STABLE PAIRS ON LOCAL K3 SURFACES

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Abstract

We prove a formula which relates Euler characteristic of moduli spaces of stable pairs on local K3 surfaces to counting invariants of semistable sheaves on them. Our formula generalizes Kawai-Yoshioka's formula for stable pairs with irreducible curve classes to arbitrary curve classes. We also propose a conjectural multiple cover formula of sheaf counting invariants which, combined with our main result, leads to an Euler characteristic version of Katz-Klemm-Vafa conjecture for stable pairs.

1. Introduction

Let S be a smooth projective K3 surface over $\mathbb{C}$, and X the total space of the canonical line bundle (i.e. trivial line bundle) on S,

$$X = S \times \mathbb{C}.$$

The space X is a non-compact Calabi-Yau 3-fold. We first state our main result, then discuss its motivation, background and outline of the proof.

1.1. Main result. Our goal is to prove a formula which relates the following two kinds of invariants on X.

(i) Stable pair invariants: The notion of stable pairs is introduced by Pandharipande-Thomas [**40**] in order to give a refined Donaldson-Thomas curve counting invariants on Calabi-Yau 3-folds. By definition, a stable pair on X consists of a pair

$$(F, s), \quad s \colon \mathcal{O}_X \to F,$$

where F is a pure one dimensional coherent sheaf on X and s is surjective in dimension one. We always assume that F is supported on the fibers of the second projection,

(1) $$X = S \times \mathbb{C} \to \mathbb{C}.$$

For $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, the moduli space of such pairs (F, s) satisfying

$$[F] = \beta, \quad \chi(F) = n,$$

is denoted by $P_n(X, \beta)$. We are interested in its topological Euler characteristic,

$$(2) \qquad\qquad \chi(P_n(X, \beta)) \in \mathbb{Z}.$$

(ii) Sheaf counting invariants: Let ω be an ample divisor on S, and take a vector

$$(3) \qquad\qquad v = (r, \beta, n) \in \mathbb{Z} \oplus H^2(S, \mathbb{Z}) \oplus \mathbb{Z}.$$

The moduli stack of ω-Gieseker semistable sheaves on X supported on the fibers of the projection (1) with Mukai vector v is denoted by

$$(4) \qquad\qquad \mathcal{M}_\omega(r, \beta, n).$$

We consider its 'Euler characteristic',

$$(5) \qquad\qquad J(r, \beta, n) = {}'\chi'(\mathcal{M}_\omega(r, \beta, n)).$$

We will see that the RHS does not depend on ω, so ω is not included in the LHS. When the vector v is primitive, then the invariant (5) is the usual Euler characteristic of the moduli space (4). However when v is not primitive, then the stack (4) may have complicated stabilizers and the definition of its 'Euler characteristic' is not obvious. In this case, we apply Joyce's theory on counting invariants, developed in [**19**], [**20**], [**21**], [**23**], [**22**] to the definition of 'χ'. Namely the invariant (5) is defined by taking the 'logarithm' in the Hall algebra and its Euler characteristic. (See Subsection 4.8.) Similar invariants have been also studied in [**44**].

Our main result is the following theorem.

Theorem 1.1. [Theorem 5.5] *The generating series*

$$\mathrm{PT}^\chi(X) := \sum_{\beta, n} \chi(P_n(X, \beta)) y^\beta z^n,$$

is written as the following product expansion,

$$\mathrm{PT}^\chi(X) = \prod_{r \geq 0, \beta > 0, n \geq 0} \exp\left((n + 2r) J(r, \beta, r + n) y^\beta z^n \right)$$

$$(6) \qquad\qquad \cdot \prod_{r > 0, \beta > 0, n > 0} \exp\left((n + 2r) J(r, \beta, r + n) y^\beta z^{-n} \right).$$

Here we have regarded $\beta \in H_2(X, \mathbb{Z})$ as an element of $H^2(S, \mathbb{Z})$, and $\beta > 0$ means that β is a Poincaré dual of an effective one cycle on S. The background of the above formula will be discussed below.

1.2. Motivation and Background. The curve counting theories on Calabi-Yau 3-folds have drawn much attention recently. Now there are three kinds of such theories: Gromov-Witten (GW) theory [**5**], [**30**], Donaldson-Thomas (DT) theory [**42**] and Pandharipande-Thomas (PT) theory [**40**]. These theories are conjectured to be equivalent by Maulik-Nekrasov-Okounkov-Pandharipande [**32**] and Pandharipande-Thomas [**40**]. Among the above three theories, DT and PT theories also count objects in the derived category of coherent sheaves on X. Based on this observation, it was speculated in [**40**] that DT/PT theories should be related by wall-crossing phenomena w.r.t. Bridgeland's space of stability conditions on the derived category of coherent sheaves [**10**]. In recent years, general theories of wall-crossing formula of DT type invariants are established by Joyce-Song [**24**] and Kontsevich-Soibelman [**29**]. By applying the wall-crossing formula, several geometric applications have been obtained, e.g. DT/PT correspondence, rationality of the generating series, flop invariance, etc. (cf. [**47**], [**43**], [**48**], [**41**], [**12**].) Our purpose is to give a further application of the wall-crossing in the derived category to the curve counting invariants on local K3 surfaces.

When $X = S \times \mathbb{C}$ for a K3 surface S, usual curve counting invariants are rather trivial. This is because that, although the curve counting invariants are unchanged under deformations of S, the K3 surface S can be deformed in a non-algebraic way so that the resulting invariants are always zero. Instead, the reduced curve counting invariants should be the correct mathematical objects to be studied. (See Subsections 6.1, 6.2.) These reduced theories are introduced and studied in [**33**], [**28**], [**34**], and unchanged under deformations of S preserving the curve class to be algebraic.

One of the goals in the study of curve counting invariants on X is to prove a conjecture by Katz-Klemm-Vafa [**26**, Section 6], which we call *KKV conjecture*. It predicts a certain evaluation of reduced curve counting invariants on X in terms of modular forms, and is derived from the duality between the M-theory on S and the heterotic string theory on T^3. Mathematically the KKV conjecture is formulated in terms of generating series of reduced GW invariants, (cf. Conjecture 6.1,) and proved for primitive curve classes by Maulik-Pandharipande-Thomas [**34**]. (A curve class $\beta \in H_2(X, \mathbb{Z})$ is *primitive* if it is not a multiple of some other element in $H_2(X, \mathbb{Z})$.) The strategy in the paper [**34**] for primitive classes is as follows:

- First prove a reduced version of GW/PT correspondence for primitive classes. Then the problem can be reduced to a computation of reduced PT invariants for primitive classes. The latter computation can be reduced to the case of irreducible curve classes by a deformation argument. (A curve class $\beta \in H_2(X, \mathbb{Z})$ is *irreducible* if it is not written as $\beta_1 + \beta_2$ for $\beta_i > 0$.)

- If the curve class is irreducible, then the reduced PT invariant coincides with the Euler characteristic of the moduli space (2). The invariants (2) for irreducible β are completely calculated by Kawai-Yoshioka [27], and apply their formula. (cf. Theorem 6.3.)

Now suppose that we try to solve KKV conjecture for arbitrary curve classes, following the above strategy. Then it is natural to try to generalize Kawai-Yoshioka's formula [27] for the invariants (2) with irreducible curve classes to arbitrary curve classes. Our main theorem (Theorem 1.1) has grown out of such an attempt. In fact we will see that the formula (6) reconstructs Kawai-Yoshioka's formula [27]. (See Subsection 6.2.) Also the formula (6) reduces the computation of the stable pair invariants (2) for arbitrary β to that of the sheaf counting invariants (5). As we will discuss in the next subsection, the latter invariants are expected to be related to the Euler characteristic of the Hilbert scheme of points on S in terms of the multiple cover formula. (cf. Conjecture 1.3.) Assuming such a multiple cover formula, the series $PT^{\chi}(X)$ is written as an infinite product similar to Borcherd's product [8], giving a complete calculation of the invariants (2) for arbitrary curve classes. (See Subsection 1.3 and the formula (9) below.)

At this moment, we don't know any relationship between reduced PT invariants and the invariants (2) when the curve class β is not irreducible. So a reduced version of GW/PT together with Theorem 1.1 do not immediately imply the KKV conjecture. Also there is a technical gap in proving a version of the formula (6) for reduced PT invariants. The issue is that, although the wall-crossing formula in [24], [29] is established for invariants expressed by the Behrend function [6], it is not clear whether reduced PT invariants are expressed in that way or not. Nevertheless, we expect that there is a close relationship between reduced PT invariants and the Euler characteristic invariants (2), and knowing the invariants (2) give a geometric intuition of the reduced PT invariants. Such an attempt, namely proving an Euler characteristic version first and then back to the virtual one, is also employed and successful in proving DT/PT correspondence and the rationality conjecture in [47], [48], [41], [12]. In fact, we will see that the reduced GW/PT correspondence, together with a conjectural version of the formula (6) for reduced PT invariants, yield KKV conjecture. (See Subsection 6.3.) In this sense, our result is expected to be a first step toward a complete proof of KKV conjecture.

1.3. Conjectural multiple cover formula. An interesting point of the formula (6) is that it gives a relationship between invariants with different features. Namely,

- A stable pair invariant (2) is easy to define and an integer. However it is not easy to relate it to the geometry of K3 surfaces, nor see any interesting dualities among the invariants (2).

- A sheaf counting invariant (5) is difficult to define and it is not necessary an integer. However it has a nice automorphic property, and seems to be related to the Euler characteristic of the Hilbert scheme of points on S.

Let us focus on the property of the invariants (5). First the invariant $J(v)$ is completely calculated when v is a primitive algebraic class. In this case, we have [27], [52],

$$(7) \qquad\qquad J(v) = \chi(\mathrm{Hilb}^{(v,v)/2+1}(S)).$$

Here $(*, *)$ is the Mukai pairing on the Mukai lattice,

$$\widetilde{H}(S, \mathbb{Z}) := \mathbb{Z} \oplus H^2(S, \mathbb{Z}) \oplus \mathbb{Z},$$

and $\mathrm{Hilb}^n(S)$ is the Hilbert scheme of n-points in S. Its Euler characteristic is computed by the Göttsche's formula [14],

$$(8) \qquad\qquad \sum_{n \geq 0} \chi(\mathrm{Hilb}^n(S))q^n = \prod_{n \geq 1} \frac{1}{(1 - q^n)^{24}}.$$

The formulas (6), (7) and (8) reconstruct Kawai-Yoshioka's formula [27] for the invariants (2) with irreducible β. (See Subsection 6.2.)

When v is not necessarily primitive, we are not able to give a complete computation of the invariant (5) at this moment. However there is an advantage of the invariant (5), as it has a certain automorphic property. Recall that the lattice $\widetilde{H}(S, \mathbb{Z})$ has a weight two Hodge structure. (See Subsection 2.1.) The following result is a refinement of [44, Corollary 5.26].

Theorem 1.2. [Theorem 4.31] *Let g be a Hodge isometry of the lattice $\widetilde{H}(S, \mathbb{Z})$. Then we have*

$$J(gv) = J(v).$$

The above result means that, if we are able to compute the invariant (5) for specific (r, β, n), e.g. $r = 0$, then we can also compute the invariant (5) for another (r, β, n) by applying a Hodge isometry g. On the other hand, the invariant of the form $J(0, \beta, n)$ is expected to satisfy a certain multiple cover formula as discussed in [24, Conjecture 6.20], [51, Theorem 6.4]. Combining these arguments, we propose the following conjecture.

Conjecture 1.3. [Conjecture 6.6] *If v is an algebraic class, then $J(v)$ is written as*

$$J(v) = \sum_{k \geq 1, k | v} \frac{1}{k^2} \chi(\mathrm{Hilb}^{(v/k, v/k)/2+1}(S)).$$

We will give some evidence of the above conjecture in Subsection 6.4. If we assume the above conjecture, then the formula (6) is written as

$$\mathrm{PT}^{\chi}(X) = \prod_{r \geq 0, \beta > 0, n \geq 0} (1 - y^{\beta} z^n)^{-(n+2r)\chi(\mathrm{Hilb}^{\beta^2/2 - r(n+r)+1}(S))}$$

$$(9) \qquad \cdot \prod_{r > 0, \beta > 0, n > 0} (1 - y^{\beta} z^{-n})^{-(n+2r)\chi(\mathrm{Hilb}^{\beta^2/2 - r(n+r)+1}(S))}.$$

The above formula, which resembles Borcherd's product [**8**], is interpreted as an Euler characteristic version of KKV conjecture for stable pairs. As we will discuss in Subsection 6.3, a similar formula for reduced PT invariants together with reduced GW/PT correspondence give a complete proof of KKV conjecture.

1.4. Outline of the proof of Theorem 1.1.

Step 1.

We compactify X as

$$\overline{X} = S \times \mathbb{P}^1,$$

and prove a formula for $\mathrm{PT}^{\chi}(\overline{X})$. (See Subsection 4.1.) The series $\mathrm{PT}^{\chi}(\overline{X})$ is related to $\mathrm{PT}^{\chi}(X)$ by, (cf. Lemma 4.3,)

$$(10) \qquad\qquad \mathrm{PT}^{\chi}(\overline{X}) = \mathrm{PT}^{\chi}(X)^2.$$

We also introduce the invariant, (cf. Definition 4.17,)

$$(11) \qquad\qquad N(r, \beta, n) \in \mathbb{Q},$$

counting certain semistable objects supported on the fibers of the projection,

$$\pi \colon \overline{X} = S \times \mathbb{P}^1 \to \mathbb{P}^1,$$

with Chern character (*not* Mukai vector) equal to (r, β, n). The invariant (11) is related to the invariant (5) by, (cf. Proposition 4.27,)

$$(12) \qquad\qquad N(r, \beta, n) = 2J(r, \beta, r + n).$$

Step 2.

In [**48**, Theorem 1.3], the author proved the following formula by using Joyce's wall-crossing formula [**23**],

$$(13) \quad \mathrm{PT}^{\chi}(\overline{X}) = \prod_{\beta > 0, n > 0} \exp\left(n N(0, \beta, n) y^{\beta} z^n\right) \left(\sum_{\beta, n} L(\beta, n) y^{\beta} z^n\right).$$

The invariant $L(\beta, n)$ counts certain objects in the derived category $D^b \mathrm{Coh}(\overline{X})$, which are $\mu_{i\omega}$-limit semistable objects in the notation of [**48**, Section 3]. (cf. Definition 3.9.) Our idea is to decompose the series of

$L(\beta, n)$ further, using the wall-crossing formula again. More precisely, we study the triangulated category,

$$\mathcal{D} := \langle \pi^* \operatorname{Pic}(\mathbb{P}^1), \operatorname{Coh}_\pi(\overline{X}) \rangle_{\mathrm{tr}} \subset D^b \operatorname{Coh}(\overline{X}).$$

Here

$$\operatorname{Coh}_\pi(\overline{X}) \subset \operatorname{Coh}(\overline{X}),$$

is the subcategory consisting of sheaves supported on the fibers of π. For a fixed ample divisor ω on S, we will construct the heart of a bounded t-structure, (cf. Definition 2.8,)

$$\mathcal{A}_\omega \subset \mathcal{D}.$$

The above heart is unchanged if ω is replaced by $t\omega$ for $t \in \mathbb{R}_{>0}$. Moreover the above heart fits into a pair,

$$\sigma_t = (Z_{t\omega}, \mathcal{A}_\omega),$$

for $t \in \mathbb{R}_{>0}$, giving a weak stability condition introduced in [**47**]. (cf. Lemma 3.4.) We will construct the invariant, (cf. Definition 4.10,)

$$\mathrm{DT}_{t\omega}^\chi(r, \beta, n) \in \mathbb{Q}$$

as an Euler characteristic version of Donaldson-Thomas type invariant, counting $Z_{t\omega}$-semistable objects $E \in \mathcal{A}_\omega$ satisfying

$$\mathrm{ch}(E) = (1, -r, -\beta, -n)$$
$$\in H^0(\overline{X}) \oplus H^2(\overline{X}) \oplus H^4(\overline{X}) \oplus H^6(\overline{X}).$$

Here r and n are regarded as integers, and β is regarded as an element of $H^2(S, \mathbb{Z})$. (See Subsection 2.3.)

Step 3.

We investigate the generating series,

$$\mathrm{DT}_{t\omega}^\chi(\overline{X}) := \sum_{r, \beta, n} \mathrm{DT}_{t\omega}^\chi(r, \beta, n) x^r y^\beta z^n,$$

which is regarded as an element of a certain topological vector space. (See Subsection 5.2.) If we consider big t, we see that, (cf. Proposition 4.16,)

$$\text{(14)} \qquad \lim_{t \to \infty} \mathrm{DT}_{t\omega}^\chi(\overline{X}) = \sum_{r, \beta, n} L(\beta, n) x^r y^\beta z^n.$$

On the other hand if we consider small t, we see that, (cf. Proposition 4.16,)

$$\text{(15)} \qquad \lim_{t \to 0} \mathrm{DT}_{t\omega}^\chi(\overline{X}) = \lim_{t \to 0} \sum_{r, \beta} \mathrm{DT}_{t\omega}^\chi(r, \beta, 0) x^r y^\beta.$$

The wall-crossing formula enables us to see how the series $\mathrm{DT}_{t\omega}^\chi(\overline{X})$ varies if we change t. Here an interesting phenomena happens: two

dimensional semistable objects on the fibers of π are involved in the wall-crossing formula. Since so far only one dimensional objects have been involved in the wall-crossing formula, e.g. the formula (13), this seems a new phenomena in this field of study.

By the wall-crossing formula, we obtain a formula relating (14) and (15). As a result, (14) is obtained by the product of (15) and the following infinite product, (cf. Corollary 5.2,)

$$(16) \qquad \prod_{\beta>0, rn>0} \exp\left((n+2r)N(r,\beta,n)x^r y^\beta z^n \right)^{\epsilon(r)},$$

where $\epsilon(r) = 1$ if $r > 0$ and $\epsilon(r) = -1$ if $r < 0$. Unfortunately the above argument is not enough to obtain the desired formula, as the RHS of (15) still remains unknown. In order to complete the proof, we focus on some abelian subcategory, (cf. Definition 3.6,)

$$\mathcal{A}_\omega(1/2) \subset \mathcal{A}_\omega.$$

We introduce finer weak stability conditions on $\mathcal{A}_\omega(1/2)$, and apply the wall-crossing formula again. Then we obtain (cf. Proposition 5.3,)

(17)
$$\lim_{t\to 0} \sum_{r,\beta} \mathrm{DT}^\chi_{t\omega}(r,\beta,0) x^r y^\beta = \prod_{r>0,\beta>0} \exp\left(2rN(r,\beta,0)x^r y^\beta \right) \cdot \sum_{r\in\mathbb{Z}} x^r.$$

Combining (10), (12), (14), (15), (16), (17), Theorem 1.2 and looking at the x^0-term, we obtain the desired formula (6). (cf. Theorem 5.5.)

1.5. Plan of the paper. In Section 2, we introduce several notation and introduce the abelian category $\mathcal{A}_\omega$. In Section 3, we construct weak stability conditions on $\mathcal{A}_\omega$ and state some results on wall-crossing phenomena. In Section 4, we introduce counting invariants via the Hall algebra of $\mathcal{A}_\omega$, their variants, and investigate the property of the invariants. In Section 5, we apply wall-crossing formula and give a proof of Theorem 1.1. In Section 6, we give some discussions toward KKV conjecture. From Section 7 to Section 10, we give several proofs postponed in the previous sections.

1.6. Notation and Convention. For a triangulated category $\mathcal{D}$, the shift functor is denoted by [1]. For a set of objects $\mathcal{S} \subset \mathcal{D}$, we denote by $\langle \mathcal{S} \rangle_{\mathrm{tr}}$ the smallest triangulated subcategory which contains $\mathcal{S}$ and $0 \in \mathcal{D}$. Also we denote by $\langle \mathcal{S} \rangle_{\mathrm{ex}}$ the smallest extension closed subcategory of $\mathcal{D}$ which contains $\mathcal{S}$ and $0 \in \mathcal{D}$. The abelian category of coherent sheaves on a variety X is denoted by $\mathrm{Coh}(X)$. We say $F \in \mathrm{Coh}(X)$ is d-dimensional if its support is d-dimensional, and we write $\dim F = d$. For a surface S, its Neron-Severi group is denoted by $\mathrm{NS}(S)$. For an element $\beta \in \mathrm{NS}(S)$, we write $\beta > 0$ if β is a Poincaré dual of an effective one cycle on S. An element $\beta \in \mathrm{NS}(S)$ with $\beta > 0$ is irreducible when

β is not written as $\beta_1 + \beta_2$ with $\beta_i > 0$. For a finitely generated abelian group Γ, an element $v \in \Gamma$ is primitive if v is not a multiple of some other element in Γ.

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2. Triangulated category of local K3 surfaces

In this section, we recall some notions used in the study of K3 surfaces. We also introduce a certain triangulated category associated to a K3 surface, and construct the heart of a bounded t-structure on it.

2.1. Generalities on K3 surfaces. Let S be a smooth projective K3 surface over $\mathbb{C}$, i.e.

$$K_S \cong \mathcal{O}_S, \quad H^1(S, \mathcal{O}_S) = 0.$$

We begin with recalling the generalities on S. The *Mukai lattice* is defined by

$$(18) \qquad \widetilde{H}(S, \mathbb{Z}) := H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z}).$$

In what follows, we naturally regard $H^0(S, \mathbb{Z})$ and $H^4(S, \mathbb{Z})$ as $\mathbb{Z}$. For two elements,

$$v_i = (r_i, \beta_i, n_i) \in \widetilde{H}(S, \mathbb{Z}), \quad i = 1, 2,$$

the *Mukai pairing* is defined by

$$(19) \qquad (v_1, v_2) := \beta_1 \beta_2 - r_1 n_2 - r_2 n_1.$$

Recall that there is a weight two Hodge structure on $\widetilde{H}(S, \mathbb{Z}) \otimes \mathbb{C}$ given by,

$$\widetilde{H}^{2,0}(S) := H^{2,0}(S), \ \widetilde{H}^{0,2}(S) := H^{0,2}(S),$$

$$\widetilde{H}^{1,1}(S) := H^{0,0}(S) \oplus H^{1,1}(S) \oplus H^{2,2}(S).$$

We define the lattice Γ_0 to be

$$\Gamma_0 := \widetilde{H}(S, \mathbb{Z}) \cap \widetilde{H}^{1,1}(S)$$

$$(20) \qquad\qquad = \mathbb{Z} \oplus \mathrm{NS}(S) \oplus \mathbb{Z}.$$

For an object $E \in D^b \mathrm{Coh}(S)$, its *Mukai vector* $v(E) \in \Gamma_0$ is defined by

$$v(E) := \mathrm{ch}(E)\sqrt{\mathrm{td}_S}$$

$$(21) \qquad\qquad = (\mathrm{ch}_0(E), \mathrm{ch}_1(E), \mathrm{ch}_0(E) + \mathrm{ch}_2(E)).$$

For any $E, F \in D^b \mathrm{Coh}(S)$, the Riemann-Roch theorem yields,

$$(22) \qquad \sum_i (-1)^i \dim \mathrm{Hom}_S(E, F[i]) = -(v(E), v(F)).$$

2.2. Local K3 surfaces. Let S be a K3 surface. We are interested in the total space of the canonical line bundle on S,

$$X = S \times \mathbb{C}.$$

We compactify X as

$$\overline{X} = S \times \mathbb{P}^1.$$

Our strategy is to deduce the geometry of X from that of $\overline{X}$. Let π be the second projection,

$$\pi \colon \overline{X} \to \mathbb{P}^1.$$

We define the abelian category

$$\mathrm{Coh}_\pi(\overline{X}) \subset \mathrm{Coh}(\overline{X}),$$

to be the subcategory consisting of sheaves supported on the fibers of π. Its derived category is denoted by $\mathcal{D}_0$,

(23) $$\mathcal{D}_0 := D^b \mathrm{Coh}_\pi(\overline{X}).$$

We introduce the following triangulated category.

Definition 2.1. We define the triangulated category $\mathcal{D}$ to be

$$\mathcal{D} := \langle \pi^* \mathrm{Pic}(\mathbb{P}^1), \mathrm{Coh}_\pi(\overline{X}) \rangle_{\mathrm{tr}} \subset D^b \mathrm{Coh}(\overline{X}).$$

Note that $\mathcal{D}_0$ is a triangulated subcategory of $\mathcal{D}$. The triangulated category $\mathcal{D}$ is not a Calabi-Yau 3 category, but close to it by the following lemma.

Lemma 2.2. *Take objects $E, F \in \mathcal{D}$, and suppose that either E or F is an object in $\mathcal{D}_0$. Then we have the isomorphism,*

$$\mathrm{Hom}_{\mathcal{D}}(E, F) \cong \mathrm{Hom}_{\mathcal{D}}(F, E[3])^\vee.$$

Proof. The result follows from the Serre duality on $\overline{X}$ and the isomorphism

$$E \otimes \omega_{\overline{X}} \cong E,$$

for $E \in \mathcal{D}_0$. q.e.d.

2.3. Chern characters on $\mathcal{D}_0$ and $\mathcal{D}$. We fix some notation on Chern characters for objects in $\mathcal{D}_0$ and $\mathcal{D}$. Let p be the first projection,

$$p \colon \overline{X} = S \times \mathbb{P}^1 \to S.$$

We define the group homomorphism cl_0 to be the composition,

(24) $$\mathrm{cl}_0 \colon K(\mathcal{D}_0) \xrightarrow{p_*} K(S) \xrightarrow{\mathrm{ch}} \Gamma_0.$$

By the definition of $\mathrm{Coh}_\pi(\overline{X})$, the push-forward $p_* E$ is an element of $K(S)$, hence the above map is well-defined. Instead of the Chern character, we can also consider the Mukai vector on $K(\mathcal{D}_0)$,

$$(25) \qquad v\colon K(\mathcal{D}_0) \overset{p_*}{\to} K(S) \overset{\mathrm{ch}\sqrt{\mathrm{td}_S}}{\to} \Gamma_0,$$

as in (21).

Remark 2.3. Although the Mukai vector is usually used in the study of K3 surfaces, we will use both of Chern characters and Mukai vectors. The reason is that, the Chern characters are useful in describing wall-crossing formula, while the Mukai vectors are useful in discussing Fourier-Mukai transforms.

Next we consider the Chern character map on $K(\overline{X})$,

$$\mathrm{ch}\colon K(\overline{X}) \to H^*(\overline{X}, \mathbb{Q}).$$

If we restrict the above map to the Grothendieck group of $\mathcal{D}$, then it factors through the subgroup,

$$\Gamma := H^0(\overline{X}, \mathbb{Z}) \oplus \left(\Gamma_0 \boxtimes H^2(\mathbb{P}^1, \mathbb{Z})\right) \subset H^*(\overline{X}, \mathbb{Q}).$$

Hence we obtain the group homomorphism,

$$\mathrm{cl} := \mathrm{ch}\colon K(\mathcal{D}) \to \Gamma.$$

We naturally identify $H^0(\overline{X}, \mathbb{Z})$ and $H^2(\mathbb{P}^1, \mathbb{Z})$ with $\mathbb{Z}$. Then Γ is identified with

$$(26) \qquad \Gamma = \mathbb{Z} \oplus \Gamma_0,$$
$$= \mathbb{Z} \oplus \mathbb{Z} \oplus \mathrm{NS}(S) \oplus \mathbb{Z}.$$

We usually write an element $v \in \Gamma$ as a vector

$$v = (R, r, \beta, n),$$

where R, r, n are integers and $\beta \in \mathrm{NS}(S)$. If $v = \mathrm{cl}(E)$ for $E \in \mathcal{D}$, then the above vector corresponds to the Chern character,

$$\mathrm{cl}(E) = (\mathrm{ch}_0(E), \mathrm{ch}_1(E), \mathrm{ch}_2(E), \mathrm{ch}_3(E)).$$

Under the above identification, we always regard $\mathrm{ch}_0(E)$, $\mathrm{ch}_1(E)$, $\mathrm{ch}_3(E)$ as integers, and $\mathrm{ch}_2(E)$ as an element of $\mathrm{NS}(S)$. Hence for instance, the intersection number $\mathrm{ch}_2(E) \cdot \omega$ for a divisor ω on S makes sense. Note that this is equal to $\mathrm{ch}_2(E) \cdot p^*\omega$ in the usual sense.

Also in the above notation, we sometimes write

$$\mathrm{rank}(v) := R, \quad \mathrm{rank}(E) := \mathrm{rank}(\mathrm{cl}(E)),$$

for $E \in \mathcal{D}$.

By the Grothendieck Riemann-Roch theorem, the maps cl and cl_0 are compatible. Namely under the identification (26), we have

$$(27) \qquad \mathrm{cl}(E) = (0, \mathrm{cl}_0(E)),$$

for $E \in \mathcal{D}_0 \subset \mathcal{D}$.

2.4. Classical stability conditions on $\mathrm{Coh}_\pi(\overline{X})$. We recall some classical notions of stability conditions on $\mathrm{Coh}_\pi(\overline{X})$. For an object $E \in \mathrm{Coh}_\pi(\overline{X})$, we write

$$\mathrm{cl}_0(E) = (r, \beta, n) \in \mathbb{Z} \oplus \mathrm{NS}(S) \oplus \mathbb{Z}.$$

For an ample divisor ω on S, the slope of E is defined to be

$$\mu_\omega(E) := \begin{cases} \infty, & \text{if } r = 0, \\ \omega \cdot \beta/r, & \text{if } r \neq 0. \end{cases}$$

Also the Hilbert polynomial of E is defined by

$$\chi_{\omega,E}(m) := \chi(\overline{X}, E \otimes \mathcal{O}_{\overline{X}}(mp^*\omega))$$
$$= a_d m^d + a_{d-1} m^{d-1} + \cdots,$$

with $a_d \neq 0$ and $d = \dim E$. The reduced Hilbert polynomial is defined to be

$$\overline{\chi}_{\omega,E}(m) := \chi_{\omega,E}(m)/a_d.$$

If $r \neq 0$, or equivalently $d = 2$, then we have

$$\overline{\chi}_{\omega,E}(m) = m^2 + \frac{2\mu_\omega(E)}{\omega^2} m + (\text{constant term}).$$

Also there is a map,

(28) $$\Gamma_0 \ni v \mapsto \overline{\chi}_{\omega,v}(m) \in \mathbb{Q}[m],$$

such that we have $\overline{\chi}_{\omega,\mathrm{cl}_0(E)}(m) = \overline{\chi}_{\omega,E}(m)$ for $E \in \mathrm{Coh}_\pi(\overline{X})$.

The total order $\succ$ on $\mathbb{Q}[m]$ is defined as follows: for $p_i(m) \in \mathbb{Q}[m]$ with $i = 1, 2$, we have $p_1(m) \succ p_2(m)$ if and only if

$$\deg p_1(m) < \deg p_2(m), \text{ or}$$
$$\deg p_1(m) = \deg p_2(m), \ p_1(m) > p_2(m), \ m \gg 0.$$

The above notions determine slope stability and Gieseker-stability on $\mathrm{Coh}_\pi(\overline{X})$.

Definition 2.4. (i) An object $E \in \mathrm{Coh}_\pi(\overline{X})$ is μ_ω-(semi)stable if for any exact sequence $0 \to F \to E \to G \to 0$ in $\mathrm{Coh}_\pi(\overline{X})$ with $F, G \neq 0$, we have

$$\mu_\omega(F) < (\leq)\mu_\omega(G).$$

(ii) An object $E \in \mathrm{Coh}_\pi(\overline{X})$ is ω-Gieseker (semi)stable if for any sub-sheaf $0 \neq F \subsetneq E$, we have

$$\overline{\chi}_{\omega,F}(m) \prec (\preceq)\overline{\chi}_{\omega,E}(m).$$

For more detail, see [**17**]. It is easy to see that if an object $E \in$ $\mathrm{Coh}_{\pi}(\overline{X})$ is μ_{ω}-(or ω-Gieseker) stable, then E is written as

$$(29) \qquad E \cong i_{p*}E', \quad p \in \mathbb{P}^1,$$

for some μ_{ω}-(or ω-Gieseker) stable sheaf E' on a fiber

$$X_p := \pi^{-1}(p) \cong S.$$

The map i_p is the inclusion $i_p \colon X_p \hookrightarrow \overline{X}$. We will use the following Bogomolov-type inequalities.

Lemma 2.5. *(i) Let $E \in \mathrm{Coh}_{\pi}(\overline{X})$ be an ω-Gieseker stable sheaf with $\mathrm{cl}_0(E) = (r, \beta, n)$. Then we have*

$$(30) \qquad \beta^2 + 2 \geq 2r(r + n).$$

(ii) If $E \in \mathrm{Coh}_{\pi}(\overline{X})$ is ω-Gieseker semistable with $\beta \cdot \omega \neq 0$, then we have

$$(31) \qquad \beta^2 + 2(\beta \cdot \omega)^2 \geq 2r(r + n).$$

Proof. If $E \in \mathrm{Coh}_{\pi}(\overline{X})$ is ω-Gieseker stable, then E is written as (29) for a μ_{ω}-stable sheaf E' on X_p. Then the inequality (31) is a well-known consequence of the Riemann-Roch theorem and the Serre duality. (cf. [**36**, Corollary 2.5].) Let $E \in \mathrm{Coh}_{\pi}(\overline{X})$ be an ω-Gieseker semistable sheaf on $\overline{X}$. Then p_*E is also an ω-Gieseker semistable sheaf on S. Let $E_1, \cdots, E_k \in \mathrm{Coh}(S)$ be ω-Gieseker stable factors of p_*E. We have

$$\dim \mathrm{Hom}_S(p_*E, p_*E) \leq \sum_{i,j} \dim \mathrm{Hom}_S(E_i, E_j)$$

$$(32) \qquad \qquad \qquad \leq k^2.$$

Also since $\beta_i \cdot \omega$ and $\beta \cdot \omega$ have the same sign and β is equal to the sum $\sum_i \beta_i$, we have $k \leq |\beta \cdot \omega|$. By (22), the Serre duality and (32), we have

$$-\beta^2 + 2r(r + n) \leq 2k^2 \leq 2(\beta \cdot \omega)^2.$$

q.e.d.

2.5. The heart of a bounded t-structure on $\mathcal{D}_0$. For an ample divisor ω on S, let us consider μ_{ω}-stability on $\mathrm{Coh}_{\pi}(\overline{X})$. For each $E \in \mathrm{Coh}_{\pi}(\overline{X})$, there is a Harder-Narasimhan filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_N = E,$$

i.e. each subquotient $F_i = E_i/E_{i-1}$ is μ_{ω}-semistable with

$$\mu_{\omega}(F_1) > \mu_{\omega}(F_2) > \cdots > \mu_{\omega}(F_N).$$

We set

$$\mu_{\omega,+}(E) := \mu_{\omega}(F_1), \quad \mu_{\omega,-}(E) := \mu_{\omega}(F_N).$$

We define the pair of full subcategories $(\mathcal{T}_\omega, \mathcal{F}_\omega)$ in $\mathrm{Coh}_\pi(\overline{X})$ to be

$$\text{(33)} \qquad \mathcal{T}_\omega := \{E \in \mathrm{Coh}_\pi(\overline{X}) : \mu_{\omega,-}(E) > 0\},$$

$$\text{(34)} \qquad \mathcal{F}_\omega := \{E \in \mathrm{Coh}_\pi(\overline{X}) : \mu_{\omega,+}(E) \le 0\}.$$

In other words, an object $E \in \mathrm{Coh}_\pi(\overline{X})$ is contained in $\mathcal{T}_\omega$, (resp. $\mathcal{F}_\omega$,) iff E is filtered by μ_ω-semistable sheaves F_i with $\mu_\omega(F_i) > 0$. (resp. $\mu_\omega(F_i) \le 0$.) The existence of Harder-Narasimhan filtrations implies that $(\mathcal{T}_\omega, \mathcal{F}_\omega)$ is a torsion pair, i.e.

- For $T \in \mathcal{T}_\omega$ and $F \in \mathcal{F}_\omega$, we have $\mathrm{Hom}(T, F) = 0$.
- For any object $E \in \mathrm{Coh}_\pi(\overline{X})$, there is an exact sequence

$$0 \to T \to E \to F \to 0,$$

with $T \in \mathcal{T}_\omega$ and $F \in \mathcal{F}_\omega$.

The associated tilting is defined in the following way. (cf. [15].)

Definition 2.6. We define the category $\mathcal{B}_\omega$ to be

$$\mathcal{B}_\omega := \langle \mathcal{F}_\omega, \mathcal{T}_\omega[-1] \rangle_{\mathrm{ex}} \subset \mathcal{D}_0.$$

The category $\mathcal{B}_\omega$ is the heart of a bounded t-structure on $\mathcal{D}_0$, hence in particular an abelian category. We note that $\mathcal{B}_\omega$ is unchanged if we replace ω by $t\omega$ for $t > 0$.

Remark 2.7. The construction of the heart $\mathcal{B}_\omega$ is an analogue of similar constructions on the derived categories of coherent sheaves on surfaces by Bridgeland [11], Arcara-Bertram [1].

2.6. The heart of a bounded t-structure on $\mathcal{D}$. Let $\mathcal{D}$ be a triangulated category defined in Definition 2.1. We define the category $\mathcal{A}_\omega$ as follows.

Definition 2.8. We define $\mathcal{A}_\omega$ to be

$$\text{(35)} \qquad \mathcal{A}_\omega := \langle \pi^* \mathrm{Pic}(\mathbb{P}^1), \mathcal{B}_\omega \rangle_{\mathrm{ex}} \subset \mathcal{D}.$$

We have the following proposition.

Proposition 2.9. *The subcategory $\mathcal{A}_\omega \subset \mathcal{D}$ is the heart of a bounded t-structure on $\mathcal{D}$.*

Proof. More precisely, we can show that there is the heart of a bounded t-structure

$$\mathcal{A}'_\omega \subset D^b \mathrm{Coh}(\overline{X}),$$

which restricts to the heart $\mathcal{A}_\omega$ on $\mathcal{D}$. The construction of $\mathcal{A}'_\omega$ will be given in Definition 7.1. The proof follows from the exactly same argument of [47, Proposition 3.6], by replacing the notation $(\mathcal{D}, \mathcal{A}, \mathcal{D}', \mathcal{D}_E, \mathcal{A}_E)$ in [47, Proposition 3.6] by

$$(D^b \mathrm{Coh}(\overline{X}), \mathcal{A}'_\omega, \mathcal{D}_0, \mathcal{D}, \mathcal{A}_\omega).$$

The only modification in the proof is that we use Lemma 7.3 instead of [**47**, Lemma 6.1]. The statement and the proof of Lemma 7.3 will be given in Subsection 7.1. q.e.d.

The heart $\mathcal{A}_\omega$ satisfies the following property.

Lemma 2.10. *For any $E \in \mathcal{A}_\omega$, we have*

$$(36) \qquad \operatorname{rank}(E) \geq 0, \quad -\operatorname{ch}_2(E) \cdot \omega \geq 0.$$

Proof. By the definitions of $\mathcal{B}_\omega$ and $\mathcal{A}_\omega$, we may assume $E \in \pi^* \operatorname{Pic}(\mathbb{P}^1)$ or $E \in \mathcal{F}_\omega$ or $E \in \mathcal{T}_\omega[-1]$. In each case, the inequalities (36) are obviously satisfied by noting (27). q.e.d.

Remark 2.11. As in [**47**, Lemma 3.5], one might expect that there is the heart of a bounded t-structure on $\langle \mathcal{O}_{\overline{X}}, \operatorname{Coh}_\pi(\overline{X}) \rangle_{\mathrm{tr}}$ given by $\langle \mathcal{O}_{\overline{X}}, \mathcal{B}_\omega \rangle_{\mathrm{ex}}$. However this is not true, since the natural map $\mathcal{O}_{\overline{X}} \to \mathcal{O}_{X_0}$ in the category $\langle \mathcal{O}_{\overline{X}}, \mathcal{B}_\omega \rangle_{\mathrm{ex}}$ does not have a kernel.

2.7. Bilinear map χ. We define the bilinear map χ

$$(37) \qquad \chi \colon \Gamma \times \Gamma_0 \to \mathbb{Z},$$

as follows:

$$(38) \qquad \chi((R, r, \beta, n), (r', \beta', n')) = R(2r' + n').$$

By the Riemann-Roch theorem and Lemma 2.2, we have

$$\chi(\operatorname{cl}(E), \operatorname{cl}_0(F)) = \dim \operatorname{Hom}_{\mathcal{D}}(E, F) - \dim \operatorname{Ext}^1_{\mathcal{D}}(E, F)$$

$$(39) \qquad\qquad + \dim \operatorname{Ext}^1_{\mathcal{D}}(F, E) - \dim \operatorname{Hom}_{\mathcal{D}}(F, E),$$

for $E \in \mathcal{A}_\omega$ and $F \in \mathcal{B}_\omega$. If we define $\chi(v, v') := -\chi(v', v)$ for $v \in \Gamma_0$ and $v' \in \Gamma$, then (39) also holds for $E \in \mathcal{B}_\omega$ and $F \in \mathcal{A}_\omega$. The above bilinear map will be used in describing the wall-crossing formula in Section 5.

Note that Γ_0 is regarded as a subgroup of Γ via $v \mapsto (0, v)$. The map χ restricts to the bilinear pairing,

$$(40) \qquad \chi|_{\Gamma_0 \times \Gamma_0} \colon \Gamma_0 \times \Gamma_0 \to \mathbb{Z},$$

which is trivial, i.e. $\chi(v, v') = 0$ for any $v, v' \in \Gamma_0$. In particular for any $E, F \in \mathcal{B}_\omega$, we have

$$\dim \operatorname{Hom}_{\mathcal{D}_0}(E, F) - \dim \operatorname{Ext}^1_{\mathcal{D}_0}(E, F)$$

$$+ \dim \operatorname{Ext}^1_{\mathcal{D}_0}(F, E) - \dim \operatorname{Hom}_{\mathcal{D}_0}(F, E) = 0.$$

2.8. Abelian categories $\mathcal{A}(r)$. Here we introduce some abelian subcategories of $\mathcal{A}_\omega$. First we introduce the following subcategory of $\operatorname{Coh}_\pi(\overline{X})$,

$$\operatorname{Coh}_\pi^{\leq 1}(\overline{X}) := \{ E \in \operatorname{Coh}_\pi(\overline{X}) : \dim E \leq 1 \}.$$

Definition 2.12. For $r \in \mathbb{Z}$, we define the category $\mathcal{A}(r)$ to be

$$\mathcal{A}(r) := \langle \pi^* \mathcal{O}_{\mathbb{P}^1}(r), \operatorname{Coh}_\pi^{\leq 1}(\overline{X})[-1] \rangle_{\mathrm{ex}} \subset \mathcal{A}_\omega.$$

The category $\mathcal{A}(r)$ has a structure of an abelian category. In fact it is essentially shown in [**47**, Lemma 3.5] that $\mathcal{A}(0)$ is the heart of a bounded t-structure on the triangulated category,

$$\langle \mathcal{O}_{\overline{X}}, \mathrm{Coh}_{\overline{\pi}}^{\leq 1}(\overline{X}) \rangle_{\mathrm{tr}} \subset D^b \mathrm{Coh}(\overline{X}).$$

Since there is an equivalence of categories,

(41) $$\otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(r) \colon \mathcal{A}(0) \xrightarrow{\sim} \mathcal{A}(r),$$

the category $\mathcal{A}(r)$ also has a structure of an abelian category.

3. Weak stability conditions on $\mathcal{D}$

In this section, we construct weak stability conditions on our triangulated category $\mathcal{D}$. The notion of weak stability conditions on triangulated categories is introduced in [**47**], generalizing Bridgeland's stability conditions [**10**]. A weak stability condition is interpreted as a limiting degeneration of Bridgeland's stability conditions, and it is a coarse version of Bayer's polynomial stability condition [**3**]. It is easier to construct examples of weak stability than those of Bridgeland stability, and the wall-crossing formula in [**23**], [**24**], [**29**] is also applied in this framework. We remark that most of the results stated in this section are technical, and their proofs will be given in Sections 8, 9 and 10.

3.1. General definition. In this subsection, we recall the definition of weak stability conditions on triangulated categories in a general setting. Let $\mathcal{T}$ be a triangulated category, and $K(\mathcal{T})$ its Grothendieck group. We fix a finitely generated free abelian group Γ and a group homomorphism,

$$\mathrm{cl} \colon K(\mathcal{T}) \to \Gamma.$$

We also fix a filtration,

$$0 = \Gamma_{-1} \subset \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_N = \Gamma,$$

such that each subquotient Γ_i/Γ_{i-1} is a free abelian group.

Definition 3.1. A *weak stability condition* on $\mathcal{T}$ consists of data

$$(Z = \{Z_i\}_{i=0}^N, \mathcal{A}),$$

such that each Z_i is a group homomorphism,

$$Z_i \colon \Gamma_i/\Gamma_{i-1} \to \mathbb{C},$$

and $\mathcal{A} \subset \mathcal{T}$ is the heart of a bounded t-structure on $\mathcal{T}$, satisfying the following conditions:

• For any non-zero $E \in \mathcal{A}$ with $\mathrm{cl}(E) \in \Gamma_i \setminus \Gamma_{i-1}$, we have

(42) $$Z(E) := Z_i([\mathrm{cl}(E)]) \in \mathbb{H}.$$

Here $[\mathrm{cl}(E)] \in \Gamma_i/\Gamma_{i-1}$ is the class of $\mathrm{cl}(E) \in \Gamma_i \setminus \Gamma_{i-1}$ and

(43) $$\mathbb{H} := \{r \exp(i\pi\phi) : r > 0, 0 < \phi \leq 1\}.$$

We say $E \in \mathcal{A}$ is Z-*(semi)stable* if for any exact sequence in $\mathcal{A}$,

$$0 \to F \to E \to G \to 0,$$

we have the inequality,

$$\arg Z(F) < (\leq) \arg Z(G).$$

- For any $E \in \mathcal{A}$, there is a filtration in $\mathcal{A}$, (Harder-Narasimhan filtration,)

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E,$$

such that each subquotient $F_i = E_i/E_{i-1}$ is Z-semistable with

$$\arg Z(F_i) > \arg Z(F_{i+1}),$$

for all i.

If we take a filtration on Γ trivial, i.e. $N = 0$, then we call a weak stability condition as a *stability condition*. In this case, the pair $(Z, \mathcal{A})$ determines a stability condition in the sense of Bridgeland [10].

We denote by $\mathrm{Stab}_{\Gamma_{\bullet}}(\mathcal{T})$ the set of weak stability conditions on $\mathcal{T}$, satisfying some technical conditions. (Local finiteness, Support property.) The detail of these properties will be recalled in Section 8. The following theorem is proved in [47, Theorem 2.15], along with the same argument of Bridgeland's theorem [10, Theorem 7.1].

Theorem 3.2. *There is a natural topology on* $\mathrm{Stab}_{\Gamma_{\bullet}}(\mathcal{T})$ *such that the map*

$$\Pi \colon \mathrm{Stab}_{\Gamma_{\bullet}}(\mathcal{T}) \to \prod_{i=0}^{N} \mathrm{Hom}_{\mathbb{Z}}(\Gamma_i/\Gamma_{i-1}, \mathbb{C}),$$

sending $(Z, \mathcal{A})$ *to* Z *is a local homeomorphism. In particular each connected component of* $\mathrm{Stab}_{\Gamma_{\bullet}}(\mathcal{T})$ *is a complex manifold.*

If $N = 0$, then the space $\mathrm{Stab}_{\Gamma_0}(\mathcal{T})$ is nothing but Bridgeland's space of stability conditions [10].

3.2. Stability conditions on $\mathcal{D}_0$**.** Let S be a K3 surface and $\overline{X} = S \times \mathbb{P}^1$ as in the previous section. In this subsection, we construct stability conditions on $\mathcal{D}_0$ where $\mathcal{D}_0$ is defined by (23). In Subsection 2.3, we constructed a group homomorphism,

$$\mathrm{cl}_0 \colon K(\mathcal{D}_0) \to \Gamma_0,$$

for $\Gamma_0 = \mathbb{Z} \oplus \mathrm{NS}(S) \oplus \mathbb{Z}$. Therefore we have the space of Bridgeland's stability conditions on $\mathcal{D}_0$,

(44) $\mathrm{Stab}_{\Gamma_0}(\mathcal{D}_0).$

We construct elements of (44) following the same arguments of [**11**], [**1**], [**46**]. For an ample divisor ω on S, we set the group homomorphism, $Z_{\omega,0} \colon \Gamma_0 \to \mathbb{C}$ to be

$$(45) \qquad Z_{\omega,0}(v) := \int_S e^{-i\omega} v, \quad v \in \Gamma_0.$$

If we write $v = (r, \beta, n) \in \mathbb{Z} \oplus \mathrm{NS}(S) \oplus \mathbb{Z}$, then (45) is written as

$$Z_{\omega,0}(v) = n - \frac{1}{2} r \omega^2 - (\omega \cdot \beta)\sqrt{-1}.$$

Let $\mathcal{B}_\omega \subset \mathcal{D}_0$ be the heart of a bounded t-structure defined in Definition 2.6. We have the following lemma.

Lemma 3.3. *For any ample divisor ω on S and $t \in \mathbb{R}_{>0}$, we have*

$$(46) \qquad (Z_{t\omega,0}, \mathcal{B}_\omega) \in \mathrm{Stab}_{\Gamma_0}(\mathcal{D}_0).$$

Proof. The same proofs as in [**11**, Proposition 7.1], [**1**, Corollary 2.1] are applied. Also see [**46**, Lemma 6.4]. q.e.d.

3.3. Constructions of weak stability conditions on $\mathcal{D}$. Let $\mathcal{D}$ be a triangulated category defined in Definition 2.1. In this subsection, we construct weak stability conditions on $\mathcal{D}$. Recall that we constructed a group homomorphism,

$$\mathrm{cl} \colon K(\mathcal{D}) \to \Gamma,$$

for $\Gamma = \mathbb{Z} \oplus \Gamma_0$ in Subsection 2.3. We take a filtration of Γ,

$$0 = \Gamma_{-1} \subset \Gamma_0 \subset \Gamma_1 := \Gamma,$$

where Γ_0 is given by (20) and the second inclusion is given by $v \mapsto (0, v)$. Now we have data which defines the space of weak stability conditions on $\mathcal{D}$. The resulting complex manifold is

$$\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}).$$

Let ω be an ample divisor on S and $t \in \mathbb{R}_{>0}$. We define the element,

$$Z_{t\omega} \in \prod_{i=0}^{1} \mathrm{Hom}(\Gamma_i / \Gamma_{i-1}, \mathbb{C}),$$

to be the following:

$$Z_{t\omega,1}(R) := R\sqrt{-1}, \quad R \in \Gamma_1 / \Gamma_0 = \mathbb{Z},$$

$$Z_{t\omega,0}(v) := \int_S e^{-it\omega} v, \quad v \in \Gamma_0.$$

Let $\mathcal{A}_\omega \subset \mathcal{D}$ be the heart defined in Definition 2.8. We have the following lemma.

Lemma 3.4. *For any ample divisor ω on S and $t \in \mathbb{R}_{>0}$, we have*

$$\sigma_{t\omega} := (Z_{t\omega}, \mathcal{A}_\omega) \in \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}).$$

Proof. For a non-zero object $E \in \mathcal{A}_\omega$, suppose that $\mathrm{rank}(E) \neq 0$. Then $\mathrm{rank}(E) > 0$ and

$$Z_{t\omega}(E) \in \mathbb{R}_{>0}\sqrt{-1},$$

by the definition of $Z_{t\omega}$. If $\mathrm{rank}(E) = 0$, then $E \in \mathcal{B}_\omega = \mathcal{B}_{t\omega}$ and we have

$$Z_{t\omega}(E) = Z_{t\omega,0}(E) \in \mathbb{H},$$

by Lemma 3.3, where $\mathbb{H}$ is given by (43). Therefore the condition (42) is satisfied. The other properties (Harder-Narasimhan property, local finiteness, support property,) will be checked in Subsection 8.2.   q.e.d.

By [**47**, Lemma 2.17], the following map is a continuous map,

$$\mathbb{R}_{>0} \ni t \mapsto \sigma_{t\omega} \in \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}).$$

Remark 3.5. The subcategory $\mathcal{B}_\omega \subset \mathcal{A}_\omega$ is closed under subobjects and quotients. In particular, an object $E \in \mathcal{B}_\omega$ is $Z_{t\omega}$-(semi)stable if and only if E is $Z_{t\omega,0}$-(semi)stable with respect to the pair $(Z_{t\omega,0}, \mathcal{B}_\omega) \in \mathrm{Stab}_{\Gamma_0}(\mathcal{D}_0)$.

Remark 3.6. By the construction, for an object $E \in \mathcal{A}_\omega$ with $\mathrm{rank}(E) \neq 0$, we have

$$\arg Z_{t\omega}(E) = \frac{\pi}{2}.$$

Therefore the $Z_{t\omega}$-semistability of E is checked by comparing $\arg Z_{t\omega,0}(F)$ with $\pi/2$ where $F \in \mathcal{B}_\omega$ is a subobject or a quotient of E in $\mathcal{A}_\omega$.

3.4. Wall and chamber structure. In this subsection, we see the wall and chamber structure on the parameter space $t \in \mathbb{R}_{>0}$, and see what happens for small t. We introduce the following notation.

$$M_{t\omega}(R, r, \beta, n) := \left\{ E \in \mathcal{A}_\omega : \begin{array}{l} E \text{ is } Z_{t\omega}\text{-semistable with} \\ \mathrm{cl}(E) = (R, r, \beta, n). \end{array} \right\}.$$

We have the following proposition.

Proposition 3.7. *For fixed $\beta \in \mathrm{NS}(S)$ and an ample divisor ω on S, there is a finite sequence of real numbers,*

$$0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = \infty,$$

such that the set of objects

$$\bigcup_{\substack{(R,r,n), \\ \arg Z_{t\omega}(R,r,\beta,n)=\pi/2}} M_{t\omega}(R, r, \beta, n),$$

is constant for each $t \in (t_{i-1}, t_i)$.

Proof. The proof will be given in Subsection 9.2.   q.e.d.

For small t, we have the following proposition.

Proposition 3.8. *In the same situation of Proposition* 3.7*, we have*

$$M_{t\omega}(R, r, \beta, n) = \emptyset,$$

for any $t \in (0, t_1)$ and $(R, r, n) \in \mathbb{Z}^{\oplus 3}$ with $R \geq 1$ and $n \neq 0$.

Proof. The proof will be given in Subsection 9.3.   q.e.d.

3.5. Comparison with $\mu_{i\omega}$-limit semistable objects. Let $\mathcal{A}(r) \subset \mathcal{A}_\omega$ be the subcategory defined in Definition 2.12. In this subsection, we relate $Z_{t\omega}$-semistable objects in $\mathcal{A}_\omega$ for $t \gg 0$ to certain semistable objects in $\mathcal{A}(r)$.

Definition 3.9. An object $E \in \mathcal{A}(r)$ with $\operatorname{rank}(E) = 1$ is $\mu_{i\omega}$-limit (semi)stable if the following conditions hold:

- For any exact sequence $0 \to F \to E \to G \to 0$ in $\mathcal{A}(r)$ with $F \in \operatorname{Coh}_\pi^{\leq 1}(\overline{X})[-1]$, we have $\operatorname{ch}_3(F) \geq 0$.
- For any exact sequence $0 \to F \to E \to G \to 0$ in $\mathcal{A}(r)$ with $G \in \operatorname{Coh}_\pi^{\leq 1}(\overline{X})[-1]$, we have $\operatorname{ch}_3(G) \leq 0$.

Note that if $E \in \mathcal{A}(r)$ satisfies $\operatorname{rank}(E) = 0$, then $E \in \operatorname{Coh}_\pi^{\leq 1}(\overline{X})[-1]$. We also call $E \in \mathcal{A}(r)$ with $\operatorname{rank}(E) = 0$ to be $\mu_{i\omega}$-*limit (semi)stable* if $E[1] \in \operatorname{Coh}_\pi(\overline{X})$ is ω-Gieseker (semi)stable.

The $\mu_{i\omega}$-limit stability coincides with the same notion discussed in [**48**, Section 3], so we have employed the same notation here. To be more precisely, we have the following lemma:

Lemma 3.10. *Take an object $E \in D^b \operatorname{Coh}(\overline{X})$ satisfying*

$$\operatorname{ch}(E) = (R, 0, -\beta, -n) \in \Gamma \subset H^*(\overline{X}, \mathbb{Q}),$$

*for $R \leq 1$. Then E is an $\mu_{i\omega}$-limit semistable object in $\mathcal{A}(0)$ iff $E[1]$ is an $\mu_{i\omega}$-limit semistable object in the sense of [**48**, Section 3].*

Proof. The notion of $\mu_{i\omega}$-stability in [**48**, Section 3] together with the proof of this result will be given in Subsection 9.5.   q.e.d.

In what follows, we use Definition 3.9 for the definition of $\mu_{i\omega}$-limit stability. For $R \leq 1$, we set

$$M_{\lim}(R, r, \beta, n) := \left\{ E \in \mathcal{A}(r) : \begin{array}{c} E \text{ is } \mu_{i\omega}\text{-limit semistable} \\ \text{with } \operatorname{cl}(E) = (R, r, \beta, n). \end{array} \right\}.$$

We have the following proposition.

Proposition 3.11. *In the same situation of Proposition* 3.7*, we have*

$$M_{t\omega}(R, r, \beta, n) = M_{\lim}(R, r, \beta, n),$$

for any $t \in (t_{k-1}, \infty)$ and $R \leq 1$ satisfying $\arg Z_{t\omega}(R, r, \beta, n) = \pi/2$.

Proof. The proof will be given in Subsection 9.6.   q.e.d.

By the equivalence (41), the following lemma is obvious.

Lemma 3.12. *For $R \le 1$, we have the following bijection of objects,*

$$\otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(r) \colon M_{\lim}(R, 0, \beta, n) \overset{1:1}{\to} M_{\lim}(R, Rr, \beta, n).$$

3.6. Abelian category $\mathcal{A}_\omega(1/2)$. In this subsection, we introduce a certain abelian category generated by $Z_{t\omega}$-semistable objects for sufficiently small t. The following is an analogue of Bayer's polynomial stability condition [**3**].

Definition 3.13. An object $E \in \mathcal{A}_\omega$ is $Z_{0\omega}$-(semi)stable if for any exact sequence $0 \to F \to E \to G \to 0$ in $\mathcal{A}_\omega$ with $F, G \ne 0$, we have

$$\arg Z_{t\omega}(F) < (\le) \arg Z_{t\omega}(G),$$

for $0 < t \ll 1$.

The same proof of Lemma 3.4 shows that there are Harder-Narasimhan filtrations with respect to $Z_{0\omega}$-stability. For $\phi \in [0, 1]$, we set

$$\mathcal{A}_\omega(\phi) := \left\langle E \in \mathcal{A}_\omega : \begin{array}{c} E \text{ is } Z_{0\omega}\text{-semistable with} \\ \lim_{t \to 0} \arg Z_{t\omega}(E) = \phi\pi \end{array} \right\rangle_{\mathrm{ex}}.$$

By the definition of $Z_{t\omega}$, we have $\mathcal{A}_\omega(\phi) \ne \{0\}$ only if $\phi \in \{0, 1/2, 1\}$. Here we focus on the case of $\phi = 1/2$. We have the following lemma.

Lemma 3.14. *(i) An object $E \in \mathcal{A}_\omega$ is $Z_{0\omega}$-(semi)stable if and only if E is $Z_{t\omega}$-(semi)stable for $0 < t \ll 1$.*
(ii) Any object $E \in \mathcal{A}_\omega(1/2)$ satisfies $\mathrm{ch}_3(E) = 0$.
(iii) The category $\mathcal{A}_\omega(1/2)$ is an abelian subcategory of $\mathcal{A}_\omega$.

Proof. The proof will be given in Subsection 10.1.   q.e.d.

We also use the following notation,

$$\mathcal{B}_\omega(1/2) := \mathcal{A}_\omega(1/2) \cap \mathcal{B}_\omega.$$

3.7. Weak stability conditions on $\mathcal{A}_\omega(1/2)$. We construct weak stability conditions on the abelian category $\mathcal{A}_\omega(1/2)$. We define finitely generated free abelian groups $\widehat{\Gamma}$, $\widehat{\Gamma}_0$ and group homomorphisms $\widehat{\mathrm{cl}}$, $\widehat{\mathrm{cl}}_0$,

$$\widehat{\mathrm{cl}} \colon K(\mathcal{A}_\omega(1/2)) \to \widehat{\Gamma} := \mathbb{Z} \oplus \mathbb{Z} \oplus \mathrm{NS}(S),$$

$$\widehat{\mathrm{cl}}_0 \colon K(\mathcal{B}_\omega(1/2)) \to \widehat{\Gamma}_0 := \mathbb{Z} \oplus \mathrm{NS}(S),$$

to be

$$\widehat{\mathrm{cl}}(E) := (\mathrm{ch}_0(E), \mathrm{ch}_1(E), \mathrm{ch}_2(E)),$$

$$\widehat{\mathrm{cl}}_0(E) := (\mathrm{ch}_1(E), \mathrm{ch}_2(E)).$$

Here as in Subsection 2.3, we have regarded $\mathrm{ch}_0(E)$, $\mathrm{ch}_1(E)$ as integers, and $\mathrm{ch}_2(E)$ as an element of $\mathrm{NS}(S)$. We take the following filtration of $\widehat{\Gamma}$,

$$0 = \widehat{\Gamma}_{-1} \subset \widehat{\Gamma}_0 \subset \widehat{\Gamma}_1 := \widehat{\Gamma}.$$

Here the embedding $\widehat{\Gamma}_0 \subset \widehat{\Gamma}$ is given by $(r, \beta) \mapsto (0, r, \beta)$. For $\theta \in (0, 1)$, we construct the element

$$\widehat{Z}_{\omega,\theta} \in \prod_{i=0}^{1} \mathrm{Hom}(\widehat{\Gamma}_i/\widehat{\Gamma}_{i-1}, \mathbb{C}),$$

as follows,

$$\widehat{Z}_{\omega,\theta,1}(R) := Re^{i\pi\theta}, \quad R \in \widehat{\Gamma}_1/\widehat{\Gamma}_0 = \mathbb{Z},$$
$$\widehat{Z}_{\omega,\theta,0}(r, \beta) := -r - (\beta \cdot \omega)\sqrt{-1}, \quad (r, \beta) \in \widehat{\Gamma}_0.$$

We have the following lemma.

Lemma 3.15. *For any ample divisor ω on S and $0 < \theta < 1$, we have*

$$(\widehat{Z}_{\omega,\theta}, \mathcal{A}_\omega(1/2)) \in \mathrm{Stab}_{\widehat{\Gamma}_\bullet}(D^b(\mathcal{A}_\omega(1/2))).$$

Proof. The same proof of Lemma 3.4 is applied, and we omit the detail.   q.e.d.

The relationship between $\widehat{Z}_{\omega,1/2}$-stability and $Z_{0\omega}$-stability is given as follows.

Lemma 3.16. *An object $E \in \mathcal{A}_\omega$ is $Z_{0\omega}$-semistable satisfying*

$$\lim_{t \to 0} \arg Z_{t\omega}(E) = \pi/2,$$

if and only if $E \in \mathcal{A}_\omega(1/2)$ and E is $\widehat{Z}_{\omega,1/2}$-semistable.

Proof. The proof will be given in Subsection 10.2.   q.e.d.

3.8. Semistable objects in $\mathcal{A}_\omega(1/2)$. We set

$$\widehat{M}_{\omega,\theta}(R, r, \beta) := \left\{ E \in \mathcal{A}_\omega(1/2) : \begin{array}{l} E \text{ is } \widehat{Z}_{\omega,\theta}\text{-semistable with} \\ \widehat{\mathrm{cl}}(E) = (R, r, \beta). \end{array} \right\}.$$

Similarly to Proposition 3.7 and Proposition 3.8, we have the following proposition.

Proposition 3.17. *For fixed $\beta \in \mathrm{NS}(S)$ and an ample divisor ω on S, there is a finite sequence,*

$$0 = \theta_k < \theta_{k-1} < \cdots < \theta_1 < \theta_0 = 1/2,$$

such that the following holds.
    (i) The set of objects

$$\bigcup_{(R,r),R\geq 1} \widehat{M}_{\omega,\theta}(R, r, \beta),$$

is constant for $\theta \in (\theta_{i-1}, \theta_i)$.
    (ii) For $0 < t \ll 1$ and any $(R, r, \beta) \in \widehat{\Gamma}$, we have

$$\widehat{M}_{\omega,1/2}(R, r, \beta) = M_{t\omega}(R, r, \beta, 0).$$

(iii) For $\theta \in (0, \theta_{k-1})$, we have

$$\widehat{M}_{\omega,\theta}(1, r, \beta) = \begin{cases} \{\pi^* \mathcal{O}_{\mathbb{P}^1}(r)\}, & \text{if } \beta = 0, \\ \emptyset, & \text{if } \beta \neq 0. \end{cases}$$

Proof. The proof will be given in Subsection 10.3.   q.e.d.

4. Counting invariants

In this section, we discuss several counting invariants on X and $\overline{X}$, which appeared in the introduction.

4.1. Stable pairs. In this subsection, we recall the notion of stable pairs introduced by Pandharipande-Thomas [**40**]. Let S be a K3 surface and $X = S \times \mathbb{C}$, as in Subsection 2.2. Note that we have the subcategory,

$$(47) \qquad \mathrm{Coh}_\pi(X) \subset \mathrm{Coh}_\pi(\overline{X}),$$

consisting of sheaves supported on fibers of the projection $\pi|_X \colon X \to \mathbb{C}$.

Definition 4.1. A stable pair on X is a pair (F, s),

$$F \in \mathrm{Coh}_\pi(X), \quad s \colon \mathcal{O}_X \to F,$$

satisfying the following conditions.
- The sheaf F is a pure one dimensional sheaf.
- The morphism s is surjective in dimension one.

For $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, the moduli space of stable pairs (F, s) satisfying

$$[F] = \beta, \quad \chi(F) = n,$$

is denoted by

$$(48) \qquad\qquad P_n(X, \beta).$$

If we replace X by $\overline{X}$ in Definition 4.1, we also have the notion of stable pairs on $\overline{X}$. By regarding β as an element of $H_2(\overline{X}, \mathbb{Z})$, we also have the similar moduli space $P_n(\overline{X}, \beta)$ and an open embedding,

$$P_n(X, \beta) \subset P_n(\overline{X}, \beta).$$

The moduli space $P_n(\overline{X}, \beta)$ is proved to be a projective scheme in [**40**], hence in particular $P_n(X, \beta)$ is a quasi-projective scheme. In what follows, we regard an algebraic class $\beta \in H_2(X, \mathbb{Z})$ as an element of $\mathrm{NS}(S)$, by the natural isomorphism $H_2(X, \mathbb{Z}) \cong H_2(S, \mathbb{Z})$ and the Poincaré duality.

We are interested in the generating series of the Euler characteristic of the moduli space (48).

Definition 4.2. We define the generating series $\mathrm{PT}^\chi(X)$ to be

$$\mathrm{PT}^\chi(X) := \sum_{\substack{\beta \in \mathrm{NS}(S), \\ n \in \mathbb{Z}}} \chi(P_n(X, \beta)) y^\beta z^n.$$

Let (F, s) be stable pair on $\overline{X}$. We remark that if we regard a pair (F, s) as a two term complex,

$$(49) \qquad\qquad I^\bullet = (\mathcal{O}_{\overline{X}} \xrightarrow{s} F) \in \mathcal{D},$$

then $P_n(\overline{X}, \beta)$ is also interpreted as a moduli space of two term complexes (49) satisfying

$$(50) \qquad\qquad \mathrm{cl}(I^\bullet) = (1, 0, -\beta, -n),$$

in the notation in Subsection 2.3. (cf. [**40**, Section 2].)

As we stated in the introduction, our goal is to give a formula relating $\mathrm{PT}^\chi(X)$ to other invariants. We first prove a formula for the generating series on $\overline{X}$,

$$\mathrm{PT}^\chi(\overline{X}) := \sum_{\substack{\beta \in \mathrm{NS}(S), \\ n \in \mathbb{Z}}} \chi(P_n(\overline{X}, \beta)) y^\beta z^n.$$

These series are related as follows.

Lemma 4.3. *We have the equality,*

$$(51) \qquad\qquad \mathrm{PT}^\chi(\overline{X}) = \mathrm{PT}^\chi(X)^2.$$

Proof. A standard $\mathbb{C}^*$-action on $\mathbb{C}$ induces $\mathbb{C}^*$-actions on $X = S \times \mathbb{C}$ and $\overline{X} = S \times \mathbb{P}^1$ via acting on the second factors. Hence $\mathbb{C}^*$ acts on the moduli spaces $P_n(X, \beta)$ and $P_n(\overline{X}, \beta)$. A stable pair (F, s) on $\overline{X}$ is $\mathbb{C}^*$-fixed if and only if

$$F \cong F_0 \oplus F_\infty, \quad s = (s_0, s_\infty) \in \Gamma(F_0) \oplus \Gamma(F_\infty),$$

where (F_0, s_0) and (F_∞, s_∞) determine $\mathbb{C}^*$-fixed stable pairs on $U_0 = S \times \mathbb{C}$ and $U_\infty = S \times (\mathbb{P}^1 \setminus \{0\})$ respectively. Since both of U_0 and U_∞ are $\mathbb{C}^*$-equivalently isomorphic to X, we have

$$P_n(\overline{X}, \beta)^{\mathbb{C}^*} \cong \coprod_{\substack{\beta_1 + \beta_2 = \beta, \\ n_1 + n_2 = n}} P_{n_1}(X, \beta_1)^{\mathbb{C}^*} \times P_{n_2}(X, \beta_2)^{\mathbb{C}^*}.$$

Taking the Euler characteristic and the $\mathbb{C}^*$-localization, we obtain (51).

$$\text{q.e.d.}$$

4.2. Product expansion formula. In the paper [**48**], the author essentially proved the following result.

Theorem 4.4. [**48**, Theorem 1.3] *For each $(\beta, n) \in \mathrm{NS}(S) \oplus \mathbb{Z}$, there are invariants,*

$$(52) \qquad N(0, \beta, n) \in \mathbb{Q}, \quad L(\beta, n) \in \mathbb{Q},$$

such that we have the following formula:

$$(53) \quad \mathrm{PT}^{\chi}(\overline{X}) = \prod_{\beta>0, n>0} \exp\left(nN(0, \beta, n)y^{\beta}z^{n}\right) \left(\sum_{\beta, n} L(\beta, n)y^{\beta}z^{n}\right).$$

Roughly speaking, the invariants (52) are given in the following way.

- The invariant $N(0, \beta, n)$ is a counting invariant of ω-Gieseker semistable sheaves $F \in \mathrm{Coh}_{\pi}(\overline{X})$, satisfying

$$\mathrm{cl}_0(F) = (0, \beta, n).$$

  If we denote the moduli space of such sheaves by $\mathcal{M}_{\omega, \overline{X}}(0, \beta, n)$, then $N(0, \beta, n)$ is given by

$$N(0, \beta, n) = `\chi'(\mathcal{M}_{\omega, \overline{X}}(0, \beta, n)).$$

- The invariant $L(\beta, n)$ is a counting invariant of $\mu_{i\omega}$-limit semistable objects $E \in \mathcal{A}(0)$, satisfying (cf. Definition 3.9,)

$$\mathrm{cl}(E) = (1, 0, -\beta, -n).$$

  If we denote the moduli space of such objects by $\mathcal{M}_{\lim}(1, 0, -\beta, -n)$, then $L(\beta, n)$ is given by

$$(54) \qquad L(\beta, n) = `\chi'(\mathcal{M}_{\lim}(1, 0, -\beta, -n)).$$

The precise definitions of $N(0, \beta, n)$ and $L(\beta, n)$ will be recalled in Definition 4.17 and Subsection 4.6 respectively. If the moduli space $\mathcal{M}_{\omega}(0, \beta, n)$ or $\mathcal{M}_{\lim}(1, 0, -\beta, -\omega)$ consists of only ω-Gieseker stable sheaves or $\mu_{i\omega}$-limit stable objects, then `$\chi'$ is the usual Euler characteristic of the moduli space. If there is a strictly semistable sheaves or objects, then the moduli space is an algebraic stack with possibly complicated stabilizers. In that case, we need to define `χ' so that the contributions of stabilizers are involved. This is worked out by Joyce [**23**] using the Hall algebra, which we discuss in the next subsection.

Remark 4.5. The invariants $N(0, \beta, n)$ and $L(0, \beta, n)$ are denoted by $N_{n,\beta}^{eu}$ and $L_{n,\beta}^{eu}$ in [**48**, Theorem 1.3] respectively.

4.3. Hall algebra of $\mathcal{A}_\omega$. In this subsection, we recall the notion of Hall algebra associated to $\mathcal{A}_\omega$. First Lieblich [**31**] constructs an algebraic stack $\mathcal{M}$ locally of finite type over $\mathbb{C}$, which parameterizes objects $E \in D^b \operatorname{Coh}(\overline{X})$ satisfying

$$\operatorname{Hom}_{\overline{X}}(E, E[i]) = 0, \ i < 0.$$

Then we have the substack,

(55) $\mathcal{O}bj(\mathcal{A}_\omega) \subset \mathcal{M},$

which parameterizes objects $E \in \mathcal{A}_\omega$. At this moment, we discuss under the assumption that $\mathcal{O}bj(\mathcal{A}_\omega)$ is also an algebraic stack locally of finite type. A necessary result will be given in Lemma 4.13 below.

Definition 4.6. We define the $\mathbb{Q}$-vector space $\mathcal{H}(\mathcal{A}_\omega)$ to be spanned by the isomorphism classes of symbols,

$$[\mathcal{X} \xrightarrow{f} \mathcal{O}bj(\mathcal{A}_\omega)],$$

where $\mathcal{X}$ is an algebraic stack of finite type over $\mathbb{C}$ with affine stabilizers, and f is a morphism of stacks. The relations are generated by

(56) $[\mathcal{X} \xrightarrow{f} \mathcal{O}bj(\mathcal{A}_\omega)] - [\mathcal{Y} \xrightarrow{f|_\mathcal{Y}} \mathcal{O}bj(\mathcal{A}_\omega)] - [\mathcal{U} \xrightarrow{f|_\mathcal{U}} \mathcal{O}bj(\mathcal{A}_\omega)],$

for a closed substack $\mathcal{Y} \subset \mathcal{X}$ and $\mathcal{U} = \mathcal{X} \setminus \mathcal{Y}$.

Here two symbols $[\mathcal{X}_i \xrightarrow{f_i} \mathcal{O}bj(\mathcal{A}_\omega)]$ for $i = 1, 2$ are isomorphic if there is an isomorphism $g \colon \mathcal{X}_1 \xrightarrow{\cong} \mathcal{X}_2$, which 2-commutes with f_i.

Let $\mathcal{E}x(\mathcal{A}_\omega)$ be the stack of short exact sequences in $\mathcal{A}_\omega$. There are morphisms of stacks,

$$p_i \colon \mathcal{E}x(\mathcal{A}_\omega) \to \mathcal{O}bj(\mathcal{A}_\omega),$$

for $i = 1, 2, 3$, sending a short exact sequence

$$0 \to E_1 \to E_2 \to E_3 \to 0$$

to the object E_i respectively.

There is an associative $*$-product on $\mathcal{H}(\mathcal{A}_\omega)$, defined by

(57) $[\mathcal{X} \xrightarrow{f} \mathcal{O}bj(\mathcal{A}_\omega)] * [\mathcal{Y} \xrightarrow{g} \mathcal{O}bj(\mathcal{A}_\omega)] = [\mathcal{Z} \xrightarrow{p_2 \circ h} \mathcal{O}bj(\mathcal{A}_\omega)],$

where $\mathcal{Z}$ and h fit into the Cartesian square,

$$
\begin{array}{ccc}
\mathcal{Z} \xrightarrow{\ \ h\ \ } \mathcal{E}x(\mathcal{A}_\omega) \xrightarrow{\ \ p_2\ \ } \mathcal{O}bj(\mathcal{A}_\omega) \\
\downarrow \qquad\qquad \downarrow{\scriptstyle (p_1, p_3)} \\
\mathcal{X} \times \mathcal{Y} \xrightarrow{f \times g} \mathcal{O}bj(\mathcal{A}_\omega)^{\times 2}.
\end{array}
$$

The above $*$-product is associative by [**20**, Theorem 5.2].

4.4. Invariants via Hall algebra. In this subsection, we construct counting invariants of $Z_{t\omega}$-semistable objects in $\mathcal{A}_\omega$ via the algebra $(\mathcal{H}(\mathcal{A}_\omega), *)$. For $v \in \Gamma$ with $\mathrm{rank}(v) \le 1$, let

$$(58) \qquad \mathcal{M}_{t\omega}(v) \subset \mathcal{O}bj(\mathcal{A}_\omega),$$

be the substack which parameterizes $Z_{t\omega}$-semistable objects $E \in \mathcal{A}_\omega$ with $\mathrm{cl}(E) = v$. For simplicity, we assume that (58) is an algebraic stack of finite type over $\mathbb{C}$. Again a necessary result will be given in Lemma 4.13. We can define the element in $\mathcal{H}(\mathcal{A}_\omega)$ to be

$$\delta_{t\omega}(v) := [\mathcal{M}_{t\omega}(v) \hookrightarrow \mathcal{O}bj(\mathcal{A}_\omega)] \in \mathcal{H}(\mathcal{A}_\omega).$$

The 'logarithm' of $\delta_{t\omega}(v)$ is defined as follows:

Definition 4.7. We define $\epsilon_{t\omega}(v) \in \mathcal{H}(\mathcal{A}_\omega)$ to be

$$(59) \qquad \epsilon_{t\omega}(v) := \sum_{\substack{l \ge 1, v_1 + \cdots + v_l = v, v_i \in \Gamma, \\ \arg Z_{t\omega}(v_i) = \arg Z_{t\omega}(v)}} \frac{(-1)^{l-1}}{l} \delta_{t\omega}(v_1) * \cdots * \delta_{t\omega}(v_l).$$

Note that the v_i in a non-zero term of the sum (59) satisfies $\mathrm{rank}(v_i) = 0$ or 1. Also we have the following lemma:

Lemma 4.8. *The sum (59) is a finite sum, hence $\epsilon_{t\omega}(v)$ is well-defined.*

Proof. The case of $\mathrm{rank}(v) = 0$ is essentially proved in [**44**, Lemma 5.12]. Suppose that $\mathrm{rank}(v) = 1$ and write $v = (1, r, \beta, n)$. Let $v_i \in \Gamma$ be an element which appears in a non-zero term of the sum (59). Then there is unique $1 \le e \le l$ such that $\mathrm{rank}(v_e) = 1$ and $\mathrm{rank}(v_i) = 0$ for $i \ne e$. We write $v_i = (0, r_i, \beta_i, n_i)$ for $i \ne e$. Since $0 < -\beta_i \cdot \omega \le -\beta \cdot \omega$, the number l in the sum (59) is bounded, and β_i^2 is bounded above by the Hodge index theorem. By the condition $\arg Z_{t\omega}(v_i) = \pi/2$, we have

$$(60) \qquad \mathrm{Re}\, Z_{t\omega}(v_i) = n_i - \frac{1}{2} r^2 t^2 \omega^2 = 0,$$

for $i \ne e$. Also there is an $Z_{t\omega,0}$-semistable object $E \in \mathcal{B}_\omega$ with $\mathrm{cl}(E_i) = v_i$, hence the same proof of Lemma 2.5 shows the inequality,

$$(61) \qquad \beta_i^2 + (\beta_i \cdot \omega)^2 \ge 2r_i(r_i + n_i),$$

for $i \ne e$. Since β_i^2 is bounded above, the equality (60) and the inequality (61) shows the boundedness of r_i and n_i. Also β_i^2 and $\beta_i \cdot \omega$ are bounded, hence there is only a finite number of possibilities for β_i. q.e.d.

There is a map, (cf. [**22**, Definition 2.1],)

$$(62) \qquad P_q : \mathcal{H}(\mathcal{A}_\omega) \to \mathbb{Q}(q^{1/2}),$$

such that if G is a special algebraic group (cf. [**22**, Definition 2.1]) acting on a variety Y, then we have

$$P_q\left(\left[[Y/G] \xrightarrow{f} \mathcal{O}bj(\mathcal{A}_\omega)\right]\right) = P_q(Y)/P_q(G),$$

where $P_q(Y)$ is the virtual Poincaré polynomial of Y. Namely if Y is smooth and projective, $P_q(Y)$ is given by

$$P_q(Y) = \sum_{i \in \mathbb{Z}} (-1)^i \dim H^i(Y, \mathbb{C}) q^{i/2},$$

and $P_q(Y)$ is defined for any Y using the relation (56) for varieties.

Theorem 4.9. [**22**, Theorem 6.2] *The following limit exists,*

$$\lim_{q^{1/2} \to 1} (q-1) P_q(\epsilon_{t\omega}(v)) \in \mathbb{Q}.$$

Using the above theorem, we can define the counting invariants. First we define the invariants of rank one objects.

Definition 4.10. For $v \in \Gamma_0$, we define $\mathrm{DT}^\chi_{t\omega}(v) \in \mathbb{Q}$ to be

(63) $\mathrm{DT}^\chi_{t\omega}(v) := \lim_{q^{1/2} \to 1} (q-1) P_q(\epsilon_{t\omega}(1, -v)).$

Here $(1, -v) \in \Gamma = \mathbb{Z} \oplus \Gamma_0$.

Remark 4.11. As we remarked in [**47**, Remark 4.10], if any object $E \in M_{t\omega}(1, -v)$ is $Z_{t\omega}$-stable, then the invariant (63) coincides with the Euler characteristic of the moduli space of objects in $M_{t\omega}(1, -v)$. However if there is a strictly semistable object $E \in M_{t\omega}(1, -v)$, then the stabilizer group $\mathrm{Aut}(E)$ contributes to the denominator of the invariant (63).

Remark 4.12. The change of the sign of v in (63) is to make the notation compatible with Chern characters of stable pairs (50).

4.5. Moduli stacks. So far we have assumed that the stacks $\mathcal{O}bj(\mathcal{A}_\omega)$ and $\mathcal{M}_{t\omega}(v)$ are algebraic stacks locally of finite type, finite type respectively. However these are too strong assumptions for our purpose. In fact it is enough to show the following lemma by discussing with the framework of Kontsevich-Soibelman [**29**, Section 3]. We remark that, the proof here is technical, and use some of the results which will be proved in later sections. The readers may skip the proof here at the first reading, and back after reading Sections 7, 9.

Lemma 4.13. *(i) The set of $\mathbb{C}$-valued points of the substack $\mathcal{O}bj(\mathcal{A}_\omega) \subset \mathcal{M}$ is a countable union of constructible subsets in $\mathcal{M}$.*
(ii) For $v \in \Gamma$ with $\mathrm{rank}(v) \leq 1$, the set of $\mathbb{C}$-valued points of the substack $\mathcal{M}_{t\omega}(v) \subset \mathcal{M}$ is a constructible subset in $\mathcal{M}$.

Proof. (i) We first note that the stack

(64) $$\mathcal{O}bj(\mathcal{B}_\omega) \subset \mathcal{M},$$

which parameterizes objects $E \in \mathcal{B}_\omega$ is an algebraic stack locally of finite type over $\mathbb{C}$. This result can be proved along with the same argument of [**44**, Lemma 4.7]. Moreover if $v \in \Gamma$ satisfies $\mathrm{rank}(v) = 0$, then the same proof of [**44**, Theorem 4.12] shows that the substack,

$$\mathcal{M}_{t\omega}(v) \subset \mathcal{O}bj(\mathcal{B}_\omega),$$

is an open substack of finite type over $\mathbb{C}$. Since any object $E \in \mathcal{B}_\omega$ has a Harder-Narasimhan filtration with respect to $Z_{t\omega,0}$-stability, the stack (64) is a countable union of constructible subsets in $\mathcal{M}$. Now we note that any object $E \in \mathcal{A}_\omega$ has a filtration with each subquotient isomorphic to either an object in $\mathcal{B}_\omega$ or in $\pi^* \mathrm{Pic}(\mathbb{P}^1)$. This fact easily implies that $\mathcal{O}bj(\mathcal{A}_\omega)$ is a countable union of constructible subsets in $\mathcal{M}$. (See the proof of [**50**, Lemma 3.2].)

(ii) Take an element $v \in \Gamma$. As we discussed in the proof of (i), the result for the case of $\mathrm{rank}(v) = 0$ is essentially proved in [**44**, Theorem 4.12]. Suppose that $\mathrm{rank}(v) = 1$, and we write it as

$$v = (1, r, \beta, n) \in \Gamma.$$

We show that $\mathcal{M}_{t\omega}(v)$ is a constructible subset in $\mathcal{M}$. Let $E \in \mathcal{A}_\omega$ be a $Z_{t\omega}$-semistable object with $\mathrm{cl}(E) = v$. By Lemma 7.4 below, there is a filtration in $\mathcal{A}_\omega$,

$$0 = E_0 \subset E_1 \subset E_2 \subset E_3 = E,$$

such that $K_i = E_i/E_{i-1}$ satisfies the condition (138). Also by the definition of $\mathcal{A}(r)$, the object $K_2 \in \mathcal{A}(r)$ has a filtration in $\mathcal{A}(r)$,

$$0 = K_{2,0} \subset K_{2,1} \subset K_{2,2} \subset K_{2,3} = K_2$$

such that $M_i = K_{2,i}/K_{2,i-1}$ satisfies

$$M_1 \in \mathrm{Coh}_\pi^{\leq 1}(\overline{X})[-1], \ M_2 \in \pi^* \mathrm{Pic}(\mathbb{P}^1), \ M_3 \in \mathrm{Coh}_\pi^{\leq 1}(\overline{X})[-1].$$

We note that the moduli stack of ω-Gieseker semistable sheaves $F \in \mathrm{Coh}_\pi(\overline{X})$ with fixed $\mathrm{cl}_0(F) \in \Gamma_0$ is an algebraic stack of finite type. Therefore it is enough to show that, for fixed $v \in \Gamma$, ample divisor ω and $t \in \mathbb{R}_{>0}$, there is only a finite number of possibilities for the numbers and numerical classes of Harder-Narasimhan factors of K_1, $K_3[1]$, $M_1[1]$ and $M_3[1]$. (See the proof of [**50**, Lemma 3.2].)

For simplicity we only show the above finiteness for K_1 and $M_1[1]$. The other cases are similarly discussed. We take Harder-Narasimhan filtrations of K_1, $M_1[1] \in \mathrm{Coh}_\pi(\overline{X})$ with respect to ω-Gieseker stability,

$$0 = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_k = K_1,$$
$$0 = B_0 \subset B_1 \subset B_2 \subset \cdots \subset B_l = M_1[1].$$

We set $C_i = A_i/A_{i-1}$ and $D_i = B_i/B_{i-1}$, and write

$$\mathrm{cl}_0(C_i) = (r_i, \beta_i, n_i), \ \mathrm{cl}_0(D_i) = (0, \beta_i', n_i').$$

Since $0 < -\omega \cdot \beta_i \leq -\omega \cdot \beta$ and $0 < \omega \cdot \beta_i' \leq -\omega \cdot \beta$ for $i \geq 2$, the numbers k and l are bounded. Moreover since $\beta_i' \geq 0$, there is only a finite number of possibilities for β_i'.

By the $Z_{t\omega}$-semistability of E, we have $\arg Z_{t\omega}(A_i) \leq \pi/2$, or equivalently

$$\sum_{j=1}^{i} \left(n_j - \frac{1}{2} r_j \omega^2 \right) \geq 0,$$

for all i. Hence by the result of Lemma 9.8 below, both of $r_1 + \cdots + r_i$ and $n_1 + \cdots + n_i$ are bounded. By the induction on i, we conclude that r_i and n_i are also bounded. Then noting that $0 \leq -\omega \cdot \beta_i \leq -\omega \cdot \beta$, Lemma 2.5 implies that β_i^2 is bounded, hence the Hodge index theorem implies that there is only a finite number of possibilities for β_i.

It remains to show the boundedness of n_i'. Again using the $Z_{t\omega}$-semistability of E, we have

$$\sum_{j=1}^{k} \left(n_j - \frac{1}{2} r_j \omega^2 \right) - \sum_{j=1}^{i} n_j' \geq 0,$$

for all i. Since k, l, r_j and n_j are bounded, we see that all n_i' are bounded above for all i. On the other hand an argument of [**47**, Lemma 3.2] shows that $M_1[1]$ is written as $\mathcal{O}_Z$ for a subscheme $Z \subset \overline{X}$ with $\dim Z \leq 1$. Therefore $\mathrm{ch}_3(M_1[1]) = \sum_{j=1}^{l} n_j'$ is bounded below by [**45**, Lemma 3.10]. Hence the boundedness of n_i' follows.   q.e.d.

4.6. Invariants $L(\beta, n)$**.** Let $L(\beta, n) \in \mathbb{Q}$ be the invariant discussed in Subsection 4.2. Here we recall the definition of $L(\beta, n)$ in [**48**, Definition 4.1], and compare it with the invariant $\mathrm{DT}_{t\omega}^{\chi}(r, \beta, n)$.

For $v = (R, r, \beta, n) \in \Gamma$ with $R \leq 1$, let

$$\mathcal{M}_{\mathrm{lim}}(v)$$

be the moduli stack of $\mu_{i\omega}$-limit semistable objects $E \in \mathcal{A}(r)$ with $\mathrm{cl}(E) = v$. (cf. Definition 3.9.) By Lemma 3.10 and [**48**, Proposition 3.17], the stack $\mathcal{M}_{\mathrm{lim}}(v)$ is an algebraic stack of finite type over $\mathbb{C}$. Hence we can define the element,

$$\delta_{\mathrm{lim}}(v) := [\mathcal{M}_{\mathrm{lim}}(v) \hookrightarrow \mathcal{O}bj\mathcal{A}_{\omega}] \in \mathcal{H}(\mathcal{A}_{\omega}),$$

and

$$\epsilon_{\mathrm{lim}}(v) := \sum_{\substack{l \geq 1, v_1, \cdots, v_l \in \Gamma, \\ 1 \leq e \leq l, v_i = (0,0,\beta_i,0), i \neq e, \\ v_1 + \cdots + v_l = v.}} \frac{(-1)^{l-1}}{l} \delta_{\mathrm{lim}}(v_1) * \cdots * \delta_{\mathrm{lim}}(v_l).$$

Then $L(\beta, n) \in \mathbb{Q}$ is defined by

(65) $$L(\beta, n) := \lim_{q^{1/2} \to 0} (q - 1) P_q(\epsilon_{\lim}(1, 0, -\beta, -n)).$$

Remark 4.14. We note that $L(\beta, n)$ is defined in the Hall algebra of $\mathcal{A}_{1/2}^p$ in the notation of [**48**]. However all the elements defining $\epsilon_{\lim}(v)$ are contained in the Hall algebra of $\mathcal{A}(0)^{\dagger}$, and since $\mathcal{A}(0)^{\dagger}$ is an extension closed subcategory of $\mathcal{A}_{1/2}^p$, the resulting invariant $L(\beta, n)$ coincides with the one defined in the Hall algebra of $\mathcal{A}_{1/2}^p$. (The notation in this remark will be recalled in Subsection 9.5.)

Remark 4.15. Noting Lemma 3.10, it is easy to check that the invariant $L(\beta, n)$ coincides with $L_{n,\beta}^{eu}$ introduced in [**48**, Definition 4.1].

We have the following proposition:

Proposition 4.16. *(i) For $t \gg 0$, we have*

$$\mathrm{DT}_{t\omega}^{\chi}(r, \beta, n) = L(\beta, n).$$

(ii) For $0 < t \ll 1$ and $n \neq 0$, we have

$$\mathrm{DT}_{t\omega}^{\chi}(r, \beta, n) = 0.$$

Proof. The result of (ii) is obvious from Proposition 3.8, and we prove (i) below. Let us take $v = (1, -r, -\beta, -n) \in \Gamma$. By Lemma 3.12 and (65), it follows that

(66) $$\lim_{q^{1/2} \to 1} (q - 1) P_q(\epsilon_{\lim}(1, -r, -\beta, -n)) = L(\beta, n).$$

Suppose that $\delta_{t\omega}(v_1) * \cdots * \delta_{t\omega}(v_l)$ appears as a non-zero term of (59). Then there is $1 \leq e \leq l$ such that $\mathrm{rank}(v_i) = 0$ for $i \neq e$ and $\mathrm{rank}(v_e) = 1$. By Proposition 3.11, we can take $t' > 0$ so that each $\delta_{t\omega}(v_i)$ coincides with $\delta_{\lim}(v_i)$ for $t > t'$. Also note that if $v_i = (0, r_i, \beta_i, n_i)$ satisfies $\delta_{\lim}(v_i) \neq 0$ and $\arg Z_{t\omega}(v_i) = \pi/2$, then we have $r_i = n_i = 0$. Therefore we have

(67) $$\epsilon_{t\omega}(1, -r, -\beta, -n) = \epsilon_{\lim}(1, -r, -\beta, -n),$$

for $t \gg 0$. Then the result of (i) follows from (67) and (66).   q.e.d.

4.7. Counting invariants of rank zero. Here we define invariants counting rank zero objects in $\mathcal{A}_{\omega}$ or $\mathcal{A}_{\omega}[1]$, and study their property. We set $C(\mathcal{B}_{\omega})$ as follows,

(68) $$C(\mathcal{B}_{\omega}) := \mathrm{Im}\,(\mathrm{cl}_0 \colon \mathcal{B}_{\omega} \to \Gamma_0)\,.$$

Definition 4.17. For $v \in \Gamma_0$, we define the invariant $N(v) \in \mathbb{Q}$ as follows.

• If $v \in C(\mathcal{B}_{\omega})$, then we define

(69) $$N(v) := \lim_{q^{1/2} \to 1} (q - 1) P_q(\epsilon_{\omega}(0, v)).$$

- If $-v \in C(\mathcal{B}_\omega)$, then we define $N(v) := N(-v)$.
- If $\pm v \notin C(\mathcal{B}_\omega)$, then we define $N(v) = 0$.

Remark 4.18. By Remark 3.5, the invariant (69) is also interpreted as a counting invariant of $Z_{\omega,0}$-semistable objects $E \in \mathcal{B}_\omega$ with $\mathrm{cl}_0(E) = v$. We also note that similar invariants on a K3 surface is already constructed and studied in [**44**].

Remark 4.19. By comparing with [**48**, Definition 4.1], the invariant of the form $N(0, \beta, n)$ in Definition 4.17 coincides with the one which appeared in the formula (53).

In defining (69), we need to choose an ample divisor ω. However it will turn out that $N(v)$ does not depend on a choice of ω. This fact follows from the same arguments of [**23**, Theorem 6.24], [**47**, Proposition-Definition 5.7] and [**44**, Theorem 1.2]. Below we explain this by introducing more general invariants counting Bridgeland semistable objects in $\mathcal{D}_0$, not necessary of the form $(Z_{\omega,0}, \mathcal{B}_\omega)$.

First we discuss the space of stability conditions on $\mathcal{D}_0$. Recall that in Lemma 3.3, we constructed stability conditions $(Z_{t\omega,0}, \mathcal{B}_\omega) \in \mathrm{Stab}_{\Gamma_0}(\mathcal{D}_0)$. These stability conditions are contained in a same connected component, which we denote by

$$\mathrm{Stab}_{\Gamma_0}^\circ(\mathcal{D}_0) \subset \mathrm{Stab}_{\Gamma_0}(\mathcal{D}_0).$$

Next let $\mathrm{Stab}(S)$ be the space of stability conditions on $D^b\mathrm{Coh}(S)$. In [**11**], Bridgeland describes a certain connected component of $\mathrm{Stab}(S)$, which we denote by

$$(70) \qquad\qquad \mathrm{Stab}^\circ(S) \subset \mathrm{Stab}(S).$$

The space of stability conditions on $\mathcal{D}_0$ and $D^b\mathrm{Coh}(S)$ are closely related. In fact, we have the following comparison result.

Theorem 4.20. *There is an isomorphism,*

$$(71) \qquad\qquad \psi\colon \mathrm{Stab}_{\Gamma_0}^\circ(\mathcal{D}_0) \xrightarrow{\sim} \mathrm{Stab}^\circ(S).$$

Proof. The result is essentially proved in [**46**, Theorem 6.5, Lemma 5.3].   q.e.d.

In the paper [**44**], the author constructed counting invariants of semistable objects in $D^b\mathrm{Coh}(S)$, motivated by Joyce's conjecture [**23**, Conjecture 6.25]. The construction itself relies on a choice of a stability condition in $\mathrm{Stab}^\circ(S)$, however it turned out that the invariant does not depend on a choice of a stability condition. Although the categories $D^b\mathrm{Coh}(S)$ and $\mathcal{D}_0$ are not equivalent, the arguments used for $D^b\mathrm{Coh}(S)$ in [**44**] is applied without any major modifications. A rough story of the arguments in [**44**] applied for $\mathcal{D}_0$ is as follows: for any element

$$\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}_{\Gamma_0}^\circ(\mathcal{D}_0),$$

we can define the invariant generalizing $N(v)$,

$$(72) \qquad\qquad N_\sigma(v) \in \mathbb{Q},$$

counting Z-semistable objects $E \in \mathcal{A}$ or $\mathcal{A}[1]$, satisfying $\mathrm{cl}_0(E) = v$. Namely we can similarly define the algebra $(H(\mathcal{A}), *)$, by replacing the stack $\mathcal{O}bj(\mathcal{A}_\omega)$ by $\mathcal{O}bj(\mathcal{A})$, the stack of objects $E \in \mathcal{A}$, in Definition 4.6. The stack of Z-semistable objects $E \in \mathcal{A}$ with $\mathrm{cl}_0(E) = v$ defines an element,

$$\delta_\sigma(v) := [\mathcal{M}_\sigma(v) \hookrightarrow \mathcal{O}bj(\mathcal{A})] \in \mathcal{H}(\mathcal{A}).$$

Also the element $\epsilon_\sigma(v) \in \mathcal{H}(\mathcal{A})$ can be defined in a way similar to (59), by replacing $\delta_{t\omega}(0, v)$ by $\delta_\sigma(v)$. The invariant (72) is defined by replacing $\mathcal{B}_\omega$, $\epsilon_{t\omega}(0, v)$ by $\mathcal{A}$, $\epsilon_\sigma(v)$ respectively in Definition 4.17. All the technical details in proving the existence of the invariant (72), e.g. the existence of moduli stacks, finiteness, etc. follow from the same arguments in [**44**]. Also if σ is a stability condition constructed in Lemma 3.3, then the invariant (72) coincides with the invariant defined in Definition 4.17. We have the following result.

Theorem 4.21. *The invariant $N_\sigma(v)$ does not depend on a choice of $\sigma \in \mathrm{Stab}^\circ_{\Gamma_0}(\mathcal{D}_0)$. In particular, the invariant $N(v)$ is independent of ω.*

Proof. The proof is same as in [**47**, Proposition-Definition 5.7], [**44**, Theorem 1.2], so we just give a sketch of the proof. We take two elements $\sigma_i \in \mathrm{Stab}^\circ_{\Gamma_\bullet}(\mathcal{D}_0)$ for $i = 0, 1$. We may assume that σ_1 is sufficiently close to σ_0. Then we can essentially apply the wall-crossing formula in an abelian category [**23**, Theorem 6.28], which describes $N_{\sigma_1}(v)$ in terms of $N_{\sigma_0}(v)$. The wall-crossing formula is described as

$$N_{\sigma_1}(v) = N_{\sigma_0}(v) + \sum_{v_1+v_2=v} a_{v_1,v_2} \chi(v_1, v_2) N_{\sigma_0}(v_1) N_{\sigma_0}(v_2) + \cdots,$$

for some $a_{v_1,v_2} \in \mathbb{Q}$ and χ is the Euler pairing on Γ_0 defined in Sub-section 2.7. All the other terms are also given by multiplications of $\chi(v_i, v_j)$, $N_{\sigma_0}(v_i)$ and some complicated coefficients. As we observed in Subsection 2.7, we have $\chi(v, v') = 0$ for $v, v' \in \Gamma_0$, all the error terms vanish, hence we have $N_{\sigma_1}(v) = N_{\sigma_0}(v)$.   q.e.d.

4.8. Sheaf counting invariants. As we discussed in the introduction, we are interested in the invariants counting semistable sheaves on the open Calabi-Yau 3-fold $X = S \times \mathbb{C}$. Let $\mathrm{Coh}_\pi(X)$ be the category given in (47). The stack

$$(73) \qquad\qquad \mathcal{C}oh_\pi(X),$$

which parameterizes objects in $\mathrm{Coh}_\pi(X)$ is known to be an algebraic stack locally of finite type over $\mathbb{C}$. By just replacing (55) by (73) in

Definition 4.6, we can define the $\mathbb{Q}$-vector space,

$$\mathcal{H}(\mathrm{Coh}_\pi(X)),$$

with a $*$-product similar to (57). Also for each $v \in \Gamma_0$, let

(74) $$\mathcal{M}_{\omega,X}(v) \subset \mathcal{C}oh_\pi(X),$$

be the substack of ω-Gieseker semistable sheaves $E \in \mathrm{Coh}_\pi(X)$ satisfying

$$v(E) = v.$$

Here $v(E)$ is the Mukai vector of E, defined by (25). The stack (74) is known to be an algebraic stack of finite type over $\mathbb{C}$.

Remark 4.22. Although we have constructed invariants (69) using the Chern character, the Mukai vector (not Chern character) is used in this subsection. The reason is that Mukai vector is useful in describing the automorphic property of the invariants. (See Theorem 4.31 below.)

The substack (74) defines an element,

$$\delta_{\omega,X}(v) := [\mathcal{M}_{\omega,X}(v) \hookrightarrow \mathcal{C}oh_\pi(X)] \in \mathcal{H}(\mathrm{Coh}_\pi(X)),$$

and its 'logarithm' defined by

(75) $$\epsilon_{\omega,X}(v) := \sum_{\substack{l \geq 1, v_1 + \cdots + v_l = v, v_i \in \Gamma_0 \\ \overline{\chi}_{\omega,v_i}(m) = \overline{\chi}_{\omega,v}(m)}} \frac{(-1)^{l-1}}{l} \delta_{\omega,X}(v_1) * \cdots * \delta_{\omega,X}(v_l).$$

Here $\overline{\chi}_{\omega,v}(m)$ is the reduced Hilbert polynomial (28). Similarly to Lemma 4.8, the sum (75) is a finite sum and $\epsilon_{\omega,X}(v)$ is well-defined. The argument is standard, so we omit the detail. We set $C(X)$ as follows,

(76) $$C(X) := \mathrm{Im}(v\colon \mathrm{Coh}_\pi(X) \to \Gamma_0).$$

We define the following sheaf counting invariant.

Definition 4.23. For $v \in \Gamma_0$, we define the invariant $J(v) \in \mathbb{Q}$ as follows.

- If $v \in C(X)$, we define

$$J(v) := \lim_{q^{1/2} \to 1} (q-1) P_q(\epsilon_{\omega,X}(v)).$$

- If $-v \in C(X)$, we define $J(v) := J(-v)$.
- If $\pm v \notin C(X)$, we define $J(v) = 0$.

Here the map

$$P_q\colon \mathcal{H}(\mathrm{Coh}_\pi(X)) \to \mathbb{Q}(q^{1/2}),$$

is defined similarly to (62). We note that a similar invariant on $\mathrm{Coh}(S)$, (not $\mathrm{Coh}_\pi(X)$,) is introduced and studied in [**23**]. Similarly to [**23**,

Theorem 6.24], the invariant $J(v)$ does not depend on a choice of ω. (Also see the proof of Theorem 4.21.)

In defining $J(v)$, we can also take ω to be an $\mathbb{R}$-ample divisor, and show that it does not depend on ω. If $v \in \Gamma_0$ is primitive and ω is in a general position in the ample cone, then the moduli stack $\mathcal{M}_{\omega,X}(v)$ is written as

$$\mathcal{M}_{\omega,X}(v) = [M_{\omega,S}(v)/\mathbb{C}^*] \times \mathbb{C},$$

where $M_{\omega,S}(v)$ is the moduli space of ω-Gieseker stable sheaves E on S satisfying $v(E) = v$, and $\mathbb{C}^*$ acts on $M_{\omega,S}(v)$ trivially. The space $M_{\omega,S}(v)$ is known to be a holomorphic symplectic manifold of dimension $(v,v)+2$, and deformation equivalent to the Hilbert scheme of $(v,v)/2+$ 1-points on S. (cf. [**52**, Theorem 0.2], [**27**, Theorem 5.151].) Therefore we have

$$J(v) = \chi(M_{\omega,S}(v))$$
$$(77) \qquad\qquad = \chi(\mathrm{Hilb}^{(v,v)/2+1}(S)),$$

where $\mathrm{Hilb}^n(S)$ is the Hilbert scheme of n-points in S. The RHS of (77) is given by the Göttsche's formula [**14**],

$$(78) \qquad \sum_{n \geq 0} \chi(\mathrm{Hilb}^n(S))q^n = \prod_{n \geq 1} \frac{1}{(1-q^n)^{24}}.$$

For a general $v \in \Gamma_0$, we will propose in Subsection 6.4 a conjectural relationship between $J(v)$ and $\chi(\mathrm{Hilb}^n(S))$ in terms of a multiple cover formula.

4.9. Comparison of $N(v)$ and $J(v)$. In [**44**, Theorem 6.6], we discussed a relationship between invariants counting semistable objects in $D^b \mathrm{Coh}(S)$ and invariants counting Gieseker semistable sheaves in $\mathrm{Coh}(S)$. A similar result is also obtained for counting invariants $N(v) \in \mathbb{Q}$ and counting invariants of Gieseker-semistable sheaves in $\mathrm{Coh}_\pi(\overline{X})$. Similarly to Definition 4.23, we can define the invariant,

$$(79) \qquad\qquad \overline{J}(v) \in \mathbb{Q},$$

counting ω-Gieseker semistable sheaves $E \in \mathrm{Coh}_\pi(\overline{X})$ with $v(E) = v \in \Gamma_0$. Namely we just replace X by $\overline{X}$ for all the ingredients in defining the invariant $J(v)$ in Definition 4.23. By the arguments similar to the proofs of [**23**, Theorem 6.24] and Theorem 4.21, we can show that $\overline{J}(v)$ does not depend on ω. We have the following result.

Theorem 4.24. *For any $v \in \Gamma_0$, we have*

$$(80) \qquad\qquad \overline{J}(v\sqrt{\mathrm{td}_S}) = N(v).$$

Proof. The proof is exactly same as in [**44**, Theorem 6.6], so we just give a sketch of the proof. For an element $v = (r, \beta, n)$, suppose that

$v \in C(X)$ where $C(X)$ is defined in (76). If $v \in C(X)$, then we can reduce the problem to the case of $\omega \cdot \beta > 0$ or $r = \beta = 0$. (See [**44**, Lemma 6.3] and the proof of [**44**, Theorem 6.6].) In these cases, the same arguments as in [**44**, Proposition 6.4], [**44**, Lemma 6.5] show that an object $E \in \mathcal{B}_\omega$ is $Z_{t\omega,0}$-semistable with $\mathrm{cl}_0(E) = v$ if and only if E is an ω-Gieseker semistable sheaf with $v(E) = v\sqrt{\mathrm{td}_S}$. This fact immediately implies the equality (80). A similar argument in the proof of [**44**, Theorem 6.6] also proves the case of $v \in -C(X)$ and $v \notin \pm C(X)$. We leave the readers to check the detail.   q.e.d.

Next we compare invariants $\overline{J}(v)$ with $J(v)$. By replacing X by $\overline{X}$ in Subsection 4.8, we have the element,

$$\delta_{\omega,\overline{X}}(v) = [\mathcal{M}_{\omega,\overline{X}}(v) \hookrightarrow \mathcal{C}oh_\pi(\overline{X})] \in \mathcal{H}(\mathrm{Coh}_\pi(\overline{X})),$$

where $\mathcal{M}_{\omega,\overline{X}}(v)$ is the moduli stack of ω-Gieseker semistable sheaves $E \in \mathrm{Coh}_\pi(\overline{X})$ with $\mathrm{cl}_0(E) = v$. For an open or closed subscheme $Z \subset \overline{X}$, we denote by $\mathcal{M}_{\omega,Z}(v) \subset \mathcal{M}_{\omega,\overline{X}}(v)$ the locus of $E \in \mathrm{Coh}_\pi(\overline{X})$ whose support is contained in Z. We set

$$\delta_{\omega,Z}(v) := [\mathcal{M}_{\omega,Z}(v) \hookrightarrow \mathcal{C}oh_\pi(\overline{X})] \in \mathcal{H}(\mathrm{Coh}_\pi(\overline{X})),$$

and define $\epsilon_{\omega,Z}(v) \in \mathcal{H}(\mathrm{Coh}_\pi(\overline{X}))$ just by replacing $\delta_{\omega,X}(v_i)$ by $\delta_{\omega,Z}(v_i)$ in (75). Also for $p \in \mathbb{P}^1$, we set

$$U_p := \overline{X} \setminus X_p.$$

We have the following lemma.

Lemma 4.25. *We have*

(81) $$\epsilon_{\omega,\overline{X}}(v) = \epsilon_{\omega,U_p}(v) + \epsilon_{\omega,X_p}(v).$$

Proof. In order to simplify the notation, we omit ω and write $\delta_{\omega,\overline{X}}(v)$ as $\delta_{\overline{X}}(v)$, etc. First we note that

$$\delta_{\overline{X}}(v) = \sum_{\substack{v_1,v_2 \in \Gamma_0, v_1+v_2=v, \\ \overline{\chi}_{\omega,v_1}(m)=\overline{\chi}_{\omega,v_2}(m)}} \delta_{U_p}(v_1) * \delta_{X_p}(v_2),$$

since any object $E \in \mathrm{Coh}_\pi(\overline{X})$ decomposes as $E_1 \oplus E_2$ with E_1 supported on U_p and E_2 supported on X_p. Since $\delta_{U_p}(v_1) * \delta_{X_p}(v_2) = \delta_{X_p}(v_2) *$

$\delta_{U_p}(v_1)$, we have

(82)

$\epsilon_{\overline{X}}(v) =$

$$\sum_{\substack{l \geq 1, v_i, v_i' \in \Gamma_0, \\ \overline{\chi}_{\omega, v_i}(m) = \overline{\chi}_{\omega, v}(m), \\ \overline{\chi}_{\omega, v_i'}(m) = \overline{\chi}_{\omega, v}(m), \\ v_1 + \cdots + v_l + v_1' + \cdots + v_l' = v}} \frac{(-1)^{l-1}}{l} \delta_{U_p}(v_1) * \cdots * \delta_{U_p}(v_l) * \delta_{X_p}(v_1') * \cdots * \delta_{X_p}(v_l').$$

Take $v_1, \cdots, v_a \in \Gamma_0$ and $v_1', \cdots, v_b' \in \Gamma_0$ with $v_i \neq 0$, $v_j' \neq 0$ for any i and j and satisfy

$$\overline{\chi}_{\omega, v_i}(m) = \overline{\chi}_{\omega, v_i'}(m) = \overline{\chi}_{\omega, v}(m),$$
$$v_1 + \cdots + v_a + v_1' + \cdots + v_b' = v.$$

If $a \geq 1$, $b \geq 1$ and $a \geq b$, then the coefficient of $\delta_{U_p}(v_1) * \cdots * \delta_{U_p}(v_a) * \delta_{X_p}(v_1') * \cdots * \delta_{X_p}(v_b')$ in (82) is

$$\sum_{l=a}^{a+b} \frac{(-1)^{l-1}}{l} \binom{l}{a} \binom{a}{a+b-l}$$
$$= \frac{(-1)^{a-1}}{b} \sum_{m=0}^{b} (-1)^m \binom{b}{m} \binom{m+a-1}{b-1}$$
$$= 0.$$

The last equality follows by taking the differentials of $x^{a-1}(x-1)^b$ by $(b-1)$-times, and substituting $x = 1$. We can similarly show the vanishing of the coefficient when $a \leq b$. Hence (81) follows. q.e.d.

We have the following lemma.

Lemma 4.26. *We have* $\overline{J}(v) = 2J(v)$.

Proof. The proof essentially follows from $\mathbb{C}^*$-localization for the invariants $\overline{J}(v)$ and $J(v)$. However a general localization formula for invariants defined via Hall algebra is not yet established. Here we give a proof assuming the terminology of [**24**].

As in the proof of Lemma 4.25, we omit ω in the notation. Let $M_{\overline{X}}(v)$ be the coarse moduli scheme of ω-Gieseker semistable sheaves $E \in \mathrm{Coh}_\pi(\overline{X})$ with $v(E) = v$. There is a natural morphism,

$$\eta \colon \mathcal{M}_{\overline{X}}(v) \to M_{\overline{X}}(v)$$

sending an ω-Gieseker semistable sheaf E to $\oplus_{i=1}^N F_i$, where $F_1, \cdots, F_N$ are ω-Gieseker stable factors of E. As in [**24**, Equation (5.9)], the

invariant $\overline{J}(v)$ can be also expressed as

$$\overline{J}(v) = \chi(M_{\overline{X}}(v), \alpha)$$
$$:= \sum_{m \in \mathbb{Z}} m \cdot \chi(\alpha^{-1}(m)),$$

for some constructible function α on $M_{\overline{X}}(v)$. In the notation of [24, Equation (5.9)], the function α is given by

(83) $$\alpha = \mathrm{CF}^{\mathrm{na}}(\eta)[\Pi_{\mathrm{CF}} \circ \overline{\Pi}^{\chi,\mathbb{Q}}_{\mathcal{M}_{\overline{X}}(v)}(\epsilon_{\overline{X}}(v))].$$

Let

$$M^{\dagger}_{\overline{X}}(v) \subset M_{\overline{X}}(v),$$

be the closed subscheme corresponding to semistable sheaves E such that $\mathrm{Supp}(E) \subset X_p$ for some $p \in \mathbb{P}^1$. Since we have the formula (81) for any $p \in \mathbb{P}^1$, the construction of α in (83) easily implies that α is zero outside $M^{\dagger}_{\overline{X}}(v)$. On the other hand, we have the natural isomorphism,

(84) $$M^{\dagger}_{\overline{X}}(v) \cong M_{X_0}(v) \times \mathbb{P}^1,$$

where $M_{X_0}(v) \subset M_{\overline{X}}(v)$ is the closed subscheme corresponding to sheaves E supported on X_0. Under the above isomorphism, we have

(85) $$\alpha|_{M_{X_0}(v) \times \{p\}} = \mathrm{CF}^{\mathrm{na}}(\eta)[\Pi_{\mathrm{CF}} \circ \overline{\Pi}^{\chi,\mathbb{Q}}_{\mathcal{M}_{\overline{X}}(v)}(\epsilon_{X_p}(v))],$$

in the notation of [24, Equation (5.9)] by Lemma 4.25.

Let $\mathrm{Coh}_{X_p}(\overline{X}) \subset \mathrm{Coh}(\overline{X})$ be the subcategory consisting of sheaves supported on X_p. For $p, q \in \mathbb{P}^1$, choose $g \in \mathrm{Aut}(\mathbb{P}^1)$ such that $g(p) = q$. Then g induces an equivalence

$$g_* \colon \mathrm{Coh}_{X_p}(\overline{X}) \overset{\sim}{\to} \mathrm{Coh}_{X_q}(\overline{X}),$$

and the induced isomorphism between the Hall algebras

(86) $$g_* \colon \mathcal{H}(\mathrm{Coh}_{X_p}(\overline{X})) \overset{\sim}{\to} \mathcal{H}(\mathrm{Coh}_{X_q}(\overline{X})).$$

The element $\epsilon_{X_p}(v)$ is regarded as an element of $\mathcal{H}(\mathrm{Coh}_{X_p}(\overline{X}))$, which is mapped to $\epsilon_{X_q}(v)$ by the isomorphism (86). Therefore by (85), we have $\alpha(x, p) = \alpha(x, q)$ for $x \in M_{X_0}(v)$ under the isomorphism (84). Hence we have

$$\overline{J}(v) = \chi(\mathbb{P}^1) \cdot \chi(M_{X_0}(v) \times \{0\}, \alpha).$$

Similarly we have

$$J(v) = \chi(\mathbb{C}) \cdot \chi(M_{X_0}(v) \times \{0\}, \alpha).$$

Since $\chi(\mathbb{P}^1) = 2$ and $\chi(\mathbb{C}) = 1$, we obtain the result. q.e.d.

We have the following corollaries:

Corollary 4.27. *For any $v \in \Gamma_0$, we have the following equality,*

$$J(v\sqrt{\mathrm{td}_S}) = \frac{1}{2}N(v).$$

Proof. The result follows by combining Theorem 4.24 and Lemma 4.26 below. q.e.d.

Corollary 4.28. *For $v = (r, \beta, n) \in \Gamma_0$ and an ample divisor ω on S, suppose that $\beta \cdot \omega \neq 0$. If $N(v) \neq 0$, then we have*

$$\beta^2 + 2(\beta \cdot \omega)^2 \geq 2r(r + n).$$

Moreover if $\beta \cdot \omega > 0$ and $rn \geq 0$, then we have $\beta > 0$.

Proof. If $N(v) \neq 0$, then Corollary 4.27 implies that there is an ω-Gieseker semistable sheaf E on $\overline{X}$ with $\mathrm{cl}_0(E) = (r, \beta, n)$ or $\mathrm{cl}_0(E) = -(r, \beta, n)$. Then the first statement follows Lemma 2.5. Suppose that $\beta \cdot \omega > 0$, and $rn \geq 0$. Let $E_1, \cdots, E_k$ be ω-Gieseker stable factors of E. If we write $\mathrm{cl}_0(E_i) = (r_i, \beta_i, n_i)$, then $\beta_i \cdot \omega > 0$, $r_i n_i \geq 0$. Applying the inequality (30) to each E_i, we see that $\beta_i^2 \geq -2$, hence $\beta_i > 0$ for all i by the Riemann-Roch theorem. Since β is a sum $\sum_i \beta_i$, we have $\beta > 0$. q.e.d.

4.10. Automorphic property of $J(v)$. In Subsection 4.8, we defined the invariant $J(v) \in \mathbb{Q}$. The invariant $J(v)$ is a counting invariant of ω-Gieseker semistable sheaves on the open Calabi-Yau 3-fold $X = S \times \mathbb{C}$. The purpose here is to observe that $J(v)$ has a certain automorphic property with respect to the group G,

$$(87) \qquad G := O_{\mathrm{Hodge}}(\widetilde{H}(S, \mathbb{Z}), (*, *))$$

consisting of isometries of the Mukai lattice $(\widetilde{H}(S, \mathbb{Z}), (*, *))$ preserving the Hodge structure on it. (See Subsection 2.1.) Note that any $g \in G$ induces an isometry of the lattice $(\Gamma_0, (*, *))$, since g preserves the Hodge structure on $\widetilde{H}(S, \mathbb{Z})$. Note that, in the previous subsection, we also defined the invariant $\overline{J}(v) \in \mathbb{Q}$ as a counting invariant of ω-Gieseker semistable sheaves on the compactification $\overline{X} = S \times \mathbb{P}^1$. Our strategy is to prove the automorphic property of $\overline{J}(v)$, and then use the result of Lemma 4.26.

The automorphic property of the invariants essentially follows by investigating the effect of the invariants under Fourier-Mukai transforms. For two K3 surfaces S, S', let Φ be a derived equivalence,

$$\Phi \colon D^b \mathrm{Coh}(S') \overset{\sim}{\to} D^b \mathrm{Coh}(S).$$

Recall that, by Orlov's theorem [38], any such an equivalence is written as

$$\Phi(-) \cong \mathbf{R}p_{2*}p_1^*(- \overset{\mathbf{L}}{\otimes} \mathcal{E}),$$

for some object $\mathcal{E} \in D^b \mathrm{Coh}(S' \times S)$, called the *kernel* of Φ. Here $p_1 \colon S' \times S \to S'$ and $p_2 \colon S' \times S \to S$ are projections. The equivalence Φ induces the Hodge isometry,

(88) $$\Phi_* \colon \widetilde{H}(S', \mathbb{Z}) \overset{\sim}{\to} \widetilde{H}(S, \mathbb{Z}),$$

given by

$$\Phi_*(-) = p_{2*}p_1^*(- \cdot \mathrm{ch}(\mathcal{E})\sqrt{\mathrm{td}_{S' \times S}}),$$

and we have the commutative diagram, (cf. [**36**, Theorem 4.9], [**38**, Proposition 3.5],)

(89)
$$
\begin{array}{ccc}
D^b \mathrm{Coh}(S') & \overset{\Phi}{\longrightarrow} & D^b \mathrm{Coh}(S) \\
\downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v} \\
\widetilde{H}(S', \mathbb{Z}) & \overset{\Phi_*}{\longrightarrow} & \widetilde{H}(S, \mathbb{Z}).
\end{array}
$$

Also the equivalence Φ induces the isomorphism,

$$\Phi_{\mathrm{St}} \colon \mathrm{Stab}(S') \overset{\sim}{\to} \mathrm{Stab}(S).$$

In order to distinguish the notation for S and S', we write $\mathcal{D}_0$, Γ_0 and $\overline{J}(v)$ as $\mathcal{D}_{0,S}$, $\Gamma_{0,S}$ and $\overline{J}_S(v)$ respectively. We have the following proposition.

Proposition 4.29. *In the above situation, suppose that Φ_{St} takes the connected component $\mathrm{Stab}^\circ(S')$ to $\mathrm{Stab}^\circ(S)$. Then for $v \in \Gamma_{0,S'}$, we have*

$$\overline{J}_{S'}(v) = \overline{J}_S(\Phi_* v).$$

Proof. Let $\mathcal{E}$ be the kernel of Φ, and $\overline{X}' := S' \times \mathbb{P}^1$. The equivalence Φ extends to the equivalence, (cf. [**39**, Assertion 1.7],)

$$\Phi^\dagger \colon D^b \mathrm{Coh}(\overline{X}') \overset{\sim}{\to} D^b \mathrm{Coh}(\overline{X}),$$

with kernel given by

$$\mathcal{E} \boxtimes \mathcal{O}_{\Delta_{\mathbb{P}^1}} \in D^b \mathrm{Coh}(S' \times S \times \mathbb{P}^1 \times \mathbb{P}^1).$$

Here we have identified $\overline{X}' \times \overline{X}$ with $S' \times S \times \mathbb{P}^1 \times \mathbb{P}^1$. It is easy to see that $\Phi^\dagger$ restricts to the equivalence between $\mathcal{D}_{0,S'}$ and $\mathcal{D}_{0,S}$. Also note that Φ_* in (88) restricts to the isomorphism between $\Gamma_{0,S'}$ and $\Gamma_{0,S}$, since Φ_* preserves the Hodge structures. Therefore by the diagram (89), we have the commutative diagram,

(90)
$$
\begin{array}{ccc}
\mathcal{D}_{0,S'} & \overset{\Phi^\dagger}{\longrightarrow} & \mathcal{D}_{0,S} \\
\downarrow{\scriptstyle \mathrm{cl}_0 \sqrt{\mathrm{td}_{S'}}} & & \downarrow{\scriptstyle \mathrm{cl}_0 \sqrt{\mathrm{td}_S}} \\
\Gamma_{0,S'} & \overset{\Phi_*}{\longrightarrow} & \Gamma_{0,S}.
\end{array}
$$

Also by the assumption and Theorem 4.20, the equivalence $\Phi^\dagger$ induces the isomorphism,

$$\Phi_{\mathrm{St}} \colon \mathrm{Stab}^\circ_{\Gamma_{0,S'}}(\mathcal{D}_{0,S'}) \xrightarrow{\sim} \mathrm{Stab}^\circ_{\Gamma_{0,S}}(\mathcal{D}_{0,S}).$$

Take $\sigma \in \mathrm{Stab}^\circ_{\Gamma_{0,S}}(\mathcal{D}_{0,S})$ and $\sigma' \in \mathrm{Stab}^\circ_{\Gamma_{0,S'}}(\mathcal{D}_{0,S'})$. Then for $v \in \Gamma_{0,S'}$, we have

$$(91) \qquad \overline{J}_S(\Phi_* v) = N_\sigma \left(\Phi_* v \cdot \sqrt{\mathrm{td}_S}^{-1} \right)$$

$$(92) \qquad = N_{\Phi_{\mathrm{St}}\sigma'} \left(\Phi_* v \cdot \sqrt{\mathrm{td}_S}^{-1} \right)$$

$$(93) \qquad = N_{\sigma'} \left(v \cdot \sqrt{\mathrm{td}_{S'}}^{-1} \right)$$

$$(94) \qquad = \overline{J}_{S'}(v).$$

Here (91) and (94) follow from Theorem 4.24, (92) follows from Theorem 4.21 and (93) follows from the commutative diagram (90). q.e.d.

Recall that we defined the group G in (87) to be the group of Hodge isometries of $\widetilde{H}(S, \mathbb{Z})$. We have the following corollary of Proposition 4.29.

Corollary 4.30. *For $v \in \Gamma_0$ and $g \in G$, we have*

$$(95) \qquad \overline{J}(gv) = \overline{J}(v).$$

Proof. For a K3 surface S, let $\mathrm{Auteq}^\circ(S)$ be the group of autoequivalences of $D^b\mathrm{Coh}(S)$, preserving the connected component $\mathrm{Stab}^\circ(S)$. Then the group homomorphism

$$\mathrm{Auteq}^\circ(S) \ni \Phi \mapsto \Phi_* \in G^+,$$

is surjective by [**18**, Corollary 4.10], [**16**, Proposition 7.9]. Here G^+ is the index two subgroup of G, consisting of $g \in G$ preserving the orientation of the positive definite four plane in $\widetilde{H}(S, \mathbb{R})$. Therefore (95) holds for $g \in G^+$ by Proposition 4.29.

Finally let $\iota \in G$ be the involution,

$$\iota = \mathrm{id}_{H^0(S,\mathbb{Z})} \oplus (-\mathrm{id}_{H^2(S,\mathbb{Z})}) \oplus \mathrm{id}_{H^4(S,\mathbb{Z})}.$$

The equality (95) for $g = \iota$ follows by applying the derived dual on $\mathcal{D}$. (Also see Proposition 9.5 below.) Since G/G^+ is generated by ι, we obtain the result. q.e.d.

By combining the above results, we have the following theorem.

Theorem 4.31. *For any $g \in G$, we have*

$$J(gv) = J(v).$$

Proof. The result follows by combining Corollary 4.30 and Lemma 4.26. q.e.d.

4.11. Invariants on $\mathcal{A}_\omega(1/2)$. In this subsection, we use notation introduced in Subsections 3.6, 3.7, 3.8. Recall that we constructed weak stability conditions $(\widehat{Z}_{\omega,\theta}, \mathcal{A}_\omega(1/2))$ in Lemma 3.15. Similarly to Subsection 4.4, we can construct counting invariants of $\widehat{Z}_{\omega,\theta}$-semistable objects in $\mathcal{A}_\omega(1/2)$. For $v \in \widehat{\Gamma}$, let

$$(96) \qquad \widehat{\mathcal{M}}_{\omega,\theta}(v),$$

be the moduli stack of $\widehat{Z}_{\omega,\theta}$-semistable objects $E \in \mathcal{A}_\omega(1/2)$ with $\widehat{\mathrm{cl}}(E) = v$. Similarly to the construction in Subsection 4.4, we define the element,

$$\widehat{\delta}_{\omega,\theta}(v) := [\widehat{\mathcal{M}}_{\omega,\theta}(v) \hookrightarrow \mathcal{O}bj(\mathcal{A}_\omega)] \in \mathcal{H}(\mathcal{A}_\omega).$$

We replace $v_i \in \Gamma$, $\delta_{t\omega}(v_i)$, $Z_{t\omega}$ in the sum (59) by $v_i \in \widehat{\Gamma}$, $\widehat{\delta}_{\omega,\theta}(v_i)$, $\widehat{Z}_{\omega,\theta}$ respectively. Then we can define the element

$$\widehat{\epsilon}_{\omega,\theta}(v) \in \mathcal{H}(\mathcal{A}_\omega),$$

for any $v \in \widehat{\Gamma}$, and the rank one invariant,

$$(97) \qquad \widehat{\mathrm{DT}}^\chi_{\omega,\theta}(v) := \lim_{q^{1/2}\to 1} (q-1)P_q(\widehat{\epsilon}_{\omega,\theta}(1,-v)),$$

for $v \in \widehat{\Gamma}_0$. Also we replace $\epsilon_{t\omega}(0,v)$, $C(\mathcal{B}_\omega)$ in Definition 4.17 by $\widehat{\epsilon}_{\omega,\theta}(0,v)$ and

$$(98) \qquad C(\mathcal{B}_\omega(1/2)) := \mathrm{Im}(\widehat{\mathrm{cl}}_0 \colon \mathcal{B}_\omega(1/2) \to \widehat{\Gamma}_0),$$

respectively. Then we have the rank zero invariant,

$$(99) \qquad \widehat{N}(v) \in \mathbb{Q},$$

counting $\widehat{Z}_{\omega,\theta}$-semistable objects $E \in \mathcal{A}_\omega(1/2)$ or $E \in \mathcal{A}_\omega(1/2)[1]$ satisfying $\widehat{\mathrm{cl}}(E) = (0,v)$. All the details in defining these invariants follow from the arguments in Subsection 4.4, so we omit the detail. Also an argument similar to the proof of Theorem 4.21 shows that $\widehat{N}(v)$ does not depend on ω and θ. The invariants (97), (99) are related to the invariants in Subsection 4.4 as follows.

Lemma 4.32. *For $v = (r,\beta) \in \widehat{\Gamma}_0$ and $0 < t \ll 1$, we have*

$$\widehat{\mathrm{DT}}^\chi_{\omega,1/2}(r,\beta) = \mathrm{DT}^\chi_{t\omega}(r,\beta,0),$$

$$\widehat{N}(r,\beta) = N(r,\beta,0).$$

Proof. The result follows from Proposition 3.17 (ii). q.e.d.

5. Wall-crossing formula

In this section, we apply the wall-crossing formula for the invariants introduced in the previous section, and give a proof of Theorem 1.1.

5.1. Joyce's formula. Joyce's wall-crossing formula [**23**, Theorem 6.28] enables us to see how the invariants $\mathrm{DT}^\chi_{t\omega}(v)$ vary if we change $t \in \mathbb{R}_{>0}$. In general, the wall-crossing formula is described in terms of Euler pairing on the (numerical) Grothendieck group of the underlying Calabi-Yau 3-fold. In our situation, the Euler pairing is not symmetric since $\overline{X}$ is not a Calabi-Yau 3-fold. Instead we use the bilinear pairing χ defined in Subsection 2.7. The existence of χ satisfying the condition (39) is enough to establish the wall-crossing formula.

If we apply the wall-crossing formula in [**23**, Equation (130)] to the invariants $\mathrm{DT}^\chi_{t\omega}(v)$, it immediately implies the following: for $t_1, t_2 > 0$ and $v \in \Gamma_0$, we have

$$\mathrm{DT}^\chi_{t_2\omega}(v) = \sum_{\substack{l \geq 1, 1 \leq e \leq l, v_i \in \Gamma_0 \\ v_1 + \cdots + v_l = v}} \sum_{\substack{G \text{ is a connected, simply connected} \\ \text{graph with vertex } \{1, \cdots, l\}, i \to j \text{ implies } i < j}}$$

$$(100) \quad \frac{1}{2^{l-1}} U(\{v_1', \cdots, v_l'\}, t_1, t_2) \prod_{i \to j \text{ in } G} \chi(v_i', v_j') \prod_{k \neq e} N(v_k) \, \mathrm{DT}^\chi_{t_1\omega}(v_e).$$

Here

$$v_i' = (0, -v_i) \text{ for } i \neq e, \quad v_e' = (1, -v_e),$$

and $U(\{v_1', \cdots, v_l'\}, t_1, t_2)$ is a certain rational number determined by the arguments of $\arg Z_{t_1\omega}(*)$ and $\arg Z_{t_2\omega}(*)$ in a combinatorial way. (cf. [**23**, Definition 4.4].) Note that a non-zero term of the RHS of (100) satisfies either $v_i' \in \Gamma_0$ or $v_j' \in \Gamma_0$, so the Euler pairing $\chi(v_i', v_j')$ makes sense. The central results in [**48**] and [**47**] provide explicit computations of the combinatorial coefficients in the RHS. The result is formulated in terms of the limiting generating series discussed in the next subsection.

5.2. Generating series. For an ample divisor ω on S and $t \in \mathbb{R}_{>0}$, we consider the following generating series,

$$\mathrm{DT}^\chi_{t\omega}(\overline{X}) := \sum_{(r, \beta, n) \in \Gamma_0} \mathrm{DT}^\chi_{t\omega}(r, \beta, n) x^r y^\beta z^n.$$

The series $\mathrm{DT}^\chi_{t\omega}(\overline{X})$ is an element of the following vector space,

$$\mathrm{DT}^\chi_{t\omega}(\overline{X}) \in \mathcal{R}_\omega := \prod_{\substack{\beta \in \mathrm{NS}(S), \\ \omega \cdot \beta \geq 0}} \mathbb{C} [\![x^{\pm 1}, z^{\pm 1}]\!] \, y^\beta.$$

The vector space $\mathcal{R}_\omega$ is a product of a countable number of copies of $\mathbb{C}$, and the Euclid topology on $\mathbb{C}$ induces a product topology on $\mathcal{R}_\omega$. By the existence of wall and chamber structure in Lemma 3.7, the following limiting series makes sense,

$$(101) \quad \lim_{t \to t_0 \pm 0} \mathrm{DT}^\chi_{t\omega}(\overline{X}) \in \mathcal{R}_\omega,$$

for any $t_0 \in \mathbb{R}_{>0}$.

On the other hand, there is no ring structure on $\mathcal{R}_\omega$, and we need to introduce a topological ring which acts on $\mathcal{R}_\omega$. We set

$$\mathcal{R}_0 := \prod_{\substack{\beta \in \mathrm{NS}(S), \\ \beta \geq 0}} \mathbb{C}[x^{\pm 1}, z^{\pm 1}]y^\beta.$$

Noting that the possible $\beta \geq 0$ with bounded $\omega \cdot \beta$ is finite, we have the natural product,

$$\mathcal{R}_0 \times \mathcal{R}_\omega \to \mathcal{R}_\omega,$$

which restricts to the ring structure on $\mathcal{R}_0$. By the same reason, the exponential for any $f \in \mathcal{R}_0$ also makes sense,

$$\exp(f) := \sum_{k \geq 0} \frac{1}{k!} f^k \in \mathcal{R}_0.$$

5.3. Wall-crossing formula of generating series. The wall-crossing formula [**23**, Theorem 6.28] describes a difference of the two limiting series (101). An argument used in [**47**, Theorem 5.8] yields the following result:

Theorem 5.1. *We have the following formula:*

$$\lim_{t \to t_0 + 0} \mathrm{DT}^\chi_{t\omega}(\overline{X})$$

(102)

$$= \prod_{\substack{\beta > 0, \\ n = \frac{1}{2} r t_0^2 \omega^2}} \exp\left((n + 2r)N(r, \beta, n)x^r y^\beta z^n \right)^{\epsilon(r)} \cdot \lim_{t \to t_0 - 0} \mathrm{DT}^\chi_{t\omega}(\overline{X}).$$

Here $\epsilon(r) = 1$ if $r > 0$, $\epsilon(r) = -1$ if $r < 0$ and $\epsilon(r) = 0$ if $r = 0$.

Proof. First we note that

$$\sum_{\substack{\beta > 0, \\ n = \frac{1}{2} r t_0^2 \omega^2}} \epsilon(r)(n + 2r)N(r, \beta, n)x^r y^\beta z^n \in \mathcal{R}_0,$$

by Lemma 4.28. Therefore the infinite product (102) makes sense by the argument in the previous subsection.

Next we note that the wall-crossing formula (100) describes the difference between two limiting series (101) in terms of χ and invariants of rank zero, i.e. $N(v) \in \mathbb{Q}$. Also the bilinear map χ restricts to zero on Γ_0, and this is exactly the same situation as in [**47**, Theorem 5.8], [**48**, Theorem 4.7]. Hence the same arguments are applied to our situation. More precisely, let W_{t_0} be the subset of Γ_0 defined by

$$W_{t_0} := \{v \in \Gamma_0 : Z_{t_0\omega,0}(v) \in \mathbb{R}_{>0}\sqrt{-1}\}.$$

Then W_{t_0} is written as $W_{t_0}^+ \cup W_{t_0}^- \cup W_{t_0}^0$,

$$W_{t_0}^+ := \left\{ (r, \beta, n) \in \Gamma_0 : n = \frac{1}{2} r t_0^2 \omega^2, \ r < 0, \omega \cdot \beta < 0 \right\},$$

$$W_{t_0}^- := \left\{ (r, \beta, n) \in \Gamma_0 : n = \frac{1}{2} r t_0^2 \omega^2, \ r > 0, \omega \cdot \beta < 0 \right\},$$

$$W_{t_0}^0 := \left\{ (r, \beta, n) \in \Gamma_0 : r = n = 0, \omega \cdot \beta < 0 \right\}.$$

For $v \in W_{t_0}^+$, we have

$$\arg Z_{(t_0 + \varepsilon)\omega, 0}(v) < \frac{\pi}{2} < \arg Z_{(t_0 - \varepsilon)\omega, 0}(v),$$

for $0 < \epsilon \ll 1$. The above inequalities are reversed for $v \in W_{t_0}^-$ and are equalities for $v \in W_{t_0}^0$. Also noting the formula (38) for χ, the arguments in [**47**, Theorem 5.8], [**48**, Theorem 4.7] imply

$$\lim_{t \to t_0 + 0} \mathrm{DT}_{t\omega}^\chi(\overline{X})$$

$$= \prod_{-(r,\beta,n) \in W_{t_0}^+} \exp\left((n + 2r) N(r, \beta, n) x^r y^\beta z^n \right)$$

$$\cdot \prod_{-(r,\beta,n) \in W_{t_0}^-} \exp\left((n + 2r) N(r, \beta, n) x^r y^\beta z^n \right)^{-1} \cdot \lim_{t \to t_0 - 0} \mathrm{DT}_{t\omega}^\chi(\overline{X})$$

$$= \prod_{-(r,\beta,n) \in W_{t_0}} \exp\left((n + 2r) N(r, \beta, n) x^r y^\beta z^n \right)^{\epsilon(r)} \cdot \lim_{t \to t_0 - 0} \mathrm{DT}_{t\omega}^\chi(\overline{X}).$$

If $-(r, \beta, n) \in W_{t_0}^+ \cup W_{t_0}^-$ satisfies $N(r, \beta, n) \neq 0$, then $\beta > 0$ follows from Corollary 4.28. Therefore we obtain the formula (5.1). q.e.d.

Let $L(\beta, n) \in \mathbb{Q}$ be the invariant, discussed in Subsection 4.2. By applying the wall-crossing formula from $t \to 0$ to $t \to \infty$, we obtain the following corollary.

Corollary 5.2. *We have the formula,*

$$\sum_{(r,\beta,n) \in \Gamma_0} L(\beta, n) x^r y^\beta z^n = \prod_{\beta > 0, rn > 0} \exp\left((n + 2r) N(r, \beta, n) x^r y^\beta z^n \right)^{\epsilon(r)}$$

$$\cdot \lim_{t \to 0} \sum_{(r,\beta,0) \in \Gamma_0} \mathrm{DT}_{t\omega}^\chi(r, \beta, 0) x^r y^\beta.$$

Proof. By Proposition 4.16, we have

$$\lim_{t \to \infty} \mathrm{DT}_{t\omega}^\chi(\overline{X}) = \sum_{(r,\beta,n) \in \Gamma_0} L(\beta, n) x^r y^\beta z^n.$$

On the other hand by Proposition 3.8, we have

$$\lim_{t \to 0} \mathrm{DT}_{t\omega}^{\chi}(\overline{X}) = \lim_{t \to 0} \sum_{(r,\beta,0)\in\Gamma_0} \mathrm{DT}_{t\omega}^{\chi}(r,\beta,0)x^r y^\beta.$$

Therefore applying the formula (102) from $0 < t \ll 1$ to $t \gg 1$, and using the same argument of [**47**, Corollary 5.11], we obtain the formula.

<div align="right">q.e.d.</div>

5.4. Wall-crossing in $\mathcal{A}_\omega(1/2)$. In this subsection, we use the notation given in Subsection 4.11. Our next step is to apply the wall-crossing formula in the subcategory $\mathcal{A}_\omega(1/2) \subset \mathcal{A}_\omega$ to prove a formula for the series $\lim_{t\to 0} \mathrm{DT}_{t\omega}^{\chi}(\overline{X})$. For $0 < \theta < 1$, we set

$$\widehat{\mathrm{DT}}_{\omega,\theta}^{\chi}(\overline{X}) := \sum_{(r,\beta)\in\widehat{\Gamma}_0} \widehat{\mathrm{DT}}_{\omega,\theta}^{\chi}(r,\beta)x^r y^\beta.$$

We note that

$$(103) \qquad \widehat{\mathrm{DT}}_{\omega,1/2}^{\chi}(\overline{X}) = \lim_{t \to 0} \sum_{(r,\beta,0)\in\Gamma_0} \mathrm{DT}_{t\omega}^{\chi}(r,\beta,0)x^r y^\beta$$

by Proposition 3.17 (iii). By the same arguments of Theorem 5.1 and Corollary 5.2, we obtain the following proposition.

Proposition 5.3. *We have the formula,*

$$(104) \qquad \widehat{\mathrm{DT}}_{\omega,1/2}^{\chi}(\overline{X}) = \prod_{r>0,\beta>0} \exp\left(2r\widehat{N}(r,\beta)x^r y^\beta\right) \cdot \sum_{r\in\mathbb{Z}} x^r.$$

Proof. Note that the bilinear map χ given in (37) restricts to a bilinear map on $\widehat{\Gamma} \times \widehat{\Gamma}_0$, given by

$$\chi((R,r,\beta),(r',\beta')) = 2Rr'.$$

The above bilinear map satisfies the same condition as in (39) for $E \in \mathcal{A}_\omega(1/2)$ and $F \in \mathcal{B}_\omega(1/2)$. Therefore the same argument of Theorem 5.1 shows that, for each $\theta_0 \in (0,1/2)$, we have

$$\lim_{\theta\to\theta_0+0} \widehat{\mathrm{DT}}_{\omega,\theta}^{\chi}(\overline{X})$$

$$(105) \qquad = \prod_{-(r,\beta)\in\widehat{W}_{\theta_0}} \exp\left(2r\widehat{N}(r,\beta)x^r y^\beta\right) \cdot \lim_{\theta\to\theta_0-0} \widehat{\mathrm{DT}}_{\omega,\theta}^{\chi}(\overline{X}).$$

Here $\widehat{W}_{\theta_0}$ is defined by

$$\widehat{W}_{\theta_0} := \{v \in \widehat{\Gamma}_0 : \widehat{Z}_{\omega,\theta_0,0}(v) \in \mathbb{R}_{>0}e^{i\pi\theta_0}\}.$$

For $(r,\beta) \in \widehat{\Gamma}_0 = \mathbb{Z} \oplus \mathrm{NS}(S)$, we have $-(r,\beta) \in \widehat{W}_{\theta_0}$ if and only if

$$r = \frac{\beta \cdot \omega}{\tan \pi\theta_0} > 0.$$

Also if $\widehat{N}(r,\beta) \neq 0$ in the formula (105), then $\beta > 0$ by Corollary 4.28 and Lemma 4.32. By applying the formula (105) from $\theta \to 0$ to $\theta \to 1/2$, we obtain

$$
\begin{aligned}
\lim_{\theta \to 1/2-0} &\widehat{\mathrm{DT}}^\chi_{\omega,\theta}(\overline{X}) \\
(106) \quad &= \prod_{r>0,\beta>0} \exp\left(2r\widehat{N}(r,\beta)x^r y^\beta\right) \cdot \lim_{\theta \to 0} \widehat{\mathrm{DT}}^\chi_{\omega,\theta}(\overline{X}).
\end{aligned}
$$

Hence the formula (104) follows from (106) and the following equalities,

$$
(107) \qquad \lim_{\theta \to 1/2-0} \widehat{\mathrm{DT}}^\chi_{\omega,\theta}(\overline{X}) = \widehat{\mathrm{DT}}^\chi_{\omega,1/2}(\overline{X}),
$$

$$
(108) \qquad \lim_{\theta \to 0} \widehat{\mathrm{DT}}^\chi_{\omega,\theta}(\overline{X}) = \sum_{r\in\mathbb{Z}} x^r.
$$

To see the equality (107), note that if $v = (r,\beta) \in \widehat{W}_{1/2}$, then $r = 0$ and $\chi((1,v'),v) = 0$ for any $v' \in \widehat{\Gamma}_0$. This implies that, by the formula given in [**23**, Theorem 6.28], there is no wall-crossing from $\theta \to 1/2 - 0$ to $\theta = 1/2$, and the generating series does not change.

Also note that Proposition 3.17 (iii) implies that

$$
\widehat{\mathcal{M}}_{\omega,\theta}(1,r,\beta) = \begin{cases} [\mathrm{Spec}\,\mathbb{C}/\mathbb{C}^*], & \text{if } \beta = 0, \\ \emptyset, & \text{if } \beta \neq 0, \end{cases}
$$

for $0 < \theta \ll 1$. Then the equality (108) follows from the definition of $\widehat{\mathrm{DT}}^\chi_{\omega,\theta}(r,\beta)$. \hfill q.e.d.

5.5. Generating series of stable pairs. Let $\mathrm{PT}^\chi(\overline{X})$ and $\mathrm{PT}^\chi(X)$ be the generating series of stable pair invariants, introduced in Subsection 4.1. By combining the results in the previous subsections, we prove formulas for these generating series.

Theorem 5.4. *We have the formula,*

$$
(109) \qquad \mathrm{PT}^\chi(\overline{X}) = \prod_{\beta>0,(r,n)\in\mathbb{S}} \exp\left((n+2r)N(r,\beta,n)y^\beta z^n\right)^{\epsilon(r+n)}.
$$

Here $\mathbb{S} \subset \mathbb{Z}^{\oplus 2}$ is given by

$$
\mathbb{S} := \{(r,n) \in \mathbb{Z}^{\oplus 2} : rn > 0 \text{ or } r = 0, n > 0, \text{ or } r > 0, n = 0\}.
$$

Proof. By Theorem 4.4, Corollary 5.2, the equality (103), Proposition 5.3 and Lemma 4.32, we obtain

$$
\mathrm{PT}^\chi(\overline{X}) \cdot \sum_{r\in\mathbb{Z}} x^r
$$

$$
= \prod_{\beta>0,(r,n)\in\mathbb{S}} \exp\left((n+2r)N(r,\beta,n)x^r y^\beta z^n\right)^{\epsilon(r+n)} \cdot \sum_{r\in\mathbb{Z}} x^r.
$$

By comparing the x^0-term, we obtain the formula (109). \hfill q.e.d.

Finally, we prove our main theorem which relates $\mathrm{PT}^\chi(X)$ to sheaf counting invariants $J(v) \in \mathbb{Q}$ for $v \in \Gamma_0$ introduced in Subsection 4.8.

Theorem 5.5. *We have the formula,*

$$\mathrm{PT}^\chi(X) = \prod_{r \geq 0, \beta > 0, n \geq 0} \exp\left((n + 2r)J(r, \beta, r + n)y^\beta z^n\right)$$

$$(110) \qquad\qquad \cdot \prod_{r > 0, \beta > 0, n > 0} \exp\left((n + 2r)J(r, \beta, r + n)y^\beta z^{-n}\right).$$

Proof. The formula (110) follows from Theorem 5.4, Lemma 4.3, Corollary 4.27, and noting that

$$J(-r, \beta, -n) = J(r, \beta, n),$$

by Theorem 4.31.   q.e.d.

6. Discussion toward Katz-Klemm-Vafa conjecture

In this section we discuss how Theorem 5.5 is related to the conjecture by Katz-Klemm-Vafa (KKV) [**26**].

6.1. KKV conjecture. Let S be a K3 surface, and $X = S \times \mathbb{C}$ as before. Let $\overline{M}_g(X, \beta)$ be the moduli stack of stable maps from genus g connected nodal curves to X with curve class $\beta \in \mathrm{NS}(S)$. Note that S has a holomorphic symplectic form, and there is a $\mathbb{C}^*$-action on X by multiplying the second factor. Therefore $\overline{M}_g(X, \beta)$ admits an equivariant reduced obstruction theory and an equivariant reduced virtual class, (see [**34**, Section 1],)

$$[\overline{M}_g(X, \beta)]^{\mathrm{red}} \in A_1^{\mathbb{C}^*}(\overline{M}_g(X, \beta), \mathbb{Q}).$$

Since $\overline{M}_g(X, \beta)^{\mathbb{C}^*}$ is compact, we can define the integration of the reduced virtual class by

$$\int_{[\overline{M}_g(X,\beta)]^{\mathrm{red}}} 1 := \int_{[\overline{M}_g(X,\beta)^{\mathbb{C}^*}]^{\mathrm{red}}} \frac{1}{e(\mathrm{Nor}^{\mathrm{vir}})} \in \mathbb{Q}(u),$$

where $\mathrm{Nor}^{\mathrm{vir}}$ is the virtual normal bundle of the embedding $\overline{M}_g(X, \beta)^{\mathbb{C}^*} \subset \overline{M}_g(X, \beta)$, and u is the equivariant parameter for the $\mathbb{C}^*$-action on X. The reduced GW invariant $R_{g,\beta} \in \mathbb{Q}$ is defined by

$$(111) \qquad\qquad R_{g,\beta} := \mathrm{Res}_{u=0} \int_{[\overline{M}_g(X,\beta)]^{\mathrm{red}}} 1.$$

The invariant (111) is unchanged under deformations of S which preserves β to be an algebraic class. The Gromov-Witten partition function

is

$$\mathrm{GW}(X) := \sum_{g \geq 0, \beta} R_{g,\beta} \lambda^{2g-2} y^{\beta}$$
$$= \sum_{\beta} \mathrm{GW}_{\beta}(X) y^{\beta},$$

where $\mathrm{GW}_{\beta}(X)$ is a series of λ. The BPS number $r_{g,\beta}$ is uniquely defined by the equation,

$$\mathrm{GW}(X) = \sum_{\beta,g,k} \frac{r_{g,\beta}}{k} \left(2 \sin \left(\frac{k\lambda}{2} \right) \right)^{2g-2} y^{k\beta}.$$

The following conjecture is a mathematical formulation of KKV conjecture [26] by Maulik-Pandharipande [33] in terms of reduced Gromov-Witten invariants.

Conjecture 6.1. [26, Section 6], [33, Conjecture 1, 2]

(i) The BPS count $r_{g,\beta}$ depends only on g and β^2. If $\beta^2 = 2h - 2$, then we set $r_{g,h} := r_{g,\beta}$.

(ii) The numbers $r_{g,h}$ are determined by the following equation,

$$(112) \qquad \sum_{g=0}^{\infty} \sum_{h=0}^{\infty} (-1)^g r_{g,h} \left(\sqrt{z} - \frac{1}{\sqrt{z}} \right)^{2g} q^{h-1} = \frac{1}{\Delta(z,q)},$$

where $\Delta(z,q)$ is

$$\Delta(z,q) = q \prod_{n=1}^{\infty} (1 - q^n)^{20} (1 - zq^n)^2 (1 - z^{-1}q^n)^2.$$

The following result is obtained by Maulik-Pandharipande-Thomas [34].

Theorem 6.2. [34, Theorem 1] *The invariants $r_{g,h}$ for primitive curve classes satisfy the equation (112).*

6.2. Reduced PT invariants. Similarly to the reduced GW theory, we can define the reduced PT invariants. Namely there is an equivariant reduced virtual class in dimension one, (cf. [34, Section 1],)

$$[P_n(X, \beta)]^{\mathrm{red}} \in A_1^{\mathbb{C}^*}(P_n(X, \beta), \mathbb{Z}),$$

and the reduced PT invariant $P_{n,\beta} \in \mathbb{Z}$ is defined by,

$$P_{n,\beta} := \mathrm{Res}_{u=0} \int_{[P_n(X,\beta)]^{\mathrm{red}}} 1.$$

The generating series is defined by,

$$\mathrm{PT}(X) := \sum_{\beta,n} P_{n,\beta} y^{\beta} z^n$$
$$= \sum_{\beta} \mathrm{PT}_{\beta}(X) y^{\beta},$$

where $\mathrm{PT}_\beta(X)$ is a series of z. If β is an irreducible curve class, then $P_{n,\beta}$ coincides with the Euler characteristic invariant,

$$(113) \qquad\qquad P_{n,\beta} = (-1)^{n-1}\chi(P_n(X,\beta)),$$

by [**34**, Lemma 8]. In this case, $P_n(X,\beta)$ depends only on n and the norm β^2 up to deformation equivalence. We write $P_n(X,\beta)$ as $P_n(X,h)$ when $\beta^2 = 2h-2$. The following result is given by Kawai-Yoshioka [**27**]. (Also see [**2**] for higher rank generalization.)

Theorem 6.3. [**27**, Theorem 5.80] *We have the formula,*

$$(114) \qquad \sum_{h=0}^{\infty}\sum_{n=1-h}^{\infty}\chi(P_n(X,h))z^n q^{h-1} = \left(\sqrt{z} - \frac{1}{\sqrt{z}}\right)^{-2}\frac{1}{\Delta(z,q)}.$$

Our formula (6) reconstructs the above result by Kawai-Yoshioka. In fact when β is irreducible and $n \geq 0$, the formula (6) implies that

$$\chi(P_n(X,\beta)) = \sum_{r\geq 0}(n+2r)J(r,\beta,r+n),$$

$$\chi(P_{-n}(X,\beta)) = \sum_{r>0}(n+2r)J(r,\beta,r+n).$$

The above formulas are nothing but specializations of [**27**, Equations (5.168), (5.170)] respectively. Using (77), we obtain

$$\chi(P_n(X,h)) = \sum_{r\geq 0}(n+2r)\chi(\mathrm{Hilb}^{h-r(r+n)}(S)),$$

$$\chi(P_{-n}(X,h)) = \sum_{r>0}(n+2r)\chi(\mathrm{Hilb}^{h-r(r+n)}(S)),$$

for $n \geq 0$. Together with some calculations involving Göttsche's formula (78), we obtain the formula (114). (See [**27**, Equations (5.171), (5.172), (5.173), (5.174)].) Note that Theorem 6.2 can be reduced to the case of irreducible curve classes by a deformation argument. Then Theorem 6.2 follows from Theorem 6.3, the formula (113) and the following reduced version of GW/PT correspondence.

Theorem 6.4. [**34**, Theorem 9] *Suppose that β is a primitive curve class. Then after the variable change $-e^{i\lambda} = z$, we have*

$$\mathrm{GW}_\beta(X) = \mathrm{PT}_\beta(X).$$

6.3. Speculation on KKV conjecture. As we discussed in the introduction, the strategy of the proof of Theorem 6.2 in [**34**] consists of two steps: prove reduced GW/PT correspondence and compute reduced PT theory. Suppose that we try to prove Conjecture 6.1 for arbitrary curve classes, along with the same strategy as in the case of primitive curve classes [**34**]. Then one might expect the following:

- The reduced GW/PT correspondence for arbitrary curve classes may hold. Namely we may have

$$\exp\left(\mathrm{GW}(X)\right) = \mathrm{PT}(X), \tag{115}$$

   by the variable change $-e^{i\lambda} = z$.
- The series $\mathrm{PT}(X)$ may be written as a similar product expansion to (6). For instance, looking at the equation (113), one may expect the following formula:

$$\mathrm{PT}(X) = \prod_{r\geq0,\beta>0,n\geq0} \exp\left((-1)^{n-1}(n+2r)J(r,\beta,r+n)y^{\beta}z^{n}\right)$$

$$\tag{116} \qquad \cdot \prod_{r>0,\beta>0,n>0} \exp\left((-1)^{n-1}(n+2r)J(r,\beta,r+n)y^{\beta}z^{-n}\right).$$

Although $J(v)$ does not involve the virtual cycle, it seems likely that $J(v)$ is invariant under deformations of S preserving v to be an algebraic class. (See Subsection 6.4 below.) Hence the formula (116) seems to make sense. At this moment we do not know whether (115), (116) hold or not. In particular it might be too strong to assume (116). However even if (116) is not true, a similar formula should be obtained if one could involve the reduced virtual cycles in the wall-crossing formula. Namely, for instance, suppose that we could relate reduced PT invariants to the weighted Euler characteristic with respect to the Behrend function [6]. Then by combining the argument in proving Theorem 1.1, work of Joyce-Song [24] and the announced result by Behrend-Getzler [7], it should be possible to prove a formula similar to (116), possibly by replacing $J(v)$ by another counting invariant which has similar properties to $J(v)$. The arguments below may also be applied after such an replacement. The following result reduces Conjecture 6.1 to the above expectations.

Theorem 6.5. *Suppose that the formulas (115) and (116) hold for any K3 surface S. Then Conjecture 6.1 is true. Furthermore we have the formula,*

$$\mathrm{PT}(X) = \prod_{r\geq0,\beta>0,n\geq0} (1+(-1)^{n-1}y^{\beta}z^{n})^{(n+2r)\chi(\mathrm{Hilb}^{\beta^{2}/2-r(n+r)+1}(S))}$$

$$\tag{117} \qquad \cdot \prod_{r>0,\beta>0,n>0} (1+(-1)^{n-1}y^{\beta}z^{-n})^{(n+2r)\chi(\mathrm{Hilb}^{\beta^{2}/2-r(n+r)+1}(S))}.$$

Proof. By a deformation argument as in [13, Section 4], [34, Section 2], we may assume that S is an elliptically fibered K3 surface $S \to \mathbb{P}^{1}$ with a section and $\mathrm{NS}(S)$ is rank two. Let

$$\mathbf{s}, \mathbf{f} \in \mathrm{NS}(S),$$

be the classes of the section and the elliptic fiber. Any $\beta \in \mathrm{NS}(S)$ is written as

$$(118) \qquad\qquad \beta = a\mathbf{s} + b\mathbf{f},$$

for some $a, b \in \mathbb{Z}$. Suppose that (115) and (116) hold. Then $\mathrm{PT}(X)$ can be described by the following Gopakumar-Vafa form, (cf. [**25**, Equation (18)],)

$$\mathrm{PT}(X) = \prod_{\beta > 0} \prod_{n=1}^{\infty} (1 + (-1)^{n-1} y^\beta z^n)^{n r_{0,\beta}}$$

$$(119) \qquad\qquad \cdot \prod_{g=1}^{\infty} \prod_{k=0}^{2g-2} (1 + (-1)^{g-k} y^\beta z^{g-1-k})^{(-1)^{g-k} r_{g,\beta} \binom{2g-2}{k}}.$$

By [**51**, Theorem 6.4], the series $\mathrm{PT}(X)$ is expressed by a Gopakumar-Vafa form (119) if and only we have the following multiple cover formula,

$$J(0, \beta, n) = \sum_{k \geq 1, k | (\beta, n)} \frac{1}{k^2} J(0, \beta/k, 1)$$

$$(120) \qquad\qquad = \sum_{k \geq 1, k | (\beta, n)} \frac{1}{k^2} \chi(\mathrm{Hilb}^{\beta^2/2k^2+1}(S)).$$

Here the second equality follows from (77). We claim that for any $v = (r, \beta, n) \in \Gamma_0$, we have the multiple cover formula,

$$(121) \qquad\qquad J(v) = \sum_{k \geq 1, k | v} \frac{1}{k^2} \chi(\mathrm{Hilb}^{(v/k, v/k)/2+1}(S)).$$

In order to prove (121), we write β as (118) for $a, b \in \mathbb{Z}$, and set

$$(r, a) = d(\overline{r}, \overline{a}),$$

where $d = \mathrm{GCD}(r, a) > 0$. By Theorem 4.31, we may assume that $r > 0$, hence $\overline{r} > 0$. Let $S' \to \mathbb{P}^1$ be the relative moduli space of stable sheaves on the fibers of the elliptic fibration $S \to \mathbb{P}^1$ with rank $\overline{r}$ and degree $\overline{a}$ on fibers. Then S' is also an elliptically fibred smooth K3 surface with a section, and we denote by $\mathbf{s}', \mathbf{f}' \in \mathrm{NS}(S')$ the classes of the section and the elliptic fiber. The universal sheaf on $S \times_{\mathbb{P}^1} S'$ induces a derived equivalence, (cf. [**38**, Theorem 3.11], [**9**, Theorem 5.3],)

$$\Phi \colon D^b \mathrm{Coh}(S') \overset{\sim}{\to} D^b \mathrm{Coh}(S).$$

As we will recall in Subsection 4.10, the equivalence Φ fits into a commutative diagram, (cf. [**36**], [**38**],)

$$
\begin{array}{ccc}
D^b \operatorname{Coh}(S') & \xrightarrow{\ \Phi\ } & D^b \operatorname{Coh}(S) \\
{\scriptstyle \operatorname{ch}\sqrt{\operatorname{td}_S}}\Big\downarrow & & \Big\downarrow{\scriptstyle \operatorname{ch}\sqrt{\operatorname{td}_S}} \\
\widetilde{H}(S',\mathbb{Z}) & \xrightarrow{\ \Phi_*\ } & \widetilde{H}(S,\mathbb{Z}),
\end{array}
$$

for an isomorphism Φ_*. By the construction of S' and Φ, we have

$$
\Phi_*^{-1}(\overline{r}, \overline{a}\mathbf{s}, 0) = (0, \mathbf{s}' + b'\mathbf{f}, m),
$$

for some $b', m \in \mathbb{Z}$. Also since

$$
\Phi_*(\mathbb{Z}[\mathbf{f}'] \oplus H^4(S',\mathbb{Z})) = \mathbb{Z}[\mathbf{f}] \oplus H^4(S,\mathbb{Z}),
$$

by the construction of Φ, it follows that

$$
\Phi_*^{-1}(r, \beta, n) = (0, \beta', n'),
$$

for some $\beta' \in \operatorname{NS}(S')$ and $n' \in \mathbb{Z}$. It is easy to see that Φ satisfies the assumption in Proposition 4.29 below. Hence combined with Lemma 4.26, we have

$$
J_S(r, \beta, n) = J_{S'}(0, \beta', n').
$$

Here we have written $J(v)$ as $J_S(v)$ in order to distinguish the invariants on S and S'. Then (121) follows from (120) for S'.

By the formula (121), we have

$$
\exp\left((-1)^{n-1}(n+2r)J(r,\beta,r+n)y^\beta z^n\right)
$$

$$
= \exp\left((-1)^{n-1}(n+2r)\sum_{\substack{k\ge 1,\\ k|(r,\beta,n)}} \frac{1}{k^2}\chi(\operatorname{Hilb}^{(r/k,\beta/k,r/k+n/k)^2/2+1}(S))y^\beta z^n\right)
$$

$$
= \exp\left(\sum_{k\ge 1} \frac{(-1)^{kn-1}}{k}(n+2r)\chi(\operatorname{Hilb}^{(r,\beta,r+n)^2/2+1}(S))y^{k\beta}z^{kn}\right)
$$

$$
= \left(1 + (-1)^{n-1}y^\beta z^n\right)^{(n+2r)\chi(\operatorname{Hilb}^{\beta^2/2-r(r+n)+1}(S))}.
$$

Therefore the formula (117) follows. Comparing (117) and (119), we see that $r_{g,\beta}$ depends only on g and β^2, i.e. Conjecture 6.1 (i) follows. Moreover the formula (117) implies that $r_{g,\beta} \ne 0$ only if $\beta^2 \ge -2$. If we write $\beta^2 = 2h - 2$ for $h \ge 0$, we have

$$
(\mathbf{s} + h\mathbf{f})^2 = \beta^2.
$$

Therefore the computation of $r_{g,\beta}$ can be reduced to the primitive case. Hence Conjecture 6.1 (ii) follows from Theorem 6.2. q.e.d.

6.4. Multiple cover formula. In the proof of Theorem 6.5, we have observed the following conjectural multiple cover formula:

Conjecture 6.6. *For $v \in \Gamma_0$, we have*

$$(122) \qquad J(v) = \sum_{k \geq 1, k|v} \frac{1}{k^2} \chi(\mathrm{Hilb}^{(v/k,v/k)/2+1}(S)).$$

The above conjecture also indicates that $J(v)$ is invariant under deformations of S preserving v to be an algebraic class. If we assume the formula (122), then the same computation in the proof of Theorem 6.5 shows that

$$\mathrm{PT}^\chi(X) = \prod_{r \geq 0, \beta > 0, n \geq 0} (1 - y^\beta z^n)^{-(n+2r)\chi(\mathrm{Hilb}^{\beta^2/2 - r(n+r)+1}(S))}$$

$$(123) \qquad \cdot \prod_{r > 0, \beta > 0, n > 0} (1 - y^\beta z^{-n})^{-(n+2r)\chi(\mathrm{Hilb}^{\beta^2/2 - r(n+r)+1}(S))}.$$

The formula (123) may be interpreted as an Euler characteristic version of KKV conjecture for stable pairs. Namely if we define $r'_{g,\beta}$ by the formula,

$$\mathrm{PT}^\chi(X) =$$

$$\prod_{\beta > 0} \prod_{n=1}^\infty (1 - y^\beta z^n)^{-n r'_{0,\beta}} \prod_{g=1}^\infty \prod_{k=0}^{2g-2} (1 - y^\beta z^{g-1-k})^{(-1)^{g-k-1} r'_{g,\beta} \binom{2g-2}{k}},$$

then $r'_{g,\beta}$ satisfies the same conditions in Conjecture 6.1. In what follows, we give some evidence of the conjectural formula (122) in some examples.

Lemma 6.7. *For $v = (0,0,n)$, we have*

$$J(0,0,n) = 24 \sum_{k \geq 1, k|n} \frac{1}{k^2}.$$

In particular the formula (122) holds.

Proof. Since $\chi(X) = 24$, the result follows from [**24**, Example 6.2], [**47**, Remark 5.14].   q.e.d.

Another evidence is as follows:

Lemma 6.8. *For $v = (r,0,r)$, we have*

$$J(r,0,r) = \frac{1}{r^2}.$$

In particular the formula (122) holds.

Proof. Let $E \in \mathrm{Coh}_\pi(X)$ be an ω-Gieseker semistable sheaf with $v(E) = (r, 0, r)$, and $E_1, \cdots, E_k$ be ω-Gieseker stable factors of E. By changing ω if necessary, we may assume that $v(E_i) = (r_i, 0, r_i)$ for some $r_i \in \mathbb{Z}$. Then Lemma 2.5 implies that $r_i = 1$, hence all the E_i is isomorphic to $\mathcal{O}_{X_p}$ for some $p \in \mathbb{C}$. By the localization argument given in Lemma 4.26 below, we may assume that $p = 0 \in \mathbb{C}$. Then $J(r, 0, r)$ is a counting invariant of objects in $\langle \mathcal{O}_{X_0} \rangle_{\mathrm{ex}}$, given by r-times extensions of $\mathcal{O}_{X_0}$. Noting that the category $\langle \mathcal{O}_{X_0} \rangle_{\mathrm{ex}}$ resembles the category of representations of a quiver with one vertex and one arrow, we can apply the same argument of [**24**, Example 7.27] to compute $J(r, 0, r)$. We leave the readers to check the detail. q.e.d.

Next we focus on the following situation. Let

$$S \to \mathbb{P}^2,$$

be a double cover branched along a general sextic. Let $H \in \mathrm{NS}(S)$ be a pull-back of a hyperplane in $\mathbb{P}^2$ to S. Note that $H^2 = 2$ and $\mathrm{NS}(S) = \mathbb{Z}[H]$. We have the following lemma.

Proposition 6.9. *In the above situation, take $v = (0, 2H, -2)$. Then we have*

(124) $$J(v) = 176337.$$

In particular the formula (122) holds.

Proof. Note that the RHS of (122) is

$$\chi(\mathrm{Hilb}^5(S)) + \frac{1}{4}\chi(\mathrm{Hilb}^2(S)) = 176256 + \frac{1}{4} \cdot 324$$
$$= 176337,$$

from the formula (78). We compute $J(v)$ directly from its definition, in the same way as in [**49**, Section 5]. In order to simplify the notation, we omit ω and X in the notation of Subsection 4.8.

We fist note that, since $v' := (0, H, -1)$ is primitive, the moduli stack $\mathcal{M}(v')$ is written as

$$\mathcal{M}(v') = [M(v')/\mathbb{C}^*],$$

for a holomorphic symplectic manifold $M(v')$ of dimension $(v', v') + 2 = 4$. Therefore $M(v')$ is deformation equivalent to $\mathrm{Hilb}^2(S)$ and

$$\chi(M(v')) = \chi(\mathrm{Hilb}^2(S))$$
$$= 324.$$

Next we observe that the moduli stack $\mathcal{M}(v)$ has a stratification,

$$\mathcal{M}(v)^{(0)} \sqcup \mathcal{M}(v)^{(1)} \sqcup \mathcal{M}(v)^{(2)} \sqcup \mathcal{M}(v)^{(3)} \sqcup \mathcal{M}(v)^{(4)},$$

where each $\mathcal{M}(v)^{(i)}$ is the following:

- $\mathcal{M}(v)^{(0)}$ corresponds to ω-Gieseker stable sheaves.

- $\mathcal{M}(v)^{(1)}$ corresponds to sheaves E which fits into a non-split exact sequence $0 \to E_1 \to E \to E_2 \to 0$ with $[E_i] \in M(v')$, and E_1 is not isomorphic to E_2.
- $\mathcal{M}(v)^{(2)}$ corresponds to sheaves E which is isomorphic to $E_1 \oplus E_2$ with $[E_i] \in M(v')$, and E_1 is not isomorphic to E_2.
- $\mathcal{M}(v)^{(3)}$ corresponds to sheaves E which fits into a non-split exact sequence $0 \to E' \to E \to E' \to 0$ with $[E'] \in M(v')$.
- $\mathcal{M}(v)^{(4)}$ corresponds to sheaves E which is isomorphic to $E'^{\oplus 2}$ with $[E'] \in M(v')$.

We compute the contributions of each strata to the invariant $J(v)$. First the strata $\mathcal{M}(v)^{(0)}$ is written as

$$\mathcal{M}(v)^{(0)} = [M(v)^{(0)}/\mathbb{C}^*],$$

for a smooth variety $M(v)^{(0)}$ of dimension $(v, v) + 2 = 10$, with a trivial $\mathbb{C}^*$-action. The variety $M(v)^{(0)}$ is birational to O'Grady's 10-dimensional symplectic manifold [**37**], and its Euler characteristic is computed by Mozgovoy [**35**, Subsection 4.3.1],

(125) $$\chi(M(v)^{(0)}) = 70956.$$

Next the contribution of $\mathcal{M}(v)^{(1)}$ to $\epsilon(v)$ is

$$\frac{1}{2} \bigcup_{\substack{(E_1,E_2)\in M(v')^{\times 2}, \\ E_1 \neq E_2}} \left[\left[\frac{\mathrm{Ext}_X^1(E_2, E_1) \setminus \{0\}}{\mathrm{Hom}(E_2, E_1) \rtimes (\mathbb{C}^*)^2} \right] \to \mathcal{C}oh_\pi(X) \right]$$

$$= \frac{1}{2} \bigcup_{\substack{(E_1,E_2)\in M(v')^{\times 2}, \\ E_1 \neq E_2}} \left[[\mathbb{P}^1/\mathbb{C}^*] \to \mathcal{C}oh_\pi(X) \right].$$

Applying $(q-1)P_t(*)$ and taking the limit $q^{1/2} \to 1$, the contribution to $J(v)$ is

(126)
$$\chi(M(v')^2) - \chi(M(v')) = 324^2 - 324$$
$$= 104652.$$

The contribution of $\mathcal{M}(v)^{(2)}$ to $J(v)$ can be shown to be zero by a similar argument of [**49**, Lemma 5.6]. The contribution of $\mathcal{M}(v)^{(3)}$ to $\epsilon(v)$ is

$$\frac{1}{2} \bigcup_{E' \in M(v')} \left[\left[\frac{\mathrm{Ext}_X^1(E', E') \setminus \{0\}}{\mathrm{Hom}(E', E') \rtimes (\mathbb{C}^*)^2} \right] \to \mathcal{C}oh_\pi(X) \right]$$

$$= \frac{1}{2} \bigcup_{E' \in M(v')} \left[\left[\frac{\mathbb{P}^4}{\mathbb{A}^1 \rtimes \mathbb{C}^*} \right] \to \mathcal{C}oh_\pi(X) \right].$$

Hence the contribution to $J(v)$ is

$$\frac{1}{2} \cdot 5 \cdot \chi(M(v')) = \frac{5}{2} \cdot 324$$

(127) $= 810.$

Finally the contribution of $\mathcal{M}(v)^{(4)}$ to $J(v)$ can be computed similarly to [**49**, Lemma 5.8],

$$-\frac{1}{4}\chi(M(v')) = -\frac{1}{4} \cdot 324$$

(128) $= -81.$

Summing up, we obtain

$$\begin{aligned}
J(v) &= (125) + (126) + (127) + (128) \\
&= 70956 + 104652 + 810 - 81 \\
&= 176337,
\end{aligned}$$

as expected.   q.e.d.

Remark 6.10. By Theorem 4.31, if $J(v)$ satisfies the formula (122), then $J(gv)$ for a Hodge isometry $g \in G$ also satisfies (122). In particular, if v is given in either Lemma 6.7 or Lemma 6.8 or Proposition 6.9, then $J(gv)$ satisfies (122) for any Hodge isometry g. For instance:

- The map $(r, \beta, n) \mapsto (n, \beta, r)$ is a Hodge isometry. In particular by Lemma 6.7, $J(n, 0, 0)$ also satisfies (122).
- The map $(r, \beta, n) \mapsto (-r, \beta, -n)$ is a Hodge isometry. In particular in the situation of Proposition 6.9, $J(0, 2H, 2)$ satisfies (122).
- For $v \in \Gamma_0$ with $(v, v) = -2$, the map on $\widetilde{H}(S, \mathbb{Z})$,

$$r_v \colon x \mapsto x + (x, v)v,$$

  is a Hodge isometry. In particular in the situation of Proposition 6.9, by applying r_v where v is

$$v = v(\mathcal{O}_S(H)) = (1, H, 2),$$

  the invariant $J(2, 0, -2)$ can be shown to satisfy (122).

7. Results on the category $\mathcal{A}_\omega$

In this section, we prove several properties on the category

$$\mathcal{A}_\omega = \langle \pi^* \operatorname{Pic}(\mathbb{P}^1), \mathcal{B}_\omega \rangle_{\mathrm{ex}} \subset \mathcal{D},$$

defined in Definition 2.8. Especially we will prove Lemma 7.3 which is used in the proof of Proposition 2.9.

7.1. Properties of $\mathcal{A}_\omega$. First we construct the heart of a certain bounded t-structure on $D^b \operatorname{Coh}(\overline{X})$. Let $\mathcal{F}'_\omega$ be the subcategory of $\operatorname{Coh}(\overline{X})$, defined by

$$\mathcal{F}'_\omega := \{E \in \operatorname{Coh}(\overline{X}) : \operatorname{Hom}(\mathcal{T}_\omega, E) = 0\}.$$

Here $\mathcal{T}_\omega$ is defined in (33). Since $\operatorname{Coh}(\overline{X})$ is noetherian, the pair $(\mathcal{T}_\omega, \mathcal{F}'_\omega)$ is a torsion pair on $\operatorname{Coh}(\overline{X})$. Also we have

(129) $$\mathcal{F}'_\omega \cap \operatorname{Coh}_\pi(\overline{X}) = \mathcal{F}_\omega,$$

where $\mathcal{F}_\omega$ is defined in (34).

Definition 7.1. We define $\mathcal{A}'_\omega$ to be

$$\mathcal{A}'_\omega := \langle \mathcal{F}'_\omega, \mathcal{T}_\omega[-1] \rangle_{\mathrm{ex}} \subset D^b \operatorname{Coh}(\overline{X}).$$

The category $\mathcal{A}'_\omega$ is the heart of a bounded t-structure on $D^b \operatorname{Coh}(\overline{X})$. It contains any line bundle on $\overline{X}$ and objects in $\mathcal{B}_\omega$.

Lemma 7.2. *(i) The subcategory $\mathcal{B}_\omega \subset \mathcal{A}'_\omega$ is closed under subobjects and quotients.*
(ii) We have

(130) $$\operatorname{Hom}(E, \pi^*\mathcal{O}_{\mathbb{P}^1}(r)) = 0,$$

for any $E \in \mathcal{B}_\omega$ and $r \in \mathbb{Z}$.
(iii) Any non-zero morphism $u \colon \pi^*\mathcal{O}_{\mathbb{P}^1}(r) \to \pi^*\mathcal{O}_{\mathbb{P}^1}(r')$ fits into an exact sequence in $\mathcal{A}_{\omega'}$,

(131) $$0 \to \pi^*\mathcal{O}_{\mathbb{P}^1}(r) \to \pi^*\mathcal{O}_{\mathbb{P}^1}(r') \to T \to 0,$$

with $T \cong \pi^*Q \in \mathcal{F}_\omega$ for a zero dimensional sheaf Q on $\mathbb{P}^1$.
(iv) For any morphism $u \colon \pi^*\mathcal{O}_{\mathbb{P}^1}(r) \to E$ with $E \in \mathcal{B}_\omega$, we have $\operatorname{Ker}(u) \in \pi^* \operatorname{Pic}(\mathbb{P}^1)$.

Proof. (i) Take an object $E \in \mathcal{B}_\omega$ and an exact sequence in $\mathcal{A}'_\omega$,

(132) $$0 \to F \to E \to G \to 0.$$

We need to show that $F, G \in \mathcal{B}_\omega$. By the definition of $\mathcal{A}'_\omega$, we have

(133) $$\mathcal{H}^1(F), \mathcal{H}^1(G) \in \mathcal{T}_\omega \subset \operatorname{Coh}_\pi(\overline{X}).$$

Also since $\mathcal{H}^0(F)$ is a subsheaf of $\mathcal{H}^0(E)$, we have $\mathcal{H}^0(F) \in \operatorname{Coh}_\pi(\overline{X})$. Hence by the long exact sequence of cohomologies associated to (132), we have $\mathcal{H}^0(G) \in \operatorname{Coh}_\pi(\overline{X})$. Since (129) holds, we have

(134) $$\mathcal{H}^0(F), \mathcal{H}^0(G) \in \mathcal{F}_\omega.$$

By (133) and (134), we conclude $F, G \in \mathcal{B}_\omega$.
(ii) By the definition of $\mathcal{B}_\omega$, we may assume that $E \in \mathcal{F}_\omega$ or $E \in \mathcal{T}_\omega[-1]$. If $E \in \mathcal{F}_\omega$, then (130) is obviously follows. Suppose that $E \in \mathcal{T}_\omega[-1]$. We may assume that, as in (29), E is isomorphic to

$i_{p*}E'[-1]$ for an ω-Gieseker stable sheaf E' on X_p for some $p \in \mathbb{P}^1$. We have

$$\operatorname{Hom}_{\overline{X}}(i_{p*}E'[-1], \pi^*\mathcal{O}_{\mathbb{P}^1}(r))$$
$$\cong \operatorname{Hom}_{X_p}(E', i_p^!\mathcal{O}_{\overline{X}}[1])$$
$$\cong \operatorname{Hom}_{X_p}(E', \mathcal{O}_{X_p}).$$

Since E' is μ_ω-stable sheaf on X_p with positive slope, we have the vanishing $\operatorname{Hom}_{X_p}(E', \mathcal{O}_{X_p}) = 0$.

(iii) If u is non-zero, then u is an injection of sheaves and the cokernel is written as π^*Q for a zero dimensional sheaf Q on $\mathbb{P}^1$. Since $\pi^*Q \in \mathcal{F}_\omega$, we have the exact sequence (131).

(iv) The morphism u factors through the subobject $\mathcal{H}^0(E) \subset E$ in $\mathcal{B}_\omega$. Let F be the image subsheaf of u in $\mathcal{H}^0(E)$,

$$\pi^*\mathcal{O}_{\mathbb{P}^1}(r) \xrightarrow{j} F \hookrightarrow \mathcal{H}^0(E).$$

Since $\mathcal{H}^0(E) \in \mathcal{F}_\omega$ and F is a subsheaf of $\mathcal{H}^0(E)$, we have $\mu_{\omega+}(F) \leq 0$. On the other hand, the surjection j factors through the surjection

$$\pi^*\mathcal{O}_{\mathbb{P}^1}(r) \twoheadrightarrow \pi^*\mathcal{O}_W \twoheadrightarrow F,$$

for some zero dimensional subscheme $W \subset \mathbb{P}^1$. This implies that $\mu_{\omega-}(F) \geq 0$, and hence F is μ_ω-semistable with $\mu_\omega(F) = 0$. If F is μ_ω-stable, the surjection $\pi^*\mathcal{O}_W \twoheadrightarrow F$ implies that $F \cong \mathcal{O}_{X_p}$ for some $p \in \mathbb{P}^1$ and the kernel of j is isomorphic to $\pi^*\mathcal{O}_{\mathbb{P}^1}(r-1)$. In general by the induction on the number of Jordan-Hölder factors of F, we can easily see that any Jordan-Hölder factor of F is isomorphic to $\mathcal{O}_{X_p}$ for some $p \in \mathbb{P}^1$ and the kernel of j is isomorphic to $\pi^*\mathcal{O}_{\mathbb{P}^1}(r')$ for some $r' \in \mathbb{Z}$. q.e.d.

Lemma 7.3. *Take $E, E' \in \mathcal{A}_\omega$ and a morphism $u \colon E \to E'$ in $\mathcal{A}'_\omega$. Then we have*

(135) $$\operatorname{Ker}(u), \ \operatorname{Cok}(u) \in \mathcal{A}_\omega.$$

Here the kernel and the cokernel are taken in the abelian category $\mathcal{A}'_\omega$.

We divide the proof into 2 steps.

Step 1. We have (135) when $E' \in \mathcal{B}_\omega$ or $E' \in \pi^* \operatorname{Pic}(\mathbb{P}^1)$.

Proof. We show the case of $E' \in \mathcal{B}_\omega$. The other case is similarly discussed. By the definition of $\mathcal{A}_\omega$, there is a filtration in $\mathcal{A}'_\omega$,

(136) $$0 = E_0 \subset E_1 \subset \cdots \subset E_N = E,$$

such that each $F_i = E_i/E_{i-1}$ is either an object in $\mathcal{B}_\omega$ or of the form $\pi^*\mathcal{O}_{\mathbb{P}^1}(r)$ for some $r \in \mathbb{Z}$. We prove (135) by the induction on N. If

$N = 1$, then (135) follows from Lemma 7.2. Suppose that (135) holds for $E = F'$, and take an exact sequence in $\mathcal{A}'_\omega$,

$$0 \to F'' \to F \to F' \to 0,$$

with F'' an object in either $\mathcal{B}_\omega$ or $\pi^* \operatorname{Pic}(\mathbb{P}^1)$. For a morphism $u \colon F \to E'$, let A be the image of the composition

$$F'' \to F \to E',$$

in $\mathcal{A}'_\omega$. By setting $B = E'/A$ in $\mathcal{A}'_\omega$, we obtain the commutative diagram of exact sequences in $\mathcal{A}'_\omega$,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F'' & \longrightarrow & F & \longrightarrow & F' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u''} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u'} & & \\
0 & \longrightarrow & A & \longrightarrow & E' & \longrightarrow & B & \longrightarrow & 0.
\end{array}
$$

Note that u'' is surjective in $\mathcal{A}'_\omega$, and $A, B \in \mathcal{B}_\omega$ by Lemma 7.2 (i). By the assumption of the induction, we have

$$\operatorname{Ker}(u''), \ \operatorname{Ker}(u'), \ \operatorname{Cok}(u') \in \mathcal{A}_\omega.$$

Therefore (135) holds for $u \colon F \to E'$ by the snake lemma.   q.e.d.

Step 2. We have (135) for any $E' \in \mathcal{A}_\omega$.

Proof. We take a N-step filtration of E' as in (136) and prove (135) by the induction on N. The case of $N = 1$ is proved in Step 1. Suppose that $N \geq 2$ and take an exact sequence in $\mathcal{A}'_\omega$,

$$0 \to A \to E' \to B \to 0,$$

with $A \in \mathcal{A}_\omega$ and B is either an object in $\mathcal{B}_\omega$ or in $\pi^* \operatorname{Pic}(\mathbb{P}^1)$. Let D be image of the composition in $\mathcal{A}'_\omega$,

$$E \xrightarrow{u} E' \to B.$$

We also denote its kernel in $\mathcal{A}'_\omega$ by C. By Step 1, we have $C \in \mathcal{A}_\omega$. We have the morphism of the exact sequences in $\mathcal{A}'_\omega$,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C & \longrightarrow & E & \longrightarrow & D & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u''} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u'} & & \\
0 & \longrightarrow & A & \longrightarrow & E' & \longrightarrow & B & \longrightarrow & 0.
\end{array}
$$

Note that u' is injective in $\mathcal{A}'_\omega$. Similarly to Step 1, (135) follows from the inductive assumption, Step 1 and the snake lemma.   q.e.d.

Now we have proved Lemma 7.3, so the proof of Proposition 2.9 is completed. In particular $\mathcal{A}_\omega$ is an abelian category, and we use this fact in what follows.

7.2. Filtrations in $\mathcal{A}_\omega$. In this subsection, we collect some results which will be used in later sections. In particular, the results here will be used in proving Proposition 3.8 in Subsection 9.3, and proving Proposition 3.11 in Subsection 9.6. Here we use the notation in Subsections 2.5, 2.6, 2.8. Let $\mathcal{T}_\omega^{\mathrm{pure}}$ be the following subcategory of $\mathcal{T}_\omega$,

$$\mathcal{T}_\omega^{\mathrm{pure}} := \{E \in \mathcal{T}_\omega : E \text{ is pure two dimensional }\} \cup \{0\}.$$

Note that $\mathcal{T}_\omega^{\mathrm{pure}}$ is a right orthogonal complement of $\mathrm{Coh}_{\overline{\pi}}^{\leq 1}(\overline{X})$ in $\mathcal{T}_\omega$.

Lemma 7.4. *For any object $E \in \mathcal{A}_\omega$ with $\mathrm{rank}(E) \leq 1$, there is a filtration in $\mathcal{A}_\omega$,*

$$(137) \qquad\qquad E_1 \subset E_2 \subset E_3 = E,$$

such that we have
$$(138)$$
$$K_1 := E_1 \in \mathcal{F}_\omega, \ \ K_2 := E_2/E_1 \in \mathcal{A}(r), \ \ K_3 := E/E_2 \in \mathcal{T}_\omega^{\mathrm{pure}}[-1],$$

for some $r \in \mathbb{Z}$. If $\mathrm{rank}(E) = 0$, we can take $K_2 \in \mathrm{Coh}_{\overline{\pi}}^{\leq 1}(\overline{X})[-1]$.

Proof. When $\mathrm{rank}(E) = 0$, then $E \in \mathcal{B}_\omega$ and the statement is obvious by the definition of $\mathcal{B}_\omega$. Suppose that $\mathrm{rank}(E) = 1$. Because $E \in \mathcal{A}'_\omega$ and $\mathcal{A}'_\omega$ is obtained as a tilting of the torsion pair $(\mathcal{T}_\omega, \mathcal{F}'_\omega)$, (cf. Definition 7.1,) we can find a filtration in $\mathcal{A}'_\omega$,

$$(139) \qquad\qquad E'_1 \subset E'_2 \subset E'_3 = E$$

satisfying

$$(140) \quad E'_1 = \mathcal{H}^0(E)_{\mathrm{tor}}, \quad E'_2/E'_1 = \mathcal{H}^0(E)_{\mathrm{fr}}, \quad E/E'_2 = \mathcal{H}^1(E)[-1].$$

Here $\mathcal{H}^0(E)_{\mathrm{tor}}$ is the maximal torsion subsheaf of $\mathcal{H}^0(E)$, and $\mathcal{H}^0(E)_{\mathrm{fr}} := \mathcal{H}^0(E)/\mathcal{H}^0(E)_{\mathrm{tor}}$. Let F be a torsion sheaf on $\overline{X}$ whose support is irreducible, and not contained in fibers of π. Then by the definition of $\mathcal{A}_\omega$, it follows that

$$\mathrm{Hom}(F, \mathcal{H}^0(E)_{\mathrm{tor}}) \subset \mathrm{Hom}(F, E) = 0.$$

Therefore $\mathcal{H}^0(E)_{\mathrm{tor}} \in \mathrm{Coh}_\pi(\overline{X})$, hence $E'_1 \in \mathcal{F}_\omega$ by (129). As for E'_2/E'_1, because $\mathcal{H}^0(E)_{\mathrm{fr}}$ is a torsion free sheaf of rank one, it is written as

$$(141) \qquad\qquad \mathcal{H}^0(E)_{\mathrm{fr}} \cong L \otimes I_Z,$$

for a line bundle L on $\overline{X}$ and a subscheme $Z \subset \overline{X}$ with $\dim Z \leq 1$. Since $E \in \mathcal{A}_\omega$, the definition of $\mathcal{A}_\omega$ yields that $L \in \pi^* \mathrm{Pic}(\mathbb{P}^1)$ and Z is supported on fibers of π. Therefore we have $E'_2/E'_1 \in \mathcal{A}(r)$ for some $r \in \mathbb{Z}$. Finally by the definition of $\mathcal{A}'_\omega$, we have $\mathcal{H}^1(E) \in \mathcal{T}_\omega$. By combining the filtration (139) with the exact sequence in $\mathcal{A}_\omega$,

$$0 \to T_1[-1] \to E/E'_2 \to T_2[-1] \to 0,$$

where $T_1 \in \mathrm{Coh}_{\overline{\pi}}^{\leq 1}(\overline{X})$ and $T_2 \in \mathcal{T}_\omega^{\mathrm{pure}}$, we obtain a desired filtration (137). q.e.d.

Another lemma we need is the following.

Lemma 7.5. *For any object $E \in \mathcal{A}_\omega$, there is an exact sequence in $\mathcal{A}_\omega$,*

(142) $$0 \to A \to E \to B \to 0,$$

such that $A \in \mathcal{B}_\omega$ and $B \in \langle \pi^* \operatorname{Pic}(\mathbb{P}^1) \rangle_{\mathrm{ex}}$.

Proof. Take an object $E \in \mathcal{A}_\omega$. If $\operatorname{rank}(E) = 0$, then $E \in \mathcal{B}_\omega$ and the result is satisfied with $B = 0$. If $\operatorname{rank}(E) > 0$, then E is written as a successive extensions of rank one objects. Hence we may assume that $\operatorname{rank}(E) = 1$.

Suppose that $\operatorname{rank}(E) = 1$. Below we use the notation in the proof of Lemma 7.4. As in (139), we can take a filtration $E'_\bullet$ of E satisfying the condition (140). As in (141), the object E'_2/E'_1 is isomorphic to $L \otimes I_Z$ for $L \in \pi^* \operatorname{Pic}(\mathbb{P}^1)$ and $Z \subset \overline{X}$ with $\dim Z \le 1$, contained in fibers of π. By combining the filtration (139) with an exact sequence in $\mathcal{A}_\omega$,

$$0 \to L \otimes \mathcal{O}_Z[-1] \to L \otimes I_Z \to L \to 0,$$

we obtain a filtration

(143) $$E''_1 \subset E''_2 \subset E''_3 = E,$$

satisfying

$$E''_1 \in \mathcal{B}_\omega, \quad E''_2/E''_1 \cong \pi^* \mathcal{O}_{\mathbb{P}^1}(r), \quad E/E''_2 \in \mathcal{T}_\omega[-1].$$

We write $E/E''_2 = A[-1]$ for $A \in \mathcal{T}_\omega$, and take a filtration

$$0 = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_N = A,$$

such that each subquotient $B_i = A_i/A_{i-1}$ is ω-Gieseker stable with

$$\overline{\chi}_{B_1,\omega}(m) \succeq \overline{\chi}_{B_2,\omega}(m) \succeq \cdots \succeq \overline{\chi}_{B_i,\omega}(m) \succeq \overline{\chi}_{B_{i+1},\omega}(m) \succeq \cdots.$$

(See Subsection 2.4.) We inductively replace the filtration (143) by another filtration

(144) $$E_1^{(j)} \subset E_2^{(j)} \subset E_3^{(j)} = E,$$

satisfying

(145) $\quad E_1^{(j)} \in \mathcal{B}_\omega, \quad E_2^{(j)}/E_1^{(j)} \in \pi^* \operatorname{Pic}(\mathbb{P}^1), \quad E/E_2^{(j)} \cong (A/A_{j-1})[-1].$

A desired exact sequence (142) is obtained by putting $j = N + 1$.

When $j = 1$, we can take a filtration (145) to be (143). For $j \ge 1$, suppose that we have a filtration (144) satisfying (145). We construct $E_2^{(j+1)}$ to be the kernel of the composition of the surjections in $\mathcal{A}_\omega$,
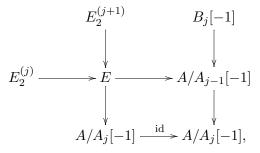
$$E \twoheadrightarrow E/E_2^{(j)} = (A/A_{j-1})[-1] \twoheadrightarrow (A/A_j)[-1].$$

Note that we have

(146) $$E/E_2^{(j+1)} \cong (A/A_j)[-1],$$

by the construction.

Next we construct $E_1^{(j+1)}$. By the diagram,

$$
\begin{array}{ccc}
E_2^{(j+1)} & & B_j[-1] \\
\downarrow & & \downarrow \\
E_2^{(j)} \longrightarrow E & \longrightarrow A/A_{j-1}[-1] \\
\downarrow & & \downarrow \\
A/A_j[-1] \xrightarrow{\ \mathrm{id}\ } A/A_j[-1],
\end{array}
$$

we have the exact sequence in $\mathcal{A}_\omega$,

$$0 \to E_2^{(j)} \to E_2^{(j+1)} \to B_j[-1] \to 0.$$

Since $E_1^{(j)} \subset E_2^{(j)}$, we also have the exact sequence in $\mathcal{A}_\omega$,

$$(147) \qquad 0 \to E_2^{(j)}/E_1^{(j)} \to E_2^{(j+1)}/E_1^{(j)} \to B_j[-1] \to 0.$$

We denote by ξ the extension class of (147). There are two cases:

The case of $\xi = 0$: In this case, we have a splitting surjection of (147),

$$E_2^{(j+1)}/E_1^{(j)} \twoheadrightarrow E_2^{(j)}/E_1^{(j)}.$$

We define $E_1^{(j+1)}$ to be the kernel of the composition

$$E_2^{(j+1)} \twoheadrightarrow E_2^{(j+1)}/E_1^{(j)} \twoheadrightarrow E_2^{(j)}/E_1^{(j)}.$$

Then we have the exact sequence in $\mathcal{A}_\omega$,

$$0 \to E_1^{(j)} \to E_1^{(j+1)} \to B_j[-1] \to 0.$$

Hence $E_1^{(j+1)} \in \mathcal{B}_\omega$. Noting (146), the filtration $E_\bullet^{(j+1)}$ satisfies the condition (145) for $j+1$.

The case of $\xi \neq 0$: By the inductive assumption, $E_2^{(j)}/E_1^{(j)}$ is isomorphic to $\pi^*\mathcal{O}_{\mathbb{P}^1}(r)$ for some $r \in \mathbb{Z}$. Also since B_j is ω-Gieseker stable, as in (29), there is $p \in \mathbb{P}^1$ and an ω-Gieseker stable sheaf B_j' on X_p such that $B_j \cong i_{p*}B_j'$. Hence the extension class ξ lies in

$$
\begin{aligned}
\xi \in \mathrm{Ext}^1_{\overline{X}}(i_{p*}B_j'[-1], \pi^*\mathcal{O}_{\mathbb{P}^1}(r)) \\
\cong \mathrm{Ext}^2_{X_p}(B_j', i_p^!\mathcal{O}_{\overline{X}}) \\
(148) \qquad \cong \mathrm{Ext}^1_{X_p}(B_j', \mathcal{O}_{X_p}).
\end{aligned}
$$

Let

$$(149) \qquad 0 \to \mathcal{O}_{X_p} \to B_j'' \to B_j' \to 0,$$

be the extension in X_p corresponding to ξ via the isomorphism (148). By Sublemma 7.6 below, we have

(150) $$i_{p*}B_j''[-1] \in \mathcal{T}_\omega[-1] \subset \mathcal{B}_\omega.$$

We have the commutative diagram,

$$
\begin{array}{ccc}
E_2^{(j+1)}/E_1^{(j)} & & \pi^*\mathcal{O}_{\mathbb{P}^1}(r+1) \\
\downarrow & & \downarrow \\
i_{p*}B_j''[-1] \longrightarrow B_j[-1] \longrightarrow & & \mathcal{O}_{X_p} \\
\xi \downarrow & & \downarrow \\
0 \longrightarrow \pi^*\mathcal{O}_{\mathbb{P}^1}(r)[1] \xrightarrow{\ \mathrm{id}\ } & & \pi^*\mathcal{O}_{\mathbb{P}^1}(r)[1].
\end{array}
$$

By the above diagram and (150), we obtain the exact sequence in $\mathcal{A}_\omega$,

$$0 \to i_{p*}B_j''[-1] \to E_2^{(j+1)}/E_1^{(j)} \to \pi^*\mathcal{O}_{\mathbb{P}^1}(r+1) \to 0.$$

We construct $E_1^{(j+1)}$ to be the kernel of the composition of surjections in $\mathcal{A}_\omega$,

$$E_2^{(j+1)} \twoheadrightarrow E_2^{(j+1)}/E_1^{(j)} \twoheadrightarrow \pi^*\mathcal{O}_{\mathbb{P}^1}(r+1).$$

Then we have the exact sequence in $\mathcal{A}_\omega$,

$$0 \to E_1^{(j)} \to E_1^{(j+1)} \to i_{p*}B_j''[-1] \to 0.$$

Therefore $E_1^{(j+1)} \in \mathcal{B}_\omega$. Noting (146), the filtration $E_\bullet^{(j+1)}$ satisfies the condition (145) for $j+1$.   q.e.d.

We have used the following sublemma.

Sublemma 7.6. *Let B_j'' be the sheaf on X_p defined by (149). Then we have*

$$i_{p*}B_j'' \in \mathcal{T}_\omega.$$

Proof. It is enough to show that

(151) $$\mathrm{Hom}(B_j'', F) = 0,$$

for any μ_ω-stable sheaf F on X_p with $\mu_\omega(F) \le 0$. Applying $\mathrm{Hom}(*, F)$ to the exact sequence (149), we have the exact sequence,

$$\mathrm{Hom}(B_j', F) \to \mathrm{Hom}(B_j'', F) \to \mathrm{Hom}(\mathcal{O}_{X_p}, F) \xrightarrow{\iota} \mathrm{Ext}^1_{X_p}(B_j', F).$$

Since B_j' is μ_ω-stable with $\mu_\omega(B_j') > 0$, we have $\mathrm{Hom}(B_j', F) = 0$. Therefore by the above sequence, (151) follows if $\mathrm{Hom}(\mathcal{O}_{X_p}, F) = 0$. Suppose that $\mathrm{Hom}(\mathcal{O}_{X_p}, F)$ is non-zero. Then F must be isomorphic to $\mathcal{O}_{X_p}$, and under the isomorphism $F \cong \mathcal{O}_{X_p}$, the image of 1 under ι is the extension class corresponding to (149). Since (149) does not split by the assumption, the map ι is injective. Hence (151) follows.   q.e.d.

8. Results on weak stability conditions

In this section, we recall some properties of weak stability conditions and complete a proof of Lemma 3.4 in Subsection 8.2.

8.1. Properties of weak stability conditions. In this subsection, we recall some technical properties of weak stability conditions. We discuss in a general situation, and use the same notation in Subsection 3.1. Let $(Z, \mathcal{A})$ be a weak stability condition on a triangulated category $\mathcal{T}$. For $0 < \phi \le 1$, the subcategory $\mathcal{P}(\phi) \subset \mathcal{T}$ is defined to be the category of Z-semistable objects $E \in \mathcal{A}$ satisfying

$$(152) \qquad Z(E) \in \mathbb{R}_{>0} \exp(i\pi\phi).$$

For other $\phi \in \mathbb{R}$, the subcategory $\mathcal{P}(\phi)$ is determined by the rule,

$$\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1].$$

The family of subcategories $\mathcal{P}(\phi)$ for $\phi \in \mathbb{R}$ determines a *slicing* introduced in [**10**, Definition 3.3]. As in [**47**, Proposition 2.13], giving a weak stability condition is equivalent to giving a data,

$$(153) \qquad \sigma = (Z = \{Z_i\}_{i=0}^{N}, \mathcal{P}),$$

where Z is as in Definition 3.1 and $\mathcal{P}$ is a slicing, satisfying the condition (152) for any non-zero $E \in \mathcal{P}(\phi)$. The subcategory $\mathcal{P}(\phi) \subset \mathcal{T}$ is called the category of σ-*semistable objects of phase* ϕ.

For an interval $I \subset \mathbb{R}$, we set

$$\mathcal{P}(I) := \langle \mathcal{P}(\phi) : \phi \in I \rangle_{\text{ex}}.$$

The following properties are required in constructing the space $\text{Stab}_{\Gamma_{\bullet}}(\mathcal{T})$.

- **(Support property):** There is a constant $C > 0$ such that for any $E \in \mathcal{P}(\phi)$ with $\text{cl}(E) \in \Gamma_i \setminus \Gamma_{i-1}$, we have

$$\|[\text{cl}(E)]\|_i \le C \cdot |Z(E)|.$$

  Here $\|*\|_i$ is a fixed norm on $(\Gamma_i/\Gamma_{i-1}) \otimes_{\mathbb{Z}} \mathbb{R}$.
- **(Local finiteness):** There is $\varepsilon > 0$ such that the quasi-abelian category $\mathcal{P}((\phi - \varepsilon, \phi + \varepsilon))$ is of finite length for any $\phi \in \mathbb{R}$.

Here we refer [**10**, Definition 4.1, Definition 5.7] for the detail on the notion of quasi-abelian categories and their finite length property. The set $\text{Stab}_{\Gamma_{\bullet}}(\mathcal{T})$ in Subsection 3.1 is defined to be the set of weak stability conditions satisfying the above two properties.

8.2. Proof of Lemma 3.4. In this subsection, we complete a proof of Lemma 3.4. Namely we prove the existence of Harder-Narasimhan filtrations, Support property and local finiteness for the pair $(Z_{t\omega}, \mathcal{A}_{\omega})$. We divide the proof into 4 steps.

Step 1. The abelian category $\mathcal{A}_{\omega}$ is noetherian.

Proof. Suppose that there is an infinite sequence of surjections in $\mathcal{A}_\omega$,

$$(154) \qquad E_1 \twoheadrightarrow E_2 \twoheadrightarrow \cdots \twoheadrightarrow E_i \twoheadrightarrow E_{i+1} \twoheadrightarrow \cdots .$$

We check that the sequence (154) terminates. By Lemma 2.10, we may assume that $\mathrm{rank}(E_i)$ and $\mathrm{ch}_2(E_i) \cdot \omega$ are constant. Also since we have surjections $\mathcal{H}^1(E_i) \twoheadrightarrow \mathcal{H}^1(E_{i+1})$ for all i, we may assume that $\mathcal{H}^1(E_i) \cong \mathcal{H}^1(E_{i+1})$ for all i. Let us take an exact sequence in $\mathcal{A}_\omega$,

$$0 \to K_i \to E_1 \to E_i \to 0.$$

We have the sequence of subsheaves,

$$\mathcal{H}^0(K_i) \subset \mathcal{H}^0(K_{i+1}) \subset \cdots \subset \mathcal{H}^0(E_1),$$

so we may assume that $\mathcal{H}^0(K_i) \cong \mathcal{H}^0(K_{i+1})$ for all i. Also since $\mathrm{rank}(K_i) = 0$, we have $K_i \in \mathcal{B}_\omega$. Furthermore since $\mathrm{ch}_2(K_i) \cdot \omega = 0$, we have

$$\dim \mathrm{Supp}\, \mathcal{H}^1(K_i) = 0.$$

Hence it is enough to bound the length of $\mathcal{H}^1(K_i)$. Setting

$$A = \mathcal{H}^0(E_1)/\mathcal{H}^0(K_1),$$

we have the exact sequence of sheaves,

$$0 \to A \to \mathcal{H}^0(E_i) \to \mathcal{H}^1(K_i) \to 0.$$

Let $A', \mathcal{H}^0(E_i)'$ be the torsion parts and $A'', \mathcal{H}^0(E_i)''$ the free parts of $A, \mathcal{H}^0(E_i)$ respectively. We have the exact sequences of sheaves,

$$(155) \qquad 0 \to A' \to \mathcal{H}^0(E_i)' \to T_i' \to 0,$$

$$(156) \qquad 0 \to A'' \to \mathcal{H}^0(E_i)'' \to T_i'' \to 0,$$

$$(157) \qquad 0 \to T_i' \to \mathcal{H}^1(K_i) \to T_i'' \to 0,$$

where T_i' and T_i'' are zero dimensional sheaves. By (155) and (156), we have the inclusions,

$$T_i' \subset (A')^{\vee\vee}/A', \quad T_i'' \subset (A'')^{\vee\vee}/A''.$$

Here for a pure two dimensional sheaf F, we set

$$F^\vee := \mathcal{E}xt^1_{\overline{X}}(F, \mathcal{O}_{\overline{X}}).$$

Therefore the length of T_i' and T_i'' are bounded. By (157), the length of $\mathcal{H}^1(K_i)$ is also bounded. q.e.d.

Step 2. There exist Harder-Narasimhan filtrations for the pair $(Z_{t\omega}, \mathcal{A}_\omega)$.

Proof. By [**47**, Proposition 2.12] and Step 1, it is enough to check that there is no infinite sequence of subobjects in $\mathcal{A}_\omega$,

$$(158) \qquad \cdots \subset E_{j+1} \subset E_j \subset \cdots \subset E_2 \subset E_1$$

with $\arg Z_{t\omega}(E_{j+1}) > \arg Z_{t\omega}(E_j/E_{j+1})$ for all j. Suppose that such a sequence exists. By Lemma 2.10, we may assume that $\mathrm{rank}(E_j)$ and $\mathrm{ch}_2(E_j) \cdot \omega$ are constant, hence

$$\mathrm{rank}(E_j/E_{j+1}) = 0, \quad \mathrm{ch}_2(E_j/E_{j+1}) \cdot \omega = 0,$$

for all j. By the definition of $\mathcal{A}_\omega$ and $Z_{t\omega}$, the above condition is equivalent to $Z_{t\omega}(E_j/E_{j+1}) \in \mathbb{R}_{<0}$. This contradicts to $\arg Z_{t\omega}(E_{j+1}) > \arg Z_{t\omega}(E_j/E_{j+1})$, hence there is no such a sequence. q.e.d.

Step 3. The pair $(Z_{t\omega}, \mathcal{A}_\omega)$ satisfies the support property.

Proof. Let $E \in \mathcal{A}_\omega$ be a $Z_{t\omega}$-semistable object with $\mathrm{cl}(E) = (R, r, \beta, n)$. If $R \neq 0$, we have

$$\frac{\|[\mathrm{cl}(E)]\|}{|Z_{t\omega}(E)|} = 1.$$

If $R = 0$, then $E \in \mathcal{B}_\omega$ and E is a $Z_{t\omega,0}$-semistable object. (cf. Remark 3.5.) Hence the support property for such E follows from that of the pair

$$(159) \qquad\qquad (Z_{t\omega,0}, \mathcal{B}_\omega) \in \mathrm{Stab}_{\Gamma_0}(\mathcal{D}_0).$$

The support property of the pair (159) follows from the same argument for the surface case. (cf. [**4**, Section 4].) q.e.d.

Step 4. The pair $(Z_{t\omega}, \mathcal{A}_\omega)$ satisfies the local finiteness.

Proof. For a pair $(Z_{t\omega}, \mathcal{A}_\omega)$, let $\{\mathcal{P}(\phi)\}_{\phi\in\mathbb{R}}$ be the corresponding slicing. For each $\phi \in \mathbb{R}$, we need to find $\varepsilon > 0$ so that $\mathcal{P}((\phi - \varepsilon, \phi + \varepsilon))$ is of finite length. Note that if $\phi \notin 1/2 + \mathbb{Z}$, then any $E \in \mathcal{P}(\phi)$ is a semistable object with respect to the pair (159). Hence the local finiteness in this case follows from that of (159), which can be proved along with the same argument for the surface case. (cf. [**11**, Lemma 4.4].) Suppose that $\phi \in 1/2 + \mathbb{Z}$. We may assume $\phi = 1/2$. In this case, it is enough to show that $\mathcal{P}((0, 1))$ is of finite length. Since $\mathcal{P}((0, 1))$ is a subcategory of $\mathcal{A}_\omega$, $\mathcal{P}((0, 1))$ is noetherian by Step 1. The proof that $\mathcal{P}((0, 1))$ is artinian follows from the same argument in Step 2 that there are no infinite sequence (158). q.e.d.

9. Results on semistable objects

In this section, we give proofs of several results on semistable objects in $\mathcal{A}_\omega$. In particular we prove some of the results stated in Section 3: Proposition 3.7 in Subsection 9.2, Proposition 3.8 in Subsection 9.3, Lemma 3.10 in Subsection 9.5 and Proposition 3.11 in Subsection 9.6.

9.1. Duality of semistable objects. In this subsection, we discuss a duality of $Z_{t\omega}$-semistable objects in $\mathcal{A}_\omega$. For an object $E \in \mathcal{D}$, note that

$$\mathbb{D}(E) := \mathbf{R}\mathcal{H}om_{\overline{X}}(E, \mathcal{O}_{\overline{X}}) \in \mathcal{D}.$$

Also note that $\mathcal{A}_\omega$ contains the following subcategory,

$$\mathcal{C}_\omega := \langle F, \mathcal{O}_x[-1] : F \in \mathcal{F}_\omega \text{ with } \mu_\omega(F) = 0, \ x \in \overline{X} \rangle_{\mathrm{ex}}.$$

We have the following lemma:

Lemma 9.1. *We have the autoequivalence,*

$$\mathbb{D} \circ [1] \colon \mathcal{C}_\omega \xrightarrow{\sim} \mathcal{C}_\omega.$$

Proof. It is enough to show that

(160) $$\mathbb{D}(\mathcal{O}_x[-1])[1] \in \mathcal{C}_\omega,$$

(161) $$\mathbb{D}(F)[1] \in \mathcal{C}_\omega,$$

for $x \in \overline{X}$ and a μ_ω-stable sheaf $F \in \mathcal{F}_\omega$ with $\mu_\omega(F) = 0$. The condition (160) follows from $\mathbb{D}(\mathcal{O}_x) = \mathcal{O}_x[-3]$. For the sheaf F as above, we write $F = i_{p*}F'$ for a μ_ω-stable sheaf on X_p as in (29). We have the distinguished triangle,

$$Q[-1] \to F \to i_{p*}F'^{\vee\vee},$$

for some zero dimensional sheaf Q. Note that $F'^{\vee\vee}$ is a locally free sheaf on X_p. Then the condition (161) follows from (160) and the fact that

$$\mathbb{D}(i_{p*}F'^{\vee\vee})[1] \cong i_{p*}F'^{\vee},$$

which is μ_ω-stable with $\mu_\omega = 0$. $\hfill$ q.e.d.

We also consider the right orthogonal complement of $\mathcal{C}_\omega$,

(162) $$\mathcal{C}_\omega^{\perp} := \{E \in \mathcal{A}_\omega : \mathrm{Hom}(\mathcal{C}_\omega, E) = 0\}.$$

The following result implies that $\mathcal{C}_\omega^{\perp}$ is also self dual.

Lemma 9.2. *We have the autoequivalence,*

$$\mathbb{D} \colon \mathcal{C}_\omega^{\perp} \xrightarrow{\sim} \mathcal{C}_\omega^{\perp}.$$

Proof. For an object $E \in \mathcal{C}_\omega^{\perp}$ and $K \in \mathcal{C}_\omega$, we have

$$\mathrm{Hom}(K, \mathbb{D}(E)) \cong \mathrm{Hom}(E, \mathbb{D}(K)[1][-1])$$
$$\cong 0,$$

since $\mathbb{D}(K)[1] \in \mathcal{C}_\omega$ by Lemma 9.1. Therefore it is enough to show that $\mathbb{D}(E) \in \mathcal{A}_\omega$. By Lemma 7.5, there is an exact sequence in $\mathcal{A}_\omega$

(163) $$0 \to A \to E \to B \to 0,$$

such that $A \in \mathcal{B}_\omega$ and $B \in \langle \pi^* \mathrm{Pic}(\mathbb{P}^1) \rangle_{\mathrm{ex}}$. Since $\mathbb{D}(B) \in \mathcal{A}_\omega$, it is enough to show that

$$\mathbb{D}(A) \in \mathcal{B}_\omega.$$

We take an exact sequence in $\mathcal{B}_\omega$,

$$0 \to F \to A \to T[-1] \to 0,$$

with $F \in \mathcal{F}_\omega$ and $T \in \mathcal{T}_\omega$. Since F is a subobject of E in $\mathcal{A}_\omega$ and $E \in \mathcal{C}_\omega^\perp$, we have $F \in \mathcal{C}_\omega^\perp$. Taking the dual of the above sequence, we obtain the distinguished triangle,

$$\mathbb{D}(T)[1] \to \mathbb{D}(A) \to \mathbb{D}(F).$$

By Lemma 9.3 below and the long exact sequence of cohomologies, we have

$$\mathcal{H}^i \mathbb{D}(A) = 0, \ i \neq 0, 1, 2,$$
$$\mathcal{H}^0 \mathbb{D}(A) \cong \mathcal{E}xt_X^1(T, \mathcal{O}_{\overline{X}}),$$
$$\dim \mathcal{H}^2 \mathbb{D}(A) = 0,$$

and the exact sequence of sheaves,

$$0 \to \mathcal{E}xt_{\overline{X}}^2(T, \mathcal{O}_{\overline{X}}) \to \mathcal{H}^1 \mathbb{D}(A) \to \mathcal{E}xt_{\overline{X}}^1(F, \mathcal{O}_{\overline{X}}) \to Q \to 0,$$

for some zero dimensional sheaf Q. Applying Lemma 9.3 again, we have

$$\mathcal{H}^0 \mathbb{D}(A) \in \mathcal{F}_\omega, \ \mathcal{H}^1 \mathbb{D}(A) \in \mathcal{T}_\omega.$$

Suppose that $\mathcal{H}^2 \mathbb{D}(A) \neq 0$. Then there is $x \in \overline{X}$ such that

$$\mathrm{Hom}(\mathbb{D}(A), \mathcal{O}_x[-2]) \neq 0.$$

Applying $\mathbb{D}$, we have

$$\mathrm{Hom}(\mathcal{O}_x[-1], A) \neq 0,$$

which contradicts to $E \in \mathcal{C}_\omega^\perp$. \hfill q.e.d.

We have used the following lemma.

Lemma 9.3. *(i) For $F \in \mathcal{C}_\omega^\perp \cap \mathcal{F}_\omega$, we have*

$$(164) \qquad \mathcal{E}xt_{\overline{X}}^i(F, \mathcal{O}_{\overline{X}}) = 0, \quad i \neq 1, 2,$$

$$(165) \qquad \mathcal{E}xt_{\overline{X}}^1(F, \mathcal{O}_{\overline{X}}) \in \mathcal{T}_\omega,$$

$$(166) \qquad \dim \mathcal{E}xt_{\overline{X}}^2(F, \mathcal{O}_{\overline{X}}) = 0.$$

(ii) For $T \in \mathcal{T}_\omega$, we have

$$(167) \qquad \mathcal{E}xt_{\overline{X}}^i(T, \mathcal{O}_{\overline{X}}) = 0, \quad i \neq 1, 2, 3,$$

$$(168) \qquad \mathcal{E}xt_{\overline{X}}^1(T, \mathcal{O}_{\overline{X}}) \in \mathcal{F}_\omega,$$

$$(169) \qquad \dim \mathcal{E}xt_{\overline{X}}^i(T, \mathcal{O}_{\overline{X}}) = 3 - i, \ i = 2, 3.$$

Proof. The properties (164), (166), (167), (169) are well-known and the proofs are standard. See [**17**] for instance. We show the property (165). The property (168) is similarly proved. Let us take $F \in \mathcal{C}_\omega^\perp \cap \mathcal{F}_\omega$. By taking Harder-Narasimhan filtration and Jordan-Hölder filtration with respect to μ_ω-stability, we may assume that $F \cong i_{p*}F'$ for some μ_ω-stable sheaf F' on X_p as in (29). The condition $F \in \mathcal{C}_\omega^\perp$ implies that $\mu_\omega(F') < 0$. Then by the adjunction, we have

$$\mathcal{E}xt_{\overline{X}}^1(F, \mathcal{O}_{\overline{X}}) \cong i_{p*}\mathcal{H}om_{X_p}(F', \mathcal{O}_{X_p})$$
$$\cong i_{p*}F'^\vee.$$

Since $F'^\vee$ is μ_ω-stable with $\mu_\omega(F'^\vee) > 0$, we have $i_{p*}F'^\vee \in \mathcal{T}_\omega$.   q.e.d.

In order to see the duality of semistable objects, we show the following lemma.

Lemma 9.4. *An object $E \in \mathcal{A}_\omega$ with $\operatorname{Im} Z_{t\omega}(E) > 0$ is $Z_{t\omega}$-semistable if and only if $E \in \mathcal{C}_\omega^\perp$ and for any exact sequence in $\mathcal{A}_\omega$*

$$(170) \qquad\qquad 0 \to F \to E \to G \to 0$$

with $F, G \in \mathcal{C}_\omega^\perp$, the inequality

$$(171) \qquad\qquad \arg Z_{t\omega}(F) \le \arg Z_{t\omega}(G)$$

is satisfied.

Proof. Take $E \in \mathcal{A}_\omega$ with $\operatorname{Im} Z_{t\omega}(E) > 0$, and suppose that E is $Z_{t\omega}$-semistable. Since $\arg Z_{t\omega}(E) < \pi$ and

$$(172) \qquad\qquad Z_{t\omega}(\mathcal{C}_\omega) \subset \mathbb{R}_{\le 0},$$

we have $E \in \mathcal{C}_\omega^\perp$ by the $Z_{t\omega}$-semistability of E. The inequality (171) with respect to the sequence (170) follows from the $Z_{t\omega}$-semistability of E.

Conversely, suppose that $E \in \mathcal{C}_\omega$ satisfies the inequality (171) with respect to any sequence (170). We take an exact sequence in $\mathcal{A}_\omega$,

$$0 \to F' \to E \to G' \to 0.$$

Since $\mathcal{A}_\omega$ is noetherian, (see Subsection 8.2,) there is an exact sequence in $\mathcal{A}_\omega$,

$$0 \to G''' \to G' \to G'' \to 0,$$

with $G''' \in \mathcal{C}_\omega$ and $G'' \in \mathcal{C}_\omega^\perp$. By composing the above sequences, we obtain the exact sequence in $\mathcal{A}_\omega$,

$$0 \to F'' \to E \to G'' \to 0,$$

with $F'', G'' \in \mathcal{C}_\omega^\perp$. Using the assumption and (172), we obtain

$$\arg Z_{t\omega}(F') \leq \arg Z_{t\omega}(F'')$$
$$\leq \arg Z_{t\omega}(G'')$$
$$\leq \arg Z_{t\omega}(G').$$

Hence E is $Z_{t\omega}$-semistable.   q.e.d.

Summarizing the above results, we obtain the following result.

Proposition 9.5. *Suppose that $R \geq 1$ or $R = 0$, $\beta \cdot \omega \neq 0$. Then we have the bijection,*

$$(173) \qquad \mathbb{D} : M_{t\omega}(R, r, \beta, n) \xrightarrow{1:1} M_{t\omega}(R, -r, \beta, -n).$$

If $R = \beta \cdot \omega = 0$, we have

$$(174) \qquad \mathbb{D} \circ [1] : M_{t\omega}(0, r, \beta, n) \xrightarrow{1:1} M_{t\omega}(0, r, -\beta, n).$$

Proof. Take an object $E \in M_{t\omega}(R, r, \beta, n)$ and suppose that $R \geq 1$ or $R = 0$, $\beta \cdot \omega \neq 0$. Then $\operatorname{Im} Z_{t\omega}(E) > 0$, hence noting Lemma 9.2, Lemma 9.4 and

$$Z_\omega(\mathbb{D}(E)) = -\overline{Z}_\omega(E),$$

we easily see that $\mathbb{D}(E)$ is a $Z_{t\omega}$-semistable object in $\mathcal{A}_\omega$. Therefore (173) follows. If $R = \beta \cdot \omega = 0$, then we have $E \in \mathcal{C}_\omega$ and (174) follows from Lemma 9.1.   q.e.d.

By applying the dualizing functor, we can also prove the following lemma.

Lemma 9.6. *For any $r \in \mathbb{Z}$, there is no non-trivial exact sequence in $\mathcal{A}_\omega$,*

$$(175) \qquad\qquad 0 \to A \to \pi^* \mathcal{O}_{\mathbb{P}^1}(r) \to B \to 0,$$

with $A, B \in \mathcal{C}_\omega^\perp$. In particular, the object $\pi^* \mathcal{O}_{\mathbb{P}^1}(r) \in \mathcal{A}_\omega$ is $Z_{t\omega}$-stable for any $t \in \mathbb{R}_{>0}$.

Proof. Since $\pi^* \mathcal{O}_{\mathbb{P}^1}(r) \in \mathcal{C}_\omega^\perp$, the $Z_{t\omega}$-stability of $\pi^* \mathcal{O}_{\mathbb{P}^1}(r)$ follows from the first statement and Lemma 9.4. Suppose that a non-trivial sequence (175) exists. Then we have $\operatorname{rank}(A) = 0$ or $\operatorname{rank}(B) = 0$, and by the duality in Lemma 9.2, we may assume that $\operatorname{rank}(B) = 0$, i.e. $B \in \mathcal{B}_\omega$. Then by Lemma 7.2 (iii), (iv), the object B is written as $\pi^* Q$ for a zero dimensional sheaf Q on $\mathbb{P}^1$. Since $\pi^* Q \in \mathcal{C}_\omega$, this contradicts to $B \in \mathcal{C}_\omega^\perp$.   q.e.d.

9.2. Proof of Proposition 3.7. In this subsection, we give a proof of Proposition 3.7, that is the existence of wall and chamber structure on $t \in \mathbb{R}_{>0}$. For the reader's convenience, we restate the proposition.

Proposition 9.7. *For fixed $\beta \in \mathrm{NS}(S)$ and an ample divisor ω on S, there is a finite sequence of real numbers,*

$$0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = \infty,$$

such that the set of objects

$$\bigcup_{\substack{(R,r,n), \\ \arg Z_{t\omega}(R,r,\beta,n)=\pi/2}} M_{t\omega}(R, r, \beta, n),$$

is constant for each $t \in (t_{i-1}, t_i)$.

Proof. We fix β, ω and take an object,

$$(176) \qquad E \in \bigcup_{\substack{(R,r,n), \\ \arg Z_{t\omega}(R,r,\beta,n)=\pi/2}} M_{t\omega}(R, r, \beta, n).$$

Suppose that $A \in \mathcal{B}_\omega$ is a subobject or a quotient of E in $\mathcal{A}_\omega$ and satisfies

$$(177) \qquad \arg Z_{t\omega}(A) = \frac{\pi}{2}.$$

If we write $\mathrm{cl}_0(A) = (r', \beta', n')$, then we have

$$(178) \qquad \mathrm{Re}\, Z_{t\omega}(A) = n' - \frac{1}{2}r' t^2 \omega^2 = 0.$$

By Lemma 7.4, there is a filtration in $\mathcal{B}_\omega$,

$$(179) \qquad 0 = A_0 \subset A_1 \subset A_2 \subset A_3 = A,$$

such that each subquotient $K_i := A_i/A_{i-1}$ satisfies the condition (138). We write $\mathrm{cl}_0(K_i) = (r_i, \beta_i, n_i)$. By the $Z_{t\omega}$-semistability of E and the condition (177), we have

$$(180) \qquad \mathrm{Re}\, Z_{t\omega}(K_1) = n_1 - \frac{1}{2}r_1 t^2 \omega^2 \geq 0,$$

$$(181) \qquad \mathrm{Re}\, Z_{t\omega}(K_3) = n_3 - \frac{1}{2}r_3 t^2 \omega^2 \leq 0.$$

Since $r_1 \geq 0$ and $r_3 \leq 0$, the inequalities (180), (181) imply that $n_1 \geq 0$ and $n_3 \leq 0$. Also by Lemma 2.10, we have $\beta \cdot \omega \leq \beta_i \cdot \omega \leq 0$. Therefore we can apply Lemma 9.8 below and conclude that (r_1, n_1), (r_3, n_3), hence $r' = r_1 + r_3$, have only a finite number of possibilities.

Suppose that $K_1 = K_3 = 0$. Then the equality (178) is satisfied only if $(r', n') = (0, 0)$. Otherwise, for instance if $K_1 \neq 0$, then $r_1 > 0$ and the inequality (180) implies that

$$n_1 - \frac{1}{2}t^2 \omega^2 r_1 \geq 0.$$

Therefore such t is bounded above. A similar argument shows the boundedness of t under the assumption $K_3 \neq 0$. Therefore the set of possible $t \in \mathbb{R}$ satisfying the equation (178) for some $A \in \mathcal{B}_\omega$, which is a subobject or a quotient of some object (176) with $(r', n') \neq 0$, is a finite set. If we denote this finite set by $0 = t_0 < t_1 < \cdots < t_k < \infty$, then $t_\bullet$ satisfies the desired condition.   q.e.d.

We have used the following lemma.

Lemma 9.8. *For fixed ample divisor ω on S and $a, b \in \mathbb{R}$, the following subsets in $\mathbb{Z}^{\oplus 2}$ are finite sets:*

(182)
$$\left\{ (r', n') : \begin{array}{l} \text{there is } T \in \mathcal{T}_\omega^{\mathrm{pure}}, \ \mathrm{cl}_0(T) = (r', \beta', n'), \\ \text{satisfying } \beta' \cdot \omega \leq a, \ n' \geq b \end{array} \right\},$$

(183)
$$\left\{ (r', n') : \begin{array}{l} \text{there is } F \in \mathcal{F}_\omega, \ \mathrm{cl}_0(F) = (r', \beta', n'), \\ \text{satisfying } \beta' \cdot \omega \geq a, \ n' \geq b \end{array} \right\}.$$

Proof. For simplicity, we prove the finiteness of (182). The finiteness of (183) is similarly proved. Take $T \in \mathcal{T}_\omega^{\mathrm{pure}}$ with $\mathrm{cl}_0(T) = (r', \beta', n')$, $\beta' \cdot \omega \leq a$ and $n' \geq b$. Taking the Harder-Narasimhan filtrations and Jordan-Hölder filtrations of T with respect to μ_ω-stability, we have a filtration of coherent sheaves,

$$0 = T_0 \subset T_1 \subset \cdots \subset T_N = T,$$

such that each $M_i := T_i / T_{i-1}$ is μ_ω-stable. We write $\mathrm{cl}_0(M_i) = (r_i, \beta_i, n_i)$. Since $\beta_i \cdot \omega > 0$ for all i, we have

$$\beta_i \cdot \omega \leq \beta' \cdot \omega \leq a, \quad 0 < N \leq a.$$

By the Hodge index theorem, there is a constant $s(a, \omega) > 0$ which depends only on a and ω such that

$$\beta_i^2 \leq s(a, \omega).$$

Also note that $r_i > 0$ for all i, since T is a pure two dimensional sheaf. Therefore applying Lemma 2.5, we have

$$n_i \leq \frac{\beta_i^2 + 2}{2 r_i} - r_i$$

$$\leq \frac{1}{2} \left(s(a, \omega) + 2 \right) - r_i.$$

Taking the sum from $i = 1$ to $i = N$, we obtain

(184)
$$n' \leq \frac{N}{2} (s(a, \omega) + 2) - r'$$

$$\leq \frac{a}{2} (s(a, \omega) + 2) - r'.$$

Combined with $n' \geq b$, we have

$$0 < r' \leq \frac{a}{2} (s(a, \omega) + 2) - b.$$

Therefore there is only a finite number of possibilities for r'. By (184) and $n' \geq b$, there is also a finite number of possibilities for n'.   q.e.d.

9.3. Proof of Proposition 3.8. In this subsection, we prove Proposition 3.8, which is restated as follows:

Proposition 9.9. *In the same situation of Proposition* 9.7*, we have*
$$M_{t\omega}(R, r, \beta, n) = \emptyset,$$
for any $t \in (0, t_1)$ and $(R, r, n) \in \mathbb{Z}^{\oplus 3}$ with $R \geq 1$ and $n \neq 0$.

Proof. Suppose that there is an object $E \in M_{t\omega}(R, r, \beta, n)$ for $t \in (0, t_1)$ and $R \geq 1$. By Proposition 9.5, we may assume that $n < 0$. By Lemma 7.5, there is an exact sequence in $\mathcal{A}_\omega$,
$$0 \to A \to E \to B \to 0,$$
such that $A \in \mathcal{B}_\omega$ and $B \in \langle \pi^* \mathrm{Pic}(\mathbb{P}^1) \rangle_{\mathrm{ex}}$. We have
$$\mathrm{cl}_0(A) = (r', \beta, n),$$
for some $r' \in \mathbb{Z}$. By the $Z_{t\omega}$-semistability of E, we have $\arg Z_{t\omega}(A) \leq \pi/2$, or equivalently

(185) $$n - \frac{1}{2}t^2\omega^2 r' \geq 0.$$

The above inequality should be satisfied for any $t \in (0, t_1)$, therefore we must have $n \geq 0$. This contradicts to $n < 0$, hence we have $M_{t\omega}(R, r, \beta, n) = \emptyset$ for $t \in (0, t_1)$, $R \geq 1$ and $n \neq 0$.   q.e.d.

9.4. Rank zero semistable objects for small t. Using the technique in the previous subsections, we give the following proposition on rank zero $Z_{t\omega}$-semistable objects for small t. This result will be used later.

Proposition 9.10. *For fixed β and ω with $\beta \cdot \omega > 0$, there is $t' > 0$ such that the following set of objects is constant for $0 < t < t'$,*
$$\bigcup_{r \in \mathbb{Z}} M_{t\omega}(0, r, \beta, 0).$$

Proof. The proof is similar to the proof of Proposition 9.7, but we need to modify the argument in some places. Let us take $E \in M_{t\omega}(0, r, \beta, 0)$ for $t \in \mathbb{R}_{>0}$. By Lemma 9.11 below, we have a finite number of possibilities for r. Hence we can take $t'' > 0$ such that

(186) $$|\mathrm{Re}\, Z_{t\omega}(E)| = \frac{1}{2}t^2 r^2 \omega^2 < 1,$$

for any $E \in M_{t\omega}(0, r, \beta, 0)$ with $0 < t < t''$. Take an object $A \in \mathcal{B}_\omega$ such that A is a subobject or quotient of E in $\mathcal{B}_\omega$ and satisfies

(187) $$\arg Z_{t\omega}(A) = \arg Z_{t\omega}(E),$$

for some $0 < t < t''$. By Lemma 7.4, there is a filtration

$$0 = A_0 \subset A_1 \subset A_2 \subset A_3 = A,$$

in $\mathcal{B}_\omega$ such that $K_i = A_i/A_{i-1}$ satisfies the condition (138). We write $\mathrm{cl}_0(K_i) = (r_i, \beta_i, n_i)$. By the $Z_{t\omega}$-semistability of E and the inequality (186), we can easily see that n_1 is bounded below and n_3 is bounded above. Hence Lemma 9.8 implies that there are only finite number of possibilities for (r_1, n_1) and (r_3, n_3).

Now we note

$$(188) \qquad |\mathrm{Re}\, Z_{t\omega}(A)| = \left| n' - \frac{1}{2} r' t^2 \omega^2 \right| < 1,$$

by the conditions (186) and (187). Since $r' = r_1 + r_3$ is bounded, the inequality (188) gives a lower bound of $t > 0$ for the existence of such object $A \in \mathcal{B}_\omega$ with $r' \neq 0$. If we denote that lower bound by t', then t' satisfies the desired condition. Note that if $r' = 0$, then (188) is only possible when $n' = 0$. However in that case $\arg Z_{t\omega}(A) = \pi/2$ for any t, and we don't need to take account of such objects.   q.e.d.

Lemma 9.11. *For fixed $a \in \mathbb{R}_{>0}$ and an ample divisor ω on S, the set of $r \in \mathbb{Z}$ such that $M_{t\omega}(0, r, \beta, 0) \neq \emptyset$ for some $t \in \mathbb{R}_{>0}$ and $0 < -\beta \cdot \omega \leq a$ is a finite set.*

Proof. By Proposition 9.5, it is enough to consider possible values $r \in \mathbb{Z}$ with $r < 0$. Let us take $E \in M_{t\omega}(0, r, \beta, 0)$ with $r < 0$. By Lemma 7.4, there is a filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset E_3 = E,$$

in B_ω such that $K_i = E_i/E_{i-1}$ satisfies the condition (138). We write $\mathrm{cl}_0(K_i) = (r_i, \beta_i, n_i)$. Since $r < 0$, we have $\arg Z_{t\omega}(E) \in (0, \pi/2)$. By the $Z_{t\omega}$-semistability of E, we have

$$\arg Z_{t\omega}(E_i) \leq \arg Z_{t\omega}(E) < \frac{\pi}{2},$$

for $i = 1, 2$. Hence we have $n_1 \geq 0$ and $n_1 + n_2 \geq 0$, therefore $n_3 = -(n_1 + n_2) \leq 0$. Since $-\beta_i \cdot \omega \leq a$, we can apply Lemma 9.8 and conclude that r_1 and r_3 are bounded. Hence $r = r_1 + r_3$ is also bounded. q.e.d.

9.5. Proof of Lemma 3.10. In this subsection, we prove Lemma 3.10, which is restated as follows:

Lemma 9.12. *Take an object $E \in D^b \mathrm{Coh}(\overline{X})$ satisfying*

$$(189) \qquad \mathrm{ch}(E) = (R, 0, -\beta, -n) \in \Gamma \subset H^*(\overline{X}, \mathbb{Q}),$$

*for $R \leq 1$. Then E is an $\mu_{i\omega}$-limit semistable object in $\mathcal{A}(0)$ iff $E[1]$ is an $\mu_{i\omega}$-limit semistable object in the sense of [**48**, Section 3].*

Proof. We only show the case of $R = 1$. The proof for the case of $R = 0$ case is easier and we omit it.

Step 1. The definition of $\mu_{i\omega}$-limit stability in [**48**, Section 3].

We first recall the notion of $\mu_{i\omega}$-limit stability in the sense of [**48**, Section 3]. In [**3**], [**45**], the notion of *polynomial stability* and *limit stability* are defined on the following category of *perverse coherent sheaves*,

$$\mathcal{A}^p := \langle \mathrm{Coh}^{\geq 2}(\overline{X})[1], \mathrm{Coh}^{\leq 1}(\overline{X}) \rangle_{\mathrm{ex}} \subset D^b \mathrm{Coh}(\overline{X}).$$

Here $\mathrm{Coh}^{\leq 1}(\overline{X})$ consists of sheaves F on $\overline{X}$ with $\dim F \leq 1$ and $\mathrm{Coh}^{\geq 2}(\overline{X})$ is the right orthogonal complement of $\mathrm{Coh}^{\leq 1}(\overline{X})$ in $\mathrm{Coh}(\overline{X})$.

By [**45**, Lemma 2.16], there exists a torsion pair $(\mathcal{A}_1^p, \mathcal{A}_{1/2}^p)$ on $\mathcal{A}^p$, defined by

$$\mathcal{A}_1^p := \langle F[1], \mathcal{O}_x : F \text{ is pure two dimensional}, x \in \overline{X} \rangle_{\mathrm{ex}},$$
$$\mathcal{A}_{1/2}^p := \{E \in \mathcal{A}^p : \mathrm{Hom}(F, E) = 0 \text{ for any } F \in \mathcal{A}_1^p\}.$$

Note that if F is a pure one dimensional sheaf on $\overline{X}$, then $F \in \mathcal{A}_{1/2}^p$.

For $E, F \in \mathcal{A}_{1/2}^p$, a morphism $u \colon E \to F$ in $\mathcal{A}^p$ is called a *strict monomorphism* if u is injective in $\mathcal{A}^p$ and $\mathrm{Cok}(u) \in \mathcal{A}_{1/2}^p$. Similarly u is called a *strict epimorphism* if u is surjective in $\mathcal{A}^p$ and $\mathrm{Ker}(u) \in \mathcal{A}_{1/2}^p$. By [**48**, Proposition 3.13], an object $E \in \mathcal{A}_{1/2}^p$ with $\mathrm{rank}(E) = -1$ is $\mu_{i\omega}$-*limit semistable* in the sense of [**48**, Section 3] iff the following conditions hold:

- For any pure one dimensional sheaf $F \neq 0$ which admits a strict monomorphism $F \hookrightarrow E$ in $\mathcal{A}_{1/2}^p$, we have $\mathrm{ch}_3(F) \leq 0$.
- For any pure one dimensional sheaf $G \neq 0$ which admits a strict epimorphism $E \twoheadrightarrow G$ in $\mathcal{A}_{1/2}^p$, we have $\mathrm{ch}_3(G) \geq 0$.

Step 2. Comparison of $\mathcal{A}_{1/2}^p$ and $\mathcal{A}(0)$.

Let $\mathcal{A}(0)$ be the category defined in Definition 2.12. The categories $\mathcal{A}_{1/2}^p$ and $\mathcal{A}(0)$ are related as follows: let $\mathcal{A}(0)^\dagger$ be the following category,

$$\mathcal{A}(0)^\dagger := \{E \in \mathcal{A}(0) : \mathrm{Hom}(\mathcal{O}_x[-1], E) = 0, x \in \overline{X}\}.$$

Then it is easy to check that

$$\mathcal{A}(0)^\dagger \subset \mathcal{A}_{1/2}^p[-1].$$

Also by replacing $\mathcal{A}_{1/2}^p$, $\mathcal{A}^p$ by $\mathcal{A}(0)^\dagger$, $\mathcal{A}(0)$ respectively, we have the notions of strict monomorphisms and strict epimorphisms in $\mathcal{A}(0)^\dagger$. Then the same proof of Lemma 9.4 shows that an object $E \in \mathcal{A}(0)$ with $\mathrm{rank}(E) = 1$ is $\mu_{i\omega}$-limit semistable in the sense of Definition 3.9 iff the following conditions hold:

- For any pure one dimensional sheaf $0 \neq F \in \mathrm{Coh}_{\overline{\pi}}^{\leq 1}(\overline{X})$ which admits a strict monomorphism $F[-1] \hookrightarrow E$ in $\mathcal{A}(0)^\dagger$, we have $\mathrm{ch}_3(F) \leq 0$.

- For any pure one dimensional sheaf $0 \neq G \in \mathrm{Coh}_{\overline{\pi}}^{\leq 1}(\overline{X})$ which admits a strict epimorphism $E \twoheadrightarrow G[-1]$ in $\mathcal{A}(0)^{\dagger}$, we have $\mathrm{ch}_3(G) \geq 0$.

Step 3. Proof of Lemma 9.12.

Now let us take $E \in \mathcal{A}_{1/2}^p[-1]$ satisfying the condition (189) for $R = 1$. By the above arguments, it is enough to show the following:

- We have $E \in \mathcal{A}(0)^{\dagger}$.
- For any strict monomorphism $F \hookrightarrow E[1]$ in $\mathcal{A}_{1/2}^p$ with F pure one dimensional sheaf, we have $F \in \mathrm{Coh}_{\overline{\pi}}^{\leq 1}(\overline{X})$ and $F[-1] \to E$ is a strict monomorphism in $\mathcal{A}(0)^{\dagger}$.
- For any strict epimorphism $E[1] \twoheadrightarrow G$ in $\mathcal{A}_{1/2}^p$ with G pure one dimensional sheaf, we have $G \in \mathrm{Coh}_{\overline{\pi}}^{\leq 1}(\overline{X})$ and $E \to G[-1]$ is a strict monomorphism in $\mathcal{A}(0)^{\dagger}$.

First we prove $E \in \mathcal{A}(0)^{\dagger}$. By [**45**, Lemma 3.2], we have $\mathcal{H}^0(E) = I_C$ for a curve $C \subset \overline{X}$ and $\mathcal{H}^1(E)$ is a one dimensional sheaf. Hence

$$\beta = [C] + [\mathcal{H}^1(E)].$$

Since $(1, 0, -\beta, -n) \in \Gamma$, the curve C is supported on fibers of π and $\mathcal{H}^1(E) \in \mathrm{Coh}_{\overline{\pi}}^{\leq 1}(\overline{X})$. This implies that $E \in \mathcal{A}(0) \cap (\mathcal{A}_{1/2}^p[-1]) = \mathcal{A}(0)^{\dagger}$.

Next we prove the second condition. The proof of the third one is similar and we omit it. Let $F \hookrightarrow E[1]$ be a strict monomorphism in $\mathcal{A}_{1/2}^p$ for a pure one dimensional sheaf F, and set $G := E[1]/F \in \mathcal{A}_{1/2}^p$. We have the exact sequence of sheaves,

$$0 \to \mathcal{H}^0(E) \xrightarrow{i} \mathcal{H}^{-1}(G) \to F \to \mathcal{H}^1(E) \xrightarrow{j} \mathcal{H}^0(G) \to 0.$$

By [**45**, Lemma 3.2], $\mathcal{H}^0(E)$ and $\mathcal{H}^{-1}(G)$ are written as I_C, $I_{C'}$ for curves C, C' in $\overline{X}$ respectively. Since $E \in \mathcal{A}(0)^{\dagger}$, C is supported on fibers of π, hence C' and $\mathrm{Cok}(i)$ are supported on fibers of π. Also since $\mathcal{H}^1(E) \in \mathrm{Coh}_{\overline{\pi}}^{\leq 1}(\overline{X})$, we have $\mathrm{Ker}(j), \mathcal{H}^0(G) \in \mathrm{Coh}_{\overline{\pi}}^{\leq 1}(\overline{X})$ by the above sequence. Therefore we have $F \in \mathrm{Coh}_{\leq 1}(\overline{X})$ and $G[-1] \in \mathcal{A}(0)$. Since $G[-1] \in \mathcal{A}_{1/2}^p[-1]$, it follows that $G[-1] \in \mathcal{A}(0)^{\dagger}$, hence $F[-1] \to E$ is a strict monomorphism in $\mathcal{A}(0)^{\dagger}$. q.e.d.

9.6. Proof of Proposition 3.11. In this subsection, we prove Proposition 3.11, which is restated as follows:

Proposition 9.13. *In the same situation of Proposition* 9.7*, we have*

$$M_{t\omega}(R, r, \beta, n) = M_{\lim}(R, r, \beta, n),$$

for any $t \in (t_{k-1}, \infty)$ *and* $R \leq 1$ *satisfying* $\arg Z_{t\omega}(R, r, \beta, n) = \pi/2$.

We take $(R, r, \beta, n) \in \Gamma$ as in the statement. For simplicity, we show the case of $R = 1$. The proof for $R = 0$ is similar and easier, so we omit it. We divide the proof into 2 steps.

Step 1. For $t > t_k$, we have

$$M_{t\omega}(1, r, \beta, n) \subset M_{\lim}(1, r, \beta, n).$$

Proof. Take an object $E \in M_{t\omega}(1, r, \beta, n)$ for $t > t_k$ and a filtration in $\mathcal{A}_\omega$,

$$E_1 \subset E_2 \subset E_3 = E,$$

given by Lemma 7.4. Suppose that $E_1 \neq 0$. Then the $Z_{t\omega}$-semistability of E implies that

(190)
$$\arg Z_{t\omega}(E_1) \leq \frac{\pi}{2}.$$

We write $\mathrm{cl}_0(E_1) = (r_1, \beta_1, n_1) \in \Gamma_0$. Then the inequality (190) is equivalent to

$$\mathrm{Re}\, Z_{t\omega}(E_1) = n_1 - \frac{1}{2}t^2\omega^2 r_1 \geq 0.$$

The above inequality should be satisfied for all $t > t_k$. However since $r_1 > 0$, the above inequality is not satisfied for $t \gg 0$. This is a contradiction, hence $E_1 = 0$. A similar argument also shows that $E/E_2 = 0$, hence $E \in \mathcal{A}(r)$ follows.

In order to show that E is $\mu_{i\omega}$-limit semistable, we take an exact sequence in $\mathcal{A}(r)$,

$$0 \to F \to E \to G \to 0.$$

If $F \in \mathrm{Coh}_{\overline{\pi}}^{\leq 1}(\overline{X})[-1]$, then the $Z_{t\omega}$-stability yields,

$$\mathrm{Re}\, Z_{t\omega}(F) = \mathrm{ch}_3(F) \geq 0.$$

Similarly if $G \in \mathrm{Coh}_{\overline{\pi}}^{\leq 1}(\overline{X})[-1]$, we obtain $\mathrm{ch}_3(G) \leq 0$. Therefore E is $\mu_{i\omega}$-limit semistable, i.e. $E \in M_{\lim}(1, r, \beta, n)$.   q.e.d.

Step 2. For $t > t_k$, we have

$$M_{\lim}(1, r, \beta, n) \subset M_{t\omega}(1, r, \beta, n).$$

Proof. Take an object $E \in M_{\lim}(1, r, \beta, n)$, and an exact sequence in $\mathcal{A}_\omega$,

(191)
$$0 \to A \to E \to B \to 0.$$

Since $\mathrm{rank}(E) = 1$, one of A or B is an object in $\mathcal{B}_\omega$. Suppose that $A \in \mathcal{B}_\omega$, and it destabilizes E with respect to $Z_{t\omega}$-stability,

(192)
$$\arg Z_{t\omega}(A) > \frac{\pi}{2}.$$

We first show that $A \in \mathcal{T}_\omega[-1]$, i.e. $\mathcal{H}^0(A) = 0$. Suppose by contradiction that $\mathcal{H}^0(A) \neq 0$. Since $\mathcal{H}^0(A)$ is a torsion sheaf on $\overline{X}$ and $E \in \mathcal{A}(r)$, the definition of $\mathcal{A}(r)$ yields that

$$\mathrm{Hom}(\mathcal{H}^0(A), E) = 0.$$

However this is a contradiction since $\mathcal{H}^0(A)$ is a subobject of E in $\mathcal{A}_\omega$. Hence $\mathcal{H}^0(A) = 0$ and $A \in \mathcal{T}_\omega[-1]$ follows.

Since $A \in \mathcal{T}_\omega[-1]$, we can take an exact sequence in $\mathcal{A}_\omega$,

$$0 \to T''[-1] \to A \to T'[-1] \to 0,$$

with $T'' \in \mathrm{Coh}_{\overline{\pi}}^{\leq 1}(\overline{X})$ and $T' \in \mathcal{T}_\omega^{\mathrm{pure}}$. The composition of injections in $\mathcal{A}_\omega$,

$$T''[-1] \hookrightarrow A \hookrightarrow E,$$

is also an injection in $\mathcal{A}(r)$ by Lemma 9.14 below. Hence the $\mu_{i\omega}$-limit semistability of E yields $\mathrm{ch}_3(T''[-1]) \geq 0$, or equivalently

$$(193) \qquad \arg Z_{t\omega}(T''[-1]) \leq \frac{\pi}{2},$$

for all $t \in \mathbb{R}_{>0}$. By (192) and (193), we have $T' \neq 0$ and

$$\arg Z_{t\omega}(T'[-1]) \geq \arg Z_{t\omega}(A)$$
$$> \frac{\pi}{2}.$$

If we write $\mathrm{cl}_0(T') = (r', \beta', n')$, then $r' > 0$ and the above inequality yields,

$$(194) \qquad -n' + \frac{1}{2}t^2\omega^2 r' < 0.$$

By applying Lemma 9.8, we see that (r', n') have only a finite number of possibilities for fixed β and ω. Therefore there is a constant $M(\beta, \omega) > 0$ which depends only on β and ω such that if $t > M(\beta, \omega)$, then the inequality (194) is violated. This means that for such t, the inequality (192) is not satisfied, i.e. $\arg Z_{t\omega}(A) \leq \pi/2$ follows.

In the case of $B \in \mathcal{B}_\omega$, we can similarly prove the inequality $\arg Z_{t\omega}(B) \geq \pi/2$ for $t > M(\beta, \omega)$, by replacing $M(\beta, \omega)$ if necessary. Therefore E is $Z_{t\omega}$-semistable for $t > M(\beta, \omega)$, hence for $t > t_k$.   q.e.d.

We have used the following lemma.

Lemma 9.14. *For $E \in \mathcal{A}(r)$ with $\mathrm{rank}(E) = 1$ and $F \in \mathrm{Coh}_{\overline{\pi}}^{\leq 1}(\overline{X})$, take an exact sequence in $\mathcal{A}_\omega$,*

$$(195) \qquad 0 \to F[-1] \to E \to G \to 0.$$

Then we have $G \in \mathcal{A}(r)$, hence the sequence (195) is an exact sequence in $\mathcal{A}(r)$.

Proof. Taking the cohomology of (195), we have the exact sequence of sheaves,

$$(196) \qquad 0 \to \mathcal{H}^0(E) \to \mathcal{H}^0(G) \to F \to \mathcal{H}^1(E) \to \mathcal{H}^1(G) \to 0.$$

Since $\mathcal{H}^1(E) \in \mathrm{Coh}_{\overline{\pi}}^{\leq 1}(\overline{X})$, we have $\mathcal{H}^1(G) \in \mathrm{Coh}_{\overline{\pi}}^{\leq 1}(\overline{X})$. In particular we have

(197) $$\mathcal{H}^1(G)[-1] \in \mathcal{A}(r).$$

Suppose that the maximal torsion subsheaf $\mathcal{H}^0(G)_{\mathrm{tor}} \subset \mathcal{H}^0(G)$ is non-zero. Then $\mathcal{H}^0(G)_{\mathrm{tor}}$ is a pure two dimensional sheaf, since $G \in \mathcal{A}_\omega \subset \mathcal{A}'_\omega$ and $\mathcal{A}'_\omega$ is a tilting by $(\mathcal{T}_\omega, \mathcal{F}'_\omega)$. (cf. Definition 7.1.) Also since $F \in \mathrm{Coh}_{\overline{\pi}}^{\leq 1}(\overline{X})$, the sequence (196) implies that $\mathcal{H}^0(E)$ is isomorphic to $\mathcal{H}^0(G)$ in codimension one. In particular, the maximal torsion subsheaf $\mathcal{H}^0(E)_{\mathrm{tor}} \subset \mathcal{H}^0(E)$ is also a two dimensional sheaf. However this contradicts to $E \in \mathcal{A}(r)$ and the definition of $\mathcal{A}(r)$. Therefore $\mathcal{H}^0(G)$ is a torsion free sheaf of rank one, and it can be written as

$$\mathcal{H}^0(G) \cong L \otimes I_Z,$$

for some $L \in \mathrm{Pic}(\overline{X})$ and $Z \subset \overline{X}$ with $\dim Z \leq 1$. By the assumptions $E \in \mathcal{A}(r)$ and $F \in \mathrm{Coh}_{\overline{\pi}}^{\leq 1}(\overline{X})$, it is easy to see from the sequence (196) that $L \in \pi^* \mathrm{Pic}(\mathbb{P}^1)$ and Z is supported on the fibers of π. Hence it follows that

(198) $$\mathcal{H}^0(G) \in \mathcal{A}(r).$$

By (197) and (198), we have $G \in \mathcal{A}(r)$.   q.e.d.

10. Results on the category $\mathcal{A}_\omega(1/2)$

In this section, we give proofs of some results on the category $\mathcal{A}_\omega(1/2)$ introduced in Subsection 3.6. In particular, we prove Lemma 3.14 in Subsection 10.1, Lemma 3.16 in Subsection 10.2 and Proposition 3.17 in Subsection 10.3. First we note that, by the existence of Harder-Narasimhan filtrations with respect to $Z_{0\omega}$-stability, (cf. Definition 3.13,) there is a filtration for any $E \in \mathcal{A}_\omega$,

(199) $$0 = E_0 \subset E_1 \subset E_2 \subset E_3 = E,$$

such that

(200) $$E_1 \in \mathcal{A}_\omega(1), \ E_2/E_1 \in \mathcal{A}_\omega(1/2), \ E/E_2 \in \mathcal{A}_\omega(0).$$

We also note that

(201) $$\mathrm{Hom}(E_1, E_2) = 0, \ E_i \in \mathcal{A}_\omega(\phi_i),$$

if $\phi_1 > \phi_2$. We note that, by setting $\mathcal{A}_\omega(\phi+1) = \mathcal{A}_\omega(\phi)[1]$, the family of subcategories $\mathcal{A}_\omega(\phi) \subset \mathcal{D}$ for $\phi \in \mathbb{R}$ determines a slicing on $\mathcal{D}$. (cf. [**10**, Definition 3.3].)

10.1. Proof of Lemma 3.14. In this subsection, we prove Lemma 3.14, which is restated as follows:

Lemma 10.1. *(i) An object $E \in \mathcal{A}_\omega$ is $Z_{0\omega}$-(semi)stable if and only if E is $Z_{t\omega}$-(semi)stable for $0 < t \ll 1$.*
(ii) Any object $E \in \mathcal{A}_\omega(1/2)$ satisfies $\mathrm{ch}_3(E) = 0$.
(iii) The category $\mathcal{A}_\omega(1/2)$ is an abelian subcategory of $\mathcal{A}_\omega$.

Proof. For $A \in \mathcal{B}_\omega$ with $\mathrm{cl}_0(A) = (r, \beta, n)$, the definition of $Z_{t\omega,0}$ yields the following:

- We have $\arg Z_{t\omega,0}(A) \to \pi/2$ for $t \to 0$ iff $n = 0$ and $\beta \cdot \omega \neq 0$.
- We have $\arg Z_{t\omega,0}(A) \to 0$ for $t \to 0$ iff $n > 0$ and $\beta \cdot \omega \neq 0$.
- We have $\arg Z_{t\omega,0}(A) \to \pi$ for $t \to 0$ iff $n < 0$ and $\beta \cdot \omega \neq 0$, or $\beta \cdot \omega = 0$.

Noting above, the results of (i) and (ii) easily follow from the definitions of $Z_{t\omega}$, $\mathcal{A}_\omega(1/2)$, and the results, proofs of Proposition 9.9, Proposition 9.10. In order to check (iii), take $E, E' \in \mathcal{A}_\omega(1/2)$ and a non-zero morphism in $\mathcal{A}_\omega$,

$$u \colon E \to E'.$$

We show that $\mathrm{Ker}(u)$, $\mathrm{Im}(u)$ and $\mathrm{Cok}(u)$ in $\mathcal{A}_\omega$ are contained in $\mathcal{A}_\omega(1/2)$. Since u is decomposed as $E \twoheadrightarrow \mathrm{Im}(u) \hookrightarrow E'$ in $\mathcal{A}_\omega$, we have

$$\mathrm{Hom}(F, \mathrm{Im}(u)) = \mathrm{Hom}(\mathrm{Im}(u), F') = 0,$$

for any $F \in \mathcal{A}_\omega(1)$ and $F' \in \mathcal{A}_\omega(0)$ by (201). This implies $\mathrm{Im}(u) \in \mathcal{A}_\omega(1/2)$ by the existence of a filtration (199) satisfying (200). Therefore we may assume that u is injective or surjective in $\mathcal{A}_\omega$. Suppose that u is surjective. Then we have $\mathrm{Hom}(F, \mathrm{Ker}(u)) = 0$ for any $F \in \mathcal{A}_\omega(1)$, hence we have

$$\mathrm{Ker}(u) \in \langle \mathcal{A}_\omega(1/2), \mathcal{A}_\omega(0) \rangle_{\mathrm{ex}}.$$

There is an exact sequence in $\mathcal{A}_\omega$,

$$0 \to A_1 \to \mathrm{Ker}(u) \to A_2 \to 0,$$

with $A_1 \in \mathcal{A}_\omega(1/2)$ and $A_2 \in \mathcal{A}_\omega(0)$. Since we have

$$\mathrm{ch}_3(\mathrm{Ker}(u)) = \mathrm{ch}_3(E) - \mathrm{ch}_3(E')$$
$$= 0,$$

and $\mathrm{ch}_3(A_1) = 0$ by (ii), we have $\mathrm{ch}_3(A_2) = 0$ if $A_2 \neq 0$. However this contradicts to $\arg Z_{t\omega}(A_2) \to 0$ for $t \to 0$. Hence $A_2 = 0$ and $\mathrm{Ker}(u) \in \mathcal{A}_\omega(1/2)$ follows. A similar argument shows that $\mathrm{Cok}(u) \in \mathcal{A}_\omega$ when u is an injection in $\mathcal{A}_\omega$. q.e.d.

10.2. Proof of Lemma 3.16. In this subsection, we prove Lemma 3.16, which is restated as follows:

Lemma 10.2. *An object $E \in \mathcal{A}_\omega$ is $Z_{0\omega}$-semistable satisfying*

$$\lim_{t \to 0} \arg Z_{t\omega}(E) = \pi/2,$$

if and only if $E \in \mathcal{A}_\omega(1/2)$ and E is $\widehat{Z}_{\omega,1/2}$-semistable.

Proof. First assume that $E \in \mathcal{A}_\omega$ is $Z_{0\omega}$-semistable with $\arg Z_{t\omega}(E) \to \pi/2$ for $t \to 0$. By the definition of $\mathcal{A}_\omega(1/2)$, we have $E \in \mathcal{A}_\omega(1/2)$. Take an exact sequence in $\mathcal{A}_\omega(1/2)$,

(202) $0 \to F \to E \to G \to 0.$

The above sequence is also an exact sequence in $\mathcal{A}_\omega$. By the $Z_{0\omega}$-(semi)stability of E, we have

(203) $\arg Z_{t\omega}(F) \leq \arg Z_{t\omega}(G),$

for $0 < t \ll 1$. Since $\mathrm{ch}_3(F) = \mathrm{ch}_3(G) = 0$ by Lemma 10.1, the above inequality implies

$$\arg \widehat{Z}_{\omega,1/2}(F) \leq \arg \widehat{Z}_{\omega,1/2}(G).$$

Therefore E is $\widehat{Z}_{\omega,1/2}$-semistable in $\mathcal{A}_\omega(1/2)$.

Conversely, suppose that $E \in \mathcal{A}_\omega(1/2)$ is $\widehat{Z}_{\omega,1/2}$-semistable, and take an exact sequence in $\mathcal{A}_\omega$,

(204) $0 \to F' \to E \to G' \to 0.$

We would like to see that

(205) $\arg Z_{t\omega}(F') \leq \arg Z_{t\omega}(G'),$

for $0 < t \ll 1$. If both of $\mathrm{rank}(F')$ and $\mathrm{rank}(G')$ are positive, then we have

$$\arg Z_{t\omega}(F') = \arg Z_{t\omega}(G') = \frac{\pi}{2},$$

for $0 < t \ll 1$. Therefore we may assume that $\mathrm{rank}(F') = 0$ or $\mathrm{rank}(G') = 0$. We discuss the case of $\mathrm{rank}(F') = 0$. The other case is similarly discussed. As in (199), we take a filtration in $\mathcal{A}_\omega$,

$$0 = F'_0 \subset F'_1 \subset F'_2 \subset F'_3 = F',$$

such that $F'_1 \in \mathcal{A}_\omega(1)$, $F'_2/F'_1 \in \mathcal{A}_\omega(1/2)$ and $F'/F'_2 \in \mathcal{A}_\omega(0)$. Since $E \in \mathcal{A}_\omega(1/2)$, we have $\mathrm{Hom}(F'_1, E) = 0$, hence $F'_1 = 0$. Suppose that $F'/F'_2 \neq 0$. Then we have $\mathrm{rank}(F') = 0$, $\mathrm{ch}_3(F') > 0$, hence $\arg Z_{t\omega}(F') \to 0$ for $t \to 0$. Since $\arg Z_{t\omega}(G') = \pi/2$, the inequality (205) is satisfied for $0 < t \ll 1$. Suppose that $F'/F'_2 = 0$. Then the

sequence (202) is an exact sequence in $\mathcal{A}_\omega(1/2)$ by Lemma 10.1. Hence by the $\widehat{Z}_{\omega,1/2}$-semistability of E, we have

$$\arg \widehat{Z}_{\omega,1/2}(F') \leq \arg \widehat{Z}_{\omega,1/2}(G'),$$

which implies the inequality (205) for $0 < t \ll 1$.   q.e.d.

10.3. Proof of Proposition 3.17. In this subsection, we prove Proposition 3.17, which is restated as follows: we have the following proposition.

Proposition 10.3. *For fixed $\beta \in \mathrm{NS}(S)$ and an ample divisor ω on S, there is a finite sequence,*

$$0 = \theta_k < \theta_{k-1} < \cdots < \theta_1 < \theta_0 = 1/2,$$

such that the following holds.
(i) The set of objects

$$\bigcup_{(R,r), R \geq 1} \widehat{M}_{\omega,\theta}(R, r, \beta),$$

is constant for $\theta \in (\theta_{i-1}, \theta_i)$.
(ii) For $0 < t \ll 1$ and any $(R, r, \beta) \in \widehat{\Gamma}$, we have

$$\widehat{M}_{\omega,1/2}(R, r, \beta) = M_{t\omega}(R, r, \beta, 0).$$

(iii) For $\theta \in (0, \theta_{k-1})$, we have

$$\widehat{M}_{\omega,\theta}(1, r, \beta) = \begin{cases} \{\pi^* \mathcal{O}_{\mathbb{P}^1}(r)\}, & \text{if } \beta = 0, \\ \emptyset, & \text{if } \beta \neq 0. \end{cases}$$

Proof. (i) Take $E \in \mathcal{B}_\omega(1/2)$ with $\widehat{\mathrm{cl}}_0(E) = (r', \beta')$. Suppose that $0 \leq -\beta' \cdot \omega \leq -\beta \cdot \omega$, and let $F_1, F_2, \cdots, F_N \in \mathcal{B}_\omega$ be the Harder-Narasimhan factors of E with respect to $Z_{0\omega}$-stability. Since $E \in \mathcal{B}_\omega(1/2)$, we have $F_i \in \mathcal{B}_\omega(1/2)$ for all i, and we write $\widehat{\mathrm{cl}}_0(F_i) = (r_i, \beta_i)$. Because $-\beta_i \cdot \omega \leq -\beta \cdot \omega$, Lemma 10.1 (i) and Lemma 9.11 imply that there is only a finite number of possibilities for r_i w.r.t. fixed β and ω. Also noting that $N \leq -\beta \cdot \omega$, the value

$$r' = \sum_{i=1}^N r_i,$$

is also bounded. Therefore for fixed β and ω, the set of $\theta \in (0, 1/2]$ satisfying

$$\widehat{Z}_{\omega,\theta}(E) = -r' - (\omega \cdot \beta')\sqrt{-1}$$

$$\in \mathbb{R}_{>0} e^{i\pi\theta},$$

for some $E \in \mathcal{B}_\omega(1/2)$ with $\widehat{\mathrm{cl}}_0(E) = (r', \beta')$, $-\beta' \cdot \omega \leq -\beta \cdot \omega$ is a finite set. If we denote this finite set by $0 = \theta_k < \theta_{k-1} < \cdots < \theta_1 < \theta_0 = 1/2$, then $\theta_\bullet$ satisfies the desired condition.

(ii) The result of (ii) is a consequence of Lemma 10.1 (i) and Lemma 10.2.

(iii) For an object $E \in \widehat{M}_{\omega,\theta}(1, r, \beta)$, take an exact sequence in $\mathcal{A}_\omega$,

$$(206) \qquad\qquad 0 \to A \to E \to B \to 0,$$

with $A \in \mathcal{B}_\omega$ and $B \in \pi^* \mathrm{Pic}(\mathbb{P}^1)$, as in Lemma 7.5. By Lemma 9.6, we have $B \in \mathcal{A}_\omega(1/2)$, hence $A \in \mathcal{B}_\omega(1/2)$ by Lemma 10.1, i.e. (206) is an exact sequence in $\mathcal{A}_\omega(1/2)$. Suppose that A is non-zero. Then we can write $\widehat{\mathrm{cl}}_0(A) = (r', \beta)$ for some $r' \in \mathbb{Z}$, and β should satisfy $\beta \cdot \omega \neq 0$. By the $\widehat{Z}_{\omega,\theta}$-semistability of E, we have

$$\arg \widehat{Z}_{\omega,\theta}(A) \leq \pi\theta.$$

The above inequality is equivalent to

$$\frac{r'}{\beta \cdot \omega} \geq \frac{1}{\tan \pi\theta}.$$

Since the RHS goes to ∞ for $\theta \to 0$, the object E is $\widehat{Z}_{\omega,\theta}$-semistable for $0 < \theta \ll 1$ only if $A = 0$, i.e. $E \cong \pi^* \mathcal{O}_{\mathbb{P}^1}(r)$. Therefore we have $\widehat{M}_{\omega,\theta}(1, r, \beta) = \emptyset$ for $\beta \neq 0$ and $0 < \theta \ll 1$, and an only possible object in $\widehat{M}_{\omega,\theta}(1, r, 0)$ for $0 < \theta \ll 1$ is $\pi^* \mathcal{O}_{\mathbb{P}^1}(r)$. On the other hand, since $\mathcal{A}_\omega(1/2) \subset \mathcal{C}_\omega^\perp$, (cf. (162),) Lemma 9.6 immediately implies that $\pi^* \mathcal{O}_{\mathbb{P}^1}(r)$ is $\widehat{Z}_{\omega,\theta}$-stable for any $\theta \in (0, 1)$. Therefore we obtain the result.   q.e.d.

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