J. DIFFERENTIAL GEOMETRY
89 (2011) 455-485

ON HYPERBOLIC GAUSS CURVATURE FLOWS

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Abstract

Contrast to the hyperbolic mean curvature flows studied in [**HKL**], [**LS**], and [**KW**], a new hyperbolic curvature flow is proposed for convex hypersurfaces. This flow is most suited when the Gauss curvature is involved. The equation satisfied by the graph of the hypersurface under this flow gives rise to a new class of fully nonlinear Euclidean invariant hyperbolic equations.

Introduction

In the mean curvature flow, one studies the motion of a hypersurface whose velocity is equal to its mean curvature along its normal direction in the Euclidean space. Many results have been obtained over the years, and one may consult the survey Huisken and Polden [**HP**] and the books Ecker [**E**], Giga [**Gi**], and Zhu [**Z**] for detailed discussions. From the point of view of differential equations, the mean curvature flow is a quasilinear parabolic equation that is invariant under the Euclidean motion. In view of the intimate relation between the heat and the wave equations, it is natural to consider the hyperbolic version of the mean curvature flow. In Yau [**Y**], it is proposed to study the motion of a hypersurface whose acceleration, instead of the velocity, is equal to its mean curvature along the normal direction. In He, Kong, and Liu [**HKL**], local solvability of this problem is established, and properties such as formation of singularities in finite time and asymptotic behavior of the flow are examined. However, this most direct analog of the mean curvature flow differs from its parabolic counterpart by not being reducible to a Euclidean invariant hyperbolic equation. In LeFloch and Smoczyk [**LS**], the motion law

$$\frac{\partial^2 X}{\partial t^2} = F\mathbf{n} - g^{ij}\left\langle \frac{\partial X}{\partial t},\ \frac{\partial^2 X}{\partial p_j \partial t}\right\rangle \frac{\partial X}{\partial p_i} \tag{1}$$

is studied. Here, F is the driving force and g^{ij} is the inverse of the induced metric on the hypersurface $X(p,t)$ in $\mathbb{R}^{n+1}$. These authors call

Received 9/30/2010.

(1) the hyperbolic mean curvature flow for the following specified choice of F:

$$F = \frac{1}{2}\Big(\Big|\frac{\partial X}{\partial t}\Big|^2 + n\Big)H,$$

where H is the mean curvature of X. This flow has the advantage of being derived from a Hamiltonian principle, and hence possesses some conservation laws. Besides, when the initial velocity is along the normal direction, the velocity of the hypersurface keeps pointing in the normal direction afterward. A flow with such property is called a normal flow. For a normal flow, the graph of the hypersurface satisfies a quasilinear Euclidean invariant hyperbolic equation. Normal flows are emphasized in [**LS**]. Subsequently, the hyperbolic curve shortening problem, that is, taking $n = 1$ and F to be the curvature of a plane curve in (1), is studied in Kong and Wang [**KW**] where several criteria on finite time blow-up for graphs are obtained. In Kong, Liu, and Wang [**KLW**], they further study the problem for closed convex curves.

Aside from the mean curvature flow, there are other curvature flows for convex hypersurfaces, notable ones including the Firey's model on worn stones [**F**] and the motion by the affine normal [**A1**] and [**ST**] which applies to image analysis. They depend on the Gauss curvature rather than the mean curvature. The reader may look up [**HP**] and [**Gi**] for more information. The differential equations derived from these flows are no longer quasilinear. Usually, they are fully nonlinear. For flows involving the Gauss curvature, they are parabolic Monge-Ampère equations.

In this paper we propose a hyperbolic version of these fully nonlinear curvature flows. For any driving force F, consider

$$\frac{\partial^2 X}{\partial t^2} = F\mathbf{n} - b^{ij}\frac{\partial F}{\partial p_i}\frac{\partial X}{\partial p_j}, \tag{2}$$

where b^{ij} is the inverse of the second fundamental form on the uniformly convex hypersurface. A flow is called normal preserving if

$$\Big\langle \frac{\partial^2 X}{\partial p_k \partial t}, \mathbf{n}\Big\rangle = 0,$$

for each $k = 1, \ldots, n$. It can be shown that if this condition is fulfilled initially, then it holds for all time under (2). For any normal preserving flow, its graph $(x, t, u(x, t))$ satisfies a fully nonlinear Euclidean invariant hyperbolic equation. For instance, taking F to be the negative reciprocal of the Gauss curvature, we obtain the hyperbolic Monge-Ampère equation

$$\det D^2_{x,t} u = -(1 + |\nabla u|^2)^{\frac{n+1}{2}},$$

and, taking it to be the Gauss curvature, we obtain the new equation

$$\det D^2_{x,t}u = \frac{(\det D^2_x u)^2}{(1+|\nabla u|^2)^{\frac{n+1}{2}}},$$

where $D^2_{x,t}u$ is the matrix $(\partial^2 u/\partial x_i \partial x_j),\ i,j=0,\ldots,n$ and $x_0=t$. It is interesting to observe that this equation relates the Monge-Ampère operator in space-time to the Monge-Ampère operator in space. It is hyperbolic, and yet the solution is convex in (x,t). Different choices of F produces many new fully nonlinear hyperbolic equations.

This paper is organized as follows. In Section 1 we present a leisure study on the reducibility of a geometric motion to a differential equation for its graph for plane curves. It serves as a motivation for the introduction of normal and normal-preserving flows. Next, we discuss the motions for hypersurfaces in Section 2. We shall show that, among other things, when expressed in terms of the support function H for the convex hypersurface, the equation for (2) becomes

$$\frac{\partial^2 H}{\partial t^2} = -F,$$

which is the exact analog of

$$\frac{\partial H}{\partial t} = -F,$$

the corresponding equation arising from

$$\frac{\partial X}{\partial t} = F\mathbf{n}.$$

In Section 3 we establish the local solvability of (2) for a large class of F based on Caffarelli, Nirenberg, and Spruck [**CNS**] theory of fully nonlinear elliptic equations. Finally, preliminary discussions on topics such as finite time blow-up and asymptotic behavior will be given in Section 4.

We would like to thank Prof. Xu-Jia Wang for drawing our attention to hyperbolic mean curvature flows and Prof. De-Xing Kong for sending us some of his works.

1. Plane Curves

We start by reviewing the reduction of the curve shortening problem to a quasilinear parabolic equation. Consider the curve shortening problem or the more general problem where a family of plane curves $\gamma(p,t)$ is driven by the motion law

$$\frac{\partial \gamma}{\partial t} = F\mathbf{n} + G\mathbf{t}, \tag{1.1}$$

where $\mathbf{n}$ and $\mathbf{t}$ are, respectively, the unit normal and tangent vectors of the curve $\gamma(\cdot,t)$, and F and G are functions depending on γ and its derivatives with respect to p. The normal $\mathbf{n}$ is the inner one when the curve is closed. Supposing for $p \in (a,b)$ and $t \in (t_0, t_1)$ the curve $\gamma(p,t)$ can be expressed in the form of a graph $(x, u(x,t)),\ x = x(p,t)$, we have

$$\gamma_t = x_t(1, u_x) + (0, u_t). \tag{1.2}$$

Taking the inner product with the choice $\mathbf{n} = (-u_x, 1)/\sqrt{1+u_x^2}$ and $\mathbf{t} = (1, u_x)/\sqrt{1+u_x^2}$, we see that (1.1) is split into two equations, namely,

$$u_t = \sqrt{1+u_x^2}\ F \tag{1.3}$$

and

$$x_t = \frac{\sqrt{1+u_x^2}\ G - u_t}{1+u_x^2}. \tag{1.4}$$

In the special case where F depends only on k, the curvature of γ, the formula $k = u_{xx}/(1+u_x^2)^{\frac{3}{2}}$ tells us that (1.3) is an evolution equation for u. In principle, one can solve (1.1) by first solving (1.3) for u and then determining x from (1.4). For instance, in the curve shortening problem $F(k) = k$ and $G \equiv 0$, so (1.3) and (1.4) become

$$u_t = \frac{u_{xx}}{1+u_x^2} \tag{1.5}$$

and

$$x_t = \frac{-u_t}{1+u_x^2}(x,t), \tag{1.6}$$

respectively. In case a solution u has been found for (1.5), x can be readily solved as the solution of the ODE (1.6). It is routine to verify that then $(x, u(x,t))$ constitutes a solution for the curve shortening problem.

Before proceeding further, we point out that for motions which only depend on the geometry of the curves, one should require the motion law to be a "geometric" one. Specifically, this means that solutions of (1.1) are preserved under any reparametrization as well as Euclidean motions. It turns out that the flow (1.1) is geometric when F and G depend only on the curvature and its derivatives with respect to the arc-length. For any geometric flow (1.1), the corresponding equation (1.3) is Euclidean invariant in the following sense. In the case under a Euclidean motion R, $(y,v) = R(x,u)$, the graphs $(x, u(x,t))$ go over to graphs $(y, v(y,t))$, and then v satisfies the same equation (1.3) with x and u replaced by y and v respectively. The reader is referred to Olver [**O**] for discussions on group invariant differential equations.

Now, consider the motion of curves where the velocity is replaced by the acceleration

$$\frac{\partial^2 \gamma}{\partial t^2} = F\mathbf{n} + G\mathbf{t}. \tag{1.7}$$

As the highest order of time derivative involved is 2, the functions F and G are allowed to depend on γ, γ_t and their derivatives with respect to p. Typical geometric flows are formed from those F and G depending on $\langle \gamma_t, \mathbf{n}\rangle, \langle \gamma_t, \mathbf{t}\rangle, \langle \gamma_t, \gamma_{ts}\rangle, k$, etc., and their derivatives with respect to the arc-length. All these are invariants under reparametrizations and Euclidean motions.

When the curves are expressed as graphs $\gamma = (x, u(x,t))$, we have

$$\gamma_{tt} = x_{tt}(1, u_x) + (0, u_{xx}x_t^2 + 2u_{xt}x_t + u_{tt}).$$

Taking inner product with $\mathbf{n}$ and $\mathbf{t}$, respectively, yields

$$u_{tt} + 2x_t u_{xt} + x_t^2 u_{xx} = \sqrt{1+u_x^2}\; F \tag{1.8}$$

and

$$x_{tt} = \frac{G - u_x F}{\sqrt{1+u_x^2}}. \tag{1.9}$$

The situation is different from (1.1). In general, (1.8) not only depends on u and its derivatives but also on x_t. In other words, (1.8) and (1.9) are coupled.

Is there some choice of x_t so that (1.8) reduces to an equation for u only? To examine this possibility, we note that from (1.8)

$$x_t = \frac{-u_{xt} \pm \sqrt{u_{xt}^2 - u_{xx}u_{tt} + u_{xx}\sqrt{1+u_x^2}\; F}}{u_{xx}}.$$

When (1.8) is reducible to an equation of the form $u_{tt} = \Psi(u_x, u_t, u_{xx}, u_{xt})$ for some function Ψ, plugging this equation into the above expression, one sees that x_t must be equal to $\Phi(u_x, u_t, u_{xx}, u_{xt})$ for some function Φ, assuming that F contains first and second derivatives of u only. Motivated by this, we introduce the following definitions. A flow (1.7) is called *reducible (to an equation)* if there exists a function $\Phi(z_1, z_2, z_3, z_4)$ such that whenever the flow is expressed as a graph $(x, u(x,t))$, $x_t = \Phi(u_x, u_t, u_{xx}, u_{xt})$ must hold. For any reducible flow, the equation obtained by substituting $x_t = \Phi$ into (1.8) is called the *associated equation of the flow.* We may assume the variables of the function F can be expressed in terms of u and its derivatives.

Two remarks are in order. First, flows that are not reducible exist. At the end of this section we will show that the flow (1.7) where $F = k$ and $G \equiv 0$ is not reducible. Second, when one is concerned with the initial value problem for (1.7), it is natural to wonder if the flow is reducible for any initial values $\gamma(0)$ and $\gamma_t(0)$. The answer is no. To see this, let us assume locally $\gamma(0) = (f_1(p), f_2(p))$ and $\gamma_t(0) = (g_1(p), g_2(p))$. As we have freedom in choosing the parameter, we may assume $x = p$, that is, f_1 is the identity map. Then the relation $x_t = \Phi$ at $t = 0$ gives the compatibility condition $g_1 = \Phi(f_2', g_2 - f_2'g_1, f_2'', (g_2 - f_2'g_1)')$. When the initial curve is fixed, that is, f_2 is given, this condition sets up a constraint between g_1 and g_2.

For a given function F, we will find two classes of "constrained" flows, namely, the normal and normal-preserving flows, and the corresponding functions G so that the flows are reducible. Our approach is based on the observation that any associated equation of a reducible flow must be Euclidean invariant, so we start by classifying all Euclidean invariant equations. Of course, this is of interest in itself. After obtaining these equations, we may compare them with (1.8) to guess what the constraint Φ should be.

We examine the quasilinear case first. Consider

$$u_{tt} = au_{xx} + bu_{xt} + c, \tag{1.10}$$

where the coefficients a, b, and c depend on x, u, u_x, and u_t.

Proposition 1.1. *Any Euclidean invariant equation* (1.10) *is of the form*

$$\begin{aligned} b &= \frac{2u_x u_t}{1+u_x^2} + \frac{\varphi(z)}{\sqrt{1+u_x^2}}, \\ a &= \frac{1}{1+u_x^2} - \frac{b^2}{4} + \frac{\chi(z)}{1+u_x^2}, \\ c &= \sqrt{1+u_x^2}\ \psi(z), \qquad z = \frac{u_t}{\sqrt{1+u_x^2}}, \end{aligned}$$

where φ, χ, and ψ are arbitrary functions.

Proof. The Euclidean group acts linearly on (x, u) and trivially on t. Its Lie algebra of infinitesimal symmetries is spanned by

$$\{\partial_x,\ \partial_u,\ -u\partial_x + x\partial_u\}.$$

According to Lie's theory of symmetries, (1.10) is Euclidean invariant if and only if

$$pr^{(2)}\mathbf{v}(u_{tt} - au_{xx} - bu_{xt} - c) = 0,$$

on $u_{tt} = au_{xx}+bu_{xt}+c$, where $\mathbf{v}$ is any infinitesimal symmetry and $pr^{(2)}\mathbf{v}$ is the second order prolongation of $\mathbf{v}$. By the prolongation formula [**O**], $pr^{(2)}\partial_x = \partial_x$, and so

$$pr^{(2)}\partial_x(u_{tt} - au_{xx} - bu_{xt} - c) = -a_x u_{xx} - b_x u_{xt} - c_x = 0,$$

which implies that a, b, c are independent of x. Similarly, they are also independent of u. Now, for the rotation $\mathbf{r} \equiv -u\partial_x + x\partial_u$, the second prolongation is given by

$$\begin{aligned} pr^{(2)}\mathbf{r} \;\; = \;\; & -u\partial_x + x\partial_u + (1+u_x^2)\partial_{u_x} + u_x u_t \partial_{u_t} + 3u_x u_{xx}\partial_{u_{xx}} \\ & +(2u_x u_{xt} + u_t u_{xx})\partial_{u_{xt}} + (u_{tt}u_x + 2u_t u_{xt})\partial_{u_{tt}}. \end{aligned}$$

Its action on (1.10) gives

$$u_{xx}\big(a_{u_x}(1+u_x^2)+a_{u_t}u_xu_t\big)+3au_xu_{xx}+u_{xt}\big(b_{u_x}(1+u_x^2)+b_{u_t}u_xu_t\big)+$$
$$b(2u_xu_{xt}+u_tu_{xx})+c_{u_x}(1+u_x^2)+c_{u_t}u_xu_t=u_{tt}u_x+2u_tu_{xt},$$

on $u_{tt}=au_{xx}+bu_{xt}+c$. We eliminate u_{tt} in this equation using (1.10). Then the variables u_{xx}, u_{xt}, u_x, and u_t become free. By setting the coefficients of u_{xx} and u_{xt} to zero, we obtain

$$a_{u_x}(1+u_x^2)+a_{u_t}u_xu_t+2au_x+bu_t=0$$

and

$$b_{u_x}(1+u_x^2)+b_{u_t}u_xu_t+bu_x-2u_t=0,$$

while the lower-order terms give

$$c_{u_x}(1+u_x^2)+c_{u_t}u_xu_t-cu_x=0.$$

These are first-order linear PDE's for the coefficients. The second and third equations are readily solved to yield

$$b=\frac{2u_xu_t}{1+u_x^2}+\frac{1}{u_t}\varphi_1\Big(\frac{u_t}{\sqrt{1+u_x^2}}\Big)$$

and

$$c=\sqrt{1+u_x^2}\,\psi\Big(\frac{u_t}{\sqrt{1+u_x^2}}\Big).$$

Plugging b into the first equation gives

$$a=\frac{1}{1+u_x^2}-\frac{b^2}{4}+\frac{1}{u_t^2}\chi_1\Big(\frac{u_t}{\sqrt{1+u_x^2}}\Big).$$

Here, φ_1, χ_1, and ψ are arbitrary functions. Clearly the proposition holds. q.e.d.

Taking $\varphi=\chi=\psi=0$, we obtain the simplest Euclidean invariant equation,

$$u_{tt}-2\frac{u_xu_t}{1+u_x^2}u_{xt}+\frac{u_x^2u_t^2}{(1+u_x^2)^2}u_{xx}=\frac{u_{xx}}{1+u_x^2}. \tag{1.11}$$

Comparing this equation with (1.8) where $F=k$, we see that $\Phi=-u_xu_t/(1+u_x^2)$. The meaning of this constraint becomes clear after using (1.2); it means that $\langle\gamma_t,\gamma_p\rangle=0$ for all time. A flow with this property is called a *normal flow*. With this constraint at hand, G could be determined from (1.9), but here we use a different reasoning that is based on the fact that (1.7) must preserve this constraint. In other words, if $\langle\gamma_t,\gamma_p\rangle=0$ at $t=0$, then it holds for all time. Keeping this in mind, we compute

$$\begin{aligned}0 &= \frac{\partial}{\partial t}\langle\gamma_t,\gamma_p\rangle\\ &= \langle\gamma_{tt},\gamma_p\rangle+\langle\gamma_t,\gamma_{pt}\rangle\\ &= G|\gamma_p|+\langle\gamma_t,\gamma_{ts}\rangle|\gamma_p|,\end{aligned}$$

from which we deduce $G = -\langle \gamma_t, \gamma_{ts} \rangle$. Note that it is independent of F. Later we will see that G depends on F for a normal preserving flow. It is routine to check that for any given F and G in (1.7), starting from an initial velocity satisfying $\langle \gamma_t(0), \gamma_p(0) \rangle = 0$, the flow (whenever it exists) is normal if and only if $G = -\langle \gamma_t, \gamma_{ts} \rangle$. In the following we show that any quasilinear Euclidean invariant equation (1.10) arises as the associated equation of some normal flow.

Proposition 1.2. *Any Euclidean invariant equation* (1.10) *is the associated equation of the normal flow*

$$\frac{\partial^2 \gamma}{\partial t^2} = F\boldsymbol{n} - \langle \gamma_t, \gamma_{ts} \rangle \boldsymbol{t}, \tag{1.12}$$

where F is of the form $F_1 + F_2 k + F_3 \langle \gamma_t, \gamma_{ts} \rangle$, and F_i, $i = 1, 2, 3$, depend on $\langle \gamma_t, \boldsymbol{n} \rangle$ only.

Proof. First, note that

$$\langle \gamma_t, \mathbf{n} \rangle = \frac{u_t}{\sqrt{1 + u_x^2}}$$

and $\langle \gamma_t, \mathbf{t} \rangle = 0$. We also claim

$$\langle \gamma_t, \gamma_{ts} \rangle = \frac{u_t u_{xt}}{(1 + u_x^2)^{3/2}} - \frac{u_x u_t^2 u_{xx}}{(1 + u_x^2)^{5/2}}.$$

To see this, we first use orthogonality to get $\gamma_t = \langle \gamma_t, \mathbf{n} \rangle \mathbf{n}$. It follows that

$$\gamma_{ts} = \langle \gamma_t, \mathbf{n} \rangle \mathbf{n}_s + (\langle \gamma_{ts}, \mathbf{n} \rangle + \langle \gamma_t, \mathbf{n}_s \rangle)\mathbf{n},$$

and so

$$\langle \gamma_t, \gamma_{ts} \rangle = \langle \gamma_t, \mathbf{n} \rangle \langle \gamma_{ts}, \mathbf{n} \rangle,$$

after using Frenet's formula. Now, $\gamma_{ts} = x_{ts}(1, u_x) + x_t(0, u_{xx})x_s + (0, u_{tx})x_s$, where $x_s = 1/(1 + u_x^2)$, hence

$$\langle \gamma_{ts}, \mathbf{n} \rangle = \frac{x_t u_{xx}}{1 + u_x^2} + \frac{u_{tx}}{1 + u_x^2},$$

and the claim follows.

Putting these into (1.8), we obtain

$$\begin{aligned} u_{tt} \quad = \quad & \sqrt{1 + u_x^2}\Big[F_1 + F_2 \frac{u_{xx}}{(1 + u_x^2)^{3/2}} + F_3\big(\frac{u_t u_{xt}}{(1 + u_x^2)^{3/2}} - \frac{u_x u_t^2 u_{xx}}{(1 + u_x^2)^{5/2}}\big)\Big] \\ & + \frac{2u_x u_t}{1 + u_x^2} u_{xt} - \frac{u_x^2 u_t^2}{(1 + u_x^2)^2} u_{xx}. \end{aligned}$$

Comparing this with Proposition 1.1, we simply take $F_3(z) = \varphi/z$, $F_2(z) = 1 - \varphi^2(z)/4 + \chi(z)$, and $F_1(z) = \psi(z)$ to obtain this proposition. q.e.d.

Next, we consider the fully nonlinear equation

$$u_{tt} = f(x, u, u_x, u_t, u_{xx}, u_{xt}). \tag{1.13}$$

Parallel to Proposition 1.1, we have the following proposition:

Proposition 1.3. *Any Euclidean invariant equation* (1.13) *is of the form*

$$u_{tt} = \frac{u_{xt}^2}{u_{xx}} + \sqrt{1+u_x^2}\ \Phi(z_1, z_2, z_3), \tag{1.14}$$

where $\Phi(z_1, z_2, z_3)$ *is an arbitrary function,* $z_1 = u_t/\sqrt{1+u_x^2}$*,* $z_2 = u_{xx}/(1+u_x^2)^{3/2}$*, and*

$$z_3 = \frac{u_{xt}}{1+u_x^2} - \frac{u_x u_t u_{xx}}{(1+u_x^2)^2}.$$

Proof. As in the proof of Proposition 1.1, f is independent of x and u by Euclidean invariance. From the action of the infinitesimal rotation, the prolongation formula gives

$$(1+u_x^2)f_{u_x} + u_x u_t f_{u_t} + 3u_x u_{xx} f_{u_{xx}} + (2u_x u_{xt} + u_t u_{xx}) f_{xt} = u_x u_{tt} + 2u_t u_{xt},$$

on $u_{tt} = f$. By eliminating u_{tt}, we can solve f from the above equation.

By a direct computation,

$$f = -\frac{u_x^2 u_t^2}{(1+u_x^2)^2} u_{xx} + \frac{2u_x u_t}{(1+u_x^2)} u_{xt} + (1+u_x^2)^{\frac{1}{2}} \Phi_1(z_1, z_2, z_3),$$

for some function Φ_1. The proposition now follows from letting $\Phi = z_3^2/z_2 + \Phi_1(z_1, z_2, z_3)$. q.e.d.

Comparing (1.14) with (1.8), we see that they are identical if we choose

$$x_t = -\frac{u_{xt}}{u_{xx}}. \tag{1.15}$$

This condition is readily checked to be equivalent to

$$\langle \gamma_{ts}, \mathbf{n} \rangle = 0. \tag{1.16}$$

A flow (1.7) is called a *normal preserving flow* if (1.16) holds for all time. To understand this definition, recall that the angle between the curve and the x-axis, α, is related to u_x by $\tan\alpha = u_x$. From

$$\sec^2\alpha \frac{\partial \alpha}{\partial t} = u_{xx} x_t + u_{xt} = 0,$$

we see that α is independent of time during the flow. As the normal angle of the curve is equal to $\alpha + \pi/2$, it is also constant in time. In other words, $\mathbf{n}(p,t)$ is equal to $\mathbf{n}(p,0)$, justifying the terminology.

As in the quasilinear case, we can determine G for a normal preserving flow. In fact,

$$\begin{aligned}\frac{\partial}{\partial t}\langle \gamma_{tp}, \mathbf{n}\rangle &= \langle \gamma_{ttp}, \mathbf{n}\rangle + \langle \gamma_{tp}, \mathbf{n}_t\rangle \\ &= \frac{\partial F}{\partial p} + |\gamma_p| k G + \langle \gamma_{tp}, \mathbf{t}\rangle\langle \mathbf{n}, \mathbf{t}_t\rangle \\ &= \frac{\partial F}{\partial p} + |\gamma_p| k G + \langle \gamma_{ts}, \mathbf{t}\rangle\langle \gamma_{tp}, \mathbf{n}\rangle.\end{aligned}$$

This is an ODE of the form $dy/dt = a + by$. Clearly, (1.7) preserves normal preserving flows if and only if $G = -k^{-1}F_s$. In fact, all fully nonlinear Euclidean invariant equations arise from this way.

Proposition 1.4. *Any Euclidean invariant equation* (1.13) *is the associated equation of a normal preserving flow*

$$\gamma_{tt} = F\boldsymbol{n} - \frac{1}{k}F_s\boldsymbol{t}, \tag{1.17}$$

where F depends on $\langle \gamma_t, \boldsymbol{n}\rangle, k,$ and $\langle \gamma_t, \gamma_{ts}\rangle$.

Proof. Plug (1.15) into (1.8) and then use Proposition 1.3. q.e.d.

Affine invariant motion laws (1.1) have been studied in connection with image processing. We may consider its hyperbolic analogs. Recall that the affine group is a subgroup of the Euclidean group whose infinitesimal symmetries are spanned by

$$\{\partial_x, \partial_u, u\partial_x, x\partial_u, x\partial_x - u\partial_u\}.$$

We have the following proposition:

Proposition 1.5. *Any affine invariant equation* (1.13) *is of the form*

$$u_{tt} = \frac{u_{xt}^2}{u_{xx}} + u_t\Phi\Big(\frac{u_{xx}}{u_t^3}\Big), \tag{1.18}$$

for some function Φ.

The proof of this proposition is similar to that of Proposition 1.3 and is omitted.

Hyperbolic versions of the curve shortening problem can be found by choosing different F and G in (1.7). In [**LS**],

$$F = \frac{1}{2}(1 + |\gamma_t|^2)k, \quad G = -\langle \gamma_t, \gamma_{ts}\rangle$$

is chosen. From the above discussion, any normal flow is reducible with associated equation given by

$$u_{tt} = \frac{1 + u_x^2 + u_t^2 - 2u_x^2u_t^2}{2(1+u_x^2)^2}u_{xx} + 2\frac{u_xu_t}{1+u_x^2}u_{xt}. \tag{1.19}$$

In [**KW**], the choice is

$$F = k, \qquad G = -\langle \gamma_t, \gamma_{ts} \rangle.$$

Again, any normal flow is reducible and its associated equation is simply given by (1.11). Both equations are quasilinear hyperbolic. Now we may take

$$F = k, \quad G = -k^{-1} k_s$$

in (1.7). Any normal-preserving flow is reducible, and its associated equation is

$$u_{tt} u_{xx} - u_{xt}^2 = \frac{u_{xx}^2}{1 + u_x^2}, \tag{1.20}$$

This is a fully nonlinear, hyperbolic equation as long as the curve is uniformly convex.

The affine curve shortening problem, which is sometimes called the fundamental equation of image processing [**AGLM**], refers to $F = k^{1/3}$ and $G = -k^{-5/3} k_s/3$ in (1.1), and is studied in [**A1**] and [**ST**]. Taking $\Phi(z) = z^{1/3}$ in (1.18), we obtain its hyperbolic version

$$u_{tt} u_{xx} - u_{xt}^2 = u_{xx}^{\frac{4}{3}}.$$

Very often, in the study of the motions of convex curves, it is useful to express the flow in terms of the support function rather than the graph (see Chou and Zhu [**CZ**]). Recall that the normal angle $\theta \in [0, 2\pi)$ of a curve satisfies

$$\mathbf{n} = -(\cos\theta,\ \sin\theta), \quad \mathbf{t} = (-\sin\theta,\ \cos\theta),$$

and the support function is a function of the normal angle given by

$$h(\theta, t) = \langle \gamma(p, t),\ -\mathbf{n} \rangle,$$

where $\gamma(p, t)$ is the point on the curve whose normal angle is equal to θ. Any closed convex curve can be determined from its support function. In fact, for $\gamma = (x, u(x, t))$, we have

$$\begin{aligned} x &= h\cos\theta - h_\theta \sin\theta, \\ u &= h\sin\theta + h_\theta \cos\theta. \end{aligned}$$

Differentiating the first of these relations in x and t, we have

$$\begin{aligned} 1 &= -(h + h_{\theta\theta})\theta_x \sin\theta, \\ 0 &= h_t \cos\theta - h_{\theta t} \sin\theta - (h + h_{\theta\theta})\theta_t \sin\theta. \end{aligned}$$

Therefore,

$$\theta_x = -\frac{k}{\sin\theta}$$

and

$$\theta_t = k\Big(\frac{h_t \cos\theta - h_{\theta t}\sin\theta}{\sin\theta}\Big),$$

after using the formula

$$k = \theta_s = \frac{1}{h_{\theta\theta} + h}.$$

By differentiating the second relation, we obtain

$$\begin{aligned}
u_x &= \frac{1}{k}\theta_x \cos\theta = -\cot\theta, \\
u_{xx} &= \frac{1}{\sin^2\theta}\theta_x = -\frac{k}{\sin^3\theta}, \\
u_{xt} &= \frac{1}{\sin^2\theta}\theta_t = -\frac{k}{\sin^3\theta}(h_t \sin\theta - h_{\theta t}\sin\theta), \\
u_t &= h_t \sin\theta + (h\cos\theta + h_{\theta\theta}\cos\theta)\theta_t + h_{\theta t}\cos\theta = \frac{h_t}{\sin\theta}, \\
u_{tt} &= \frac{h_{tt}}{\sin\theta} + \Big(\frac{h_{t\theta}}{\sin\theta} - \frac{h_t\cos\theta}{\sin^2\theta}\Big)\theta_t = \frac{h_{tt}}{\sin\theta} \\
&\quad - \frac{k}{\sin^3\theta}(h_{t\theta}\sin\theta - h_t\cos\theta)^2.
\end{aligned}$$

Using these formulas, we can express equations (1.19), (1.11), and (1.20) in terms of the support function. For (1.19) and (1.11), the equations are

$$h_{tt} = \frac{2h_{t\theta}^2 - 1 - h_t^2}{2(h_{\theta\theta} + h)}$$

and

$$h_{tt} = \frac{h_{t\theta}^2 - 1}{h_{\theta\theta} + h},$$

respectively. As for (1.20), the equation is

$$h_{tt} = -\frac{1}{h_{\theta\theta} + h},$$

which is the exact analog of the curve shortening problem when expressed in terms of the support function

$$h_t = -\frac{1}{h_{\theta\theta} + h}.$$

In concluding this section, let us show that the flow (1.7) is not reducible when $F(k) = k$ and $G \equiv 0$. To formulate the result, put the constraint $x_t = \Phi(u_x, u_t, u_{xx}, u_{xt})$ into (1.8) to get

$$u_{tt} + 2\Phi u_{xt} + \Phi^2 u_{xx} = \frac{u_{xx}}{1 + u_x^2}. \tag{1.21}$$

Equation(1.9) now reads as

$$x_{tt} = \frac{-u_x u_{xx}}{(1 + u_x^2)^2}. \tag{1.22}$$

Proposition 1.6. *There is no such smooth function $\Phi(z_1, z_2, z_3, z_4)$ satisfying (i) that* (1.21) *is solvable locally in space and time for arbitrary smooth initial data $u(0)$ and $u_t(0)$, and (ii) that the constraint $x_t = \Phi(u_x, u_t, u_{xx}, u_{xt})$ fulfills* (1.22).

Proof. From the constraint we have

(1.23)
$$x_{tt} = \Phi_{z_1}(u_{xx}\Phi + u_{xt}) + \Phi_{z_2}(u_{xt}\Phi + u_{tt}) + \Phi_{z_3}(u_{xxx}\Phi + u_{xxt}) + \Phi_{z_4}(u_{xxt}\Phi + u_{xtt}).$$

On the other hand, from (1.21) we have

$$\begin{aligned} u_{xtt} &= \frac{u_{xxx}}{1+u_x^2} - \frac{2u_x u_{xx}^2}{(1+u_x^2)^2} - 2u_{xxt}\Phi - u_{xxx}\Phi^2 \\ &\quad -(2u_{xt} + 2u_{xx}\Phi)(\Phi_{z_1}u_{xx} + \Phi_{z_2}u_{xt} + \Phi_{z_3}u_{xxx} + \Phi_{z_4}u_{xxt}). \end{aligned}$$

Eliminating the term u_{xtt} in (1.23) by this equation and then identifying it with (1.22), we obtain a relation of the form $Au_{xxt} + Bu_{xxx} + C = 0$, between $u_x, u_t, u_{xx}, u_{xt}, u_{xxt}$, and u_{xxx}. By our assumption (i), all these variables are free. It follows that $A = B = 0$, that is,

$$\Phi_{z_3} + \Phi_{z_4}[-\Phi - (2z_4 + 2z_3\Phi)\Phi_{z_4}] = 0, \tag{1.24}$$

and

$$\Phi_{z_3}\Phi + \Phi_{z_4}\Big[-\Phi^2 - (2z_4 + 2z_3\Phi)\Phi_{z_3} + \frac{1}{1+z_1^2}\Big] = 0, \tag{1.25}$$

for all (z_1, z_2, z_3, z_4). The lower-order term C also vanishes, but we do not need it.

We solve for Φ_{z_3} from (1.24) and plug it into (1.25) to get

$$\Phi_{z_4}\Big[\big(1-(2z_4+2z_3\Phi)\Phi_{z_4}\big)\big(\Phi+(2z_4+2z_3\Phi)\Phi_{z_4}\big) - \big(-\Phi^2 + \frac{1}{1+z_1^2})\Big] = 0.$$

Thus, either $\Phi_{z_4} = 0$ or

$$\Phi_{z_4} = \pm\frac{1}{(2z_4 + 2z_3\Phi)\sqrt{1+z_1^2}}.$$

If $\Phi_{z_4} = 0$, then $\Phi_{z_3} = 0$ and $x_t = \Phi(u_x, u_t)$. It is easy to see that this is impossible. We take

$$\Phi_{z_4} = \frac{1}{(2z_4 + 2z_2\Phi)\sqrt{1+z_1^2}}\ . \tag{1.26}$$

(The other case can be treated similarly.) From (1.24), we have

$$\Phi_{z_3} = \frac{\Phi\sqrt{1+z_1^2} + 1}{(2z_4 + 2z_3\Phi)(1+z_1^2)}\ . \tag{1.27}$$

By differentiating (1.26), we have

$$\begin{aligned}
\Phi_{z_4 z_3} &= -\frac{2\Phi + 2z_3\Phi_{z_3}}{(2z_4+2z_3\Phi)^2\sqrt{1+z_1^2}} \\
&= -\frac{2\Phi}{(2z_4+2z_3\Phi)^2\sqrt{1+z_1^2}} \\
&\quad -\frac{2z_3}{(2z_4+2z_3\Phi)^2\sqrt{1+z_1^2}}\frac{\Phi\sqrt{1+z_1^2}+1}{(2z_4+2z_3\Phi)(1+z_1^2)}.
\end{aligned}$$

And similarly from (1.27),

$$\begin{aligned}
\Phi_{z_3 z_4} &= \frac{\Phi_{z_4}}{(2z_4+2z_3\Phi)\sqrt{1+z_1^2}} - \frac{(\Phi\sqrt{1+z_1^2}+1)(2+2z_3\Phi_{z_4})}{(2z_4+2z_3\Phi)^2(1+z_1^2)} \\
&= \frac{1}{(2z_4+2z_3\Phi)(1+z_1^2)} - \frac{2\Phi}{(2z_4+2z_3\Phi)^2\sqrt{1+z_1^2}} \\
&\quad -\frac{2}{(2z_4+2z_3\Phi)(1+z_1^2)} - \frac{2z_3(\Phi\sqrt{1+z_1^2}+1)}{(2z_4+2z_3\Phi)^3(1+z_1^2)}\ .
\end{aligned}$$

We find

$$\Phi_{z_4 z_3} - \Phi_{z_3 z_4} = \frac{1}{(2z_4+2z_3\Phi)(1+z_1^2)} \neq 0;$$

the contradiction holds, so the flow (1.7) ($F = k$ and $G \equiv 0$) is not reducible. q.e.d.

2. Hypersurfaces

In this section, we study the geometric motion of hypersurfaces given by

$$\frac{\partial^2 X}{\partial t^2} = F\mathbf{n} + G^j\frac{\partial X}{\partial p_j}, \tag{2.1}$$

where $X(\cdot, t)$ is a hypersurface in $\mathbb{R}^{n+1}$ at each t. The notion of a normal flow extends trivially to all dimensions, namely, $X(p,t)$ is a *normal flow* if $X_t(p,t)$ is orthogonal to the hypersurface at $X(p,t)$ for each t.

Proposition 2.1. *The flow* (2.1) *is normal if and only if it is given by* (1) *and*

$$\left\langle X_t, \frac{\partial X}{\partial p_j}\right\rangle = 0, \qquad j = 1, \ldots, n,$$

at $t = 0$.

Proof. From (2.1) we have

$$\frac{\partial}{\partial t}\langle X_t, X_k\rangle = G^j g_{jk} + \langle X_t, X_{tk}\rangle, \tag{2.2}$$

where $X_k \equiv \partial X/\partial p_k$. From this, it is readily seen that the proposition holds. q.e.d.

Now, we write down the equation for the graph of the flow. Let $X(p,t) = (x, u(x,t))$, where $x = (x^1, \ldots, x^n)$ depends on (p,t). We have

$$\frac{\partial X}{\partial t} = \Big(\frac{\partial x}{\partial t}, \frac{\partial u}{\partial x_i}\frac{\partial x^i}{\partial t} + \frac{\partial u}{\partial t}\Big)$$

and

$$\frac{\partial^2 X}{\partial t^2} = \Big(\frac{\partial^2 x}{\partial t^2}, \frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial x^i}{\partial t}\frac{\partial x^j}{\partial t} + \frac{\partial u}{\partial x_i}\frac{\partial^2 x^i}{\partial t^2} + 2\frac{\partial^2 u}{\partial x_i \partial t}\frac{\partial x^i}{\partial t} + \frac{\partial^2 u}{\partial t^2}\Big).$$

Taking inner product of the last expression with $\mathbf{n}$ yields

$$u_{tt} + 2u_{jt}x_t^j + u_{ij}x_t^i x_t^j = F\sqrt{1+|\nabla u|^2}. \tag{2.3}$$

To determine x_t, we use the orthogonality condition $\langle X_t, X_k\rangle = 0$ to get

$$x_t^i g_{ki} + u_t u_k = 0,$$

for each k. Using $g_{ki} = \delta_{ki} + u_k u_i$ and $g^{ki} = \delta_{ki} - u_k u_i/(1+|\nabla u|^2)$, we have

$$x_t^k = -g^{ki}u_i u_t = -\Big(\delta_{ki} - \frac{u_k u_i}{1+|\nabla u|^2}\Big)u_i u_t.$$

So, the associated equation is

$$u_{tt} - \frac{2u_t u_i}{1+|\nabla u|^2}u_{it} + \frac{u_t^2 u_i u_j}{(1+|\nabla u|^2)^2}u_{ij} = F\sqrt{1+|\nabla u|^2}. \tag{2.4}$$

When $F = A + BH$, where H is the mean curvature of $X(\cdot,t)$ and A, B depend on X up to its first order derivatives, (2.4) is hyperbolic if and only if B is positive.

When it comes to the fully nonlinear case, we consider uniformly convex hypersurfaces only. A family of (uniformly convex) hypersurfaces $X(\cdot,t)$ is called *normal preserving* if its normal at $X(p,t)$ is equal to its normal at $X(p,0)$, or equivalently, $\partial\mathbf{n}/\partial t = 0$.

Proposition 2.2. *Let $X(\cdot,t)$ be a family of uniformly convex hypersurfaces satisfying* (2.1). *It is normal preserving if and only if it is given by* (2) *and*

$$\Big\langle \frac{\partial X_t}{\partial p_j}, \boldsymbol{n}\Big\rangle = 0, \qquad j = 1, \ldots, n,$$

at $t = 0$.

Proof. As $\partial\mathbf{n}/\partial t$ is always orthogonal to $\mathbf{n}$, the flow is normal preserving if and only if

$$\Big\langle \frac{\partial \mathbf{n}}{\partial t}, \frac{\partial X}{\partial p_k}\Big\rangle = 0, \quad k = 1, \ldots, n.$$

We compute

$$\begin{aligned}\frac{\partial}{\partial t}\langle \mathbf{n}_t, X_k\rangle &= -\frac{\partial}{\partial t}\langle \mathbf{n}, X_{kt}\rangle \\ &= -\langle \mathbf{n}_t, X_{kt}\rangle - \langle \mathbf{n}, X_{ktt}\rangle \\ &= -\langle \mathbf{n}_t, X_{kt}\rangle - F_k - G^i\langle X_{ki}, \mathbf{n}\rangle.\end{aligned}$$

Using

$$\frac{\partial \mathbf{n}}{\partial t} = g^{ij}\Big\langle \frac{\partial \mathbf{n}}{\partial t}, \frac{\partial X}{\partial p_j}\Big\rangle \frac{\partial X}{\partial p_i},$$

we have

$$\frac{\partial}{\partial t}\Big\langle \frac{\partial \mathbf{n}}{\partial t}, \frac{\partial X}{\partial p_k}\Big\rangle = -g^{ij}\Big\langle \frac{\partial X}{\partial p_i}, \frac{\partial^2 X}{\partial p_k \partial t}\Big\rangle \Big\langle \frac{\partial \mathbf{n}}{\partial t}, \frac{\partial X}{\partial p_j}\Big\rangle - \frac{\partial F}{\partial p_k} - G^i\Big\langle \frac{\partial^2 X}{\partial p_k \partial p_i}, \mathbf{n}\Big\rangle.$$

This is a system of ODE of the form

$$\frac{\partial}{\partial t} y = By + a,$$

where $y = (y^1, \ldots, y^n)$, $y^k = \langle \mathbf{n}_t, X_k\rangle$, and $a^k = -F_k - G^i\langle X_{ki}, \mathbf{n}\rangle$. Now it is clear that the flow is normal preserving if and only if $a^k \equiv 0$ for all k. The proposition follows from the Weigarten equation

$$b_{ij} = \Big\langle \mathbf{n}, \frac{\partial^2 X}{\partial p_i \partial p_j}\Big\rangle.$$

q.e.d.

To obtain the equation for the graph of a normal-preserving flow, we use the normal-preserving condition to obtain $u_{ij}x_t^j + u_{it} = 0$ for each i. It follows that

$$x_t^i = -u^{ij}u_{jt}, \ \ i = 1, \ldots, n.$$

Plugging this into (2.3) yields

$$u_{tt} - u^{ij}u_{it}u_{jt} = \sqrt{1 + |\nabla u|^2}\ F.$$

We claim that this equation can be rewritten as

$$\det D^2_{x,t}u = \det D^2_x u \sqrt{1 + |\nabla u|^2}\ F. \tag{2.5}$$

For, first of all, using $u^{ij} = c_{ij}(\det D_x^2 u)^{-1}$, where c_{ij} is the $(i,j)-$cofactor of $D_x^2 u$, it suffices to show

$$\det D^2_{x,t}u = u_{tt}\det D^2_x u - c_{ij}u_{it}u_{jt}.$$

Denoting $x_0 = t$, we compute the determinant of the Hessian matrix $D^2_{x,t}u$ by expanding it along the first column

$$\det D^2_{x,t}u = \sum_{j=0}^{n}(-1)^j u_{j0}m_{j0}$$

where m_{j0} is the $(j,0)$-minor of $D^2_{x,t}u$. By expanding along the first row $(u_{01}, u_{02}, \dots, u_{0n})$ of the $n \times n$-matrix obtained from $D^2_{x,t}u$ by deleting its 0th column and jth row, we have

$$m_{j0} = (-1)^{j+1} u_{0i} c_{ij}.$$

It follows that

$$\begin{aligned}
\det D^2_{x,t}u &= \sum_{j=0}^{n} (-1)^j u_{j0} m_{j0} \\
&= u_{00} \det D^2_x u + \sum_{1}^{n} (-1)^j u_{j0} (-1)^{j+1} u_{0i} c_{ij} \\
&= u_{00} \det D^2_x u - c_{ij} u_{i0} u_{j0},
\end{aligned}$$

and (2.5) holds.

The equation for the support function of a normal-preserving flow assumes a simple form.

Recall that for any convex hypersurface X in $\mathbb{R}^{n+1}$, its support function H is a function of homogeneous one defined in $\mathbb{R}^{n+1}/\{0\}$ satisfying

$$H(z) = \langle z, X(p) \rangle, \quad |z| = 1,$$

where $X(p)$ is any point on the hypersurface whose unit outer normal is z. It is well known that any uniformly convex hypersurface can be recovered by its support function via the formula

$$X^i(z) = \frac{\partial H}{\partial z_i}(z), \quad i = 1, \dots, n,$$

where the unit outer normal z is used to parametrize the hypersurface.

Consider now $X(., t)$, a family of uniformly convex, closed hypersurfaces that is normal preserving. We may parametrize the initial hypersurface by its unit outer normal z. By the normal-preserving property, z is always the unit outer normal at the point $X(z, t)$ for all t. In particular, we have $\mathbf{n} = -z$. By taking inner product of (2.1) with z, we have

$$\begin{aligned}
F &= \left\langle \frac{\partial^2 X}{\partial t^2}, -z \right\rangle \\
&= -\sum_{1}^{n+1} z_j \frac{\partial}{\partial z_j} \frac{\partial^2 H}{\partial t^2} \\
&= -\frac{\partial^2 H}{\partial t^2},
\end{aligned}$$

after using Euler's identity for homogeneous functions. We have the following equation for the support function of a normal-preserving flow

$$\frac{\partial^2 H}{\partial t^2} = -F. \tag{2.6}$$

3. Local Solvability

We will establish the local solvability for the normal-preserving flow (2) where F is a function depending on the principal curvatures of the hypersurface. This will be achieved by reducing it to a fully nonlinear hyperbolic equation. Local solvability for fully nonlinear hyperbolic equations is more or less standard. Here, we present it in a way so that the regularity requirement on extending the solution further in time is clear. We will not discuss local solvability for normal flows, which is related to quasilinear hyperbolic equations. The reader may consult [**LS**] for typical results.

Consider the initial value problem for the flow (2), that is,

$$\begin{cases} \dfrac{\partial^2 X}{\partial t^2} = F\mathbf{n} - b^{ij}\dfrac{\partial F}{\partial z_i}\dfrac{\partial X}{\partial z_j}, \\ X(0) \text{ and } X_t(0) \quad \text{are given,} \end{cases} \tag{3.1}$$

for a normal-preserving flow. Due to the definition of a normal-preserving flow, we may always take the independent variable z to be the unit outer normal of $X(\cdot,t)$. Here, F is a curvature function. Following the formulation in Urbas [**U1**], which is based on Caffarelli, Nirenberg, and Spruck's theory of fully nonlinear elliptic equations [**CNS**], we take it to be a function $f(R_1,\ldots,R_n)$, where $R_1,\ldots,R_n$ are the principal radii of curvature for a uniformly convex hypersurface in $\mathbb{R}^{n+1}$. The smooth function f is symmetric in the positive cone $\Gamma^+ = \{R = (R_1,\ldots,\ R_n) : R_i > 0,\ i = 1,\ldots,n\}$. Moreover, it satisfies the monotonicity condition

$$\frac{\partial f}{\partial R_j}(R_1,\ldots,R_n) < 0, \quad j = 1,\ldots,n, \quad (R_1,\ldots,R_n) \in \Gamma^+. \tag{3.2}$$

Theorem 3.1. *Consider* (3.1) *under* (3.2)*, where* $X(0)$ *is a uniformly convex hypersurface in* $\mathbb{R}^{n+1}$ *parametrized by its unit outer normal and* $X_t(0)$ *satisfies* $\langle \partial X_t(0)/\partial z_j, \boldsymbol{n}\rangle = 0,\ j = 1,\ldots,n$. *Suppose* $X(0) \in H^k(S^n)$ *and* $X_t(0) \in H^{k-1}(S^n), k > n/2+2$. *There exists a positive* $T \leq \infty$ *such that* (3.1) *has a unique solution* $X(t)$ *in* $C([0,T),H^k(S^n)) \bigcap C^1([0,T),H^{k-1}(S^n))$ *that is parametrized by its unit outer normal and uniformly convex at each* t. *It is smooth provided* $X(0)$ *and* $X_t(0)$ *are smooth. Moreover, it is maximal in the sense that if* T *is finite, either the minimum of the principal radii of curvatures of* $X(t)$ *tends to zero or*

$$||X(t)||_{C^2(S^n)} \to \infty,$$

as t *approaches* T.

To prove this theorem, we look at the initial value problem for the associated equation satisfied by the support functions $H(z,t)$ of the hypersurfaces. By (2.6),

$$\begin{cases} \dfrac{\partial^2 H}{\partial t^2} = -f(R_1, \ldots, R_n), & (z,t) \in S^n \times [0,T), \\ H(0) \text{ and } H_t(0) \text{ are given} , \end{cases} \tag{3.3}$$

where $H(0)$ is the support function of a uniformly convex hypersurface and $H_t(0)$ is of homogeneous degree 1. Our first job is to express the right-hand side of the equation in (3.3) in terms of the support function and its derivatives.

Let X be a uniformly convex hypersurface with support function H that is of homogenous degree 1. We may fix a point on S^n and consider the restriction of H on the tangent space through this point. For a typical choice, take the point to be the south pole $(0, \ldots, 0, -1)$ and set, for $x \in \mathbb{R}^n$,

$$u(x_1, \ldots, x_n) = H(x_1, \ldots, x_n, -1) = \sqrt{1+|x|^2}\, H\Big(\frac{x_1, \ldots, x_n, -1}{\sqrt{1+|x|^2}}\Big).$$

The second fundamental form of the hypersurface at the point $X(z)$ is given by

$$b_{ij}(x) = \frac{u_{ij}(x)}{\sqrt{1+|x|^2}}, \qquad z = \frac{(x,-1)}{\sqrt{1+|x|^2}}.$$

The radii of principal curvatures are the eigenvalues of the induced metric of X with respect to the second fundamental form, i.e., $\det\{g_{ij} - Rb_{ij}\} = 0$. It turns out they are the eigenvalues of the matrix (s_{ij}) given by

$$s_{ij} = (1+|x|^2)^{\frac{1}{2}} (\delta_{ik} + x_i x_k) u_{jk};$$

see [**U1**]. This matrix is not symmetric. However, observing that the symmetric matrix given by

$$\hat{s}_{ij} = \Big(\delta_{ik} + \frac{x_i x_k}{1+(1+|x|^2)^{\frac{1}{2}}}\Big)\Big(\delta_{jl} + \frac{x_j x_l}{1+(1+|x|^2)^{\frac{1}{2}}}\Big) u_{kl} \tag{3.4}$$

shares the same eigenvalues with (b_{ij}) [**CNS**], we know there exists a smooth function F such that

$$F(\hat{s}_{ij}) = f(R_1, \ldots, R_n)$$

by our assumptions on f. The eigenvalues of the matrix $(\partial F/\partial z_{ij})$ are given precisely by $\partial f/\partial R_1, \ldots, \partial f/\partial R_n$ [**CNS**], and so (3.2) is equivalent to

$$\frac{\partial F}{\partial z_{ij}}(A) < 0, \tag{3.5}$$

on all positive definite matrices A.

Restricting on the hyperplane $x_{n+1} = -1$, (3.3) becomes

$$
\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = -(1+|x|^2)F(\hat{s}_{ij}), \\ u(0) \text{ and } u_t(0) \text{ are given }, \end{cases} \tag{3.6}
$$

where $(\hat{s}_{ij})$ is in (3.4).

The above discussion leads us to the general fully nonlinear hyperbolic equation

$$
\begin{cases} \dfrac{\partial^2 v}{\partial t^2} = \phi(x, D_x^2 v), \quad (x,t) \in \mathbb{R}^n \times [0,T), \\ v(0) = f, \ \dfrac{\partial v}{\partial t}(0) = g, \end{cases} \tag{3.7}
$$

where the smooth function $\phi(x, z_{ij})$ satisfies $\phi(x, z_{ij}) = \phi(x, z_{ji})$ and the ellipticity condition: There exists a symmetric matrix Z_0 such that for any symmetric matrix Z satisfying $Z_0 + Z$ is positive definite,

$$
\frac{\partial \phi}{\partial z_{ij}}(x, Z) > 0. \tag{3.8}
$$

Clearly, this condition is satisfied for (3.6) under (3.5) for v being $u - u(0)$ and Z_0 the Hessian of $u(0)$.

We would like to solve (3.7) locally in time. To do this we first reduce it to a quasilinear system of second order equations. In fact, for each $k = 1, \dots, n$, $v_k = \partial v/\partial x_k$ satisfies

$$
\begin{cases} \dfrac{\partial^2 v_k}{\partial t^2} = a^{ij} \dfrac{\partial^2 v_k}{\partial x_i \partial x_j} + b^k, \\ v_k(0) = \dfrac{\partial f}{\partial x_k}, \ \dfrac{\partial v_k}{\partial t}(0) = \dfrac{\partial g}{\partial x_k}, \end{cases}
$$

where $a^{ij} = \phi_{z_{ij}}(x, D_x^2 v)$ and $b^k = \phi_{x_k}(x, D_x^2 v)$. Let us consider a second-order, quasilinear system for $\mathbf{v} = (v^1, \dots, v^n)$,

$$
\begin{cases} \dfrac{\partial^2 v^k}{\partial t^2} = a^{ij} \dfrac{\partial^2 v^k}{\partial x_i \partial x_j} + b^k, \\ \mathbf{v}(0) \text{ and } \dfrac{\partial \mathbf{v}}{\partial t}(0) \ \text{ are given}, \end{cases} \tag{3.9}
$$

where $a^{ij} = a^{ij}(x, D_x \mathbf{v})$, $b^k = b^k(x, D_x \mathbf{v})$, and $D_x \mathbf{v} = (\nabla v^1, \dots, \nabla v^n)$. Clearly, $\mathbf{v} = (\partial v/\partial x_1, \dots, \partial v/\partial x_n)$ solves (3.9) whenever v is a solution of (3.7). On the other hand, we assert that if $\mathbf{v}$ solves (3.9) with $\mathbf{v}(0) = \nabla f$ and $\mathbf{v}_t(0) = \nabla g$, then a solution to (3.7) can be found.

We differentiate (3.9) in x_l to obtain

$$
v^k_{ltt} = \phi_{z_{ij}} v^k_{lij} + \phi_{z_{ij} z_{mn}} v^m_{ln} v^k_{ij} + \phi_{z_{ij} x_l} v^k_{ij} + \phi_{x_k z_{ij}} v^i_{lj} + \phi_{x_k x_l}.
$$

It follows that

$$\begin{aligned}(v_l^k - v_k^l)_{tt} &= \phi_{z_{ij}}(v_l^k - v_k^l)_{ij} + \phi_{z_{ij}z_{mn}}(v_{ln}^m v_{ij}^k - v_{kn}^m v_{ij}^l) \\ &\quad +\phi_{z_{ij}x_l} v_{ij}^k - \phi_{z_{ij}x_k} v_{ij}^l + \phi_{z_k z_{ij}} v_{lj}^i - \phi_{x_l z_{ij}} v_{kj}^i \\ &= \phi_{z_{ij}}(v_l^k - v_k^l)_{ij} + \phi_{z_{ij}z_{mn}}[v_{lj}^i(v_m^k - v_k^m)_n + v_{kn}^m(v_l^i - v_i^l)_j] \\ &\quad +\phi_{z_{ij}x_l}(v_i^k - v_k^i)_j + \phi_{z_{ij}x_k}(v_l^i - v_i^l)_j,\end{aligned}$$

after using

$$\phi_{z_{ij}z_{mn}} v_{ln}^m v_{ij}^k = \phi_{z_{ij}z_{mn}} v_{lj}^i v_{mn}^k.$$

Thus, $\omega^{kl} \equiv v_l^k - v_k^l$ satisfies

$$\begin{cases} \dfrac{\partial^2 \omega^{kl}}{\partial t^2} = a^{ij} \dfrac{\partial^2 \omega^{kl}}{\partial x_i \partial x_j} + c_{ij}^{klm} \dfrac{\partial \omega^{ij}}{\partial x_m} \\ \omega^{kl}(0) = 0, \ \dfrac{\partial \omega^{kl}}{\partial t}(0) = 0, \end{cases} \tag{3.10}$$

for some functions c_{ij}^{klm}. One can show that the solution to this linear system is identically zero (see the remark below), so $v_l^k = v_k^l$ for each k, l and there exists a potential function $\tilde{v}$ such that $\partial \tilde{v}/\partial x_k = v^k$. Consequently,

$$\frac{\partial^2 \tilde{v}}{\partial t^2} = \phi(x, D_x^2 \tilde{v}) + c(t)$$

holds for some function $c(t)$. At $t = 0$

$$\tilde{v}(x,0) = f(x) + c_1 \quad \text{and} \quad \tilde{v}_t(x,0) = g(x) + c_2,$$

for some constants c_1 and c_2. A solution for (3.7) is found by taking $v(x,t) = \tilde{v}(x,t) + \chi(t)$ where χ solves $\chi'' = -c(t)$, $\chi(0) = -c_1$, and $\chi'(0) = -c_2$.

We have reduced the solvability of (3.7) to that of (3.9). A further step is to reduce (3.9) to a first-order, quasilinear system.

Consider the following system for an $\mathbb{R}^{(n+2)n}$-valued function $\mathbf{w}$

$$\begin{cases} \dfrac{\partial w^k}{\partial t} = w^{k0}, \\ \dfrac{\partial w^{k0}}{\partial t} = a^{ij} \dfrac{\partial w^{ki}}{\partial x_j} + b^k, \\ \dfrac{\partial w^{kj}}{\partial t} = \dfrac{\partial w^{k0}}{\partial x_j}, \\ \mathbf{w}(0) \ \text{ given }, \end{cases} \tag{3.11}$$

where $\mathbf{w} = (w^1, w^{10}, w^{11}, \dots, w^{1n}, \dots, w^n, w^{n0}, w^{n1}, \dots, w^{nn})$ and the coefficients a^{ij} and b^k are evaluated at $(x, w^{11}, \dots, w^{1n}, \dots, w^{n1}, \dots, w^{nn})$. When $\mathbf{w}(0) = (v^1(0), v_t^1(0), v_1^1(0), \dots, v_1^n(0), \dots, v^n(0), v_t^n(0), v_1^n(0), \dots,$

$v^n_n(0)$) and the initial data $\mathbf{v}(0)$ and $\mathbf{v}_t(0)$ are given in (3.9), the function $\mathbf{w} = (v^1, v^1_t, v^1_1, \ldots, v^1_n, \ldots, v^n, v^n_t, v^n_1, \ldots, v^n_n)$ solves (3.11). Conversely, let $\mathbf{w}$ be a solution of (3.11) satisfying these special initial values. Then, for $k, l = 1, \ldots, n$,

$$\frac{\partial}{\partial t}\Big(w^{lk} - \frac{\partial w^l}{\partial x_k}\Big) = \frac{\partial w^{l0}}{\partial x_k} - \frac{\partial}{\partial x_k}\frac{\partial w^l}{\partial t} = 0,$$

whence $\mathbf{v} \equiv (w^1, \ldots, w^n)$ solves (3.9).

To solve (3.11), we note that for each k, $\mathbf{w}^k \equiv (w^k, w^{k0}, \ldots, w^{kn})$ satisfies

$$\frac{\partial \mathbf{w}^k}{\partial t} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a^{11} & \cdots & a^{n1} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \frac{\partial \mathbf{w}^k}{\partial x_1} + \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a^{12} & \cdots & a^{n2} \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \frac{\partial \mathbf{w}^k}{\partial x_2} +$$

$$\cdots + \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a^{1n} & \cdots & a^{nn} \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix} \frac{\partial \mathbf{w}^k}{\partial x_n} + \mathbf{b},$$

where $\mathbf{b} = \mathbf{b}(x, \mathbf{w})$. By multiplying this system with the matrix $\mathcal{R}$ that is the the n-copies direct sum of the $(n+2) \times (n+2)$ matrix

$$R = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \cdot & \cdot & & a^{ij} & \\ 0 & 0 & & & \end{bmatrix},$$

we obtain

$$\mathcal{R}\frac{\partial \mathbf{w}}{\partial t} = A^j \frac{\partial \mathbf{w}}{\partial x_j} + \mathbf{c}, \tag{3.12}$$

where A^j's are now symmetric and $\mathbf{c} \equiv \mathcal{R}\mathbf{b}$. When (3.12) is derived from (3.9), $\mathcal{R}$ is positive definite under (3.8), and so this is a quasilinear symmetric hyperbolic system.

Remark 3.1. We sketch how to show that (3.10) only admits the trivial solution. By introducing the new variable $W = (w^{kl}, w^{kl}_t, w^{kl}_j)$, we can make (3.10) into a first-order linear system for W with zero initial data. By multiplying this system with the matrix that is the n^2 copies direct sum of R given above, we can turn it into a linear hyperbolic system. By the energy estimate, we deduce the Grownall's inequality $d||W||^2_{L^2}/dt \leq C||W||_{L^2}$. It follows that W vanishes identically.

The theory of quasilinear symmetric hyperbolic systems is well known. Consider a general system (3.12) where $\mathcal{R}, A^j$, and $\mathbf{c}$ are smooth functions of $(x, \mathbf{w}) \in \mathbb{R}^n \times \mathcal{U}$, $\mathcal{U}$ an open set in $\mathbb{R}^N$ for some N. Moreoever, $\mathcal{R}$ and A^j's are symmetric $N \times N$ matrices, and all eigenvalues of $\mathcal{R}$ are positive in $\mathbb{R}^n \times \mathcal{U}$. The following facts can be found or derived easily from Taylor [**T**].

Fact (a). For any $\mathbf{w}(0) \in H^k(\mathbb{R}^n)$, $k > N/2+1$, with $\mathbf{w} \in \mathcal{V}$ where $\mathcal{V}$ is an open set compactly contained in $\mathcal{U}$, (3.12) has a unique classical solution $\mathbf{w}$ defined on some interval $[0,T)$, $T > 0, \mathbf{w}(t) \in \mathcal{V}$, which belongs to $C([0,T), H^k(\mathbb{R}^n)) \bigcap C^1([0,T), H^{k-1}(\mathbb{R}^n))$,

Fact (b). Suppose $||\mathbf{w}(t)||_{C^1}$ is uniformly bounded for $t \in [0,T)$. Then there exists $T_1 > T$ such that the solution extends to $C([0,T_1), H^k(\mathbb{R}^n))$ with $\mathbf{w}(t)$ in $\mathcal{U}$.

Fact (c). $\mathbf{w}(x,t)$ is smooth in $\mathbb{R}^n \times [0,T)$ if $\mathbf{w}(0)$ is smooth at $t=0$.

By choosing a suitable $\mathcal{U}$ for our situation, we deduce from these facts that there exists a unique solution to (3.12) on a maximal interval $[0, T_{max}), T_{max} \leq \infty$, in the sense that when T_{max} is finite, either $\lambda(t)$, the lowest eigenvalue of $(w^{ij}(t)) + Z_0, i,j = 1, \ldots, n$ where Z_0 is the Hessian of $u(0)$—see (3.8)—satisfies

$$\inf_t \lambda(t) \to 0$$

or

$$\sup_t ||\mathbf{w}(t)||_{C^1} \to \infty,$$

as $t \uparrow T_{max}$.

Proof of Theorem 3.1.

Set $v = u - u(0)$ in (3.6) and consider the problem

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2} = -(1+|x|^2)F(\hat{s}_{ij}), \\ v(0) = 0 \text{ and } v_t(0) \text{ is given,} \end{cases} \tag{3.13}$$

where

$$\hat{s}_{ij} = \Big(\delta_{ik} + \frac{x_i x_k}{1+(1+|x|^2)^{\frac{1}{2}}}\Big)\Big(\delta_{jl} + \frac{x_j x_l}{1+(1+|x|^2)^{\frac{1}{2}}}\Big)(u(0)_{ij} + v_{ij})$$

and $v_t(0)$ is a function compactly supported in $\mathbb{R}^n$ that is equal to $u_t(0)$ in the unit ball $B_1 = \{x \in \mathbb{R}^n : |x| < 1\}$. From our discussion for $u(0) \in H^{k+2}(\mathbb{R}^n)$ and $v_t(0) \in H^{k+1}(\mathbb{R}^n)$, $k > n/2+1$, problem (3.13) has a solution $v(t)$, $t \in [0,T)$. Then $u = v + u(0)$ solves the equation in (3.6). By finite speed of propagation of solutions (Mizohata [**M**]) for hyperbolic equations, there exists a time $T > 0$ such that the values of $v(x,t), (x,t) \in B_{1/2} \times [0,T)$ depend only on the initial values in B_1. Hence, u solves (3.6) in $B_{1/2} \times [0,T)$. Passing through the tangent space at each point z on the unit sphere, we can obtain a similar solution in $[0, T(z))$. The balls obtained by projecting all $B_{1/2}$

on these tangent spaces to the sphere form an open cover of the sphere. We can choose finitely many balls to cover the sphere. Letting $T = \min\{T(z_1), \ldots, T(z_N)\}$ where $z_j, j = 1, \ldots, N$, are centers of these balls, it is clear that one can construct $H(z,t)$ on $[0,T)$ by putting these solutions u together. We have shown that (3.3) is locally solvable.

Letting $X^k(z,t) = \partial H/\partial z_k$, we have

$$\Big\langle \frac{\partial X}{\partial z_k}, \frac{\partial \mathbf{n}}{\partial t}, \Big\rangle = -\Big\langle \frac{\partial^2 X}{\partial z_k \partial t}, \mathbf{n} \Big\rangle = -\sum_{j=1}^{n} z^j \frac{\partial}{\partial z_j}\frac{\partial H_t}{\partial z_k} = 0,$$

by Euler's identity for homogeneous functions. It follows that X satisfies the normal-preserving condition. By Proposition 2.2, it solves (3.1) on $[0,T)$.

The assertion on smoothness of X follows from Fact (c) above. Finally, from the expression relating X and H we see that the C^3-norm of H is controlled by the C^2-norm of X. The proof of Theorem 3.1 is completed.

4. Finite Time Blow-up

After establishing the local solvability for some normal-preserving flows driven by curvatures, we turn to other properties of the flows such as the formation of finite-time singularities and long-time behavior. In the literature, numerous results concerning these topics are available for fully nonlinear parabolic flows. As a preliminary study, we shall focus on the Gauss curvature flow. We take $F = K^\alpha, \alpha > 0$, where K is the Gauss curvature of the hypersurface in (3.1), and we call the resulting flow the contracting Gauss curvature flow. Its parabolic counterpart has been studied by several authors, including [**F**], [**T**], [**C**], [**A3**], and [**A4**]. A common feature is, for any closed uniformly convex hypersurface $X(0)$, that $X(t)$ contracts to a point in finite time and its ultimate shape is largely known when α is less than or equal to $1/n$. To examine the same question for the hyperbolic case, we first consider a special case, namely, the initial hypersurface is a sphere and its initial velocity is given by $R_1\mathbf{n}$, for some real number R_1. Under these assumptions, this flow reduces to an ODE for $R(t)$, the radius of the sphere at time t,

$$\begin{cases} R'' = -\dfrac{1}{R^{n\alpha}}, \\ R(0) = R_0 > 0, \;\; R'(0) = R_1. \end{cases}$$

The following proposition is easily proved.

Proposition 4.1. *Let $c = R_1^2/2 - R_0^{1-n\alpha}/(n\alpha - 1)$. For $\alpha \in (1/n, \infty)$,*
(a) when $R_1 < 0$ and $c \in \mathbb{R}$, the sphere contracts to a point in finite time,

(b) when $R_1 > 0$ and $c < 0$, the sphere expands first and then contracts to a point in finite time; when $c \geq 0$, it expands to ∞ and

$$R(t) = \begin{cases} O(t), & c > 0, \\ O(t^{\frac{2}{n\alpha+1}}), & c = 0, \end{cases}$$

as $t \to \infty$.

For $\alpha \in (0, 1/n]$, c is always positive, and
(c) when $R_1 \geq 0$, the sphere expands first and then contracts to a point in finite time,
(d) when $R_1 < 0$, the sphere contracts to a point in finite time.

Thus, unlike the parabolic case, inward acceleration does not necessarily mean contraction for the hypersurface. The initial velocity plays a role. Nevertheless, for $\alpha \in (0, 1/n]$, although the sphere may expand for a while, it eventually contracts to its center in finite time. For uniformly convex hypersurfaces with general initial data, we have the following proposition:

Proposition 4.2. *Any normal preserving contracting Gauss curvature flow blows up in finite time for for $\alpha \in (0, 1/n]$.*

Proof. Let $H(\cdot, t)$ be the support function of this flow. By (2.6), it satisfies

$$\begin{cases} \dfrac{\partial^2 H}{\partial t^2} = -K^\alpha, \\ H(0) \ \text{ and } H_t(0) \ \text{ are given.} \end{cases}$$

Let us assume it exists for all time and draw a contradiction. First of all, we have

$$\begin{aligned} \sigma_n &= \int_X K ds \\ &\leq \Big(\int_X K^{\alpha+1} ds\Big)^{\frac{1}{\alpha+1}} \Big(\int_X ds\Big)^{\frac{\alpha}{\alpha+1}} \\ &= \Big(\int_{S^n} K^\alpha dz\Big)^{\frac{1}{\alpha+1}} A(t)^{\frac{\alpha}{\alpha+1}}, \end{aligned}$$

where $\sigma_n = |S^n|$ and $A(t)$ is the surface area of $X(t)$. On the other hand, from

$$H(z,t) = H(z,0) + \int_0^t \frac{\partial H}{\partial t}(z,s) ds \leq H(z,0) + \sup_z \frac{\partial H}{\partial t}(z,0) t,$$

we see that the growth of the support function is at most linear. Therefore, the surface area satisfies

$$A(t) \leq C(1 + t^n),$$

for some constant C. It follows that

$$\begin{aligned}\int H_{tt} &= -\int K^{\alpha} \\ &\leq -\frac{\sigma_n^{\alpha+1}}{A(t)^{\alpha}} \\ &\leq -\frac{\sigma_n^{\alpha+1}}{C(1+t^n)} \\ &\leq C_1 - C_2 t^{1-\alpha n},\end{aligned}$$

for some constants C_1 and C_2. When $n\alpha = 1$, the term $C_2 t^{1-n\alpha}$ should be replaced by $C_2 \log t$. Therefore,

$$\begin{aligned}\int H(z,t)dz &= \int H(z,0)dz + \int_0^t \int H_t(z,s)dzds \\ &\leq \int H(z,0)dz + C_1 t - \frac{C_2}{2-n\alpha} t^{2-n\alpha},\end{aligned}$$

becomes negative for large time. The same conclusion holds when $n\alpha = 1$. However, the integral of $H(t)$ is the mean width of the convex body enclosed by $X(t)$ (see [**S**]) and it cannot be negative. To see this we note that when the origin is contained inside the convex body, the support function is nonnegative everywhere, and so this integral is nonnegative. When one uses different coordinates to represent the support functions, they differ from each other only by a linear function; hence the integrals are the same. Thus, we have arrived at a contradiction. We conclude that the solution of (3.1) cannot exist for all time when $n\alpha$ is less than or equal to 1. q.e.d.

A natural question is: Could the hypersurface develop a singularity before it contracts to a point under this contracting flow? We believe this is possible, although an example is out of our hand. Nevertheless, we present a noncompact example where an isolated singularity develops in finite time for α in $(0, 1/n)$.

Let C be a convex cone based at the origin in $\mathbb{R}^{n+1}$ whose cross section is bounded by a closed, uniformly convex hypersurface. According to Urbas [**U1**], there exists a uniformly convex hypersurface X^* sitting inside C and asymptotic to its boundary at ∞ satisfying

$$\left\langle X^*, \mathbf{n} \right\rangle = K^{\alpha}.$$

Consider the ODE for $\alpha \in (0, 1/n)$,

$$\phi'' = \frac{1}{\phi^{n\alpha}}, \quad \phi(0) = 1,\ \phi'(0) = \phi_1 < 0.$$

When ϕ_1 satisfies $\phi_1^2 > 2/(1-n\alpha)$, it is easy to see that it has a solution in $[0, T)$ for some T and $\phi(t) \to 0$, as $t \uparrow T$. Letting $X(t) = \phi(t)X^*$, it is readily verified that $X(t)$ solves the contracting Gauss curvature flow with $X(0) = X^*$, and geometrically it collapses to the boundary of C as t approaches T. We see that the curvature blows up only at the origin. Away from the origin, the hypersurface remains smooth, while its Gauss curvature vanishes.

Next we present a necessary condition for the existence of global normal preserving flows (3.1) when F is positive. It leads to a criterion for finite time blow-up for special initial velocity.

Proposition 4.3. *Let X be a normal preserving flow solving* (3.1) *in $S^n \times [0,\infty)$ where $F > 0$. Then its support function $H(z,t)$ must satisfy*

$$H_t(z,0) + H_t(-z,0) \geq 0, \quad \textit{for all } z.$$

Proof. Let X be a global normal-preserving solution of (3.1). Then $\tilde{X} \equiv (t, X(t))$ is a hypersurface in $[0,\infty) \times \mathbb{R}^{n+1}$. When expressed locally as a graph of some function, the Gauss curvature of $\tilde{X}$ is of the same sign as the determinant of the Hessian matrix of this function, which is positive by (2.5) when F is positive. Therefore, $\tilde{X}$ is a uniformly convex hypersurface in $[0,\infty) \times \mathbb{R}^{n+1}$. In a coordinate system, $\tilde{X}$ is expressed as the union of the graphs of two uniformly convex functions $u(x,t)$, and $v(x,t)$ defined in the closure of some convex domain Ω satisfying $v < u$ in Ω. Given a point $X(z_0, 0)$ on the initial hypersurface, we may choose a coordinate system such that this point is $(x_0, u(x_0,0))$ and its unit outer normal is $(0,\ldots,0,1)$; that is, $\nabla u(x_0,0) = (0,\ldots,0)$ holds. Let $(y_0, v(y_0,0))$ be the unique point on $X(0)$ satisfying $\nabla v(y_0,0) = (0,\ldots,0)$. Its unit outer normal is given by $(0,\ldots,0,-1)$. So the tangent hyperplanes at $(x_0, u(x_0,0))$ and $(y_0, v(y_0,0))$ are parallel in $\mathbb{R}^{n+1}$.

The tangent hyperplanes of $\tilde{X}$ at $(0, x_0, u(x_0))$ and $(0, y_0, v(y_0))$ are given, respectively, by

$$P_1 = \{(t,x,u) : \quad u_t(x_0,0)t = u - u(x_0,0)\}$$

and

$$P_2 = \{(t,x,u) : \quad v_t(y_0,0)t = u - v(y_0,0)\}.$$

When $\tilde{X}$ is global, P_1 always sits above P_2, so they never intersect. It means that these two hyperplanes either do not intersect or they intersect at negative time. When the latter happens, the intersection time is given

$$T = \frac{v(y_0,0) - u(x_0,0)}{u_t(x_0,0) - v_t(y_0,0)} < 0.$$

It follows that

$$u_t(x_0,0) \geq v_t(y_0,0) \tag{4.1}$$

must hold.

We express (4.1) in terms of the support function. By differentiating the relation $X(0) = (x, u(x,0))$, we have $X_t = (x_t, u_t + u_j x_t^j)$. As the outer normal of $X(0)$ at z_0 is $(0,\ldots,0,1)$, $u_t(x_0,0) = \langle X_t(z_0,0), z_0\rangle =$

$H_t(z_0, 0)$. Similarly, we have $H_t(-z_0, 0) = -v_t(y_0, 0)$, and hence $H_t(z_0, 0) + H_t(-z_0, t) \geq 0$ from (4.1). q.e.d.

Condition (4.1) can be rewritten as $\langle X_t(z, 0), z\rangle + \langle X_t(-z, 0), -z\rangle \geq 0$ for all outer normal z. Noting that $\langle X_t(z, 0), z\rangle$ is the "outer normal speed" along z, the sum of the inner normal speed along z and $-z$ may be called the "net outer normal speed" along z. This condition implies the following criterion for finite time blow-up: The solution cannot exist for all time when the "net outer normal speed" is negative for some z. In fact, an upper bound on its life span is given by

$$\inf\left\{\frac{-\text{the width along } z}{\text{the net outer normal speed along } z} :\ z \in P\right\},$$

where P is the subset of the upper hemisphere consisting of all z along which the net inner normal speed is negative. Note that the width along z is given by $H(z, 0) + H(-z, 0)$ and is equal to $u(x_0, 0) - v(y_0, 0)$ in the above proof.

Finally, we consider the expanding Gauss curvature flow by taking $F = -K^{-\beta}, \beta > 0$, in (3.1). Results on parabolic expanding Gauss curvature flows can be found in, for instance, Urbas [**U1**, **U3**] and Chow and Tsai [**CT**]. The hypersurface expands to infinity in infinite time and becomes round when β is less than or equal to $1/n$. When $\beta = 1$ and $n = 2$, it is known that the surface expands to infinity like a sphere in finite time by Schnürer [**S**]. In the hyperbolic case, we examine the motion of a sphere first. Indeed, when $X(0)$ is a sphere of radius R_0 and $X_t(0)$ has constant normal speed R_1, we have the following proposition:

Proposition 4.4. *Let $c = R_1^2/2 - R_0^{1+n\beta}/(n\beta + 1)$. For $\beta > 0$:*
(a) When $R_1 > 0$ and $c \in \mathbb{R}$, the sphere expands to infinity as $t \uparrow T$, where T is finite when $\beta \in (1/n, \infty)$ and is infinite when $\beta \in (0, 1/n]$. In fact, as $t \to \infty$,

$$R(t) = \begin{cases} O(t^{\frac{2}{1-n\beta}}), & \beta < \dfrac{1}{n}, \\ O(e^t), & \beta = \dfrac{1}{n}. \end{cases}$$

(b) When $R_1 < 0$ and $c < 0$, the sphere first contracts and then expands to infinity behaving like in (a); when $R_1 < 0$ and $c \geq 0$, the sphere contracts to a point in finite time.

There is a special case, namely, $\beta = 1$ and $n = 1$, where a rather complete analysis is possible. In this case, the expanding flow becomes, in terms of its support function, a linear problem

$$\begin{cases} h_{tt} = h_{\theta\theta} + h, \\ h(0) = f,\ h_t(0) = g. \end{cases} \tag{4.2}$$

The solution can be represented by the cosine series, namely,

$$\begin{aligned} h(\theta,t) &= \frac{a_0-a_0'}{2}e^{-t}+\frac{a_0+a_0'}{2}e^{t}+(a_1+a_1't)\cos\theta \\ &\quad+\sum_2^\infty\left(a_j\cos\sqrt{j^2-1}\ t+\frac{a_j'}{\sqrt{j^2-1}}\sin\sqrt{j^2-1}\ t\right)\cos j\theta, \end{aligned}$$

provided

$$f(\theta)=a_0+\sum_1^\infty(a_j\cos j\theta+b_j\sin j\theta)$$

and

$$g(\theta)=a_0'+\sum_1^\infty(a_j'\cos j\theta+b_j'\sin j\theta).$$

For some choice of f and g, we show that a uniformly convex initial curve may develop an isolated singularity in finite time. If we take $a_0=2, a_2=-1/3, a_0'=-4$, all other coefficients vanish in the above expression for h. Then we have

$$(h_{\theta\theta}+h)(\theta,t)=3e^{-t}-e^{t}+\cos\sqrt{3}t\cos 2\theta,$$

which is positive at $t=0$. However, there exists a time $T\sim 0.3$ such that $h_{\theta\theta}+h$ is positive on $[0,T)$ but vanishes at $(\pm\pi/2,T)$. As the curvature of the solution curve is given by the reciprocal of $h_{\theta\theta}+h$, the flow is regular in $[0,T)$ and develops two isolated singularities at T.

On the other hand, the flow behaves nicely for a class of initial values.

Proposition 4.5. *Consider* (4.2) *where the initial values are smooth and satisfy* $f_{\theta\theta}+f,\ g_{\theta\theta}+g>0$. *Then the flow remains smooth and expands to infinity like a circle.*

Proof. It suffices to show that $h_{\theta\theta}+h$ is positive for all t. We note that $\varphi=h_{\theta\theta}+h$ satisfies the one-dimensional wave equation with a zeroth order term $\varphi_{tt}=\varphi_{\theta\theta}+\varphi$, and the initial values $\varphi(0)=f_{\theta\theta}+f,\ \varphi_t(0)=g_{\theta\theta}+g$ are positive. Therefore, we may apply the maximum principle for one-dimensional wave equation; see section 2 in chapter 4 of Protter and Weinberger [**PW**] to obtain the desired conclusion. The asymptotic behavior of the flow can be read off from the formula of the support function. q.e.d.

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