J. DIFFERENTIAL GEOMETRY
91 (2012) 215-232

SURFACES WITH PARALLEL MEAN CURVATURE VECTOR IN COMPLEX SPACE FORMS

Dorel Fetcu

Abstract

We consider surfaces with parallel mean curvature vector (pmc surfaces) in complex space forms and introduce a holomorphic differential on such surfaces. When the complex dimension of the ambient space is equal to two we find a second holomorphic differential and then determine those pmc surfaces on which both differentials vanish. We also provide a reduction of codimension theorem and prove a non-existence result for pmc 2-spheres in complex space forms.

1. Introduction

Sixty years ago, H. Hopf was the first to use a quadratic form in order to study surfaces immersed in a 3-dimensional Euclidean space. He proved, in 1951, that any such surface which is homeomorphic to a sphere and has constant mean curvature is actually isometric to a round sphere (see [**13**]). This result was extended by S.-S. Chern to surfaces immersed in 3-dimensional space forms (see [**8**]) and by U. Abresch and H. Rosenberg to surfaces in simply connected, homogeneous 3-dimensional Riemannian manifolds, whose group of isometries has dimension 4 (see [**1**, **2**]). Very recently, H. Alencar, M. do Carmo and R. Tribuzy have made the next step by obtaining Hopf-type results in spaces with dimension higher than 3, namely in product spaces $M^n(c) \times \mathbb{R}$, where $M^n(c)$ is a simply connected n-dimensional space form with constant sectional curvature $c \neq 0$ (see [**3**, **4**]). They have considered the case of surfaces with parallel mean curvature vector, as a natural generalization of those with constant mean curvature in a 3-dimensional ambient space. We also have to mention a recent paper of F. Torralbo and F. Urbano, which is devoted to the study of surfaces with parallel mean curvature vector in $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$.

Minimal surfaces and surfaces with parallel mean curvature vector in complex space forms have been also a well studied subject in the

The author was supported by a Postdoctoral Fellowship ”Pós-Doutorado Júnior (PDJ) – 150138/2010–5” offered by CNPq, Brazil.
Received 07/05/2010.

last two decades (see, for example, [**5**, **7**, **9**, **10**, **12**, **15**, **16**, **17**, **18**]). In all these papers the Kähler angle proved to play a decisive role in understanding the geometry of immersed surfaces in a complex space form, and, in several of them, important results were obtained when this angle was supposed to be constant (see [**5**, **16**, **18**]).

The main goal of our paper is to obtain some characterization results concerning surfaces with parallel mean curvature vector in complex space forms by using as a principal tool holomorphic quadratic forms defined on these surfaces. The paper is organized as follows. In Section 2 we introduce a quadratic form Q on surfaces of an arbitrary complex space form and prove that its $(2,0)$-part is holomorphic when the mean curvature vector of the surface is parallel. In Section 3 we work in the complex space forms with complex dimension equal to 2 and find another quadratic form Q' with holomorphic $(2,0)$-part. Then we determine surfaces with parallel mean curvature vector on which both $(2,0)$-part of Q and $(2,0)$-part of Q' vanish. As a by-product we reobtain a result in [**12**]. More precisely, we prove that a 2-sphere can be immersed as a surface with parallel mean curvature vector only in a flat complex space form and it is a round sphere in a hyperplane in $\mathbb{C}^2$. In Section 4 we deal with surfaces in $\mathbb{C}^n$ with parallel mean curvature vector, and we prove that the $(2,0)$-part of Q vanishes on such a surface if and only if it is pseudo-umbilical. The main result of Section 5 is a reduction theorem, which states that a surface in a complex space form, with parallel mean curvature vector, either is totally real and pseudo-umbilical or it is not pseudo-umbilical and lies in a complex space form with complex dimension less or equal to 5. The last Section is devoted to the study of the 2-spheres with parallel mean curvature vector and constant Kähler angle. We prove that there are no non-pseudo-umbilical such spheres in a complex space form with constant holomorphic sectional curvature $\rho \neq 0$.

Acknowledgements. The author wants to thank Professor Harold Rosenberg for suggesting this subject, useful comments and discussions and constant encouragement, and the IMPA in Rio de Janeiro for providing a very stimulative work environment during the preparation of this paper.

2. A quadratic form

Let Σ^2 be an immersed surface in $N^n(\rho)$, where N is a complex space form with complex dimension n, complex structure $(J, \langle , \rangle)$, and with constant holomorphic sectional curvature ρ; which is either $\mathbb{C}P^n(\rho)$, or $\mathbb{C}^n$, or $\mathbb{C}H^n(\rho)$, as $\rho > 0$, $\rho = 0$, and $\rho < 0$, respectively. Let us define a quadratic form Q on Σ^2 by

$$Q(X,Y) = 8|H|^2\langle \sigma(X,Y), H\rangle + 3\rho\langle JX, H\rangle\langle JY, H\rangle,$$

where σ is the second fundamental form of the surface and H is its mean curvature vector field. Assume that H is parallel in the normal bundle of Σ^2, i.e. $\nabla^\perp H = 0$, the normal connection $\nabla^\perp$ being defined by the equation of Weingarten

$$\nabla^N_X V = -A_V X + \nabla^\perp_X V,$$

for any vector field X tangent to Σ^2 and any vector field V normal to the surface, where ∇^N is the Levi-Civita connection on N and A is the shape operator.

We shall prove that the $(2,0)$-part of Q is holomorphic. In order to do that, let us first consider the isothermal coordinates (u,v) on Σ^2. Then $ds^2 = \lambda^2(du^2+dv^2)$ and define $z = u+iv$, $\bar z = u-iv$, $dz = \frac{1}{\sqrt{2}}(du+idv)$, $d\bar z = \frac{1}{\sqrt{2}}(du-idv)$ and

$$Z = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\Big), \quad \bar Z = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\Big).$$

We also have $\langle Z, \bar Z\rangle = \langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u}\rangle = \langle \frac{\partial}{\partial v}, \frac{\partial}{\partial v}\rangle = \lambda^2$.

In the following we shall calculate

$$\bar Z(Q(Z,Z)) = \bar Z(8|H|^2\langle \sigma(Z,Z), H\rangle + 3\rho\langle JZ, H\rangle^2).$$

First, we get

$$\begin{aligned}\bar Z(\langle \sigma(Z,Z), H\rangle) &= \langle \nabla^N_{\bar Z}\sigma(Z,Z), H\rangle + \langle \sigma(Z,Z), \nabla^N_{\bar Z} H\rangle\\ &= \langle \nabla^\perp_{\bar Z}\sigma(Z,Z), H\rangle + \langle \sigma(Z,Z), \nabla^\perp_{\bar Z} H\rangle\\ &= \langle (\nabla^\perp_{\bar Z}\sigma)(Z,Z), H\rangle + \langle \sigma(Z,Z), \nabla^\perp_{\bar Z} H\rangle,\end{aligned}$$

where we have used that

$$(\nabla^\perp_{\bar Z}\sigma)(Z,Z) = \nabla^\perp_{\bar Z}\sigma(Z,Z) - 2\sigma(\nabla_{\bar Z}Z, Z) = \nabla^\perp_{\bar Z}\sigma(Z,Z)$$

since, from the definition of the connection ∇ on the surface, we easily get $\nabla_{\bar Z}Z = 0$.

Now, from the Codazzi equation, we obtain

(2.1)
$$\begin{aligned}\bar Z(\langle \sigma(Z,Z), H\rangle) &= \langle (\nabla^\perp_Z\sigma)(\bar Z, Z), H\rangle + \langle (R^N(\bar Z, Z)Z)^\perp, H\rangle\\ &\quad + \langle \sigma(Z,Z), \nabla^\perp_{\bar Z} H\rangle\\ &= \langle (\nabla^\perp_Z\sigma)(\bar Z, Z), H\rangle + \langle R^N(\bar Z, Z)Z, H\rangle\\ &\quad + \langle \sigma(Z,Z), \nabla^\perp_{\bar Z} H\rangle.\end{aligned}$$

From the expression of the curvature tensor field of N

$$R^N(U,V)W = \tfrac{\rho}{4}\{\langle V,W\rangle U - \langle U,W\rangle V + \langle JV,W\rangle JU - \langle JU,W\rangle JV + 2\langle JV,U\rangle JW\},$$

it follows

$$\langle R^N(\bar{Z},Z)Z,H\rangle = \frac{3\rho}{4}\langle \bar{Z},JZ\rangle\langle H,JZ\rangle. \tag{2.2}$$

We also have the following

Lemma 2.1.

$$\langle(\nabla^{\perp}_{Z}\sigma)(\bar{Z},Z),H\rangle = \langle\bar{Z},Z\rangle\langle\nabla^{\perp}_{Z}H,H\rangle. \tag{2.3}$$

Proof. By using the definition of $(\nabla^{\perp}_{Z}\sigma)(\bar{Z},Z)$ one obtains

$$\begin{aligned}(\nabla^{\perp}_{Z}\sigma)(\bar{Z},Z) &= \nabla^{\perp}_{Z}\sigma(\bar{Z},Z) - \sigma(\nabla_Z\bar{Z},Z) - \sigma(\bar{Z},\nabla_Z Z)\\ &= \nabla^{\perp}_{Z}\sigma(\bar{Z},Z) - \sigma(\bar{Z},\nabla_Z Z)\end{aligned}$$

since $\nabla_Z\bar{Z} = 0$.

Next, let us consider the unit vector fields e_1 and e_2 corresponding to $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$, respectively, and $E = \frac{1}{\sqrt{2}}(e_1 - ie_2)$. Then we have $Z = \lambda E$ and

$$\sigma(\bar{Z},Z) = \frac{\lambda^2}{2}\sigma(e_1 - ie_2, e_1 + ie_2) = \frac{\lambda^2}{2}(\sigma(e_1,e_1) + \sigma(e_2,e_2)) = \langle\bar{Z},Z\rangle H.$$

Since $\nabla_Z Z$ is tangent it follows that $\nabla_Z Z = aZ + b\bar{Z}$ and then $0 = \langle\nabla_Z Z, Z\rangle = b\lambda^2$, where we have used the fact that $\langle Z,Z\rangle = 0$, and $a = \frac{1}{\lambda^2}\langle\nabla_Z Z,\bar{Z}\rangle$.

In conclusion

$$\begin{aligned}\langle(\nabla^{\perp}_{Z}\sigma)(\bar{Z},Z),H\rangle &= \langle\nabla^{N}_{Z}(\langle\bar{Z},Z\rangle H),H\rangle - \langle\nabla_Z Z,\bar{Z}\rangle\langle H,H\rangle\\ &= \langle\nabla_Z\bar{Z},Z\rangle\langle H,H\rangle + \langle\nabla_Z Z,\bar{Z}\rangle\langle H,H\rangle\\ &\quad + \langle\bar{Z},Z\rangle\langle\nabla^{\perp}_{Z}H,H\rangle - \langle\nabla_Z Z,\bar{Z}\rangle\langle H,H\rangle\\ &= \langle\bar{Z},Z\rangle\langle\nabla^{\perp}_{Z}H,H\rangle.\end{aligned}$$

q.e.d.

Lemma 2.2.

$$\bar{Z}(\langle JZ,H\rangle^2) = 2\langle JZ,H\rangle\langle(JZ)^{\perp},\nabla^{\perp}_{\bar{Z}}H\rangle - 2|H|^2\langle\bar{Z},JZ\rangle\langle JZ,H\rangle \tag{2.4}$$

Proof. From the definitions of the Kähler structure and of the Levi-Civita connection we have

$$\begin{aligned}\bar{Z}(\langle JZ,H\rangle^2) &= 2\langle JZ,H\rangle\{\langle\nabla^N_{\bar{Z}}JZ,H\rangle+\langle JZ,\nabla^N_{\bar{Z}}H\rangle\}\\ &= 2\langle JZ,H\rangle\{\langle\bar{Z},Z\rangle\langle JH,H\rangle-\langle(JZ)^\top,A_H\bar{Z}\rangle\\ &\quad+\langle(JZ)^\perp,\nabla^\perp_{\bar{Z}}H\rangle\}\\ &= 2\langle JZ,H\rangle\{\langle(JZ)^\perp,\nabla^\perp_{\bar{Z}}H\rangle-\langle\sigma((JZ)^\top,\bar{Z}),H\rangle\}\\ &= 2\langle JZ,H\rangle\{\langle(JZ)^\perp,\nabla^\perp_{\bar{Z}}H\rangle-\langle JZ,\bar{Z}\rangle|H|^2\},\end{aligned}$$

where we have used $\nabla^N_{\bar{Z}}Z=\sigma(\bar{Z},Z)=\langle\bar{Z},Z\rangle H$, as we have seen in the proof of the previous Lemma, and $(JZ)^\top=\frac{1}{\lambda^2}\langle JZ,\bar{Z}\rangle Z$, that can be easily checked. q.e.d.

Replacing (2.2), (2.3) and (2.4) into (2.1) we obtain that $\bar{Z}(Q(Z,Z))$ vanishes and then we come to the conclusion that

Proposition 2.3. *If Σ^2 is an immersed surface in a complex space form $N^n(\rho)$, with parallel mean curvature vector field, then the $(2,0)$-part of the quadratic form Q, defined on Σ^2 by*

$$Q(X,Y)=8|H|^2\langle\sigma(X,Y),H\rangle+3\rho\langle JX,H\rangle\langle JY,H\rangle,$$

is holomorphic.

3. Quadratic forms and 2-spheres in 2-dimensional complex space forms

In this section we shall define a new quadratic form on a surface Σ^2 immersed in a complex space form $N^2(\rho)$, with parallel mean curvature vector field $H\neq 0$, and prove that its $(2,0)$-part is holomorphic. Then, by using these two quadratic forms, we shall classify the 2-spheres with nonzero parallel mean curvature vector.

3.1. Another quadratic form. Let us consider an oriented orthonormal local frame $\{\widetilde{e}_1,\widetilde{e}_2\}$ on the surface and denote by θ the Kähler angle function defined by $\langle J\widetilde{e}_1,\widetilde{e}_2\rangle=\cos\theta$. The immersion $x:\Sigma^2\to N$ is said to be holomorphic if $\cos\theta=1$, anti-holomorphic if $\cos\theta=-1$, and totally real if $\cos\theta=0$. In the following we shall assume that x is neither holomorphic or anti-holomorphic.

Next, we take $e_3=-\frac{H}{|H|}$ and let e_4 be the unique unit normal vector field orthogonal to e_3 compatible with the orientation of Σ^2 in N. Since e_3 is parallel in the normal bundle so is e_4, and, as the Kähler angle is independent of the choice of the orthonormal frame on the surface (see, for example, [**9**]), we have $\langle Je_4,e_3\rangle=\cos\theta$.

Now, we can consider the vector fields

$$e_1 = \cot\theta e_3 - \frac{1}{\sin\theta} J e_4, \quad e_2 = \frac{1}{\sin\theta} J e_3 + \cot\theta e_4$$

tangent to the surface and get an orthonormal frame field $\{e_1, e_2, e_3, e_4\}$ adapted to Σ^2 in N.

We define a quadratic form Q' on Σ^2 by

$$Q'(X, Y) = 8i|H|\langle\sigma(X, Y), e_4\rangle + 3\rho\langle JX, e_4\rangle\langle JY, e_4\rangle$$

and again consider the isothermal coordinates (u, v) on Σ^2 and the tangent complex vector fields Z and $\bar{Z}$. In the same way as in the case of Q, using the Codazzi equation, the fact that H and e_4 are parallel and the expression of the curvature tensor field of N, we get

$$\bar{Z}(\langle\sigma(Z, Z), e_4\rangle) = \frac{3\rho}{4}\langle\bar{Z}, JZ\rangle\langle JZ, e_4\rangle. \tag{3.1}$$

On the other hand, we have

$$\begin{aligned}
\bar{Z}(\langle JZ, e_4\rangle^2) &= 2\langle JZ, e_4\rangle\{\langle\nabla^N_{\bar{Z}} JZ, e_4\rangle + \langle JZ, \nabla^N_{\bar{Z}} e_4\rangle\} \\
&= 2\langle JZ, e_4\rangle\{\langle\bar{Z}, Z\rangle\langle JH, e_4\rangle - \langle(JZ)^\top, A_{e_4}\bar{Z}\rangle\} \\
&= -2|H|\langle JZ, e_4\rangle\langle\bar{Z}, Z\rangle\langle Je_3, e_4\rangle \\
&\quad -2\langle JZ, e_4\rangle\langle\sigma((JZ)^\top, \bar{Z}), e_4\rangle \\
&= 2|H|\langle JZ, e_4\rangle\langle\bar{Z}, Z\rangle\cos\theta - 2\langle JZ, e_4\rangle\langle JZ, \bar{Z}\rangle\langle H, e_4\rangle \\
&= 2|H|\langle JZ, e_4\rangle\langle\bar{Z}, Z\rangle\cos\theta,
\end{aligned}$$

where we have used

$$\nabla^N_{\bar{Z}} Z = \sigma(\bar{Z}, Z) = \langle\bar{Z}, Z\rangle H, \quad (JZ)^\top = \frac{1}{\lambda^2}\langle JZ, \bar{Z}\rangle Z$$

and $\langle Je_4, e_3\rangle = \cos\theta$. We have

$$\langle\bar{Z}, JZ\rangle = -i\langle\bar{Z}, Z\rangle\langle e_1, Je_2\rangle = i\langle\bar{Z}, Z\rangle\cos\theta,$$

and, therefore, one obtains

$$\bar{Z}(\langle JZ, e_4\rangle^2) = -2i|H|\langle\bar{Z}, JZ\rangle\langle JZ, e_4\rangle. \tag{3.2}$$

Hence, from (3.1) and (3.2), one obtains $\bar{Z}(Q'(Z, Z)) = 0$, which means that the $(2, 0)$-part of the quadratic form Q' is holomorphic.

3.2. 2-spheres in 2-dimensional complex space forms. In order to classify the 2-spheres in 2-dimensional complex space forms, we shall need a result of T. Ogata in [**16**], which we will briefly recall in the following (see also [**12**] and [**15**]). Consider a surface Σ^2 isometrically immersed in a complex space form $N^2(\rho)$, with parallel mean curvature vector field $H \neq 0$. Using the frame field on $N^2(\rho)$ adapted to Σ^2, defined above, and considering isothermal coordinates (u, v) on the surface, Ogata proved that there exist complex-valued functions a and c on Σ^2 such that θ, λ, a and c satisfy

$$\begin{cases} \frac{\partial\theta}{\partial z} = \lambda(a+b) \\ \frac{\partial\lambda}{\partial\bar{z}} = -|\lambda|^2(\bar{a}-b)\cot\theta \\ \frac{\partial a}{\partial\bar{z}} = \bar{\lambda}\Big(2|a|^2 - 2ab + \frac{3\rho\sin^2\theta}{8}\Big)\cot\theta \\ \frac{\partial c}{\partial z} = 2\lambda(a-b)c\cot\theta \\ |c|^2 = |a|^2 + \frac{\rho(3\sin^2\theta-2)}{8} \end{cases} \tag{3.3}$$

where $z = u + iv$ and $|H| = 2b$; and also the converse: if ρ is a real constant, b a positive constant, Σ^2 a 2-dimensional Riemannian manifold, and there exist some functions θ, a and c on Σ^2 satisfying (3.3), then there is an isometric immersion of Σ^2 into $N^2(\rho)$ with parallel mean curvature vector field of length equal to $2b$ and with the Kähler angle θ. The second fundamental form of Σ^2 in N w.r.t. $\{e_1, e_2, e_3, e_4\}$ is given by

$$\sigma^3 = \begin{pmatrix} -2b - \Re(\bar{a}+c) & -\Im(\bar{a}+c) \\ -\Im(\bar{a}+c) & -2b + \Re(\bar{a}+c) \end{pmatrix}$$

and

$$\sigma^4 = \begin{pmatrix} \Im(\bar{a}-c) & -\Re(\bar{a}-c) \\ -\Re(\bar{a}-c) & -\Im(\bar{a}-c) \end{pmatrix}$$

and the Gaussian curvature of Σ^2 is $K = 4b^2 - 4|c|^2 + \frac{\rho}{2}$ (see also [**12**]).

Assume now that the $(2,0)$-part of Q and the $(2,0)$-part of Q' vanish on the surface Σ^2. It follows, from the expression of the second fundamental form, that $\bar{c} + a \in \mathbb{R}$, $\bar{c} - a \in \mathbb{R}$ and

$$32b(\bar{c}+a) - 3\rho\sin^2\theta = 0, \quad 32b(\bar{c}-a) + 3\rho\sin^2\theta = 0.$$

Therefore $c = 0$ and $a = \frac{3\rho\sin^2\theta}{32b}$ and, from the fifth equation of (3.3), it follows

$$9\rho^2\sin^4\theta + 128\rho b^2(3\sin^2\theta - 2) = 0. \tag{3.4}$$

We have to split the study of this equation in two cases. First, if $\rho = 0$ then the above equation holds and $a = 0$. Next, if $\rho \neq 0$, we get that function θ is a constant. This, together with the first equation of (3.3), leads to $a = \frac{3\rho\sin^2\theta}{32b} = -b$. By replacing in equation (3.4) we obtain

$\rho = -12b^2$ and then $\sin^2\theta = \frac{8}{9}$. We note that in both cases the Gaussian curvature of Σ^2 is given by $K = 4b^2 + \frac{\rho}{2} = \text{constant}$ (see [**12**]). Thus, by using Theorem 1.1 in [**12**], we have just proved that

Theorem 3.1. *If the* $(2,0)$*-part of* Q *and the* $(2,0)$*-part of* Q' *vanish on a surface* Σ^2 *isometrically immersed in a complex space form* $N^2(\rho)$*, with parallel mean curvature vector field of length* $2b > 0$*, then either*

1) $N^2(\rho) = \mathbb{C}H^2(-12b^2)$ *and* Σ^2 *is the slant surface in* [**6**, Theorem 3(2)];
2) $N^2(\rho) = \mathbb{C}^2$ *and* Σ^2 *is a part of a round sphere in a hyperplane in* $\mathbb{C}^2$.

Since the Gaussian curvature K is nonnegative only in the second case of the Theorem, we have also reobtained the following result of S. Hirakawa in [**12**].

Corollary 3.2. *If* $\mathbb{S}^2$ *is an isometrically immersed sphere in a* 2*-dimensional complex space form, with nonzero parallel mean curvature vector, then it is a round sphere in a hyperplane in* $\mathbb{C}^2$.

4. A remark on pmc 2-spheres in $\mathbb{C}^n$

Proposition 4.1. *Let* Σ^2 *be an isometrically immersed surface in* $\mathbb{C}^n$*, with nonzero parallel mean curvature vector. Then the* $(2,0)$*-part of the quadratic form* Q *vanishes on* Σ^2 *if and only if the surface is pseudo-umbilical, i.e.* $A_H = |H|^2\,\mathrm{I}$.

Proof. It can be easily seen that if Σ^2 is pseudo-umbilical then the $(2,0)$-part of Q vanishes and, therefore, we have to prove only the necessity.

From $Q(Z,Z) = \frac{\langle Z,\bar{Z}\rangle^2}{2} Q(e_1 - ie_2, e_1 - ie_2) = 0$ it follows

$$\langle \sigma(e_1,e_1) - \sigma(e_2,e_2), H\rangle = 0$$

and

$$\langle \sigma(e_1,e_2), H\rangle = 0.$$

But, since $\langle \sigma(e_1,e_1) + \sigma(e_2,e_2), H\rangle = 2|H|^2$, we obtain, for each $i \in \{1,2\}$,

$$\langle A_H e_i, e_i\rangle = \langle \sigma(e_i,e_i), H\rangle = |H|^2.$$

Therefore $A_H = |H|^2\,\mathrm{I}$, i.e. Σ^2 is pseudo-umbilical. q.e.d.

S.-T. Yau proved, in [**21**, Theorem 4], that if Σ^2 is a surface with parallel mean curvature vector H in a manifold N with constant sectional curvature, then either Σ^2 is a minimal surface of an umbilical hypersurface of N or Σ^2 lies in a 3-dimensional umbilical submanifold of N with constant mean curvature, as H is an umbilical direction or the second fundamental form of Σ^2 can be diagonalized simultaneously.

We note that, in the first case, the mean curvature vector field of Σ^2 in $\mathbb{C}^n$ is orthogonal to the hypersurface.

Applying this result, together with Proposition 4.1, to the 2-spheres in $\mathbb{C}^n$, and using the Gauss equation of a hypersurface in $\mathbb{C}^n$, we get

Proposition 4.2. *If $\mathbb{S}^2$ is an isometrically immersed sphere in $\mathbb{C}^n$, with nonzero parallel mean curvature vector field H, then it is a minimal surface of a hypersphere $\mathbb{S}^{2n-1}(|H|) \subset \mathbb{C}^n$.*

5. Reduction of codimension

Let $x : \Sigma^2 \to N^n(\rho)$, $n \geq 3$, $\rho \neq 0$, be an isometric immersion of a surface Σ^2 in a complex space form, with parallel mean curvature vector field $H \neq 0$.

Lemma 5.1. *For any vector V normal to Σ^2, which is also orthogonal to $JT\Sigma^2$ and to JH, we have $[A_H, A_V] = 0$, i.e. A_H commutes with A_V.*

Proof. The statement follows easily, from the Ricci equation

$$\langle R^\perp(X,Y)H, V\rangle = \langle [A_H, A_V]X, Y\rangle + \langle R^N(X,Y)H, V\rangle,$$

since

$$\begin{aligned} \langle R^N(X,Y)H, V\rangle &= \tfrac{\rho}{4}\{\langle JY, H\rangle\langle JX, V\rangle - \langle JX, H\rangle\langle JY, V\rangle \\ &\quad +2\langle JY, X\rangle\langle JH, V\rangle\} \\ &= 0 \end{aligned}$$

and $R^\perp(X,Y)H = 0$. q.e.d.

Remark 5.2. If $n = 3$ and $H \perp JT\Sigma^2$ do not hold simultaneously, then there exists at least one normal vector V as in Lemma 5.1. This can be proved by using the basis of the tangent space TN along Σ^2 defined in [**17**], which construction we shall briefly explain in the following. Let us consider a local orthonormal frame $\{e_1, e_2\}$ of vector fields tangent to Σ^2. Since we have assumed that $H \neq 0$ it follows that Σ^2 is not holomorphic or antiholomorphic, which means that $\cos^2\theta = 1$ only at isolated points, and we shall work in the open dense set of points where $\cos^2\theta \neq 1$, where θ is the Kähler angle function. The next step is to define two normal vectors by

$$e_3 = -\cot\theta e_1 - \frac{1}{\sin\theta}Je_2 \quad \text{and} \quad e_4 = \frac{1}{\sin\theta}Je_1 - \cot\theta e_2.$$

Thus $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis in $\operatorname{span}\{e_1, e_2, Je_1, Je_2\}$. Moreover, we can set

$$\widetilde{e}_1 = \cos\Big(\frac{\theta}{2}\Big)e_1 + \sin\Big(\frac{\theta}{2}\Big)e_3, \quad \widetilde{e}_2 = \cos\Big(\frac{\theta}{2}\Big)e_2 + \sin\Big(\frac{\theta}{2}\Big)e_4$$

$$\widetilde{e}_3=\sin\left(\frac{\theta}{2}\right)e_1-\cos\left(\frac{\theta}{2}\right)e_3,\quad \widetilde{e}_4=-\sin\left(\frac{\theta}{2}\right)e_2+\cos\left(\frac{\theta}{2}\right)e_4$$

and obtain a J-canonical basis of span$\{e_1,e_2,Je_1,Je_2\}$, i.e. $J\widetilde{e}_{2i-1}=\widetilde{e}_{2i}$. Finally, let us consider a J-basis of TN along Σ^2, of the form $\{\widetilde{e}_1,\widetilde{e}_2,\widetilde{e}_3,\widetilde{e}_4,\widetilde{e}_5,\widetilde{e}_6=J\widetilde{e}_5,\dots,\widetilde{e}_{2n-1},\widetilde{e}_{2n}=J\widetilde{e}_{2n-1}\}$. Now, three situations can occur:

1) $H\in(JT\Sigma^2)^\perp$, and then $\widetilde{e}_5\perp JT\Sigma^2$ and $\widetilde{e}_5\perp JH$, where we have denoted by $(JT\Sigma^2)^\perp=\{(JX)^\perp : X \text{ tangent to } \Sigma^2\}$;
2) $H\perp JT\Sigma^2$, and then, if we choose $\widetilde{e}_5=H$ and $\widetilde{e}_6=JH$, we have $\widetilde{e}_7\perp JT\Sigma^2$ and $\widetilde{e}_7\perp JH$ (obviously, this case can occur only if $n>3$);
3) $H\notin(JT\Sigma^2)^\perp$ and H is not orthogonal to $JT\Sigma^2$. In this case we may consider the vector u, the projection of H on the complementary space of $(JT\Sigma^2)^\perp$ in TN (along Σ^2) and set $\widetilde{e}_5=\frac{u}{|u|}$. It follows that $\widetilde{e}_5\perp JT\Sigma^2$ and $\widetilde{e}_5\perp JH$.

If $n=3$ and $H\perp JT\Sigma^2$ it is easy to see that

$$\langle R^N(X,Y)H,e_3\rangle=\langle R^N(X,Y)H,e_4\rangle=0$$

for any vector fields X and Y tangent to Σ^2, and then that A_H commutes with A_{e_3} and A_{e_4}.

Conclusively, we get the following

Corollary 5.3. *Either H is an umbilical direction or there exists a basis that diagonalizes simultaneously A_H and A_V, for all normal vectors satisfying $V\perp JH$, if $n=3$ and $H\perp JT\Sigma^2$, or the conditions in Lemma* 5.1*, otherwise.*

Lemma 5.4. *Assume that H is nowhere an umbilical direction. Then there exists a parallel subbundle of the normal bundle which contains the image of the second fundamental form σ and has dimension less or equal to* 8.

Proof. We consider the following subbundle L of the normal bundle

$$L=\operatorname{span}\{\operatorname{Im}\sigma\cup(J\operatorname{Im}\sigma)^\perp\cup(JT\Sigma^2)^\perp\},$$

and we will show that L is parallel.

First, we shall prove that, if V is orthogonal to L, then $\nabla^\perp_{e_i}V$ is orthogonal to $JT\Sigma^2$ and to JH, where $\{e_1,e_2\}$ is an orthonormal frame w.r.t. which we have $\langle\sigma(e_1,e_2),V\rangle=\langle\sigma(e_1,e_2),H\rangle=0$. Indeed, we get

$$\begin{aligned}\langle(JH)^\perp,\nabla^\perp_{e_i}V\rangle &= \langle(JH)^\perp,\nabla^N_{e_i}V\rangle=-\langle\nabla^N_{e_i}(JH)^\perp,V\rangle\\ &= -\langle\nabla^N_{e_i}JH,V\rangle+\langle\nabla^N_{e_i}(JH)^\top,V\rangle\\ &= \langle JA_He_i,V\rangle+\langle\sigma(e_i,(JH)^\top),V\rangle\\ &= 0\end{aligned}$$

and

$$\begin{aligned}\langle (Je_j)^\perp, \nabla^\perp_{e_i} V\rangle &= -\langle \nabla^N_{e_i}(Je_j)^\perp, V\rangle\\ &= -\langle \nabla^N_{e_i} Je_j, V\rangle + \langle \nabla^N_{e_i}(Je_j)^\top, V\rangle\\ &= -\langle J\nabla_{e_i} e_j, V\rangle - \langle J\sigma(e_i,e_j), V\rangle\\ &\quad + \langle \sigma(e_i,(Je_j)^\top), V\rangle\\ &= 0.\end{aligned}$$

Next, we shall prove that if a normal subbundle S is orthogonal to L, then so is $\nabla^\perp S$, i.e.

$$\langle \sigma(e_i,e_j), \nabla^\perp_{e_k} V\rangle = 0, \quad \langle J\sigma(e_i,e_j), \nabla^\perp_{e_k} V\rangle = 0 \quad \text{and} \quad \langle Je_i, \nabla^\perp_{e_k} V\rangle = 0$$

for any $V \in S$ and $i,j,k \in \{1,2\}$. Since we have just proved the last property, it remains only to verify the first two of them.

We denote $A_{ijk} = \langle \nabla^\perp_{e_k}\sigma(e_i,e_j), V\rangle$ and, since σ is symmetric, we have $A_{ijk} = A_{jik}$. We also obtain $A_{ijk} = -\langle \sigma(e_i,e_j), \nabla^\perp_{e_k} V\rangle$, since V is orthogonal to L. We get

$$\begin{aligned}\langle (\nabla^\perp_{e_k}\sigma)(e_i,e_j), V\rangle &= \langle \nabla^\perp_{e_k}\sigma(e_i,e_j), V\rangle - \langle \sigma(\nabla_{e_k} e_i, e_j), V\rangle\\ &\quad -\langle \sigma(e_i, \nabla_{e_k} e_j), V\rangle\\ &= \langle \nabla^\perp_{e_k}\sigma(e_i,e_j), V\rangle,\end{aligned}$$

and, from the Codazzi equation,

$$\begin{aligned}\langle (\nabla^\perp_{e_k}\sigma)(e_i,e_j), V\rangle &= \langle (\nabla^\perp_{e_i}\sigma)(e_k,e_j) + (R^N(e_k,e_i)e_j)^\perp, V\rangle\\ &= \langle (\nabla^\perp_{e_j}\sigma)(e_k,e_i) + (R^N(e_k,e_j)e_i)^\perp, V\rangle\\ &= \langle (\nabla^\perp_{e_i}\sigma)(e_k,e_j), V\rangle = \langle (\nabla^\perp_{e_j}\sigma)(e_k,e_i), V\rangle.\end{aligned}$$

We have just proved that $A_{ijk} = A_{kji} = A_{ikj}$.

Next, since $\nabla^\perp_{e_k} V$ is orthogonal to $JT\Sigma^2$ and to JH, it follows that the frame field $\{e_1,e_2\}$ diagonalizes $A_{\nabla^\perp_{e_k} V}$ and we get

$$A_{ijk} = -\langle \sigma(e_i,e_j), \nabla^\perp_{e_k} V\rangle = -\langle e_i, A_{\nabla^\perp_{e_k} V} e_j\rangle = 0$$

for any $i \neq j$. Hence, we have obtained that $A_{ijk} = 0$ if two indices are different from each other.

Finally, we only have to prove that $A_{iii} = 0$. Indeed, we have

$$\begin{aligned}A_{iii} &= -\langle \sigma(e_i,e_i), \nabla^\perp_{e_i} V\rangle = -\langle 2H, \nabla^\perp_{e_i} V\rangle + \langle \sigma(e_j,e_j), \nabla^\perp_{e_i} V\rangle\\ &= \langle 2\nabla^\perp_{e_i} H, V\rangle - A_{jji} = 0.\end{aligned}$$

It is easy to see that if V is orthogonal to L, then JV is normal and orthogonal to L. It follows that

$$\begin{aligned}
\langle (J\sigma(e_i,e_j))^\perp, \nabla^\perp_{e_k} V\rangle &= -\langle \nabla^N_{e_k}(J\sigma(e_i,e_j))^\perp, V\rangle \\
&= -\langle \nabla^N_{e_k} J\sigma(e_i,e_j), V\rangle \\
&\quad +\langle \nabla^N_{e_k}(J\sigma(e_i,e_j))^\top, V\rangle \\
&= \langle JA_{\sigma(e_i,e_j)}e_k, V\rangle - \langle J\nabla^\perp_{e_k}\sigma(e_i,e_j), V\rangle \\
&\quad +\langle \sigma(e_k,(J\sigma(e_i,e_j))^\top), V\rangle \\
&= \langle \nabla^\perp_{e_k}\sigma(e_i,e_j), JV\rangle = 0.
\end{aligned}$$

Thus, we come to the conclusion that the subbundle L is parallel.

q.e.d.

When H is umbilical we can use the quadratic form Q to prove the following

Lemma 5.5. *Let Σ^2 be an immersed surface in a complex space form $N^n(\rho)$, $\rho \neq 0$, with nonzero parallel mean curvature vector H. If H is an umbilical direction everywhere, then Σ^2 is a totally real pseudo-umbilical surface of N.*

Proof. Since H is umbilical it follows that $\langle \sigma(Z,Z), H\rangle = 0$, which implies that Σ^2 is pseudo-umbilical and that $Q(Z,Z) = 3\rho\langle JZ, H\rangle^2$.

Next, as the $(2,0)$-part of Q is holomorphic, we have $\bar{Z}(Q(Z,Z)) = 0$, and further

$$0 = \bar{Z}(\langle JZ, H\rangle^2) = -2|H|^2\langle JZ, H\rangle\langle JZ, \bar{Z}\rangle,$$

as we have seen in a previous section. Hence, $\langle JZ, \bar{Z}\rangle = 0$ or $\langle JZ, H\rangle = 0$. Assume that the set of zeroes of $\langle JZ, \bar{Z}\rangle = 0$ is not the entire Σ^2. Then, by analyticity, it is a closed set without interior points and its complement is an open dense set in Σ^2. In this last set we have $\langle JZ, H\rangle = 0$ and then, since H is parallel and Σ^2 is pseudo-umbilical,

$$\begin{aligned}
0 = \bar{Z}(\langle JZ, H\rangle) &= \langle J\nabla^N_{\bar{Z}} Z, H\rangle + \langle JZ, \nabla^N_{\bar{Z}} H\rangle \\
&= -\langle \bar{Z}, Z\rangle\langle JH, H\rangle - \langle JZ, A_H\bar{Z}\rangle \\
&= -|H|^2\langle JZ, \bar{Z}\rangle,
\end{aligned}$$

which means that Σ^2 is also totally real. q.e.d.

Remark 5.6. Some kind of a converse result was obtained by B.–Y. Chen and K. Ogiue since they proved in [**7**] that if a unit normal vector field to a 2-sphere, immersed in a complex space form as a totally real surface, is parallel and isoperimetric, then it is umbilical.

Remark 5.7. In [**19**] N. Sato proved that, if M is a pseudo-umbilical submanifold of a complex projective space $\mathbb{C}P^n(\rho)$, with nonzero parallel mean curvature vector field, then it is a totally real submanifold. Moreover, the mean curvature vector field H is orthogonal to JTM. Therefore, if M is a surface, it follows that the $(2,0)$-part of Q vanishes on M.

Remark 5.8. In order to show that only the two situations exposed in Lemma 5.4 and Lemma 5.5 can occur, we shall use an argument similar to that in Remark 5 in [**4**]. Thus, since the map $p \in \Sigma^2 \to (A_H - \mu\,\mathrm{I})(p)$, where μ is a constant, is analytic, it follows that if H is an umbilical direction, then this either holds on Σ^2 or only for a closed set without interior points. In this second case H is not an umbilical direction in an open dense set, and then Lemma 5.4 holds on this set. By continuity it holds on Σ^2.

By using Lemma 5.4 and Lemma 5.5 we can state

Proposition 5.9. *Either H is everywhere an umbilical direction, and Σ^2 is a totally real pseudo-umbilical surface of N, or H is nowhere an umbilical direction, and there exists a subbundle of the normal bundle that is parallel, contains the image of the second fundamental form and its dimension is less or equal to* 8.

Now, from Proposition 5.9 and a result of J. H. Eschenburg and R. Tribuzy [**11**, Theorem 2], it follows

Theorem 5.10. *Let Σ^2 be an isometrically immersed surface in a complex space form $N^n(\rho)$, $n \geq 3$, $\rho \neq 0$, with nonzero parallel mean curvature vector. Then, one of the following holds*:

1) Σ^2 *is a totally real pseudo-umbilical surface of $N^n(\rho)$, or*
2) Σ^2 *is not pseudo-umbilical and it lies in a complex space form $N^r(\rho)$, where $r \leq 5$.*

Remark 5.11. The case when $\rho = 0$ is solved by Theorem 4 in [**21**].

Remark 5.12. We have seen (Remark 5.6) that if Σ^2 is a totally real 2-sphere then it is pseudo-umbilical and therefore the second case of the previous Theorem cannot occur for such surfaces.

6. 2-spheres with constant Kähler angle in complex space forms

This section is devoted to the study of immersed surfaces Σ^2 in a complex space form $N^n(\rho)$, $n \geq 3$, $\rho \neq 0$, with nonzero non-umbilical parallel mean curvature vector H and constant Kähler angle, on which the $(2,0)$-part of Q vanishes. We shall compute the Laplacian of the function $|A_H|^2$ for such a surface and show that there are no 2-spheres with these properties.

Let $\{e_1, e_2\}$ be an orthonormal frame on Σ^2 such that $H \perp Je_1$. The fact that the $(2,0)$-part of the quadratic form Q vanishes can be written as

(6.1)
$$\begin{cases} 8|H|^2\langle\sigma(e_1,e_1)-\sigma(e_2,e_2),H\rangle = -3\rho(\langle Je_1,H\rangle^2-\langle Je_2,H\rangle^2) \\ 8|H|^2\langle\sigma(e_1,e_2),H\rangle = 3\rho\langle Je_1,H\rangle\langle Je_2,H\rangle, \end{cases}$$

and, from the second equation, we see that $\langle\sigma(e_1,e_2),H\rangle = 0$. It follows that the frame $\{e_1, e_2\}$ diagonalizes simultaneously A_H and A_V, for all normal vectors V as in Corollary 5.3, since we are in the second case of Theorem 5.10.

Next, since Σ^2 is not holomorphic or anti-holomorphic, we have that $\cos\theta \neq \pm 1$ on an open dense set and then we can consider again the normal vectors

$$e_3 = -\cot\theta e_1 - \frac{1}{\sin\theta}Je_2 \quad \text{and} \quad e_4 = \frac{1}{\sin\theta}Je_1 - \cot\theta e_2$$

and get an orthonormal frame $\{e_1, e_2, e_3, e_4\}$ in $\operatorname{span}\{e_1, e_2, Je_1, Je_2\}$, where θ is the Kähler angle on Σ^2.

It is easy to see that if $H \perp JT\Sigma^2$ it results that the surface is pseudo-umbilical, which is a contradiction.

On the other hand, if we assume that $H \in \operatorname{span}\{e_3, e_4\}$ it follows $H = \pm|H|e_3$, since $Je_1 \perp H$, and then e_3 is parallel. Also, since all normal vectors but e_4 verify conditions in Corollary 5.3 we have $\sigma(e_1, e_2) \parallel e_4$. By using these facts and the expression of e_3 we obtain that $\sigma(e_i, e_j) \in \operatorname{span}\{e_3, e_4\}$ for $i, j \in \{1, 2\}$, and then $\dim L = 2$, where L is the subbundle in Lemma 5.4. Therefore, again by the meaning of Theorem 2 in [**11**], we get that Σ^2 lies in a complex space form $N^2(\rho)$, which case was studied earlier in this paper.

In the following, we shall assume that $H \notin \operatorname{span}\{e_3, e_4\}$, and, as we also know that H is not orthogonal to $JT\Sigma^2$, one obtains that the mean curvature vector can be written as

$$H = |H|(\cos\beta e_3 + \sin\beta e_5)$$

where β is a real-valued function defined locally on Σ^2 and e_5 is a unit normal vector field such that $e_5 \perp JT\Sigma^2$. We consider the orthonormal frame field $\{e_1, e_2, e_3, e_4, e_5, e_6 = Je_5, \ldots, e_{2n-1}, e_{2n} = Je_{2n-1}\}$ on N and its dual frame $\{\theta_i\}_{i=1}^{2n}$. These are well defined at the points of Σ^2 where $\sin(2\beta) \neq 0$, which, due to our assumptions, form an open dense set in Σ^2. The structure equations of the surface are

$$d\phi = -i\theta_{12}\wedge\phi \quad \text{and} \quad d\theta_{12} = -\frac{i}{2}K\phi\wedge\bar{\phi},$$

where $\phi = \theta_1 + i\theta_2$, the real 1-form θ_{12} is the connection form of the Riemannian metric on Σ^2 and K is the Gaussian curvature.

A result of T. Ogata in [**17**], together with $H \perp e_i$ for any $i \geq 4$, $i \neq 5$, implies that, w.r.t. the above orthonormal frame, the components of the second fundamental form are given by

$$\sigma^3 = \begin{pmatrix} |H|\cos\beta - \Re(\bar{a}+c) & -\Im(\bar{a}+c) \\ -\Im(\bar{a}+c) & |H|\cos\beta + \Re(\bar{a}+c) \end{pmatrix}$$

$$\sigma^4 = \begin{pmatrix} \Im(\bar{a}-c) & -\Re(\bar{a}-c) \\ -\Re(\bar{a}-c) & -\Im(\bar{a}-c) \end{pmatrix}$$

$$\sigma^5 = \begin{pmatrix} |H|\sin\beta - \Re(\bar{a}_3+c_3) & -\Im(\bar{a}_3+c_3) \\ -\Im(\bar{a}_3+c_3) & |H|\sin\beta + \Re(\bar{a}_3+c_3) \end{pmatrix}$$

$$\sigma^6 = \begin{pmatrix} \Im(\bar{a}_3-c_3) & -\Re(\bar{a}_3-c_3) \\ -\Re(\bar{a}_3-c_3) & -\Im(\bar{a}_3-c_3) \end{pmatrix}$$

and

$$\sigma^{2\alpha-1} = \begin{pmatrix} -\Re(\bar{a}_\alpha+c_\alpha) & -\Im(\bar{a}_\alpha+c_\alpha) \\ -\Im(\bar{a}_\alpha+c_\alpha) & \Re(\bar{a}_\alpha+c_\alpha) \end{pmatrix}$$

$$\sigma^{2\alpha} = \begin{pmatrix} \Im(\bar{a}_\alpha-c_\alpha) & -\Re(\bar{a}_\alpha-c_\alpha) \\ -\Re(\bar{a}_\alpha-c_\alpha) & -\Im(\bar{a}_\alpha-c_\alpha) \end{pmatrix}$$

where a, c, a_α, c_α, with $\alpha \in \{3, \ldots, n\}$, are complex-valued functions defined locally on the surface Σ^2. We note that, since $\sigma(e_1, e_2) \perp H$ and $\sigma(e_1, e_2) \perp e_5$, it follows $\sigma(e_1, e_2) \perp e_3$. Moreover, since $\sigma(e_1, e_2) \perp e_i$ for any $i \in \{1, \ldots, 2n\} \setminus \{4, 6\}$, we have $\bar{a} + c \in \mathbb{R}$, $\bar{a}_3 + c_3 \in \mathbb{R}$ and $a_\alpha = c_\alpha$ for any $\alpha \geq 4$.

In the same paper [**17**], amongst others, the author computed the differential of the Kähler angle function θ for a minimal surface. In the same way, this time for our surface, we get

$$d\theta = \Big(a - \frac{|H|}{2}\cos\beta\Big)\phi + \Big(\bar{a} - \frac{|H|}{2}\cos\beta\Big)\bar{\phi}.$$

The next step is to determine the connection form θ_{12} and the differential of the function β, by using the property of H being parallel. We have

$$\nabla^\perp_{e_i} H = (-\sin\beta e_3 + \cos\beta e_5)d\beta(e_i) + \cos\beta\nabla^\perp_{e_i} e_3 + \sin\beta\nabla^\perp_{e_i} e_5 = 0 \tag{6.2}$$

for $i \in \{1,2\}$, and then

$$\cos\beta\langle\nabla^N_{e_i}e_3,e_4\rangle+\sin\beta\langle\nabla^N_{e_i}e_3,e_4\rangle=0,\quad i\in\{1,2\}$$

from where, by using the expressions of e_3 in the first term, of e_4 in the second one and of the second fundamental form of Σ^2, we get

$$\theta_{12}(e_1)=\cot\theta\Im(\bar{a}-c)-\frac{\tan\beta}{\sin\theta}\Im(\bar{a}_3-c_3)$$

$$\theta_{12}(e_2)=-|H|\frac{\cot\theta}{\cos\beta}-2\cot\theta\Re a+\tan\beta\Big(\tan\Big(\frac{\theta}{2}\Big)\Re a_3-\cot\Big(\frac{\theta}{2}\Big)\Re c_3\Big)$$

and finally $\theta_{12}=f_1\phi+\bar{f}_1\bar{\phi}$, where

$$f_1=\frac{i}{2}\Big(|H|\frac{\cot\theta}{\cos\beta}+2\cot\theta a-\frac{\tan\beta}{\sin\theta}(a_3-\bar{c}_3)+\cot\theta\tan\beta(a_3+\bar{c}_3)\Big). \tag{6.3}$$

Now, from equation (6.2), we also obtain

$$d\beta(e_i)+\langle\nabla^N_{e_i}e_3,e_5\rangle=0,\quad i\in\{1,2\}$$

and then, replacing e_3 by its expression and also using the expression of the second fundamental form, we get

$$d\beta(e_1)=|H|\cot\theta\sin\beta+\tan\Big(\frac{\theta}{2}\Big)\Re a_3-\cot\Big(\frac{\theta}{2}\Big)\Re c_3$$

and

$$d\beta(e_2)=\frac{1}{\sin\theta}\Im(\bar{a}_3-c_3).$$

Hence the differential of β is given by $d\beta=f_2\phi+\bar{f}_2\bar{\phi}$, where

$$f_2=\frac{1}{2}\Big(|H|\cot\theta\sin\beta+\frac{1}{\sin\theta}(a_3-\bar{c}_3)-\cot\theta(a_3+\bar{c}_3)\Big). \tag{6.4}$$

We note that if the Kähler angle θ is constant, then $a=\bar{a}=\frac{|H|}{2}\cos\beta$, and, from (6.3), it results

$$f_1=\frac{i}{2}\Big\{|H|\cot\theta\Big(\cos\beta+\frac{1}{\cos\beta}\Big)-\frac{\tan\beta}{\sin\theta}(a_3-\bar{c}_3)+\cot\theta\tan\beta(a_3+\bar{c}_3)\Big\}. \tag{6.5}$$

Let us now return to the first equation of (6.1), which can be rewritten as

$$\mu_1-\mu_2=\frac{3}{8}\rho\sin^2\theta\cos^2\beta,$$

where $A_He_i=\mu_ie_i$. Since $\mu_1+\mu_2=2|H|^2$ we have $\mu_1=|H|^2+\frac{3}{16}\rho\sin^2\theta\cos^2\beta$ and $\mu_2=|H|^2-\frac{3}{16}\rho\sin^2\theta\cos^2\beta$. Thus

$$|A_H|^2=\mu_1^2+\mu_2^2=2|H|^4+\frac{9}{128}\rho^2\sin^4\theta\cos^4\beta. \tag{6.6}$$

In the following, we shall assume that the Kähler angle of the surface Σ^2 is constant and then the Laplacian of $|A_H|^2$ is given by

$$\Delta|A_H|^2=\frac{9}{128}\rho^2\sin^4\theta\Delta(\cos^4\beta).$$

In order to compute the Laplacian of $\cos^4\beta$ we need the following formula, obtained by using (6.4) and (6.5),

$$\begin{aligned} d(\cos^4\beta) &= -4\sin\beta\cos^3\beta d\beta = -4\sin\beta\cos^3\beta(f_2\phi+\bar{f}_2\bar{\phi}) \\ &= -4\cos^4\beta\Big\{\Big(if_1+|H|\tfrac{\cot\theta}{\cos\beta}\Big)\phi+\Big(-i\bar{f}_1+|H|\tfrac{\cot\theta}{\cos\beta}\Big)\bar{\phi}\Big\}. \end{aligned}$$

We also have $dd^c(\cos^4\beta)=\frac{i}{2}(\Delta(\cos^4\beta))\phi\wedge\bar{\phi}$ and

$$d^c(\cos^4\beta)=-4i\cos^4\beta\Big\{\Big(-i\bar{f}_1+|H|\frac{\cot\theta}{\cos\beta}\Big)\bar{\phi}-\Big(if_1+|H|\frac{\cot\theta}{\cos\beta}\Big)\phi\Big\}.$$

After a straightforward computation, we get

$$\Delta(\cos^4\beta)=4\cos^4\beta\Big(K+4|f_1|^2+12\Big|if_1+|H|\frac{\cot\theta}{\cos\beta}\Big|^2\Big)$$

and then

$$\Delta|A_H|^2=\frac{9}{32}\rho^2\sin^4\theta\cos^4\beta\Big(K+4|f_1|^2+12\Big|if_1+|H|\frac{\cot\theta}{\cos\beta}\Big|^2\Big).$$

Assume now that Σ^2 is complete and has nonnegative Gaussian curvature. It follows, from a result of A. Huber in [**14**], that Σ^2 is parabolic. Then, from the above formula, we get that $|A_H|^2$ is a subharmonic function, and, since $|A_H|^2$ is bounded (due to (6.6)), it results $K=0$, which, together with the Gauss-Bonnet Theorem, leads to the following non-existence result.

Theorem 6.1. *There are no 2-spheres with nonzero non-umbilical parallel mean curvature vector and constant Kähler angle in a non-flat complex space form.*

References

[1] U. Abresch & H. Rosenberg, *A Hopf differential for constant mean curvature surfaces in* $\mathbb{S}^2\times\mathbb{R}$ *and* $\mathbb{H}^2\times\mathbb{R}$, Acta Math. **193** (2004), 141–174, MR 2134864, Zbl 1078.53053.

[2] U. Abresch & H. Rosenberg, *Generalized Hopf differentials*, Mat. Contemp. **28** (2005), 1–28, MR 2195187, Zbl 1118.53036.

[3] H. Alencar, M. do Carmo & R. Tribuzy, *A theorem of Hopf and the Cauchy–Riemann inequality*, Comm. Anal. Geom. **15** (2007), 283–298, MR 2344324, Zbl 1134.53031.

[4] H. Alencar, M. do Carmo & R. Tribuzy, *A Hopf Theorem for ambient spaces of dimensions higher than three*, J. Differential Geometry **84** (2010), 1–17, MR 2629507, Zbl 1197.53073.

[5] J. Bolton, G. R. Jensen, M. Rigoli & L. M. Woodward, *On conformal minimal immersions of* $\mathbb{S}^2$ *into* $\mathbb{C}P^n$, Math. Ann. **279** (1988), 599–620, MR 0926423, Zbl 0642.53063.

[6] B.–Y. Chen, *Special slant surfaces and a basic inequality*, Results Math. **33** (1998), 65–78, MR 1610123, Zbl 0893.53022.

[7] B.–Y. Chen & K. Ogiue, *On totally real submanifolds*, Trans. Amer. Math. Soc. **193** (1974), 257–266, MR 0346708, Zbl 0286.53019.

[8] S.–S. Chern, *On surfaces of constant mean curvature in a three–dimensional space of constant curvature*, Geometric dynamics (Rio de Janeiro, 1981), Lecture Notes in Math. **1007**, Springer, Berlin, 1983, 104–108, MR 0730266, Zbl 0521.53006.

[9] S.–S. Chern & J. Wolfson, *Minimal surfaces by moving frames*, Amer. J. Math **105** (1983), 59–83, MR 0692106, Zbl 0521.53050.

[10] J. H. Eschenburg, I. V. Guadalupe & R. Tribuzy, *The fundamental equations of minimal surfaces in* $\mathbb{C}P^2$, Math. Ann. **270** (1985), 571–598, MR 0776173, Zbl 0536.53056.

[11] J. H. Eschenburg & R. Tribuzy, *Existence and uniqueness of maps into affine homogeneous spaces*, Rend. Sem. Mat. Univ. Padova **89** (1993), 11–18, MR 1229038, Zbl 0803.53032.

[12] S. Hirakawa, *Constant Gaussian curvature surfaces with parallel mean curvature vector in two-dimensional complex space forms*, Geom. Dedicata **118** (2006), 229–244, MR 2239458, Zbl 1101.53036.

[13] H. Hopf, *Differential geometry in the large*, Lecture Notes in Math. **1000**, Springer-Verlag, 1983, MR 0707850, Zbl 0669.53001.

[14] A. Huber, *On subharmonic functions and differential geometry in the large*, Comment. Math. Helv. **32** (1957), 13–71, MR 0094452, Zbl 0080.15001.

[15] K. Kenmotsu & D. Zhou, *The classification of the surfaces with parallel mean curvature vector in two-dimensional complex space forms*, Amer. J. Math. **122** (2000), 295–317, MR 1749050, Zbl 0997.53006.

[16] T. Ogata, *Surfaces with parallel mean curvature vector in* $P^2(C)$, Kodai Math. J. **18** (1995), 397–407, MR 1362916, Zbl 0865.53007.

[17] T. Ogata, *Curvature pinching theorem for minimal surfaces with constant Kähler angle in complex projective spaces*, Tôhoku Math. J. **43** (1991), 361–374, MR 1117210, Zbl 0739.53048.

[18] Y. Ohnita, *Minimal surfaces with constant curvature and Kähler angle in complex space forms*, Tsukuba J. Math. **13** (1989), 191–207, MR 1003602, Zbl 0678.53055.

[19] N. Sato, *Totally real submanifolds of a complex space form with nonzero parallel mean curvature vector*, Yokohama Math. J. **44** (1997), 1–4, MR 1453345, Zbl 0879.53036.

[20] F. Torralbo & F. Urbano, *Surfaces with parallel mean curvature vector in* $\mathbb{S}^2\times\mathbb{S}^2$ *and* $\mathbb{H}^2\times\mathbb{H}^2$, Trans. Amer. Math. Soc. **364** (2012), 785–813.

[21] S.–T. Yau, *Submanifolds with constant mean curvature*, Amer. J. Math. **96** (1974), 346–366, MR 0370443, Zbl 0304.53041.

Gheorghe Asachi Technical University of Iasi
Department of Mathematics and Informatics
11 Carol I Blvd., 700506 Iasi, Romania
E-mail address: dfetcu@math.tuiasi.ro