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ON QUADRATIC ORTHOGONAL BISECTIONAL CURVATURE

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Abstract

In this article we study compact Kähler manifolds satisfying a certain nonnegativity condition on the bisectional curvature. Under this condition, we show that the scalar curvature is nonnegative and that the first Chern class is positive assuming local irreducibility. We also obtain a partial classification of possible de Rham decompositions of the universal cover under this condition.

1. Introduction

We begin with the following definition.

Definition 1.1. A Kähler manifold (M, g) of complex dimension n is said to have *nonnegative quadratic orthogonal bisectional curvature (NQOBC)* at $p \in M$ if $n \geq 2$ and: *for any* unitary frame $\{e_1, \ldots, e_n\}$ of $T_p^{(1,0)}(M)$ and *any* real numbers $\xi_1, ..., \xi_n$ we have

$$\sum_{i,j=1}^{n} R_{i\bar{i}j\bar{j}}(\xi_i - \xi_j)^2 \geq 0. \tag{1.1}$$

We will say that a manifold (M, g) has NQOBC provided it does at every point.

One can similarly define the notion of nonpositive orthogonal quadratic bisectional curvature. Note that if a product of Kähler manifolds $M_1 \times M_2$ has NQOBC then so must each factor M_1 and M_2. The reverse implication however may be false in general: M_1 and M_2 may both have NQOBC while $M_1 \times M_2$ may not. For example, take $M_1 = \mathbb{C}$ and M_2 as in Example 1.2 in [**8**].

The condition of NQOBC is weaker than requiring M to have nonnegative orthogonal bisectional curvature: $R(V, \bar{V}, W, \bar{W}) \geq 0$ for any orthogonal unitary pair $V, W \in T^{(1,0)}(M)$, while on complex surfaces the two conditions are equivalent. The first example of a Kähler manifold

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with NQOBC which does not support a Kähler metric with nonnegative orthogonal bisectional curvature was provided in [**10**] by Li-Wu-Zheng. (The example in [**11**] was one of the classical Kähler C-spaces with $b_2 = 1$. After the preparation of this article, following Li-Wu-Zheng, in [**5**] the authors produced families of such examples and classified the Kähler C-spaces having $b_2 = 1$ and NQOBC.)

The structure of compact Kähler manifolds with nonnegative orthogonal bisectional curvature, is now completely understood by the works of Chen [**7**] and Gu-Zhang [**9**]. Their results extend those on the generalized Frankel conjecture for compact manifolds with nonnegative holomorphic bisectional curvature, formulated by Yau [**17**, p.677], and established through the works of Howard-Smyth-Wu [**10**], Wu [**15**] and eventually by Bando [**1**] in three-dimension and Mok [**12**] for all dimensions. Frankel's conjecture for compact manifolds with positive holomorphic bisectional curvature was established by Mori [**13**] and Siu-Yau [**14**] independently.

The NQOBC condition arises naturally in the proof of the following fact which appeared implicitly in an earlier work [**3**]:

Lemma 1.1. *If* (M^n, g) *has NQOBC, then all harmonic* $(1,1)$ *forms are parallel.*

The proof uses the Wietzenböck identity for $(1,1)$ forms. The lemma is implicit from earlier works though was only used in these under the stronger assumptions of positive and nonnegative holomorphic bisectional curvature (see for example [**3**, **8**, **10**] and references therein). We will make use of Lemma 1.1 at various points.

The NQOBC condition was first considered explicitly by Wu-Yau-Zheng in [**16**] where the authors studied the boundary of the Kähler cone of manifolds with NQOBC.

Motivated by the work in [**16**], in this paper we try to understand more on this class of Kähler manifolds. We first prove that: *If a Kähler manifold has NQOBC at a point p, then the scalar curvature is nonnegative at p and is zero if and only if it is flat at p, provided the complex dimension is at least 3.* See Theorem 3.1 for more details. Using this result we prove: *If* (M, g) *is a compact Kähler manifold with NQOBC which is locally irreducible, then* $c_1(M) > 0$. This generalizes [**9**, Theorem 2.1]. This naturally leads us to consider the deRham factorization of the universal cover of M for which we prove the following. *All compact factors have positive first Chern class. There is at most one non-compact factor being either: complex Euclidean space (in which case all compact factors will have quasi positive Ricci curvature), or non Euclidean one complex dimensional (in which case all compact factors will have positive Ricci curvature).* This gives a slight refinement of [**9**, Theorem 1.3 (2)]. For a more detailed description, see Theorem 4.2.

The organization of the paper is as follows. In Section 2, we will give a quick proof of a result in [**16**] on the boundary of the Kähler cone of manifolds with NQOBC which is similar to but independent of the proof in [**19**]. In Section 3, we prove that Kähler manifolds with NQOBC must have nonnegative scalar curvature. In Section 4 we prove the above mentioned structure results for compact Kähler manifolds with NQOBC.

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2. Boundary of the Kähler cone of manifolds with NQOBC

The Kähler cone of a compact Kähler manifold (M, g) with NQOBC turns out to be rather special: every boundary class α contains a smooth nonnegative representative $\eta \in \alpha$. This was first proved in [**16**] by solving a degenerate complex Monge-Ampère equation under the condition of NQOBC. In fact, the following holds:

Theorem 2.1. *Suppose (M^n, g) is a compact complex n-dimensional Kähler manifold with NQOBC. Let α be in the closure of the Kähler cone of M and η be the unique harmonic representative in α. Then η is nonnegative. Moreover, η is positive if and only if $\alpha^n[M] > 0$.*

A short proof of Theorem 2.1 was provided in [**19**] and also independently in [**4**]. We provide details of the proof here for the sake of completeness. We follow the presentation in [**4**] which we think is more concise. We refer to [**4**] where it was also shown that a rather straight forward observation on the proof in [**16**] leads to another proof of Theorem 2.1.

Given a complex manifold M, recall that a real class $\alpha \in H^{(1,1)}(M)$ is called a Kähler class if α contains a smooth positive definite representative η. The space of Kähler classes is a convex cone in $H^{(1,1)}(M)$ referred to as the Kähler cone which we denote by $\mathcal{K}$. We say that α is in the closure of $\mathcal{K}$ if $[(1-t)\omega + t\eta] \in \mathcal{K}$ for any smooth $\eta \in \alpha$, $\omega \in \mathcal{K}$ and $t \in [0, 1)$. Finally, given any real $\alpha \in H^{(1,1)}(M)$ we use $\alpha^n[M]$ to denote the integral $\int_M \alpha^n$.

Proof of Theorem 2.1. Let η be as in Theorem 2.1 and let ω_0 be the Kähler form for (M, g). By Lemma 1.1, η is parallel and thus has constant real eigenvalues $a_1, ..., a_n$ on M with respect to ω_0. Also, $[(1-t)\omega_0 + t\eta] \in \mathcal{K}$ for every $t \in [0, 1)$.

In other words, for each $t \in [0, 1)$ there exists $f_t \in C^\infty(M)$ and $\omega_t \in \mathcal{K}$ such that $(1-t)\omega_0 + t\eta = \omega_t + dd^c f_t$, giving

$$\mathrm{Vol}_g(M) \prod_{i=1}^{n} (1 - t + ta_i) = \int_M ((1-t)\omega_0 + t\eta)^n = \int_M (\omega_t + dd^c f_t)^n > 0$$

for all $t \in [0,1)$. On the other hand, if $a_k < 0$ for some k then $1-t+ta_k$ and thus the product on the LHS above would vanish for some $t_0 \in (0,1)$ giving a contradiction. Thus a_i must be nonnegative for each i, in other words η is nonnegative. In particular, we have $\int_M \eta^n \geq 0$ with strict inequality if and only if η is positive. This completes the proof. q.e.d.

We end this section with the following description of the boundary of $\mathcal{K}$ from [**4**]. Let $\widetilde{M}$ be the universal cover of M with projection $\pi : \widetilde{M} \to M$. Then by the de Rham decomposition Theorem for Kähler manifolds, we may write

$$(\widetilde{M}, \widetilde{\omega}_0) = (\widetilde{M}_0, \widetilde{\sigma}_0) \times (\widetilde{M}_1, \widetilde{\sigma}_1) \times \cdots \times (\widetilde{M}_k, \widetilde{\sigma}_k)$$

where $\widetilde{\omega}_0 = \pi^*(\omega_0)$, $\widetilde{M}_0$ is flat and each $\widetilde{M}_i$, for $i \geq 1$, is nonflat irreducible and Kähler and the decomposition is unique up to permutation.

Corollary 2.1. *Let η be a real harmonic (1,1) form on M, and let $\widetilde{\eta} = \pi^*(\eta)$. Then $\widetilde{\eta} = \widetilde{\eta}_0 \times \prod_{i=1}^k a_i \widetilde{\sigma}_i$ for some real constants a_i and $\widetilde{\eta}_0$ is a parallel harmonic form on $\widetilde{M}_0$. In particular, the boundary of $\mathcal{K}$ can be identified with the space of harmonic $(1,1)$ forms $\widetilde{\eta}$ on $\widetilde{M}$ satisfying: $\widetilde{\eta}$ is equivariant with respect to $\pi_1(M)$ and $\widetilde{\eta} = \widetilde{\eta}_0 \times \prod_{i=0}^k a_i \widetilde{\sigma}_i$, where $\widetilde{\eta}_0 \geq 0$ and $a_i \geq 0$ for all i with equality holding for some i.*

Proof. Since η is parallel, then $\widetilde{M}$ is a product of Kähler manifolds $\widetilde{N}_i$ such that at each point, the $T^{(1,0)}(\widetilde{N}_i)'s$ are eigenspaces of $\widetilde{\eta}$. By the uniqueness and irreducibility of the $\widetilde{M}_i$'s for $i \geq 1$, the result follows. q.e.d.

3. Scalar curvature of manifolds with NQOBC

Let (M^n, g) be a Kähler manifold. If M has nonnegative holomorphic bisectional curvature, then the Ricci curvature of M is nonnegative. If on the other hand M only has NQOBC, then the Ricci curvature may have negative eigenvalues at some point, see [**9**, Example 1.2]. However, we have the following:

Theorem 3.1. *Suppose (M^n, g) has nonnegative (resp. nonpositive) QOBC at p. Then the scalar curvature $S(p)$ is nonnegative (resp. nonpositive) and $S(p) = 0$ if and only if for all unitary pairs V, W we have $R(V, \bar{V}, W, \bar{W}) = 0$ and $R(V, \bar{V}, V, \bar{V}) + R(W, \bar{W}, W, \bar{W}) = 0$. If $n \geq 3$, $S(p) = 0$ also implies $R(V, \bar{V}, V, \bar{V}) = 0$ for all $V \in T_p^{(1,0)}(M)$ and hence M is flat at p.*

The nonnegativity (nonpositivity) of the scalar curvature $S(p)$ in the theorem follows from the following slightly more general result.

Lemma 3.1. *Let (M^n, g) be a Kähler manifold and $p \in M$.*

(a) *Suppose there exists an $n \times n$ real matrix a_{ij} with the properties:*

$$\sum_{i \neq j} a_{ij} + 2 \sum_i a_{ii} > 0$$

and for any unitary frame $\{e_1, \dots, e_n\}$ of $T_p^{(1,0)}(M)$,

$$\sum_{i,j=1}^n R_{i\bar{i}j\bar{j}} a_{ij} \geq 0. \tag{3.1}$$

Then $S(p)$ is nonnegative. If in addition, there is a unitary frame such that (3.1) is a strict inequality then $S(p) > 0$.

(b) *Suppose there exists an $n \times n$ real matrix a_{ij} with the properties:*

$$\sum_{i \neq j} a_{ij} + 2 \sum_i a_{ii} > 0$$

and for any unitary frame $\{e_1, \dots, e_n\}$ of $T_p^{(1,0)}(M)$,

$$\sum_{i,j=1}^n R_{i\bar{i}j\bar{j}} a_{ij} \leq 0. \tag{3.2}$$

Then $S(p)$ is nonpositive. If in addition, there is a unitary frame such that (3.2) is a strict inequality, then $S(p) < 0$.

Proof. To prove (a), let $\mathcal{F}$ be the set of all ordered unitary bases in $T_p^{(1,0)}(M)$. For any frame $\alpha = \{e_1, \dots, e_n\} \in \mathcal{F}$ and $\mathfrak{u} = (u_{ij}) \in U(n)$, we define the frame

$$\mathfrak{u}\alpha := \left\{ \sum_j u_{j1} e_j, \dots, \sum_j u_{jn} e_j \right\} \in \mathcal{F}.$$

In other words, $\mathfrak{u}$ is just the change of basis matrix from α to $\mathfrak{u}\alpha$. Now we choose some frame $\alpha' \in \mathcal{F}$ to be fixed throughout the proof.

Define functions $F(ij)$ (for any i, j) and $G(i)$ (for any i) on $U(n)$ by

$$\begin{aligned} F(ij)(\mathfrak{u}) &:= R(f_i, \bar{f}_i, f_j, \bar{f}_j) \\ G(i)(\mathfrak{u}) &:= R(f_i, \bar{f}_i, f_i, \bar{f}_i) \end{aligned} \tag{3.3}$$

where $\{f_1, \dots, f_n\} = \mathfrak{u}\alpha'$. Now we establish the following

Claim: Let μ be the left invariant Haar measure on $U(n)$. Then we have $2 \int_{U(n)} F(ij)(\mathfrak{u}) d\nu(\mathfrak{u}) = \int_{U(n)} G(k)(\mathfrak{u}) d\nu(\mathfrak{u})$ for any $i \neq j$ and for any k. In particular, the LHS is independent of i, j and the RHS is independent of k.

For $i \neq j$, let $\mathfrak{u}_\circ(ij) \in U(n)$ be the unitary matrix satisfying: for any $\{f_1, \dots, f_n\} \in \mathcal{F}$, $\{h_1, \dots, h_n\} = \mathfrak{u}_\circ(ij)\{f_1, \dots, f_n\}$ satisfies

$$\begin{aligned} h_i &= \tfrac{1+\sqrt{-1}}{2} f_i + \tfrac{1-\sqrt{-1}}{2} f_j \\ h_j &= \tfrac{1-\sqrt{-1}}{2} f_i + \tfrac{1+\sqrt{-1}}{2} f_j \end{aligned} \tag{3.4}$$

and $h_k = f_k$ if $k \neq i, j$. Also let $\mathfrak{v}_\circ(ij) \in U(n)$ be the unitary matrix satisfying: for any $\{f_1, \dots, f_n\} \in \mathcal{F}$, $\{h_1, \dots, h_n\} = \mathfrak{v}_\circ(ij)\{f_1, \dots, f_n\}$ satisfies

$$\begin{aligned} h_i &= \tfrac{1}{\sqrt{2}}(f_i + f_j) \\ h_j &= \tfrac{1}{\sqrt{2}}(f_i - f_j) \end{aligned} \tag{3.5}$$

and $h_k = f_k$ if $k \neq i, j$.

A straight forward computation gives the following for any $i \neq j$ and $\mathfrak{u} \in U(n)$

$$F(ij)(\mathfrak{u}) + F(ij)(\mathfrak{u}_\circ(ij)\mathfrak{u}) = \frac{1}{2}\left(G(i)(\mathfrak{v}_\circ(ij)\mathfrak{u}) + G(j)(\mathfrak{v}_\circ(ij)\mathfrak{u})\right). \tag{3.6}$$

Thus for $i \neq j$ we have

$$\begin{aligned} 4\int_{U(n)} & F(ij)(\mathfrak{u})d\nu(\mathfrak{u}) \\ &= 2\int_{U(n)} \left(F(ij)(\mathfrak{u}) + F(ij)(\mathfrak{u}_\circ\mathfrak{u})\right) d\nu(\mathfrak{u}) \\ &= \int_{U(n)} \left(G(i)(\mathfrak{v}_\circ(ij)\mathfrak{u}) + G(j)(\mathfrak{v}_\circ(ij)\mathfrak{u})\right) d\nu(\mathfrak{u}) \\ &= \int_{U(n)} \left(G(i)(\mathfrak{u}) + G(j)(\mathfrak{u})\right) d\nu(\mathfrak{u}). \end{aligned} \tag{3.7}$$

On the other hand, for $i \neq j$ if we let $\mathfrak{w}_\circ \in U(n)$ be such that for any $\alpha \in \mathcal{F}$, the frame $\mathfrak{w}_\circ\alpha \in \mathcal{F}$ is simply obtained by switching the i and jth elements in α, then $G(i)(\mathfrak{u}) = G(j)(\mathfrak{w}_\circ\mathfrak{u})$ and hence

$$\int_{U(n)} G(i)(\mathfrak{u})d\nu(\mathfrak{u}) = \int_{U(n)} G(j)(\mathfrak{u})d\nu(\mathfrak{u}),$$

and (3.7) thus implies:

$$\begin{aligned} 2\int_{U(n)} F(ij)(\mathfrak{u})d\nu(\mathfrak{u}) &= \int_{U(n)} G(i)(\mathfrak{u})d\nu(\mathfrak{u}) \\ &= \int_{U(n)} G(1)(\mathfrak{u})d\nu(\mathfrak{u}) = K(p) \end{aligned} \tag{3.8}$$

for some $K(p)$ depending only on p. In particular, the LHS does not depend on i, j. This establishes the Claim.

For any $\mathfrak{u} \in U(n)$ we may write

$$S(p) = \sum_{i \neq j} F(ij)(\mathfrak{u}) + \sum_i G(i)(\mathfrak{u}). \tag{3.9}$$

Hence

$$\begin{aligned} S(p) &= \int_{U(n)} \left(\sum_{i \neq j} F(ij)(\mathfrak{u}) + \sum_i G(i)(\mathfrak{u}) \right) d\nu(\mathfrak{u}) \\ &= \frac{n(n+1)}{2} K(p) \end{aligned} \tag{3.10}$$

The equality in (3.10) may be interpreted as saying that on any Kähler manifold $S(p)$ is either an average of holomorphic sectional curvatures at p, a known fact from [**2**] (also see [**20**, p189]), or an average of orthogonal holomorphic bisectional curvatures at p. In particular, if either the holomorphic sectional curvatures or orthogonal bisectional curvatures are positive (nonnegative) respectively at p, then $S(p)$ is automatically positive (nonnegative).

Now let a_{ij} be as in part (a) of the lemma. Then

(3.11)

$$\begin{aligned} &\int_{U(n)} \sum_{i,j=1}^{n} F(ij)(\mathfrak{u}) a_{ij} d\mu(\mathfrak{u}) \\ &\qquad = \sum_{i \neq j} a_{ij} \int_{U(n)} F(ij)(\mathfrak{u}) d\mu(\mathfrak{u}) + \sum_i a_{ii} \int_{U(n)} G(i)(\mathfrak{u}) d\mu(\mathfrak{u}) \\ &\qquad = \frac{K(p)}{2} \left(\sum_{i \neq j} a_{ij} + 2 \sum_i a_{ii} \right). \end{aligned}$$

It follows that if (3.1) is true for all unitary frames, then $K(p) \geq 0$ as $\sum_{i \neq j} a_{ij} + 2 \sum_i a_{ii} > 0$. If in addition (3.1) is a strict inequality for some unitary frame, then

$$\sum_{ij} F(ij)(\mathfrak{u}) a_{ij} > 0$$

for some $\mathfrak{u} \in U(n)$. By continuity and the fact that (3.1) is true for all unitary frames, we conclude that $K(p) > 0$ and thus $S(p) > 0$ by (3.10).

The proof of (b) is similar.

q.e.d.

Proof of Theorem 3.1. The first assertion of the theorem follows immediately from Lemma 3.1. We now prove the other assertions in the Theorem in the case of nonnegative QOBC while the proof of nonpositive QOBC is similar.

We show $S(p)=0$ iff $R(V,\bar V,W,\bar W)=0$ for all unitary pairs U,V: Suppose $R(V,\bar V,W,\bar W)=0$ for all unitary pairs. Then if $K(p)$ is as in the proof Lemma 3.1 we have $K(p)=0$, and thus $S(p)=0$ by (3.10). Conversely, suppose $S(p)=0$. By Lemma 3.1(a), (1.1) is then an equality for all frames and choices of ξ. Thus for any unitary frame $\{e_1,\dots,e_n\}$, if we let $b_{ij}=R_{i\bar i j\bar j}=b_{ji}$ then the function f of $\xi=(\xi_1,\dots,\xi_n)\in\mathbb{R}^n$ given by

$$f(\xi)=\sum_{i\neq j} b_{ij}(\xi_i-\xi_j)^2$$

is identically zero. Hence for all $k\neq l$,

$$0=\frac{\partial^2 f}{\partial\xi_k\partial\xi_l}=-2b_{kl}-2b_{lk}=-4b_{kl}.$$

Hence $R_{k\bar k l\bar l}=0$. Since $\{e_i\}$ was arbitrary it follows that for any unitary pair V,W, $R(V,\bar V,W,\bar W)=0$. By (3.6), we also have $R(V,\bar V,V,\bar V)+R(W,\bar W,W,\bar W)=0$.

We now consider the case when $n\geq 3$. If $n\geq 3$, then for any unitary pair V,W we can find U such that U,V,W forms a unitary triple. Then by the above we have

$$0=R(U,\bar U,U,\bar U)+R(W,\bar W,W,\bar W)=-2R(V,\bar V,V,\bar V).$$

This completes the proof of the Theorem since V was arbitrary. q.e.d.

In case $n=2$, the last assertion of the theorem may not be true. This can be seen by Example 1.2 in [**9**]. Namely, if we let Σ be a compact Riemann surface with constant curvature -1 and let $\mathbb{CP}^1$ be the standard sphere. Then $\Sigma\times\mathbb{CP}^1$ has NQOBC. But the scalar curvature is zero everywhere.

4. Irreducible and reducible manifolds with NQOBC

We first use Theorem 3.1 to prove that a locally irreducible Kähler manifold with NQOBC must have positive first Chern class. This generalizes Theorem 2.1 in [**9**]. Note that by Lemma 1.1, $h^{1,1}(M)=1$ for such a manifold.

Theorem 4.1. *Let (M^n,g), $n\geq 2$, be compact Kähler manifold with NQOBC and Kähler form ω. Suppose $h^{1,1}(M)=1$. Then $c_1(M)=l[\omega]$ for some $l>0$.*

Proof. Since $h^{1,1}(M)=1$ we have $c_1(M)=l[\omega]$ for some l. By Theorem 3.1, the scalar curvature S of M is nonnegative. Hence

$$0\leq\int_M S\omega^n=\int_M \mathrm{Ric}\wedge\omega^{n-1}=l\,\mathrm{Vol}(M)$$

and thus $l\geq 0$. We now show that $l>0$. Suppose otherwise, and that $l=0$ and thus $S=0$ everywhere.

Suppose $n \geq 3$. Then Theorem 3.1 implies M is flat which is impossible since $h^{1,1}(M) = 1$. Suppose $n = 2$. At any point we may choose local coordinates z^i such that $\{\frac{\partial}{\partial z^i}\}$ are unitary and eigenvectors of Ric. Let $a_i := R_{i\bar{i}}$. Then as $S = 0$ we have $a_1 = -a_2$ and we further calculate

$$
\begin{aligned}
\mathrm{Ric} \wedge \mathrm{Ric} &= \left(\sum_{i=1}^{2} a_i dz^i \wedge d\bar{z}^i\right) \wedge \left(\sum_{j=1}^{2} a_j dz^j \wedge d\bar{z}^j\right) \\
&= 2a_1 a_2 dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2 \\
&= -2a_1^2 dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2.
\end{aligned} \tag{4.1}
$$

Since $l = 0$,

$$\int_M \mathrm{Ric}^2 = 0$$

which implies $a_1 = a_2 = 0$ everywhere. On the other hand, Theorem 3.1 implies $R_{1\bar{1}2\bar{2}} = 0$ and thus $0 = a_i = R_{i\bar{i}} = R_{i\bar{i}i\bar{i}}$ for $i = 1, 2$. Thus M is flat which is impossible since $h^{1,1}(M) = 1$. This concludes the proof of the Theorem by contradiction. The above argument for $n = 2$ can actually be used for all $n \geq 2$. q.e.d.

Next we want to study the case when (M, g) is possibly reducible. We will obtain a partial classification of the possible deRham factorizations of the universal cover of M (Theorem 4.2). Let $\widetilde{M}$ be the universal cover of M with covering map π. Let

$$\widetilde{M} = \widetilde{M}_0 \times \widetilde{M}_1 \times \cdots \times \widetilde{M}_k$$

be the deRham decomposition of $\widetilde{M}$ where $\widetilde{M}_0$ is Euclidean and has dimension possibly zero, and $\widetilde{M}_i$, $i \geq 1$ (with positive dimension) are irreducible non-flat factors. Then the product of any subcollection of $\tilde{M}_i$'s still has NQOBC provided the product has dimension at least 2. By Lemma 1.1, $c_1(M)$ can be represented by a parallel harmonic (1,1) form η so that $\mathrm{Ric} = \eta + \sqrt{-1}\partial\bar{\partial} f$ for some smooth function f on M. Then this pullls back to $\tilde{M}$ to give

$$\widetilde{\mathrm{Ric}} = \widetilde{\eta} + \sqrt{-1}\partial\bar{\partial}\tilde{f} \tag{4.2}$$

where $\tilde{f} = \pi^* f$ and $\widetilde{\eta} = \pi^*\eta$. Moreover, by Corollary 2.1 $\widetilde{\eta}$ has the form

$$\widetilde{\eta} = \widetilde{\eta}_0 \times l_1\tilde{\omega}_1 \times \ldots \cdots \times l_k\tilde{\omega}_k$$

where $\tilde{\omega}_i$ is the Kähler form of $\tilde{M}_i$, l_i are constants and $\widetilde{\eta}_0$ is parallel in $\widetilde{M}_0$. In the following, for a Kähler manifold (N, h) we use the notation $\Delta_N u = h^{i\bar{j}} u_{i\bar{j}}$.

Lemma 4.1. *We have $\widetilde{\eta}_0 = 0$. Also, if $i \geq 1$ and $\dim \widetilde{M}_i \geq 2$ then $l_i > 0$.*

Proof. Suppose $l_1 \leq 0$ and $\dim \widetilde{M}_1 \geq 2$ say. Let

$$\tilde{p} = (\tilde{p}_0, \tilde{p}_1, \ldots, \tilde{p}_k) \in \widetilde{M}_0 \times \widetilde{M}_1 \times \cdots \times \widetilde{M}_k = \widetilde{M}$$

be such that $f(\pi\tilde{p}) = \max_M f$. Consider the function

$$h(\tilde{q}) = \tilde{f}(\tilde{p}_0, \tilde{q}, \ldots, \tilde{p}_k)$$

on $\widetilde{M}_1$ for $\tilde{q} \in \widetilde{M}_1$. Then $h(\tilde{p}_1) = \max_{\widetilde{M}_1} h$. On the other hand, $\widetilde{\mathrm{Ric}}_{(1)} = l_1\tilde{\omega}_1 + \sqrt{-1}\partial\bar{\partial}h$. By Theorem 3.1, the scalar curvature of $\widetilde{M}_1$ is nonnegative because $\dim \widetilde{M}_1 \geq 2$ and we conclude that

$$\Delta_{\widetilde{M}_1} h \geq 0.$$

By the strong maximum principle, it follows that h is constant and thus $\widetilde{\mathrm{Ric}}_{(1)} = l_1\omega_1$. By Theorem 3.1 again we have $l_1 \geq 0$, and so $l_1 = 0$ and $\widetilde{\mathrm{Ric}}_{(1)} = 0$. In particular, the scalar curvature of $\widetilde{M}_1$ is zero. By Theorem 3.1 and the fact that $\widetilde{\mathrm{Ric}}_{(1)} = 0$, we conclude that $\widetilde{M}_1$ in fact has zero holomorphic sectional curvature everywhere. This contradicts the fact that $\widetilde{M}_1$ is nonflat.

To prove that $\widetilde{\eta}_0 = 0$, first note that for the function

$$\tilde{f}_0(\tilde{q}) = \tilde{f}(\tilde{q}, \tilde{p}_1, \ldots, \tilde{p}_k)$$

on $\widetilde{M}_0$, with $\tilde{p}_i$ being fixed, we have

$$\Delta_{\widetilde{M}_0} \tilde{f}_0 = C$$

for some constant C because $\widetilde{M}_0$ is flat and $\widetilde{\eta}_0$ is parallel. Since $\tilde{f}_0$ is bounded and $\widetilde{M}_0$ is flat, we may conclude that $\tilde{f}_0$ is constant. Hence $\widetilde{\eta}_0 = 0$. q.e.d.

Lemma 4.2. *Suppose M has NQOBC at p and $M = M_1 \times M_2$ with $\dim(M_1) = 1$. Let $e_1, \ldots, e_n$ be a unitary frame at p such that e_1 is tangent to M_1 and $e_2, \ldots, e_n$ are tangent to M_2. Then*

$$\sum_{j=2}^{n} R(e_2, \bar{e}_2, e_j, \bar{e}_j) \geq -R(e_1, \bar{e}_1, e_1, \bar{e}_1).$$

Proof. Let $f_1 = \frac{e_1+e_2}{\sqrt{2}}$, $f_2 = \frac{e_1-e_2}{\sqrt{2}}$, $f_j = e_j$ for $j \geq 3$. Then

$$R(f_1, \bar{f}_1, f_2, \bar{f}_2) = \frac{1}{4}\left(R(e_1, \bar{e}_1, e_1, \bar{e}_1) + R(e_2, \bar{e}_2, e_2, \bar{e}_2)\right),$$

and for $j \geq 3$

$$R(f_1, \bar{f}_1, f_j, \bar{f}_j) = \frac{1}{2}R(e_2, \bar{e}_2, e_j, \bar{e}_j) = R(f_2, \bar{f}_2, f_j, \bar{f}_j).$$

Let $\xi_1 = 1$, $\xi_2 = -1$ and $\xi_j = 0$ for $j \geq 3$, then

$$
\begin{aligned}
0 &\leq \sum_{i<j} R(f_i, \bar{f}_i, f_j, \bar{f}_j)(\xi_i - \xi_j)^2 \\
&= 4R(f_1, \bar{f}_1, f_2, \bar{f}_2) + \sum_{j=3}^{n} R(f_1, \bar{f}_1, f_j, \bar{f}_j) + \sum_{j=3}^{n} R(f_2, \bar{f}_2, f_j, \bar{f}_j) \qquad (4.3) \\
&= R(e_1, \bar{e}_1, e_1, \bar{e}_1) + R(e_2, \bar{e}_2, e_2, \bar{e}_2) + \sum_{j=3}^{n} R(e_2, \bar{e}_2, e_j, \bar{e}_j).
\end{aligned}
$$

From this the result follows. q.e.d.

Theorem 4.2. *We have the following mutually exclusive cases:*

(i) $\dim \widetilde{M}_0 \geq 1$ *and for each* $i \geq 1$*:* $\widetilde{M}_i$ *is compact with* $l_i > 0$ *and quasi positive Ricci curvature, i.e., the Ricci curvature of* $\widetilde{M}_i$ *is nonnegative and is positive at some point. In particular, if* $\dim \widetilde{M}_i = 1$ *then* $\widetilde{M}_i$ *is biholomorphic to* $\mathbb{CP}^1$.

(ii) $\dim \widetilde{M}_0 = 0$ *and for each* $i \geq 1$*:* $\widetilde{M}_i$ *is compact with* $l_i > 0$*. In particular, if* $\dim \widetilde{M}_i = 1$ *then* $\widetilde{M}_i$ *is biholomorphic to* $\mathbb{CP}^1$ *and all other factors have nonnegative (positive, respectively) Ricci curvature provided* $\widetilde{M}_i$ *has nonpositive (negative, respectively) Gauss curvature somewhere.*

(iii) $\dim \widetilde{M}_0 = 0$ *and for some* $i \geq 1$, $\widetilde{M}_i$ *is biholomorphic either to* $\mathbb{C}$ *with* $l_i = 0$ *or to the unit disk in* $\mathbb{C}$ *with* $l_i < 0$*, and all other factors are compact and have* $l_j > 0$ *with Ricci curvature bounded below by a positive constant.*

Proof. If $\dim \widetilde{M}_0 \geq 1$, by Lemma 4.2 and the fact that $\widetilde{M}_0$ is flat, we conclude that the Ricci curvature of $\widetilde{M}_i$ is nonnegative for all $i \geq 1$. Hence M has nonnegative Ricci curvature. By [**6**], $\widetilde{M}_i$ is compact for all $i \geq 1$. If $\dim \widetilde{M}_i \geq 2$, then $l_i > 0$ by Lemma 4.1. If $\dim \widetilde{M}_i = 1$, then it is biholomorphic to $\mathbb{CP}^1$ because it is compact and has nonnegative Gaussian curvature. Since $\mathrm{Ric} = \eta + \sqrt{-1}\partial\bar{\partial} f$, the Gauss-Bonnet theorem implies:

$$4\pi = \int_{\widetilde{M}_i} K_i = l_i \,\mathrm{Vol}(\widetilde{M}_i)$$

where K_i is the Gaussian curvature of $\widetilde{M}_i$. Hence $l_i > 0$. Now let

$$\tilde{p} = (\tilde{p}_0, \tilde{p}_1, \ldots, \tilde{p}_k) \in \widetilde{M}_0 \times \widetilde{M}_1 \times \cdots \times \widetilde{M}_k = \widetilde{M}$$

be such that $f(\pi\tilde{p}) = \min_M f$. Then at $\tilde{p}$,

$$\widetilde{\mathrm{Ric}} \geq \widetilde{\eta}.$$

from which it is easy to see that $\widetilde{M}_i$ has positive Ricci curvature at $\tilde{p}_i$. Thus $\tilde{M}_i$ has quasi positive Ricci curvature for each $i \geq 1$. Hence we are in the situation of case (i) in the Theorem.

Suppose $\dim \widetilde{M}_0 = 0$ and $l_i > 0$ for all $i \geq 1$. By Yau's theorem on the Calabi conjecture [**18**], there is a real-valued function u on M such that $g_{i\bar{j}} + u_{i\bar{j}} > 0$ is a Kähler metric with Ricci form being η. Pulling this metric back by π gives a Kähler metric on $\widetilde{M}$ with Ricci form being $\widetilde{\eta}$ which is positive and is bounded away from 0, because $\dim \widetilde{M}_0 = 0$ and $l_i > 0$. Hence $\widetilde{M}$ is compact by Myers theorem. Using Lemma 4.2, we are thus in the situation of case (ii) in the Theorem.

Suppose $\dim \widetilde{M}_0 = 0$ and $l_1 \leq 0$, say. Then $\dim \widetilde{M}_1 = 1$ by Lemma 4.1. The induced metric on $\widetilde{M}_1$ is of the form $e^{2\lambda}|dz|^2$ where $|dz|^2$ is the standard Euclidean metric on $\mathbb{C}$ or the unit disk in $\mathbb{C}$. Let $h(\tilde{q}) = \tilde{f}(\tilde{q}, \tilde{p}_2, \ldots, \tilde{p}_k)$. Then the Gaussian curvature of $\widetilde{M}_1$ satisfies:

$$K_1 = l_1 + \Delta_{\widetilde{M}_1} h = l_1 + \frac{1}{2} e^{-2\lambda} \Delta_0 h$$

by (4.2) and where Δ_0 is the Euclidean Laplacian. On the other hand, we also have

$$K_1 = -\Delta_{\widetilde{M}_1} \lambda = -\frac{1}{2} e^{-2\lambda} \Delta_0 \lambda$$

Hence

$$l_1 = -\frac{1}{2} e^{-2\lambda} \Delta_0 (\lambda + h)$$

which is just the Gaussian curvature of the metric $e^{2\lambda+2h}|dz|^2$. Since h is bounded, the metric $e^{2\lambda+2h}|dz|^2$ is complete. Hence $l_1 = 0$ if and only if $\widetilde{M}_1$ is biholomorphic to $\mathbb{C}$ and $l_1 < 0$ if and only if $\widetilde{M}_1$ is biholomorphic to the unit disk in $\mathbb{C}$. In case $l_1 = 0$, the Gaussian curvature of $\widetilde{M}_1$ must be negative somewhere. Otherwise, $\widetilde{M}_1$ must be flat by the proof of [**6**, Theorem 3]. This is impossible, because $\dim \widetilde{M}_0 = 0$. In case $l_1 < 0$, it is easy to see that the Gaussian curvature of $\widetilde{M}_1$ is negative somewhere. Using Lemma 4.2, we are thus in the situation of case (iii) in the Theorem.

q.e.d.

References

1. S. Bando, *On the classification of three-dimensional compact Kähler manifolds of nonnegative bisectional curvature*, J. Differential Geom. **19** (1984), no. 2, 283–297, MR 0755227, Zbl 0547.53034.
2. M. Berger, *Sur les variétés d'Einstein compactes*, C.R. IIIe Réunion Math. Expression latine, Namur 1965, 35–55, MR 0238226.
3. R.L. Bishop & S.I. Goldberg, *On the second cohomology group of a Kähler manifold of positive curvature*, Proc. Amer. Math. Soc. **16** (1965), 119–122, MR 0172221, Zbl 0125.39403.
4. A. Chau & L.F. Tam, *A note on harmonic forms and the boundary of the Kähler cone*, arXiv:1105.2913.
5. A. Chau & L.F. Tam, *Kähler C-spaces of classical type and quadratic bisectional curvature*, arXiv:1202.4542.

6. J. Cheeger, & D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Differential Geom. **6** (1971/72), 119–128, MR 0303460, Zbl 0223.53033.
7. X.X. Chen, *On Kähler manifolds with positive orthogonal bisectional curvature*, Adv. Math. **215** (2007), no. 2, 427–445, MR 2355611, Zbl 1131.53038.
8. S.I. Goldberg & S. Kobayashi, *Holomorphic bisectional curvature*, J. Differential Geom. **1** (1967), 225–233, MR 0227901, Zbl 0169.53202.
9. H.L. Gu & Z.H. Zhang, *An Extension of Mok's Theorem on the Generalized Frankel Conjecture*, Sci. China Math. **53** (2010), no. 5, 1253–1264, MR 2653275, Zbl 1204.53058.
10. A. Howard, B. Smyth & H. Wu, *On compact Kähler manifolds of nonnegative bisectional curvature, I,* Acta Math. **147** (1981), 51–56, MR 0631087, Zbl 0473.53055.
11. Q. Li, D. Wu & F. Zheng, *An example of compact Kähler manifold with nonnegative quadratic bisectional curvature*, arXiv:1110.1749.
12. N. Mok, *The uniformization theorem for compact Kähler manifolds of non- negative bisectional curvature*, J. Differential Geom. **27** (1988), no. 2, 179–214, MR 0925119, Zbl 0642.53071.
13. S. Mori, *Projective manifolds with ample tangent bundles*, Ann. of Math. (2) **110** (1979), 593–606, MR 0554387, Zbl 0423.14006.
14. Y.T. Siu & S.T. Yau, *Complex Kähler manifolds of positive bisectional curvature*, Invent. Math. **59** (1980), 189–204.
15. H. Wu, *On compact Kähler manifolds of nonnegative bisectional curvature II*, Acta Math. **147** (1981), 57–70, MR 0631088, Zbl 0473.53056.
16. D. Wu, S.T. Yau & F. Zheng, *A degenerate Monge Ampère equation and the boundary classes of Kähler cones*, Math. Res. Lett. **16** (2009), no.2, 365–374, MR 2496750, Zbl 1183.32018.
17. S.-T. Yau (ed), *Seminar on differential geometry*, Princeton University Press; Tokyo: University of Tokyo Press, 1982, MR 0645728, Zbl 0471.00020.
18. S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I,* Comm. Pure and Appl. Math. **31** (1978), 339–441, MR 0480350, Zbl 0369.53059.
19. X. Zhang, *On the boundary of Kähler cones*, Proc. of A.M.S., posted on June 21, 2011, PII S 0002-9939(2011)10929-8 (to appear in print).
20. F. Zheng, *Complex differential geometry*, American Mathematical Soc., International Press, 2000, MR 1777835, Zbl 0984.53002.

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