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AN EXTENSION OF SCHÄFFER'S DUAL GIRTH CONJECTURE TO GRASSMANNIANS

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Abstract

In this note we introduce a natural Finsler structure on convex surfaces, referred to as the quotient Finsler structure, which is dual in a sense to the inclusion of a convex surface in a normed space as a submanifold. It has an associated quotient girth, which is similar to the notion of girth defined by Schäffer. We prove the analogs of Schäffer's dual girth conjecture (proved by Álvarez-Paiva) and the Holmes–Thompson dual volumes theorem in the quotient setting. We then show that the quotient Finsler structure admits a natural extension to higher Grassmannians, and prove the corresponding theorems in the general case. We follow Álvarez-Paiva's approach to the problem, namely, we study the symplectic geometry of the associated co-ball bundles. For the higher Grassmannians, the theory of Hamiltonian actions is applied.

In the following, we will be concerned with certain invariants of Finsler manifolds that are associated naturally to real, finite-dimensional normed spaces. For a survey of Schäffer's work on the subject, and some related facts from convex geometry, see [**12**].

Consider a normed space V. Let $M \subset V$ be a closed hypersurface. As a submanifold of V, M inherits a natural Finsler structure (that is, a norm on every tangent space), denoted ψ_V. We call (M, ψ_V) the *immersion* Finsler structure on M, and write $\psi_V(v) = \|v\|_V$ for $v \in T_mM$.

However, if M is the boundary of a strictly star-shaped body with a center at the origin, then there is another, equally natural Finsler structure ϕ_V on M, induced by V (here by strictly star-shaped we mean that $m \notin T_mM$ for all $m \in M$). Namely, we identify the tangent space T_mM with the quotient space $V/\langle m\rangle$, where $\langle m\rangle$ is the 1-dimensional linear space spanned by m. We call (M, ϕ_V) the *quotient* Finsler structure on M, and may write for $v \in T_mM$

$$\phi_V(v) = \inf_{t\in\mathbb{R}} \|v + tm\|_V$$

If $M = S(V)$ is the unit sphere of V, what we get is a Finsler structure on $M = S^{n-1}$ (as smooth manifolds), intrinsically attached to the

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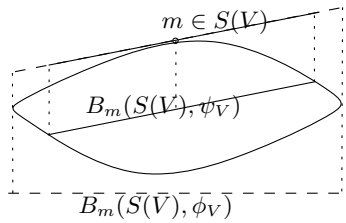
normed space V. It is the quotient structure and its generlizations that we will study throughout the paper. In Figure 1, below, the two Finsler structures on $S(V)$ are depicted simultaneously, and B_q denotes the unit ball of the corresponding norm.

Figure 1. Immersion and quotient unit balls



The girth of a Finsler structure on a sphere is the length of the shortest symmetric curve. We may already state our first theorem:

Theorem. *Let V be a normed space. Then the girth of $(S(V), \phi_V)$ equals the girth of $(S(V^*), \phi_{V^*})$.*

This should be compared with Schäffer's dual girth conjecture [**11**], proved by Álvarez-Paiva in [**1**]:

Theorem. *(***Schäffer, Álvarez-Paiva***). Let V be a normed space. Then the girth of $(S(V), \psi_V)$ equals the girth of $(S(V^*), \psi_{V^*})$.*

Next, we consider a more general scenario. Let V be a linear space, and $K, L \subset V$ convex bodies (by a convex body we always mean the unit ball of some symmetric norm). Let V_L denote the normed space V with unit ball L. Then ∂K has two Finsler structures induced on it from V_L: the immersion structure ψ_L, and the quotient structure ϕ_L. We prove a theorem concerning the quotient structure that parallels the generalized dual girth conjecture proved in [**1**] and the Holmes–Thompson dual volumes theorem [**6**], which in turn concern the immersion Finsler structure. More precisely, we prove the following:

Theorem. *Let $K, L \subset V$ be convex bodies. Then $(\partial K, \phi_L)$ and $(\partial L^o, \phi_{K^o})$ have equal girth and Holmes–Thompson volume. If K, L are smooth and strictly convex, then the spectra and the symmetric length spectra coincide.*

For the definitions of Holmes-Thompson volume, length spectrum, etc., see section 1.

As it turns out, the quotient Finsler structure admits a natural extension to higher Grassmannians, together with the duality theorems stated above (for the immersion structure, no such extension is known). Consider the oriented Grassmannian $\tilde{G}(V, k)$, which is the set of oriented

k-dimensional subspaces of the n-dimensional space V, with the natural smooth manifold structure. Assume some norm β is given on the space of linear operators $\mathrm{Hom}(V, V)$. We describe an associated Finsler structure on $\tilde{G}(V, k)$, denoted ϕ_β, which is defined for every $1 \leq k \leq n-1$; for $k = 1$ and β the nuclear norm on $\mathrm{Hom}(V_K, V_L)$, it reduces to the quotient Finsler structure $(\partial K, \phi_L)$. We then study some of its invariants, such as the girth and Holmes–Thompson volume. Also, we associate a certain natural number to each geodesic, called its rank. We prove the following:

Theorem. *For any norm β on Hom(V, V), the manifolds $(\tilde{G}(V, k), \phi_\beta)$ and $(\tilde{G}(V, n-k), \phi_\beta)$ have equal Holmes–Thompson volumes and equal girth. If β is smooth and strictly convex, the length spectra and symmetric length spectra of the two manifolds coincide, alongside with the corresponding ranks of geodesics.*

In contrast to the case $k = 1$, where the result follows from the existence of a natural diffeomorphism between the co-sphere bundles that preserves the canonic 1-form, for general k we are only able to construct such a morphism on a dense open subset. We then make use of a natural decomposition of the cotangent bundles and apply an implicit existence lemma from linear algebra to construct a correspondence between the geodesics of the two manfifolds.

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1. Background from Finsler and symplectic geometry

For completeness, we recall the basic definitions of Finsler geometry and its relation to symplectic geometry. We also fix notation that will later be used. For more details, consult [**2**].

1.1. General properties of girth. Recall that a smooth Finsler manifold (M, ϕ) is a smooth manifold M, together with a function $\phi : TM \to \mathbb{R}$ that is smooth outside the zero section, and restricts to a strictly convex norm on $T_m M$ for every $m \in M$. The notions of length of a curve and distance on M are identical to their Riemannian manifolds counterpart.

Remark 1.1. By strictly convex, we mean that the gaussian curvature is strictly positive. In the following, we will also make use of the term *weakly strictly convex*, which means that the boundary of the body does not contain straight segments.

Remark 1.2. We sometimes relax the requirement for ϕ to be a smooth function with strictly convex restrictions to tangent spaces, settling only for continuity and convex restrictions. This allows us to treat stable invariants of Finsler manifolds, such as girth and Holmes–Thompson volume (see definitions below), under more general conditions.

Definition 1.3. Let (M, R) be a smooth compact manifold with an involutive diffeomorphism $R : M \to M$ that has no fixed points. We refer to R as the antipodal map, and also denote the natural extension $R : T^*M \to T^*M$ by the same symbol. We say that a Finsler metric ϕ on M is symmetric if R is an isometry of (M, ϕ). The girth $g_\phi(M)$ is the length of the shortest symmetric (i.e., R-invariant) geodesic.

Remark 1.4. Note that $g_\phi(M) = 2\min\{\mathrm{dist}(p, Rp) : p \in M\}$ for the following reason: Take the closest pair of antipodal points (p, Rp), with $d = \mathrm{dist}(p, Rp)$. Take a curve γ between them with $L(\gamma) = d$. Then $R\gamma$ also joins (p, Rp) and $L(R\gamma) = d$, and for any $q \in \gamma$, we get two curves between q and Rq, which have length d. Thus $\mathrm{dist}(q, Rq) = d$ by minimality assumption, and so those curves are geodesic. In particular, $\gamma \cup R\gamma$ is geodesic around p, i.e., a symmetric closed geodesic. Therefore, $g_\phi(M) \le 2\min\{\mathrm{dist}(p, Rp) : p \in M\}$. The reverse inequality is obvious.

Lemma 1.5. *Let ϕ_1, ϕ_2 be symmetric Finsler structures on (M, R), such that $(1+\epsilon)^{-1}\phi_1 \le \phi_2 \le (1+\epsilon)\phi_1$. Then $(1+\epsilon)^{-1} g_{\phi_1}(M) \le g_{\phi_2}(M) \le (1+\epsilon) g_{\phi_1}(M)$.*

Proof. Denote $g_j = g_{\phi_j}(M)$. Consider the ϕ_1-shortest curve $\gamma \subset \tilde{G}(V, k)$ with $\gamma(1) = R\gamma(0)$. Then $|L(\gamma; \phi_2) - L(\gamma; \phi_1)| \le \int_0^1 |\phi_2(\dot\gamma) - \phi_1(\dot\gamma)| dt \le \epsilon L(\gamma; \phi_1)$. Therefore, $g_2 \le 2L(\gamma, \phi_2) \le 2(1+\epsilon)L(\gamma; \phi_1) = g_1(1+\epsilon)$. Similarly, $g_1 \le g_2(1+\epsilon)$. q.e.d.

Remark 1.6. The lemma often allows one to omit assumptions of smoothness and strict convexity of the Finsler metric when studying the girth of spaces.

1.2. Symplectic invariants. Let (M, ϕ) be a smooth Finsler manifold. A curve γ on M is a geodesic if it locally minimizes distance. The set of lengths of closed geodesics on M is called the length spectrum, and if M is equipped with an antipodal map, the symmetric length spectrum is the set of lengths of closed symmetric geodesics. The girth is then the minimal element of the symmetric length spectrum.

Every cotangent space T_m^*M is equipped with the dual norm ϕ_m^*. Define the co-ball bundle by $B^*M = \{(m, \xi) \in T^*M : \phi_m^*(\xi) \le 1\}$,

and similarly S^*M is the co-sphere bundle. Denote also by T_0^*M the cotangent bundle with the zero-section excluded. One has the Legendre duality map $\mathcal{L}_m : T_mM \to T_m^*M$ (see subsection 2.1 for the definition of Legendre transform). For a curve $\gamma \in M$ we define its lift to T^*M, denoted $\mathcal{L}\gamma$, by $\mathcal{L}\gamma(t) = (\gamma(t), \mathcal{L}_{\gamma(t)}(\dot\gamma(t)))$. We denote the canonic 1-form and symplectic form on T^*M by α and $\omega = d\alpha$. Also, we define the associated Hamiltonian function $H_M(m,\xi) = \frac{1}{2}\phi^*(\xi)^2$ on T^*M.

There are several natural volume densities on a Finsler manifold. We will use exclusively the Holmes–Thompson volume v_{HT}, defined as the push-forward under the natural projection $B^*M \to M$ of the Liouville volume form $\frac{1}{n!}\omega^n$ on B^*M. Note that one only needs the Finsler structure to be continuous in order to define the Holmes–Thompson density, and that $v_{HT}(M) = \frac{1}{n!}\int_{B^*M}\omega^n$ is continuous as a function of $\phi \in C(TM,\mathbb{R})$.

Proposition 1.7. *Let M be a geodesically complete Finsler manifold. For a curve $\gamma(t) \in M$, denote $\Gamma(t) = \mathcal{L}\gamma(t) \in T^*M$. The following are equivalent:*

1) *$\gamma(t)$ is a geodesic with arc-length parametrization.*
2) *$\Gamma(t)$ is a flow curve for the associated Hamiltonian H_M of constant energy $H_M = \frac{1}{2}$.*
3) *$\Gamma(t)$ is a flow curve for the Reeb vector field on S^*M.*

Moreover, a curve $\Gamma(t)$ satisfying either 2 or 3 is necessarily the lift of a geodesic $\gamma(t)$.

For completeness, we sketch the proof of this well-known fact.

Proof. A geodesic between $a = \gamma(t_0)$ and $b = \gamma(t_1)$ with a parametrization proportional to arc-length if and only if it is a minimizer of the energy functional $E(\gamma(t)) = \frac{1}{2}\int_{t_0}^{t_1}\phi(\dot\gamma(t))^2 dt$ (see [**8**], Ch. 12). By the Lagrangian-Hamiltonian duality, those lift precisely to the flow curves of H_M on T^*M. The parametrization is arc-length $\iff \phi^*(\mathcal{L}\dot\gamma) \equiv 1 \iff H(\mathcal{L}\gamma) = \frac{1}{2}$. The equivalence of 2 and 3 follows from the 2-homogeneity of H_M. q.e.d.

Slightly abusing the standard terminology, we will call a curve $\Gamma(t)$ satisfying either condition 2 or 3 a *characteristic curve.*

Corollary 1.8. *Suppose M and N are two Finsler manifolds, and either*

- *$\Phi : T_0^*M \to T_0^*N$ is a symplectomorphism such that $\Phi^*H_N = H_M$, or*
- *$\Phi : S^*M \to S^*N$ is a diffeomorphism s.t. $\Phi^*\alpha_N = \alpha_M$.*

Then M and N have equal length spectrum. If M and N both possess antipodal maps R_M, R_N and $\Phi R_M = R_N\Phi$ then the symmetric length spectra of M and N coincide as well.

Remark 1.9. It is easy to verify that, given two 2-homogeneous Hamiltonian functions H_M, H_N on T^*M and T^*N, respectively, and a symplectomorphism between open conic subsets $\Phi : C_M \to C_N$, where $C_M \subset T^*M$ and $C_N \subset T^*N$, such that $\Phi^* H_N = H_M$, then it must necessarily preserve the canonic one-form: $\Phi^* \alpha_N = \alpha_M$.

2. Quotient girth

2.1. Definitions and basic properties. Let us first introduce some notation. For a normed space V, $S(V)$ denotes the unit sphere, and $B = B(V)$ the unit ball of V. For $q \in S(V)$, the Legendre transform $\mathcal{L}_V(q) \in S(V^*)$ denotes the unique covector ξ for which $\{x \in V : \xi(x) = 1\}$ is the tangent hyperplane to $S(V)$ at q. When no confusion can arise, we write $\mathcal{L}$ instead of $\mathcal{L}_V$. By a convex body, we will always mean the unit ball of some symmetric norm on V. For a convex body $K \subset V$, V_K will denote the normed space V with unit ball K. Denote by $\mathcal{K}(n)$ the set of convex bodies in $\mathbb{R}^n$ equipped with the Hausdorff metric.

Definition 2.1. For a pair of convex bodies $K, L \subset V$, denote by ψ_L the immersion Finsler structure, induced on ∂K by the embedding $\partial K \subset V_L$, and by ϕ_L the quotient Finsler structure on ∂K, given by the obvious identification $T_q(\partial K) = V_L/\langle q \rangle$. The immersion girth $g_i(K; L)$ and the quotient girth $g_q(K; L)$ of K with respect to L is the length of the shortest closed symmetric curve on ∂K with the corresponding metric.

Remark 2.2. For a normed space V with unit ball B, denote $\phi_V = \phi_{B,B}$ and $g_q(V) = g_q(B, B)$. The Finsler manifold $(S(V), \phi_V)$ is an intrinsic invariant of normed spaces. Moreover, for a subspace $U \subset V$, the intrinsic Finsler structure ϕ_U on $S(U)$ coincides with the one inherited by the inclusion $S(U) \subset S(V)$.

Remark 2.3. The notions of quotient girth and quotient Holmes–Thompson volume extend easily to any pair of convex bodies K, L, without any smoothness or strict convexity assumptions (see also Remark 1.6) Thus all results stated below for the girth and Holmes–Thompson volume extend to the general case by continuity.

Let us begin by comparing the immersion and quotient Finsler metrics. Recall that for a 2-dimensional convex body $K \subset V$, its isoperimetrix $I_K \subset V$ is (up to homothety) the dual body K^o, under the identification of V and V^* by the volume form on V.

Proposition 2.4. *Let K, L be smooth and weakly strictly convex. For a pair of convex bodies $K, L \subset V$, one has the inequality $\phi_L \leq \psi_L$ on ∂K. We can describe the case of equality:*

1) *In dimension* $n = 2$, $(\partial K, \phi_L)$ *and* $(\partial K, \psi_L)$ *coincide if and only if* L *is homothetic to* I_K. *In particular,* $\phi_V = \psi_V$ *on* $S(V)$ *if and only if* $S(V)$ *is a Radon curve.*
2) *In dimension* $n \geq 3$, $(\partial K, \phi_L)$ *and* $(\partial K, \psi_L)$ *coincide if and only if* K *and* L *are homothetic ellipsoids.*

Proof. The inequality is obvious. Let us show that $\phi_L = \psi_L$ if and only if for all pairs $x \in \partial K$, $y \in \partial L$, $y \in T_x(\partial K) \Leftrightarrow x \in T_y(\partial L)$.

Fix $x \in \partial K$, $y \in \partial L$ s.t. $y \in T_x(\partial K)$. Then $\psi_L(y) = \|y\|_L$, $\phi_L(x) = \inf_{t\in\mathbb{R}} \|tx + y\|_L$. Then $\psi_L(x) = \phi_L(x)$ if and only if $\|y + tx\|_L \geq \|y\|_L$ for all t, i.e., $x \in T_y(\partial L)$. Since this is true for all pairs x, w, by the weak strict convexity of ∂L the reverse implication also follows. Note that the condition on (K, L) is symmetric.

When $n = 2$, it is easy to see that the equality condition guarantees uniqueness of a body L corresponding to K: one can write a differential equation on the polar representation of L. Taking $L = I_K$ shows existence, proving the case $n = 2$.

Now assume $n \geq 3$. If both K, L are ellipsoids, the lemma above applies. In the other direction, it follows from the lemma that for any $q \in \partial K$, the shadow boundary of L in the direction q lies in the hyperplane $T_q\partial K \subset V$. By Blaschke's Theorem, L is an ellipsoid. By symmetry, so is K. Thus K and L define two Euclidean structures that induce the same orthogonality relation. Therefore, they are homothetic. q.e.d.

Proposition 2.5. *For a normed space with* $\dim V = 2$ *and* $B = B(V)$, *the quotient girth satisfies* $g_q(V) \geq \frac{2}{\pi}\frac{M(B)}{vr(B^o)^2}$, *where* $M(B) = |B \times B^o|$ *is the Mahler volume and* vr *denotes the volume ratio.*

Proof. By continuity of all magnitudes in the inequality in $\mathcal{K}(n)$, we may assume $B(V)$ is smooth and strictly convex. Choose some ellipsoid $E \supset B$, which defines a Euclidean structure. Fix some orthonormal coordinates in V. Denote $\partial B = \gamma(\alpha) = (x(\alpha), y(\alpha))$, where α is the angle measured counterclockwise from some reference direction. Let $\beta = \beta(\alpha)$ be the angle of the point on γ such that $\dot\gamma(\beta)||(-\gamma(\alpha))$ (positively parallel). Solving $\dot\gamma(\alpha) = s(\alpha)\gamma(\beta(\alpha)) + t(\alpha)\gamma(\alpha)$, we get

$$\phi_V(\dot\gamma(\alpha)) = s(\alpha) = \frac{\dot y(\alpha)x(\alpha) - \dot x(\alpha)y(\alpha)}{y(\beta)x(\alpha) - x(\beta)y(\alpha)}$$

Both numerator and denominator are positive. Thus

$$g_q(V) = \int_0^{2\pi} s(\alpha)d\alpha = \int_0^{2\pi} \frac{\det(\gamma(\alpha), \dot\gamma(\alpha))}{\det(\gamma(\alpha), \gamma(\beta))}d\alpha$$

The denominator is bounded from above by $|\det(\gamma(\alpha), \gamma(\beta))| \leq |\gamma(\alpha)||\gamma(\beta)| \leq 1$. Therefore,

$$g_q(V) \geq \int_0^{2\pi} \det(\gamma(\alpha), \dot\gamma(\alpha))d\alpha = 2Area_E(B)$$

where $Area_E$ denotes the Lebesgue measure, normalized so that $Area_E(E) = \pi$. It remains to choose the optimal E:

$$g_q(V) \geq 2\pi \max_{E \supset B} \frac{|B|}{|E|} = 2\pi \max_{E^o \subset B^o} \frac{|B \times B^o|}{|E \times E^o|} \frac{|E^o|}{|B^o|} = \frac{2}{\pi} \frac{M(B)}{vr(B^o)^2}$$

q.e.d.

Corollary 2.6. *For all 2-dimensional normed spaces V, $4 < g_q(V) < 8$.*

Proof. One inequality is obvious: $g_q(V) \leq g_i(V) \leq 8$ and $g_i(V) = 8$ only for the parallelogram [**11**], which has quotient girth equal to $8 \log 2$ (see Appendix A.3), so in fact $g_q(V) < 8$. For the other inequality, note that by Mahler's conjecture for the plane [**7**, **9**], $M(B)$ is uniquely minimized by the square, while by Ball's theorem [**4**] (and since the dual of a square is a square), $vr(B^o)$ is maximized for the square, and the inequality above is strict for the square. Thus,

$$g_q(V) > \frac{2}{\pi} \frac{8}{4/\pi} = 4$$

q.e.d.

Remark 2.7. It seems plausible that in fact $8 \log 2 \leq g_q(V) \leq 2\pi$ when $\dim V = 2$, the extremal cases being the square (see A.3 for the computation of its quotient girth) and the circle.

Remark 2.8. It is shown by Bernig in [**5**] that the upper bound of 2π does indeed hold and is attained uniquely by an ellipsoid. It follows that an n-dimensional normed space V that has maximal possible quotient girth must have all its 2-dimensional subspaces Euclidean. Therefore, V itself is Euclidean.

From Lemma 1.5 we get the following:

Corollary 2.9. *$g_q(K; L)$ is continuous on $\mathcal{K}(n) \times \mathcal{K}(n)$.*

2.2. Main theorems.

Theorem 2.10. *Let $K, L \subset V$ be smooth and strictly convex bodies. Then there is a diffeomorphism $\Phi : S^*(\partial K, \phi_L) \to S^*(\partial L^o, \phi_{K^o})$ respecting the canonic 1-form up to sign. Also, Φ respects the antipodal map.*

Proof. The construction of the diffeomorphism between the corresponding co-sphere bundles is reminiscent of the one in [**1**]. Observe that for $q \in \partial K$ we have the isometric embedding $T_q^*(\partial K) = (V_L/q)^* \hookrightarrow V_{L^o}^*$. In particular, $S_q^*(\partial K) \hookrightarrow \partial L^o$ and in fact $S_q^*(\partial K) = \{p \in \partial L^o : \langle p, q \rangle = 0\}$. Define $Z \subset V \times V^*$ by $Z = \{(q, p) : p(q) = 0\}$. We then can identify $S^*(\partial K) \simeq (\partial K \times \partial L^o) \cap Z$, and by symmetry also $S^*(\partial L^o) \simeq (\partial K \times \partial L^o) \cap Z$. Now observe that the forms $\alpha_1 = pdq$ and

$\alpha_2 = qdp$ defined on $V \times V^*$ satisfy $\alpha_1\Big|_Z + \alpha_2\Big|_Z = 0$ and restrict to the canonic 1-forms on $S^*(\partial K)$ and $S^*(\partial L^o)$, respectively. Thus $\Phi(q,p) = (p,q)$ is the required map. q.e.d.

Remark 2.11. We see from the proof above that ψ is in some sense dual to ϕ: one has a natural isometric isomorphism of the normed bundles $T(\partial K, \psi_L) \to T^*(\partial K^o, \phi_{L^o})$, given by the Legendre transform and the fiberwise isometric identification $T_q(\partial K, \psi_L) = T^*_{\mathcal{L}(q)}(\partial K^o, \phi_{L^o})$.

As a corollary we get the following theorem:

Theorem 2.12. *(Dual spheres have equal quotient girth) Let $K, L \subset V$ be convex bodies. Then $(\partial K, \phi_L)$ and $(\partial L^o, \phi_{K^o})$ have equal girth and Holmes–Thompson volume. If K, L are smooth and strictly convex, then their length spectra and symmetric length spectra coincide. In particular, for any normed space V, we get $g_q(V) = g_q(V^*)$.*

2.3. The associated double fibration. The following is a geometric observation relating the immersion and quotient settings, which is not used elsewhere in the paper.

In the proof of the original girth conjecture in [**1**], the following double fibration appears naturally:

$$\begin{tikzcd} & T \arrow[dl] \arrow[dr] & \\ \partial K & & \partial L^o \end{tikzcd}$$

where $T \subset \partial K \times \partial L^o$ consists of all pairs (q,p) such that the pairing $T_q\partial K \times T_p\partial L^o \to \mathbb{R}$ is degenerate.

In the quotient girth setting, a different fibration appears:

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ \partial L & & \partial K^o \end{tikzcd}$$

where $P = \{(q,p) \in \partial L \times \partial K^o : \langle q,p\rangle = 0\}$ (note that we exchanged the roles of K, L here). There is in fact a natural diffeomorphism of the double fibrations: Let $B : \partial K \times \partial L^o \to \partial L \times \partial K^o$ be given by $B(q,p) = (\mathcal{L}p, \mathcal{L}q)$. One has then $T_q\partial K \times T_p\partial L^o \to \mathbb{R}$ degenerate $\iff$ $\mathcal{L}q \in T_p\partial L^o \iff \langle \mathcal{L}q, \mathcal{L}p\rangle = 0$. Thus $B : T \to P$ is a diffeomorphism, and it respects the double fibration structure.

3. The oriented Grassmannian

3.1. Background. We begin by recalling some basic constructions and fixing notation.

3.1.1. Oriented Grassmannians. The oriented Grassmannian $\tilde{G}(V,k)$ is the set of oriented k-dimensional subspaces of V. It is naturally a smooth manifold. We also write $\tilde{\mathbb{P}}(V) = \tilde{G}(V,1)$, which is the projective space of oriented lines in V. In the following, we always assume that V is an oriented vector space.

Recall the Plücker embedding $i : \tilde{G}(V,k) \to \tilde{\mathbb{P}}(\wedge^k V)$ given by $i(\Lambda) = p(\wedge^k \Lambda)$, where $p : (\wedge^k V)\backslash 0 \to \tilde{\mathbb{P}}(\wedge^k V)$ is the canonic projection. Take $\Lambda \in \tilde{G}(V,k)$, fix a basis $e_1, ..., e_k$ of Λ, and identify $T_\Lambda \tilde{G}(V,k) = T_{i\Lambda} i\big(\tilde{G}(V,k)\big) \simeq \mathrm{Hom}(\Lambda, V/\Lambda)$ by assigning $\dot{\gamma}_f(0) \in T_{i\Lambda} i\big(\tilde{G}(V,k)\big)$ to $f \in \mathrm{Hom}(\Lambda, V/\Lambda)$ through the correspondence $\gamma_f : [0,1] \to i\big(\tilde{G}(V,k)\big)$, $\gamma_f(t) = p\Big((e_1 + tf(e_1)) \wedge ... \wedge (e_k + tf(e_k))\Big)$. Clearly this identification is independent of the choice of $e_1, ..., e_k$. Thus, there is a canonic identification $T_\Lambda \tilde{G}(V,k) \simeq \mathrm{Hom}(\Lambda, V/\Lambda)$.

3.1.2. Norms on spaces of operators. Let A, B be two linear spaces. When given an arbitrary norm β on $\mathrm{Hom}(A,B)$, one immediately obtains a norm $\overline{\beta}$ on $\mathrm{Hom}(B^*, A^*)$ by letting $T \mapsto T^*$ be an isometry, and the dual norm β^* on $\mathrm{Hom}(B,A)$, defined by trace duality. It is immediate that $\overline{\beta^*} = \overline{\beta}^*$ on $\mathrm{Hom}(A^*, B^*)$. Note that for a normed space V, the nuclear (projective) norm $\beta = \| \bullet \|_N$ on $\mathrm{Hom}(V,V)$ satisfies that $\overline{\beta}$ is again the nuclear norm on $\mathrm{Hom}(V^*, V^*)$.

Given a norm β on $\mathrm{Hom}(V,V)$ and a subspace $\Lambda \subset V$, one has the natural inclusion map $\mathrm{Hom}(V/\Lambda, \Lambda) \subset \mathrm{Hom}(V,V)$ and a quotient map $\mathrm{Hom}(V,V) \twoheadrightarrow \mathrm{Hom}(\Lambda, V/\Lambda)$. Denote the induced subspace and quotient space norms by β_i and β_π, respectively. It is immediate that $\overline{\beta_\pi} = \overline{\beta}_\pi$ on $\mathrm{Hom}((V/\Lambda)^*, \Lambda^*)$, while $\overline{\beta_i} = \overline{\beta}_i$ on $\mathrm{Hom}(\Lambda^*, (V/\Lambda)^*)$, $(\beta_i)^* = (\beta^*)_\pi$ on $\mathrm{Hom}(\Lambda, V/\Lambda)$, and $(\beta_\pi)^* = (\beta^*)_i$ on $\mathrm{Hom}(V/\Lambda, \Lambda)$.

3.2. Definition of the Finsler structure.

Definition 3.1. For an arbitrary norm β on $\mathrm{Hom}(V,V)$, $(\tilde{G}(V,k), \phi_\beta)$ is the Finsler manifold that has the quotient norm β_π on the tangent spaces $T_\Lambda \tilde{G}(V,k) = \mathrm{Hom}(\Lambda, V/\Lambda)$. If β is smooth and strictly convex, we get a smooth Finsler manifold. Orientation reversal on subspaces defines an antipodal map on $\tilde{G}(V,k)$ that is an isometry of ϕ_β.

Remark 3.2. By trace duality, the cotangent spaces

$$T^*_\Lambda \tilde{G}(V,k) = \mathrm{Hom}(V/\Lambda, \Lambda) \subset \mathrm{Hom}(V,V)$$

are equipped with the dual norm, which is $(\beta^*)_i$. In the following, we will often consider a cotangent vector $T \in T^*_\Lambda \tilde{G}(V,k)$ simply as an element of $\mathrm{Hom}(V,V)$.

Remark 3.3. As before, when we deal with the girth or Holmes–Thompson volume of the $(\tilde{G}(V,k), \phi_\beta)$, one can omit smoothness and strict convexity assumptions on β.

Example 3.4. For a Euclidean space V and $\beta = \|\|_{HS}$ the Hilbert–Schmidt norm on $\mathrm{Hom}(V,V)$, $(\tilde{G}(V,k);\phi_{HS})$ is the standard $SO(n)$-invariant Riemannian structure.

Remark 3.5. One can consider also the following more functorial construction: Let $K, L \subset V$ be two symmetric convex bodies. For a subspace $\Lambda \subset V$, denote by Λ_K the normed space Λ with unit ball $\Lambda \cap K$. Consider some uniform crossnorm α, that is, an assignment of a norm to $A^* \otimes B \simeq \mathrm{Hom}(A,B)$ for pairs of isometry classes of finite-dimensional normed spaces A, B (see A.1 for a review of crossnorms). Important examples are the operator (injective) norm $\|\|_{Op}$ and the nuclear (projective) norm $\|\|_N$. Equipping the spaces $\mathrm{Hom}(\Lambda_K, V_L/\Lambda)$ with α, $(\tilde{G}(V,k);\phi_{\alpha,K,L})$ becomes a Finsler manifold. If V is a given normed space, we take $K = L = \{\|x\| \le 1\}$ and denote $\phi_{\alpha,K,L} = \phi_{\alpha,V}$.

Example 3.6. For $k = 1$, the Finsler structure on $(\tilde{G}(V,1);\phi_{\alpha,K,L})$ is independent of α, and the induced Finsler structure on $\tilde{\mathbb{P}}(V)$ will be denoted $\phi_{K,L}$. It also coincides with $(\tilde{G}(V,1),\phi_\beta)$ where β is the nuclear norm (or any other projective crossnorm) on $\mathrm{Hom}(V_K, V_L)$.

For a smooth convex body $K \subset V$, $\partial K \simeq \tilde{\mathbb{P}}(V)$ as smooth manifolds. We will show that the Finsler manifold constructed above generalizes the quotient Finsler structure of Section 2.

Proposition 3.7. *As Finsler manifolds, one has* $(\tilde{\mathbb{P}}(V),\phi_{K,L}) = (\partial K,\phi_L)$.

Proof. Fix $q \in \partial K$, and denote $M = \langle q \rangle$, $\xi = \mathcal{L}(q)$, $W = \{\xi = 0\}$. A linear function $f : M \to V/M$ is uniquely defined by $v = f(q)$. Let $w \in W$ be the unique vector with $v = \mathrm{Pr}_{V/M}(w)$. Then $\phi_{K,L}(f) = \|f\| = \|v\|_{V_L/M}$. The curve γ_f on the sphere is given by $\gamma_f(t) = \frac{q+tw}{\|q+tw\|}$ and

$$\dot{\gamma}_f(0) = w - q\frac{d}{dt}\bigg|_{t=0} \|q+tw\| = w$$

so that $\phi_L(\dot{\gamma}_f(0)) = \|Pr_{V/M}w\|_{V_L/M} = \|v\|_{V_L/M}$. q.e.d.

3.3. Main theorems. Recall that V is an oriented vector space.

Proposition 3.8. *For any norm β on Hom(V,V), the Finsler manifolds $(\tilde{G}(V,n-k),\phi_\beta)$ and $(\tilde{G}(V^*,k),\phi_{\overline{\beta}})$ are canonically isometric.*

Proof. The natural identification $A : \tilde{G}(V,n-k) \to \tilde{G}(V^*,k)$ defined by $\Lambda \mapsto (V/\Lambda)^*$ (with the orientation on $(V/\Lambda)^*$ induced by that of Λ) has the differential $D_\Lambda(A) : \mathrm{Hom}(\Lambda, V/\Lambda) \to \mathrm{Hom}((V/\Lambda)^*, \Lambda^*)$ given by $(D_\Lambda A)(f) = f^*$, which is by definition an isometry. q.e.d.

Corollary 3.9. *Assume α is a symmetric crossnorm, $K, L \subset V$ convex symmetric bodies. Then $(\tilde{G}(V, n-k), \phi_{\alpha,K,L})$ and $(\tilde{G}(V^*, k), \phi_{\alpha,L^o,K^o})$ are canonically isometric.*

Once we have made those observations, we will be concerned from now on with various correspondences between $\tilde{G}(V, k)$ and $\tilde{G}(V, n-k)$. We will assume without loss of generality that $2k \leq n$.

Theorem 3.10. *Fix any norm β on $Hom(V, V)$. Then*

$$v_{HT}(\tilde{G}(V, k), \phi_\beta) = v_{HT}(\tilde{G}(V, n-k), \phi_\beta)$$

where v_{HT} denotes the Holmes–Thompson volume.

Proof. First, assume that β is smooth and strictly convex.

For any $1 \leq l < n$, let $s = \min(l, n-l)$ and consider the filtration $T^*(\tilde{G}(V, l)) = C_s^l \supset C_{s-1}^l \supset \cdots \supset C_1^l \supset C_0^l$ where $C_r^l = \{(\Lambda, T) \in T^*\tilde{G}(V, l) : \mathrm{rank}(T : V \to V) \leq r\}$ are closed submanifolds. Also, define $E_r^l = C_r^l \backslash C_{r-1}^l$—the points of rank r in $T^*\tilde{G}(V, l)$, which is an open submanifold in C_r^l . It is easy to see that C_r^l is connected for all r, and E_s^l is open and dense.

We identify $T \in T_\Lambda^*\tilde{G}(V, l) = \mathrm{Hom}(V/\Lambda, \Lambda)$ with $T : V \to V$ such that $\mathrm{Im}(T) \subset \Lambda \subset \mathrm{Ker}(T)$. The group $\mathrm{GL}(V)$ acts on $\tilde{G}(V, l)$. Therefore (see A.2.2), we get an induced Hamiltonian action of $\mathrm{GL}(V)$ on $T^*\tilde{G}(V, l)$, which is given explicitly by $U(\Lambda, T) = (U\Lambda, UTU^{-1})$.

We claim that the $\mathrm{GL}(V)$-orbits are precisely E_r^l, $0 \leq r \leq l$. Indeed, all operators $T : V \to V$ satisfying $T^2 = 0$ and $\mathrm{rank}(T) = r$ are conjugate to each other, so it remains to show that (Λ, T) and (Λ', T) lie in the same orbit. Choosing some complement $\mathrm{Ker}(T) \oplus W = V$, define $U : V \to V$ so that $U|_{\mathrm{Im}(T)} = Id$, $U(\Lambda) = \Lambda'$, $U(\mathrm{Ker}(T)) = \mathrm{Ker}(T)$ and $U|_W = Id$. Then obviously $U \in \mathrm{GL}(V)$, and $UTU^{-1} = T$, proving the claim. The corresponding orbits of the coadjoint action of $\mathrm{GL}(V)$ on $\mathfrak{gl}(V)^* \backsimeq \mathfrak{gl}(V)$ are simply $A_r = \{T : T^2 = 0, \mathrm{rank}(T) = r\}$, which are equipped with Kirillov's symplectic form.

The momentum map $\mu : T^*\tilde{G}(V, l) \to \mathfrak{gl}(V)^*$ is given for $X \in \mathfrak{gl}(V)$ by

$$\langle \mu(\Lambda, T), X \rangle = tr(T\underline{X}_\Lambda)$$

where $\underline{X}_\Lambda$ denotes the infinitesimal action (fundamental vector field) of X at Λ. Thus after identifying $\mathfrak{gl}(V)^*$ with $\mathfrak{gl}(V)$ by trace duality, $\mu(\Lambda, T) = T$.

It follows from A.8 that $\mu_l : E_s^l \to A_s$ is locally a symplectomorphism, which is clearly surjective and 2-to-1 (for instance, if $l \leq n/2$ then $\mu_l(\mathrm{Im}(T), T) = T$, and there are two possible orientations for $\mathrm{Im}(T)$).

We would like to find a diffeomorphism Φ that makes the following diagram commutative:

$$\begin{tikzcd} E_k^k \arrow[rr, "\Phi"] \arrow[dr, "\mu_k"'] & & E_k^{n-k} \arrow[dl, "\mu_{n-k}"] \\ & A_k & \end{tikzcd}$$

It would immediately follow that Φ is a symplectomorphism. Define

$$\Phi(\mathrm{Im}(T), T) = (\mathrm{Ker}(T), T)$$

taking the orientation on $\mathrm{Ker}(T)$ so that $T : V/\mathrm{Ker}(T) \to \mathrm{Im}(T)$ is orientation preserving. It is straightforward to verify that Φ satisfies all conditions, so $\Phi : E_k^k \to E_k^{n-k}$ is an isomorphism of symplectic manifolds. Moreover, it obviously preserves the norm ϕ_β^*. This proves equality of volumes, since for any Finsler metric ϕ,

$$v_{HT}(\tilde{G}(V,l),\phi) = \frac{1}{(l(n-l))!}\int_{B^*(\tilde{G}(V,l),\phi)} \omega^{l(n-l)} =$$

$$= \frac{1}{(l(n-l))!}\int_{B^*(\tilde{G}(V,l),\phi)\cap E_s^l} \omega^{l(n-l)}$$

Finally, the result for arbitrary norms β follows by approximation and by continuity of the Holmes–Thompson volume w.r.t. β. q.e.d.

Remark 3.11. It follows by Remark 1.9 that $\Phi^*\alpha_2 = \alpha_1$.

Remark 3.12. In the case $k = 1$, it follows from the proof that the cotangent bundles are symplectomorphic outside the zero section, and the associated Hamiltonians are respected. It follows by Corollary 1.8 that the length spectra, as well as the symmetric length spectra, coincide. Together with Proposition 3.8, this generalizes Theorems 2.10 and 2.12 to arbitrary norms β on $\mathrm{Hom}(V,V)$.

Remark 3.13. It is worth noting that Φ cannot be extended continuously outside E_k^k when $k < n/2$: for any $(\Lambda_0, T_0) \in T^*_{\Lambda_0}\tilde{G}(V,k)$ with $\mathrm{rank}(T) < k$, one can always find two nearby points $(\Lambda_0, T_1), (\Lambda_0, T_2)$ s.t. $\mathrm{rank}(T_1) = \mathrm{rank}(T_2) = k$ while $\mathrm{Ker}(T_1)$ and $\mathrm{Ker}(T_2)$ are far apart on $\tilde{G}(V, n-k)$. When $k = n/2$, Φ is just the identity map.

Before we proceed to study the girth of Grassmannians, let us briefly recall some terminology. For a Finsler manifold (M,ϕ) and a curve $\gamma_t \in M$, we call the curve $\Gamma_t = (\gamma_t, \mathcal{L}(\dot{\gamma}_t)) \in T^*M$ its *lift* to T^*M. A curve Γ_t of such form we call a *lift curve.*

Lemma 3.14. *For a smooth, strictly convex norm β on Hom(V,V), the geodesics in $(\tilde{G}(V,k),\phi_\beta)$ lift to curves of constant rank in $T^*\tilde{G}(V,k)$.*

Proof. We use the notation of the proof of Theorem 3.10.

The level sets of $\mu : T^*\tilde{G}(V,l) \to \mathfrak{gl}(V)$ are $Z_T = \{(\Lambda, T) : \mathrm{Im}(T) \subset \Lambda \subset \mathrm{Ker}(T)\} \subset T^*\tilde{G}(V,l)$. It follows (see A.2) that the skew-orthogonal space $(T_{(\Lambda,T)}Z_T)^\perp = T_{(\Lambda,T)}E_r^l$ where $r = \mathrm{rank}(T)$.

The Hamiltonian $H = \frac{1}{2}\phi_\beta^{*2}$ is constant on Z_T, and so $X_H\Big|_{E_r^k} \in TE_r^k$. Since C_r^k and C_{r-1}^k are closed, the flow defined by X_H leaves E_j^k invariant, which concludes the proof. q.e.d.

This motivates the following definition:

Definition 3.15. Fix a smooth, strictly convex norm β on $\mathrm{Hom}(V,V)$. The *rank* of a geodesic in $(\tilde{G}(V,k), \phi_\beta)$ is the constant rank of its lift to $T^*\tilde{G}(V,k)$.

Example 3.16. The girth of $(\tilde{G}(V,k), \phi_{HS})$ is attained on a geodesic of rank 1 (which can be visualized as a rotation of a two dimensional plane, while fixing all orthogonal directions).

We will make use of the following general fact:

Lemma 3.17. *Let X be a normed space, and $Y \subset X$ a subspace. Then for all $y \in Y$, $\mathcal{L}_Y(y) = \mathrm{Pr}_{Y^*}(\mathcal{L}_X(y))$.*

Proof. We may assume that $\|y\| = 1$. Let us check that $\mathrm{Pr}_{Y^*}(\mathcal{L}_X(y))(y) = 1$, and $\mathrm{Pr}_{Y^*}(\mathcal{L}_X(y))(z) \le 1$ for all $z \in Y$ with $\|z\| = 1$. Since $\mathrm{Pr}_{Y^*}(\mathcal{L}_X(y))(z) = \mathcal{L}_X(y)(z)$ for all $z \in Y$, both properties obviously hold. q.e.d.

We will also need a lemma from linear algebra:

Lemma 3.18. *Let V be an n-dimensional real vector space, and $T \in GL(V)$. Suppose $\Lambda \subset V$ is a subspace with $\dim \Lambda = k$ and $T(\Lambda) = \Lambda$. Then there is a subspace $\Omega \subset V$ s.t. $\dim \Omega = n - k$, $T(\Omega) = \Omega$ and $\det T|_\Lambda \det T|_\Omega = \det T$.*

Proof. Simply observe that for $T^* \in \mathrm{GL}(V^*)$ and $\Lambda^\perp = (V/\Lambda)^*$, $T^*(\Lambda^\perp) = \Lambda^\perp$. It is well known that T^* and T are conjugate over $\mathbb{R}$, i.e., $T^* = UTU^{-1}$ for some invertible $U : V \to V^*$. Thus $\Omega = U^{-1}\Lambda^\perp$ is invariant for T, and $\det T = \det T|_\Lambda \det T|_{V/\Lambda} = \det T|_\Lambda \det T^*|_{\Lambda^\perp}$. q.e.d.

Theorem 3.19. *Fix a smooth, strictly convex norm β on $Hom(V,V)$. Then there exists a bijection between the closed geodesics of $(\tilde{G}(V,k), \phi_\beta)$ and those of $(\tilde{G}(V,n-k), \phi_\beta)$ that respects length and rank. Moreover, symmetric geodesics correspond to symmetric geodesics.*

Proof. We again use the notation of the proof of Theorem 3.10. Suppose that for some $1 \le r \le k$, $\gamma_t = (\Lambda_t, T_t) \subset E_r^k$, $0 \le t \le L$ is a characteristic curve, i.e., the lift of a geodesic Λ_t of length L and rank r in $(\tilde{G}(V,k), \phi_\beta)$ with arc-length parametrization, such that $\Lambda_L = \Lambda_0$

(closed geodesic, referred to as the first case) or $\Lambda_L = \overline{\Lambda_0}$ (half of a closed symmetric geodesic, referred to as the second case). In the second case, the extension to a full symmetric geodesic is given by $\Lambda_t = \overline{\Lambda_{t-L}}$ for $L \leq t \leq 2L$. The parameter t is taken mod L in the first case, and mod $2L$ in the second case. In both cases, $T_{t+L} = T_t$. Denote $I_t = \mathrm{Im}(T_t)$, $K_t = \mathrm{Ker}(T_t)$, and define $S_t = \mathcal{L}_{\mathrm{Hom}(V,V)}(T_t) \in \mathrm{Hom}(V,V)$. Then $\beta(S_t) = 1$, and it follows from Lemma 3.17 that $\dot{\Lambda}_t = \mathrm{Pr}_{\mathrm{Hom}(\Lambda_t, V/\Lambda_t)}(S_t)$. Define $B_t \in \mathrm{Hom}(V,V)$ by the differential equation $\dot{B}_t = S_t B_t$, with $B_0 = Id$. Note that $B_t \in \mathrm{GL}^+(V)$ for all t, and $B_{t+L} = B_t B_L$.

First, observe that $B_t(\Lambda_0) = \Lambda_t$. This is evident by taking $e_1(0), ..., e_k(0)$ a basis of Λ_0, and $e_l(t) = B_t e_l(0)$. Then $\dot{e}_l(t) = S_t e_l(t)$, as required.

We next claim that $B_t(I_0) = I_t$ and $B_t(K_0) = K_t$. This can be seen as follows: By Corollary A.9, $\mu_k(\Gamma_t)$ is a flow curve in A_r for the Hamiltonian $H(T) = \frac{1}{2}\beta^*(T)^2$. Since A_r and E_r^r are locally symplectomorphic through μ_r, the curve $(I_t, T_t) \in E_r^r$ is characteristic (strictly speaking, one first has to fix some orientation on I_0, but we only consider local properties of the curve such as it being characteristic, or a lift curve). In particular, it is a lift curve, so $T_t = \mathcal{L}_{\mathrm{Hom}(I_t, V/I_t)}(\dot{I}_t) \Rightarrow \dot{I}_t = \mathrm{Pr}_{\mathrm{Hom}(I_t, V/I_t)}(\mathcal{L}_{\mathrm{Hom}(V,V)} T_t) = \mathrm{Pr}_{\mathrm{Hom}(I_t, V/I_t)} S_t$. This readily implies, as above, that $I_t = B_t(I_0)$. The proof that $K_t = B_t(K_0)$ is identical.

Denote by $\tilde{B}_t : K_0/I_0 \to K_t/I_t$ the operator induced from $B_t : V \to V$, and fix some orientation on I_0. Then K_0 inherits an orientation, and so do $K_t = B_t(K_0)$, $I_t = B_t(I_0) = V/K_t$ (equalities of oriented spaces). In particular, K_L/I_L and K_0/I_0 coincide as oriented vector spaces (in fact, this orientation is independent of the one on I_0), and $\tilde{B}_L \in \mathrm{GL}^+(K_0/I_0)$. Now consider $\tilde{\Lambda}_0 = \Lambda_0/I_0 \subset K_0/I_0$, which inherits an orientation from (Λ_0, I_0), and $\dim \tilde{\Lambda}_0 = k - r$.

Apply Lemma 3.18 to conclude the existence of a subspace $\tilde{\Omega}_0 \subset K_0/I_0$ s.t. $\dim \tilde{\Omega}_0 = n - k - r$ and $\tilde{B}_L(\tilde{\Omega}_0) = \tilde{\Omega}_0$ as unoriented spaces. Moreover, one can write

$$\mathrm{sign}\det \tilde{B}_L|_{\tilde{\Omega}_0} = \mathrm{sign}\det \tilde{B}_L|_{\tilde{\Lambda}_0} = \mathrm{sign}\det B_L|_{\Lambda_0} \mathrm{sign}\det B_L|_{I_0}$$

Fix some orientation on $\tilde{\Omega}_0$, and define $\Omega_0 = \mathrm{Pr}^{-1}_{K_0/I_0}(\tilde{\Omega}_0)$ with the induced orientation. Note that

$$B_L(\Omega_0) = B_L(\mathrm{Pr}^{-1}_{K_0/I_0}(\tilde{\Omega}_0)) = \mathrm{Pr}^{-1}_{K_0/I_0}(\tilde{B}_L \tilde{\Omega}_0) = \Omega_0$$

(ignoring orientations), while

$$\mathrm{sign}\det B_L|_{\Omega_0} = \mathrm{sign}\det B_L|_{\tilde{\Omega}_0} \mathrm{sign}\det B_L|_{I_0} = \mathrm{sign}\det B_L|_{\Lambda_0}$$

i.e., $B_L(\Omega_0) = \Omega_0$ (first case) or $B_L(\Omega_0) = \overline{\Omega_0}$ (second case).

Let $\Gamma_t = (\Omega_t, T'_t)$ be the unique characteristic curve through (Ω_0, T_0). According to Lemma A.9, Γ_t is mapped by μ_k to $T_t \in A_r$ so $T'_t = T_t$; and Γ_t is also a lift curve, so $\dot{\Omega}_t = \mathrm{Pr}_{\mathrm{Hom}(\Omega_t, V/\Omega_t)}(S_t)$ and $\Omega_t = B_t(\Omega_0)$ as

before. We conclude that Ω_t, $0 \le t \le L$ is a geodesic in $(\tilde{G}(V, n-k), \phi_\beta)$ of length L and rank r, which is closed (in the first case) or constitutes half of the closed symmetric geodesic $\Omega_t = B_t\Omega_0$, $0 \le t \le 2L$ (second case). Finally, it remains to note that we may choose $\Omega_0 = \Omega_0(\Lambda_t)$ in a shift invariant manner, i.e., in such a way that $\Omega_0(\Lambda_{t+T}) = \Omega_T$ for all T. This concludes the proof. q.e.d.

Remark 3.20. The same proof shows also the existence of a correspondence between geodesics joining antipodal points.

Corollary 3.21. *For any norm β on $Hom(V, V)$, $(\tilde{G}(V, k), \phi_\beta)$ and $(\tilde{G}(V, n-k), \phi_\beta)$ have equal girth.*

Corollary 3.22. *Assume α is a projective crossnorm, $K, L \subset V$ convex symmetric bodies. Then $(\tilde{G}(V, k), \phi_{\alpha,K,L})$ and $(\tilde{G}(V, n-k), \phi_{\alpha,K,L})$ have equal Holmes–Thompson volume, and equal girth. If α on $Hom(V_K, V_L)$ is smooth and strictly convex, the length spectra and symmetric length spectra coincide together with rank.*

Corollary 3.23. *Assume α is a symmetric and projective crossnorm, and V a normed space. Then $(\tilde{G}(V, k), \phi_{V,\alpha})$ and $(\tilde{G}(V^*, k), \phi_{V^*,\alpha})$ have equal Holmes–Thompson volume and equal girth.*

Appendix A.

A.1. A brief overview of tensor norms: For further details we refer to [**10**]. A uniform crossnorm α is an assignment of a norm to $A^* \otimes B \simeq \mathrm{Hom}(A, B)$ for pairs of isometry classes of finite-dimensional normed spaces A, B, s.t. rank 1 operators have the natural norm: $\alpha(a^*b) = \|a^*\|\|b\|$. Important examples are the operator (injective) norm $\|\|_{Op}$ and the nuclear (projective) norm $\|\|_N$. One can also introduce the dual uniform crossnorm α^*, given by trace duality: $T \in (\mathrm{Hom}(X, Y), \alpha)$ and $S \in (\mathrm{Hom}(Y, X), \alpha^*)$ are paired by $(T, S) \mapsto tr(ST)$. For $\alpha = \|\|_{Op}$, one has $\alpha^* = \|\|_N$.

1) A uniform crossnorm α is symmetric if the adjoint map
$$^* : (\mathrm{Hom}(A, B), \alpha) \to (\mathrm{Hom}(B^*, A^*), \alpha)$$
is an isometry for all pairs A, B. If α is symmetric, so is α^*.
2) α is injective if for all quadruples $(A_1 \subset A, B_1 \subset B)$, the natural injection $\mathrm{Hom}(A/A_1, B_1) \hookrightarrow \mathrm{Hom}(A, B)$ is an injection of α-normed spaces.
3) α is projective if for all quadruples $(A_1 \subset A, B_1 \subset B)$, the natural projection $\mathrm{Hom}(A, B) \twoheadrightarrow \mathrm{Hom}(A_1, B/B_1)$ is a projection of α-normed spaces.
4) α is projective if and only if α^* is injective.
5) Thus $\|\|_{Op}$ is symmetric and injective, while $\|\|_N$ is symmetric and projective.

A.2. The momentum map.

A.2.1. Generalities on momentum map. Most of the following can be found in any textbook on Hamiltonian dynamics; see, for instance, [**3**]. It appears here to make the exposition self-contained, to fix notation, and also to prove several lemmas that we couldn't find in the precise form which we need to apply.

We are given a Lie group G, its Lie algebra $\mathfrak{g}$, and a symplectic manifold W with a symplectic action of G.

Claim A.1. Given maps $\mu : W \to \mathfrak{g}^*$ and $\tilde{\mu} : \mathfrak{g} \to C^\infty(W)$ satisfying $\tilde{\mu}(X)(p) = \langle \mu(p), X\rangle$, then $\omega(\underline{X}_p, \bullet) = \langle D_p\mu(\bullet), X\rangle$ for all $X \in \mathfrak{g}$ if and only if $\tilde{\mu}(X)$ is a Hamiltonian map for $\underline{X}$, i.e., $d_p(\tilde{\mu}(X)) = \omega(\underline{X}_p, \bullet)$.

Proof. Calculation: $\mu(p)(X) = \tilde{\mu}(X)(p)$, after differentiating by p one has $\langle D_p\mu(\bullet), X\rangle = d_p(\tilde{\mu}(X))$. q.e.d.

Definition A.2. Under these conditions, the action of G is *Hamiltonian*, and μ is called a *momentum map*. In particular, for $H = \langle X, \mu(\bullet)\rangle = \tilde{\mu} \circ X : W \to \mathbb{R}$, $X_H = \underline{X}$.

For a general (not necessarily equivariant) momentum map, we can describe the image of the differential $D_p\mu$: Since $\langle D_p\mu(v), X\rangle = \langle v, i_{\underline{X}_p}\omega\rangle$, $D_p\mu : T_pW \to \mathfrak{g}^*$ and $i_{\underline{X}_p} : \mathfrak{g} \to T_p^*W$ are dual maps, so

$$\mathrm{Im}(D_p\mu) = \mathrm{Ann}(\mathrm{Ker}(i_{\underline{X}_p})) = \mathrm{Ann}(\{X : \underline{X}_p = 0\})$$

and if $\mu(p) = \xi$,

$$T_p(\mu^{-1}(\xi)) = \mathrm{Ker}(D_p\mu) = \mathrm{Ann}(\mathrm{Im}(i_{\underline{X}_p})) = \{v : \omega(v, \underline{X}_p) = 0\} = T_p(Gp)^\perp$$

In particular, $\mathrm{rank}(D_p\mu) = \mathrm{codim}(\{X : \underline{X}_p = 0\}) = \dim(Gp)$.

Definition A.3. If the fundamental vector fields on W and $\mathfrak{g}^*$ for the coadjoint action satisfy $\underline{X}_{\mu(p)} = D_p\mu(\underline{X}_p)$ for all $p \in W$, the momentum map is called *equivariant*.

Claim A.4. The momentum map is equivariant if and only if $\omega(\underline{X}_p, \underline{Y}_p) = \langle \mu(p), [X, Y]\rangle$ for all $X, Y \in \mathfrak{g}$, $p \in W$.

Proof. Take any $Z \in \mathfrak{g}$. Then the co-adjoint infinitesimal action is given by

$$\langle \underline{X}_{\mu(p)}, Z\rangle = \langle \mu(p), [Z, X]\rangle$$

while by definition of momentum map

$$\langle D_p\mu(\underline{X}_p), Z\rangle = \omega(\underline{Z}_p, \underline{X}_p)$$

Thus, equivariance of μ amounts to equality of the two right-hand sides. q.e.d.

Corollary A.5. *Suppose the momentum map μ commutes with the action of G: $\mu(gp) = Ad_g^*(\mu(p))$. Then μ is an equivariant momentum map. If G is connected, the reverse implication also holds.*

Proof. Take $g(t) = exp(tX)$ and differentiate: we get $D_p\mu(\underline{X}_p) = \underline{X}_{\mu(p)}$, and the claim above applies. Connectedness allows to integrate this equation. q.e.d.

Remark A.6. When $\mu(gp) = Ad_g^*(\mu(p))$, we refer to μ as a G-equivariant momentum map. By the corollary, G-equivariance implies equivariance, and the two notions coincide for connected groups G.

Fact A.7. The co-adjoint orbit $G\xi$ is naturally a symplectic manifold, with Kirillov's symplectic form given by $\omega_\xi(\underline{X}_\xi, \underline{Y}_\xi) = \langle \xi, [X, Y]\rangle$ for $X, Y \in \mathfrak{g}$. The action of G on $G\xi$ is Hamiltonian, with G-equivariant momentum map given by the inclusion $G\xi \subset G$.

Corollary A.8. *Let W be a symplectic manifold equipped with a Hamiltonian action of G and an equivariant momentum map μ. Suppose $\mu(p) = \xi$ where $p \in W$, $\xi \in \mathfrak{g}^*$. Then $\omega_p(\underline{X}_p, \underline{Y}_p) = \omega_\xi(\underline{X}_\xi, \underline{Y}_\xi)$ for all $X, Y \in \mathfrak{g}$. If, moreover, the action of G is transitive at p (i.e., $T_p(Gp) = T_pW$), then $\mu^*\omega_\xi = \omega_p$*

Proof. By equivariance of μ, we get $\omega_p(\underline{X}_p, \underline{Y}_p) = \langle \xi, [X, Y]\rangle = \omega_\xi(\underline{X}_\xi, \underline{Y}_\xi)$. The last part amounts to the verification $\omega_p(\underline{X}_p, \underline{Y}_p) = \omega_\xi(D_p\mu\underline{X}_p, D_p\mu\underline{Y}_p)$ for all $X, Y \in \mathfrak{g}$, which follows from the first part again by equivariance of μ. q.e.d.

Corollary A.9. *Let G be a Lie group, $H : \mathfrak{g}^* \to \mathbb{R}$ any smooth function, and fix $\xi \in \mathfrak{g}^*$. Let W be a symplectic manifold equipped with a Hamiltonian action of G and momentum map μ. Suppose $\mu(p) = \xi$. Consider the Hamiltonian $\mu^*H = H \circ \mu$ on W. Denote the μ^*H-flow on W by ϕ_t, and the H-flow on $G\xi$ by ψ_t . Then:*
(1) $\phi_t(p) \in Gp$.
(2) If μ is equivariant, then $\mu\phi_t(p) = \psi_t(\xi)$.

Proof. (1) Simply note that μ^*H is constant along level sets of μ, so $X_{\mu^*H} \in T_p(\mu^{-1}(\xi))^\perp = T_p(Gp)$. For (2), we should verify that

$$D_p\mu(X_{\mu^*H}) = X_H$$

Take any $Y \in \mathfrak{g}$ and verify that $\langle X_H, Y\rangle = \langle D_p\mu(X_{\mu^*H}), Y\rangle$. We know that

$$\langle D_p\mu(X_{\mu^*H}), Y\rangle = -\omega(X_{\mu^*H}, \underline{Y}_p) = -d_p(H \circ \mu)(\underline{Y}_p) =$$

$$= -dH(D_p\mu\underline{Y}_p) = -\omega_\xi(X_H, D_p\mu\underline{Y}_p)$$

Now by equivariance, $D_p\mu\underline{Y}_p = \underline{Y}_\xi$, while $X_H = \underline{X}_\xi$ for some $X \in \mathfrak{g}$ (since it is tangent to the co-adjoint orbit of ξ), so we need only check that $-\omega_\xi(\underline{X}_\xi, \underline{Y}_\xi) = \langle \underline{X}_\xi, Y\rangle$, and indeed both sides equal $\langle \xi, [Y, X]\rangle$. q.e.d.

A.2.2. A canonic structure on the cotangent bundle. Given a Lie group G acting on a smooth manifold M, one can naturally extend the action of G to T^*M: $\hat{g}(q,p) = (gq, v \mapsto p(dg^{-1}(v)))$. Then one easily verifies that this action preserves the canonic 1-form α: since for $\xi \in T_{(q,p)}(T^*M)$ one has $dqd\hat{g}(\xi) = dgdq(\xi)$,

$$\alpha_{\hat{g}(q,p)}(d\hat{g}(\xi)) = p'(dqd\hat{g}(\xi)) = p(dg^{-1}dqd\hat{g}(\xi)) = p(dq(\xi)) = \alpha(\xi)$$

Then given $X \in \mathfrak{g}$, $\tilde{\mu}(X)(q,p) = p(\underline{X}_q)$ is a Hamiltonian function for $\underline{X}_q$:

$$\omega(\bullet, \hat{\underline{X}}_q) = i_{\hat{\underline{X}}_q}(d\alpha) = L_{\hat{\underline{X}}_q}\alpha + d(i_{\hat{\underline{X}}_q}\alpha)$$

The first summand is 0 since the action of G preserves α, and $i_{\hat{\underline{X}}_q}\alpha = \alpha(\hat{\underline{X}}_q) = p(\underline{X}_q)$; so $\omega(\bullet, \hat{\underline{X}}_q) = d(\tilde{\mu}(X))$ as required. The momentum map itself can therefore be defined by

$$\langle \mu(q,p), X\rangle = p(\underline{X}_q)$$

This momentum map is G-equivariant:

$$\langle \mu(g(q,p)), X\rangle = g_*p(\underline{X}_{gq}) = p(dg^{-1}(\underline{X}_{gq}))$$

while

$$\langle Ad_g^*\mu(q,p), X\rangle = \langle \mu(q,p), Ad_{g^{-1}}X\rangle = p(\underline{Ad_{g^{-1}}X}_q)$$

and $\underline{Ad_{g^{-1}}X}_q = \frac{d}{dt}(g^{-1}exp(tX)gq) = dg^{-1}(\underline{X}_{gq})$. So, $\mu(g(q,p)) = Ad_g^*\mu(q,p)$ and in particular by Corollary A.5, μ is equivariant.

A.3. The quotient girth of the square. Fix a coordinate system in $\mathbb{R}^2$, and let $Q = [-1,1] \times [-1,1]$. Let γ be the boundary curve parametrized by angle $0 \le \alpha \le 2\pi$, so $\gamma = (1, \tan\alpha)$ and $\dot{\gamma} = (0, \frac{1}{\cos^2\alpha})$ for $0 \le \alpha \le \pi/4$. One immediately calculates that

$$\begin{aligned}\frac{1}{8}g_q(Q) &= \int_0^{\pi/4} \frac{\det(\gamma(\alpha), \dot{\gamma}(\alpha))}{\det(\gamma(\alpha), \gamma(3\pi/4))} d\alpha \\ &= \int_0^{\pi/4} \frac{d\alpha}{\cos\alpha(\sin\alpha + \cos\alpha)} = \log 2\end{aligned}$$

So the quotient girth of the square equals $8\log 2 \approx 5.545$.

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