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ALGEBRAIC HYPERBOLICITY OF RAMIFIED COVERS OF $\mathbb{G}_m^2$ (AND INTEGRAL POINTS ON AFFINE SUBSETS OF $\mathbb{P}_2$)

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Abstract

Let X be a smooth affine surface, $X \to \mathbb{G}_m^2$ be a finite morphism. We study the affine curves on X, with bounded genus and number of points at infinity, obtaining bounds for their degree in terms of Euler characteristic.

A typical example where these bounds hold is represented by the complement of a three-component curve in the projective plane, of total degree at least 4.

The corresponding results may be interpreted as bounding the height of integral points on X over a function field. In the language of Diophantine Equations, our results may be rephrased in terms of bounding the height of the solutions of $f(u, v, y) = 0$, with u, v, y over a function field, u, v S-units.

It turns out that all of this contain some cases of a strong version of a conjecture of Vojta over function fields in the split case. Moreover, our method would apply also to the nonsplit case.

We remark that special cases of our results in the holomorphic context were studied by M. Green already in the seventies, and recently in greater generality by Noguchi, Winkelmann, and Yamanoi [**12**]; however, the algebraic context was left open and seems not to fall in the existing techniques.

1. Introduction

Let κ be an algebraically closed field of characteristic zero, X be a smooth affine surface over κ and $\pi : X \to \mathbb{G}_m^2$ be a finite morphism. We are interested in bounding the degree of curves on X, in terms of their Euler characteristic, under suitable hypotheses on X (or on π).

We observe that such a problem constitutes an affine analogue of an issue treated by Bogomolov for compact surfaces in high generality [**1**].

As usual, these problems have an analogue in the holomorphic context, where one wants to prove, for the same surfaces, the degeneracy of entire curves. In this direction, we remark that special cases were

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studied already in the seventies by M. Green, who considered the equation $y^2(z) = \exp(\phi(z)) + \exp(\psi(z)) + 1$ in entire functions y, ϕ, ψ (he worked under the condition that the functions ϕ, ψ be of finite order). The solutions correspond to entire curves on the surface X obtained as complement of a particular configuration of a conic and two lines in $\mathbb{P}_2$; the main point of Green's proof consists in exploiting a projection $X \to \mathbb{G}_m^2$.

Recently Noguchi, Winkelmann, and Yamanoi [**12**] vastly generalized Green's result, again working over varieties endowed with finite maps to tori (either compact or not).

However, the algebraic case (and certainly the arithmetic case) remained pratically completely open and does not appear to fall into existing techniques, even in the simplest instances in dimension two. (We know here only of a paper of Lu [**11**], in the compact case, which only bounds the degree of complete curves of genus 1, i.e., Euler characteristic 0. This is a severe limitation which that paper does not overcome.)

Some progress came from our paper [**5**], where we employed a new type of *abcd* result, and in particular solved the case of the complement of a conic and two lines in the projective plane. In the present paper, we vastly extend such results.

The whole context is deeply related to Vojta's conjecture on integral points, which we now briefly formulate in the split function field case: Let $\tilde{X}$ be a smooth complete surface, $K_{\tilde{X}}$ be a canonical divisor of $\tilde{X}$, and D be a reduced normal crossing divisor on $\tilde{X}$; let $X = \tilde{X} \setminus D$ be the complement of the support of D. Following general convention, we say that X is of *log-general type*, if $D + K_{\tilde{X}}$ is big: This means that $h^0(n(D + K)) \gg n^2$; see [**7**], Chap. 1, for other equivalent condition for bigness. Then Vojta's conjecture, in the split function field case, predicts a bound for the degree of affine curves in X, in terms of their Euler characteristic.

For instance, Bogomolov's theorem cited above essentially proved such bounds for empty D—i.e., for compact X.

In this paper, we prove such bounds in the case where $X = \tilde{X} \setminus D$ is an affine surface that dominates the torus $\mathbb{G}_m^2$; we need mild hypotheses on normal crossing, expressed in Theorems 1 and 2 below.

Remark. SPLIT AND NONSPLIT CASE. We stress at once that our method would also apply to the nonsplit case—namely, when X is defined over the same function field where we seek the points. In other words, in these cases X is a threefold fibered over a curve, and we study the cross sections.

In fact, the sources of the present methods are mainly [**5**] (and a function field version of [**3**] is also useful in one of the possible approaches). Now, the results of [**5**] partly generalize to the nonsplit case (see, for instance, Step 3 and following arguments in §3 in [**5**]).

The simplest and most natural case arises when $\tilde{X} = \mathbb{P}_2$ is the projective plane; so $X = \mathbb{P}_2 \setminus D$ where D is a plane curve. In this case, Vojta's previously mentioned conjecture predicts a bound for the degree of curves on X, in terms of their Euler characteristic, provided $\deg D \geq 4$ (and D has normal-crossing singularities). It has long been known that in the case where D has four components the sought bound holds. Instead we shall treat here the three-component case.

Now we shall state in detail our results. Consider a smooth projective curve $\tilde{\mathcal{C}}$ of genus g, and a finite nonempty subset $S \subset \tilde{\mathcal{C}}$, and let $\mathcal{C} = \tilde{\mathcal{C}} \setminus S$ be the corresponding affine curve. We also put

$$\chi_{\mathcal{C}} = 2g - 2 + \sharp(S)$$

and call it the Euler characteristic of the affine curve $\mathcal{C}$. By analogy with the number field case, we set $\mathcal{O}_S = \kappa[\mathcal{C}]$, the ring of regular functions on the affine curve $\mathcal{C}$. The integral points $X(\mathcal{O}_S)$ on the surface X thus correspond to curves Y (not necessarily smooth) lying on X, parametrized by the smooth curve $\mathcal{C}$, and we set $\chi_Y = \chi_{\mathcal{C}}$.

Theorem 1. *[Vojta's conjecture for $\mathbb{P}_2$ minus three divisors]. Let $D = D_1 + \ldots + D_r \subset \mathbb{P}_2$ be a curve whose components D_i meet transversally at each point of intersection. Suppose $r \geq 3$ and $\deg(D) \geq 4$. Then there exists a number $C_1 = C_1(D)$ such that for every curve $Y \subset \mathbb{P}_2 \setminus D$, we have the following bound:*

$$\deg(Y) \leq C_1 \cdot \max\{1, \chi_Y\}. \tag{1}$$

Remark. The condition on the degree of D—namely, $\deg D \geq 4$—is best possible. Actually, if $\deg D \leq 3$, no bound like (0.1) can generally hold (see [**5**], §4 and its *Addendum*, §(ii)).

The condition on the normal-crossing singularities is weaker than in the usual Vojta's conjecture: for instance, each component of D can have arbitrary singularities at any point where it does not intersect any other component.

It will be clear from the proof that our bounds are effective and that in principle one may determine all the curves of given Euler characteristic on the affine surface $\mathbb{P}_2 \setminus D$.

As we mentioned, the case when D is a configuration of a conic and two lines in general position was open until recently and solved in [**5**]. Its analogue in complex analysis (Nevanlinna theory) was conjectured (actually for a special configuration) by M. Green in the seventies, and was solved only recently by Noguchi, Winkelmann, and Yamanoi in much greater generality [**12**].

Note that under the hypotheses of Theorem 1, the surface X admits a dominant map $\pi : X \to \mathbb{G}_m^2$; if $F_i = 0$ are equations for the components D_i, then the map $(F_1^{\deg F_3}/F_3^{\deg F_1}, F_2^{\deg F_3}/F_3^{\deg F_2})$ takes values in $\mathbb{G}_m^2$. Actually, if the number of components of D is exactly three, every regular map from X to a semi-abelian variety factors through such map π (some authors say that the "logarithmic irregularity" is 2, or that the generalized Albanese variety of X is the two-dimensional torus $\mathbb{G}_m^2$). We note that if the logarithmic irregularity is higher than 2, the results improve, since we can apply the general theorems of Faltings and Vojta.

Theorem 1 will be deduced from a more general result, i.e., Theorem 2, below, which applies to surfaces admitting such maps to $\mathbb{G}_m^2$. We shall comment about the classification of these surfaces after Theorem 2.

As above, X is a smooth affine surface and $\pi : X \to \mathbb{G}_m^2$ a finite map to the torus. We suppose that $\mathbb{G}_m^2$ is embedded in $\mathbb{P}_2$ as the complement of the divisor $UVW = 0$ (U, V, W being homogeneous coordinates in $\mathbb{P}_2$).

Theorem 2. *Let $Z \subset X$ be the ramification divisor of the finite map $\pi : X \to \mathbb{G}_m^2$. Assume that the closure of $\pi(Z)$ in $\mathbb{P}_2$ does not intersect the set of singular points of the boundary of $\mathbb{G}_m^2$ in $\mathbb{P}_2$—i.e., $(0:0:1), (0:1:0), (1:0:0)$. If X is of log-general type, then there exists a number $C_2 = C_2(X, \pi)$ such that for every curve $Y \subset X$ of Euler characteristic χ_Y the following inequality holds:*

$$\deg Y \le C_2 \cdot \max\{1, \chi_Y\}. \tag{0.1}$$

Remark: Classification of the relevant surfaces. We recall that for a surface X endowed with a finite map to the torus $\mathbb{G}_m^2$, it can be proved that

(1) X itself is a torus (so it has log-Kodaira dimension 0, which happens if and only if π is unramified), or

(2) it is a $\mathbb{G}_m$-bundle over a curve $\mathcal{C}$ with Euler characteristic > 0 (for instance a product $\mathcal{C} \times \mathbb{G}_m$), so it has log-Kodaira dimension 1, or

(3) it is of log-general type (log-Kodaira dimension 2).

This classification can be easily obtained from a theorem of Kawamata ([**8**], Theorem 27; see also [**12**], Theorem 2.12).

In cases (1) and (2), no bound of the form (0.1) above may hold for general curves on X; in the third case, our Theorem 2 provides such bounds (under some natural hypothesis on the compactification of X). Note that in case (2) one can easily prove a bound for the degree of curves of Euler characteristic 0, i.e., those uniformized by $\mathbb{G}_m$; for such curves, in case (3) one could prove their finiteness, as a corollary of our results. This fact should be compared with the recent results of Lu [**11**] in the compact case, where he proves the finiteness of genus one curves on complete surfaces of general type dominating an abelian surface.

Remark. In the statement of Theorem 2, by $\deg Y$ we mean the degree under a projective embedding of a completion $\tilde{X}$ of the surface X in a projective space (i.e., on the choice of an ample line bundle on $\tilde{X}$). Hence it is meant, here and in the sequel, that the number C_2 also depends on such embeddings. Inspection of the proof will show that the numbers $C_1, C_2, \ldots$ appearing in the statements will depend only on the degree of $\tilde{X}$ and the degree of π. Also, they will be effectively computable in terms of $\deg \tilde{X}, \deg \pi$.

In the particular case considered in Theorem 1, the hypothesis on the closure of $\pi(Z)$ shall turn out to be a consequence of the normal-crossing hypothesis on D. As shown in Example 3, this latter hypothesis cannot be removed. Nevertheless, for every surface of log-general type endowed with a dominant map to $\mathbb{G}_m^2$, we can prove a weaker statement in the direction of Vojta's conjecture, which needs no further hypothesis. Namely, we can prove the following:

Given a smooth surface X of positive log-Kodaira dimension endowed with a finite dominant map $\pi : X \to \mathbb{G}_m^2$, the bound (0.1) holds for every curve $Y \subset X$ such that the Zariski closure in $\mathbb{P}_2$ of $\pi(Y)$ does not contain any of the singular points of the boundary of $\mathbb{G}_m^2$ in $\mathbb{P}_2$.

The next result (Theorem 3) will be presented from a more algebraic view point, so we introduce the relevant notation (see also the next section). For a fixed (abstract) projective curve $\tilde{\mathcal{C}}$ and a finite set of points $S \subset \tilde{\mathcal{C}}$, we denote by $\mathcal{C} = \tilde{\mathcal{C}} \setminus S$ the complement of S, and by $\mathcal{O}_S = \kappa[\mathcal{C}]$ the corresponding algebra of regular functions. Its elements will be called S-integers, in analogy with the number field case. The group of S-units $\mathcal{O}_S^*$ accordingly consists of the rational functions on $\tilde{\mathcal{C}}$ having all their zeros and poles in S.

Let us return to our surface X. Every morphism $\mathcal{C} \to X$ corresponds to an integral point in $X(\mathcal{O}_S)$. Hence, to study the curves on X parametrized by the abstract curve $\mathcal{C}$, one is led to consider integral points on the surface. All of our previous results could be reformulated in these terms.

Up to birationality, the surface X is defined in $\mathbb{G}_m^2 \times \mathbb{A}^1$ by an equation of the form $f(u, v, y) = 0$. For such a surface, we may often control the integral points over $\mathcal{O}_S$ also by another method. In this respect we shall prove the following theorem, in which we do not even assume the smoothness of the surface implicitly appearing in the statement.

Theorem 3. *Let $f(U, V, Y) \in \kappa[U, V, Y]$ be an irreducible polynomial, monic in Y. Suppose that the discriminant $\Delta(U, V) \in \kappa[U, V]$ of f with respect to Y has no multiple nonmonomial factors. Then, for an affine curve $\mathcal{C}$ as above, one of the following cases occurs:*

(a) There exist numbers C_3, C_4, effectively computable in terms of f and $\chi = \chi_{\mathcal{C}}$, such that the solutions $(u, v, y) \in \mathcal{O}_S^* \times \mathcal{O}_S^* \times \mathcal{O}_S$ of the equation $f(u, v, y) = 0$ satisfy either

$$\max\{\deg(u), \deg(v), \deg(y)\} \le C_3$$

or a multiplicative dependence relation $u^a = \lambda \cdot v^b$, for a pair $(a, b) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ with $\max\{|a|, |b|\} \le C_4$ and $\lambda \in \kappa^*$.

(b) After an automorphism of $\mathbb{G}_m^2$, there is a $Q \in \kappa[U]$ such that $\Delta(U, V) = Q(U)V^a$ and $f(U, V, Y) = V^l P(U, V^m Y + A(U, V))$, where

$P \in \kappa[U^{\pm 1}, W]$, $A(U,V) \in \kappa[U^{\pm 1}, V^{\pm 1}]$, $a, l, m \in \mathbb{Z}$, $a \geq 0$. Also, either $f(U,V,Y) = (Y + A(U,V))^d - bV^nU^p$, where $b \in \kappa^*$, d, n, p are positive integers, or $\deg(u) \leq C_5$, and the number of possible u is finite, bounded only in terms of $\deg f$ and χ.

We note that the assumption on the discriminant is "generically" true; i.e., the polynomials of given degree for which it is not verified form a proper algebraic subset of the space of all such polynomials.

Also, in alternative (b), the (possibly singular) surface $\mathcal{E}$ defined by f in $\mathbb{G}_m^2 \times \mathbb{A}^1$ is isomorphic to a product $\mathcal{C}' \times \mathbb{G}_m$, where $\mathcal{C}'$ is a curve, and moreover it is a cover of $\mathbb{G}_m^2$ ramified at most above finitely many translates of a same one-dimensional algebraic subgroup of $\mathbb{G}_m^2$ (i.e., the subgroup defined by $U = 1$ in the new coordinates). Actually, in the first of cases (b), this surface is unramified above $\mathbb{G}_m^2$, whereas in the second case we can have only finitely many possibilities for u, with effectively bounded height.

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2. Preliminaries on heights

All of the present results may be formulated in two different, though equivalent, languages, either "geometrical" or "diophantine," so to say. Namely, either we bound the degree of the image $Y \subset X$ of a map $\mathcal{C} \to X$, or we bound the degree of the functions on $\mathcal{C}$ that express this map; in turn, this last degree may be seen as a height function. Hence we start by detailing this (elementary) connection; we believe that this will add clarity to our exposition and also stress the geometrical-diophantine link.

Morphisms and integral points. Let κ be, as before, an algebraically closed field of characteristic 0, $\tilde{\mathcal{C}}$ a smooth complete curve defined over κ, of genus $g = g(\tilde{\mathcal{C}})$, $S \subset \tilde{\mathcal{C}}$ a finite nonempty set of points of $\tilde{\mathcal{C}}$. We shall consider nonconstant morphisms $\varphi : \mathcal{C} \to X$; note that φ corresponds to an integral point on $X(\mathcal{O}_S) = X(\kappa[\mathcal{C}])$, and $\pi \circ \varphi$ to a S-unit point $(u, v) \in \mathbb{G}_m^2(\kappa[\mathcal{C}])$. We fix a projective embedding $\tilde{X} \hookrightarrow \mathbb{P}_M$ of a smooth complete surface $\tilde{X}$ (recall $X = \tilde{X} \setminus D$), so that we can speak of the degree $\deg Y$ of a curve $Y \subset X$.

Note that in Theorems 1 and 2 we have a bound for the **degree of the image** $Y = \varphi(\mathcal{C})$, whereas in Theorem 3 we bound the **degree of the functions** u, v defined on $\mathcal{C}$; this degree is a **height**. Note also that this last bound is valid, however, only excepting some possibly infinite families of multiplicative dependent pairs of functions (u, v), which in fact appear in the statement of Theorem 3, whereas no exception appears in the previous statements.

In fact, we can explicitly explain these differences and analogies by relating the two kind of degrees as follows.

We may write $\pi \circ \varphi = (u, v)$, where u, v are rational functions on $\mathcal{C}$, with zeros and poles only in S. In one direction, it is easy to see that a bound for the degree of the functions u, v implies a bound for $\deg(\varphi(\mathcal{C}))$, on taking intersections with lines $au + bv + c = 0$ (see, for instance, Lemma 3.2 of [**5**]).

In the inverse direction, given (as in Theorems 1 and 2) a bound on the degree of the image $Y = \varphi(\mathcal{C})$, we can recover as follows complete information for the u, v and their degrees. We give a few details.

First, note that $\max(\deg(u), \deg(v)) \le \deg(\varphi) \cdot \deg Y$: it suffices to count, for example, zeros, or poles. So, if φ is birational on its image, the maximum of the degrees of u, v is bounded by the degree of the image.

If instead $\varphi : \mathcal{C} \to Y$ is not birational, then we contend that only two things may happen:

(a) The degree of u, v is bounded uniformly by $C \deg Y$ where C only depends on the curve $\mathcal{C}$, or

(b) $\chi_Y = 0$ and φ factors through $\mathbb{G}_m$ as $\mathcal{C} \to \mathbb{G}_m \to Y$. In this case, we can view the pair (u, v) as a map $\mathbb{G}_m \to \mathbb{G}_m^2$; since the invertible regular functions on $\mathbb{G}_m$ form a cyclic group modulo constants, u, v must be multiplicatively dependent. Now, after composing with endomorphisms $x \mapsto x^k$ of $\mathbb{G}_m$, one obtains solutions u, v of arbitrarily large degree.

To prove that this alternative necessarily holds, we use the following known fact, whose proof shall be recalled below in a moment.

Fact. *Let $f : \mathcal{C}_1 \to \mathcal{C}_2$ be a morphism of degree $d \ge 1$ between the smooth affine curves $\mathcal{C}_1, \mathcal{C}_2$. Then $\chi_{\mathcal{C}_1} \ge d \cdot \chi_{\mathcal{C}_2}$.*

Given this, take $\mathcal{C}_1 = \mathcal{C}$, and let $\varphi_2 : \mathcal{C}_2 \to Y$ be the desingularization of Y. Then φ factors as $\varphi_2 \circ f$, for a morphism $f : \mathcal{C}_1 \to \mathcal{C}_2$ of smooth curves. If $\chi_{\mathcal{C}_2} > 0$, then, by the Fact, $\deg(\varphi) \le \chi_{\mathcal{C}_1}/\chi_{\mathcal{C}_2}$ and we fall into (a), because $\max(\deg(u), \deg(v)) \le \deg(\varphi) \deg Y \le C_3 \chi_{\mathcal{C}_1} \deg Y$. Otherwise, $\chi_{\mathcal{C}_2} = 0$, so $\mathcal{C}_2 = \mathbb{G}_m$, whence u, v are necessarily powers of a single function on $\mathcal{C}$, and this gives the multiplicative dependence relation mentioned in (b) and in Theorem 3.

To conclude this pause, here is the promised easy proof of the Fact: Let $\tilde{f} : \tilde{\mathcal{C}}_1 \to \tilde{\mathcal{C}}_2$ be the continuation of f to the complete smooth models for $\mathcal{C}_1, \mathcal{C}_2$; for $i = 1, 2$, denote by g_i the genus of $\tilde{\mathcal{C}}_i$ and let $S_i = \tilde{\mathcal{C}}_i \setminus \mathcal{C}_i$. By hypothesis $\tilde{f}^{-1}(S_2) \subset S_1$. By Riemann-Hurwitz formula we have $2g_1 - 2 = d(2g_2 - 2) + \sum_{P \in \mathcal{C}_2} (d - \rho(P))$ where $\rho(P) := \sharp(\tilde{f}^{-1}(P))$. Then

$$\begin{array}{rcl} \chi_{\mathcal{C}_1} & = & d\chi_2 + \sharp(S_1) - d\sharp(S_2) + \sum_{P \in \mathcal{C}_2} (d - \rho(P)) \\ & \ge & d\chi_{\mathcal{C}_2} + \tilde{f}^{-1}(S_2) - d\sharp(S_2) + \sum_{P \in S_2} (d - \rho(P)) \\ & \ge & d\chi_{\mathcal{C}_2}. \end{array}$$

In the sequel, $c_1, \ldots$ will denote computable numbers that, unless otherwise stated, will depend only on X, π (and actually only on their degree).

Heights. As usual, for a rational function $a \in \kappa(\tilde{\mathcal{C}})$, we let $h(a) = h_{\tilde{\mathcal{C}}}(a)$ be its height—i.e., its degree as a morphism $a : \tilde{\mathcal{C}} \to \mathbb{P}_1$. For $n \geq 2$ and elements $u_1, \ldots, u_n \in \kappa(\tilde{\mathcal{C}})$ not all zero, we denote by $h_{\tilde{\mathcal{C}}}(u_1 : \ldots : u_n)$, or simply $h(u_1 : \ldots : u_n)$, the projective height

$$h(u_1 : \ldots : u_n) = h_{\tilde{\mathcal{C}}}(u_1 : \ldots : u_n) = -\sum_{v \in \tilde{\mathcal{C}}} \min\{v(u_1), \ldots, v(u_n)\}.$$

Recall that $\tilde{X}$ is embedded in $\mathbb{P}_M$, so a morphism $\varphi : \tilde{\mathcal{C}} \to \tilde{X}$ is given by a point $\mathbb{P}_M(\kappa(\tilde{\mathcal{C}}))$ and we define its projective height accordingly—namely, if φ is given by $\varphi(p) = (u_1(p) : \ldots : u_n(p))$,

$$H = H(\varphi) = h_{\tilde{\mathcal{C}}}(u_1 : \ldots : u_n).$$

Start of the proofs. We denote by $Z \subset X$ the ramification divisor of $\pi : X \to \mathbb{G}_m^2$. The Euler characteristic of the affine curve $\mathcal{C} = \tilde{\mathcal{C}} \setminus S$ will be denoted by χ.

In the sequel, with the purpose of proving Theorems 1 and 2, we shall suppose that $\varphi : \mathcal{C} \to X$ is nonconstant and that the curve $\varphi(\mathcal{C})$ is not contained in the ramification divisor Z, for otherwise we have automatically a bound for its degree.

We start by proving a crucial proposition that shows that the contribution of the ramification divisor to the height of φ is negligible. This is at the heart of the present method, and strongly depends on a bound obtained in [**5**, Theorem 1.2] for the multiple zeros of a polynomial $A(u, v)$ in S-unit functions (u, v) on a curve.

Proposition 1. *. Let X, Z be as above; for every $\epsilon > 0$ and every integer χ there exists a number $C = C(X, \pi, \epsilon, \chi) = C'(X, \pi, \epsilon) \cdot \max(1, \chi)$ such that for every morphism $\varphi : \mathcal{C} \to X$ (with $\chi_{\mathcal{C}} = \chi$) of height $H > C$, with $\varphi(\mathcal{C}) \not\subset Z$, the degree of the divisor $\varphi^*(Z)$ satisfies*

$$\deg(\varphi^*(Z)) \leq \epsilon H. \tag{2}$$

In particular, the number of points $p \in \mathcal{C}$ such that $\varphi(p) \in Z$ is bounded by ϵH.

We stress that we consider the morphism $\varphi : \mathcal{C} \to X$ from the noncomplete curve $\mathcal{C}$. It may be that the points in S are sent to the points at infinity of $\tilde{Z}$ with high multiplicity, so that the inequality like (1.1) can fail (for small ϵ) if $\varphi^*(Z)$ is replaced by $\tilde{\varphi}^*(\tilde{Z})$. This certainly happens if $\tilde{Z}$ is big, which occurs in the most interesting cases.

Proof. We can suppose that Z is nonempty. By arguing on each irreducible component of Z, we can also suppose that Z is irreducible.

Since X is smooth and Z has codimension 1, we may cover X by a finite number of Zariski open subsets U such that Z has a local equation $f_U = 0$ in each of the subsets U. The divisor $\pi(Z)$ will be defined in $\mathbb{G}_m^2$ by an irreducible polynomial equation $A(X,Y) = 0$. Note that A is not a monomial, because Z is nonempty. The polynomial A induces a regular function $\pi^*(A)$ on X, through pull-back by π. Since Z is inside the ramification locus of π, $\pi^*(A)$ is divisible by f_U^2 in the local ring of Z in U, so we can write $\pi^*(A) = f_U^2 g$, where $g = g_U$ is regular on U, since A is regular on $\pi(U)$ (in fact on the whole $\mathbb{G}_m^2$).

Now let $p \in \mathcal{C}$ be a point with $\varphi(p) \in Z \cap U$. Write $\pi \circ \varphi = (u, v)$, where u, v are S-unit functions on $\mathcal{C}$. Then the contribution of p to $\varphi^*(Z)$ is $a_p := \mathrm{ord}_p(f_U \circ \varphi)$. Since $\pi^*(A) = f_U^2 g$, with g regular at p, we have

$$\mathrm{ord}_p(A(u,v)) \geq 2a_p.$$

Then, taking into account all the open sets in the cover and summing over the points p in $\varphi^{-1}(Z)$, we obtain

$$\text{(3)} \qquad \deg \varphi^*(Z) = \sum_{\varphi(p)\in Z} \mathrm{ord}_p f_U \circ \varphi \leq \sum_{\varphi(p)\in Z} (\mathrm{ord}_p(A(u,v)) - 1).$$

To estimate the right-hand side, we note that the summation includes only points $p \in \mathcal{C} = \tilde{\mathcal{C}} \setminus S$, and we distinguish two cases:

Suppose first that u, v are multiplicatively independent modulo κ^*; we then apply Theorem 1.2 from [**5**], to obtain directly from (3) a bound

$$\deg \varphi^*(Z) \leq \epsilon H$$

provided H is larger then some number of the shape $C'(X, \pi, \epsilon) \cdot \max(1, \chi)$ (because the polynomial $A(X,Y)$ depends only on X, π); this proves the sought conclusion in this case.

Suppose now that u, v are multiplicatively dependent modulo constants, so they satisfy a so-called generating relation $u^r v^s \in \kappa^*$, with r, s coprime integers. This case could also be treated by using the results of [**5**], as in the first case, but the statement of Theorem 1.2 in [**5**] was not completely explicit for this situation; so we give a direct simple argument here.

Interchanging u, v if necessary we may write $u = t^s, v = \alpha t^r$, with $\alpha \in \kappa^*$ and a suitable S-unit function $t \in \mathcal{O}_S^*$. We have to bound the multiple zeros of $B(t) = B_{r,s,\alpha}(t) := A(t^s, \alpha t^r)$ and to show that $\sum_{p\in\mathcal{C}}(\mathrm{ord}_p B(t) - 1) \leq \epsilon H$ if r, s are large enough with respect to ϵ and χ.

Consider the Laurent polynomial $B(T) := A(T^s, \alpha T^r) \in \kappa[T, T^{-1}]$ and factor it as a product of a power of T times coprime factors $(T - \beta)^m$, where $\beta \in \kappa^*$ and $m = m(\beta)$ is a positive integer.

The power of T gives no contribution because t has no zeros on $\mathcal{C}$.

The total number of multiple zeros of terms of the form $t-\beta$, each zero counted with a weight equal to "multiplicity -1," for a given S-unit t and variable $\beta \in \kappa^*$, is bounded by the degree of the zero divisor of the differential dt/t; however, this degree is bounded by the degree of the pole divisor $+2g-2$, so by χ only, since t is an S-unit. Also, $m(\beta)$ is bounded by $(\deg A)^2$, in view of Hajos Lemma [[**13**], Lemma 1, p. 187]. Hence, to conclude it suffices to bound the number of β such that $m(\beta) > 1$. Let us call Ω the set of such β.

For this task note that each such β is a common zero of $B(T)$ and $B'(T)$. Using the definition of $B(T)$, we find that $(\beta^s, \alpha\beta^r)$ is a common zero of $A(X,Y)$ and $A^*(X,Y) := sX\frac{\partial A}{\partial X} + rY\frac{\partial A}{\partial Y}$. Since $A(X,Y)$ is irreducible, these last polynomials can have common factors only if $A(X,Y)$ divides $A^*(X,Y)$; comparing degrees, this implies that A^* is a scalar multiple of A. But then, by differentiation, we immediately find that $B'(T)$ is a scalar multiple of $B(T)/T$, which in turn implies that $B(T)$ is a constant times a power of T. In this case, as observed, there would be no multiple zeros of $B(t)$.

Hence we may suppose that A, A^* are coprime, and by Bezout's Theorem the number of common zeros is bounded by $(\deg A)^2$.

We have proved that the cardinality of the set $\{(\beta^s, \alpha\beta^r) : \beta \in \Omega\}$ is bounded by $(\deg A)^2$. Since, for fixed α, the map $\beta \mapsto (\beta^s, \alpha\beta^r)$ is injective, due to the coprimality of r, s, the same bound holds for the cardinality of Ω, concluding the proof. q.e.d.

§2 Proof of Theorem 2.

Let us first outline the strategy of our proof, which makes essential use of Proposition 1 from last paragraph (and which ultimately rests on the gcd estimates of [**5**]). Let $\tilde{X} \supset X$ be a smooth projective closure of X such that the map π extends to a regular map $\tilde{\pi} : \tilde{X} \to \mathbb{P}_2$. Note that we do not require that $\tilde{\pi}$ be finite (so $\tilde{X}$ is not necessarily its normal completion, as defined, for instance, in [**12**]).

Let $Y \subset X$ be an irreducible curve, and let $\tilde{Y} \subset \tilde{X}$ be its closure in $\tilde{X}$. The hypothesis that X is of log-general type will be used to prove that the ramification divisor $\tilde{Z} \subset \tilde{X}$ is big; so, if $\tilde{Y}$ is not a component of $\tilde{Z}$ (in which case its degree would be bounded in terms of the surface only), the intersection product $\tilde{Y}.\tilde{Z}$ is at least $c\deg(Y)$, for some positive number c. On the other hand, the degree of $\varphi^*(Z)$, which measures the degree of intersection of $\tilde{Y}$ and $\tilde{Z}$ *outside* the points at infinity of X, is bounded by $\epsilon \deg Y$ by Proposition 1. So $\tilde{Y}$ and $\tilde{Z}$ must intersect with high multiplicity at some point at infinity. This will be shown to be impossible, by *abc*-like inequalities.

Let us now go to the details of this proof.

To prove that $\tilde{Z}$ is big, we shall use the following lemma.

Lemma 1. *Let $\tilde{\pi} : \tilde{X} \to \mathbb{P}_2$, $K_{\tilde{X}}$ and $\tilde{Z}$ be as before. Let L_1, L_2, L_3 be three lines in general position in $\mathbb{P}_2$, and let D be the support of $\tilde{\pi}^*(L_1 + L_2 + L_3)$ (i.e., the sum of the components of $\tilde{\pi}^*(L_1 + L_2 + L_3)$ counted with multiplicity 1), and suppose it has normal-crossing intersections. Then $D + K_{\tilde{X}}$ is linearly equivalent to $\tilde{Z}$. In particular, if X is of log-general type, then $\tilde{Z}$ is big.*

Proof. By [**7**] (1.11), the canonical class on $\tilde{X}$ may be computed as

$$K_{\tilde{X}} = \tilde{\pi}^*(K_{\mathbb{P}_2}) + Ram,$$

where Ram is the ramification divisor of $\tilde{\pi}$. On the one hand, $K_{\mathbb{P}_2}$ is the class of $-(L_1 + L_2 + L_3)$; on the other hand, Ram decomposes as $Ram = \tilde{Z} + R_D$, where R_D is the contribution coming from the support contained in D. Also, $\tilde{\pi}^*(L_1 + L_2 + L_3) = D + R_D$. Substituting, we obtain the lemma. q.e.d.

To obtain a lower bound for the intersection multiplicity $\tilde{Y}.\tilde{Z}$, we use the following well-known general fact:

Lemma 2. *. Let $\tilde{X}$ be a complete surface embedded in a projective space. Let $\tilde{Z} \subset \tilde{X}$ be a big divisor. Then there exist a finite set of curves $Y_1, \ldots, Y_r$ in $\tilde{X}$ and a positive number c such that for every curve $\tilde{Y} \subset \tilde{X}$ with $\tilde{Y} \notin \{Y_1, \ldots, Y_r\}$ the following inequality holds:*

$$\tilde{Y}.\tilde{Z} \geq c \cdot \deg \tilde{Y}. \tag{4}$$

Proof. Let H be a hyperplane section of $\tilde{X}$ and $\tilde{Z}$ a big divisor. By Kodaira's lemma [[**9**], Prop. 2.2.6], there exists an integer $n > 0$ and an effective divisor E such that $n\tilde{Z}$ is linearly equivalent to $H + E$ (see also [**7**]). Let $\tilde{Y}$ be a curve not contained in the support $Y_1 \cup \ldots \cup Y_r$ of E. Then

$$\tilde{Y}.\tilde{Z} = \frac{1}{n}(H + E).\tilde{Y} \geq \frac{1}{n} H.\tilde{Y} = \frac{1}{n} \deg Y,$$

so the lemma is proved with $c = 1/n$. q.e.d.

We now proceed to prove Theorem 2. Let $L_1 + L_2 + L_3$ be the boundary of $\mathbb{G}_m^2$ in $\mathbb{P}_2$; more precisely, using coordinates $(U : V : W)$ in $\mathbb{P}_2$, let $L_1 : U = 0$, $L_2 : V = 0$ and $L_3 : W = 0$ be the equations for the three lines at infinity. By hypothesis, now the image $\tilde{\pi}(\tilde{Z}) \subset \mathbb{P}_2$ of the closure of the ramification divisor of π does not contain any of the points $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)$.

Let $Y \subset X$ be an irreducible curve; suppose it is defined by a birational proper morphism $\varphi : \mathcal{C} \to Y \subset X$, from a smooth affine curve $\mathcal{C} = \tilde{\mathcal{C}} \setminus S$ as before. Let u, v again be S-units such that the morphism $\pi \circ \varphi$ is given as (u, v) (or, in homogeneous coordinates, as $(u : v : 1)$). Let $H = \max\{h(u), h(v)\}$ be the height of (u, v). Suppose that Y is distinct from any of the curves $Y_1, \ldots, Y_r$ appearing in Lemma 2 (otherwise, a bound for its degree follows at once). Then by Lemma 2 we have

$\tilde{Y}.\tilde{Z} \geq c \deg Y$, for a number c depending only on X and π. Letting $f(U,V,W) = 0$ be an equation for $\tilde{\pi}(\tilde{Z})$, this fact can also be stated as an inequality of the form

$$\sum_{\nu \in \tilde{\mathcal{C}}} \max(0, \mathrm{ord}_\nu f(u,v,1)) \geq c_2 H$$

for a suitable c_2. Proposition 1 can also be stated as an inequality of that form, but in the other direction. Namely, for every $\epsilon > 0$, the inequality

$$\sum_{\nu \in \tilde{\mathcal{C}} \setminus S} \max(0, \mathrm{ord}_\nu f(u,v,1)) < \epsilon H$$

holds, provided H is large with respect to ϵ. Chosing $\epsilon = c_2/2$, we have

$$\sum_{\nu \in S} \max(0, \mathrm{ord}_\nu f(u,v,1)) \geq \frac{c_2}{2} H.$$

The hypothesis of Theorem 2 that $\tilde{\pi}(\tilde{Z})$ does not contain any of the singular points of $L_1 + L_2 + L_3$ implies that $f(0,0,1)$ does not vanish, so $f(u,v,1)$ is a sum of monomials in u, v with nonzero constant term. Then the above displayed inequality implies, via the generalized *abc* inequality of Brownawell and Masser [**2**], that either u, v satisfy some fixed algebraic relation or have bounded height, concluding the proof.

3. Proof of Theorem 1

We shall first deduce Theorem 1 from Theorem 2; then we shall briefly sketch an alternative proof, which no longer depends on Theorem 2 but instead uses tools from the arithmetic case treated in [**3**].

Proof. Assume the hypotheses of Theorem 1. Put $X = \mathbb{P}_2 \setminus (D_1 \cup \ldots \cup D_r)$, $\tilde{X} = \mathbb{P}_2$.

Let F_1, F_2, F_3 be forms without multiple factors defining, respectively, the divisors $D_1, D_2, D_3 + \ldots + D_r$ of degrees a_1, a_2, a_3. Note that $a_1 + a_2 + a_3 \geq 4$. By the normal-crossing condition appearing in Theorem 1, the three forms have no common zero in $\mathbb{P}_2$.

Define $\bar{\pi} : \tilde{X} \to \mathbb{P}_2$ as

$$\bar{\pi}(x : y : z) = (F_1(x,y,z)^{a_2 a_3} : F_2(x,y,z)^{a_1 a_3} : F_3(x,y,z)^{a_2 a_1}),$$

and note that it is everywhere well defined and finite, because of our assumptions that not all the F_i vanish at a same point. We embed $\mathbb{G}_m^2 \hookrightarrow \mathbb{P}_2$ in the usual way so that $X = \bar{\pi}^{-1}(\mathbb{G}_m^2) = \bar{X} \setminus |D|$, where $D = D_1 + \ldots + D_r$. We let π be the restriction of $\bar{\pi}$ to X.

By Lemma 4, $\tilde{Z}$ is linearly equivalent to $D + K_{\mathbb{P}_2} = (\deg D - 3)L$, where L is line in $\mathbb{P}_2$. In particular, the ramification divisor $\tilde{Z}$ is ample (so by Lemma 2.1 X is of log-general type). Hence we have only to show that the ramification divisor $\tilde{Z}$ does not meet D at any point P such that $\tilde{\pi}(P)$ is a singular point of $L_1 + L_2 + L_3$. For this we argue locally at P

and we suppose without loss that $P = (0 : 0 : 1)$. Suppose for simplicity that $F_1(P) = F_2(P) = 0$, so $F_3(P) \neq 0$. Since the D_i are supposed to intersect transversely at P, and since F_1, F_2 do not have multiple factors, the functions $\xi := F_1/z^{a_1}, \eta := F_2/z^{a_2}$ are local parameters at P. Now the map $\tilde{\pi}$ locally at P is defined as $(\xi, \eta) \mapsto (\xi^{a_2 a_3}\alpha, \eta^{a_1 a_3}\beta)$, for functions α, β defined and not zero at P. The ramification divisor is calculated locally by means of the differentials, which we express in the coordinates ξ, η. The determinant of the Jacobian matrix is of the shape

$$\xi^{a_2 a_3 - 1}\eta^{a_1 a_3 - 1}\gamma,$$

where the regular function γ does not vanish at P. Actually, γ is the determinant of the matrix with rows given by $(a_2 a_3\alpha + \xi\alpha_\xi, \xi\alpha\eta)$, $(\eta\beta\xi, a_1 a_3\beta + \eta\beta\eta)$. This proves that the only ramification component passing through P lies in the support of D, which means that the closure of Z does not contain P.

Now Theorem 2 can be applied, and its conclusion implies Theorem 1. q.e.d.

Alternative proof. We only sketch another possible approach. As we repeatedly remarked, the nonconstant maps $\mathcal{C} \to X = \mathbb{P}_2 \setminus (D_1 \cup \ldots \cup D_r)$ correspond to S-integral points on $\mathbb{P}_2$ with respect to the divisor $D = D_1 + \ldots + D_r$. Whenever D has at least four components, it is well known how to prove the degeneracy of the set of integral points, both over number fields and over function fields. In the crucial case when D has just three components, Proposition 1 provides an extra divisor Z with respect to which the points in question are "almost integral." We then dispose of *four* ample divisors, no three of them intersecting, as noticed. Now, the proof of the Main Theorem of [**3**] (or Theorem 2.1 of [**4**], but actually Theorem 1(a) in [**3**] would suffice in this case) applies with almost no modification to this situation, after replacing Schmidt's Subspace Theorem by its function field analogue proved by Wang [**15**]. Since the latter is effective, this proof also leads to effective estimates.

Note moreover that Wang's Subspace Theorem is proved also in the nonconstant case—i.e., for linear forms with nonconstant coefficients in a given function field. So this second proof too extends to the nonsplit case.

4. Proof of Theorem 3

In the sequel we shall use constants $C_1, \ldots$ (whose dependencies shall be indicated), but not necessarily with the same meaning as in the statement.

Let $d := \deg_Y f$; since f is monic in Y, d cannot be zero. We denote by U, V indeterminates over κ and we let α satisfy $f(U, V, \alpha) = 0$. Note that α has degree d over $\kappa(U,V)$; we denote by $\alpha = \alpha_1, \dots, \alpha_d$ its conjugates.

Let $\Delta = \Delta(U,V) \in \kappa[U,V]$ be the discriminant of f with respect to Y, so we have $\Delta = \prod_{i<j}(\alpha_i - \alpha_j)^2$ (and we interpret $\Delta = 1$ if $d = 1$). Our assumption says that Δ does not have multiple factors as a polynomial in U, V, apart possibly from monomials in U, V.

Now let $h := f(U,V,Y)/(Y-\alpha) = \prod_{1<i\le d}(Y - \alpha_i)$. The division algorithm shows that $h \in \kappa[U,V,\alpha,Y]$. Hence the discriminant of h with respect to Y is a polynomial $g(U,V,\alpha) \in \kappa[U,V,\alpha]$, and we have $g(U,V,\alpha) = \prod_{1<i<j\le d}(\alpha_i - \alpha_j)^2$.

Hence, taking into account that $f'(U,V,\alpha) = \prod_{i>1}(\alpha - \alpha_i)$ (where we have denoted with a dash differentiation with respect to Y), we get

$$\Delta(U,V) = f'^2(U,V,\alpha)g(U,V,\alpha).$$

This is an identity in $\kappa[U,V,\alpha] \cong \kappa[U,V,Y]/(f)$. Hence it can be evaluated at our solutions $(u,v,y) \in \mathcal{O}_S^* \times \mathcal{O}_S^* \times \mathcal{O}_S$, which in fact satisfy $f(u,v,y) = 0$. We obtain

$$\Delta(u,v) = f'^2(u,v,y)g(u,v,y). \tag{5}$$

We now argue similarly to Proposition 1 above, which in fact may be described as a more abstract "discriminant argument." We view $f'^2(u,v,y)$ and $g(u,v,y)$ as rational functions on $\tilde{\mathcal{C}}$. Since they are polynomials in their arguments $u,v,y \in \mathcal{O}_S$, they in fact lie in $\mathcal{O}_S$.

Let $p \in \tilde{\mathcal{C}} \setminus S$ be a zero of $f'^2(u,v,y)$. Then p is not a pole of $g(u,v,y)$, and so (4.1) shows that p is a multiple zero of $\Delta(u,v)$.

Now we invoke Theorem 1.2 in [**5**], which bounds the number of such multiple zeros outside S, provided u,v are multiplicatively independent. As in the above proofs (through Proposition 1), this is a most crucial tool in the argument.

More precisely, by exactly the same arguments as in Proposition 1 (applied to $\Delta(u,v)$ in place of the $A(u,v)$ appearing therein), we deduce from Theorem 1.2 of [**5**] that for every $\epsilon > 0$ one of the following alternatives holds:

(i) The number of zeros of $f'(u,v,y)$ outside S is small—namely,

$$\sum_{p\in\tilde{\mathcal{C}}\setminus S} \mathrm{ord}_p(f'(u,v,y)) \le \epsilon H(u,v). \tag{6}$$

Here we are using the notation about heights introduced in §1. This says that "$f'(u,v,y)$ *is an almost* S*-unit*."

(ii) The S-units u,v satisfy a relation $u^a = \lambda v^b$ with $\lambda \in \kappa^*$ and a,b integers not both zero and such that $|a| + |b| \le C_1(f,\chi,\epsilon)$, where C_1 depends only on f, χ, and ϵ.

In a moment we shall deal with (i), on choosing some small ϵ depending only on f and $|S|$. Taking into account this future choice of ϵ, we can at once dispense with (ii), by saying that if it holds, we are in one of the cases of the conclusion.

So, it is sufficient to deal with (i), for a given small enough ϵ. Since $\kappa(U, V, \alpha)$ has transcendence degree 2 over κ, there exists a fixed nontrivial irreducible polynomial relation $R(U, V, f'(U, V, \alpha)) = 0$, from which we derive

$$R(u, v, f'(u, v, y)) = 0. \tag{7}$$

Setting $w := f'(u, v, y)$, we view (7) as a linear combination over κ of monomial terms $u^a v^b w^c$, for certain integer vectors $(a, b, c) \in \mathbb{Z}^3$. We want to apply a theorem of Brownawell and Masser on relations in S-units (or almost S-units); for this however we need that there are no vanishing proper subsums. Let us analyze this.

Suppose that (7) contains some proper vanishing subsum, expressed, say, by an equation $P(u, v, w) = 0$; these subsums are finite in number and depending only on R, and hence only on f. Since R is irreducible, we can eliminate from the equations $R = P = 0$ the variable w and obtain a nontrivial relation $T(u, v) = 0$. This relation gives rise to a map from $\mathcal{C}$ to the curve defined by T. By the preliminaries explained in the paragraph "Morphisms and Integral Points," at the beginning of §1, we conclude from such a relation either a bound for $H(u, v)$ or a multiplicative relation of the sought type and with the sought bound for the degree. (Here we could also apply [**2**] with no problems because u, v are S-units.) The bounds shall depend on T, but, as remarked, there are only finitely many possibilities for T, depending only on f. (Note that the number of these relations and the degree of the resulting equation $T = 0$ are bounded in terms only of $\deg f$.)

Therefore, we shall suppose from now on that no proper subsum of (7) vanishes.

On dividing by a monomial term in U, V, we may suppose that R is a Laurent polynomial in u, v that contains the term 1; with this normalization, let us denote by Σ the set of vectors (a, b, c) that correspond to monomials appearing in R. The number of terms in R is bounded dependently only on f, so by [BM], Theorem 1, we get:

$$\max_{(a,b,c)\in\Sigma} H(u^a v^b w^c) \leq C_2(f)(\chi + \epsilon H(u, v)). \tag{8}$$

Here the last term on the right arises from (6), taking into account that w may be not quite an S-unit, but that by (6) it becomes an S'-unit for a set S' containing at most $|S| + \epsilon H(u, v)$ points.

To exploit (8) we consider again some cases. A first case occurs when the set Σ contains three linearly independent vectors. In this case, by multiplicative elimination we may bound $H(u), H(v)$ by $C_3(f)$ times the right side of (8). Hence, for ϵ small enough in terms of f, we obtain that $H(u,v)$ is bounded by $C_4(f)\chi$, and we fall into one of the cases of the sought conclusion.

If Σ does not even contain two independent vectors, then the relation may be written as $R_1(u^l v^m w^n) = 0$ for a nonzero polynomial R_1. This falls as a special instance of relation (9) below and can be treated by the arguments to follow (9).

If Σ contains two but not three independent vectors, we may find a basis for the lattice they generate, and by a further 2×2 unimodular transformation we may assume that such a basis has the shape $(l,m,0),(a,b,c)$. Note that $c \neq 0$, because otherwise $R(u,v) = 0$ would yield an algebraic relation between U,V.

Replacing the lattice by the primitive lattice defining the same $\mathbb{Q}$-vector space, we may even assume that $\gcd(l,m) = 1$ and by completing (l,m) to a basis of $\mathbb{Z}^2$, we may assume that $(l,m) = (1,0)$: note in fact that these lattice transformations correspond to automorphisms of $\mathbb{G}_m^2$, contemplated in the statement.

Let us then perform such an automorphism, expressing our functions in terms of the new coordinates on $\mathbb{G}_m^2$. (Now the polynomials will become Laurent polynomials with respect to U,V.) From now on we shall refer to these new coordinates.

After these transformations, the relation $R = 0$ may be written as

$$R_2(U, M f'^c(U,V,\alpha)), \tag{9}$$

where M is a suitable monomial in U,V. We may conjugate α over $\kappa(U,V)$ in all possible ways and obtain that $M f'^c(U,V,\beta)$ is algebraic over $\kappa(U)$ for all β such that $f(U,V,\beta) = 0$. Taking the product over β and recalling that $\Delta(U,V) = \prod_\beta f'(U,V,\beta)$, we deduce that $M^d\Delta^c(U,V)$ is algebraic over $\kappa(U)$. Therefore, differentiation with respect to V sends $V^n\Delta^c$ to zero, for a suitable integer n. Hence $\Delta(U,V) = Q(U)V^{-n/c}$, where Q is a polynomial (so c divides n). Geometrically, this means that the original cover of $\mathbb{G}_m^2$ is ramified precisely above a finite union of cosets of certain algebraic subgroups (in the new coordinates, they are defined by $U = \rho$, where $Q(\rho) = 0$).

Now, the equation $f(U,V,Y) = 0$ may be seen as defining a (possibly reducible) curve over $K := \overline{\kappa(U)}$ (that is, in the variables V,Y); by the previous conclusion on the discriminant, in particular each component of this curve is an unbranched cover of $\mathbb{G}_m$, through the V-map. It is then well known that the defining equation of each component gives a cyclic extension of $K(V)$. Actually, by the conclusion on the discriminant, this entails that the polynomial f may be written as a product of m (irreducible) factors of the shape $(Y - A(V))^e - bV^s$, where $A \in K[V^{\pm 1}]$, $b \in K$ and e,s are coprime integers, $e > 0$. Not to insert the (known) argument inside the main reasoning, we pause to recall a proof of this fact in a remark.

Remark. We prove the following claim:

Let K be an algebraically closed field of characteristic 0 and let $g \in K[V,Y]$ be an irreducible polynomial monic and of degree e in Y such that its discriminant with respect to Y is a constant times a power of V. Then $g(V,Y) = (Y - a(V))^e - bV^s$ where $a \in K[V]$, $b \in K^*$ and s is an integer prime to e.

Proof. If $e = 1$, the conclusion holds trivially, so we suppose $e > 1$. In view of the condition on the discriminant, the extension L of $K(V)$ defined by g is ramified only above 0 and ∞ (above the V-line). The Hurwitz genus formula for $L/K(V)$ shows that L/K must have genus 0 and be totally ramified (i.e., to order e) above both points. By Lüroth's theorem we have $L = K(T)$ for some $T \in L$, so $V = p(T)$ for some rational function p in one variable over K, of degree e. The above condition says that L is totally ramified above $V = \infty$, say at T_0, and by a homography we can assume $T_0 = \infty$. This allows us to assume that p is a polynomial of degree e. The condition that this is totally ramified above 0 says that $p(T) = l(T - T_1)^e$ for some $T_1 \in K$ and now we may assume $T_1 = 0$ and $l = 1$ after a translation and dilation, so $T^e = V$.

The roots of g in Y lie in $K(T)$ and actually in $K[T]$ since g is monic in Y (and $K[T]$ is integrally closed); so they may be written uniquely as $a_0 + a_1\theta T + \ldots + a_m\theta^m T^m$, where $a_i \in K$ are fixed and θ runs through the eth roots of unity. The difference of any two roots must be a constant times a power of T because of the assumption on the discriminant. This immediately shows that, among the a_i such that $i \not\equiv 0 \pmod{e}$, all but one vanish—say, $a_s \neq 0$ for a single s not multiple of e. Since the roots are pairwise distinct we have that actually s is prime to e. Then the sum of the other terms is a polynomial in $K[T^e] = K[V]$, which we denote by $a(V)$. The roots of g may then be written as $a(V) + a_s\theta^s T^s$, and this clearly proves the assertion, with $b = a_s^e$.

Let us now go back to the argument we had interrupted. Since f is irreducible over $\kappa(U,V)$, necessarily the A, b that occur form a complete system of conjugates of any of these pairs, over $\kappa(U)$; let us denote them as A^σ, b^σ, for automorphisms σ of $K/\kappa(U)$. This also proves that e, s may be chosen so not to depend on the factor, which in turn yields $me = d$.

If $e = 1$, we may absorb bV^s into $A(V)$ and suppose $b = 0$. From our conclusion on the discriminant, we deduce that $A^\sigma = A + c_\sigma V^{r_\sigma}$, for $c_\sigma \in K$ and integers r_σ. If an integer r is among the r_σ, then a term cV^r must appear in $A(V)$. Clearly V^r appears in $A^\sigma - A$ if and only if $c^\sigma \neq c$, which happens precisely when σ is outside a certain subgroup of automorphisms of $\mathrm{G}al(K/\kappa(U))$. Suppose that for distinct integers r, r' the corresponding such subgroup is proper. Then there exist automorphisms σ not lying in any of the two subgroups, and so such that both $V^r, V^{r'}$ appear in $A^\sigma - A$, a contradiction with the fact that $A^\sigma - A = c_\sigma V^{r_\sigma}$. Hence the said subgroup is proper for at most one integer r; this implies that $r_\sigma = r$ for all σ, and that $A(V) = cV^r + B(V)$ where $c \in K$ and $B \in \kappa(U)[V^{\pm 1}]$. However, since $f \in \kappa[U^{\pm 1}, V^{\pm 1}, Y]$ (recall that we have performed an automorphism of $\mathbb{G}_m^2$), it is also clear that $B = B(U, V) \in \kappa[U^{\pm 1}, V^{\pm 1}]$.

In this case, after a substitution $Y = Z + B(U, V)$, we have the following expression for f, where $b \in K$ and b^σ runs through its conjugates over $\kappa(U)$:

$$f(U, V, Y) = \prod_\sigma (Z - b^\sigma V^r). \tag{10}$$

Now let $e > 1$. The discriminant of the said product contains the nonzero factors $A(V) - A^\sigma(V) + V^r(\zeta b^{1/e} - \theta(b^\sigma)^{1/e})$, for a given choice of the e-th roots of V, b, b^σ and all choices of e-th roots of unity ζ, θ, where $r = s/e$. As before, we deduce that each factor must be (as a polynomial in $V^{1/e}$) a multiple of a (fractional) power of V. Also, we have $b \neq 0$ and more than a single choice for ζ, θ, which forces this power to be V^r. But then all the differences $A - A^\sigma$ must also be of the shape $c_\sigma V^r$, for $c_\sigma \in K$. In particular, we may write $A = B + aV^r$, where $B \in \kappa(U)[V^{\pm 1}]$ and $a \in K$. As in the case $e = 1$, in fact we must have $B = B(U, V) \in \kappa[U^{\pm 1}, V^{\pm 1}]$. Also, since A is a Laurent polynomial in V, either $a = 0$ or r is integral. But this last fact cannot happen, because e, s are coprime and $e > 1$. Hence $a = 0$, and after the same substitution $Y = Z + B(U, V)$ as before, we have

$$f(U, V, Y) = \prod_\sigma (Z^e - b^\sigma V^s), \qquad \gcd(e, s) = 1, \tag{11}$$

a formula which includes (4.6) as a special case. This formula also shows that

$$f(U, V, Y) = V^{sm} P(U, Z/V^s), \qquad Z = Y - B(U, V), \tag{12}$$

where $P \in \kappa[U^{\pm 1}, W]$ is an irreducible (Laurent) polynomial and $B \in \kappa[U^{\pm 1}, V^{\pm 1}]$.

This already shows that our surface defined by $f = 0$ is a product of $\mathbb{G}_m$ by a curve (which corresponds to $P = 0$ and to the subfield $\kappa(U, b^{1/e})$ of K).

In practice, this is the real content of the theorem, in that we have reduced to the case of curves rather than surfaces, with which we started. So we could stop here; however, we shall complete the picture and go further, analyzing again our solutions (u, v, y) with the new information provided by (12). In fact, we have the solutions $P(u, w) = 0$, where $u \in \mathcal{O}_S^*$ and where $w = (y - B(u, v))/v^s \in \mathcal{O}_S$.

We could now proceed in several ways: either invoking or reproving the needed special case of Siegel's theorem for the function field $\kappa(\tilde{\mathcal{C}})$, or repeating the arguments at the beginning, in this simplified context. We choose this last possibility. In fact, we can do this because we again have the needed assumptions on the discriminant: the discriminant $\Delta_P = \Delta_P(U)$ of $P(U,W)$ with respect to W is a Laurent polynomial in U, without multiple zeros in $\mathbb{G}_m$—actually, by (12) this discriminant is just the above $Q(U)$, up to a factor that is a monomial in U.

The case of constant $u = \alpha \in \kappa^*$ is a special case of a multiplicative dependence of the sought type. Let us then consider only the solutions with nonconstant u.

Then we have the equation (5) above, with P in place of f and Δ_P in place of Δ (and some polynomial in $\kappa[u^{\pm 1}, w]$, in place of g). As above, note that every zero $p \in \mathcal{C}$ of $P'(u,w)$ is either in S or a multiple zero of $\Delta_P(u)$. On the other hand, we have just noticed that $\Delta_P(U)$ has no multiple factors $(U-\alpha)^l$, $l > 1$, with $\alpha \in \kappa^*$. Hence the number of zeros of $P'(u,w)$ outside S does not exceed the number of multiple zeros in $\tilde{\mathcal{C}} \setminus S$ of a fixed product of pairwise factors of the shape $u - \alpha$. To estimate this last number, we no longer need the recourse to Theorem 1.2 from [**5**] (which we crucially needed above in this proof, before (6)). We merely need the usual "abc" estimate, as in the second easier part of the proof of Proposition 1; to avoid any reference, we shall reproduce that few-lines argument: For nonconstant u, the total number of zeros of multiplicity > 1, counted with weight equal to its multiplicity -1, of $u - \alpha$, even for varying $\alpha \in \kappa^*$, is bounded by the degree of the zero divisor of the differential du/u; however, this degree equals the pole-degree $+2g-2$, so is bounded by χ, since u is an S-unit.

This leads to a sharp analogue of (6)—i.e.,

$$\sum_{p \in \tilde{\mathcal{C}} \setminus S} \mathrm{ord}_p(P'(u,w)) \le \chi. \tag{13}$$

As in the derivation of (4.3) above, setting $P(U,\xi) = 0$, by transcendence degree, there exists a fixed nontrivial irreducible polynomial relation $R_3(U, P'(U,\xi)) = 0$; this implies

$$R_3(u, P'(u,w)) = 0.$$

But in view of (13), $P'(u,w)$ becomes an S_1-unit, for a set $S_1 \in \tilde{\mathcal{C}}(\kappa)$ containing S and with at most $\#S + \chi$ elements. Hence we may apply the Brownawell-Masser Theorem to the last displayed algebraic relation, with S_1 in place of S. Of course, we must again take into account possible vanishing subsums. By the irreducibility of R_3, any such subsum would lead (by elimination) to a constant u, a case already considered. On the other hand, if there are no vanishing subsums, let us suppose on dividing by a monomial in U that R_3 contains a constant term. As above let us denote by Σ the set of integer pairs (a,b) such that $u^a P'(u,w)^b$ appears in $R_3 = 0$. We have, by [**2**],

$$\max_{(a,b)\in\Sigma} H(u^a P'(u,w)^b) \leq C_5(\deg f)(|S| + \chi). \tag{14}$$

If Σ does not contain two independent vectors, then we have a relation $R_1(U^l P'(U,\beta)^m) = 0$, whence $U^l P'(U,\beta)^m$ is constant. Conjugating β among the possible roots of $P(U,\beta) = 0$ and multiplying, we obtain that the discriminant Δ_1 is a power of U. As in a previous part of the proof, P defines a cyclic unramified cover of $\mathbb{G}_m$—i.e., we may write, for suitable $A_1, B_1 \in \kappa[U^{\pm 1}]$ and $b_1 \in \kappa^*$,

$$P(U,W) = (W - A_1(U))^d - b_1 U^h B_1(U)^d.$$

This yields the following shape for f,

$$f(U,V,Y) = (Y - A_2(U,V))^d - b_1 V^{sm} U^h B_1(U)^d, \tag{15}$$

showing that our surface is a cyclic cover of $\mathbb{G}_m^2$. The assumption on Δ also shows that B_1 may be taken constant, as in the statement.

If Σ contains two independent vectors, we may perform a multiplicative elimination and obtain $H(u) \leq C_6 = C_6(\deg f, \chi)$.

Then we may write $u = c\tilde{u}$, where $c \in \kappa^*$ and $\tilde{u}$ assumes only finitely many values; the set of possible values depends only on C_6, S, and $\kappa(\tilde{\mathcal{C}})$ (and of course on our choice of representatives modulo κ^*), whereas their number depends only on C_6 and χ.

For fixed $\tilde{u}$, a solution w of $P(c\tilde{u}, w) = 0$ defines a function field $\kappa(\tilde{u}, w)$, whose branch points on $\kappa(\tilde{u})$ are of the shape $c^{-1}R$, where R is the set of branch points of the curve $P(u,w) = 0$ above the u-line. Since $\kappa(\tilde{u}, w) \subset \kappa(\tilde{\mathcal{C}})$, such a set is contained in the set R_1 of branch points of $\kappa(\tilde{\mathcal{C}})/\kappa(\tilde{u})$. Let $R_2 := R \setminus \{0, \infty\}$. Then $c^{-1}R_2 \subset R_1$. Also, $|R_1|$ is bounded by $C_7 = C_7(\deg f, C_6)$. We conclude that if there are more than C_7 relevant values of c, then $R \subset \{0, \infty\}$ therefore, as before, P defines a cyclic unramified cover of $\mathbb{G}_m$ (above the U-line), leading again to (15), with B_1 constant.

This completes the description of the various possibilities and concludes the proof.

5. Examples

Example 1. We begin by showing that inequality (1) is the best possible for what concerns the type of dependence on χ_Y. Namely, there exists a number C_2' such that for curves $Y \subset X$ of arbitrarily high degree,

$$\max(1, \chi_Y) \leq C_2' \deg(Y).$$

We take for Y a component of $\pi^{-1}(W)$, where $W \subset \mathbb{G}_m^2$ is a rational curve of high degree d. Since W intersects the branch divisor of π in at most $\ll d$ points, the genus of Y is also $\ll d$. (Here the implied constants depend only on X and π.) The number of places at infinity is also $\ll d$, and so is χ_Y.

Example 2. We give an explicit application of Theorems 1 and 3. Consider the case when D_1, D_2 are lines, defined by, say, $Z = 0$ and $X = 0$ respectively, and D_3 is an irreducible curve of degree $d \geq 2$, defined by a homogeneous equation $G(X, Y, Z) = 0$. We suppose that D_3 is not tangent to D_1 or D_2 at any point of intersection and that $D_1 \cap D_2 \cap D_3 = \emptyset$. This last condition amounts to $G(0, 1, 0) \neq 0$. We seek the S-integral points for $\mathbb{P}_2 \setminus D$ over a function field $\kappa(\tilde{\mathcal{C}})$. A morphism $\varphi : \mathcal{C} \to \mathbb{P}_2 \setminus |D|$ is given by $(u_0 : u_1 : u_2)$ where u_0, u_1, u_2 are rational functions on $\tilde{\mathcal{C}}$. The condition that $\varphi(\mathcal{C})$ omits the divisors D_1, D_2 corresponds to u_0/u_2, u_2/u_0, u_1/u_2 being regular on the affine curve $\mathcal{C} = \tilde{\mathcal{C}} \setminus S$. Then put $u := u_0/u_2$, which will be a unit in the ring $\mathcal{O}_S$, and $y = u_1/u_2$, which is regular on $\mathcal{C}$. Also, put $g(x, y) = G(x, y, 1)$. Then the condition that $\varphi(\mathcal{C})$ does not meet D_3 amounts to $g(u, y)$ being a unit in $\mathcal{O}_S$. Hence we arrive at the equation $g(u, y) = v$. The assumptions of our Theorem 1 are satisfied so we deduce that the solutions $(u, v, y) \in \mathcal{O}_S^* \times \mathcal{O}_S^* \times \mathcal{O}_S$ to the equation

$$g(u, y) = v$$

have bounded degree, outside possible "trivial" families. (We have explained in §1 how inequalities of the form (0.1) give bounds for the degree (or height) of the solutions to the corresponding Diophantine equations, up to possible trivial families. Such families only arise when the morphism $\varphi : \mathcal{C} \to X$ factors through $\mathbb{G}_m$, and u, v are multiplicatively dependent modulo constants.)

Let us also see how to apply Theorem 3, at least in certain cases. Putting $f(u, v, y) := g(u, y) - v$, we arrive at the equation

$$f(u, v, y) = 0 \qquad u, v \in (\mathcal{O}_S)^*, \quad y \in \mathcal{O}_S.$$

Since $G(0, 1, 0) \neq 0$ (which is a consequence of the fact that $D_1 \cap D_2 \cap D_3 = \emptyset$), the polynomial $f(U, V, Y)$ is monic in Y, as required by the hypothesis of Theorem 3.

Now, if the discriminant of $f(U, V, Y)$ with respect to the variable Y has no repeated factors, we can apply Theorem 3 and conclude that if u, v are multiplicatively independent modulo constants, their degree is $\ll \chi_{\mathcal{C}}$, where the implied constant depends only on f. Hence the conclusion of Theorem 1 is recovered again. However, although the condition on the discriminant is "generically" satisfied, it is not automatically satisfied even for the special polynomials arising from Example 2. Take for instance for D_3 the Fermat cubic of equation $X^3 + Y^3 + Z^3 = 0$; the corresponding polynomial will be $f(U, V, Y) = Y^3 + U^3 + 1 - V$, whose discriminant is a square. In that case Theorem 1 (or Theorem 2) applies, but not Theorem 3. On the other hand, there are cases where Theorem

3 can be used but Theorem 2 does not (directly) apply—e.g., when the surface defined by the equation $f(U,V,Y)=0$ is singular.

Remark. The case $d=2$ of Example 2 was treated in [**5**]; the arithmetic analogue is still unknown, but see [**6**] for special cases, like the equation $g(a^m,y)=b^n$, for fixed $a,b\in\mathbb{Z}$, to be solved in integers m,n.

Example 3. We now modify Example 2 and obtain a counter-example, showing that the condition on the normal crossing of the divisor $D_1+\ldots+D_r$ in Theorem 1 cannot be omitted. Consider the sum of a conic and two lines intersecting on the conic. Supposing one line is the line at infinity of the projective plane, so that its complement is identified with $\mathbb{A}^2$, with affine coordinates x,y, let $\{x=0\}$ be the second line and $(x-1)y+1=0$ be the equation for the conic. Then the S-integral points on the complement of such configuration corresponds to the pairs $(u,y)\in\mathcal{O}_S^*\times\mathcal{O}_S$ such that $(u-1)y+1=: v$ is a unit. Hence they correspond to the pairs of units $(u,v)\in(\mathcal{O}_S^*)^2$ such that $(u-1)$ divides $(v-1)$ in the ring $\mathcal{O}_S$. Now, for every unit u and integer n, we have a solution $(u,v)=(u,u^n)$ and they form a Zariski-dense set.

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