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CRITICAL POINTS OF GREEN'S FUNCTIONS ON COMPLETE MANIFOLDS

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Abstract

We prove that the number of critical points of a Li–Tam Green's function on a complete open Riemannian surface of finite type admits a topological upper bound, given by the first Betti number of the surface. In higher dimensions, we show that there are no topological upper bounds on the number of critical points by constructing, for each nonnegative integer N, a Riemannian manifold diffeomorphic to $\mathbb{R}^n$ ($n \geqslant 3$) whose minimal Green's function has at least N non-degenerate critical points. Variations on the method of proof of the latter result yield contractible n-manifolds whose minimal Green's functions have level sets diffeomorphic to any fixed codimension 1 compact submanifold of $\mathbb{R}^n$.

1. Introduction

Let (M, g) be a noncompact, complete Riemannian n-manifold without boundary and let us denote by $\mathcal{G} : (M \times M)\backslash \operatorname{diag}(M \times M) \to \mathbb{R}$ a symmetric Green's function of (M, g), which satisfies

$$\Delta_g \mathcal{G}(\cdot, y) = -\delta_y \tag{1.1}$$

for each $y \in M$. We will find it notationally convenient to fix a point $y \in M$, once and for all, and consider a Green's function $G := \mathcal{G}(\cdot, y)$ with pole y, which is smooth and harmonic in $M\backslash\{y\}$.

The study of the Green's functions of the Laplacian in a complete Riemannian manifold is a classical problem in geometric analysis and partial differential equations. Consequently, there is a vast literature on this topic covering, among many other aspects, the existence of positive Green's functions [**8**, **24**, **25**], upper and lower bounds, gradient estimates and asymptotics [**28**, **26**, **9**, **19**], and the connection between Green's functions and the heat kernel [**37**, **27**, **16**].

In this paper we shall focus on the study of the critical points of Green's functions on a complete Riemannian manifold. The chief difficulty lies in the fact that, generally speaking, Green's function estimates

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are not sufficiently fine to elucidate whether the gradient of G vanishes in a certain region. Moreover, it is well known that the codimension of the critical set of G is at least 2 [**17**], which introduces additional complications in the analysis. For this reason, our approach is based on a combination of techniques from differential topology and second-order elliptic PDEs.

Let us state our main results. Our first theorem asserts that there is a topological upper bound for the number of critical points of any Li–Tam Green's function on a surface of finite type. Here by Li–Tam Green's function we mean a Green's function that arises from the exhaustion procedure described by Li and Tam in [**24**], which will be briefly recalled in Section 2. Any Li–Tam Green's function coincides with the minimal one when the latter exists.

Theorem 1.1. *Let (M, g) be a smooth open Riemannian surface of finite type. The number of critical points of any Li–Tam Green's function G on M is not larger than the first Betti number $b_1(M)$, and if this upper bound is attained, then G is Morse.*

Theorem 1.1 is a substantial extension of the classical result [**38**] that the Dirichlet Green's function of a simply or doubly connected domain in the Euclidean plane respectively has zero or one critical points and, to our best knowledge, is the first general finiteness result for critical points in noncompact manifolds. The proof of the theorem combines global ideas (namely, uniformization and Hopf's index theorem), local index bounds, and some classical harmonic function theory. We will also discuss how the topological upper bound can be slightly refined using the conformal geometry of the surface (Remark 2.3) and an extension to higher-dimensional axisymmetric manifolds (Theorem 2.6).

Our second result shows that, contrary to what happens in the case of surfaces, the number of critical points of the Green's function of a manifold of dimension $n \geqslant 3$ does not admit a topological upper bound. In fact, we provide a procedure for constructing analytic metrics in $\mathbb{R}^n$ ($n \geqslant 3$) whose minimal Green's functions have level sets of prescribed topology and as many non-degenerate critical points as one wishes, which allows us to prove the following

Theorem 1.2. *Let N be a positive integer and Σ a smooth codimension 1 closed submanifold of $\mathbb{R}^n$. For any $n \geqslant 3$ there exist real analytic complete Riemannian manifolds (M_j, g_j) $(1 \leqslant j \leqslant 3)$ diffeomorphic to $\mathbb{R}^n$ such that:*

(i) *The minimal Green's function of (M_1, g_1) has at least N non-degenerate critical points.*
(ii) *The minimal Green's function of (M_2, g_2) has a level set diffeomorphic to Σ.*

(iii) *The critical set of the minimal Green's function of* (M_3, g_3) *has codimension at most* 3.

In all cases, the Green's functions tend to zero at infinity.

Additional motivation for this theorem comes from a question of Kawohl [**22**], recently solved in [**12**], concerning the possible level sets and critical points of the solution to an exterior boundary problem in Euclidean space. In the context of Riemannian geometry, the natural analogue of Kawohl's problem is whether the minimal Green's function (when it exists) of a manifold diffeomorphic to $\mathbb{R}^n$ necessarily has no critical points and all its level sets are homeomorphic to spheres. The first statement in Theorem 1.2 is reminiscent of results of Morse and Sheldon [**30**, **35**] on the existence of Morse harmonic functions in bounded domains of $\mathbb{R}^2$ and $\mathbb{R}^3$ with an arbitrary number of non-degenerate critical points, but these authors' constructions are fundamentally different in scope and cannot be modified to deal with the problem studied in this paper.

The article is organized as follows. In Section 2 we give the proof of Theorem 1.1 and, using similar ideas, show that if a Green's function of an axisymmetric manifold diffeomorphic to $\mathbb{R}^n$ tends to zero at infinity, then it does not have any critical points. In Section 3 we prove that the Dirichlet Green's function of a bounded domain in a Riemannian manifold is generically Morse, a result we utilize in the proof of Theorem 1.2. This statement cannot be deduced from Albert's, Uhlenbeck's, or Bando–Urakawa's analogous theorems for the eigenfunctions of the Laplacian [**36**, **2**, **4**] or Damon's results for filtered differential operators [**10**]. Finally, in Section 4 we present the proof of Theorem 1.2.

We conclude this section by introducing some standard notation. We shall denote by Δ_g, ∇_g, $\mathrm{d}V_g$, $|\cdot|_g$, and dist_g, respectively, the Laplacian, gradient operator, Riemannian measure, norm, and distance function in (M, g), and we shall reserve the notation Δ, ∇, $\mathrm{d}x$, $|\cdot|$, and dist for the corresponding objects in Euclidean space $(\mathbb{R}^n, g_0)$. We shall use the notation $B_g(x, r)$ and $B(x, r)$, in each case, for the open geodesic balls in (M, g) and in $(\mathbb{R}^n, g_0)$ of center x and radius r. The *critical set* of a C^1 function f is defined as the set of points x in the domain of f such that $\mathrm{d}f(x) = 0$. Throughout this paper, the Riemannian manifold (M, g) will be assumed to be connected, oriented, and of class C^∞.

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2. Green's functions on surfaces

In this section we will prove a topological upper bound for the number of critical points of a Green's function on a C^∞ Riemannian surface (M, g). We assume that the surface is of *finite type*, i.e., that its fundamental group is finitely generated.

We will also assume that the Green's function satisfies certain convenient conditions at infinity in order to control some global properties of its level sets; in particular, a standard choice is to restrict one's attention to the minimal Green's function whenever it exists. More generally, we shall assume in what follows that the Green's function has been obtained through an exhaustion procedure as in Li and Tam's paper [**24**].

Let us summarize the main facts about Li–Tam Green's functions, which are obviously true in any dimension. We will henceforth denote by $\mathcal{G}_\Omega : (\Omega \times \Omega)\backslash \operatorname{diag}(\Omega \times \Omega) \to \mathbb{R}$ the symmetric Dirichlet Green's function of a bounded domain $\Omega \subset M$, which is defined by

$$\Delta_g \mathcal{G}_\Omega(\cdot, y) = -\delta_y \quad \text{in } \Omega\,, \qquad \mathcal{G}_\Omega(\cdot, y) = 0 \quad \text{on } \partial\Omega\,, \tag{2.1}$$

and set $G_\Omega := \mathcal{G}_\Omega(\cdot, y)$. Denoting by $\Omega_1 \subset \Omega_2 \subset \cdots$ an exhaustion of M by bounded domains, it was proved in [**24**] that there exists a sequence of nonnegative real numbers $(a_j)_{j=1}^\infty$ such that $G_{\Omega_j} - a_j$ converges uniformly on compact sets of $M\backslash\{y\}$ to a Green's function G with pole y, and that it coincides with the minimal one whenever the latter exists. Throughout this paper we will refer to the class of Green's functions that arise through this procedure as *Li–Tam Green's functions*. Li–Tam Green's functions are generally non-unique, but in any case a Li–Tam Green's function has the following properties [**24**]:

(i) G is decreasing in the sense that
$$\sup_{M\backslash B_g(y,r)} G = \max_{\partial B_g(y,r)} G$$
for all $r > 0$, $B_g(y, r)$ being the geodesic ball centered at the pole y of radius r.

(ii) $\mathcal{G}$ is a symmetric function, that is, $\mathcal{G}(x, y) = \mathcal{G}(y, x)$.

(iii) If H is an amenable isometry group of (M, g), one can assume that $\mathcal{G}$ is invariant under H, that is, $\mathcal{G}(h(x), h(y)) = \mathcal{G}(x, y)$ for all $x, y \in M$ and $h \in H$ (cf. e.g. [**11**]).

The proof of Theorem 1.1 is based on an index-theoretical argument that combines a local analysis and a global one. The local part consists of the calculation of the Hopf index of a critical point of the Green's function using a blow-up technique. The global part exploits the conformal symmetry of the equation to show that the critical set of G is finite and analyze the asymptotic behavior of the Green's function in each end of the surface.

Let us begin with the following local lemma, where we estimate the index that a critical point of G can have. We will prove it in slightly greater generality than we would need for Theorem 1.1 because we will also use it in the proof of Theorem 2.6 below. This result is analogous to Cheng's local analysis of the nodal set of the eigenfunctions on a surface [**7**], but our proof relies on a blow-up argument instead of the Kuiper–Kuo theorem. The result being local, we will state it in local coordinates (x_1, x_2).

Lemma 2.1. *Let $z \in \mathbb{R}^2$ be a critical point of a function u that satisfies an equation of the form $g^{ij}\,\partial_i\partial_j u + b^i\,\partial_i u = 0$ in a neighborhood of z. We assume the functions g^{ij} and b^i are smooth and the symmetric matrix (g^{ij}) is invertible, and call $m \geqslant 2$ the degree of the lowest nonzero homogeneous term in the Taylor expansion of $u - u(z)$ at z. Then z is an isolated zero of $\nabla_g u$ and its index is* $\operatorname{ind}(z) = 1 - m$.

Proof. There is no loss of generality in assuming that $z = 0$ and that, after making a linear change of coordinates if necessary, $g^{ij}(0) = \delta^{ij}$. The unique continuation theorem [**23**] implies that the function $u - u(0)$ can vanish only up to finite order at 0, so m is necessarily finite. By a theorem of Bers [**5**], there exists a homogeneous polynomial h_m of degree m such that

$$u(x) - u(0) = h_m(x) + O(|x|^{m+1})\,, \tag{2.2a}$$

$$\nabla_g u(x) = \nabla h_m(x) + O(|x|^m)\,, \tag{2.2b}$$

where in these coordinates $\nabla_g u$ is the vector of components $g^{ij}\,\partial_j u$. Furthermore, the polynomial h_m is harmonic with respect to the Euclidean metric ($\Delta h_m = 0$), which implies that the origin is an isolated critical point of h_m. It follows that the critical point of u must be isolated too.

Let us now consider polar coordinates $(r, \theta) \in \mathbb{R}^+ \times \mathbb{S}^1$ defined by $(x_1, x_2) = (r\cos\theta, r\sin\theta)$. As h_m is harmonic, it readily follows that in these coordinates one has $h_m(r, \theta) = C_0\, r^m \cos(m\theta - \theta_0)$ for some constants C_0 and θ_0. There is obviously no loss of generality in setting $\theta_0 = 0$. We define the polar blow up [**33**] of the gradient $\nabla_g u$ at z using polar coordinates as the vector field

$$X := \frac{1}{C_0 m\, r^{m-2}}\, \nabla_g u = \frac{1}{C_0 m\, r^{m-2}} \left(\nabla h_m + O(r^m)\right),$$

where we have used Eq. (2.2b). The blown-up trajectories are then given by

$$\dot{r} = r \cos m\theta + O(r^2)\,, \tag{2.3a}$$

$$\dot{\theta} = -\sin m\theta + O(r)\,. \tag{2.3b}$$

The blown-up critical points are thus $(0, \theta_k)$, with $\theta_k := k\pi/m$ and $k = 1, \dots, 2m$. The Jacobian matrix of X at $(0, \theta_k)$ is

$$DX(0, \theta_k) = \begin{pmatrix} (-1)^k & 0 \\ 0 & (-1)^{k+1} \end{pmatrix}, \tag{2.4}$$

so these critical points are hyperbolic saddles. By blowing down, we immediately find that a deleted neighborhood of 0 consists exactly of $2m$ hyperbolic sectors of the vector field X.

Since the field X is proportional to the gradient field $\nabla_g u$ through a nonvanishing factor, the well known Bendixson formula for the index of a planar vector field (cf. e.g. [**33**, Theorem 3.12.7]) asserts that the index of the critical point is

$$\operatorname{ind}(0) = 1 - \frac{\text{number of hyperbolic sectors}}{2} = 1 - m\,,$$

as claimed. q.e.d.

Let us now focus on global aspects of the problem. Since (M, g) is a smooth Riemannian surface of finite type, it is well known (cf. e.g. [**20**]) that there is a compact surface Σ endowed with a metric of constant curvature $\bar{g}$, a certain number $\lambda_1 \geqslant 0$ of isolated points $p_i \in \Sigma$, and another number $\lambda_2 \geqslant 0$ of topological disks $D_i \subset \Sigma$ such that (M, g) is conformally isometric to the interior of

$$\overline{M} := \Sigma \backslash \bigg(\bigcup_{i=1}^{\lambda_1} \{p_i\} \cup \bigcup_{j=1}^{\lambda_2} D_j \bigg)$$

with the metric $\bar{g}$. Equivalently, there exists a diffeomorphism $\Phi : M \to \operatorname{int} \overline{M}$ and a smooth positive function f on M such that $\Phi^* \bar{g} = f g$. Moreover, the boundaries ∂D_i can be taken analytic without loss of generality. Denoting the genus of Σ by ν and setting $\lambda := \lambda_1 + \lambda_2 \geqslant 1$, it is clear that the pair (ν, λ) determines the surface M uniquely up to diffeomorphism.

In the following lemma we prove several basic properties of the Li–Tam Green's function G, some of which are well known. In terms of the diffeomorphism Φ considered above, it is convenient to introduce the notation $\overline{G} := G \circ \Phi^{-1}$ for the Green's function on (the interior of) $\overline{M}$ and use the notation $\bar{y} := \Phi(y)$ for the image of the pole of the Green's function under the aforementioned diffeomorphism.

Lemma 2.2. *The Green's function has the following properties:*

(i) *G is positive if and only if $\lambda_2 \geqslant 1$. In this case, $\overline{G}$ tends to zero at the boundary of each disk ∂D_j. $\overline{G}$ can be extended so as to satisfy $\Delta_{\bar{g}}\overline{G} = 0$ in a neighborhood of each isolated point p_i and each circle ∂D_j, and the gradient $\nabla_{\bar{g}}\overline{G}$ is orthogonal to (and nonzero at) ∂D_j.*

(ii) *If $\lambda_2 = 0$, $\overline{G}$ tends to $-\infty$ at, at least, one of the points p_i, and satisfies the equation $\Delta_{\bar{g}}\overline{G} = 0$ in a neighborhood of any point p_j where $\overline{G}$ is bounded from below.*

(iii) *$\overline{G}$ has a finite number of critical points in the interior of $\overline{M}\backslash\{\bar{y}\}$.*

Proof. A short calculation using the conformal properties of the Laplacian in two dimensions shows that $\overline{G}$ satisfies the equation

$$\Delta_{\bar{g}}\overline{G} = -\delta_{\bar{y}}$$

in the interior of $\overline{M}$. When the number λ_2 of disks is at least 1, it is standard (for example, due to the existence of bounded positive harmonic functions in the interior of $\overline{M}$ [**24**]) that G is the minimal Green's function of (M, g), which corresponds to the unique solution $\overline{G}$ of the boundary problem

$$\Delta_{\bar{g}}\overline{G} = -\delta_{\bar{y}} \quad \text{in } \Sigma\backslash\bigcup_{j=1}^{\lambda_2}\overline{D_j}\,, \qquad \overline{G} = 0 \quad \text{on } \partial D_j \text{ for all } j.$$

Moreover, the fact that $\bar{g}$ and ∂D_j are analytic ensures [**29**] that $\overline{G}$ is analytic and harmonic in a neighborhood of each circle ∂D_j.

When $\lambda_2 = 0$, the function $\overline{G}$ satisfies the equation $\Delta_{\bar{g}}\overline{G} = 0$ but at the pole $\bar{y}$ and the isolated points p_i. By property (i) of Li–Tam Green's functions, $\overline{G}$ is upper bounded at each point p_i. If it is also lower bounded, it is standard that p_i is a removable singularity [**14**] and $\Delta_{\bar{g}}\overline{G} = 0$ in a neighborhood of p_i. If $\overline{G}$ is not lower bounded, p_i is an isolated singularity of $\overline{G}$, and the fact that $\overline{G}$ is upper bounded readily implies that $\Delta_{\bar{g}}\overline{G} = c_i\delta_{p_i}$ in a neighborhood of p_i for some positive constant c_i. Hence

$$\Delta_{\bar{g}}\overline{G} = -\delta_{\bar{y}} + \sum_{i=1}^{\lambda_1} c_i\,\delta_{p_i}$$

in the closed manifold Σ, so obviously $\sum_i c_i = 1$.

Let us now prove that the number of critical points of G is finite. As the boundary ∂D_j is analytic and $\overline{G}$ is positive (when $\lambda_2 > 0$), by Hopf's boundary point lemma [**15**] there is a neighborhood U_j of each disk $\overline{D_j}$ such that $\overline{G}$ does not have any critical points in U_j. The boundary ∂D_j being a level curve of $\overline{G}$, this ensures the gradient $\nabla_{\bar{g}}\overline{G}$ is orthogonal to this set. If p denotes either an isolated point p_i where $\overline{G}$ tends to $-\infty$

or the pole $\bar{y}$, the well known asymptotics [**14**]
(2.5)
$$\big|G(x)+C_p\log\operatorname{dist}_{\bar g}(x,p)\big|+\big|\nabla_{\bar g}\overline{G}(x)+C_p\nabla_{\bar g}\log\operatorname{dist}_{\bar g}(x,p)\big|_{\bar g}=O(1)$$

for the Green's function ensures that $\overline{G}$ does not have any critical points in a neighborhood of p. Here C_p is to be interpreted as $-c_i/2\pi$ if $p=p_i$ and $1/2\pi$ if $p=\bar{y}$. Since the critical points of $\overline{G}$ are isolated and cannot accumulate at the isolated points p_i where $\overline{G}$ is bounded due to the fact that $\overline{G}$ is harmonic at these points and Lemma 2.1, it follows that $\overline{G}$ has a finite number of critical points, as claimed. q.e.d.

We are now ready to prove Theorem 1.1. In the proof we will assume that the Green's function $\overline{G}$ has been extended to the points p_i where it is bounded and to the circles ∂D_j as in Lemma 2.2. We recall that the first Betti number of the manifold M is given by

$$b_1(M)=2\nu+\lambda-1\,,$$

and that a smooth function is *Morse* if its Hessian matrix has maximal rank at all its critical points. It should be noted too that obviously $\operatorname{Cr}(G)=\Phi^{-1}(\operatorname{Cr}(\overline{G}))$, where $\operatorname{Cr}(f)$ denotes the critical set of a function f.

Proof of Theorem 1.1. Let $D_{\bar y}$ be a small disk in $\overline{M}$ that contains the pole $\bar{y}$. For concreteness, using the asymptotics for the Green's function (2.5) we can assume that the boundary of this disk is the level curve $\overline{G}^{-1}(c_0)$ for some large positive constant c_0. By construction, the gradient $\nabla_{\bar g}\overline{G}$ is transverse (in fact, orthogonal) to $\partial D_{\bar y}$. By the asymptotics for the gradient (2.5) it is clear that $\overline{G}$ does not have any critical points in the closure of $D_{\bar y}$.

Suppose that the number λ_2 of removed disks is at least one and consider the manifold with boundary

$$\Sigma':=\Sigma\backslash\left(D_{\bar y}\cup\bigcup_{j=1}^{\lambda_2}D_j\right).$$

By item (i) in Lemma 2.2 the vector field $\nabla_{\bar g}\overline{G}$ is smooth in Σ' and transverse to its boundary. Let us denote by $z_1,\dots,z_N$ the critical points of $\overline{G}$ in $\overline{M}$, which are finite in number by item (iii) in Lemma 2.2. In addition to this, some of the isolated points p_i can be critical points of $\overline{G}$ too; without loss of generality we can order them so that these critical points are $p_1,\dots,p_{\lambda_1'}$ with $0\leqslant\lambda_1'\leqslant\lambda_1$.

If we now apply Hopf's index theorem [**31**] to the vector field $\nabla_{\bar g}\overline{G}$ in the manifold with boundary Σ' and realize that in dimension 2 it does not matter whether the vector field points inward or outward at the boundary, we get that the sum of the indices of the zeros of $\nabla_{\bar{g}}\overline{G}$ equals the Euler characteristic of Σ':

$$\sum_{k=1}^{N} \operatorname{ind}(z_k) + \sum_{i=1}^{\lambda_1'} \operatorname{ind}(p_i) = \chi(\Sigma') \,.$$

Since $\chi(\Sigma')$ can be readily shown to be $1 - 2\nu - \lambda_2$ and the index of each critical point is smaller than or equal to -1 by Lemma 2.1, we find (2.6)

$$-\sum_{k=1}^{N} \operatorname{ind}(z_k) = 2\nu + \lambda_2 - 1 + \sum_{i=1}^{\lambda_1'} \operatorname{ind}(p_i) \leqslant 2\nu + \lambda_2 - \lambda_1' - 1 \leqslant 2\nu + \lambda - 1 \,.$$

This implies that $N \leqslant 2\nu + \lambda - 1$, and the equality is not satisfied but perhaps when $\operatorname{ind}(z_k) = -1$ for all k, that is, when $\overline{G}$ is Morse. (The fact that the critical point z_k is non-degenerate when $\operatorname{ind}(z_k) = -1$ is an immediate consequence of Lemma 2.1.)

Consider now the case where $\lambda_2 = 0$. We have seen in item (ii) of Lemma 2.2 that there are some points p_j where $\overline{G}$ tends to $-\infty$, and that $\overline{G}$ is harmonic at the other points p_k. Among the points where the function is harmonic, some can be critical points of $\overline{G}$. Without loss of generality we can label these critical points as $p_1, \dots, p_{\lambda_1'}$. Let λ_1'' be the number of points where $\overline{G}$ tends to $-\infty$; for concreteness, we can assume that these points are $p_{\lambda_1'+1}, \dots, p_{\lambda_1'+\lambda_1''}$. Notice that necessarily $\lambda_1'' \geqslant 1$.

As we did with the pole $\bar{y}$, let us take small disks D_{p_j} containing the points p_j, with $\lambda_1' + 1 \leqslant j \leqslant \lambda_1' + \lambda_1''$. As before, the boundary circles can be chosen as the λ_1'' components of the level curve $\overline{G}^{-1}(-c_0)$ for large enough c_0. The asymptotics for the Green's function given by (2.5) ensures that $\overline{G}$ does not have any critical points in the closure of these disks and $\nabla_{\bar{g}}\overline{G}$ is transverse (and orthogonal by construction) to the boundary ∂D_{p_j}.

We can now apply Hopf's index theorem to the vector field $\nabla_{\bar{g}}\overline{G}$ in the manifold with boundary

$$\Sigma'' := \Sigma \backslash \left(D_{\bar{y}} \cup \bigcup_{j=\lambda_1'+1}^{\lambda_1'+\lambda_1''} D_{p_j} \right) .$$

The Euler characteristic of Σ'' is $1 - 2\nu - \lambda_1''$, so denoting the critical points of $\overline{G}$ in $\overline{M}$ again by $z_1, \dots, z_N$ and arguing as above, one arrives at the inequality

$$-\sum_{k=1}^{N} \operatorname{ind}(z_k) = 2\nu + \lambda_1'' - 1 + \sum_{i=1}^{\lambda_1'} \operatorname{ind}(p_i) \leqslant 2\nu + \lambda_1'' - \lambda_1' - 1 \leqslant 2\nu + \lambda - 1 \,.$$

This yields $N \leqslant 2\nu + \lambda - 1$, the inequality being saturated at most when $\operatorname{ind}(z_k) = -1$ for all k (that is, when $\overline{G}$ is Morse). q.e.d.

Remark 2.3. Although the bound $N \leqslant 2\nu + \lambda - 1$ is purely topological, it should be emphasized that the proof of the theorem shows that knowledge of the conformal structure of the manifold allows us to improve the bound. In particular, when $\lambda_2 \geqslant 1$ it stems from (2.6) that $N \leqslant 2\nu + \lambda_2 - 1$. This can be paraphrased as saying that only hyperbolic ends (i.e., the removed disks) contribute to the upper bound. Again, if this bound is saturated the Green's function must be Morse. The converse implication does not hold.

It is clear that a Li–Tam Green's function of a surface diffeomorphic to $\mathbb{R}^2$ does not have any critical points. To conclude this section, we will illustrate how the approach that we have followed in the proof of Theorem 1.1 can be used to show the absence of critical points in certain classes of higher-dimensional manifolds and certain choices of the pole y. The kinds of spaces we will consider are those of axisymmetric manifolds, and we will make the additional assumption that there is a Green's function G (which is necessarily minimal) tending to zero at infinity. It is known that this assumption is satisfied, e.g., when the ends of the manifold are large and it has nonnegative Ricci curvature [**28**] or asymptotically nonnegative sectional curvature [**26**].

Before starting our treatment of axisymmetric manifolds, let us prove an easy lemma that will be also of use in forthcoming sections:

Lemma 2.4. *Let G be the minimal Green's function with pole y of a Riemannian manifold $(\mathbb{R}^n, g)$, with $n \geqslant 2$. Suppose that G tends to zero at infinity. Then each level set $G^{-1}(c)$ is compact and connected, and the relatively compact domain bounded by $G^{-1}(c)$ contains the pole y.*

Proof. The level sets of G are necessarily compact because G is positive and tends to zero at infinity. Suppose that $G^{-1}(c)$ had more than one connected component. As $\mathbb{R}^n$ is contractible, there would be at least two disjoint relatively compact sets, S_1 and S_2, such that G would be constant on their boundaries. As the pole y does not belong to at least one of these sets, say S_1, it follows from the maximum principle that G is constant in S_1, contradicting the unique continuation theorem [**23**]. q.e.d.

Remark 2.5. The lemma and its proof remain valid if we replace G by the Dirichlet Green's function G_Ω of a domain $\Omega \subset (\mathbb{R}^n, g)$ with connected boundary.

We recall that a Riemannian n-dimensional manifold is *axisymmetric* if it is diffeomorphic to $\mathbb{R}^n$ and there is a subgroup of isometries H isomorphic to $\mathrm{SO}(n-1)$. To simplify the notation, we will introduce global coordinates $x = (x_1, \dots, x_n)$ taking values in $\mathbb{R}^n$ and assume without loss of generality that H acts via rotations of the last $n-1$ coordinates $x' := (x_2, \dots, x_n)$. The metric is then necessarily of the form

$$g = \alpha\, \mathrm{d}x_1^2 + \beta \left(\mathrm{d}x_2^2 + \cdots + \mathrm{d}x_n^2\right) + \gamma \left(x_2\, \mathrm{d}x_2 + \cdots + x_n\, \mathrm{d}x_n\right)^2 , \tag{2.7}$$

where α, β, γ are smooth functions of x_1 and $|x'|^2$, α and β are positive, and

$$\beta(x_1, |x'|^2) + |x'|^2 \gamma(x_1, |x'|^2) > 0 .$$

The fixed set of H is then the x_1-axis, and the action of H is proper and free on its complement.

Theorem 2.6. *Let $(\mathbb{R}^n, g)$ be an axisymmetric manifold whose Green's function G tends to zero at infinity and suppose that the position of the pole y is fixed by the action of the rotation group H. Then G does not have any critical points.*

Proof. Without loss of generality, we can then assume that y is located at the origin $x = 0$. The pole being invariant under H, property (iii) of Li–Tam Green's functions ensures that the Green's function is invariant under H, which means that G only depends on x_1 and $|x'|^2$. By the form of the metric, $\nabla_g G$ is obviously tangent to the plane

$$\Pi := \left\{x_3 = \cdots = x_n = 0\right\} .$$

The vector field Y is then defined as the pullback of $\nabla_g G$ to this invariant plane, and can be regarded (through the coordinates (x_1, x_2)) as a vector field in $\mathbb{R}^2$. In fact, if we denote by

$$\bar{g} := \alpha(x_1, x_2^2)\, \mathrm{d}x_1^2 + \left(\beta(x_1, x_2^2) + x_2^2\, \gamma(x_1, x_2^2)\right) \mathrm{d}x_2^2 \tag{2.8}$$

the pullback of the metric to the plane Π, it is obvious that $Y = \nabla_{\bar{g}} \overline{G}$, with $\overline{G}$ the restriction of G to the plane Π. Note that the functions α, β, γ that appear in (2.8) are the same as in Eq. (2.7) (but they are of course evaluated at different arguments).

By the axisymmetry of G, it is obviously enough to ensure that G does not have any critical points in the plane Π, which is tantamount to saying that $\overline{G}$ does not have any critical points. A short computation using the metric (2.7) shows that, when we evaluate the equation $\Delta_g G = 0$ at any point of the plane Π other than the pole, we get

$$0 = (\Delta_g G)\big|_{\Pi \setminus \{0\}} = \bar{g}^{ij}\, \partial_i \partial_j \overline{G} + b^i\, \partial_i \overline{G} ,$$

where the indices i, j range over $\{1, 2\}$ and $b^i(x_1, x_2)\partial_i$ is a smooth vector field in Π. Therefore, Lemma 2.1 guarantees that the critical points of $\overline{G}$ are isolated and their indices are not greater than -1.

By Sard's theorem [**1**], there is a sequence of regular values $(c_k)_{k=1}^{\infty} \searrow 0$ of $\overline{G}$, so that the gradient of $\overline{G}$ does not vanish on $\Gamma_k := \overline{G}^{-1}(c_k)$.

Each of these level sets is diffeomorphic to a circle, with the pole lying in its interior Ω_k (that is, in the relatively compact domain bounded by Γ_k). To prove this, it is enough to notice that Γ_k, which is obviously smooth, is compact and connected because $G^{-1}(c_k)$ is (this is immediate by Lemma 2.4). Moreover, this lemma ensures that the pole is contained in the domain Ω_k. It should be noticed that, as $\overline{G}$ is positive and tends to zero at infinity, the domains Ω_k exhaust the plane Π in the sense that $\Omega_k \subset \Omega_{k+1}$ and $\Pi = \bigcup_{k=1}^{\infty} \Omega_k$.

Take a small disk D around the point 0 in the plane Π. From the asymptotics of the Green's function G near the pole [**14**]

$$G(x) = \frac{1}{|\mathbb{S}^{n-1}| \operatorname{dist}_g(x,y)^{n-2}} + O\big(\operatorname{dist}_g(x,y)^{3-n}\big)\,, \tag{2.9a}$$

$$|\nabla_g G(x)| = \frac{n-2}{|\mathbb{S}^{n-1}| \operatorname{dist}_g(x,y)^{n-1}} + O\big(\operatorname{dist}_g(x,y)^{2-n}\big)\,, \tag{2.9b}$$

we infer that $\overline{G}$ does not have any critical points in D. It is clear that one can take $D = \overline{G}^{-1}(c)$ for some large constant c. By construction, $\nabla_{\bar g}\overline{G}$ is then orthogonal to the boundaries ∂D and Γ_k.

Let us now apply the index theorem for manifolds with boundary to the vector field $\nabla_{\bar g}\overline{G}$ in the region $\overline{\Omega_k}\backslash D$. The critical points $z_1, \dots, z_{N_k}$ of $\overline{G}$ in Ω_k are necessarily finite in number because the set $\overline{\Omega_k}\backslash D$ is compact and we have already seen that critical points are isolated. Hence, Hopf's index theorem yields

$$\sum_{j=1}^{N_k} \operatorname{ind}(z_j) = \chi\big(\overline{\Omega_k}\backslash D\big) = 0\,.$$

Since $\operatorname{ind}(z_j) \leqslant -1$, this implies that $N_k = 0$ for all k. As the domains Ω_k exhaust the whole plane, the result follows. q.e.d.

3. Dirichlet Green's functions with non-degenerate critical points

In this section we aim to prove that the Dirichlet Green's function of a "generic" bounded domain Ω of a smooth Riemannian manifold (M, g) of dimensions $n \geqslant 2$ is Morse, which will be instrumental in the proof of Theorem 1.2. The usefulness of Morse functions for our purposes lies in the fact that it ensures that the function is locally smoothly conjugate to its second order Taylor expansion at the critical point [**18**], so that the structure of the neighboring level sets can be easily controlled. Notice we are not assuming that the boundary $\partial\Omega$ is connected.

Before proving the main result of this section, we find it convenient to introduce the following definition.

Definition 3.1. Given any positive integer k, two C^∞ bounded domains $\Omega, \Omega' \subset M$ are said to be *ϵ-close in C^k* if there exists a C^∞ diffeomorphism Φ of M mapping $(\Omega, \partial\Omega)$ onto $(\Omega', \partial\Omega')$ and such that $\Phi - \mathrm{id}$ is compactly supported and satisfies $\|\Phi - \mathrm{id}\|_{C^k} < \epsilon$.

As is well known, we can define the above C^k norm, e.g. using the covariant derivative. The precise way in which one defines the C^k norm is inessential for our purposes because $\Phi - \mathrm{id}$ has compact support. We are now ready to prove the desired generic non-degeneracy result:

Theorem 3.2. *Let Ω be a smooth bounded domain of (M, g) and fix a point $y \in \Omega$. Then for any $\epsilon > 0$ and any positive integer k there is a smooth domain Ω' ϵ-close to Ω in C^k whose Green's function $G_{\Omega'}$ with pole y is Morse. Furthermore, there is some smaller $\delta > 0$ such that the Green's function with pole y of any C^k domain δ-close to Ω' in C^k is also Morse.*

Remark 3.3. Equivalently, the theorem can be restated as follows: Given any bounded domain $\Omega \subset M$, $G_{\Phi(\Omega)}$ is Morse for a C^k-generic (i.e., belonging to a set open and dense in the C^k norm) embedding $\Phi : \overline{\Omega} \to M$.

Proof. Let us divide the proof in two parts.

Density. We shall show that there exists another domain $\Omega' \subset M$ of class C^∞ which is ϵ-close in C^k to Ω and such that its associated Green's function $G_{\Omega'}$ with pole y is Morse in $\overline{\Omega'}\backslash\{y\}$.

We start by noticing that G_Ω is smooth in $\overline{\Omega}\backslash\{y\}$, and that the gradient of G_Ω is nonzero on $\partial\Omega$ by the Hopf boundary point lemma [**15**]. Take the level set $G_\Omega^{-1}(\eta)$ for a small positive constant η, so that the gradient of G_Ω does not vanish on it by continuity. The same argument we used in the proof of Lemma 2.4 shows here that there is a unique subset Ω_0 of Ω whose boundary consists of components of $G_\Omega^{-1}(\eta)$, and that Ω_0 is automatically connected, contains y, and satisfies $\partial\Omega_0 = G_\Omega^{-1}(\eta)$. Choosing η small enough, Ω_0 is $\frac{\epsilon}{2}$-close in C^k to Ω.

The critical set of G_Ω being compact, we can cover it by finitely many open patches $V_a \subset \Omega\backslash\{y\}$ $(1 \leqslant a \leqslant N)$ of harmonic coordinates $(x_1^a, \dots, x_n^a)$ [**13**], which satisfy the equation $\Delta_g x_j^a = 0$ in the closure $\overline{V_a}$ (that is, in an open neighborhood of $\overline{V_a}$). As $\overline{\Omega}\backslash V_a$ is connected, a Runge-type theorem proved by Bagby and Blanchet [**3**] ensures that these local harmonic coordinates x_j^a can be approximated in the C^k norm by harmonic functions X_j^a, which satisfy $\Delta_g X_j^a = 0$ in $\overline{\Omega}$, so that the norm $\|x_j^a - X_j^a\|_{C^k(\overline{V_a})}$ is arbitrarily small. (Actually, only C^0 approximation is stated in this reference, but the fact that it can be promoted to C^k approximation is obvious by standard Schauder estimates [**15**].)

We shall prove in what follows that there exist constants λ_j^a arbitrarily close to zero such that the function

$$f_N := G_\Omega - \sum_{a=1}^{N} \sum_{j=1}^{n} \lambda_j^a X_j^a$$

is Morse in the closure of $\mathcal{V}_N := \bigcup_{a=1}^{N} V_a \subset \Omega$. As f_N obviously approximates G_Ω in the C^k norm in any compact subset of $\Omega \backslash \{y\}$ and $\nabla_g G_\Omega \neq 0$ both in $\partial\Omega$ and in a neighborhood of y, it will immediately follow from the former claim and the boundedness of Ω that f_N is Morse in $\overline{\Omega_0} \backslash \{y\}$ for sufficiently small values of λ_a^j.

We shall prove the above claim by induction. Let us begin by showing that there exist constants $\lambda_1^1, \dots, \lambda_n^1$ in an arbitrarily small neighborhood of 0 such that

$$f_1 := G_\Omega - \sum_{j=1}^{n} \lambda_j^1 X_j^1$$

is Morse in $\overline{V_1}$. Sard's theorem [**1**] ensures that there exists an open and dense subset $\Lambda_1 \subset \mathbb{R}^n$ such that the Hessian of the expression of the Green's function G_Ω in the coordinates $(x_1^1, \dots, x_n^1)$ is nonsingular on the points of $\overline{V_1}$ where

$$\frac{\partial G_\Omega}{\partial x_j^1} = \lambda_j^1$$

for $1 \leqslant j \leqslant n$ and all $(\lambda_1^1, \dots, \lambda_n^1) \in \Lambda_1$. This simply means that the function

$$G_\Omega - \sum_{j=1}^{n} \lambda_j^1 \, x_j^1$$

is Morse in $\overline{V_1}$ for all $(\lambda_1^1, \dots, \lambda_n^1) \in \Lambda_1$. By the stability of Morse functions and the fact that the function X_j^1 approximates the harmonic coordinate x_j^1 in the C^k norm, the function f_1 is Morse too in $\overline{V_1}$.

Let us next assume as the induction hypothesis that the function

$$f_{m-1} := G_\Omega - \sum_{a=1}^{m-1} \sum_{j=1}^{n} \lambda_j^a X_j^a$$

is Morse in the closure of $\mathcal{V}_{m-1} := \bigcup_{a=1}^{m-1} V_a$ for an open and dense set of values of (λ_j^a). If we now invoke the same Sard-type argument we used for f_1 replacing G_Ω by f_{m-1}, V_1 by V_m, and λ_j^1 by λ_j^m, we immediately derive that the function

$$f_m := f_{m-1} - \sum_{j=1}^{n} \lambda_j^m X_j^m$$

is Morse in $\overline{V_m}$ for an open and dense set $\Lambda_m \subset \mathbb{R}^n$ of values of $(\lambda_1^m, \dots, \lambda_n^m)$. Moreover, the C^2-openness of Morse functions in $\overline{\mathcal{V}_{m-1}}$ guarantees that f_m is also Morse in $\overline{\mathcal{V}_{m-1}}$ provided that the new parameters λ_j^m are small enough. By induction in m, this proves that there exist arbitrarily small $\lambda_j^a \in \mathbb{R}$ such that f_N is Morse in $\overline{\mathcal{V}_N}$.

By construction, f_N is harmonic in $\Omega \backslash \{y\}$. Besides, Thom's isotopy theorem [**1**, Section 20.2] ensures that the level set $f_N^{-1}(\eta) \subset \Omega$ and the boundary $\partial\Omega_0$ are $\frac{\epsilon}{2}$-close in C^k, provided that the parameters λ_j^a are chosen close enough to zero. Hence the first part of the theorem now follows by defining Ω' to be the bounded domain enclosed by $f_N^{-1}(\eta)$, so that $G_{\Omega'} = f_N - \eta$.

Openness. Suppose that G_Ω is Morse. It is clear that for any $\epsilon > 0$ there exists $\delta_1 > 0$ such that

$$\max_{\partial(\Omega\cap\Omega')} G_\Omega < \epsilon \,, \qquad \max_{\partial(\Omega\cap\Omega')} G_{\Omega'} < \epsilon$$

for any C^k domain Ω' which is δ_1-close in C^k to Ω. In particular,

$$\big| G_\Omega - G_{\Omega'} \big| < 2\epsilon \tag{3.1}$$

in $\partial(\Omega \cap \Omega')$. The function $G_\Omega - G_{\Omega'}$ being harmonic in $\Omega \cap \Omega'$, the estimate (3.1) must hold in $\Omega \cap \Omega'$ as well by virtue of the maximum principle, and hence it follows that, as Ω' becomes closer and closer to Ω, the Green's function $G_{\Omega'}$ approximates G_Ω in the C^0 (and, by Schauder estimates [**15**], C^k) norm in compact subsets of $(\Omega \cap \Omega') \backslash \{y\}$. Therefore a simple stability argument ensures that $G_{\Omega'}$ is Morse in any compact subset of $(\Omega \cap \Omega') \backslash \{y\}$. As the gradient of $G_{\Omega'}$ does not vanish either in a neighborhood of y by the asymptotics (2.5) and (2.9) or at $\partial\Omega$ by Hopf's boundary point lemma, the theorem follows. q.e.d.

4. Green's functions with prescribed behavior

In this section we prove Theorem 1.2, which follows from Corollaries 4.4-4.5 and Theorem 4.7 below. Roughly speaking, the content of Theorem 1.2 is that there are well-behaved metrics in $\mathbb{R}^n$ ($n \geqslant 3$) having Green's functions with good fall-off at infinity (in particular, tending to zero) that have as many critical points as one wishes, and level sets of arbitrarily complicated topology. This is in strong contrast with the 2-dimensional situation (cf. Theorem 1.1), where a Li–Tam Green's function does not have any critical points.

In fact, the proof of Theorem 1.2 provides a method for constructing metrics whose Green's functions have prescribed critical and level sets, and can be painlessly extended to other topologies. The reason why we concentrate on metrics in $\mathbb{R}^n$ is that a Green's function that tends to zero at infinity automatically has critical points unless the manifold is homeomorphic to $\mathbb{R}^n$, as we show in the following

Proposition 4.1. *Let G be a Green's function on a smooth manifold (M, g), and suppose that G tends to zero at infinity. Then if G has no critical points, M is homeomorphic to $\mathbb{R}^n$.*

Proof. The well known asymptotics for the Green's function near the pole (2.9) shows that the level sets $G^{-1}(c)$ are homeomorphic to the sphere $\mathbb{S}^{n-1}$ for sufficiently large c. Consider the local flow ϕ_t of the vector field $\frac{\nabla_g G}{|\nabla_g G|_g^2}$, which is smooth in $M\backslash\{y\}$ because G does not have any critical points. Since the level sets $G^{-1}(c)$ are compact (because G is positive and tends to zero at infinity) and obviously

$$\frac{\mathrm{d}}{\mathrm{d}t} G(\phi_t x) = 1$$

for all x, it is immediate that the local flow ϕ_t is a homeomorphism mapping

$$\phi_t\big(G^{-1}(c)\big) = G^{-1}(c+t)$$

whenever c and $c + t$ are positive. Hence all the level sets of G must be homeomorphic to $\mathbb{S}^{n-1}$. Let us now take a decreasing sequence $c_j \searrow 0$ and consider the domain B_j containing y whose boundary is $G^{-1}(c_j)$, which we have shown to be homeomorphic to a ball. Then M is the monotone union of the balls B_j, so a theorem of Brown now ensures [**6**] that M is then homeomorphic to $\mathbb{R}^n$. q.e.d.

The proof of our results, which is of interest in itself, is based on the construction of a metric in $\mathbb{R}^n$ whose minimal Green's function tends to zero and approximates the Euclidean Green's function of a prescribed domain Ω with Dirichlet boundary conditions (Theorem 4.2). We will see that this metric, which can be taken analytic, is obtained by multiplying the Euclidean metric by a conformal factor that is approximately 1 in the domain Ω and very large in its complement. Suitable decay conditions on the curvature ensure the existence of a Green's function with the desired decay at infinity, which permits us to complete the proof of the theorem using variational methods:

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth connected boundary, and let G_Ω be its Euclidean Green's function with pole y. Given a compact subset S of $\Omega\backslash\{y\}$, a positive integer k, and $\delta > 0$, there is a complete analytic metric in $\mathbb{R}^n$ whose minimal Green's function G with pole y approximates G_Ω in the $C^k(S)$ norm as $\|G - G_\Omega\|_{C^k(S)} < \delta$. Furthermore, G tends to zero at infinity.*

Proof. One can take global coordinates $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and assume without loss of generality that y is located at the origin $x = 0$ and the order of approximation k is at least 3. For any positive integer j, let $\tilde{\varphi}_j : \mathbb{R}^n \to [1,\infty)$ be a smooth function such that $\tilde{\varphi}_j(x) = 1$ if $x \in \Omega$ and $\tilde{\varphi}_j(x) = j$ if $\operatorname{dist}(x,\Omega) > \frac{1}{j}$. By Whitney's approximation theorem [**32**, Section 1.6.5], we can approximate each $\tilde{\varphi}_j$ by an analytic function $\varphi_j : \mathbb{R}^n \to \mathbb{R}$ in such a way that

$$\sum_{|\alpha|\leqslant k} \big|D^\alpha \varphi_j(x) - D^\alpha \tilde{\varphi}_j(x)\big| < \frac{\mathrm{e}^{-|x|}}{j}\,. \tag{4.1}$$

There is no loss of generality in assuming that $\varphi_j(0) = 1$ for all $j \in \mathbb{N}$.

Let us now define the conformally flat analytic metrics $g_j := \varphi_j g_0$ on $\mathbb{R}^n$, where g_0 denotes the Euclidean metric in the coordinates $(x_1,\dots,x_n)$. For notational simplicity we shall denote by a subscript j the geometric quantities corresponding to the metric g_j, e.g. $\mathrm{d}V_j$, ∇_j, and Δ_j. It is clear that the end of $(\mathbb{R}^n, g_j)$ is large because this manifold has Euclidean volume growth. Moreover, the approximation (4.1) ensures that the Riemann tensor is bounded by

$$\big|\operatorname{Rm}_j(x)\big| < C\,\mathrm{e}^{-|x|}\,,$$

so that $(\mathbb{R}^n, g_j)$ has asymptotically nonnegative curvature. This implies that $(\mathbb{R}^n, g_j)$ has a unique minimal positive Green's function G_j, which satisfies $\Delta_j G_j = -\delta_0$ and falls off at infinity as [**21**, **26**]

$$G_j(x) < \int_{\operatorname{dist}_j(x,0)}^\infty \frac{t\,\mathrm{d}t}{\operatorname{Vol}_j(B_j(0,t))} < \frac{C}{j^{n-1}|x|^{n-2}}\,. \tag{4.2}$$

In the last inequality we have only used the properties of the function φ_j and elementary identities.

It is therefore standard [**34**] that one can express G_j in terms of the unique solution G_j^R to the boundary problem

$$\Delta_j G_j^R = 0 \qquad \text{in } A^R := \big\{x \in \mathbb{R}^n : 1/R < |x| < R\big\}\,, \tag{4.3a}$$

$$G_j^R\big|_{|x|=1/R} = \frac{R^{n-2}}{|\mathbb{S}^{n-1}|}\,, \qquad G_j^R\big|_{|x|=R} = 0, \tag{4.3b}$$

on the annulus A^R of center 0, inner radius R^{-1}, and outer radius R. Indeed, we will next show that

$$G_j = \big(1 + o(1)\big)\, G_j^R + o(1)\,, \tag{4.4}$$

so that we clearly have uniform convergence in any compact set $K \subset \mathbb{R}^n\backslash\{0\}$:

$$\big\|G_j - G_j^R\big\|_{C^0(K)} = o(1)\,. \tag{4.5}$$

Here and in what follows, $o(1)$ denotes a quantity that tends to 0 as $R \to \infty$ uniformly in j (in the previous equation, this quantity obviously depends on the compact set K too). To prove Eq. (4.4), it suffices to note that the asymptotics for the Green's function near the pole (see Eq. (2.9)) ensures that

$$G_j = \big(1+o(1)\big)\, G_j^R \quad \text{on } |x| = 1/R\,,$$

while $G_j|_{|x|=R} = o(1)$ by (4.2). Therefore one can choose a small positive constant η, of order $o(1)$, such that

$$\pm\big(G_j - (1\pm\eta)G_j^R\big) < 0$$

on $|x| = 1/R$ and

$$\big|G_j - (1\pm\eta)G_j^R\big| < \eta$$

on $|x| = R$. Since $G_j - (1\pm\eta)G_j^R$ is harmonic in A^R, the maximum principle ensures that

$$(1-\eta)G_j^R - \eta < G_j < (1+\eta)G_j^R + \eta\,,$$

which is equivalent to (4.4).

Let us now fix a large value of R such that $B(0,1/R) \subset S \subset \Omega \subset B(0,R)$, where S is the compact set that appears in the statement of the theorem, and study the behavior of G_j^R for large j. To this end, we shall use [**15**] that G_j^R is the unique minimizer of the functional

$$E_j^R[F] := \int_{A^R} \big|\nabla_j F\big|_j^2 \,\mathrm{d}V_j = \int_{A^R} \varphi_j(x)^{\frac{n}{2}-1} \big|\nabla F(x)\big|^2 \,\mathrm{d}x\,,$$

defined on the space of Lipschitz functions

$$\mathcal{C}^R := \Big\{F\in C^{0,1}(A^R) : F\big|_{|x|=\frac{1}{R}} = \tfrac{R^{n-2}}{|\mathbb{S}^{n-1}|},\ F\big|_{|x|=R} = 0\Big\}\,.$$

It is apparent that $\inf E_j^R$ is bounded uniformly in j. Indeed, if F_0 belongs to the set

$$\mathcal{C}(\Omega;R) := \Big\{F\in C^{0,1}(A^R\cap\Omega) : F\big|_{|x|=\frac{1}{R}} = \tfrac{R^{n-2}}{|\mathbb{S}^{n-1}|},\ F\big|_{\partial\Omega} = 0\Big\}\,,$$

it immediately follows from the inclusion $\mathcal{C}(\Omega;R)\subset\mathcal{C}^R$ that

$$\inf E_j^R \leqslant E_j^R[F_0] \leqslant \big(1+\tfrac{1}{j}\big)^{\frac{n}{2}-1} \int_{A^R\cap\Omega} |\nabla F_0|^2\,\mathrm{d}x\,.$$

In particular, by letting F_0 vary over $\mathcal{C}(\Omega;R)$, it stems from this that

$$\inf E_j^R \leqslant \big(1+\tfrac{1}{j}\big)^{\frac{n}{2}-1} \inf \mathcal{E}_\Omega^R = \inf\mathcal{E}_\Omega^R + o_R(1)\,, \tag{4.6}$$

where $\mathcal{E}_\Omega^R : \mathcal{C}(\Omega;R)\to\mathbb{R}$ denotes the energy functional

$$\mathcal{E}_\Omega^R[F] := \int_{A^R\cap\Omega} |\nabla F|^2\,\mathrm{d}x$$

and the symbol $o_R(1)$ henceforth stands for a quantity that tends to zero as $j\to\infty$ but is not necessarily uniform in R.

Let us now suppose that there exist some $\epsilon_1, \epsilon_2 > 0$ and a subsequence $(j_s)_{s=1}^\infty \nearrow \infty$ such that the Lebesgue measure of the set

$$U_s^R := \big\{x \in A^R \backslash \overline{\Omega} : \big|\nabla G_{j_s}^R(x)\big| \geqslant \epsilon_1\big\}$$

is at least ϵ_2 for all s. In this case

$$\begin{aligned}\limsup_{j\to\infty} \inf E_j^R &\geqslant \lim_{s\to\infty} \inf E_{j_s}^R = \lim_{s\to\infty} E_{j_s}^R[G_{j_s}^R] \\ &\geqslant \lim_{s\to\infty} (j_s - 1)^{\frac{n}{2}-1} \epsilon_1^2 \epsilon_2 = +\infty\,,\end{aligned}$$

contradicting the fact that $\inf E_j^R$ is bounded in j. It then follows that ∇G_j^R tends to zero in measure in $A^R\backslash\overline{\Omega}$ as $j \to \infty$. The uniform (in j) estimate (4.6) for

$$\inf E_j^R = E_j^R[G_j^R] \geqslant \frac{1}{2}\int_{A^R\backslash\Omega} |\nabla G_j^R|^2\,\mathrm{d}x$$

then ensures, by Vitali's theorem, that

$$\int_{A^R\backslash\Omega} |\nabla G_j^R|^2\,\mathrm{d}x = o_R(1)\,, \tag{4.7}$$

i.e., that the L^2 norm of the gradient of G_j^R becomes very small outside Ω for large j.

Now let K be any compact subset of $\overline{A^R}\backslash\overline{\Omega}$. Standard Schauder estimates for the differential equation $\Delta_j G_j^R = 0$ yield the C^2 bound [**15**]

$$\|G_j^R\|_{C^2(K)} \leqslant C_K \|G_j^R\|_{C^0(A^R)} \leqslant \frac{C_K R^{n-2}}{|\mathbb{S}^{n-1}|}\,, \tag{4.8}$$

which is uniform in j. The maximum principle has been used to derive the second inequality. The constant C_K can be chosen to depend on the compact set K and the dimension n but not on j [**15**] because the principal symbol of the elliptic operator $j\Delta_j$ at any point $x \in K$ is given by

$$j\,\varphi_j^{-1}(x)\,\mathrm{id} = \big(1 + O(j^{-1})\big)\,\mathrm{id}\,.$$

In view of the C^2 bound (4.7), the L^2 estimate (4.5) then implies that ∇G_j^R converges pointwise to zero as $j \to \infty$ uniformly in each compact set $K \subset \overline{A^R}\backslash\overline{\Omega}$. Since $G_j^R|_{|x|=R} = 0$ for all j, this immediately yields the C^1 estimate

$$\big\|G_j^R\big\|_{C^1(K)} = o_R(1) \tag{4.9}$$

for all K as above.

Let us consider thickenings of the domain Ω of the form

$$\Omega_\epsilon := \big\{x \in \mathbb{R}^n : \mathrm{dist}(x, \Omega) < \epsilon\big\}\,,$$

with ϵ a small positive number, and take a smooth function $\chi_\epsilon : \mathbb{R}^n \to \mathbb{R}$ with $\chi_\epsilon = 1$ in $\Omega_{\epsilon/2}$ and $\chi_\epsilon = 0$ in $A^R\backslash\Omega_\epsilon$. Let us now set $v_{\epsilon,j}^R := \chi_\epsilon G_j^R$, which is a function that clearly belongs to the space $\mathcal{C}(\Omega_\epsilon; R)$. With the obvious definition of the energy functional $\mathcal{E}^R_{\Omega_\epsilon}$, a short calculation shows

$$\mathcal{E}^R_{\Omega_\epsilon}[v^R_{\epsilon,j}] = \int_{\Omega_{\epsilon/2}\cap A^R} |\nabla G^R_j|^2 \, \mathrm{d}x + \int_{\Omega_\epsilon\backslash\Omega_{\epsilon/2}} \big|\chi_\epsilon \nabla G^R_j + G^R_j \nabla\chi_\epsilon\big|^2 \, \mathrm{d}x \,.$$

By Eq. (4.7), the first summand is at most $\inf E^R_j + o_{\epsilon,R}(1)$, while the second term is obviously bounded by $C_\epsilon \|G^R_j\|_{C^1(\Omega_\epsilon\backslash\Omega_{\epsilon/2})}$. Using the estimate (4.9), this yields

$$\inf \mathcal{E}^R_{\Omega_\epsilon} \leqslant \mathcal{E}^R_{\Omega_\epsilon}[v^R_{\epsilon,j}] \leqslant \inf E^R_j + o_{\epsilon,R}(1) \,. \tag{4.10}$$

Here we are using the notation $o_{\epsilon,R}(1)$ for a quantity that tends to zero as $j\to\infty$ but is not uniform in ϵ or R. Let us hereafter use the notation $\tilde{o}_R(1)$ to denote a quantity that goes to 0 as $\epsilon\to 0$ (*not* as $j\to\infty$) and is uniform in j but not necessarily in R. It is clear then that we have

$$\inf \mathcal{E}^R_{\Omega_\epsilon} = \inf \mathcal{E}^R_{\Omega} + \tilde{o}_R(1) \,,$$

so that Eqs. (4.6) and (4.10) yield

$$\mathcal{E}^R_{\Omega_\epsilon}[v^R_{\epsilon,j}] \leqslant \inf \mathcal{E}^R_{\Omega_\epsilon} + o_{\epsilon,R}(1) + \tilde{o}_R(1) \,. \tag{4.11}$$

Let us now compare the function $v^R_{\epsilon,j}$ with the unique solution $G^R_{\Omega_\epsilon}$ of the boundary value problem

$$\Delta G^R_{\Omega_\epsilon} = 0 \quad \text{in } A^R\cap\Omega_\epsilon \,, \qquad G^R_{\Omega_\epsilon}\big|_{|x|=R^{-1}} = \frac{R^{n-2}}{|\mathbb{S}^{n-1}|} \,, \qquad G^R_{\Omega_\epsilon}\big|_{\partial\Omega_\epsilon} = 0 \,. \tag{4.12}$$

Since the function $\frac{1}{2}(G^R_{\Omega_\epsilon} + v^R_{\epsilon,j})$ belongs to the space $\mathcal{C}(\Omega_\epsilon; R)$, it is now immediate that

$$\begin{aligned}
\int \frac{|\nabla G^R_{\Omega_\epsilon} - \nabla v^R_{\epsilon,j}|^2}{4} &= \int \frac{|\nabla G^R_{\Omega_\epsilon}|^2 + |\nabla v^R_{\epsilon,j}|^2}{2} - \int \left|\nabla\left(\frac{G^R_{\Omega_\epsilon} + v^R_{\epsilon,j}}{2}\right)\right|^2 \\
&= \frac{\mathcal{E}^R_{\Omega_\epsilon}[G^R_{\Omega_\epsilon}] + \mathcal{E}^R_{\Omega_\epsilon}[v^R_{\epsilon,j}]}{2} - \mathcal{E}^R_{\Omega_\epsilon}\left[\frac{G^R_{\Omega_\epsilon} + v^R_{\epsilon,j}}{2}\right] \\
&\leqslant \inf \mathcal{E}^R_{\Omega_\epsilon} - \mathcal{E}^R_{\Omega_\epsilon}\left[\frac{G^R_{\Omega_\epsilon} + v^R_{\epsilon,j}}{2}\right] + o_{\epsilon,R}(1) + \tilde{o}_R(1) \\
&\leqslant o_{\epsilon,R}(1) + \tilde{o}_R(1), \qquad (4.13)
\end{aligned}$$

where all integrals are taken over $\Omega_\epsilon\cap A^R$ with respect to the Lebesgue measure dx and we have used Eq. (4.11) to estimate $\mathcal{E}^R_{\Omega_\epsilon}[v^R_{\epsilon,j}]$.

Let

$$K' := \left\{x \in \overline{A^R}\cap\Omega : \operatorname{dist}(x,\partial\Omega) \leqslant \epsilon'\right\} ,$$

with ϵ' small enough so that the set S is contained in K'. In view of the definition of the conformal factor φ_j, standard Schauder estimates and the maximum principle for the equations $\Delta_j v^R_{\epsilon,j}=0$ and $\Delta G^R_{\Omega_\epsilon}=0$ show that in the compact set K' we have the uniform C^2 bounds

$$\|G^R_{\Omega_\epsilon}\|_{C^2(K')}\leqslant C_{K'}\|G^R_{\Omega_\epsilon}\|_{C^0(A^R\cap\Omega_\epsilon)}\leqslant\frac{C_{K'}R^{n-2}}{|\mathbb{S}^{n-1}|}\,,$$
$$\|v^R_{\epsilon,j}\|_{C^2(K')}\leqslant C_{K'}\|v^R_{\epsilon,j}\|_{C^0(A^R\cap\Omega_\epsilon)}\leqslant\frac{C_{K'}R^{n-2}}{|\mathbb{S}^{n-1}|}\,,$$

with $C_{K'}$ depending only on K' and the dimension n. With these bounds, the L^2 estimate (4.13) implies that

$$\nabla G^R_{\Omega_\epsilon}-\nabla v^R_{\epsilon,j}=o_{\epsilon,R}(1)+\tilde{o}_R(1)$$

pointwise in K'. Since $G^R_{\Omega_\epsilon}-v^R_{\epsilon,j}=0$ on $|x|=1/R$, this yields the C^1 bound

$$\big\|G^R_{\Omega_\epsilon}-v^R_{\epsilon,j}\big\|_{C^1(K')}=o_{\epsilon,R}(1)+\tilde{o}_R(1)\,. \tag{4.14}$$

Let G^R_Ω be the solution to the boundary value problem (4.12) with Ω_ϵ replaced by Ω. Since $G^R_{\Omega_\epsilon}$ converges to G^R_Ω for small ϵ, i.e.,

$$\lim_{\epsilon\to0^+}G^R_{\Omega_\epsilon}=G^R_\Omega \tag{4.15}$$

pointwise in K', and $v^R_{\epsilon,j}=G^R_j$ in K', it stems from this that

$$\big\|G^R_j-G^R_\Omega\|_{C^0(K')}=o_R(1)\,. \tag{4.16}$$

To see this, note that for any $\eta>0$ one can take a very small ϵ such that $|G^R_{\Omega_\epsilon}-G^R_\Omega|<\eta/2$ in K' by (4.15), and if ϵ is small enough Eq. (4.14) guarantees that there is a large j such that $|G^R_{\Omega_\epsilon}-v^R_{\epsilon,j}|<\eta/2$ in K'. Hence for this j, $|G^R_j-G^R_\Omega|<\eta$, which implies the claim.

Let K be any compact subset of $\Omega\backslash\{0\}$ whose interior contains S. Since obviously

$$\lim_{R\to\infty}\big\|G_\Omega-G^R_\Omega\big\|_{C^0(K)}=0\,,$$

Eqs. (4.5) and (4.16) readily imply that for any $\delta'>0$ one can choose large values of R and j so that

$$\big\|G_j-G_\Omega\big\|_{C^0(K)}<\delta'\,. \tag{4.17}$$

Notice that

$$\varphi_j\Delta_j(G_j-G_\Omega)=-\varphi_j\Delta_jG_\Omega=-\Delta G_\Omega+Y_j\cdot\nabla G_\Omega=Y_j\cdot\nabla G_\Omega\,, \tag{4.18}$$

in $\Omega\backslash\{0\}$, with the vector field $Y_j:=-(\frac{n}{2}-1)\varphi_j^{-1}\nabla\varphi_j$ bounded in K as

$$\|Y_j\|_{C^{k-1}(K)}<\frac{C_{k,K}}{j} \tag{4.19}$$

by the property (4.1) of the conformal factor φ_j. Schauder estimates for the equation (4.18) and the bounds (4.17) and (4.19) yield

$$\begin{aligned}\|G_j - G_\Omega\|_{C^k(S)} &\leqslant C_1\big(\|G_j - G_\Omega\|_{C^0(K)} + \|\nabla G_\Omega\|_{C^{k-1}(K)}\|Y_j\|_{C^{k-1}(K)}\big)\\ &\leqslant C_1\delta' + \frac{C_2}{j}\,,\end{aligned}$$

where again the constants C_1, C_2 depend on the number of derivatives k and the compact sets K and S, but not on δ' or j. Therefore one can choose δ' small enough and j large enough so that the difference between G_Ω and the Green's function of the analytic metric g_j in the compact S, $\|G_j - G_\Omega\|_{C^k(S)}$, is smaller than any given positive number δ, completing the proof of the theorem. q.e.d.

The fact that Theorem 4.2 yields a C^k approximation ($k \geqslant 2$) is crucial for relating the topological properties of the level sets of the Green's function of the curved manifold to those of the Euclidean Green's function of the domain. In the rest of this section we shall present several concrete applications of this idea as corollaries of the previous theorem.

The following easy corollary provides a somewhat more concrete way of relating the Green's function of a "generic" domain G_Ω to that of $(\mathbb{R}^n, g)$, where the metric g is constructed as in Theorem 4.2. By "generic" we mean that the Green's function G_Ω is assumed to be Morse, which is indeed true for an open and dense set (with respect to any C^k topology) of smooth domains by Theorem 3.2:

Corollary 4.3. *Let Ω be a domain with C^∞ connected boundary and consider a compact set $K \subset \Omega$ and a small neighborhood B of a point $y \in \Omega$. If the Green's function G_Ω with pole y is Morse, there is a C^∞ diffeomorphism Θ of $\mathbb{R}^n$, arbitrarily close to the identity in the C^k norm, and an analytic metric g on $\mathbb{R}^n$ whose minimal Green's function G tends to zero at infinity and satisfies*

$$G(x) = (G_\Omega \circ \Theta)(x) \tag{4.20}$$

for all $x \in K\backslash B$.

Proof. By Theorem 4.2, one can choose an analytic metric g so that G and G_Ω are arbitrarily close in the $C^{k+1}(K\backslash B)$ norm. Given any $\epsilon > 0$, the structural stability of Morse functions implies [**1**] that there exists a C^∞ diffeomorphism Θ of $K\backslash B$ onto its image with $\|\Theta - \mathrm{id}\|_{C^k(K\backslash B)} < \epsilon$ and satisfying (4.20) for all $x \in K\backslash B$. Since Θ is close to the identity, it is standard that it can be extended to a diffeomorphism of $\mathbb{R}^n$, which we still denote by Θ, so that $\|\Theta - \mathrm{id}\|_{C^k(\mathbb{R}^n)} < 2\epsilon$. q.e.d.

In the following corollary we show that the level sets of a minimal Green's function in $\mathbb{R}^n$ can have highly nontrivial topologies:

Corollary 4.4. *Let $\Sigma \subset \mathbb{R}^n$ be a compact, codimension 1 submanifold without boundary of class C^∞, and let y be a point in the domain bounded by Σ. Then, for any positive integer k and any $\epsilon > 0$, there exist a compactly supported diffeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ with $\|\Phi - \mathrm{id}\|_{C^k} < \epsilon$ and a complete analytic Riemannian manifold $(\mathbb{R}^n, g)$ whose minimal Green's function with pole y tends to zero at infinity and has $\Phi(\Sigma)$ as a level set.*

Proof. Let Ω be the bounded domain enclosed by Σ and let G_Ω be its Green's function with pole y. By the Hopf boundary point lemma [**15**], the gradient of G_Ω does not vanish on Σ. The level sets of G_Ω being connected by Lemma 2.4 and Remark 2.5, it then follows that the C^ω submanifold $\Sigma' := G_\Omega^{-1}(c)$ is ϵ-close in C^k to Σ for small enough $c > 0$.

By Theorem 4.2 one can choose an analytic metric g on $\mathbb{R}^n$ such that its minimal positive Green's function G is arbitrarily close to G_Ω in the C^k norm on compact subsets of $\Omega\backslash\{y\}$ and tends to zero at infinity. As the level sets of G are connected by Lemma 2.4, it follows by Thom's isotopy theorem [**1**] that $G^{-1}(c)$ is diffeotopic to Σ', so that $G^{-1}(c) = \Phi(\Sigma)$ for some ambient diffeomorphism close to the identity. q.e.d.

Next we shall apply Theorem 4.2 to construct analytic metrics in $\mathbb{R}^n$ with many critical points. Additionally, we can ensure that each of these critical points is non-degenerate and control its *Morse index*, which is the number of negative eigenvalues of its Hessian matrix at the critical point. In order to see that this corollary actually shows that the Green's function can have as many critical points as one wishes, it is enough to apply the statement, for instance, to a domain Ω in $\mathbb{R}^n$ whose boundary is the connected sum of N copies of $\mathbb{S}^1 \times \mathbb{S}^{n-2}$, whose first Betti number is $b_1(\overline{\Omega}) = N$.

Corollary 4.5. *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^∞ connected boundary and let $b_p(\overline{\Omega})$ be the Betti numbers of its closure. Then there exists an analytic metric in $\mathbb{R}^n$ whose minimal Green's function with pole y tends to zero at infinity and has at least $b_p(\overline{\Omega})$ non-degenerate critical points of Morse index $n - p$, for $1 \leqslant p \leqslant n - 2$.*

Proof. By translating and slightly deforming the domain if necessary, Theorem 3.2 ensures that one can take $y \in \Omega$ such that the Green's function G_Ω of Ω with pole y is Morse. By the Hopf boundary point lemma [**15**], the gradient of G_Ω does not vanish on the boundary of Ω. Let us consider the auxiliary function $f : \overline{\Omega} \to \mathbb{R}$, of class C^2, defined by

$$f(x) := \begin{cases} -1/(G_\Omega(x) + 1)^2 & \text{if } x \neq y\,, \\ 0 & \text{if } x = y\,. \end{cases}$$

Since the gradient $\nabla_g f$ obviously points inwards on the boundary $\partial\Omega$, by the Morse theory for manifolds with boundary [**31**] it follows that f has at least $b_p(\overline{\Omega})$ critical points of index $n-p$, for all $p = 0, \dots, n$. It is a straightforward computation that all the critical points of f other than y are also critical points of G_Ω and they have the same Morse indices. Therefore Corollary 4.3 guarantees that there exists an analytic metric whose minimal Green's function has the same number of critical points in Ω as G_Ω and of the same Morse type. q.e.d.

Remark 4.6. The reason why one only needs to take $1 \leqslant p \leqslant n-2$ in the statement is the following. For a manifold with boundary $\overline{\Omega}$, $b_{n-1}(\overline{\Omega}) = b_n(\overline{\Omega}) = 0$, so it is enough to consider $p \leqslant n-2$. The zeroth Betti number is $b_0(\overline{\Omega}) = 1$, but the associated critical point of the auxiliary function f simply reflects that y is the maximum of f, but y is clearly not related to a critical point of G_Ω but to its pole.

To conclude the proof of Theorem 1.2, we shall next show how to construct an analytic metric in $\mathbb{R}^n$ whose Green's function has a critical set of codimension at most 3. (Of course, the statement ensures that *some* of the components of the critical set have codimension at most 3, not that all of them do.)

Theorem 4.7. *There exists an analytic metric in $\mathbb{R}^n$ whose minimal Green's function with pole y tends to zero at infinity and has a critical set of codimension at most* 3.

Proof. First of all, let us take Cartesian coordinates $(x_1, \dots, x_n)$ in $\mathbb{R}^n$ so that the pole of G is given by $y = 0$. Let us consider the polynomial

$$Q(x_1, x_2) := \prod_{k=0}^{N-1} \left[(x_1 - 2k - 1)^2 + x_2^2 - 1\right]^2,$$

where N is a positive integer, and define the domain in $\mathbb{R}^3$

$$\Omega_0 := \left\{x \in \mathbb{R}^3 : Q(x_1, x_2) + x_3^2 < a\,,\ x_1 > 0\right\}.$$

This domain is obviously diffeomorphic to a solid torus of genus N if the constant $a > 0$ is sufficiently small.

By Corollary 4.5 it suffices to consider the case where $n \geqslant 4$. We will identify $\mathbb{R}^3$ with the 3-plane in $\mathbb{R}^n$ corresponding to the coordinates (x_1, x_2, x_3). Let H be the group of rotations of the $n-2$ coordinates $(x_1, x_4, \dots, x_n)$ (i.e., the group generated by the vector fields $x_1\partial_j - x_j\partial_1$ with $4 \leqslant j \leqslant n$). We define a domain $\Omega_1 \subset \mathbb{R}^n$ as the interior of the set $H \cdot \Omega_0$ (that is, we regard Ω_0 as a subset of $\mathbb{R}^n$, consider its orbit under the action of the group H and define Ω_1 as the interior of this orbit).

The boundary of the domain Ω_1 is not smooth, as it has corners on its intersection with the 2-plane $\{x_1 = x_4 = \cdots = x_n = 0\}$. Hence we start by defining a smooth domain, which we call Ω, by rounding off the corners of Ω_1. This new domain can be taken arbitrarily close in C^0 to Ω_1. The domain Ω_1 is invariant under the group H and under the reflections σ_k along all the hyperplanes $\{x_k = 0\}$ (when $k = 2, 3$ it follows from the definition of Ω_0, not from the fact that we are considering an orbit of the group H). Therefore, we can take the new domain Ω to be invariant under these transformations as well. For later convenience, we will call H_0 the group generated by the reflections σ_2 and σ_3.

Let G_Ω be the Euclidean Dirichlet Green's function of Ω with pole at 0. The Green's function G_Ω is invariant under the Euclidean isometries which preserve both the pole $y = 0$ and the domain Ω, so $G_\Omega \circ \sigma_k = G_\Omega$ for $k = 1, \dots, n$. In particular, the gradient ∇G_Ω is tangent to the x_1-axis.

Our goal now is to study the critical points of G_Ω on the x_1-axis. Clearly the intersection of Ω with the x_1-axis consists of $2N + 1$ connected components L_α, which are line segments we label with an integer $-N \leqslant \alpha \leqslant N$. These components are ordered in the natural way, so that the value of the coordinate x_1 is greater in L_α than in $L_{\alpha'}$ if and only if $\alpha > \alpha'$. The restriction of the function G_Ω, which we call Ψ_Ω, is smooth in

$$L := \left(\bigcup_{\alpha=-N}^{N} L_\alpha \right) \backslash \{0\}$$

and vanishes on the endpoints of each ∂L_α by the boundary conditions. Moreover, Ψ_Ω is everywhere positive because so is G_Ω, which implies that Ψ_Ω has a local maximum in L_α for all $\alpha \neq 0$ by Rolle's theorem. Since the gradient ∇G_Ω is tangent to the x_1-axis, it follows that the aforementioned maxima of Ψ_Ω correspond to critical points of G_Ω, which are necessarily of saddle type. Let z be one of the above critical points of G_Ω, located on L. As G_Ω is invariant under the group of rotations H, the critical set of G_Ω must contain the H-orbit passing through z, which has dimension $n - 3$. Thus the critical set of G_Ω has codimension at most 3.

We shall now construct a sequence of analytic metrics g_j on $\mathbb{R}^n$ such that the minimal Green's functions G_j in $(\mathbb{R}^n, g_j)$ with pole 0 approximate G_Ω in the C^k norm in compact sets of $\Omega \backslash \{0\}$. The construction is based on a straightforward modification of the proof of Theorem 4.2 that takes into account the symmetries of the domain Ω. Again we denote by $\tilde{\varphi}_j : \mathbb{R}^n \to [1, \infty)$ a smooth function such that $\tilde{\varphi}_j(x) = 1$ if $x \in \Omega$ and $\tilde{\varphi}_j(x) = j$ if $\operatorname{dist}(x, \Omega) > \frac{1}{j}$. The domain Ω being invariant under the action of the group $H \oplus H_0$, we can obviously take $\tilde{\varphi}_j$ invariant under $H \oplus H_0$.

By Whitney's approximation theorem [**32**, Section 1.6.5], for any positive integer j there exists an analytic function $\hat{\varphi}_j : \mathbb{R}^n \to \mathbb{R}$ such that

$$\sum_{|\alpha|\leqslant 3} |D^\alpha \hat{\varphi}_j(x) - D^\alpha \tilde{\varphi}_j(x)| < \frac{\mathrm{e}^{-|x|}}{j}\,. \tag{4.21}$$

We can obviously assume that $\hat{\varphi}_j(0) = 1$. Observe that both the set

$$\Lambda_j := \left\{x \in \mathbb{R}^n : \operatorname{dist}(x, \partial\Omega) > \tfrac{1}{j}\right\}$$

and the majorating function $\mathrm{e}^{-|x|}/j$ in (4.21) are invariant under the compact group $H \oplus H_0$. Consequently, we can define an analytic symmetrization φ_j of the function $\hat{\varphi}_j$ by

$$\varphi_j(x) := \frac{1}{4} \sum_{\sigma\in H_0} \int_H \hat{\varphi}_j(\sigma \circ h(x))\, \mathrm{d}h\,,$$

where $\mathrm{d}h$ denotes the normalized Haar measure of H. Since $D^\alpha \tilde{\varphi}_j = 0$ in the set Λ_j, it readily follows from Eq. (4.21) that for all $x \in \mathbb{R}^n$ with $\operatorname{dist}(x, \partial\Omega) > \frac{1}{j}$ we have the estimate:

$$\begin{aligned}\sum_{|\alpha|\leqslant 3} |D^\alpha(\varphi_j - \tilde{\varphi}_j)(x)| &\leqslant \frac{1}{4} \sum_{\sigma\in H_0} \int_H \sum_{|\alpha|\leqslant 3} \left|D^\alpha \hat{\varphi}_j(\sigma h x) - D^\alpha \tilde{\varphi}_j(x)\right| \mathrm{d}h \\ &\leqslant \frac{C\mathrm{e}^{-|x|}}{j}\,,\end{aligned}$$

where C does not depend on j.

If we now define the complete metric $g_j := \varphi_j g_0$, it follows that g_j has asymptotically nonnegative curvature and Euclidean volume growth, so $(\mathbb{R}^n, g_j)$ admits a minimal positive Green's function G_j that tends to zero at infinity [**21**, **26**]. Following verbatim the proof of Theorem 4.2, one infers that G_j approximates G_Ω in any C^l norm in compact subsets of $\Omega\backslash\{0\}$. By construction $H \oplus H_0$ is a group of isometries of $(\mathbb{R}^n, g_j)$, so, by property (iii) of Li–Tam Green's functions and the fact that the pole is at the origin, the gradient field $\nabla_j G_j$ is tangent to the x_1-axis. As in the proof of Theorem 4.2, we are denoting by a subscript j the objects corresponding to the metric g_j.

Let us use the notation Ψ_j for the restriction of the Green's function G_j to the set L. As Ψ_j tends to Ψ_Ω uniformly on compact subsets of each segment L_α and Ψ_Ω has at least $2N$ local maxima, it follows that Ψ_j also has at least $2N$ local maxima if j is large enough. Rolle's theorem and the fact that Ψ_j tends to $+\infty$ at 0 ensure that Ψ_j also has at least $2N$ local minima. By symmetry, these local extrema of Ψ_j correspond to critical points of G_j, and the invariance of G_j under the group of isometries H implies that the critical set of G_j also contains the H-orbit passing through each of these critical points. Since these orbits have dimension $n-3$, the result follows for large enough j. q.e.d.

Remark 4.8. The proof of Theorem 4.7 relies on the construction of a metric in $\mathbb{R}^n$ with an $\mathrm{SO}(n-2)$ isometry subgroup leaving a point y invariant and whose Green's function G with pole y has a nonempty critical set. However, we saw in Theorem 2.6 that the existence of an $\mathrm{SO}(n-1)$ isometry group automatically implies that the Green's function has no critical points, so this construction cannot be adapted to obtain critical sets of codimension 2. The question of whether there are Green's functions on $(\mathbb{R}^n, g)$ tending to zero at infinity with critical sets of codimension 2 remains open.

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