduce an iteration for an ordinary differential equation, which is the linearized equation of (1.7), by regarding $\tilde{x}$ as parameters. The iteration improves the regularity of solutions step by step until the $C^{n+3-\epsilon}$ regularity. It also applies to the linear elliptic equation with singular coefficients near the boundary,

$$\sum\nolimits_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} + \sum\nolimits_{i=1}^{n} \frac{b_i(x)}{d(x)} u_{x_i} + c(x) u = f, \tag{1.11}$$

and yields the global $C^{\beta+1}$ regularity if $\beta > 0$ is not an integer, where $\beta = \frac{b\cdot\gamma}{a_{ij}\gamma_i\gamma_j}$, γ is the unit outer normal of $\partial\Omega$, and where $d(x)$ is the distance from x to the boundary $\partial\Omega$. If β is an integer, our iteration gives a necessary and sufficient *compatibility condition* for regularity higher than $C^{\beta+1}$. This compatibility condition implies that the regularity in Theorem 1.3 is optimal.

An interesting and related problem is the boundary value problem for the equation of Fefferman [**12**]

$$\begin{aligned} (-1)^n \det \begin{pmatrix} v, & v_{\bar{z}_j} \\ v_{z_i}, & v_{z_i \bar{z}_j} \end{pmatrix} &= 1 \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.12}$$

The equation is obtained by the change $v = e^{-u}$ from the complex Monge-Ampère equation [**12**]

$$\det u_{z_i\bar z_j} = e^{(n+1)u} \quad \text{in } \Omega, \qquad u = \infty \quad \text{on } \partial\Omega, \tag{1.13}$$

where Ω is a strictly pseudoconvex domain in $\mathbb{C}^n$. If u is a solution to (1.13), then $\sum \frac{\partial^2 u}{\partial z_i \partial \bar z_j} dz_i d\bar z_j$ is a complete Kähler-Einstein metric on Ω [**12**], in a way similar to the real case (1.10) considered in [**24**], where $\frac{-1}{u}\sum u_{x_ix_j}dx_idx_j$ is a Hilbert metric in Ω.

By computing the formal power series expansion, Fefferman observed that the solution hypersurface to (1.12) should be $C^{n+2-\delta}$ smooth up to the boundary, and Cheng and Yau [**8**] were able to prove the $C^{n+3/2-\delta}$ regularity for any small $\delta > 0$. Lee and Melrose [**19**] finally obtained the optimal $C^{n+2-\delta}$ regularity result observed in [**12**].

Another boundary value problem for the Monge-Ampère equation with singularity at the boundary was studied by Urbas [**32**, **33**]. He proved that there exists a positive constant K such that when $\partial\Omega \in C^\infty$ is uniformly convex, there is a unique (up to a constant) convex solution to

$$\frac{\det D^2 u}{(1+|Du|^2)^{(n+2)/2}} = K \quad \text{in } \Omega, \qquad |Du| = \infty \quad \text{on } \partial\Omega. \tag{1.14}$$

The solution itself is not smooth at the boundary, but its graph is a C^∞ smooth hypersurface up to the boundary.

We would like to point out that equations (1.7) or (1.10), (1.12), and (1.14) contain different singularities and the regularity of solutions to these equations is quite different. For equation (1.14), by the elliptic regularity theory and using the support function of the convex hypersurface, higher regularity follows readily from the $C^{2,\alpha}$ regularity. The higher regularity for (1.7) and (1.12) is more complicated. For equation (1.12), which becomes $v\Delta v = |Dv|^2 - 1$ in $\mathbb{C}^1$, Lee and Melrose [**19**] proved the following interesting singularity profile. There exist functions $\psi_j = \phi_0^{(n+1)j} a_j$, $j \geq 1$, $a_j \in C^\infty(\overline{\Omega})$, such that for all integers $N \geq 1$,

$$u - \sum_{j=0}^{N} \psi_j [\log(-\phi_0)]^j \in C^{(n+1)N-1}(\overline{\Omega}), \tag{1.15}$$

where ϕ_0 is a defining function for the domain Ω, namely $-\phi_0(x) \approx d_x$ when x is close to $\partial\Omega$, where d_x is the distance from x to $\partial\Omega$.

By a formal computation in dimension 2, a solution to equation (1.7) does not seem to have the singularity profile (1.15). Our argument implies that there exists a smooth function $a(\tilde x)$ such that

$$u(x) = p(x) + a(\tilde x) x_1^{n+3} \log x_1 + q(x) \tag{1.16}$$

where p is a polynomial of order $n+3$ and q is a higher order term, namely $q(x) = o(|x|^{n+3})$ near $x = 0$. By our compatibility condition, if $a \equiv 0$, then $q \in C^\infty(\overline{\Omega})$.

We would like to point out that the transform (1.2) and its properties in Section 2 below were found by the authors in 2005. In the last few years, the authors asked several researchers in the area whether the transform (1.2) is new. In early 2009, John Loftin told us that the transform coincides with the conormal map in affine geometry, which transforms a hyperbolic affine sphere to its dual sphere [**23**]; see Example 4 below. The authors would like to thank John Loftin for this.

This paper is arranged as follows. In Section 2 we discuss in some detail the transform (1.2) and prove Theorem 1.1. In Section 3 we prove Theorem 1.2. In Sections 4 and 5 we prove Theorem 1.3 for the cases $k = 2$ and $k \geq 3$, respectively.

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2. A transform for convex functions

To make the notation simpler, we write $h = u^\#$ and the transform (1.2) as

$$\begin{aligned} y &= \frac{x}{u(x)}, \\ h(y) &= \frac{1}{u(x)}. \end{aligned} \tag{2.1}$$

Obviously the inverse transform is given by

$$\begin{aligned} x &= \frac{y}{h(y)}, \\ u(x) &= \frac{1}{h(y)}. \end{aligned} \tag{2.2}$$

Hence $(u^\#)^\# = u$; namely, the transform of $u^\#$ is u itself.

Geometric interpretation. The transform (2.1) has a clear geometric interpretation. Let $\mathcal{M}_u = \{(x, u(x), x \in \Omega\}$ be the graph of the convex function u. For any point $X = (x, u(x)) \in \mathcal{M}_u$, consider a ray $\mathcal{R}$ from the origin O, given by $\mathcal{R} = \{tX : \ t > 0\}$. If $u(x) > 0$, then the ray $\mathcal{R}$ intersects with the hyperplane $P = \{x_{n+1} = 1\}$ at a point $Y = (y, 1) \in P$, where $y = x/u(x)$. We have two similar triangles OxX and OyY, where x, y are regarded as points on the plane $\{x_{n+1} = 0\}$. Then the value of h at y is equal to the ratio

$$\frac{|y|}{|x|} = \frac{1}{u(x)} = \frac{|Y|}{|X|}.$$

The following properties of the transform are easy to verify.

(i) If $u(0) = 0$, $Du(0) = 0$, and u is uniformly convex, then h is asymptotically quadratic.

(ii) If Ω is a convex domain containing the origin and $u < 0$ is a convex function in Ω, vanishing on $\partial\Omega$, then $-h$ is an entire convex function in $\mathbb{R}^n$, asymptotic to a convex cone determined by Ω.

(iii) For any vector $\xi \neq 0$, like the transform $x \to x/|x|^2$ in (1.4), the transform (2.1) maps the ray $R_\xi = \{t\xi : \ t > 0\}$ to the ray itself.

(iv) From the above geometric interpretation, we have the following monotonicity of the transform. Let $u, P, \mathcal{R}, X, Y$ be as above. If $\hat{u}$ is another positive convex function such that $\hat{X} = (\hat{x}, \hat{u}(\hat{x}))$ is a point on $\mathcal{R}$ between X and Y, then $\hat{h}(y) < h(y)$. As a consequence, for any positive constants C_1 and C_2, we have

$$\begin{aligned} &\text{if } u(x) \geq C_1|x|^2, \text{ then } h(y) \leq C_1|y|^2; \\ &\text{if } 0 \leq u(x) \leq C_2|x|^2, \text{ then } h(y) \geq C_2|y|^2. \end{aligned}$$

The transform has some other interesting properties. For example, it transforms linear functions to linear functions (Example 1), and transforms the quadratic function $u(x) = \sum a_{ij}x_ix_j$ to itself (Example 2). For relation with polar set of a convex body, see Example 4 below. Now we compute

$$\frac{\partial y_k}{\partial x_i} = \frac{\delta_{ki}}{u} - \frac{x_k u_{x_i}}{u^2}$$

and

$$\begin{aligned} u_{x_i} &= -\frac{h_{y_k}}{h^2} \frac{\partial y_k}{\partial x_i} \\ &= -\frac{h_{y_k}}{h^2}\left(\frac{\delta_{ki}}{u} - \frac{x_k u_{x_i}}{u^2}\right) \\ &= -\frac{h_{y_k}}{h} + \frac{y_k h_{y_k} u_{x_i}}{h}. \end{aligned}$$

Hence

$$u_{x_i} = \frac{h_{y_i}}{y_k h_{y_k} - h} = \frac{h_{y_i}}{h^*} \tag{2.3}$$

and

$$\begin{aligned} u^* &= x_i u_{x_i} - u \\ &= \frac{y_i}{h} \frac{h_{y_i}}{h^*} - \frac{1}{h} \\ &= \frac{1}{h^*}. \end{aligned} \tag{2.4}$$

Lemma 2.1. *We have*

$$(u^*)^{\#} = (u^{\#})^*. \tag{2.5}$$

Proof. By (2.4), we have

$$h^* = \frac{1}{u^*}.$$

By definition (2.1), we have

$$(u^*)^\# = \frac{1}{u^*}.$$

Hence $(u^*)^\# = h^* = (u^\#)^*$. q.e.d.

Next we compute the second derivatives.

$$\begin{aligned}
u_{x_i x_j} &= \Big(\frac{h_{y_i}}{h^*}\Big)_{y_l} \frac{\partial y_l}{\partial x_j} \\
&= \Big(\frac{h_{y_i y_l}}{h^*} - \frac{y_k h_{y_i} h_{y_k y_l}}{{h^*}^2}\Big)\Big(\frac{\delta_{lj}}{u} - \frac{x_l u_{x_j}}{u^2}\Big) \\
&= \frac{h}{h^*}\big(h_{il} - \frac{y_k}{h^*} h_{y_i} h_{y_k y_l}\big)\big(\delta_{lj} - \frac{y_l}{h^*} h_{y_j}\big) \\
&= \frac{h}{h^*}\big(h_{ij} - \frac{y_k}{h^*} h_i h_{kj} - \frac{y_l}{h^*} h_j h_{li} + \frac{y_k y_l}{{h^*}^2} h_i h_j h_{kl}\big) \\
&= \frac{h}{h^*}\big(\delta_{ik} - a_i b_k\big)\{h_{kl}\}\big(\delta_{lj} - b_l a_j\big),
\end{aligned} \tag{2.6}$$

where

$$a_i = h_{y_i}, \qquad b_k = \frac{y_k}{h^*}.$$

We have

$$\begin{aligned}
\det(\delta_{ij} - a_i b_j) &= 1 - \sum a_k b_k \\
&= 1 - \frac{y_k h_{y_k}}{h^*} \\
&= -\frac{h}{h^*}.
\end{aligned}$$

Hence

$$\begin{aligned}
\det D^2 u &= \frac{h^n}{{h^*}^n} \det D^2 h \,|\det(\delta_{ij} - a_i b_j)|^2 \\
&= \frac{h^{n+2}}{{h^*}^{n+2}} \det D^2 h.
\end{aligned} \tag{2.7}$$

In particular, we have (see Remark 2.2 below)

$$u^{n+2} \det D^2 u = \frac{1}{{h^*}^{n+2} \det D^2 h^*}. \tag{2.8}$$

Proof of Theorem 1.1. Part (b) of Theorem 1.1 was proved in Lemma 2.1. Part (a) was verified in (2.6). Part (c) follows from (2.8). □

Remark 2.1. In order to keep h convex, we may let

$$y = \frac{x}{u(x)},$$
$$h(y) = -\frac{1}{u(x)}$$

if $u^*/u < 0$.
Remark 2.2. In comparison to (2.8), we recall the corresponding formula for the Legendre transform (1.1). For the Legendre transform (1.1), by differentiating either $x = Du^*(y)$ or $y = Du(x)$ we have

$$\{D^2u^*(y)\} \cdot \{D^2u(x)\} = I,$$

namely D^2u^* is the inverse of D^2u. It follows that

$$\det D^2u = 1/\det D^2u^*. \tag{2.9}$$

Let us compute a few examples.
Example 1.

$$u(x) = a \cdot x + b \qquad (b \neq 0).$$

The graph of u is a linear function. Then by $y = \frac{x}{u(x)} = \frac{x}{a \cdot x + b}$, we have $x = \frac{by}{1 - a \cdot y}$ and

$$h(y) = \frac{1}{b}(1 - a \cdot y).$$

Example 2.

$$u(x) = \sum a_{ij}x_ix_j,$$

where (a_{ij}) is positive definite. To compute $h = u^{\#}$, let ξ be a unit vector in $\mathbb{R}^n$. By the geometric interpretation, it suffices to compute the transform of u restricted to the 2-plane spanned by the vectors ξ and $e_{n+1} = (0, \ldots, 0, 1)$. Therefore it suffices to compute the transform for $\tilde{u}(t) = ct^2$, where $t \in \mathbb{R}^1$ and c is a constant. By direct computation we have $\tilde{h}(s) = cs^2$. Hence

$$h(y) = \sum a_{ij}y_iy_j.$$

Namely, $u^{\#} = u$, the transform of u is itself.
Example 3.

$$u(x) = \frac{1}{2}(a + |x|^2).$$

By $y = \frac{x}{u}$ we have $|y| = \frac{2|x|}{a+|x|^2}$. Hence $|x| = \frac{1}{|y|}\big(1 \pm \sqrt{1-a|y|^2}\big)$. Therefore

$$\begin{aligned} h(y) &= \frac{1}{u(x)} = \frac{|y|}{|x|} \\ &= \frac{|y|^2}{1 \pm \sqrt{1-a|y|^2}} \\ &= \frac{1}{a}\big(1 \mp \sqrt{1-a|y|^2}\big) =: h^{\pm} \quad \text{if } a \neq 0. \end{aligned}$$

We obtain

if $a > 0$, then $h^{\pm}$ are two parts of the same sphere;
if $a = 0$, then $h \equiv u$;
if $a < 0$, then $h^{\pm}$ are two hyperboloids.

Note that in the above example, if $|x|^2$ is replaced by $\sum a_{ij}x_ix_j$ in u, then $|y|^2$ in h can simply be replaced by $\sum a_{ij}y_iy_j$.

Example 4. Let u be an affine hyperbolic sphere asymptotic to a convex cone with vertex at the origin. Then u^* (the Legendre transform of u) is a convex function that satisfies

$$\begin{aligned} |u^*|^{n+2}\det D^2u^* &= c_0 \qquad \text{in } \Omega, \\ u^* &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{2.10}$$

for some constant $c_0 > 0$. By (2.8), $h = u^{\#}$ satisfies

$$\begin{aligned} |h|^{n+2}\det D^2h &= 1/c_0 \qquad \text{in } \Omega^{\#}, \\ h &= 0 \qquad \text{on } \partial\Omega^{\#}, \end{aligned} \tag{2.11}$$

where $\Omega^{\#}$ is the polar body of Ω, given by [**30**]

$$\Omega^{\#} = \{x \in \mathbb{R}^n : \ \langle x, y\rangle < 1 \ \forall\ y \in \Omega\}.$$

Hence h^* also defines an affine hyperbolic sphere. It is the dual affine hyperbolic sphere defined by the conormal map of u, given in [**14**]. For affine hyperbolic spheres, the explicit formula (2.1) for the dual affine sphere was first given by J. Loftin in [**23**].

There are a number of geometric problems that involve the following equation:

$$\det(\nabla^2 H + HI) = \frac{\eta(X)}{H^{n+2}} \quad \text{on } S^n. \tag{2.12}$$

For example, if $\eta \equiv 1$, then H is the support function of an elliptic affine sphere. For general η, (2.12) is the equation for the p-Minkowski problem [**25**] or for the Minkowski problem in the centro-affine geometry [**10**].

The equation is also closely related to the Blaschke-Santalo inequality [**26**, **30**],

$$\inf_{\xi\in K} V(K)\int_{S^n}\frac{1}{(H-\xi\cdot x)^{n+1}}\le\frac{\omega_n^2}{n+1}, \tag{2.13}$$

where K is a convex body, H is its support function, and

$$V(K)=\frac{1}{n+1}\int_{S^n} H\det(\nabla^2 H+HI)\,d\sigma$$

is the volume of K. The infimum is taken over all ξ, satisfying $H-\xi\cdot x>0$. For symmetric convex hypersurfaces, (2.12) is the Euler equation of the Blaschke-Santalo inequality.

Let u be the projection of H on $\{x_{n+1}=-1\}$, given by

$$u(x)=\sqrt{1+|x|^2}H(X),\qquad X=\Big(\frac{x}{\sqrt{1+|x|^2}},\frac{-1}{\sqrt{1+|x|^2}}\Big). \tag{2.14}$$

By direct computation,

$$\det D^2u=(1+|x|^2)^{-\frac{n+2}{2}}\det(\nabla^2H+HI).$$

Hence u satisfies (1.5).

3. A Bernstein theorem

In this section we employ transform (1.2) to prove a Bernstein theorem (Theorem 1.2) for the problem

$$\det D^2 u = \Big(\frac{u_{x_1}}{x_1}\Big)^{n+2} \quad \text{in } \mathbb{R}^{n,+}:=\mathbb{R}^n\cap\{x_1>0\}, \tag{3.1}$$

$$u = \frac{1}{2}|\tilde{x}|^2 \quad \text{on } \{x_1=0\}. \tag{3.2}$$

That is, a smooth convex solution to (3.1)–(3.2) is either $u(x)\equiv\frac12|x|^2$ or $u(x)\equiv\frac12|\tilde{x}|^2$. For the equation

$$\det D^2u=1 \quad \text{in } \mathbb{R}^n, \tag{3.3}$$

it is well known that an entire convex solution must be a quadratic function, proved by Jörgens [**17**] for $n=2$, Calabi [**5**] for $n\le 5$, and for all $n\ge 2$ by Pogorelov [**27**]. See [**9**] for a different proof. A more general result was proved by Caffarelli and Li in [**3**]. For semilinear elliptic equations in half space, a Liouville type theorem was proved by Li and Zhu in [**20**].

Here we use transform (1.2) and a method of moving planes to prove the Bernstein theorem for (3.1)–(3.2). As the reader will see below, technically our proof for the Monge-Ampère equation is quite different from that for the semilinear elliptic equation [**20**]. First we point out a nice property of equation (3.1), namely its invariance under transforms (1.1) and (1.2).

Lemma 3.1. *Let u be a smooth convex solution to (3.1). Then both u^* and $u^\#$ satisfy (3.1).*

Proof. By (1.1) and (2.9), it is apparent that the Legendre transform u^* satisfies (3.1).

Let $h := u^\#$ be the transform of u, given in (1.2). By (2.2) and (2.3) we have

$$\Big(\frac{u_{x_1}}{x_1}\Big)^{n+2} = \Big(\frac{h_{y_1}}{h^*}\Big/\frac{y_1}{h(y)}\Big)^{n+2} = \Big(\frac{h_{y_1}}{h^*}\frac{h(y)}{y_1}\Big)^{n+2},$$

where h^* is the Legendre transform of h. By (2.7),

$$\det D^2 u = \frac{h^{n+2}}{h^{*n+2}} \det D^2 h.$$

Hence h satisfies (3.1). q.e.d.

Lemma 3.2. *Let u be a smooth convex solution to (3.1). Then $u_{x_1} = 0$ on $\{x_1 = 0\}$.*

Proof. If $|u_{x_1}(0)| > 0$, then $|u_{x_1}(x)| \ge \frac{1}{2}|u_{x_1}(0)|$ for $x \in B_r^+(0)$, where $B_r^+(0) = B_r(0) \cap \{x_1 > 0\}$ and $r > 0$ is small, depending on u. Consider the integral

$$\int_{B_r^+} \det D^2 u = \int_{B_r^+} \Big(\frac{u_{x_1}}{x_1}\Big)^{n+2}.$$

The left hand side is equal to $Du(B_r^+(0))$, but the right hand side is equal to ∞. The contradiction implies Lemma 3.2. q.e.d.

Lemma 3.3. *Let u be a smooth convex solution to (3.1)–(3.2). Then either $u \equiv \frac{1}{2}|\tilde{x}|^2$ or $u_{x_1} > 0$ when $x_1 > 0$.*

Proof. By Lemma 3.2 and the convexity of u, we have $u_{x_1} \ge 0$. Hence $\forall\ \tilde{x} \in \mathbb{R}^{n-1}$ and $x_1 > 0$ we have $u(x_1, \tilde{x}) \ge \frac{1}{2}|\tilde{x}|^2$. We claim that if $u \not\equiv \frac{1}{2}|\tilde{x}|^2$, then $\forall\ \tilde{x} \in \mathbb{R}^{n-1}$ and there exists $x_1 > 0$ such that $u(x_1, \tilde{x}) > \frac{1}{2}|\tilde{x}|^2$. Indeed, if there exists $\tilde{x}_0 \in \mathbb{R}^{n-1}$ such that $u(x_1, \tilde{x}_0) = \frac{1}{2}|\tilde{x}_0|^2$ for all $x_1 > 0$, by a translation of $\tilde{x}$ and subtraction of a linear function of $\tilde{x}$, we assume $\tilde{x}_0 = 0$. Then $u(x_1, 0) = 0$ for all $x_1 > 0$. Extend u to $\mathbb{R}^n$ such that it is even in x_1. Then u is a convex function in $\mathbb{R}^n$ and $u(x_1, 0) \equiv 0$. The convexity of u then implies that u is a function of $\tilde{x}$, independent of x_1. By the boundary condition (3.2) we have $u \equiv \frac{1}{2}|\tilde{x}|^2$. The claim is proved.

Note that if there is a sequence $\tilde{x}_k \to \tilde{x}_0$ and $x_{1,k} \to \infty$ such that $u(x_{1,k}, \tilde{x}_k) = \frac{1}{2}|\tilde{x}_k|^2$, by convexity it means $u(x_1, \tilde{x}_0) = \frac{1}{2}|\tilde{x}_0|^2$ for all $x_1 > 0$, which is ruled out by the above claim. Hence $\forall\ R > 0$, $\exists\ M > 0$ such that $u(x_1, \tilde{x}) > \frac{1}{2}|\tilde{x}|^2$ whenever $x_1 > M$ and $|\tilde{x}| < R$.

To prove $u_{x_1}(x_1, \tilde{x}) > 0$ when $x_1 > 0$, it suffices to prove it for $\tilde{x} = 0$. Let $v(x) = \frac{1}{2}\epsilon|x|^2$. Then

$$\det D^2 v > \Big(\frac{v_{x_1}}{x_1}\Big)^{n+2}.$$

By the above claim, we can choose $\epsilon > 0$ small and $R > 1$ large such that $v < u$ on $\partial B_R^+(0)$. By the comparison principle, we obtain $v \le u$ in $B_R^+(0)$. Hence $u_{x_1}(0) \ge 0$ and $u_{x_1}(x_1, 0) > 0$ when $x_1 > 0$. q.e.d.

In the following proof of Theorem 1.2, we always assume $u \not\equiv \frac{1}{2}|\tilde{x}|^2$.

Lemma 3.4. *Let u be a smooth convex solution to (3.1)–(3.2). Then for any $R > 0$, there exist constants $C_1, C_2 > 0$ such that*

$$C_1 x_1^2 \le u(x) - \frac{1}{2}|\tilde{x}|^2 \le C_2 x_1^2 \quad \forall\, x \in B_R^+(0), \tag{3.4}$$

where C_1, C_2 are allowed to depend on $\tilde{x}$ and R.

Proof. For any $\tilde{x}_0 \in \mathbb{R}^{n-1}$, by a translation of the coordinates $\tilde{x} = (x_2, \dots, x_n)$, we may assume $x_0 = 0$. By subtracting a linear function of $\tilde{x}$, we may also assume that $u(0) = 0$ and $D_{\tilde{x}} u(0) = 0$.

In the previous lemma, we have shown that $v(x) = \frac{1}{2}\epsilon|x|^2$ is a sub-barrier. Hence the first inequality was proved. Let $w(x) = \frac{1}{2}M|x|^2$. Then

$$\det D^2 w < \big(\frac{w_{x_1}}{x_1}\big)^{n+2}$$

if we choose M large such that $w > u$ on $\partial B_R^+(0)$. By the comparison principle, we obtain $w \ge u$ in $B_R^+(0)$. Hence w is an upper barrier and we obtain the second inequality of (3.4). q.e.d.

Lemma 3.5. *Let u be a smooth convex solution to (3.1)–(3.2). Then $u_{11} \equiv 1$ on $\{x_1 = 0\}$, where we denote $u_{ij} = u_{x_i x_j}$.*

Proof. By Lemma 3.2, we have $u_{x_1} = 0$ on $\{x_1 = 0\}$. Hence by differentiating in $\tilde{x}$ we have

$$u_{1k}(x) = 0 \quad \text{on} \quad \{x_1 = 0\}. \tag{3.5}$$

From equation (3.1), it then follows:

$$u_{11} \det D^2_{\tilde{x}} u = u_{11}^{n+2} \quad \text{on } \{x_1 = 0\}. \tag{3.6}$$

Namely,

$$u_{11}^{n+1} = \det D^2_{\tilde{x}} u \quad \text{on } \{x_1 = 0\}.$$

By (3.2), $\det D^2_{\tilde{x}} u \equiv 1$ on $\{x_1 = 0\}$. q.e.d.

By Lemma 3.5, we have, for $k \ge 1$,

$$D^k_{\tilde{x}} u_{11} = 0 \quad \text{on} \quad \{x_1 = 0\}. \tag{3.7}$$

By Lemma 3.2, we also have

$$D^k_{\tilde{x}} u_{x_1} = 0 \quad \text{on} \quad \{x_1 = 0\}. \tag{3.8}$$

The boundary condition also implies that if $k \ge 3$,

$$D^k_{\tilde{x}} u = 0 \quad \text{on} \quad \{x_1 = 0\}. \tag{3.9}$$

Lemma 3.6. *Let u be a smooth convex solution to (3.1)–(3.2). Then $u_{111} \equiv 0$ on $\{x_1 = 0\}$.*

Proof. Write equation (3.1) as

$$u_{11} = \{(\frac{u_{x_1}}{x_1})^{n+2} + \tilde{g}\}/\det D^2_{\tilde{x}}u, \tag{3.10}$$

where

$$\tilde{g} = -\sum_{i,j>1} r_{1i}r_{j1}\,\frac{\partial^2 \det r}{\partial r_{1i}\partial r_{j1}} \quad \text{at } r = D^2u. \tag{3.11}$$

Differentiating (3.10) in $\tilde{x}$, we have

$$u_{111} = H(\frac{u_{11}}{x_1} - \frac{u_{x_1}}{x_1^2}) + G \tag{3.12}$$

where

$$H = (n+2)[\frac{u_{x_1}}{x_1}]^{n+1}/\det D^2_{\tilde{x}}u, \tag{3.13}$$

$$G = (\frac{\tilde{g}}{\det D^2_{\tilde{x}}u})_{x_1} + (\frac{1}{\det D^2_{\tilde{x}}u})_{x_1}(\frac{u_{x_1}}{x_1})^{n+2}. \tag{3.14}$$

By (3.5), (3.7)–(3.9), and Lemma 3.5 we have

$$G(0,\tilde{x}) = 0, \quad H(0,\tilde{x}) = n+2.$$

Hence by Taylor expansion and (3.12) we have $u_{111} = (n+2)u_{111}$ on $\{x_1 = 0\}$, which implies the desired result. q.e.d.

Lemma 3.7. *Let $u \in C^{3,1}(B_r(0))$ be a smooth convex function. Suppose at the origin 0, u has the expansion*

$$u(x) = \frac{1}{2}|x|^2 + a(x) + b(x) \tag{3.15}$$

where

$$a(x) = \sum_{i,j,k} u_{ijk}(0)x_ix_jx_k,$$
$$|D^kb(x)| = O(|x|^{4-k}) \quad \textit{for } 0 \le k \le 4.$$

Let $h = u^{\#}$ be the transform of u. Then as $y \to \infty$,

$$h(y) = \frac{1}{2}|y|^2 + 2\frac{a(y)}{|y|^2} + b^*(y), \tag{3.16}$$

with

$$b^*(y) = O(1),$$
$$Db^*(y) = O(\frac{1}{|y|}) \quad \textit{as } |y| \to \infty.$$

Proof. By $y = \frac{x}{u(x)}$ we have

$$\begin{aligned} y_i &= \frac{2x_i}{|x|^2}\frac{|x|^2}{|x|^2+2a(x)+2b(x)} \\ &= \frac{2x_i}{|x|^2}\left[1-\frac{2a(x)+2b(x)}{|x|^2+2a(x)+2b(x)}\right] \\ &= \frac{2x_i}{|x|^2}\left[1-\frac{2a(x)}{|x|^2}+O(|x|^2)\right]. \end{aligned}$$

Hence $y_i = \frac{2x_i}{|x|^2}\big(1+O(|x|)\big)$ and $x_i = \frac{2y_i}{|y|^2}\big(1+O(\frac{1}{|y|})\big)$. It follows that

$$\begin{aligned} |x| &= \frac{2}{|y|}\frac{1}{1+2a(x)/|x|^2+2b(x)/|x|^2} \\ &= 2|y|^{-1}\left[1-2a(x)/|x|^2-2b'(x)/|x|^2\right] \\ &= 2|y|^{-1}\left[1-4a(y)/|y|^4+\tilde{b}(y)\right], \end{aligned}$$

where b' satisfies $|D^k b'(x)| = O(|x|^{4-k})$ for $0 \le k \le 4$, and $\tilde{b}$ satisfies

$$|\tilde{b}(y)| \le C|y|^{-2}, \quad |D\tilde{b}(y)| \le C|y|^{-3} \quad \text{as } y \to \infty.$$

Hence

$$\begin{aligned} u(x) &= \frac{1}{2}|x|^2 + a(x) + b(x) \\ &= 2|y|^{-2}\big[1-4a(y)/|y|^4+\tilde{b}(y)\big]^2 + 8a(y)/|y|^6 + \tilde{b}(y)/|y|^2 \\ &= 2|y|^{-2}\big\{\big[1-4a(y)/|y|^4+\tilde{b}(y)\big]^2 + 4a(y)/|y|^4 + \tilde{b}(y)\big\} \\ &= 2|y|^{-2}\big\{1-4a(y)/|y|^4+\tilde{b}(y)\big\}, \end{aligned}$$

where $\tilde{b}$ changes from line to line but all satisfy the above asymptotic behavior. We thus obtain

$$\begin{aligned} h(y) &= \frac{1}{u(x)} = \frac{|y|^2/2}{1-4a(y)/|y|^4+\tilde{b}(y)} \\ &= \frac{1}{2}|y|^2\big(1+4a(y)/|y|^4+\tilde{b}(y)\big) \\ &= \frac{1}{2}|y|^2 + 2a(y)/|y|^2 + b^*(y). \end{aligned}$$

q.e.d.

We are now ready to use the method of moving planes to prove Theorem 1.2. Denote

$$\begin{aligned} \Sigma_\lambda &= \{x_n = \lambda\}, \\ D_{\lambda,R} &= B'_R(0) \times \{\lambda < x_n < \lambda + R\}, \\ D^*_{\lambda,R} &= B'_R(0) \times \{\lambda - R < x_n < \lambda\}, \end{aligned}$$

where Σ_λ is a plane, $B'_R(0) = \{x' \in \mathbb{R}^{n-1} : \ |x'| < R, \ x_1 > 0\}$, where $x' = (x_1, \ldots, x_{n-1})$. The domain $D_{\lambda,R}$ is a cylinder in $\{x_1 > 0\}$, and $D^*_{\lambda,R}$ is the reflection of $D_{\lambda,R}$ in the plane Σ_λ. For a point $x =$

$(x_1, \cdots, x_n) \in D_{\lambda,R}$, we denote by $x_\lambda = (x_1, \ldots, x_{n-1}, 2\lambda - x_n)$ the reflection of x in Σ_λ.

By the moving planes we want to prove that for any $\lambda > 0$ and all large $R > 0$,

$$h(x) > h(x_\lambda) \quad \forall\, x \in D_{\lambda,R}. \tag{3.17}$$

In the following we use x to denote the variables of h. If (3.17) holds, then by sending $\lambda \to 0$ and $R \to \infty$, we obtain

$$h(x_1, \ldots, x_{n-1}, x_n) \geq h(x_1, \ldots, x_{n-1}, -x_n).$$

As the axis x_n can be chosen in any direction perpendicular to x_1, hence h, and also u, are symmetric in $\tilde{x}$. Note that we can choose any point $\tilde{x}_0$ as a center. Hence the above proof implies that u is symmetric with respect to $\tilde{x}$ (with center at $\tilde{x}_0$). We can then use the following lemma for $u(x_1, \tilde{x})$, for any fixed x_1, to conclude that u is a quadratic function.

Lemma 3.8. *Let u be a convex function in $\mathbb{R}^n$. Suppose that at any point $x_0 \in \mathbb{R}^n$, $u - Du(x_0)(x - x_0)$ is symmetric with respect to x_0. Then up to a constant multiplication, $u(x) = \frac{1}{2}|x|^2 + \ell(x)$, where ℓ is a linear function.*

Proof. By subtracting a linear function we assume that $u(0) = 0$ and $Du(0) = 0$. By assumption, u is symmetric with respect to the origin 0. Hence $u(x) = u(|x|)$. Therefore it suffices to consider u as a radial function defined on $[0, \infty)$. Hence the condition in the lemma implies that for any $a > 0$,

$$u(a+t) - u(a-t) = [u(a+t) - u'(a)t] - [u(a-t) + u'(a)t] + 2u'(a)t = 2u'(a)t.$$

If $u \in C^3$, by the Taylor expansion at a, for sufficiently small t, we find that the third derivative of u must vanish everywhere. Hence u is a quadratic function.

If u is not C^3 smooth, consider the case $a = 0$, and by subtracting a linear function we assume $u'(0) = 0$. Then the above formula means u is even, so its mollification is also even. Hence its third derivative exists and vanishes at $a = 0$. As a is arbitrary, u is a quadratic function. q.e.d.

From Lemma 3.8 and by (3.17) (or the above discussions), we see that for any given x_1, $u(x_1, \tilde{x})$ is a quadratic function of $\tilde{x}$, i.e.

$$u(x_1, \tilde{x}) = \frac{1}{2}|\tilde{x}|^2 + a(x_1)\tilde{x} + b(x_1),$$

where $a(x_1) = (a_2(x_1), \ldots, a_n(x_1))$. Hence

$$\begin{aligned} \det D^2 u &= b'' - \sum\nolimits_{i=2}^{n} (a_i')^2, \\ \left(\frac{u_{x_1}}{x_1}\right)^{n+2} &= \left|\frac{a'\tilde{x} + b'}{x_1}\right|^{n+2}, \end{aligned}$$

where b' and b'' are the first and second derivatives of b. Notice that $\det D^2 u$ is independent of $\tilde{x}$. Hence the right hand sides of the above two formulas are the same and we have $a' = 0$,

$$b'' = \left|\frac{b'}{x_1}\right|^{n+2}.$$

Noting that $b'(0) = 0$, we have $b' = x_1$ and so $u = \frac{1}{2}|x|^2$.

It remains to prove (3.17) for any $\lambda > 0$ and all large R. This will be achieved in the following two steps.

(i) We show that (3.17) holds for all large $R > 0$ when $\lambda > \lambda_0$, where $\lambda_0 > 0$ depends only on the upper bound of $|u_{ijk}(0)|$. This step is for any $h = u^{\#}$, provided u satisfies the expression (3.15).

(ii) When $u_{ijk}(0) = 0$, by the maximum principle we show that $\lambda_0 = 0$.

Step (i) readily follows from (3.16). First note that by the boundary condition (3.2), we have

$$h(x) = \frac{1}{2}|\tilde{x}|^2 \quad \text{on} \quad \{x_1 = 0\}.$$

By (3.16) we have

$$\begin{aligned} h(x) - h(x_\lambda) &= \frac{1}{2}\big[x_n^2 - (x_n - 2\lambda)^2\big] + 2\big[\frac{a(x)}{|x|^2} - \frac{a(x_\lambda)}{|x_\lambda|^2}\big] \\ &+ \big[b^*(x) - b^*(x_\lambda)\big] \\ &= 2\lambda[x_n - \lambda] + 2\big[\frac{a(x)}{|x|^2} - \frac{a(x_\lambda)}{|x_\lambda|^2}\big] + \big[b^*(x) - b^*(x_\lambda)\big]. \end{aligned} \tag{3.18}$$

Note that

$$|D_{x_n} a^*(x)| \leq C\sum |u_{ijk}(0)| \quad \text{when } |x| >> 1, \tag{3.19}$$

$$|D_{x_n} b^*(x)| = O(|x|^{-1}), \tag{3.20}$$

where $a^*(x) = \frac{a(x)}{|x|^2}$. Hence $\exists\ \lambda_0$ such that if $\lambda > \lambda_0$ and $x_n > \lambda$, $h(x) - h(x_\lambda) > 0$.

For step (ii), first we observe that equation (3.1) is invariant in reflection, namely if $h(x)$ satisfies (3.1), so does

$$h_\lambda(x) := h(x', 2\lambda - x_n).$$

Therefore both h and h_λ satisfy the equation

$$\det D^2 h = \big(\frac{h_{x_1}}{x_1}\big)^{n+2} \quad \text{in } D_{\lambda,R}. \tag{3.21}$$

Hence by the strong maximum principle or the comparison principle, it follows that if

$$h \geq h_\lambda \quad \text{on} \quad \partial D_{\lambda,R}, \tag{3.22}$$

then

$$\text{either} \quad h > h_\lambda \quad \text{in} \quad D_{\lambda,R}, \tag{3.23}$$

$$\text{or} \quad h \equiv h_\lambda \quad \text{in} \quad D_{\lambda,R}. \tag{3.24}$$

In the former case, (3.17) is proved. In the latter case we have $h = h_\lambda$ in the whole $\{x_n > \lambda\}$, and hence h is symmetric with respect to $\{x_n = \lambda\}$. By the boundary condition (3.2), λ must be zero.

Therefore it suffices to verify (3.22) for all large $R > 0$. The boundary $\partial D_{\lambda,R}$ consists of the following parts, $\partial D_{\lambda,R} = \bigcup_{i=1}^4 \Gamma_i$, where

$$\begin{aligned}
\Gamma_1 &= \partial D_{\lambda,R} \cap \{x_n = \lambda\}, \\
\Gamma_2 &= \partial D_{\lambda,R} \cap \{x_1 = 0\}, \\
\Gamma_3 &= \partial D_{\lambda,R} \cap \{|x'| = R\}, \\
\Gamma_4 &= \partial D_{\lambda,R} \cap \{x_n = \lambda + R\}.
\end{aligned}$$

On Γ_1, we obviously have $h = h_\lambda$.

On Γ_2, by the boundary condition we obviously have $h > h_\lambda$, provided $\lambda > 0$.

On Γ_3, by (3.7)–(3.9) and Lemma 3.6, we have $u_{ijk}(0) = 0$. Hence for any given $\lambda > 0$, if $R > 1$ is sufficiently large, by (3.19) and (3.20), we obtain from (3.18) that

$$\begin{aligned}
h(x) - h_\lambda(x) &= h(x) - h(x_\lambda) \\
&\geq 2\lambda(x_n - \lambda) - o(x_n - \lambda) > 0.
\end{aligned}$$

On Γ_4, (3.22) follows from (3.18) and (3.20) as $a^* \equiv 0$ in (3.19).

Therefore we have verified (3.22), and so also (3.17) by the maximum principle, for all $\lambda > 0$. Hence by Lemma 3.8 and the discussion before it, we conclude that u is a quadratic function. Theorem 1.2 is proved.

Remark 3.1. For any $\alpha > 0$, the above moving plane argument also implies that a smooth convex solution to

$$\det D^2 u = \Big(\frac{u_{x_1}}{x_1}\Big)^{n+2+\alpha} \quad \text{in } \mathbb{R}^{n,+}, \tag{3.25}$$

$$u = \frac{1}{2}|\tilde{x}|^2 \qquad \text{on } \{x_1 = 0\} \tag{3.26}$$

must be $u(x) = \frac{1}{2}|x|^2$ or $u(x) = \frac{1}{2}|\tilde{x}|^2$. Indeed, by the transform (2.1), equation (3.2) is changed to

$$\det D^2 h = \Big(\frac{h}{h^*}\Big)^\alpha \Big(\frac{h_{x_1}}{x_1}\Big)^{n+2+\alpha}. \tag{3.27}$$

When $\alpha \geq 0$, the comparison principle is applicable to the functions h and h_λ, and from (3.22) we can still infer (3.23) or (3.24).

Remark 3.2. By the moving plane, we see that a smooth convex solution to

$$\det D^2 u = \Big(\frac{u_{x_1}}{x_1}\Big)^{n+2} \quad \text{in } \mathbb{R}^{n,+} = \{x_1 > 0\}, \tag{3.28}$$

$$u = \phi(\tilde{x}) \quad \text{on } \{x_1 = 0\} \tag{3.29}$$

is symmetric with respect to $\tilde{x}$ if ϕ is. One can easily verify that (3.7)–(3.9) and Lemma 3.6 hold at $x = 0$, and hence $u_{ijk}(0) = 0$.

An interesting question is whether one can use the transform (1.2) to prove Bernstein theorems for other Monge-Ampère type equations,

$$\det D^2 u = f(x, u, Du) \quad \text{in } \mathbb{R}^n, \tag{3.30}$$

such as the case $f \equiv 1$. Let u be a solution to (3.30) and let $h = u^{\#}$ be the transform of u. Another interesting question is the singularity removability for h near the origin. Note that if h is smooth at the origin, then u is asymptotic to a quadratic function as infinity.

4. $C^{2,\alpha}$ regularity

In this section we prove the global $C^{2,\alpha}$ regularity for the problem

$$\begin{aligned} \det D^2 u = \frac{\eta(x)}{|u|^{n+2}} \quad &\text{in } \ \Omega, \\ u = 0 \quad &\text{on } \partial\Omega. \end{aligned} \tag{4.1}$$

Namely, the graph of u is $C^{2,\alpha}$ smooth up to the boundary. In this section we assume that Ω is a bounded, uniformly convex domain with $C^{k,\alpha}$ boundary, and η is positive and $C^{k-2,\alpha}$ smooth, $k \geq 3$. By Cheng-Yau [**7**] and Caffarelli [**2**, **16**], there is a unique convex solution $u \in C^{k,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ to (4.1). By a translation of coordinates, we assume that the origin $0 \in \partial\Omega$, and $e_1 = (1, 0, \ldots, 0)$ is the inner normal of $\partial\Omega$ at 0. We make the rotation of the coordinates

$$\begin{aligned} &y_1 = -x_{n+1}, \\ &y_k = x_k, \quad k = 2, \ldots, n, \\ &y_{n+1} = x_1. \end{aligned}$$

In the new coordinates, the graph of u near the origin can be represented as

$$y_{n+1} = v(y).$$

Since $u_{x_1} < 0$ near the origin, we have $v_{y_1} > 0$ near the origin.

From equation (4.1), the Gauss curvature

$$\begin{aligned} K &= \frac{\det D^2 u}{[1 + |Du|^2]^{(n+2)/2}} \\ &= \frac{\eta}{|u|^{n+2}[1 + |Du|^2]^{(n+2)/2}}. \end{aligned}$$

Hence

$$\det D^2 v = K[1+|Dv|^2]^{(n+2)/2} = \eta\left[\frac{1+|Dv|^2}{y_1^2(1+|Du|^2)}\right]^{\frac{n+2}{2}}.$$

Claim:

$$\left[\frac{1+|Dv|^2}{1+|Du|^2}\right]^{\frac{n+2}{2}} = v_{y_1}^{n+2}.$$

Indeed, note that the tangent plane is given by

$$x_{n+1} - u_{x_1}x_1 - \cdots - u_{x_n}x_n = 0.$$

After the above rotation of coordinates, it is given by

$$y_{n+1} + \frac{1}{u_{x_1}}y_1 + \frac{u_{x_2}}{u_{x_1}}y_2 + \cdots + \frac{u_{x_n}}{u_{x_1}}y_n = 0.$$

Hence

$$\begin{aligned} v_{y_1} &= -1/u_{x_1}, \\ v_{y_2} &= -u_{x_2}/u_{x_1}, \\ &\cdots \\ v_{y_n} &= -u_{x_n}/u_{x_1}, \end{aligned}$$

and the claim follows.

Therefore equation (4.1) can be written as (from now on we write v and y as u and x)

$$\det D^2 u = \eta\left[\frac{u_{x_1}}{x_1}\right]^{n+2} \quad \text{in} \quad B_R^+, \tag{4.2}$$

where $\eta = \eta(u, x_2, \ldots, x_n)$. By a rescaling, we assume that

$$\eta(0) = 1$$

and near 0, the boundary $\partial\Omega$ is given by $\{x_1 = \phi(x_2, \ldots, x_n)\}$ in the original coordinates. Then the boundary condition for (4.2) is

$$u = \phi \quad \text{on} \quad \{x_1 = 0\}. \tag{4.3}$$

Moreover, we have

$$u(0) = 0, \quad u \geq 0 \text{ and } u_{x_1} > 0 \text{ near } 0.$$

By Lemma 3.2,

$$u_{x_1} = 0 \quad \text{on} \quad \{x_1 = 0\}.$$

It follows that for any $k \geq 0$,

$$\begin{aligned} &D_{\tilde{x}}^k u_{x_1} = 0 \quad \text{on} \quad \{x_1 = 0\}, \\ &D_{\tilde{x}}^k u = D_{\tilde{x}}^k \phi, \end{aligned} \tag{4.4}$$

where $\tilde{x} = (x_2, \ldots, x_n)$. In particular, we have $u_{1k} = 0$ on $x_1 = 0$, for any $k \geq 2$. Therefore at any boundary point, we can make a rotation of the axes $x_2, \ldots, x_n$ such that D^2u is diagonal. By a unimodular linear transform of $\tilde{x}$, we may assume that $u_{ii}(0) = u_{jj}(0)$ for all $2 \leq i, j \leq n$. Since ϕ is uniformly convex, there is no loss in assuming that $u_{ii}(0) = 1$ for all $2 \leq i \leq n$. By (3.6) we can then assume that

$$D^2 u(0) = I \tag{4.5}$$

is the unit matrix.

To prove the regularity of the solution to (4.1), we employ the method of continuity, as was used in [**8**] for equation (1.12). It is worth noting that one may use the method of continuity in different ways. For example, we can write problems (4.2)–(4.3) as a Dirichlet problem for the support function of the solution on the south hemisphere $S^{n,-}$ and apply the method of continuity to the Dirichlet problem in the space $C^{3,\alpha}(S^{n,-})$. We can apply the method of continuity to $v = u^2$ in $C^{3,\alpha}(\overline{\Omega})$. Instead of fixing the domain Ω and allowing η to vary, we can also fix an η and allow the domain Ω to vary. That is, letting Ω_t be a family of uniformly convex domains such that Ω_0 is the unit ball and $\Omega_1 = \Omega$, apply the method of continuity to Ω_t.

Remark 4.1. (i) The $C^{2,\alpha}$ regularity at boundary was obtained in [**22**]. At a boundary point 0, one can construct a lower-barrier w_1 and an upper-barrier w_2 such that

$$0 \leq w_2(x) - w_1(x) \leq C|x|^{2+\alpha}. \tag{4.6}$$

From (4.6), one can obtain the boundary $C^{2,\alpha}$ estimate as follows. By rescaling, one can first prove the strict convexity of solutions and use the interior second derivative estimate of Pogorelov [**28**] to get the $C^{1,1}$ estimate, such that equation (4.2) is uniformly elliptic. Then by (4.6) one can prove that for any ball $B_r(z)$, there is a quadratic polynomial P_z such that $\sup_{B_r(z)} |u(x) - P_z(x)| \leq Cr^{2+\alpha}$. By using Campanato's space, the $C^{2,\alpha}$ regularity follows as in [**1**, **29**].

(ii) Unaware of the work [**22**], in early 2008 we found a similar proof. We also constructed upper and lower barriers satisfying (4.6). Our barriers are as follows: Considering a boundary point 0, by a linear transform of $\tilde{x}$, we may assume that $\phi(\tilde{x}) = \frac{1}{2}|\tilde{x}|^2 + O(|\tilde{x}|^{2+\alpha})$ for some $\alpha \in (0,1)$. Then it is straightforward to verify that

$$\begin{aligned} w_1(x) &= \frac{1}{2}|x|^2 + a|x|^{2+\alpha}, \\ w_2(x) &= \frac{1}{2}|x|^2 - a|x|^{2+\alpha} \end{aligned}$$

are upper and lower barriers provided a is sufficiently large. Instead of using Camponato's space, we observed that the $C^{2,\alpha}$ estimate can be obtained from the interior regularity theory of Evans-Krylov [**11**, **18**] by a rescaling argument. That is, for any points $x, y \in \mathbb{R}^{n,+}$ with $|x-y| < \frac{1}{2}\min(d_x, d_y)$, where $d_x = \text{dist}(x, \partial\mathbb{R}^{n,+})$, by rescaling and the interior regularity theory of Evans-Krylov we have $|D^2u(x) - D^2u(y)| \le C|x - y|^\alpha$. For any two points $x, y \in \mathbb{R}^{n,+}$, assuming $d_y \le d_x$, we choose a sequence of points (x_k) on the line segment $\overline{xy}$ such that $x_0 = x$ and $x_N = y$ such that $|x_i - x_{i+1}| < \frac{1}{2}\min(d_{x_i}, d_{x_{i+1}})$ for $i = 0, 1, \ldots, N$. Then

$$\begin{aligned}|D^2u(x) - D^2u(y)| &\le \sum_{i=1}^{N} |D^2u(x_i) - D^2u(x_{i-1})| \\ &\le \sum C|x_i - x_{i-1}|^\alpha \le C|x - y|^\alpha.\end{aligned}$$

Note that $N = \infty$ if y is a boundary point. Note also that D^2u is $C^{2,\alpha}$ smooth on $\{x_1 = 0\}$, which follows from $u_{x_1x_k} = 0$ (Lemma 3.2) and (3.6), and so we may assume that the segment $\overline{xy}$ is parallel to the x_1-axis.

(iii) Note that if one applies the regularity in [**1**, **11**, **18**, **29**] directly, one may get the $C^{2,\alpha}$ regularity for a small $\alpha > 0$ only. But for the Monge-Ampère equation (4.2), if η is sufficiently smooth, we have the C^∞ interior regularity and the above argument implies the $C^{2,\alpha}$ regularity, for any $\alpha > 0$.

In the following, we first use the method of moving planes to prove a continuity estimate for D^2u at the boundary, then prove the $C^{2,\alpha}$ regularity, for any $\alpha \in (0, 1)$. The proof is more complicated than the one in Remark 4.1, and also we need to assume $\partial\Omega \in C^{3,\alpha}$. We present the proof here because the technique is new and should be of some interest in the area. Moreover, some estimates will be needed in the next section for the proof of the higher regularity, which is more complicated. Our main interest is to obtain the optimal regularity, as with Fefferman's equation (1.12), mentioned in the introduction.

4.1. Continuity estimate for D^2u. Let u be a smooth convex solution to (4.2) and (4.3). We assume $\partial\Omega \in C^{3,\alpha}$ and $\eta \in C^{1,\alpha}$. We will first prove for simplicity the regularity of solutions to (4.2)–(4.3) for the case $\eta \equiv 1$. We then explain that our argument also applies to solutions with general smooth and positive η.

To start with, we first recall the estimate in Lemma 3.4,

$$\phi(\tilde{x}) + C_1x_1^2 \le u(x) \le \phi(\tilde{x}) + C_2x_1^2 \quad \text{in } B_R^+(0),$$

obtained by proper construction of barriers, which also follows from (4.6). Hence by the uniform convexity of ϕ we have

$$C_1|x|^2 \le u(x) \le C_2|x|^2 \quad \text{in } B_R^+(0), \tag{4.7}$$

for two positive constants C_1, C_2. From (4.7) and the convexity of u we also have

$$\partial_{x_1}u > 0 \ \text{ in } \ B_R^+(0). \tag{4.8}$$

To prove that D^2u is continuous at the boundary, it suffices to prove that $D^2u(x) \to D^2u(0)$. Namely, for an arbitrary sequence $p_m \to 0$, we want to prove that $D^2u(p_m) \to D^2u(0) = I$ by (4.5).[1] By a translation of $\tilde{x}$, we may assume that p_m lies on the x_1-axis such that $p_m = (\delta_m, 0, \dots, 0)$. Make the dilation $x \to x/\delta_m$ and $u \to u/\delta_m^2 =: u_m$. Then by the interior regularity of the Monge-Ampère equation (as in Remark 4.1), u_m is locally uniformly smooth near $e_1 = (1, 0, \dots, 0)$ and so $D^2u_m(e_1)$ converges. Moreover, u_m satisfies the equation

$$(4.9) \qquad \begin{aligned} \det D^2 u &= \Big[\frac{u_{x_1}}{x_1}\Big]^{n+2} \quad \text{in } B^+_{R/\delta_m}, \\ u &= \phi_m \quad \text{on } \{x_1 = 0\}, \end{aligned}$$

where $\phi_m(\tilde{x}) = \delta_m^{-2}\phi(\delta_m x)$. Note that (4.7) is also invariant under the dilation. Since $\phi_m \to \frac{1}{2}|\tilde{x}|^2$ locally smoothly, as in (3.7)–(3.9) and Lemma 3.6 we have

$$(4.10) \qquad \begin{aligned} D^3_{x_1} u_m(0) &\to 0, \\ D_{\tilde{x}} D^2_{x_1} u_m(0) &\to 0, \\ D^2_{\tilde{x}} D_{x_1} u_m(0) &= 0, \\ D^3_{\tilde{x}} u_m(0) &\to 0, \end{aligned}$$

where the equality in the third line of (4.10) is due to Lemma 3.2, which also holds for solutions to (4.2) and (4.3).

Let $h_m = u_m^{\#}$ be the transform of u_m. By (4.7), the function h_m is defined in $\mathbb{R}^{n,+} \backslash B_{C\delta_m}(0)$. By the monotonicity property (iv) of the transform in Section 2, we also have

$$(4.11) \qquad h_m \le C\delta_m^2 \quad \text{on } \partial B^+_{C\delta_m}(0).$$

We can extend h_m to $B^+_{C\delta_m}(0)$ such that h_m is convex in $\mathbb{R}^{n,+}$ and $h_m \le C\delta_m^2$ in $B^+_{C\delta_m}(0)$.

With the above preparation, we now apply the moving plane argument in Section 3 to h_m. By step (i) of Section 3 (details after Lemma 3.8), there exists $\lambda_0 > 0$, independent of m, such that

$$(4.12) \qquad h_m(x) \ge h_{m,\lambda}(x) \quad \forall\, x \in D_{\lambda,R}$$

for all $\lambda \ge \lambda_0$ and large $R > 0$, where $h_{m,\lambda}(x) = h_m(x_\lambda)$.

For step (ii), by (4.10) and (4.11), one can easily verify that for each m, there exists $\lambda_m > 0$, with $\lambda_m \to 0$ as $m \to \infty$, such that (4.12)

[1] Since we use the continuity method, the solution is assumed to be smooth and $D^2u(p_m) \to D^2u(0)$ holds automatically. Our blow-up argument below gives a uniform convergence rate for the sequence $D^2u(p_m)$, which implies an estimate for the modulus of continuity of D^2u at the boundary. More generally the blow-up argument also applies to a sequence of solutions $\{u_m\}$, and one obtains an estimate, uniformly in m, for the modulus of continuity of D^2u_m at the boundary.

holds for any $\lambda > \lambda_m$ and large $R > 1$. Hence the limit functions $h = \lim_{m\to\infty} h_m$ and $h_\lambda = \lim_{m\to\infty} h_{m,\lambda}$ satisfy

$$(4.12)' \qquad h(x) \ge h_\lambda(x) \quad \forall\, x \in \mathbb{R}^{n,+}.$$

Therefore h is rotationally symmetric in $\tilde{x}$. By Lemma 3.8 and our boundary condition, we have $h \equiv \frac{1}{2}|x|^2$. Changing back, we obtain $D^2u(p_m) \to I$.

Next we show that the above moving plane argument also applies to (4.2) for any positive and continuous η. Indeed, examining the above proof, we see that the argument for (4.7)–(4.11) is fine, and the argument for (4.12) and (4.12)′ needs change. When η is not a constant, the functions h_m and $h_{m,\lambda}$ satisfy respectively the equation

$$\det D^2 h = \eta(\cdot)\big(\tfrac{h_{x_1}}{x_1}\big)^{n+2}$$

$$\det D^2 h_\lambda = \eta_\lambda(\cdot)\big(\tfrac{(h_\lambda)_{x_1}}{x_1}\big)^{n+2}$$

in $D_{\lambda,R}$. For a given $\lambda \in (0,\lambda_0)$, we don't have the monotonicity $\eta \le \eta_\lambda$ and cannot infer the monotonicity (4.12) directly. But we have a weak form of (4.12); namely, for any given $\lambda_0 > 0$ and $R_0 > 1$, there exists $\epsilon_m \to 0$ such that

$$(4.13) \qquad h_m(x) \ge h_{m,\lambda}(x) - \epsilon_m \quad \forall\, x \in D_{\lambda,R}$$

for all $\lambda \ge \lambda_0$ and large $R \ge R_0$. Indeed, if this is not true, by step (i) we can move the plane Σ_λ such that when $m >> 1$,

$$(4.13)' \qquad h_m(x) \ge h_{m,\lambda}(x) - \epsilon_0 \quad \forall\, x \in D_{\lambda,R}$$

for all $\lambda \ge \lambda_0$, and there is a point $x_0 \in D_{\lambda,R}$ such that

$$(4.13)'' \qquad h_m(x_0) \le h_{m,\lambda}(x_0) - \epsilon_0/2,$$

where $\epsilon_0 > 0$ is a small constant. We should point out that the functions h_m and $h_{m,\lambda}$ behave nicely at infinity, and (4.13) holds when $|x|$ is sufficiently large, as shown in step (i). Sending $m \to \infty$ and noticing that both η and η_λ converge to $\eta(0)$ locally uniformly, we infer that (4.12)′ holds for the limits $h = \lim_{m\to\infty} h_m$ and $h_\lambda = \lim_{m\to\infty} h_{m,\lambda}$, which is in contradiction with (4.13)″. Hence the limit function h is rotationally symmetric, and so we also obtain a continuity estimate for D^2u.

We also remark that the modulus of continuity of D^2u depends on n, $\inf \eta$, $\|\eta\|_{C^{1,\alpha}}$, and $\|\phi\|_{C^{3,\alpha}}$. The $C^{3,\alpha}$ regularity of ϕ and the $C^{1,\alpha}$ regularity of η are such that (4.10) holds. But for the argument in §4.1, it suffices to assume η is continuous. Note that in Lemma 3.7, we need $u \in C^{3,1}$, but by approximation it suffices to assume that $u \in C^{3,\alpha}$.

4.2. $C^{1,\alpha}$**-estimate for** $u_{\tilde{x}}$**.** Differentiating equation (4.2) in x_k, $k \geq 2$, we get

$$L[w] = (n+2)\tilde{\eta}\Big[\frac{u_{x_1}}{x_1}\Big]^{n+1}\frac{w_{x_1}}{x_1} + \tilde{\eta}_{x_k}\Big[\frac{u_{x_1}}{x_1}\Big]^{n+2}, \tag{4.14}$$

where $\tilde{\eta}(x) = \eta(u(x), \tilde{x})$, $w = u_{x_k}$, and $L = \sum a_{ij}(x)\partial_{x_i}\partial_{x_j}$ is the linearized operator of $\det D^2 u$. It is a linear, uniformly elliptic operator with continuous coefficients. By the boundary condition (4.3), we have $w(0, \tilde{x}) = \phi_{x_k}(\tilde{x})$.

We construct proper barriers to show that

$$|w(x) - \phi_{x_k}(\tilde{x})| \leq Cx_1^2 \quad \text{in } G_r, \tag{4.15}$$

where

$$G_r = \{|\tilde{x}| < r, 0 < x_1 < r\}.$$

First by subtracting a linear function of $\tilde{x}$, we assume that w satisfies

$$|w| \leq C|\tilde{x}|^2 \quad \text{on } \{x_1 = 0\}.$$

Let

$$z = a|\tilde{x}|^2 + bx_1^2.$$

Obviously

$$z \geq w \quad \text{on } \ \partial G_r, \tag{4.16}$$

when a, b are suitably large. To verify

$$Lz \leq (n+2)\tilde{\eta}\Big[\frac{u_{x_1}}{x_1}\Big]^{n+1}\frac{z_{x_1}}{x_1} + \tilde{\eta}_{x_k}\Big[\frac{u_{x_1}}{x_1}\Big]^{n+2} \quad \text{in } G_r, \tag{4.17}$$

by (4.5), the continuity of D^2u, and recalling that $\tilde{\eta}(0) = 1$, we have

$$\begin{aligned} \text{the matrix } \{a_{ij}\} &= I + o(1), \\ \frac{u_{x_1}}{x_1} &= u_{11}(0) + o(1) = 1 + o(1), \\ \tilde{\eta} &= 1 + o(1) \end{aligned}$$

near the origin. Hence

$$\begin{aligned} Lz = a_{11}z_{11} + C &= 2b[1 + o(1)] + C, \\ (n+2)\tilde{\eta}\Big[\frac{u_{x_1}}{x_1}\Big]^{n+1}\frac{z_{x_1}}{x_1} + \tilde{\eta}_{x_k}\Big[\frac{u_{x_1}}{x_1}\Big]^{n+2} &\geq 2b(n+2)\big[1 - o(1)\big] - C, \end{aligned}$$

where C depends on a but not b. Therefore (4.17) holds when b is chosen large.

From (4.16) and (4.17) and by the comparison principle, we have $z \geq w$. It follows that on the x_1-axis, $w(x_1, 0) \leq bx_1^2$ on the x_1-axis. By the same argument we also have $w(x_1, 0) \geq -bx_1^2$ on the x_1-axis. Similarly, for any $\tilde{x}$, we have $|w(x) - \phi_{x_k}(\tilde{x})| \leq bx_1^2$. Therefore (4.15) holds.

Observe that (4.15) implies $\frac{w_{x_1}}{x_1} \in L^\infty(G_r)$. By (4.14) and the $W^{2,p}$ estimate for the linear elliptic equation, we obtain

$$\|u_{\tilde{x}}\|_{W^{2,p}(G_r)} \leq C_1, \tag{4.18}$$

where C_1 depends on n, r, p, ϕ, η and the modulus of continuity of D^2u. By the Sobolev embedding, it follows that for any $\alpha \in (0,1)$,

$$||D_{\tilde{x}}u||_{C^{1,\alpha}(\bar{G}_r)} \leq C, \tag{4.19}$$

where C depends on C_1 and α.

4.3. $C^{2,\alpha}$-estimate for u. It remains to prove $u_{11} \in C^\alpha$. Usually the regularity of u_{11} follows from the regularity of u_{ij} $(i + j > 2)$ and the equation. However, this is not the case for equation (4.2), due to the singular term u_{x_1}/x_1.

Denote

$$\begin{aligned} f(t) &= u_{x_1}(t, \tilde{x}), \\ h(t) &= \tilde{\eta}/\det D^2_{\tilde{x}}u(t, \tilde{x}), \\ g(t) &= \tilde{g}(t, \tilde{x})/\det D^2_{\tilde{x}}u(t, \tilde{x}), \end{aligned}$$

where $\tilde{g}$ is given in (3.11). Then we can write (3.10) as an ordinary differential equation

$$f'(t) = h(t)\big[\frac{f}{t}\big]^{n+2} + g(t), \tag{4.20}$$

regarding $\tilde{x}$ as parameters. By (4.4) and (4.19), we have $g, h \in C^\alpha(\bar{G}_r)$ and

$$\begin{aligned} &g(0) = 0, \quad |g(t)| \leq Ct^\alpha, \\ &|h(t) - h(0)| \leq Ct^\alpha. \end{aligned} \tag{4.21}$$

By (4.5) and the C^2 continuity of u, we have

$$\frac{1}{2} < \frac{t}{f(t)} < \frac{3}{2} \quad \text{when} \quad (t, \tilde{x}) \in G_r \tag{4.22}$$

for some $r > 0$. To prove $u_{11} \in C^\alpha$, by equation (3.10) it suffices to prove $\frac{u_{x_1}}{x_1} \in C^\alpha$.

We first show that $\frac{u_{x_1}}{x_1}$ is C^α in x_1; namely, $\frac{f}{t}$ is C^α in t. Solving the ordinary differential equation (4.20), we have

$$[f(t)]^{-n-1} = [f(r)]^{-n-1} + (n+1)\int_t^r \left[\frac{h(s)}{s^{n+2}} + \frac{g(s)}{f^{n+2}(s)}\right] ds.$$

Namely,

(4.23)

$$\big[\frac{t}{f(t)}\big]^{n+1} = \big[\frac{t}{f(r)}\big]^{n+1} + (n+1)\big[G_1(t) + G_2(t)\big] + \frac{1 - (tr^{-1})^{n+1}}{n+1} h(0),$$

where

$$G_1(t) = t^{n+1}\int_t^r \frac{h(s)-h(0)}{s^{n+2}}ds, \tag{4.24}$$

$$G_2(t) = t^{n+1}\int_t^r \frac{g(s)}{f^{n+2}(s)}ds. \tag{4.25}$$

Differentiating G_1 and G_2, we get

$$\begin{aligned} G_1'(t) &= (n+1)t^n\int_t^r \frac{h(s)-h(0)}{s^{n+2}}ds - t^{n+1}\frac{h(t)-h(0)}{t^{n+2}},\\ G_2'(t) &= (n+1)t^n\int_t^r \frac{g(s)}{f^{n+2}(s)}ds - t^{n+1}\frac{g(t)}{f^{n+2}(t)}. \end{aligned}$$

By (4.21) and (4.22), we obtain $|G_i'(t)| < Ct^{\alpha-1}$ for $t \in (0,r)$ and $i = 1,2$. Hence $G_i \in C^\alpha[0,r]$ and so is $[\frac{t}{f(t)}]^{n+1}$. It follows that the right hand side of (4.20) is in $C^\alpha[0,r]$. Hence $u_{11} = f' \in C^\alpha[0,r]$, i.e., u_{11} is Hölder in x_1, and its Hölder norm is controlled by that of u_{ij} for $i+j > 2$, obtained in §4.2.

Next we show that $\frac{u_{x_1}}{x_1}$ is C^α continuous in $\tilde{x}$. Indeed, regard $\tilde{x}$ as parameters, and recall that f, g, h in the ordinary differential equation (4.20) all depend on the parameters $\tilde{x}$, and are Hölder continuous in $\tilde{x}$. We have

$$\begin{aligned} |G_1(t,\tilde{x}_1) - G_1(t,\tilde{x}_2)| &\le t^{n+1}\int_t^r \frac{|h(s,\tilde{x}_1)-h(s,\tilde{x}_2)| + |h(0,\tilde{x}_1)-h(0,\tilde{x}_2)|}{s^{n+2}}ds\\ &\le Ct^{n+1}\int_t^r \frac{|\tilde{x}_1-\tilde{x}_2|^\alpha}{s^{n+2}}ds\\ &\le C|\tilde{x}_1-\tilde{x}_2|^\alpha. \end{aligned}$$

Next we estimate G_2. Note that if $|\tilde{x}_1-\tilde{x}_2| > t$, then

$$\begin{aligned} |G_2(t,\tilde{x}_1) - G_2(t,\tilde{x}_2)| &\le |G_2(t,\tilde{x}_1)| + |G_2(t,\tilde{x}_2)|\\ &\le Ct^\alpha \le C|\tilde{x}_1-\tilde{x}_2|^\alpha. \end{aligned}$$

If $|\tilde{x}_1-\tilde{x}_2| < t$, then

$$|G_2(t,\tilde{x}_1) - G_2(t,\tilde{x}_2)| \le t^{n+1}\int_t^r \Big|\frac{g(s,\tilde{x}_1)}{f^{n+2}(s,\tilde{x}_1)} - \frac{g(s,\tilde{x}_2)}{f^{n+2}(s,\tilde{x}_2)}\Big|ds$$

$$\le t^{n+1}\Big[\int_t^r \Big|\frac{g(s,\tilde{x}_1)-g(s,\tilde{x}_2)}{f^{n+2}(s,\tilde{x}_1)}\Big|ds + \int_t^r \Big|\frac{g(s,\tilde{x}_2)}{f^{n+2}(s,\tilde{x}_1)} - \frac{g(s,\tilde{x}_2)}{f^{n+2}(s,\tilde{x}_2)}\Big|ds\Big].$$

The first integral

$$\le Ct^{n+1}\int_t^r \frac{|\tilde{x}_1-\tilde{x}_2|^\alpha}{s^{n+2}}ds \le C|\tilde{x}_1-\tilde{x}_2|^\alpha.$$

The second one

$$
\begin{aligned}
&\le Ct^{n+1}\int_t^r s^\alpha \Big|\frac{f^{n+2}(s,\tilde{x}_1)-f^{n+2}(s,\tilde{x}_2)}{f^{n+2}(s,\tilde{x}_1)f^{n+2}(s,\tilde{x}_2)}\Big|ds \\
&\le Ct^{n+1}\int_t^r \frac{s^\alpha}{s^{2(n+2)}}\big|f^{n+2}(s,\tilde{x}_1)-f^{n+2}(s,\tilde{x}_2)\big|ds \\
&\le Ct^{n+1}\int_t^r \frac{s^\alpha}{s^{n+3}}\big|f(s,\tilde{x}_1)-f(s,\tilde{x}_2)\big|ds \\
&\le Ct^{n+1}\int_t^r \frac{s^\alpha}{s^{n+3}}\big|\tilde{x}_1-\tilde{x}_2\big|ds \\
&\le C|\tilde{x}_1-\tilde{x}_2|^\alpha.
\end{aligned}
$$

We have therefore obtained by (4.23) that $\left[\frac{t}{f(t)}\right]^{n+1}$ is Hölder in $\tilde{x}$. Hence, u_{11} is Hölder in $\tilde{x}$ by (4.20).

5. Higher regularity

In §4 we have proved the boundary $C^{2,\alpha}$ regularity for solutions to (4.2) and (4.3). But due to the singularity $\frac{u_{x_1}}{x_1}$ in (4.2), it does not imply higher regularity. The following examples show that a solution to an elliptic equation with singularity u_{x_1}/x_1 is usually not smooth at the boundary.

Example 5.1. The function $u = x_1^{1+\alpha}$, $\alpha \in \mathbb{R}^1$, satisfies the equation

$$\Delta u = \alpha \frac{u_{x_1}}{x_1} \quad \text{in } \mathbb{R}^{n,+},$$

but $u \notin C^{1,\alpha+\epsilon}$ $\forall\ \epsilon > 0$ if α is not a positive integer.

Example 5.2. The function $u(x) = x_1^{k+2}\log x_1$ ($\forall\ k \ge 0$) is a solution to

$$\Delta u = (k+1)\frac{u_{x_1}}{x_1} + (k+2)x_1^k \quad \text{in } \mathbb{R}^{n,+},$$

but $u \notin C^{k+2}$.

The above examples show that the regularity is determined by the coefficient of $\frac{u_{x_1}}{x_1}$. The first example shows that u cannot be C^∞ smooth if the coefficient of $\frac{u_{x_1}}{x_1}$ is not an integer. In the second example, the solution is not $C^{1,1}$ if $k = 0$. In this case, there is no $C^{1,1}$ super-solution to the equation. In fact, we will prove for the ordinary differential equation

$$f'(t) = h(t)\frac{f(t)}{t} + g(t), \qquad t \in [0, t_0), \tag{5.1}$$

that if $h(0) = 1$, $h, g \in C^\alpha$ for some $\alpha \in (0,1)$, then the solution to (5.1) is $C^{1,\alpha}$ at $t = 0$ if and only if $g(0) = 0$.

For the linearized equation of (4.2), the coefficient of $\frac{u_{x_1}}{x_1}$ is equal to $n+2$ at $x_1 = 0$; see Remark 5.3 below. In this section we introduce an iteration to prove the $C^{n+2,\alpha}$ regularity. We point out that our iteration does not assume a priori the regularity of the solution at the boundary.

We prove directly that a $C^{2,\alpha}$ solution is $C^{n+2,\alpha}$ near the boundary. An interesting consequence of our iteration is a compatibility condition for the general elliptic equation with the singularity u_{x_1}/x_1, which implies our regularity is optimal.

5.1. A lemma. Our iteration and higher regularity are based on the following lemma.

Lemma 5.1. *Assume* $h, g \in C^\alpha(\bar{G}_r)$, $f \in C^\alpha(\bar{G}_r) \cap C^1(G_r)$, *where* $\alpha \in (0,1)$. *Assume*

$$h(0,\tilde{x}) = N \quad \forall\ |\tilde{x}| < r$$

for some positive integer N. *If* $N = 1$, *we also assume that* $g(0,\tilde{x}) = 0$. *Then if*

$$f'(t,\tilde{x}) = h(t,\tilde{x})\frac{f(t,\tilde{x})}{t} + g(t,\tilde{x}) \quad (t,\tilde{x}) \in G_r, \tag{5.2}$$

where f' *denotes a derivative of* f *in* t, *we have* $f(0,\tilde{x}) = 0$ *for* $|\tilde{x}| < r$ *and for any* $\epsilon > 0$, $f \in C^{1,\alpha-\epsilon}(\bar{G}_r)$ *and*

$$||f||_{C^{1,\alpha-\epsilon}(\bar{G}_r)} \leq C_\epsilon$$

for a constant C_ϵ *depending only on* ϵ *and the Hölder norms of* g *and* h.

Proof. Let $z = t^{-N} f$. Then equation (5.2) is changed to

$$z'(t,\tilde{x}) = \frac{h(t,\tilde{x}) - N}{t} z(t,\tilde{x}) + \frac{g(t,\tilde{x})}{t^N}.$$

Hence

$$z(t,\tilde{x}) = e^{-H(t,\tilde{x})}\left[z(r,\tilde{x}) - \int_t^r \frac{G(s,\tilde{x})}{s^N} ds\right],$$

where

$$\begin{aligned} H(t,\tilde{x}) &= \int_t^r \frac{h(s,\tilde{x}) - N}{s} ds, \\ G(s,\tilde{x}) &= g(s,\tilde{x}) e^{H(s,\tilde{x})}. \end{aligned}$$

We obtain

$$\frac{f(t,\tilde{x})}{t} = e^{-H(t,\tilde{x})}\big[\frac{f(r,\tilde{x})}{r^N} t^{N-1} - \tilde{G}(t,\tilde{x})\big], \tag{5.3}$$

where

$$\begin{aligned} \tilde{G}(t,\tilde{x}) &= t^{N-1}\int_t^r \frac{G(s,\tilde{x})}{s^N} ds \\ &= t^{N-1}\left[\int_t^r \frac{G(0,\tilde{x})}{s^N} ds + \int_t^r \frac{G(s,\tilde{x}) - G(0,\tilde{x})}{s^N} ds\right]. \end{aligned} \tag{5.3$'$}$$

By assumption, N is an integer and $G(0,\tilde{x}) = 0$ if $N = 1$. Hence the first integral on the right hand side is a smooth function. Therefore there exists a constant $C > 0$ such that

$$\begin{aligned} |H(t,\tilde{x})| &\leq C, \\ |H'(t,\tilde{x})| &\leq Ct^{\alpha-1}, \\ |\tilde{G}(t,\tilde{x})| &\leq C, \\ |\tilde{G}'(t,\tilde{x})| &\leq Ct^{\alpha-1} \end{aligned}$$

for all $(t,\tilde{x}) \in G_r$. Namely $H(t,\cdot), e^{-H(t,\cdot)}, G(t,\cdot)$ and $\tilde{G}(t,\cdot)$, and so also $\frac{f(t,\cdot)}{t}$, are all in $C^\alpha[0,r]$ in t and their Hölder norms are independent of $\tilde{x}$. It means by (5.2) that $f'(t,\cdot) \in C^\alpha[0,r]$ in t, and so $f(t,\cdot) \in C^{1,\alpha}[0,r]$ and its $C^{1,\alpha}$ norm is independent of $\tilde{x}$.

Multiplying both sides of (5.2) by t and by the $C^{1,\alpha}$ regularity of f, we obtain that $f(0,\tilde{x}) = 0$ for all $|\tilde{x}| < r$.

To show that f is $C^{1,\alpha-\epsilon}$ in $\tilde{x}$, observe that

$$H(0,\tilde{x}) = \int_0^r \frac{h(s,\tilde{x}) - N}{s} ds = \int_0^r \frac{h(s,\tilde{x}) - h(0,\tilde{x})}{s} ds.$$

The Hölder continuity of h implies that

$$|H(t,\tilde{x}) - H(0,\tilde{x})| \leq Ct^\alpha$$

for all $(t,\tilde{x}) \in G_r$; namely, H is Hölder continuous in t. To show that H is Hölder continuous in $\tilde{x}$, note that

$$\begin{aligned} |h(t,\tilde{x}_1) - h(t,\tilde{x}_2)| &\leq |h(t,\tilde{x}_1) - N| + |h(t,\tilde{x}_2) - N| \\ &= |h(t,\tilde{x}_1) - h(0,\tilde{x}_1)| + |h(t,\tilde{x}_2) - h(0,\tilde{x}_1)| \\ &\leq Ct^\alpha. \end{aligned}$$

If $|\tilde{x}_1 - \tilde{x}_2| \geq t$, then for any $\epsilon > 0$,

$$\begin{aligned} |H(t,\tilde{x}_1) - H(t,\tilde{x}_2)| &\leq \int_t^r \left|\frac{h(s,\tilde{x}_1) - h(s,\tilde{x}_2)}{s}\right| ds \\ &\leq C\int_t^r s^{\alpha\epsilon} \frac{|h(s,\tilde{x}_1) - h(s,\tilde{x}_2)|^{1-\epsilon}}{s} ds \\ &\leq C|\tilde{x}_1 - \tilde{x}_2|^{\alpha(1-\epsilon)} \int_t^r s^{\alpha\epsilon-1} ds \\ &\leq C_\epsilon |\tilde{x}_1 - \tilde{x}_2|^{\alpha(1-\epsilon)}. \end{aligned}$$

If $|\tilde{x}_1 - \tilde{x}_2| < t$, then for any $\epsilon > 0$,

$$\begin{aligned} |H(t,\tilde{x}_1) - H(t,\tilde{x}_2)| &\leq \int_t^r \Big|\frac{h(s,\tilde{x}_1) - h(s,\tilde{x}_2)}{s}\Big| ds \\ &\leq C\int_t^r \frac{|\tilde{x}_1 - \tilde{x}_2|^\alpha}{s} ds \\ &\leq C|\tilde{x}_1 - \tilde{x}_2|^\alpha |\log t| \\ &\leq C_\epsilon |\tilde{x}_1 - \tilde{x}_2|^{\alpha-\epsilon}. \end{aligned}$$

Similarly, one can estimate the Hölder continuity of $\tilde{G}(t,\tilde{x})$. q.e.d.

Remark 5.1. By the function $u = Ct^2 \log t$ and letting $f = u_{x_1}$, we see from Example 5.2 that Lemma 5.1 is not true if $N = 1$ and $g(0,\tilde{x}) \neq 0$. Namely, the solution f of (5.2) is not Lipschitz if $g(0,\tilde{x}) \neq 0$. This is due to the integration in (5.3)$'$,

$$t^{N-1}\int_t^r \frac{G(0,\tilde{x})}{s^N} ds = G(0,\tilde{x}) \log s\big|_t^r,$$

which is not continuous at $t = 0$ when $N = 1$. More precisely, if $N = 1$, then by (5.3), the solution can be expressed as

$$f(t,\tilde{x}) = f_0(t,\tilde{x}) + g(0,\tilde{x}) \log t,$$

where $f_0 \in C^{1,\alpha}$. The solution cannot be Lipschitz continuous at $t = 0$ if $g(0,\tilde{x}) \neq 0$.

Therefore, if $h(0) = 1$, by Lemma 5.1 and Remark 5.1 we see that the solution f is $C^{1,\alpha}$ near $t = 0$ if and only if $g(0) = 0$.

Remark 5.2. In Lemma 5.1 we proved that f is $C^{1,\alpha}$ in t and $C^{1,\alpha-\epsilon}$ in $\tilde{x}$. However, for the solution u to (4.2)–(4.3), if $u = \phi \in C^{k,\alpha}$ on $\{x_1 = 0\}$ and u is $C^{k,\alpha}$ in x_1, then by the rescaling argument in Remark 4.1(ii), u is $C^{k,\alpha}$ smooth in all variables x. Therefore for the regularity of u, the Hölder exponent in $\tilde{x}$ is the same as in x_1.

5.2. $C^{2,\alpha}$ estimate for $D_{\tilde{x}}^j u$. In §4.2 we have shown that $u_{\tilde{x}} \in C^{1,\alpha}$ for all $\alpha \in (0,1)$. Now we show that $u_{\tilde{x}} \in C^{1,1}$ up to the boundary $\{x_1 = 0\}$. Indeed, consider the point $(\delta, 0)$ on the x_1-axis, where $\delta > 0$ is a small constant. We make the scaling $v(y) = \delta^{-2} w(x)$ and $y = x/\delta$, where $w = D_{\tilde{x}} u$ as in (4.14). Then by (4.19) and the argument in §4.2, v satisfies the uniformly elliptic equation (4.14) in $B_{2/3}(e_1)$ with Hölder continuous coefficients, where $e_1 = (1, 0, \dots, 0)$. By (4.15), v is uniformly bounded. Hence, by the interior regularity for the equation (4.14), we have $\|v\|_{C^2(B_{1/2}(e_1))} \leq C$. Scaling back, we see that the second derivatives of w are uniformly bounded. Hence we obtain

$$\|u_{\tilde{x}}\|_{C^{1,1}(\bar{G}_r)} \leq C. \tag{5.4}$$

To prove that $u_{\tilde{x}\tilde{x}} \in C^{1,1}$, we differentiate equation (4.14) in $\tilde{x}$ again to get

$$L(w) = (n+2)\tilde{\eta}\big(\frac{u_{x_1}}{x_1}\big)^{n+1}\frac{w_{x_1}}{x_1} + g \tag{5.5}$$

where $w = D^2_{\tilde{x}}u$, $g = \hat{g} + \tilde{g}$, $\hat{g}$ arises in differentiating the coefficients of the operator L, and $\tilde{g}$ arises in differentiating the right hand side of (4.14),

$$\tilde{g} = (n+2)(n+1)\tilde{\eta}\big(\frac{u_{x_1}}{x_1}\big)^{n}\big(\frac{u_{1\tilde{x}}}{x_1}\big)^{2} + 2(n+2)\tilde{\eta}_{\tilde{x}}\big(\frac{u_{x_1}}{x_1}\big)^{n+1}\frac{u_{1\tilde{x}}}{x_1} + \tilde{\eta}_{x_k\tilde{x}}\big(\frac{u_{x_1}}{x_1}\big)^{n+2}.$$

By the $C^{1,1}$ estimate for $u_{\tilde{x}}$, we have $\hat{g} \in L^\infty(G_r)$. By (4.19) and (4.10) (or (5.4)), we have $\frac{u_{x_1}}{x_1}$, $\frac{u_{1\tilde{x}}}{x_1} \in L^\infty$, which implies $\tilde{g} \in L^\infty$. Hence, by the $W^{2,p}$-estimate we have the estimate $\|w\|_{W^{2,p}(G_r)} \le C$, which implies $w \in C^{1,\alpha}(\bar{G}_r)$ $\forall\, \alpha \in (0,1)$. By the rescaling argument in Remark 4.1(ii), we obtain $w \in C^{1,1}(\bar{G}_r)$, i.e.,

$$\|D^2_{\tilde{x}}u\|_{C^{1,1}(\bar{G}_r)} \le C.$$

Repeating the above argument, we obtain

$$||D^j_{\tilde{x}}u||_{C^{1,1}(\bar{G}_r)} \le C \tag{5.6}$$

for any $j \ge 1$.

To prove the $C^{2,\alpha}$ estimate for $D^j_{\tilde{x}}u$, by (5.6) it remains to prove that $D^j_{\tilde{x}}u_{11} \in C^\alpha(\overline{G}_r)$ for $j \ge 1$.

Fix a $k \ge 2$ and denote $f(t,\tilde{x}) = u_{1k}(t,\tilde{x})$. We write equation (4.14) in the form

$$f'(t,\tilde{x}) = h(t,\tilde{x})\frac{f(t,\tilde{x})}{t} + g(t,\tilde{x}), \tag{5.7}$$

where

$$\begin{aligned} h(t,\tilde{x}) &= \frac{n+2}{a_{11}(t,\tilde{x})}\tilde{\eta}(t,\tilde{x})\big[\frac{u_{x_1}(t,\tilde{x})}{t}\big]^{n+1}, \\ g(t,\tilde{x}) &= -\sum_{i+j>2}\frac{a_{ij}(t,\tilde{x})}{a_{11}(t,\tilde{x})}u_{ijk}(t,\tilde{x}) + \frac{\tilde{\eta}_{x_k}}{a_{11}(t,\tilde{x})}\big(\frac{u_{x_1}}{x_1}\big)^{n+2}. \end{aligned}$$

Note that $h, g \in C^\alpha(\bar{G}_r)$ by (5.6), and

$$h(0,\tilde{x}) = n+2 \tag{5.8}$$

by Remark 5.3. Applying Lemma 5.1 to equation (5.7), we obtain $f' \in C^{\alpha-\epsilon}(\bar{G}_r)$ for any $\epsilon > 0$. Therefore $u_{11k} \in C^{\alpha-\epsilon}(\bar{G}_r)$ for $k = 2, 3, \dots, n$ and we have the estimate

$$\|u_{11k}\|_{C^{\alpha-\epsilon}(\bar{G}_r)} \le C. \tag{5.9}$$

As noted in Remark 5.2, the $C^{\alpha-\epsilon}$ norm can be replaced by the C^α norm. To prove

$$\|D^j_{\tilde{x}}u\|_{C^{2,\alpha}(\bar{G}_r)} \le C \quad \forall\, j \ge 1, \tag{5.10}$$

we use induction. Denote

$$\begin{aligned} f_j(t,\tilde{x}) &= D_{\tilde{x}}^{j+1}u_1(t,\tilde{x}),\\ g_1(t,\tilde{x}) &= \partial_{\tilde{x}}g(t,\tilde{x}) + \partial_{\tilde{x}}h(t,\tilde{x})\frac{f(t,\tilde{x})}{t},\\ g_{j+1}(t,\tilde{x}) &= \partial_{\tilde{x}}g_j(t,\tilde{x}) + \partial_{\tilde{x}}h(t,\tilde{x})\frac{f_j(t,\tilde{x})}{t}. \end{aligned}$$

Note that the subscripts in f_k, g_k, and h_k mean index and the subscripts in u_{1k} mean partial derivatives. By (5.7), we have the equation

$$f_j'(t,\tilde{x}) = h(t,\tilde{x})\frac{f_j(t,\tilde{x})}{t} + g_j(t,\tilde{x}). \tag{5.11}$$

From (5.9), we have

$$\begin{aligned} &\|g_1\|_{C^{\alpha-\epsilon}(\bar{G}_r)} \le C,\\ &\|\frac{u_{1k}}{x_1}\|_{C^{\alpha-\epsilon}(\bar{G}_r)} \le C. \end{aligned}$$

By induction, assume that for some $j \ge 1$ and any small $\epsilon > 0$ (such that $j\epsilon << 1$),

$$\begin{aligned} &\|D_{\tilde{x}}^j u_{11}\|_{C^{\alpha-j\epsilon}(\bar{G}_r)} \le C,\\ &\|g_j\|_{C^{\alpha-j\epsilon}(\bar{G}_r)} \le C. \end{aligned} \tag{5.12}$$

We want to prove that (5.12) holds for $j+1$. Applying Lemma 5.1 to equation (5.11) for $j+1$, we obtain that $f_{j+1}' \in C^{\alpha-(j+1)\epsilon}(\bar{G}_r)$. Namely,

$$\|D_{\tilde{x}}^{j+1}u_{11}\|_{C^{\alpha-(j+1)\epsilon}(\bar{G}_r)} \le C,$$

which together with (5.6) implies

$$\|g_{j+1}\|_{C^{\alpha-(j+1)\epsilon}(\bar{G}_r)} \le C.$$

Therefore, we have proved that (5.12) holds for all $j \ge 1$. Hence (5.10) is proved. Note that by Taylor expansion, (5.10) implies

$$\|\frac{D_{\tilde{x}}^j u_1(x_1,\tilde{x})}{x_1}\|_{C^{\alpha}(\bar{G}_r)} \le C, \quad \forall j \ge 1. \tag{5.13}$$

Remark 5.3. We claim that

$$h(0,\tilde{x}) \equiv n+2.$$

Indeed, from (4.2) and (4.4), we have

$$a_{11}u_{11} = \eta u_{11}^{n+2} \quad \text{on} \quad \{x_1 = 0\}.$$

Hence $u_{11}^{n+1} = a_{11}/\eta$ on $\{x_1 = 0\}$. We obtain

$$h(0,\tilde{x}) = \lim_{t\to 0}\frac{n+2}{a_{11}}\eta\,\big[\frac{u_{x_1}(t,\tilde{x})}{t}\big]^{n+1} = \frac{n+2}{a_{11}}\eta\, u_{11}^{n+1} = n+2.$$

5.3. $C^{3,\alpha}$ estimate for u and $D^j_{\tilde{x}}u$. We first show $u_{111} \in C^\alpha(\bar{G}_r)$. Write equation (4.2) in the form (3.10), namely,

$$u_{11} = \frac{\tilde{\eta}}{\tilde{d}}(\frac{u_{x_1}}{x_1})^{n+2} + \frac{\tilde{g}}{\tilde{d}},$$

where $\tilde{g}$ is given in (3.11) and $\tilde{d} = \det D^2_{\tilde{x}}u$. From §5.2, $\tilde{d} \in C^{2,\alpha}(\bar{G}_r)$ and $\tilde{g} \in C^{1,\alpha}(\bar{G}_r)$. Differentiating the above equation, we have

$$u_{111}(t,\tilde{x}) = h(t,\tilde{x})(\frac{u_{11}}{t} - \frac{u_{x_1}}{t^2}) + g(t,\tilde{x}) \tag{5.14}$$

where t denotes x_1,

$$h = \frac{n+2}{\tilde{d}}\tilde{\eta}\big[\frac{u_{x_1}}{t}\big]^{n+1},$$
$$g = (\frac{\tilde{g}}{\tilde{d}})_{x_1} + (\frac{u_{x_1}}{t})^{n+2}(\frac{\tilde{\eta}}{\tilde{d}})_{x_1}.$$

By Taylor expansion at $t = 0$ and recalling that $D^j_{\tilde{x}}D_{x_1}u = 0$ at $\{x_1 = 0\}$, we have $\frac{u_{x_1}}{t} \in C^\alpha(\bar{G}_r)$. Hence $g, h \in C^\alpha(\bar{G}_r)$.

Denote

$$f_0(t,\tilde{x}) = u_{11}(t,\tilde{x}) - \frac{u_{x_1}(t,\tilde{x})}{t}.$$

Then (5.14) can be written as

$$f_0'(t,\tilde{x}) = h_0(t,\tilde{x})\frac{f_0(t,\tilde{x})}{t} + g_0(t,\tilde{x}), \tag{5.15}$$

where

$$\begin{aligned} g_0(t,\tilde{x}) &= g(t,\tilde{x}), \\ h_0(t,\tilde{x}) &= h(t,\tilde{x}) - 1, \\ h_0(0,\tilde{x}) &= n+1. \end{aligned} \tag{5.16}$$

Applying Lemma 5.1 to equation (5.15), we obtain $f_0' \in C^{\alpha-\epsilon}(\bar{G}_r)$. Observing that (5.14) and (5.15) imply

$$\begin{aligned} u_{111}(t,\tilde{x}) &= h(t,\tilde{x})\frac{f_0(t,\tilde{x})}{t} + g(t,\tilde{x}) \\ &= \big[h\frac{f_0' - g_0}{h_0}\big](t,\tilde{x}) + g(t,\tilde{x}), \end{aligned} \tag{5.17}$$

we obtain $u_{111}(t,\tilde{x}) \in C^{\alpha-\epsilon}(\bar{G}_r)$. Note that $\alpha \in (0,1)$ is arbitrary and $\epsilon > 0$ can be sufficiently small. We have thus proved

$$\|u_{111}\|_{C^\alpha(\bar{G}_r)} \le C. \tag{5.18}$$

Next we show that $D^j_{\tilde{x}}u \in C^{3,\alpha}(\bar{G}_r)$ for $j \ge 1$, namely $D^j_{\tilde{x}}u_{111} \in C^\alpha(\bar{G}_r)$. By (5.17), it suffices to prove that $D^j_{\tilde{x}}f_0' \in C^\alpha(\bar{G}_r)$. By differentiating (5.15) and applying Lemma 5.1, this can be proved similarly as (5.10). Note that equation (5.15) is of the same type as (5.7). We leave the details to the reader.

5.4. $C^{k,\alpha}$ estimate for u and $D^j_{\tilde{x}}u$ ($4 \le k \le n+2$). To prove the $C^{k,\alpha}$ estimate for u and $D^j_{\tilde{x}}u$, we use the induction argument. Assume that for some $3 \le k \le n+1$ and any $j \ge 1$,

$$\begin{aligned} &||u||_{C^{k,\alpha}(\bar{G}_r)} \le C, \\ &||D^j_{\tilde{x}}u||_{C^{k,\alpha}(\bar{G}_r)} \le C. \end{aligned} \tag{5.19}$$

We want to show that (5.19) holds for $k+1$.

We first prove that

$$||\partial^{k+1}_{x_1}u||_{C^{\alpha}(\bar{G}_r)} \le C. \tag{5.20}$$

For this purpose, we introduce an iteration as follows. Let f_0 be as given in (5.15). For $i = 1, 2, \ldots$, suppose f_{i-1} satisfies

$$f'_{i-1}(t,\tilde{x}) = h_{i-1}(t,\tilde{x})\frac{f_{i-1}(t,\tilde{x})}{t} + g_{i-1}(t,\tilde{x}). \tag{5.21}$$

We introduce

$$f_i = f'_{i-1} - \frac{f_{i-1}}{t}. \tag{5.22}$$

Then

$$(\frac{f_{i-1}}{t})' = \frac{f_i}{t} \tag{5.23}$$

and f_i satisfies the equation

$$f'_i = h_i\frac{f_i}{t} + g_i, \tag{5.24}$$

where

$$\begin{aligned} &h_i = h_{i-1} - 1, \\ &g_i = g'_{i-1} + h'_{i-1}\tfrac{f_{i-1}}{t}. \end{aligned} \tag{5.25}$$

From (5.19) and using the Taylor expansion, we have $h_0, g_0 \in C^{k-2,\alpha}(\bar{G}_r)$. Therefore $h_{k-2} \in C^{k-2,\alpha}(\bar{G}_r)$ and $g_{k-2} \in C^{\alpha}(\bar{G}_r)$. Note that $h_{k-2}(0,\tilde{x}) = n-k+3$ for $k \le n+1$. Applying Lemma 5.1 to equation (5.24), we obtain $f'_{k-2} \in C^{\alpha-\epsilon}(\bar{G}_r)$ for any $\epsilon > 0$.

Observe that (5.22)–(5.24) imply

$$\begin{aligned} f''_{i-1} &= f'_i + (\frac{f_{i-1}}{t})' \\ &= (h_i+1)\frac{f_i}{t} + g_i \\ &= (h_i+1)\frac{f'_i - g_i}{h_i} + g_i. \end{aligned} \tag{5.26}$$

Hence $f'_{k-2} \in C^{\alpha-\epsilon}(\bar{G}_r)$ implies that $f_0^{(k-1)} \in C^{\alpha-\epsilon}(\bar{G}_r)$ for any $\epsilon > 0$, where we denote $\phi^{(k)}(t) = \frac{d^k}{dt^k}\phi$. From (5.17) we then infer that $\partial^{k+1}_{x_1}u(x_1,\tilde{x}) \in C^{\alpha}(\bar{G}_r)$, and (5.20) is thus proved.

From (5.19) (the induction assumption), we see that for any $j \geq 1$, $D_{\tilde{x}}^j h_{k-2} \in C^{k-2,\alpha}(\bar{G}_r)$ and $D_{\tilde{x}}^j g_{k-2} \in C^{\alpha}(\bar{G}_r)$. Differentiating equation (5.24) and repeating the argument from (5.11) to (5.13), we infer that for any $j \geq 1$,

$$\|D_{\tilde{x}}^j f'_{k-2}(t, \tilde{x})\|_{C^\alpha(\bar{G}_r)} \leq C.$$

Hence by (5.26) and (5.17), as above we conclude that

$$\|D_{\tilde{x}}^j \partial_{x_1}^{k+1} u\|_{C^\alpha(\bar{G}_r)} \leq C. \tag{5.27}$$

Combining (5.20) and (5.27), we obtain (5.19) for $k+1$. We have thus proved the global $C^{k,\alpha}$ estimate for u and $D_{\tilde{x}}^j u$ for any $k \leq n+2$ and $j \geq 1$. Theorem 1.3 is proved.

Remark 5.4. In Theorem 1.3 we assume that the boundary $\partial\Omega \in C^\infty$. This condition can be weakened. For the $C^{2,\alpha}$ regularity, it suffices to assume $\partial\Omega \in C^{2,\alpha}$ by the argument in Remark 4.1 (but note that our moving plane argument assumes $\partial\Omega \in C^{3,\alpha}$). For the $C^{k,\alpha}$ regularity, $k \geq 3$, it suffices to assume that $\partial\Omega \in C^{k,1}$, as we have used the $W^{2,p}$ estimate between (5.5) and (5.6).

5.5. A compatibility condition. For the regularity higher than $C^{n+2,\alpha}$, we need to apply Lemma 5.1 to f_n. Note that $h_n(0, \tilde{x}) \equiv 1$. By Lemma 5.1 and Remark 5.1, $f_n \in C^{1,\alpha}$, or equivalently $u \in C^{n+3,\alpha}$, if and only if the condition (5.28) holds,

$$g_n(0, \tilde{x}) = 0 \quad \forall\ \tilde{x}. \tag{5.28}$$

This is a compatibility condition, and if it holds, then by the iteration (5.22)–(5.25), $h_{n+1}(0, \tilde{x}) \equiv 0$. Hence we can write (5.24) in the form

$$f'_{n+1} = \frac{h_{n+1}}{t} f_{n+1} + g_{n+1},$$

where $\frac{h_{n+1}}{t}$ is smooth and we obtain higher regularity for u. In other words, $u \in C^\infty$ if and only if (5.28) holds. By Remark 5.1, the solution has the expression (1.16) in general.

The compatibility condition (5.28) does not hold in general. Indeed, consider the simplest equation

$$\begin{aligned} \Delta u &= N \frac{u_{x_1}}{x_1} + \tilde{g} \quad \text{in } \mathbb{R}^{n,+}, \\ u &= \phi \quad \text{on } \{x_1 = 0\}, \end{aligned} \tag{5.29}$$

where N is a positive integer, and ϕ and $\tilde{g}$ are C^∞ smooth functions. Denote $f_0 = u_{x_1}$ and write the above equation in the form

$$f'_0(t, \tilde{x}) = N \frac{f_0(t, \tilde{x})}{t} + g_0(t, \tilde{x}), \tag{5.30}$$

where $t = x_1$ and $g_0 = \tilde{g} - \sum_{i=2}^n u_{ii}$. The argument in §5.2–5.4 implies that $u \in C^{N,\alpha}$. By Lemma 5.1 and Remark 5.1, $u \in C^{N+1,\alpha}$ if and only if

$g_{N-1}(0) = 0$. By (5.25) and since $h \equiv N$ for (5.29), we have $g_{N-1}(0, \tilde{x}) = \frac{d^{N-1}}{dt^{N-1}} g_0(t, \tilde{x}) \mid_{t=0}$. Hence we need the compatibility condition

$$\frac{d^{N-1}}{dt^{N-1}} g_0(t, \tilde{x}) \mid_{t=0} = 0. \tag{5.31}$$

If $N = 1, 2, 3$, (5.31) can be written explicitly as

$$\begin{aligned} &\tilde{g}(0, \tilde{x}) - \sum_{i=2}^{n} \phi_{x_i x_i}(\tilde{x}) = 0, \\ &\tilde{g}_{x_1}(0, \tilde{x}) = 0, \\ &\tilde{g}_{x_1 x_1}(0, \tilde{x}) + \frac{1}{2} \sum_{i \geq 2} \tilde{g}_{x_i x_i}(0, \tilde{x}) - \frac{1}{2} \sum_{i,j \geq 2} \phi_{x_i x_i x_j x_j}(\tilde{x}) = 0. \end{aligned} \tag{5.32}$$

We note that when $N < 0$, the global regularity of solutions to (5.29) was obtained in [**13**].

To conclude this paper, we note that the argument in this section applies not only to the linear equation (1.11) but also to fully nonlinear, uniformly elliptic equations of the form

$$F(D^2 u, \frac{u_{x_1}}{x_1}, Du, \frac{u}{x_1^2}, x) = 0 \ \text{ in } \mathbb{R}^{n,+}. \tag{5.33}$$

If F is independent of u/x_1^2, $F \in C^\infty$, and F satisfies $F_q/F_{M_{11}} \geq N$ on $\{x_1 = 0\}$, we can prove $u \in C^{N+\alpha}$ up to the boundary for $\alpha \in (0, 1)$, where $N > 0$ is a constant. If $F_q/F_{M_{11}} \equiv N$ is an integer, then $u \in C^{N+1,\alpha}$ (up to the boundary) if and only if the compatibility condition $g_{N-1}(0, \tilde{x}) = 0$ on $\{x_1 = 0\}$ is satisfied, as in (5.28) or (5.31). If the operator F depends on u/x_1^2, the above iteration does not work and we need a different one to get higher regularity. These results are of interest themselves, and we plan to consider them in a separate work.

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