J. DIFFERENTIAL GEOMETRY
89 (2011) 529-551

A DEFORMATION OF PENNER'S SIMPLICIAL COORDINATE

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Abstract

We find a one-parameter family of coordinates $\{\Psi_h\}_{h\in\mathbb{R}}$ which is a deformation of Penner's simplicial coordinate of the decorated Teichmüller space of an ideally triangulated punctured surface (S,T) of negative Euler characteristic. If $h \geqslant 0$, the decorated Teichmüller space in the Ψ_h coordinate becomes an explicit convex polytope $P(T)$ independent of h, and if $h < 0$, the decorated Teichmüller space becomes an explicit bounded convex polytope $P_h(T)$ so that $P_h(T) \subset P_{h'}(T)$ if $h < h'$. As a consequence, Bowditch-Epstein and Penner's cell decomposition of the decorated Teichmüller space is reproduced.

1. Introduction

Decorated Teichmüller space of a punctured surface was introduced by Penner in [15] as a fiber bundle over the Teichmüller space of complete hyperbolic metrics with cusp ends. He also gave a cell decomposition of the decorated Teichmüller space invariant under the mapping class group action. To give the cell decomposition, Penner used the convex hull construction and introduced the simplicial coordinate Ψ in which the cells can be easily described. In [4], Bowditch-Epstein obtained the same cell decomposition using the Delaunay construction. The corresponding results for the Teichmüller space of a surface with geodesic boundary have also been obtained. Using Penner's convex hull construction, Ushijima [19] found a mapping class group invariant cell decomposition, and following the approach of Bowditch-Epstein [4], Hazel [10] obtained a natural cell decomposition of the Teichmüller space of a surface with fixed geodesic boundary lengths. As a counterpart of Penner's simplical coordinate Ψ, Luo [12] introduced a coordinate Ψ_0 on the Teichmüller space of an ideally triangulated surface with geodesic boundary, and Mondello [14] pointed out that the Ψ_0 coordinate gave a natural cell decomposition of the Teichmüller space.

Part of the work is supported by an NSF research fellowship.
Received 04/16/2011.

In [13], Luo deformed the Ψ_0 coordinate to a one-parameter family of coordinates $\{\Psi_h\}_{h\in\mathbb{R}}$ of the Teichmüller space of a surface with geodesic boundary, and proved that, for $h \geqslant 0$, the image of Ψ_h is an explicit open convex polytope independent of h. For $h < 0$, Guo [6] proved that the image of Ψ_h is an explicit bounded open polytope. It is then a natural question to ask if there is a corresponding deformation of Penner's simplicial coordinate Ψ. The purpose of this paper is to provide an affirmative answer to this question. We give a one-parameter family of coordinates $\{\Psi_h\}_{h\in\mathbb{R}}$ of the decorated Teichmüller space of an ideally triangulated punctured surface so that Ψ_0 coincides with Penner's simplicial coordinate Ψ (Theorem 1.1). We also describe the image of Ψ_h (Theorem 1.2) and show that Ψ_h is the unique possible deformation of Ψ (Theorem 5.1). As an application, Bowditch-Epstein and Penner's cell decomposition of the decorated Teichmüller space is reproduced using the Ψ_h coordinate (Corollary 1.4). The main results of this paper can be considered as a counterpart of the work of [6], [13] and [8].

To be precise, let $\overline{T}$ be a triangulation of a closed surface $\overline{S}$ and let V, E and F be the set of vertices, edges and triangles of $\overline{T}$ respectively. We call $T = \{\sigma - V \mid \sigma \in F\}$ an ideal triangulation of the punctured surface $S = \overline{S} - V$, and V the set of ideal vertices (or cusps) of S. As a convention in this paper, S is assumed to have negative Euler characteristic. Let $T_c(S)$ be the Teichmüller space of complete hyperbolic metrics with cusp ends on S. According to Penner [15], a *decorated hyperbolic metric* $(d, r) \in T_c(S) \times \mathbb{R}^V_{>0}$ on S is the equivalence class of a hyperbolic metric d in $T_c(S)$ such that each cusp v is associated with a horodisk B_v centered at v so that the length of ∂B_v is r_v. The space of decorated hyperbolic metrics $T_c(S) \times \mathbb{R}^V_{>0}$ is the *decorated Teichmüller space.*

Let us recall Penner's simplicial coordinate Ψ. Let $(d, r) \in T_c(S) \times \mathbb{R}^V_{>0}$ be a decorated hyperbolic metric and let e be an edge of T. If a and a' are the generalized angles (see Section 2) facing e, and b, b', c and c' are the generalized angles adjacent to e, then Penner's simplicial coordinate $\Psi\colon\ T_c(S) \times \mathbb{R}^V_{>0} \to \mathbb{R}^E$ is defined by

$$\Psi(d, r)(e) = \frac{b + c - a}{2} + \frac{b' + c' - a'}{2}.$$

An edge path $(t_0, e_1, t_1, \ldots, e_n, t_n)$ in a triangulation T is an alternating sequence of edges e_i with $e_i \neq e_{i+1}$ for $i = 1, \ldots, n-1$ and triangles t_i so that adjacent triangles t_{i-1} and t_i share the same edge e_i for any $i = 1, \ldots, n$. An *edge loop* is an edge path with $t_n = t_0$. A *fundamental edge path* is an edge path so that each edge in the triangulation appears at most once, and a *fundamental edge loop* is an edge loop so that each edge in the triangulation appears at most twice. In [15], Penner proved

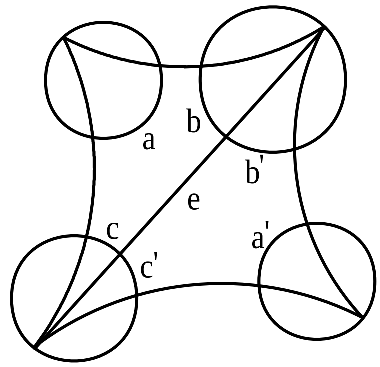

Figure 1. Penner's simplicial coordinate.

that for any vector $z \in \mathbb{R}^E_{\geqslant 0}$ such that $\sum_{i=1}^k z(e_i) > 0$ for any fundamental edge loop $(e_1, t_1, \ldots, e_k, t_k)$, there exists a unique decorated complete hyperbolic metric (d, r) on S so that $\Psi(d, r) = z$. By using a variational principle on decorated ideal triangles, Guo and Luo [7] were able to prove that Penner's simplicial coordinate $\Psi\colon\ T_c(S) \times \mathbb{R}^V_{>0} \to \mathbb{R}^E$ is a smooth embedding with image the convex polytope

$$P(T) = \Big\{ z \in \mathbb{R}^E \mid \sum_{i=1}^k z(e_i) > 0 \text{ for any fundamental edge loop } (e_1, t_1, \ldots, e_k, t_k) \Big\}.$$

Let (S, T) be an ideally triangulated punctured surface. To deform Penner's simplicial coordinate, we define for each $h \in \mathbb{R}$ a map $\Psi_h\colon\ T_c(S) \times \mathbb{R}^V_{>0} \to \mathbb{R}^E$ by

$$\Psi_h(d, r)(e) = \int_0^{\frac{b+c-a}{2}} e^{ht^2} dt + \int_0^{\frac{b'+c'-a'}{2}} e^{ht^2} dt,$$

where a and a' are the generalized angles facing e, and b, b', c and c' are the generalized angles adjacent to e as in Figure 1. The main theorems of this paper are the following

Theorem 1.1. *Suppose that (S, T) is an ideally triangulated punctured surface. Then for all $h \in \mathbb{R}$, the map $\Psi_h\colon\ T_c(S) \times \mathbb{R}^V_{>0} \to \mathbb{R}^E$ is a smooth embedding.*

Theorem 1.2. *For $h \in \mathbb{R}$ and an ideally triangulated punctured surface (S, T), let $P_h(T)$ be the set of points $z \in \mathbb{R}^E$ such that*

(a) $z(e) < 2\int_0^{+\infty} e^{ht^2} dt$ for each edge $e \in E$,

(b) $\sum_{i=1}^n z(e_i) > -2\int_0^{+\infty} e^{ht^2} dt$ for each fundamental edge path $(t_0, e_1, t_1, \ldots, e_n, t_n)$,

(c) $\sum_{i=1}^{n} z(e_i) > 0$ *for each fundamental edge loop* $(e_1, t_1, \ldots, e_n, t_n)$*.*
Then we have $\Psi_h(T_c(S) \times \mathbb{R}^V_{>0}) = P_h(T)$*. Furthermore, if* $h \geqslant 0$*, then conditions (a) and (b) become trivial, and the image of* Ψ_h *is the open convex polytope* $P(T)$*, hence independent of* h*; and if* $h < 0$*, then the image* $P_h(T)$ *is a bounded open convex polytope so that* $P_h(T) \subset P_{h'}(T)$ *if* $h < h'$*.*

Clearly Ψ_0 coincides with Penner's simplicial coordinate Ψ and Ψ_h is a deformation of Ψ. Theorem 1.1 is proved in Section 2 using the strategy of Guo-Luo [7]. We set up a variational principle from the derivative cosine law of decorated ideal triangles whose energy function V_h is strictly concave. For $i = 1, \ldots, |E|$, each variable of V_h is a smooth monotonic function of the edge length l_i in the decorated hyperbolic metric (d, r), and Ψ_h is the gradient of V_h, hence is a smooth embedding. We study various degenerations of decorated ideal triangles in Section 3 with which we will prove Theorem 1.2 in Section 4. We will also prove that $\{\Psi_h\}_{h \in \mathbb{R}}$ is the unique possible deformation of Penner's simplicial coordinate by using a variational principle (Theorem 5.1).

The Delaunay cell decomposition of a decorated hyperbolic surface will be reviewed in Section 6 and we will prove the following

Theorem 1.3. *Suppose* (S, T) *is an ideally triangulated punctured surface, and* $(d, r) \in T_c(S) \times \mathbb{R}^V_{>0}$ *is a decorated hyperbolic metric so that the horodisks associated to the ideal vertices do not intersect. Then for all* $h \in \mathbb{R}$*, the corresponding Delaunay decomposition* $\Sigma_{d,r}$ *coincides with the ideal triangulation* T *if and only if* $\Psi_h(d, r)(e) > 0$ *for each* $e \in E$*.*

Bowditch-Epstein [4] and Penner [15] showed that there is a natural cell decomposition of the decorated Teichmüller space $T_c(S) \times \mathbb{R}^V_{>0}$ invariant under the mapping class group action. One interesting consequence of Theorems 1.1, 1.2 and 1.3 is the following. Let $A(S) - A_\infty(S)$ be the fillable arc complex as in [9], and let $|A(S) - A_\infty(S)|$ be its underlying space. Penner [15] provided a mapping class group equivariant homeomorphism

$$\Pi\colon\ T_c(S) \times \mathbb{R}^V_{>0} \to |A(S) - A_\infty(S)| \times \mathbb{R}_{>0}$$

so that the restriction of Π to each simplex of maximum dimension is given by the simplicial coordinate Ψ. Using Penner's method, we have the following

Corollary 1.4. *Suppose that* S *is a punctured surface of negative Euler characteristic.*
(a) For all $h > 0$*, there is a homeomorphism*

$$\Pi_h\colon\ T_c(S) \times \mathbb{R}^V_{>0} \to |A(S) - A_\infty(S)| \times \mathbb{R}_{>0}$$

equivariant under the mapping class group action so that the restriction of Π_h *to each simplex of maximum dimension is given by the* Ψ_h *coordinate.*

(b) The cell structures for various $h > 0$ *are the same as Penner's.*

Acknowledgments. The author would like to thank Feng Luo for instructive discussions on this subject and suggestions for improving this paper, Shiu-chun Wong for making several crucial suggestions on polishing the writing of this paper, Ren Guo, Julien Roger and Jian Song for useful suggestions and Tianling Jin for helpful discussions. The author is also very grateful to the referee for the carefully reading and making many valuable suggestions on both the mathematics and the writing of this paper.

2. A variational principle on decorated ideal triangles

Let (S, T) be an ideally triangulated punctured surface with a set of ideal vertices V and a set of edges E. We assume that S has negative Euler characteristic. The proof of Theorem 1.1 goes as follows. By Penner [15], there is a smooth parametrization of the decorated Teichmüller space $T_c(S) \times \mathbb{R}^V_{>0}$ by $\mathbb{R}^E$ using the edge lengths. From the cosine law of decorated ideal triangles [15], we construct for each $h \in \mathbb{R}$ a smooth strictly convex function V_h on a convex subset of $\mathbb{R}^E$ so that its gradient is Ψ_h. By a variational principle, for each $h \in \mathbb{R}$, the map $\Psi\colon T_c(S) \times \mathbb{R}^V_{>0} \to R^E$ is a smooth embedding. This variational principle, whose proof is elementary, is: *If* X *is an open convex set in* $\mathbb{R}^n$ *and* $f\colon X \to \mathbb{R}$ *is smooth strictly concave, then the gradient* $\nabla f\colon X \to \mathbb{R}^n$ *is injective. Furthermore, if the Hessian of* f *is negative definite for all* $x \in X$*, then* ∇f *is a smooth embedding.*

A *decorated ideal triangle* Δ in the hyperbolic plane $\mathbb{H}^2$ is an ideal triangle such that each ideal vertex v is associated with a horodisk B_v centered at v. If e_1 and e_2 are two edges adjacent to an ideal vertex v of Δ, then the *generalized angle* of Δ at v is defined to be the length of the intersection of ∂B_v and the cusp region enclosed by e_1 and e_2. (In [15], Penner called the generalized angles the h-lengths of a decorated ideal triangle, and in [7], Guo and Luo defined the generalized angle to be twice of the generalized angle defined here.) If e is an edge of Δ with ideal vertices u and v, then the *generalized edge length* (or *edge length* for simplicity) of e in Δ is the signed hyperbolic distance between the intersection of e and ∂B_u and the intersection of e and ∂B_v (Figure 2 (a)). Note that if $B_u \cap B_v \neq \emptyset$, then the generalized edge length of e is either zero or negative (Figure 2 (b)). In a decorated hyperbolic metric $(d, r) \in T_c(S) \times \mathbb{R}^V_{>0}$, each triangle σ in T is isometric to an ideal triangle and the decoration $r \in \mathbb{R}^V_{>0}$ induces a decoration on σ. If $e \in E$ is an edge and σ is an ideal triangle adjacent to e, then the *generalized* *edge length* $l_{d,r}(e)$ of e is defined to be the generalized edge length of e in σ. It is clear that $l_{d,r}(e)$ does not depend on the choice of σ.

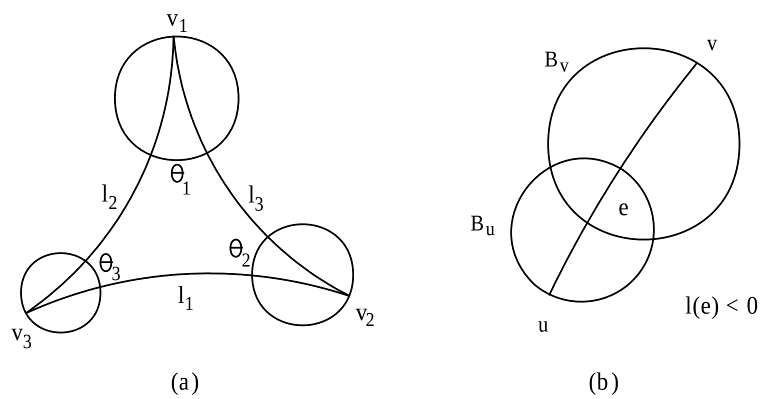

Figure 2. Generalized angles and edge lengths.

Penner [15] defined the length parametrization

$$\begin{aligned} L\colon\ T_c(S)\times\mathbb{R}^V_{>0} &\to \mathbb{R}^E \\ (d,r) &\mapsto l_{d,r} \end{aligned}$$

and showed that L is a diffeomorphism. (The exponential of half of the generalized edge length, which is called the λ-length in [15], is sometimes called Penner's coordinate in the literature.) Penner also proved the following cosine law of decorated ideal triangles. Suppose that Δ is a decorated ideal triangle with edge lengths l_1, l_2 and l_3 and opposite generalized angles θ_1, θ_2 and θ_3. For $i,j,k=1,2,3$,

(1) $$\theta_i = e^{\frac{l_i-l_j-l_k}{2}} \quad\text{and}\quad e^{l_i}=\frac{1}{\theta_j\theta_k}.$$

As a consequence, there is the sine law of decorated triangles:

(2) $$\frac{\theta_1}{e^{l_1}}=\frac{\theta_2}{e^{l_2}}=\frac{\theta_3}{e^{l_3}}.$$

For $i,j,k=1,2,3$ and $x_i=\frac{\theta_j+\theta_k-\theta_i}{2}$, let $\mu(x_i)=\int_0^{x_i}e^{ht^2}dt$ and $u_i=\int_0^{l_i}e^{-he^{-t}}dt$. Denote by $U\subset\mathbb{R}^3$ the set of all possible values of $u=(u_1,u_2,u_3)$.

Lemma 2.1. *For each $h\in\mathbb{R}$, the differential 1-form $\omega_h=\sum_{i=1}^3\mu(x_i)du_i$ is closed in U and the function F_h defined by the integral $F_h(u)=\int_0^u\omega_h$ is strictly concave in U. Furthermore,*

$$\frac{\partial F_h}{\partial u_i}=\int_0^{x_i}e^{ht^2}dt.$$

Proof. Consider the matrix $H = [\frac{\partial \mu(x_i)}{\partial u_j}]_{3\times 3}$. The closedness of ω_h is equivalent to that H is symmetric, and the strict concavity of F_h will follow from the negative definiteness of H. It follows from the partial derivatives of (1) that $\frac{\partial x_i}{\partial l_i} = -\frac{x_i+x_j+x_k}{2}$ and $\frac{\partial x_i}{\partial l_j} = \frac{x_k}{2}$. We have

$$\frac{\partial \mu(x_i)}{\partial u_i} = \frac{e^{hx_i^2}}{e^{-he^{-l_i}}}\frac{\partial x_i}{\partial l_i} = -\frac{x_i+x_j+x_k}{2}e^{h\left(\frac{\theta_i^2+\theta_j^2+\theta_k^2}{4}+\frac{3\theta_j\theta_k-\theta_i\theta_k-\theta_i\theta_j}{2}\right)},$$

and for $i \neq j$, we have

$$\frac{\partial \mu(x_i)}{\partial u_j} = \frac{e^{hx_i^2}}{e^{-he^{-l_j}}}\frac{\partial x_i}{\partial l_j} = \frac{x_k}{2}e^{h\left(\frac{\theta_i^2+\theta_j^2+\theta_k^2}{4}+\frac{\theta_j\theta_k+\theta_i\theta_k-\theta_i\theta_j}{2}\right)},$$

from which we see that H is symmetric. Let

$$c = \frac{1}{2}e^{h\left(\frac{\theta_i^2+\theta_j^2+\theta_k^2}{4}-\frac{\theta_j\theta_k+\theta_i\theta_k+\theta_i\theta_j}{2}\right)} > 0$$

and let D be the diagonal matrix whose (i,i)-th entry is $e^{h\theta_j\theta_k}$. The matrix H can be written as $cDMD$, where

$$M = \begin{bmatrix} -(x_1+x_2+x_3) & x_3 & x_2 \\ x_3 & -(x_1+x_2+x_3) & x_1 \\ x_2 & x_1 & -(x_1+x_2+x_3) \end{bmatrix}.$$

The negative definiteness of H is equivalent to that of M, i.e., the positive definiteness of $-M$. This follows from the direct calculation that each leading principal minor is positive using Sylvester's criterion. q.e.d

Proof of Theorem 1.1. For a decorated hyperbolic metric $(d,r) \in T_c(S) \times \mathbb{R}^V_{>0}$, let $l_{d,r} \in \mathbb{R}^E$ be its length parameter. The integral $u(e) = \int_0^{l_{d,r}(e)} e^{-he^{-t}}dt$ is a smooth monotonic function of $l_{d,r}(e)$, and the possible values of u form an open convex cube U in $\mathbb{R}^E$. With $u_i = u(e_i)$, the energy function $V_h\colon U \to \mathbb{R}$ is defined by

$$V_h(u) = \sum_{\{e_i,e_j,e_k\}} F_h(u_i,u_j,u_k),$$

in which the summation is taken over all of the decorated ideal triangles. By Lemma 2.1, V_h is smooth and strictly concave in U and

$$\frac{\partial V_h}{\partial u_i} = \Psi_h(e_i),$$

i.e., $\nabla V_h = \Psi_h$. By the variational principle, the map $\Psi_h = \nabla V_h\colon U \to \mathbb{R}^E$ is a smooth embedding. q.e.d

3. Degenerations of decorated ideal triangles

To describe the image of Ψ_h, we study degenerations of decorated ideal triangles. Suppose Δ is a decorated ideal triangle with edge lengths l_1, l_2 and l_3 and opposite generalized angles θ_1, θ_2 and θ_3.

Lemma 3.1.

(I) If $\{(l_1, l_2, l_3)\}$ converges to $(-\infty, c_2, c_3)$ with $c_2, c_3 \in (-\infty, +\infty]$, then $\{\theta_1\}$ converges to 0, and we can take a subsequence so that at least one of $\{\theta_2\}$ and $\{\theta_3\}$ converges to $+\infty$.

(II) If $\{(l_1, l_2, l_3)\}$ converges to $(-\infty, -\infty, c_3)$ with $c_3 \in (-\infty, +\infty]$, then $\{\theta_3\}$ converges to $+\infty$, and we can take a subsequence so that at least one of $\{\theta_1\}$ and $\{\theta_2\}$ converges to a finite number.

(III) If $\{(l_1, l_2, l_3)\}$ converges to $(-\infty, -\infty, -\infty)$, then we can take a subsequence such that at least two of $\{\theta_1\}$, $\{\theta_2\}$ and $\{\theta_3\}$ converge to $+\infty$.

Proof. For (I), if $\{(l_1, l_2, l_3)\}$ converges to $(-\infty, c_2, c_3)$, then $\{\frac{l_1-l_2-l_3}{2}\}$ converges to $-\infty$. By cosine law (1), $\{\theta_1\} = \{e^{\frac{l_1-l_2-l_3}{2}}\}$ converges to 0. Let $a_2 = \frac{l_2-l_1-l_3}{2}$ and $a_3 = \frac{l_3-l_1-l_2}{2}$, so $\{a_2 + a_3\} = \{-l_1\}$ converges to $+\infty$. Thus, by taking a subsequence if necessary, at least one of $\{a_2\}$ and $\{a_3\}$, say $\{a_2\}$, converges to $+\infty$, and $\{\theta_2\} = \{e^{a_2}\}$ converges to $+\infty$. For (II), if $\{(l_1, l_2, l_3)\}$ converges to $(-\infty, -\infty, c_3)$, then $\{\frac{l_3-l_1-l_2}{2}\}$ converges to $+\infty$, and $\{\theta_3\} = \{e^{\frac{l_3-l_1-l_2}{2}}\}$ converges to $+\infty$. Letting $a_1 = \frac{l_1-l_2-l_3}{2}$ and $a_2 = \frac{l_2-l_1-l_3}{2}$, we have $\{a_1 + a_2\} = \{-l_3\}$ converges to $-c_3$. Thus, either both $\{a_1\}$ and $\{a_2\}$ converge to a finite number, or by taking a subsequence if necessary, at least one of $\{a_1\}$ and $\{a_2\}$, say $\{a_1\}$, converges to $-\infty$. In the former case, both $\{\theta_1\} = \{e^{a_1}\}$ and $\{\theta_2\} = \{e^{a_2}\}$ converge to a finite number, and in the latter case, $\{\theta_1\} = \{e^{a_1}\}$ converges to 0. For (III), we have by cosine law (1) that $\{\theta_1\theta_2\} = \{e^{-l_3}\}$ converges to $+\infty$. Thus, by taking a subsequence if necessary, at least one of $\{\theta_1\}$ and $\{\theta_2\}$, say $\{\theta_1\}$, converges to $+\infty$. Since $\{\theta_2\theta_3\} = \{e^{-l_1}\}$ converges to $+\infty$ as well, by taking a subsequence, at least one of $\{\theta_2\}$ and $\{\theta_3\}$ converges to $+\infty$. q.e.d

We call a converging sequence of decorated ideal triangles in (I), (II) and (III) of Lemma 3.1 a *degenerated decorated ideal triangle of type I, II and III* respectively. If a is the generalized angle facing an edge e in a decorated triangle Δ, and b and c are the generalized angles adjacent to e, then we call $x(e) = \frac{b+c-a}{2}$ the *x-invariant* of e in Δ.

Corollary 3.2. *If Δ is a degenerated decorated ideal triangle of type I, II or III, then by taking a subsequence if necessary, there is an edge e of Δ such that $\{l(e)\}$ converges to $-\infty$ and $\{x(e)\}$ converges to $+\infty$.*

Proof. If Δ is of type I and $\{l_1\}$ converges to $-\infty$, then by Lemma 3.1 (I), $\{x_1\} = \{\frac{\theta_2+\theta_3-\theta_1}{2}\}$ converges to $+\infty$. If Δ is of type II and $\{(l_1, l_2, l_3)\}$ converges to $(-\infty, -\infty, c_3)$, then by Lemma 3.1 and taking a subsequence if necessary, at least one of $\{\theta_1\}$ and $\{\theta_2\}$, say $\{\theta_1\}$, converges to a finite number, and $\{\theta_3\}$ converges to $+\infty$. Thus, $\{l_1\}$ converges to $-\infty$ and $\{x_1\} = \{\frac{\theta_2+\theta_3-\theta_1}{2}\}$ converges to $+\infty$. If Δ is of type III, then there are at least two of $\{\theta_1\}$, $\{\theta_2\}$ and $\{\theta_3\}$ that converge to $+\infty$. Suppose $\{\theta_3\}$ is one of the two that converge to $+\infty$. Since $\{x_1+x_2\} = \{\theta_3\}$ converges to $+\infty$, by taking a subsequence if necessary, at least one of $\{x_1\}$ and $\{x_2\}$, say $\{x_1\}$, converges to $+\infty$. Thus, $\{l_1\}$ converges to $-\infty$ and $\{x_1\}$ converges to $+\infty$. q.e.d

We call an edge e as in Corollary 3.2 where $l(e) \to -\infty$ and $x(e) \to +\infty$ a *bad edge* of Δ, and otherwise, e is a *good edge*. Note that there may be more than one bad edge in a degenerated ideal triangle.

Lemma 3.3. *Let $\{\Delta^{(m)}\}$ be a sequence of decorated ideal triangles that converges to a degenerated decorated ideal triangle Δ of type I, II or III. Then we can take a subsequence so that for m sufficiently large, the length of each bad edge of $\Delta^{(m)}$ is strictly less than the length of each good edge.*

Proof. If Δ is of type I, then by Lemma 3.1, the length of the only bad edge converges to $-\infty$ and the length of other two edges converge to a finite number. For m sufficiently large, the length of the bad edge is less than the lengths of the good edges.

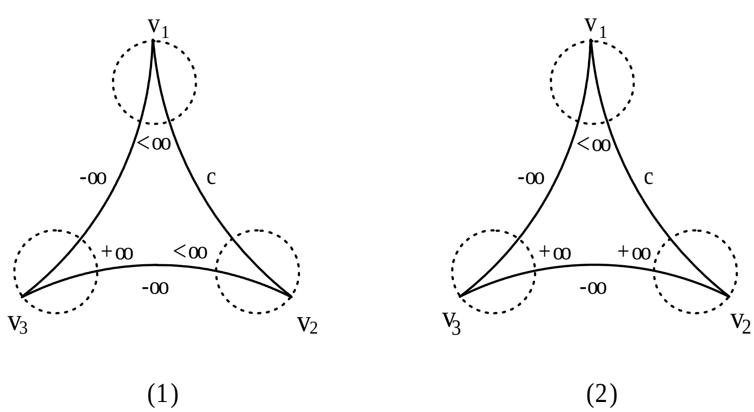

Figure 3. Type II.

If Δ is of type II, we may assume that $\{(l_1^{(m)}, l_2^{(m)}, l_3^{(m)})\}$ converges to $(-\infty, -\infty, c)$ with $c \in (-\infty, +\infty]$. By Lemma 3.1, there are two cases to be considered (Figure 3).

Case 1. Suppose that $\theta_3^{(m)}$ converges to $+\infty$ and both $\theta_1^{(m)}$ and $\theta_2^{(m)}$ converge to a finite number. In this case, both l_1 and l_2 are bad and converge to $-\infty$. The only good edge length l_3 converges to $c \in (-\infty, +\infty]$. Hence for m sufficiently large, $l_1^{(m)} < l_3^{(m)}$ and $l_2^{(m)} < l_3^{(m)}$.

Case 2. Suppose that $\theta_3^{(m)}$ converges to $+\infty$, and one of $\theta_1^{(m)}$ and $\theta_2^{(m)}$, say $\theta_2^{(m)}$, converges to $+\infty$ and $\theta_1^{(m)}$ converges to a finite number. In this case l_1 is bad. If l_2 is also bad, then both l_1 and l_2 converge to $-\infty$, and l_3 converges to $c \in (-\infty, +\infty]$. Hence for m sufficiently large, $l_1^{(m)} < l_3^{(m)}$ and $l_2^{(m)} < l_3^{(m)}$. If l_2 is good, then $\theta_1^{(m)} < \theta_2^{(m)}$ for m sufficiently large, since $\theta_1^{(m)}$ converges to a finite number and $\theta_2^{(m)}$ converges to $+\infty$. By sine law (2), $l_1^{(m)} < l_2^{(m)}$.

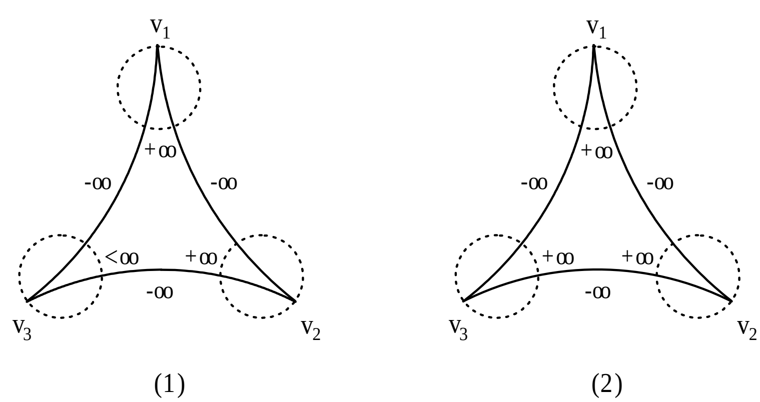

Figure 4. Type III.

If Δ is of type III, then by Lemma 3.1, we also consider two cases (Figure 4).

Case 1. Two of $\theta_1^{(m)}$, $\theta_2^{(m)}$ and $\theta_3^{(m)}$, say $\theta_1^{(m)}$ and $\theta_2^{(m)}$ converge to $+\infty$, and $\theta_3^{(m)}$ converges to a finite number. In this case, l_3 is bad. Since $\theta_3^{(m)} < \theta_1^{(m)}$ and $\theta_3^{(m)} < \theta_2^{(m)}$ for m sufficiently large, by sine law (2), $l_3^{(m)} < l_1^{(m)}$ and $l_3^{(m)} < l_2^{(m)}$. If one of l_1 and l_2, say l_2, is also bad, then $x_2^{(m)} = \frac{\theta_1^{(m)}+\theta_3^{(m)}-\theta_2^{(m)}}{2}$ converges to $+\infty$. Since $\theta_3^{(m)}$ converges to a finite number, $\theta_2^{(m)} < \theta_1^{(m)}$ for m sufficiently large. By sine law (2), $l_2^{(m)} < l_1^{(m)}$.

Case 2. All of $\theta_1^{(m)}$, $\theta_2^{(m)}$ and $\theta_3^{(m)}$ converge to $+\infty$. In this case, since $x_i^{(m)} + x_j^{(m)} = \theta_k^{(m)}$ converges to $+\infty$, by taking a subsequence if necessary, at least two of $x_1^{(m)}$, $x_2^{(m)}$ and $x_3^{(m)}$, say $x_1^{(m)}$ and $x_2^{(m)}$, converge to $+\infty$. Therefore, l_3 is the only possible good edge length, and $x_3^{(m)}$ converges to a finite number. For m sufficiently large, $\theta_1^{(m)} = x_2^{(m)} + x_3^{(m)} < x_1^{(m)} + x_2^{(m)} = \theta_3^{(m)}$ and $\theta_2^{(m)} = x_1^{(m)} + x_3^{(m)} < x_1^{(m)} + x_2^{(m)} = \theta_3^{(m)}$. By sine law (2), $l_1^{(m)} < l_3^{(m)}$ and $l_2^{(m)} < l_3^{(m)}$. q.e.d

Lemma 3.4.

(a) If $\{(l_1, l_2, l_3)\}$ converges to $(+\infty, f_2, f_3)$ with $f_2, f_3 \in \mathbb{R}$, then $\{(\theta_1, \theta_2, \theta_3)\}$ converges to $(+\infty, 0, 0)$.

(b) If $\{(l_1, l_2, l_3)\}$ converges to $(+\infty, +\infty, f_3)$ with $f_3 \in \mathbb{R}$, then $\{\theta_3\}$ converges to 0.

(c) If $\{(l_1, l_2, l_3)\}$ converges to $(+\infty, +\infty, +\infty)$, then we can take a subsequence such that at least two of $\{\theta_1\}$, $\{\theta_2\}$ and $\{\theta_3\}$ converge to 0.

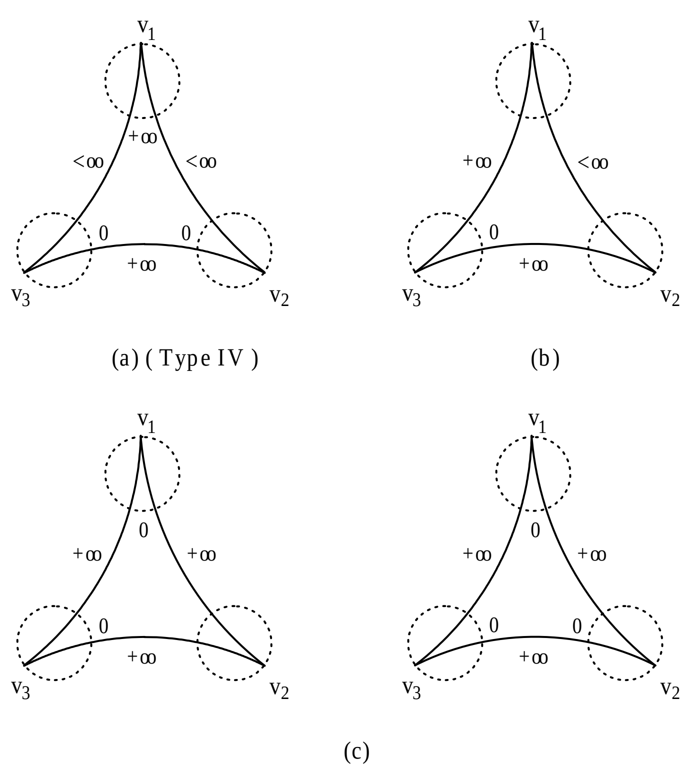

Figure 5. Type IV and other types.

We call a converging sequence of decorated ideal triangles in (a) of Lemma 3.4 a *degenerated decorated ideal triangle of type IV* (Figure 5).

Proof. For (a), if $\{(l_1,l_2,l_3)\}$ converges to $(+\infty,f_2,f_3)$, then by cosine law (1), $\{\theta_1\}=\{e^{\frac{l_1-l_2-l_3}{2}}\}$ converges to $+\infty$, $\{\theta_2\}=\{e^{\frac{l_2-l_1-l_3}{2}}\}$ converges to 0, and $\{\theta_3\}=\{e^{\frac{l_3-l_1-l_2}{2}}\}$ converges to 0. For (b), if $\{(l_1,l_2,l_3)\}$ converges to $(+\infty,+\infty,f_3)$, then $\{\frac{l_3-l_1-l_2}{2}\}$ converges to $-\infty$, and $\{\theta_3\}=\{e^{\frac{l_3-l_1-l_2}{2}}\}$ converges to 0. For (c), if $\{(l_1,l_2,l_3)\}$ converges to $(+\infty,+\infty,+\infty)$, then we have by cosine law (1) that $\{\theta_1\theta_2\}=\{e^{-l_3}\}$ converges to 0. Thus, by taking a subsequence if necessary, at least one of $\{\theta_1\}$ and $\{\theta_2\}$, say $\{\theta_1\}$, converges to 0. Since $\{\theta_2\theta_3\}=\{e^{-l_1}\}$ converges to 0 as well, by taking a subsequence, at least one of $\{\theta_2\}$ and $\{\theta_3\}$ converges to 0. q.e.d

4. The image of Ψ_h

The image of Ψ_h is described in Theorem 1.2. The main task of this section is to give a proof of this theorem. To show that the image of Ψ_h is indeed $P_h(T)$, we make use of the following propositions which are proved in this section.

Proposition 4.1. *$\Psi_h(T_c(S)\times\mathbb{R}^V_{>0})\subset P_h(T)$ for all $h\in\mathbb{R}$.*

Proposition 4.2. *For all $h\in\mathbb{R}$, the image $\Psi_h(T_c(S)\times\mathbb{R}^V_{>0})$ is closed in $P_h(T)$.*

Proof of Theorem 1.2. Let $P(T)$ be defined as in Theorem 1.2. For $h\geqslant 0$, $P(T)=P_h(T)$ is determined by finitely many strict linear inequalities corresponding to the fundamental edge loops and hence is an open convex polytope independent of h. For $h<0$, $P_h(T)$ is likewise determined by fundamental edge loops and fundamental edge paths. Moreover, since each edge e can be regarded as a fundamental edge path, conditions (a) and (b) imply that $-2\int_0^{+\infty}e^{ht^2}dt<z(e)<2\int_0^{+\infty}e^{ht^2}dt$ for each $e\in E$. Thus, $P_h(T)$ is bounded. The monotonicity of the function $f(h)=\int_0^{+\infty}e^{ht^2}dt$ implies that $P_h(T)\subset P_{h'}(T)$ if $h<h'$, and the fact that $\lim_{h\to-\infty}f(h)=\lim_{h\to-\infty}\sqrt{\frac{\pi}{-2h}}=0$ implies that $\bigcap_{h\in\mathbb{R}_{<0}}P_h(T)=\emptyset$. By Theorem 1.1 and the Invariance of Domain Theorem, $\Psi_h(T_c(S)\times\mathbb{R}^V_{>0})$ is open in $P_h(T)$. By Proposition 4.2, $\Psi_h(T_c(S)\times\mathbb{R}^V_{>0})$ is closed in $P_h(T)$. Connectedness of $P_h(T)$ therefore implies that $\Psi_h(T_c(S)\times\mathbb{R}^V_{>0})=P_h(T)$. q.e.d

The following Lemma 4.3 will be used in the proof of Propositions 4.1 and 4.2.

Lemma 4.3. *If $r\in\mathbb{R}$ and $x>0$, then* *(a) for each* $h \in \mathbb{R}$,

$$\int_0^{x+r} e^{ht^2} dt + \int_0^{x-r} e^{ht^2} dt > 0,$$

(b) for each $h \geqslant 0$,

$$\int_0^{x+r} e^{ht^2} dt + \int_0^{x-r} e^{ht^2} dt \geqslant 2 \int_0^{x} e^{ht^2} dt.$$

Proof. For (a), let $f(x) = \int_0^{x+r} e^{ht^2} dt + \int_0^{x-r} e^{ht^2} dt$. Since $f'(x) = e^{h(x+r)^2} + e^{h(x-r)^2} > 0$, the function f is strictly increasing, hence $f(x) > f(0) = 0$ for $x > 0$. For (b), let $g(x) = \int_0^{x+r} e^{ht^2} dt + \int_0^{x-r} e^{ht^2} dt - 2\int_0^{x} e^{ht^2} dt$. We have that $g(0) = 0$ and $g'(x) = e^{h(x+r)^2} + e^{h(x-r)^2} - 2e^{hx^2} \geqslant 0$. The last inequality follows from the convexity of the function $F(t) = e^{ht^2}$ for $h \geqslant 0$. Since g is increasing, $g(x) \geqslant g(0) = 0$ for $x > 0$. q.e.d

Proof of Proposition 4.1. For $h \geqslant 0$, fix a decorated hyperbolic metric $(d, r) \in T_c(S) \times \mathbb{R}^V_{>0}$. For any fundamental edge loop $(e_1, t_1, \ldots, e_k, t_k)$, let a_i be the generalized angle adjacent to e_i and e_{i+1} (where $e_{k+1} = e_1$). Let the generalized angles of t_i facing e_i and e_{i+1} respectively be b_i and c_i. By definition, the contribution of $\sum_{i=1}^{k} z(e_i)$ from t_i is

$$\int_0^{\frac{a_i+b_i-c_i}{2}} e^{ht^2} dt + \int_0^{\frac{a_i+c_i-b_i}{2}} e^{ht^2} dt,$$

which is strictly larger than 0 from Lemma 4.3 (a) since $a_i > 0$.

For $h < 0$, let e be any edge in the ideal triangulation T, and let a and a' be the generalized angles facing e. Let b, c, b' and c' be the generalized angles adjacent to e. Then

$$\Psi_h(d, r)(e) = \int_0^{\frac{b+c-a}{2}} e^{ht^2} dt + \int_0^{\frac{b'+c'-a'}{2}} e^{ht^2} dt < 2 \int_0^{+\infty} e^{ht^2} dt.$$

Thus, condition (a) in the definition of $P_h(T)$ is satisfied. Given a fundamental edge path $(t_0, e_0, t_1, \ldots, e_n, t_n)$, let θ_i be the generalized angle in t_i adjacent to e_i and e_{i+1} for $i = 1, \ldots, n-1$, and let β_i and γ_i respectively be the generalized angles of t_i facing e_i and e_{i+1}. Denote by a_0 the generalized angle of t_0 facing e_0, and by a_n the generalized angle of t_n facing e_n. Let b_0 and c_0 be the generalized angles of t_0 adjacent to e_0, and let b_n and c_n be the generalized angles of t_n adjacent to e_n. We have

$$
\begin{aligned}
&\sum_{i=1}^{n}\Psi_h(d,r)(e_i)\\
&=\int_0^{\frac{b_0+c_0-a_0}{2}} e^{ht^2}dt+\sum_{i=1}^{n-1}\left(\int_0^{\frac{\theta_i+\gamma_i-\beta_i}{2}} e^{ht^2}dt+\int_0^{\frac{\theta_i+\beta_i-\gamma_i}{2}} e^{ht^2}dt\right)\\
&\quad+\int_0^{\frac{b_n+c_n-a_n}{2}} e^{ht^2}dt\\
&>\int_0^{\frac{b_0+c_0-a_0}{2}} e^{ht^2}dt+\int_0^{\frac{b_n+c_n-a_n}{2}} e^{ht^2}dt\\
&>-2\int_0^{+\infty} e^{ht^2}dt,
\end{aligned}
$$

where the first inequality is by Lemma 4.3 (a). Thus, condition (b) is satisfied. Given a fundamental edge loop $(e_1, t_1, \ldots, e_n, t_n)$ with $e_{n+1} = e_1$, let θ_i for $i = 1, \ldots, n$ be the generalized angle in t_i adjacent to e_i and e_{i+1}, and let β_i (resp. γ_i) be the generalized angle in t_i facing e_i (resp. e_{i+1}). Again by Lemma 4.3 (a),

$$
\sum_{i=1}^{n}\Psi_h(d,r)(e_i)=\sum_{i=1}^{n}\left(\int_0^{\frac{\theta_i+\gamma_i-\beta_i}{2}} e^{ht^2}dt+\int_0^{\frac{\theta_i+\beta_i-\gamma_i}{2}} e^{ht^2}dt\right)>0.
$$

Thus, condition (c) is satisfied, and $\Psi_h(T_c(S)\times\mathbb{R}^V_{>0})\subset P_h(T)$. q.e.d

To prove Proposition 4.2, we use Penner's length parametrization. For each sequence $\{l^{(m)}\}$ in $\mathbb{R}^E$ such that $\{\Psi_h(l^{(m)})\}$ converges to a point $z \in P(T)$, we claim that $\{l^{(m)}\}$ contains a subsequence converging to a point in $\mathbb{R}^E$. Let $\theta^{(m)}$ be the generalized angles of the decorated ideal triangles in (S, T) in the decorated hyperbolic metric $l^{(m)}$. By taking a subsequence if necessary, we may assume that $\{l^{(m)}\}$ converges in $[-\infty, +\infty]^E$ and that for each generalized angle θ_i, the limit $\lim_{m\to\infty}\theta_i^{(m)}$ exists in $[0, +\infty]$. In the case that $h \geqslant 0$, we need the following

Lemma 4.4. *If $h \geqslant 0$, then $\lim_{m\to\infty}\theta_i^{(m)} \in [0, +\infty)$ for all i.*

Proof. Suppose to the contrary that $\lim_{m\to\infty}\theta_1^{(m)} = +\infty$ for some generalized angle θ_1. Let e_2 and e_3 be the edges adjacent to θ_1 in the triangle t_1, and θ_2 and θ_3 respectively be the generalized angles facing e_2 and e_3. Take a fundamental edge loop $(e_{n_1}, t_{n_1}, \ldots, e_{n_k}, t_{n_k})$ containing (e_2, t_1, e_3). By Lemma 4.3, we have

$$\begin{aligned}\sum_{i=1}^{k} z(e_{n_i}) &= \lim_{m\to\infty}\sum_{i=1}^{k}\Psi_h(l^{(m)})(e_{n_i})\\ &\geqslant \lim_{m\to\infty}\left(\int_0^{\frac{\theta_1^{(m)}+\theta_2^{(m)}-\theta_3^{(m)}}{2}} e^{ht^2}dt + \int_0^{\frac{\theta_1^{(m)}+\theta_3^{(m)}-\theta_2^{(m)}}{2}} e^{ht^2}dt\right)\\ &\geqslant \lim_{m\to\infty} 2\int_0^{\frac{\theta_1^{(m)}}{2}} e^{ht^2}dt\\ &= +\infty.\end{aligned}$$

This contradicts the assumption that $z \in P(T)$. q.e.d

Proof of Proposition 4.2. For $h \geqslant 0$, by taking a subsequence of $\{l^{(m)}\}$, we may assume that $\lim_{m\to\infty} l^{(m)} = l \in [-\infty, +\infty]^E$. If l were not in $\mathbb{R}^E$, then there would exist an edge $e \in E$ so that $l(e) = \pm\infty$. Let Δ be a decorated ideal triangle adjacent to e, and let $\theta_1^{(m)}$ and $\theta_2^{(m)}$ be the generalized angles in Δ adjacent to e in the metric $l^{(m)}$. By (1),

$$e^{l^{(m)}(e)} = \frac{1}{\theta_1^{(m)}\theta_2^{(m)}},$$

and $\theta_i^{(m)} \in (0, +\infty)$ for $i = 1, 2$.

Case 1 If $l(e) = -\infty$, then $e^{l(e)} = 0$. By the identity above, one of $\lim_{m\to\infty}\theta_i^{(m)}$ for $i = 1, 2$ must be $+\infty$. This contradicts Lemma 4.4.

Case 2 If $l(e) = +\infty$, then $e^{l(e)} = +\infty$. By the identity above, one of $\lim_{m\to\infty}\theta_i^{(m)}$ for $i = 1, 2$ must be zero. Suppose without loss of generality that $\lim_{m\to\infty}\theta_1^{(m)} = 0$. Let e_1 be the edge in the decorated ideal triangle Δ opposite to θ_2, and let θ_3 be the generalized angle in Δ facing e. By (1), we have

$$e^{l^{(m)}(e_1)} = \frac{1}{\theta_1^{(m)}\theta_3^{(m)}}.$$

By Lemma 4.4, $\theta_3^{(m)}$ is bounded above, hence $l(e_1) = +\infty$. For any decorated ideal triangle Δ adjacent to e with $l(e) = +\infty$, we have an edge e_1 in Δ and a generalized angle θ_1 adjacent to e and e_1 so that $l(e_1) = +\infty$ and $\lim_{m\to\infty}\theta_1^{(m)} = 0$. Applying this logic to e_1 and the decorated ideal triangle Δ_1 adjacent to e_1 other than Δ, we obtain the next angle θ_2 and edge e_2 in Δ_1 so that $l(e_2) = +\infty$ and $\lim_{m\to\infty}\theta_2^{(m)} = 0$. Since there are only finitely many edges and triangles, this yields a fundamental edge loop $(e_k, \Delta_k, \dots, e_n, \Delta_n)$ in T such that $l(e_i) = +\infty$ for $i = k, \dots, n$ and $\lim_{m\to\infty} \theta_i^{(m)} = 0$, where θ_i is the generalized angle in Δ_{i-1} adjacent to e_{i-1} and e_i. Denote respectively by β_i and γ_i the generalized angles of Δ_{i-1} facing e_{i-1} and e_i, and let $\bar\beta_i = \lim_{m\to\infty} \beta_i^{(m)}$ and $\bar\gamma_i = \lim_{m\to\infty} \gamma_i^{(m)}$. By Lemma 4.4, both $\bar\beta_i$ and $\bar\gamma_i$ are finite real numbers, and we have

$$\begin{aligned}\sum_{i=k}^{n} z(e_i) &= \lim_{m\to\infty} \sum_{i=k}^{n} \Psi_h(l^{(m)})(e_i)\\ &= \lim_{m\to\infty} \sum_{i=k}^{n} \left(\int_0^{\frac{\theta_i^{(m)}+\beta_i^{(m)}-\gamma_i^{(m)}}{2}} e^{ht^2} dt + \int_0^{\frac{\theta_i^{(m)}+\gamma_i^{(m)}-\beta_i^{(m)}}{2}} e^{ht^2} dt \right)\\ &= \sum_{i=k}^{n} \left(\int_0^{\frac{\bar\beta_i-\bar\gamma_i}{2}} e^{ht^2} dt + \int_0^{\frac{\bar\gamma_i-\bar\beta_i}{2}} e^{ht^2} dt \right)\\ &= 0.\end{aligned}$$

This contradicts the assumption that $z \in P(T)$.

For $h < 0$ and each sequence $\{l^{(m)}\}$ in $\mathbb{R}^E$ so that $\{\Psi_h(l^{(m)})\}$ converges to a point $z \in P_h(T)$, we claim that $\{l^{(m)}\}$ contains a subsequence converging to a point in $\mathbb{R}^E$. By taking a subsequence if necessary, we may assume that $\{l^{(m)}\}$ converges to $l \in [-\infty, +\infty]^E$. If l were not in $\mathbb{R}^E$, there would exist an edge e so that $l(e) = \pm\infty$.

Case 1. If $l(e) = -\infty$ for some $e \in E$, then there is a degenerated decorated ideal triangle Δ of type I, II or III. By Corollary 3.2, there is a bad edge e_1 in Δ. Let Δ_1 be the other decorated ideal triangle adjacent to e_1, and let x_0 and x_1 respectively be the x-invariants of e_1 in Δ and Δ_1. If e_1 is bad in Δ_1, then

$$\begin{aligned} z(e_1) &= \lim_{m\to\infty} \Psi_h(l^{(m)})(e_1) = \lim_{m\to\infty} \left(\int_0^{x_0^{(m)}} e^{ht^2} dt + \int_0^{x_1^{(m)}} e^{ht^2} dt \right)\\ &= 2\int_0^{+\infty} e^{ht^2} dt,\end{aligned}$$

which contradicts the assumption that $z \in P_h(T)$. Therefore e_1 has to be a good edge in Δ_1. Since $l(e_1) = -\infty$, the decorated triangle Δ_1 is degenerated of type I, II or III. By Corollary 3.2, there is a bad edge e_2 in Δ_1. For the same reason, e_2 has to be good in the other decorated ideal triangle Δ_2 adjacent to e_2, and there is a bad edge e_3 in Δ_2. Serially applying this logic and using that there are finitely many edges, we find an edge loop $(e_k, \Delta_k, \ldots, e_n, \Delta_n)$ with $e_{n+1} = e_k$ so that for each $i = k, \ldots, n$ the edge e_i is good in Δ_i and the edge e_{i+1} is bad in Δ_i. By Lemma 3.3, we can take a subsequence so that $l^{(m)}(e_i) > l^{(m)}(e_{i+1})$ for m sufficiently large. Thus, we have $l^{(m)}(e_k) > l^{(m)}(e_{n+1})$, which contradicts that $e_{n+1} = e_k$.

In light of Case 1, we may assume that $l \in (-\infty, +\infty]^E$.

Case 2. If $l(e) = +\infty$ for some $e \in E$, let Δ_1 be a decorated ideal triangle adjacent to e. If Δ_1 is not of type IV, then by Lemma 3.4, there is an edge e_1 of Δ_1 and an generalized angle θ_1 adjacent to e and e_1 so that $l(e_1) = +\infty$ and $\lim_{m\to\infty} \theta_1^{(m)} = 0$ (see Figure 5). The other decorated ideal triangle Δ_2 adjacent to e_1 is either of type IV or contains an edge e_2 and a generalized angle θ_2 adjacent to e_1 and e_2 so that $l(e_2) = +\infty$ and $\lim_{m\to\infty} \theta_2^{(m)} = 0$. Again, the serial application of this procedure terminates with an edge e_p and a decorated ideal triangle Δ_{p+1} adjacent to e_p so that $l(e_p) = +\infty$ and Δ_{p+1} is of type IV, or since there are only finitely many edges, produces a fundamental edge loop $(e_k, \Delta_k, \ldots, e_n, \Delta_n)$ such that $l(e_i) = +\infty$ for $i = k, \ldots, n$ and $\lim_{m\to\infty} \theta_i^{(m)} = 0$, where θ_i is the generalized angle in Δ_i adjacent to e_i and e_{i+1}. If it yields such a fundamental edge loop $(e_k, \Delta_k, \ldots, e_n, \Delta_n)$, denote by β_i (resp. γ_i) the generalized angle in Δ_i facing e_i (resp. e_{i+1}) for $i = k, \ldots, n$. Let $\bar\beta_i = \lim_{m\to\infty} \beta_i^{(m)}$ and $\bar\gamma_i = \lim_{m\to\infty} \gamma_i^{(m)}$, so that

$$\begin{aligned}\sum_{i=k}^{n} z(e_i) &= \lim_{m\to\infty} \sum_{i=1}^{k} \Psi_h(l^{(m)})(e_i) \\ &= \lim_{m\to\infty} \sum_{i=1}^{k} \left(\int_0^{\frac{\theta_i^{(m)}+\beta_i^{(m)}-\gamma_i^{(m)}}{2}} e^{ht^2} dt + \int_0^{\frac{\theta_i^{(m)}+\gamma_i^{(m)}-\beta_i^{(m)}}{2}} e^{ht^2} dt \right) \\ &= \sum_{i=1}^{k} \left(\int_0^{\frac{\bar\beta_i-\bar\gamma_i}{2}} e^{ht^2} dt + \int_0^{\frac{\bar\gamma_i-\bar\beta_i}{2}} e^{ht^2} dt \right) \\ &= 0,\end{aligned}$$

which contradicts the assumption that $z \in P_h(T)$. If it terminates with e_p and Δ_{p+1} of type IV, then we consider the other decorated ideal triangle Δ_0 adjacent to e. If Δ_0 is not of type IV, then it contains an edge e_{-1} and a generalized angle θ_0 adjacent to e_{-1} and e so that $l(e_{-1}) = +\infty$ and $\lim_{m\to\infty} \theta_0^{(m)} = 0$. As before, either there is a fundamental edge loop, contradicting the assumption that $z \in P_h(T)$, or the procedure terminates with an edge e_{-q} and a decorated ideal triangle Δ_{-q} adjacent to e_{-q} so that $l(e_{-q}) = +\infty$ and Δ_{-q} is of type IV. If the procedure stops at e_{-q} and Δ_{-q} of type IV, we get a fundamental edge path $(\Delta_{-q}, e_{-q}, \ldots, e_p, \Delta_{p+1})$, where $e_0 = e$, such that Δ_{-q} and Δ_p are of type IV with $l(e_{-q}) = +\infty$ and $l(e_p) = +\infty$, and $\lim_{m\to\infty} \theta_i^{(m)} = 0$, where θ_i is the generalized angle of Δ_i adjacent to e_{i-1} and e_i for $i = 1-q, \ldots, p$. Denote by a_{-q} the generalized angle of Δ_{-q} facing e_{-q}, and by a_p the generalized angle of Δ_{p+1} facing e_p. Let b_{-q} and c_{-q} be the generalized angles of Δ_{-q} adjacent to e_{-q}, and let b_p and c_p be the generalized angles of Δ_{p+1} adjacent to e_p. We find

$$\begin{aligned}
\sum_{i=-q}^{p} z(e_i) &= \lim_{m\to\infty} \sum_{i=-q}^{p} \Psi_h(l^{(m)})(e_i) \\
&= \lim_{m\to\infty} \left(\int_0^{\frac{b_{-q}^{(m)}+c_{-q}^{(m)}-a_{-q}^{(m)}}{2}} e^{ht^2} dt + \int_0^{\frac{b_p^{(m)}+c_p^{(m)}-a_p^{(m)}}{2}} e^{ht^2} dt \right. \\
&\quad \left. + \sum_{i=1-q}^{p} \left(\int_0^{\frac{\theta_i^{(m)}+\beta_i^{(m)}-\gamma_i^{(m)}}{2}} e^{ht^2} dt + \int_0^{\frac{\theta_i^{(m)}+\gamma_i^{(m)}-\beta_i^{(m)}}{2}} e^{ht^2} dt \right) \right) \\
&= \int_0^{-\infty} e^{ht^2} dt + \int_0^{-\infty} e^{ht^2} dt \\
&\quad + \sum_{i=1-q}^{p} \left(\int_0^{\frac{\bar\beta_i-\bar\gamma_i}{2}} e^{ht^2} dt + \int_0^{\frac{\bar\gamma_i-\bar\beta_i}{2}} e^{ht^2} dt \right) \\
&= -2 \int_0^{+\infty} e^{ht^2} dt,
\end{aligned}$$

which contradicts the assumption that $z \in P_h(T)$. q.e.d

5. Uniqueness of the energy function

Let Δ be a decorated ideal triangle with edge lengths l_1, l_2, l_3 with opposite generalized angles θ_1, θ_2, θ_3 and set $x_i = \frac{\theta_j+\theta_k-\theta_i}{2}$ for $i, j, k = 1, 2, 3$. The following theorem shows that Ψ_h is the unique possible deformation of Penner's simplicial coordinate by using the variational principle stated in Section 2.

Theorem 5.1. *Let μ and u be two non-constant smooth functions. Up to an overall scale, there is a unique closed 1-form $\omega = \sum_{i=1}^3 \mu(x_i) du(l_i)$ which is given by*

$$w_h = \sum_{i=1}^{3} \int^{x_i} e^{ht^2} dt d \left(\int^{l_i} e^{-he^{-t}} dt \right)$$

for some $h \in \mathbb{R}$.

The proof of Theorem 5.1 makes use of the following lemma.

Lemma 5.2. *Let f and g be two non-constant smooth functions on $\mathbb{R}$. If $\frac{f(x_i)}{g(l_j)}$ is symmetric in $i,j=1,2$, then there are constants h, c_1 and c_2 so that*

$$f(t)=e^{ht^2+c_1} \quad \textit{and} \quad g(t)=e^{-he^{-t}+c_2}.$$

Proof. By taking $\frac{\partial}{\partial l_k}$ in the equality $\frac{f(x_i)}{g(l_j)}=\frac{f(x_j)}{g(l_i)}$, we have $\frac{f'(x_i)}{g(l_j)}\frac{\partial x_i}{\partial l_k}=\frac{f'(x_j)}{g(l_i)}\frac{\partial x_j}{\partial l_k}$ for $i,j,k=1,2,3$. We deduce from (1) that $\frac{\partial x_i}{\partial l_j}=\frac{x_k}{2}$, so $\frac{f'(x_i)}{g(l_j)}\frac{x_j}{2}=\frac{f'(x_j)}{g(l_i)}\frac{x_i}{2}$. Thus, $\frac{f'(x_i)}{f'(x_j)}\frac{x_j}{x_i}=\frac{g(l_j)}{g(l_i)}=\frac{f(x_i)}{f(x_j)}$, which implies $\frac{f'(x_i)}{f(x_i)}\frac{1}{x_i}=\frac{f'(x_j)}{f(x_j)}\frac{1}{x_j}$ and $\frac{f'(t)}{f(t)}\frac{1}{t}=2h_1$ for some $h_1\in\mathbb{R}$. Solving this ordinary differential equation for f, we find

$$f(t)=e^{h_1t^2+c_1}$$

for some $c_1\in\mathbb{R}$. By taking $\frac{\partial}{\partial x_k}$ in the equality $\frac{g(l_i)}{f(x_j)}=\frac{g(l_j)}{f(x_i)}$, we have $\frac{g'(l_i)}{f(x_j)}\frac{\partial l_i}{\partial x_k}=\frac{g'(l_j)}{f(x_i)}\frac{\partial l_j}{\partial x_k}$ for $i,j,k=1,2,3$. From (1) again, we deduce that $\frac{\partial l_i}{\partial x_j}=-\frac{1}{\theta_k}$, so $-\frac{g'(l_i)}{f(x_j)}\frac{1}{\theta_j}=-\frac{g'(l_j)}{f(x_i)}\frac{1}{\theta_i}$. Thus, $\frac{g'(l_i)}{g'(l_j)}\frac{e^{l_i}}{e^{l_j}}=\frac{g'(l_i)}{g'(l_j)}\frac{\theta_i}{\theta_j}=\frac{f(x_j)}{f(x_i)}=\frac{g(l_i)}{g(l_j)}$, which implies $\frac{g'(l_i)}{g(l_i)}e^{l_i}=\frac{g'(l_j)}{g(l_j)}e^{l_j}$ and $\frac{g'(t)}{g(t)}e^t=h_2$ for some $h_2\in\mathbb{R}$. Solving this ordinary differential equation for g, we find

$$g(t)=e^{-h_2e^{-t}+c_2}$$

for some $c_1\in\mathbb{R}$. From $f(t)=e^{h_1t^2+c_1}$ and the equality $\frac{f(x_i)}{g(l_j)}=\frac{f(x_j)}{g(l_i)}$, we conclude that $h_1=h_2$. q.e.d

Proof of Theorem 5.1. The differential 1-form $\omega=\sum_{i=1}^3\mu(x_i)du(l_i)$ is closed if and only if $\frac{\partial\mu(x_i)}{\partial u(l_j)}=\frac{\mu'(x_i)}{u'(l_j)}\frac{\partial x_i}{\partial l_j}$ is symmetric in i and j. Since $\frac{\partial x_i}{\partial l_j}=\frac{\partial x_j}{\partial l_i}=\frac{x_k}{2}$, ω is closed if and only if $\frac{\mu'(x_i)}{u'(l_j)}$ is symmetric in i and j. By Lemma 5.2, if $\frac{\mu'(x_i)}{u'(l_j)}$ is symmetric in i and j, then $\mu'(x_i)=e^{hx_i^2+c_1}$ and $u'(l_i)=e^{-he^{-l_i}+c_2}$ for some constants h, c_1 and c_2. q.e.d

6. Ψ_h and the Delaunay decomposition

We first review the construction of the Delaunay decomposition associated to a decorated hyperbolic metric following Bowditch-Epstein [4]. Suppose S is a punctured surface with a set of ideal vertices V, and let (d,r) be a decorated hyperbolic metric on S so that the horodisks associated to the ideal vertices do not intersect. Let B_v be the horodisks associated to the ideal vertex v, and let $B=\bigcup_{v\in V}B_v$. The *spine* $\Gamma_{d,r}$ of S is the set of points in S which have at least two distinct shortest geodesics to ∂B. The spine $\Gamma_{d,r}$ is shown [4] to be a graph whose edges are geodesic arcs on S.

Let $e_1^*, \dots, e_N^*$ be the edges of $\Gamma_{d,r}$. By construction each interior point of an edge e_i^* has exactly two distinct shortest geodesics to ∂B. For each edge e_i^*, there are two horodisks B_1 and B_2 (possibly coincide) so that points in the interior of e_i^* have precisely two shortest geodesics to ∂B_1 and ∂B_2. Let e_i be the shortest geodesic from ∂B_1 to ∂B_2. It is known that e_i intersects e_i^* perpendicularly, and $\{e_1, \dots, e_N\}$ are disjoint. The components of $S \setminus \{e_1, \dots, e_N\}$ consists of decorated polygons (ideal polygons with horodisks associated to the ideal vertices) which are the 2-cells of the *Delaunay decomposition* $\Sigma_{d,r}$. The 1-cells of $\Sigma_{d,r}$ consist of the edges $\{e_1, \dots, e_N\}$ and the arcs on ∂B which are the intersection of ∂B with the ideal polygons. For a generic decorated hyperbolic metric (d, r), each 2-cell of $\Sigma_{d,r}$ is a decorated ideal triangle, and $\Sigma_{d,r}$ is a decorated ideal triangulation of S.

Let D be a 2-cell of $\Sigma_{d,r}$. We call the hyperbolic circle on S tangent to all arcs of $D \cap \partial B$ the *inscribed circle* of D. By the construction of the Delaunay decomposition, for each 2-cell D of $\Sigma_{d,r}$, there is exactly one vertex v^* of the spine $\Gamma_{d,r}$ lying in the interior of D. Moreover, v^* is of equal distance to all arcs of $D \cap \partial B$, hence is the center of the inscribed circle of D. Thus, the center of the inscribed circle of each 2-cell D of the Delaunay decomposition is in the interior of D. We need the following proposition of Penner [17] whose proof is included here to the convenience of the readers.

Lemma 6.1. ([17]) *Suppose Δ is a decorated ideal triangle with edge lengths $l_i > 0$ and opposite generalized angles θ_i for $i = 1, 2, 3$. Then $x_i = \frac{\theta_j + \theta_k - \theta_i}{2} > 0$ for $i = 1, 2, 3$ if and only if the center of the inscribed circle of Δ is in the interior of Δ.*

Proof. For $i = 1, 2, 3$ let B_i be the horodisks associated to the ideal vertices of Δ, and let Z_i be the point of tangency of the inscribe circle of Δ and ∂B_i. Label the intersection of the horodisks and the edges of Δ by X_1, Y_1, X_2, Y_2, X_3 and Y_3 cyclically as in Figure 6(a). For two points A and B in the hyperbolic plane $\mathbb{H}^2$, let AB be the geodesic segment connecting A and B, and let $|AB|$ the length of AB. If the center v of the inscribed circle is in the interior of Δ, then $x_i = |X_i Z_{i+1}| > 0$ for $i = 1, 2, 3$. If v is on $X_i Y_i$, or v and Δ are on different sides of $X_i Y_i$ for some $i \in \{1, 2, 3\}$, then $x_i = -|X_i Z_{i+1}| \leqslant 0$. See Figure 6 (b). q.e.d

Proof of Theorem 1.3. Let (d, r) be a decorated hyperbolic metric so that the associated Delaunay decomposition $\Sigma_{d,r}$ is a decorated ideal triangulation of S. For each edge e of $\Sigma_{d,r}$, let Δ and Δ' be the decorated ideal triangles adjacent to e, and let θ_1 and θ_1' respectively be the

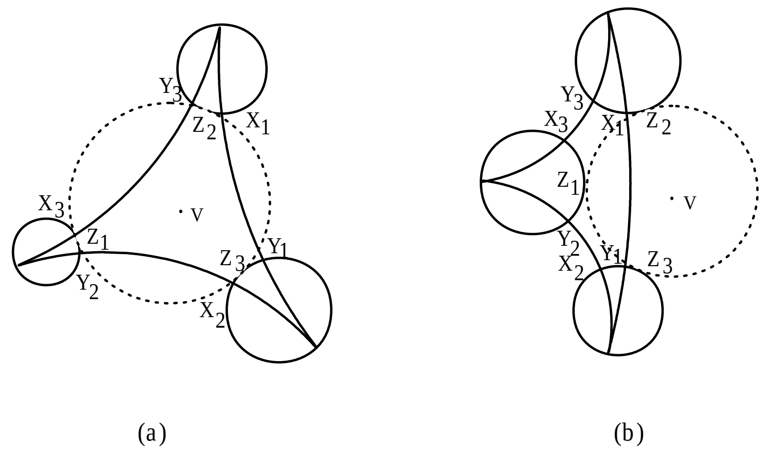

Figure 6. The inscribed circle.

generalized angles of Δ and Δ' facing e, and θ_2, θ_3, θ'_2 and θ'_3 be the generalized angles adjacent to e. Let $x(e) = \frac{\theta_2+\theta_3-\theta_1}{2}$ and $x'(e) = \frac{\theta'_2+\theta'_3-\theta'_1}{2}$. From Lemma 6.1 and the fact that the center of the inscribed circle of each 2-cell of the Delaunay decomposition is in the interior of the 2-cell, we conclude that $x(e)$ and $x'(e)$ are positive, and

$$\Psi_h(d,r)(e) = \int_0^{x(e)} e^{ht^2} dt + \int_0^{x'(e)} e^{ht^2} dt > 0.$$

On the other hand, if T is an ideal triangulation of S such that $\Psi_h(d,r)(e) \leqslant 0$ for some edge e, then at least one of $x(e)$ and $x'(e)$, say $x(e)$, is less than or equal to zero. By Lemma 6.1, the center of the inscribed circle of Δ is not in the interior of Δ. Since the center of the inscribed circle of each 2-cell of the Delaunay decomposition has to be in the interior of the 2-cell, T cannot be the Delaunay decomposition $\Sigma_{d,r}$ of S. q.e.d

7. Further questions

Suppose Δ is a decorated ideal triangle with edge lengths l_1, l_2 and l_3 and opposite generalized angles θ_1, θ_2 and θ_3. For each $h \neq -1$, the differential 1-form $\omega_h = \sum_{i=1}^{3} \theta_i^{h+1} de^{-(h+1)l_i}$ is closed in $\mathbb{R}^3$. However, the primitive $F_h(u) = \int_0^u \omega_h$ is not strictly concave on $\mathbb{R}^3$. Let (S,T) be an ideally triangulated punctured surface. For each $h \neq -1$, we define a map $\Phi_h\colon\ T_c(S) \times \mathbb{R}^V_{>0} \to \mathbb{R}^E$ by

$$\Phi_h(d,r)(e) = \theta^{h+1} + \theta'^{h+1},$$

where θ and θ' are the generalized angles facing e. To the best of the author's knowledge, there is no counterexample to the following

Conjecture 7.1. *The map $\Phi_h\colon T_c(S) \times \mathbb{R}^V_{>0} \to \mathbb{R}^E$ is a smooth embedding, and the image of Φ_h is a convex polytope.*

The motivation of this conjecture is as follows. Penner's simplical coordinate Ψ and its deformation Ψ_h are in some sense analogues to Colin de Vedière's invariant [5] for circle packings in a different setting, and the quantities Φ_h are the corresponding analogues to Rivin's invariant [18] for the polyhedra surfaces in this setting, see also [1] and [11].

By Corollary 1.4, for each $h \geqslant 0$, there is a homeomorphism

$$\Pi_h\colon T_c(S) \times \mathbb{R}^V_{>0} \to |A(S) - A_\infty(S)| \times \mathbb{R}_{>0}$$

equivariant under the mapping class group action. If $h \neq h'$, then $\Pi^{-1}_{h'}\Pi_h$ is a self-homeomorphism of the decorated Teichmüller space equivariant under the mapping class group action. These self-homeomorphisms deserve a further study. We do not know yet if these self-homeomorphisms are smooth on the decorated Teichmüller space. As suggested by the referee of this article, it also seems natural to ask if these self-homeomorphisms have bounded distortion.

The Weil-Pertersson Kähler form on the Teichmüller space was computed in the length coordinates in [16]. How to express the Weil-Petersson symplectic form on the decorated Teichmüller space in terms of the simplicial coordinate Ψ and in terms of the Ψ_h coordinate, and how to relate the Ψ_h coordinate to the quantum Teichmüller space are interesting problems ([2], [3], [14] and [17]).

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