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ON THE TRANSVERSE INVARIANT FOR BINDINGS OF OPEN BOOKS

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Abstract

Let $T \subset (Y, \xi)$ be a transverse knot which is the binding of some open book, (T, π), for the ambient contact manifold (Y, ξ). In this paper, we show that the transverse invariant $\widehat{\mathcal{T}}(T) \in \widehat{\mathrm{HFK}}(-Y, K)$, defined in [**LOSSz09**], is nonvanishing for such transverse knots. This is true regardless of whether or not ξ is tight. We also prove a vanishing theorem for the invariants $\mathcal{L}$ and $\mathcal{T}$. As a corollary, we show that if (T, π) is an open book with connected binding, then the complement of T has no Giroux torsion.

1. Introduction

In a recent paper by Lisca, Ozsváth, Stipsicz, and Szabó [**LOSSz09**], the authors define invariants of null-homologous Legendrian and transverse knots. These invariants live in the knot Floer homology groups of the ambient space with reversed orientation, and generalize previously defined Heegaard Floer invariants of contact structures on 3-manifolds which are either closed or compact with convex boundary. They have been useful in constructing new examples of knot types which are not transversally simple (see [**LOSSz09**, **OS10**]), and play an important role in the classification of Legendrian and transverse twist knots (see [**ENV10**]).

In this paper, we investigate properties of these invariants for a certain important class of transverse knots.

Theorem 1. *Let $T \subset (Y, \xi)$ be a transverse knot which can be realized as the binding of an open book (T, π) compatible with the contact structure ξ. Then, the transverse invariant $\widehat{\mathcal{T}}(T)$ is nonvanishing.*

Remark 1.1. In [**LOSSz09**], it is shown that if $c(Y, \xi) \neq 0$, then $\mathcal{T}(T) \neq 0$ for any transverse knot $T \subset (Y, \xi)$. In Theorem 1, no restrictions are made on the ambient contact structure ξ. In particular, the theorem is true even when ξ is overtwisted. Moreover, the nonvanishing of the invariant $\widehat{\mathcal{T}}$ implies the nonvanishing of the invariant $\mathcal{T}$.

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Let L be a null-homologous Legendrian knot in (Y,ξ). It is shown in [**LOSSz09**] that the invariant $\mathcal{L}(L)$ inside $\mathrm{HFK}^-(-Y,L)$ remains unchanged under negative stabilization, and therefore yields an invariant of transverse knots. If T is a null-homologous transverse knot in (Y,ξ) and L is a Legendrian approximation of T, then $\mathcal{T}(T) := \mathcal{L}(L)$. We will generally state results only in the Legendrian case, even though the same results are also true in the transverse case.

Remark 1.2. There is a natural map $\mathrm{HFK}^-(-Y,L) \to \widehat{\mathrm{HFK}}(-Y,L)$, induced by setting $U = 0$. Under this map, $\mathcal{L}(L)$ is sent to $\widehat{\mathcal{L}}(L)$. Therefore, if $\widehat{\mathcal{L}}(L)$ is nonzero, then $\mathcal{L}(L)$ must also be nonzero. Similarly, $\mathcal{L}(L)$ vanishing implies that $\widehat{\mathcal{L}}(L)$ must also vanish.

In addition to understanding when these invariants are nonzero, we are also interested in circumstances under which they vanish. It is shown in [**LOSSz09**] that if the complement of a Legendrian knot contains an overtwisted disk, then the Legendrian invariant for that knot vanishes. Here, we generalize this result by proving:

Theorem 2. *Let L be a Legendrian knot in a contact manifold (Y,ξ). If the complement $Y-L$ contains a compact submanifold N with convex boundary such that $c(N,\xi|_N) = 0$ in $SFH(-N,\Gamma)$, then the Legendrian invariant $\mathcal{L}(L)$ vanishes.*

Since I-invariant neighborhoods of convex overtwisted disks have vanishing contact invariant (Example 1 of [**HKM09b**]), Theorem 2 generalizes the vanishing theorem from [**LOSSz09**].

In [**GHV07**], Ghiggini, Honda, and Van Horn–Morris show that a closed contact manifold with positive Giroux torsion has vanishing contact invariant. They show this by proving that the contact element for a 2π-torsion layer vanishes in sutured Floer homology. Thus, as an immediate corollary to Theorem 2, we have:

Corollary 3. *Let L be a Legendrian knot in a contact manifold (Y,ξ). If the complement $Y-L$ has positive Giroux torsion, then the Legendrian invariant $\mathcal{L}(L)$ vanishes.*

Remark 1.3. A similar result has been established for the weaker invariant $\widehat{\mathcal{L}}$ by Stipsicz and Vértesi [**SV09**] using slightly different arguments.

Remark 1.4. Theorem 2 and Corollary 3 are true in the transverse case as well.

Combining the transverse version of Corollary 3 with Theorem 1, we conclude the following interesting fact about complements of connected bindings:

Theorem 4. *Let (T,π) be an open book with a single binding component. Then the complement of T has no Giroux torsion.*

As Giroux torsion is presently the only known mechanism for a 3-manifold to admit more than a finite number of tight contact structures, it is important to understand the relationship between tight contact structures with positive Giroux torsion and the open books which support them. Of course, Theorem 4 only applies to connected bindings of open books, leading one to conjecture that the same should be true for arbitrary open book decompositions. We prove this with Etnyre in [**EV10**] using different methods.

Theorem 1.5 (Etnyre-V, [**EV10**])**.** *Let (B, π) be an open book for a contact manifold (Y, ξ). Then the complement of B has no Giroux torsion.*

This paper is organized as follows: in Section 2, we briefly review some of the basic concepts in contact geometry and knot Floer homology. Section 3 is devoted to proving Theorem 1. In Section 4, we conclude with a proof of Theorem 2.

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2. Background

2.1. Contact Geometry Preliminaries. Recall that a contact structure on an oriented 3-manifold is a plane field ξ satisfying a certain nonintegrability condition. We assume that our plane fields are cooriented, and that ξ is given as the kernel of some global 1-form: $\xi = \ker(\alpha)$ with $\alpha(N_p) > 0$ for each oriented normal vector N_p to ξ_p. Such an α is called a *contact form for* ξ. In this case, the nonintegrability condition is equivalent to the statement $\alpha \wedge d\alpha > 0$.

A primary tool used in the study of contact manifolds has been Giroux's correspondence between contact structures on 3-manifolds and open book decompositions up to an equivalence called positive stabilization [**Gir02**]. An *open book decomposition* of a contact 3-manifold (Y, ξ) is a pair (B, π), where B is an oriented, fibered, transverse link and $\pi : (Y - B) \to S^1$ is a fibration of the complement of B by oriented surfaces whose oriented boundary is B.

An open book is said to be *compatible* with a contact structure ξ if, after an isotopy of the contact structure, there exists a contact form α for ξ such that:

1) $\alpha(v) > 0$ for each (nonzero) oriented tangent vector v to B, and
2) $d\alpha$ restricts to a positive area form on each page of the open book.

Given an open book decompositon of a 3-manifold Y, Thurston and Winkelnkemper [**TW75**] show how one can produce a compatible contact structure on Y. Giroux proves in [**Gir02**] that two contact structures which are compatible with the same open book are, in fact, isotopic as contact structures. Giroux also proves that two contact structures are isotopic if and only if they are compatible with open books which are related by a sequence of positive stabilizations.

For more on open book decompositions and their properties, we refer the interested reader to [**Etn06**, **Col08**].

Definition 2.1. A knot L smoothly embedded in a contact manifold (Y, ξ) is called *Legendrian* if $T_pL \subset \xi_p$ for all p in L.

Definition 2.2. An oriented knot T smoothly embedded in a contact manifold (Y, ξ) is called *transverse* if it always intersects the contact planes transversally with each intersection positive.

We say that two Legendrian knots are *Legendrian isotopic* if they are isotopic through Legendrian knots; similarly, two transverse knots are *transversally isotopic* if they are isotopic through transverse knots. Given an oriented Legendrian knot L, one can produce a canonical transverse knot nearby to L, called the transverse pushoff of L. On the other hand, given a transverse knot T, there are many distinct (oriented) Legendrian knots whose transverse pushoffs are T. According to Etnyre and Honda [**EH01**], any two Legendrian approximations of the same transverse knot can be related by a sequences of negative stabilizations (see also [**EFM01**]). In that sense, these two constructions are inverses to one another, up to the ambiguity involved in choosing a Legendrian approximation of a given transverse knot.

As a consequence of the discussion above, if I is an invariant of Legendrian knots which remains unchanged under negative stabilization, then I is also an invariant of transverse knots: if T is a transverse knot and L is a Legendrian approximation of T, define $I(T)$ to be equal to the invariant $I(L)$ of the Legendrian knot L. This is how the authors define the transverse invariants $\mathcal{T}(T)$ and $\widehat{\mathcal{T}}(T)$ in [**LOSSz09**].

We encourage individuals eager to learn more about Legendrian and transverse knots to read Etnyre's survey [**Etn05**].

2.2. Heegaard Floer Preliminaries. Heegaard Floer homology is, broadly speaking, an invariant of 3-manifolds and their subspaces originally defined by Ozsváth and Szabó in [**OSz04a**]. This paper is primarily concerned with a specialization of this general theory which yields invariants of (null-homologous) knots inside closed 3-manifolds. Knot Floer homology, as it is called, was defined by Ozsváth and Szabó in [**OSz04c**] and independently by Rasmussen in [**Ras03**]. It comes in several flavors, but we will only make use of the "minus" and "hat" versions, denoted $\mathrm{HFK}^-(Y, K)$ and $\widehat{\mathrm{HFK}}(Y, K)$ respectively.

In knot Floer homology, the basic input is a doubly-pointed Heegaard diagram; that is, a Heegaard diagram (Σ, α, β), together with two basepoints $w, z \in \Sigma - (\alpha \cup \beta)$, in the complement of the α- and β-curves. These diagrams are required to satisfy either a strong or weak admissibility condition depending on which version of the theory one is working with. Given a doubly pointed Heegaard diagram, one can produce a knot in the resulting 3-manifold. To do this, connect z to w by an arc in the complement of the α-curves, and w to z by an arc in the complement of the β-curves. After depressing the interiors of these arcs into the α- and β-handlebodies, respectively, the result is an oriented knot inside the closed 3-manifold specified by the Heegaard diagram (Σ, α, β). Using a bit of elementary Morse theory, one can show that any knot in any closed 3-manifold can be represented by a doubly-pointed Heegaard diagram.

If the genus of our Heegaard surface Σ is g, then the chain groups for $\mathrm{HFK}^-(Y, K)$ are generated as a $\mathbb{Z}/2[U]$-module by the intersection points between the two g-dimensional subtori $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g$ inside $\mathrm{Sym}^g(\Sigma)$. On the Heegaard diagram, such an intersection point corresponds to an (unordered) g-tuples of intersections $(x_1, \ldots, x_g)$, where $x_i \in \alpha_i \cap \beta_{\sigma(i)}$ for some permutation $\sigma \in S_g$. Given a complex structure on Σ, $\mathrm{Sym}^g(\Sigma)$ inherits a natural complex structure from the projection $\times_g \Sigma \to \mathrm{Sym}^g(\Sigma)$. The boundary map counts certain rigid holomorphic disks in $\mathrm{Sym}^g(\Sigma)$, with boundary lying on $\mathbb{T}_\alpha \cup \mathbb{T}_\beta$, connecting these intersection points:

$$\partial^- \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}),\, \mu(\phi) = 1, \\ n_z(\phi) = 0}} \#\widehat{\mathfrak{M}}(\phi) \cdot U^{n_w(\phi)} \cdot \mathbf{y}.$$

Here $n_v(\phi)$ is equal to the algebraic number of times the disk ϕ intersects the subspace $\{v\} \times \mathrm{Sym}^{g-1}(\Sigma)$; $\pi_2(\mathbf{x}, \mathbf{y})$ is the set of homotopy classes of disks connecting $\mathbf{x}$ to $\mathbf{y}$ with boundaries lying on $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$.

The chain groups for $\widehat{\mathrm{HFK}}(Y, K)$ are generated as a $\mathbb{Z}/2$-vector space again by the intersection points between $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ in $\mathrm{Sym}^g(\Sigma)$. In this case, the boundary map counts rigid holomorphic disks in $\mathrm{Sym}^g(\Sigma)$, with boundaries lying on $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$, now missing both z and w:

$$\widehat{\partial} \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}),\, \mu(\phi) = 1, \\ n_z(\phi) = 0,\, n_w(\phi) = 0}} \#\widehat{\mathfrak{M}}(\phi) \cdot \mathbf{y}.$$

For more information on Heegaard Floer homology and knot Floer homology, we refer the reader to [**OSz04a**, **Lip06**] and [**OSz04c**, **Ras03**], respectively.

2.3. Invariants of Legendrian and Transverse Knots. Let L be a Legendrian knot with knot type K, and let T be a transverse knot in

the same knot type. In [**LOSSz09**], the authors define invariants $\mathcal{L}(L)$ and $\mathcal{T}(T)$ in $\mathrm{HFK}^-(-Y,K)$ and $\widehat{\mathcal{L}}(L)$ and $\widehat{\mathcal{T}}(T)$ in $\widehat{\mathrm{HFK}}(-Y,K)$. These invariants are produced in a similar fashion to the contact invariants in [**HKM09a**, **HKM09b**]. Below we describe how to construct the invariant for a Legendrian knot.

Following the discussion in Section 3 of [**LOSSz09**], let $L \subset (Y,\xi)$ be a null-homologous Legendrian knot. Consider an open book decomposition of (Y,ξ) containing L on a page S. Choose a basis $\{a_0,\dots,a_n\}$ for S (i.e a collection of disjoint, properly embedded arcs $\{a_0,\dots,a_n\}$ such that $S-\bigcup a_i$ is homeomorphic to a disk) with the property that L intersects only the basis element a_0, and does so transversally in a single point. Let $\{b_0,\dots,b_n\}$ be a collection of properly embedded arcs obtained from the a_i by applying a small isotopy so that the endpoints of the arcs are isotoped according to the induced orientation on ∂S and so that each b_i intersects a_i transversally in the single point x_i. If $\phi : S \to S$ is the monodromy map representing the chosen open book decomposition, then a Heegaard diagram for the ambient 3-manifold is given by

$$(\Sigma,\alpha,\beta) = (S_{1/2} \cup -S_1, (a_i \cup a_i), (b_i \cup \phi(b_i))).$$

The first basepoint, z, is placed on the page $S_{1/2}$ in the complement of the thin strips of isotopy between the a_i and b_i. The second basepoint, w, is placed on the page $S_{1/2}$ inside the thin strip of isotopy between a_0 and b_0. The two possible placements of w correspond to the two possible orientations of L.

The Lengendrian invariants $\mathcal{L}(L)$ and $\widehat{\mathcal{L}}(L)$ are defined, up to isomorphism, to be the element $[\mathbf{x}] = [(x_0,\dots,x_n)]$ in $\mathrm{HFK}^-(\Sigma,\beta,\alpha,w,z)$ and $\widehat{\mathrm{HFK}}(\Sigma,\beta,\alpha,w,z)$ respectively. A picture of this construction in the case at hand is given in Figure 4. It is shown in [**LOSSz09**] that $\mathcal{L}(L)$ and $\widehat{\mathcal{L}}(L)$ are fixed under negative stabilizations of L, allowing one to define transverse invariants. If T is a transverse knot, the transverse invariants $\mathcal{T}(T)$ and $\widehat{\mathcal{T}}(T)$ are defined to be the Legendrian invariants of a Legendrian approximation of T.

One interesting property of these invariants is that they do not necessarily vanish for knots in an overtwisted contact manifolds; this is why we do not need to assume tightness in Theorem 1. Another property, which will be useful in Section 3, is that these invariants are natural with respect to contact $(+1)$-surgeries.

Theorem 2.1 (Ozsváth-Stipsicz, [**OS10**])**.** *Let $L \subset (Y,\xi)$ be a Legendrian knot. If (Y_B,ξ_B,L_B) is obtained from (Y,ξ,L) by contact $(+1)$-surgery along a Legendrian knot B in (Y,ξ,L), then under the natural map*

$$F_B : \mathrm{HFK}^-(-Y,L) \to \mathrm{HFK}^-(-Y_B,L_B),$$

$\mathcal{L}(L)$ is mapped to $\mathcal{L}(L_B)$.

Ding and Geiges showed in [**DG04**] that every contact $(+1)$-surgery along a Legendrian knot B can be "canceled" by a contact (-1)-surgery (also called a Legendrian surgery) along a Legendrian pushoff of B (and vice versa). Thus, we have the following immediate corollary to the above theorem:

Corollary 2.2. *Let $L \subset (Y,\xi)$ be a Legendrian knot, and suppose that (Y_B,ξ_B,L_B) is obtained from (Y,ξ,L) by Legendrian surgery along a Legendrian knot B in (Y,ξ,L). If $\mathcal{L}(L) \neq 0$ in $\mathrm{HFK}^-(-Y,L)$, then $\mathcal{L}(L_B) \neq 0$ in $\mathrm{HFK}^-(-Y_B,L_B)$.*

Remark 2.3. Theorem 2.1 and Corollary 2.2 are also true for the invariant $\widehat{\mathcal{L}}(L)$ and for the invariants $\mathcal{T}(T)$ and $\widehat{\mathcal{T}}(T)$ in the case of a transverse knot.

By using Honda, Kazez and Matić's alternate construction of the Ozsváth-Szabó contact invariant [**HKM09a**], we see that under the natural map $\mathrm{HFK}^-(-Y,L) \to \widehat{\mathrm{HF}}(-Y)$ induced by setting $U=1$, $\mathcal{L}(L)$ maps to $c(Y,\xi)$, the contact invariant of the ambient contact manifold.

We encourage the interested reader to look at [**LOSSz09**, **OS10**] to learn about other properties of these invariants.

3. Proof of Theorem 1

Let $T \subset (Y,\xi)$ be a transverse knot. Recall that Theorem 1 states that if T is the binding for some open book (T,π) for (Y,ξ), then the transverse invariant $\widehat{\mathcal{T}}(T) \in \widehat{\mathrm{HFK}}(-Y,T)$ is nonvanishing.

In this section, we prove Theorem 1 in three steps. In Section 3.1 we construct an open book on which a Legendrian approximation L of the transverse knot T sits. Then we show in Section 3.2 that the Heegaard diagram obtained in Section 3.1 is weakly admissible. Finally, in Section 3.3, we prove the theorem in the special case where the monodromy map ϕ_n consists of a product of n negative Dehn twists along a boundary-parallel curve.

An arbitrary monodromy map differs from some such ϕ_n by a sequence of positive Dehn twists, or Legendrian surgeries, along curves contained in pages of the open book. By Corollary 2.2, since the transverse invariant is nonvanishing for the monodromy maps ϕ_n, it must also be nonvanishing for an arbitrary monodromy map.

3.1. Obtaining the pointed diagram. By hypothesis, T is the binding of an open book (T,π) for (Y,ξ). To compute the transverse invariant $\widehat{\mathcal{T}}(T)$, we need to find a Legendrian approximation L of T, realized as a curve on a page of an open book for (Y,ξ).

In Figure 1(a), we see a page of the open book (T,π). Here, T appears as the binding $\partial S = T$. Assuming the curve γ could be realized as a

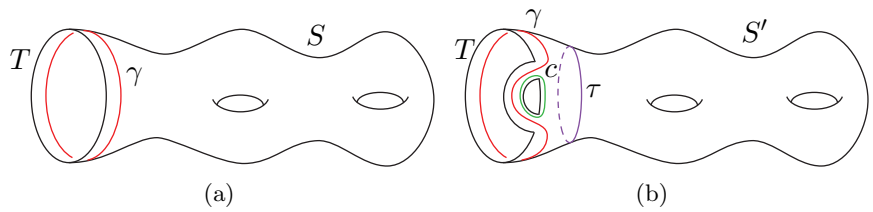


Figure 1

Legendrian curve, it would be the natural choice for the Legendrian approximation L. Unfortunately, since γ is zero in the homology of the page, it is, in Honda's terminology [**Hon00**], isolating, and thus we cannot use his realization principle to make it Legendrian.

To fix this problem, we stabilize the diagram along a boundary parallel arc. The result of such a stabilization is illustrated in Figure 1(b). To see that this solves the problem, we prove the following lemma:

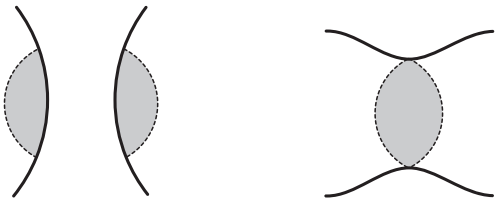
Lemma 3.1. *The boundary parallel stabilization depicted in Figure* 1(b) *can be performed while fixing T as the "outer" boundary component.*

Assume the truth of Lemma 3.1 for the moment. Then the curve γ depicted in Figure 1(b) is no longer isolating, as it now represents a nonzero element in the homology of the page. According to Honda's Legendrian realization principle, it can therefore be made Legendrian on the page S'. By construction, if we orient this Legendrian coherently with T, then T is the transverse pushoff of γ.

Proof. Consider S^3 with its standard tight contact structure. Let (H_+, π_+) be the open book for (S^3, ξ_{std}) whose binding consists of two perpendicular Hopf circles and whose pages consist of negative Hopf bands connecting these two curves. In this case, each binding component is a transverse unknot with self-linking number equal to -1.

Let T be a transverse knot contained in a contact manifold (Y, ξ) and let U be a transverse unknot in (S^3, ξ_{std}) with self-linking number equal to -1. Observe that the complement of a standard neighborhood of a point contained in U is itself a standard neighborhood of a point contained in a transverse knot. Therefore, according to Etnyre and Honda [**EH03**], if we perform a transverse connected sum of T with the transverse unknot of self-linking number equal to -1 in (S^3, ξ_{std}), we do not change the transverse knot type of T.

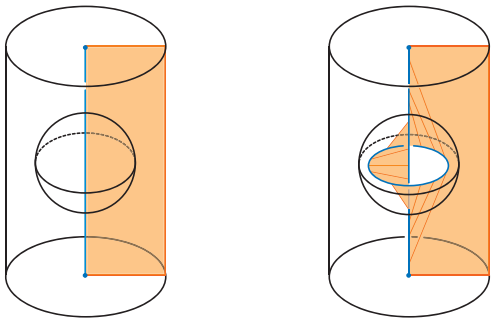
Let (B, π) be an open book with connected binding for a contact manifold (Y, ξ). Consider the contact manifold obtained from (Y, ξ) by stabilizing along a boundary parallel arc, as depicted in Figure 1(b). This has the effect of Murasugi summing the open book (B, π) with the open book (H_+, π_+) along bigon regions bounded by boundary-parallel



(a) (b)

Figure 2

arcs contained in pages of the respective open books. The summing process is depicted in Figure 2. Figure 2(a) shows the open books before the Murasugi sum, while Figure 2(b) shows the resulting open book after the sum.



(a) (b)

Figure 3

The Murasugi sum operation has the effect of performing a contact connected sum of (Y, ξ) with (S^3, ξ_{std}) and a transverse connected sum of the binding component B with one of the binding components of H_+ (see [**Tor00**]). Before and after pictures of this operation are shown in Figures 3(a) and 3(b), respectively. Since both of the binding components of the open book (H_+, π_+) are transverse unknots with self-linking number equal to -1, this connected sum has no effect on the transverse knot type of the "outer" boundary component of the open book in Figure 1. q.e.d.

Since the curve γ can now be Legendrian realized and approximates T as desired, we will denote γ by L from this point forward. The new monodromy map $\phi' : S' \to S'$ is equal to the old monodromy map, ϕ, composed with one positive Dehn twist along the curve c shown in Figure 1(b). For notational ease, we continue denoting the monodromy map by ϕ and the page by S.

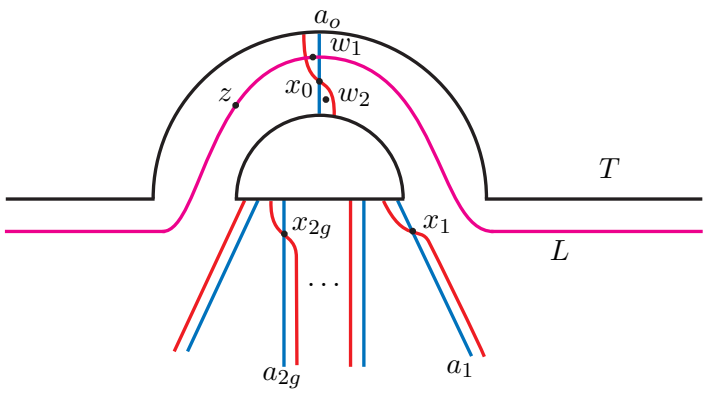


Figure 4

We choose a basis for our surface whose local picture near the stabilization is depicted in Figure 4. There are two possible choices for the placement of the second basepoint w: w_1 and w_2. In order for L to be oriented coherently with T, we must choose $w = w_1$.

3.2. Admissibility. Our goal here is to construct a weakly admissible, doubly-pointed Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ from the open book described in Section 3.1.

Before we continue, let us discuss some notation. We are concerned with open book decompositions whose pages are twice-punctured surfaces. We picture a genus g surface as a $4g$-sided polygon with certain boundary edges identified. We choose the standard identification scheme, where the first and third edges are identified, as are the second and fourth edges, the fifth and seventh edges, and so on. For convenience, we always assume that the first edge appears in the 12 o'clock position, at the top of each diagram.

Our punctures are always situated so that one of the punctures is in the center of the polygon, with the other close by. We choose our basis elements, $a_1, \ldots, a_{2g}$, to be straight arcs emanating from the center of the polygon and passing out the corresponding edge. If we were to forget about the identifications being made at the boundary, the basis element a_i would break into two straight arcs emanating from the center of the diagram. For ease of exposition in what follows, we label the first segment that we see as we move clockwise around the diagram $a_{i,I}$, and the second $a_{i,F}$, where the subscript I stands for "initial", while the subscript F stands for "final".

Up to isotopy, we may assume that the second boundary component of our surface lies (pictorially) in the region between the curve segments $a_{2g,F}$ and $a_{1,I}$, as shown in Figure 5. The last basis element a_0 is a straight line segment connecting the two boundary components of the surface.

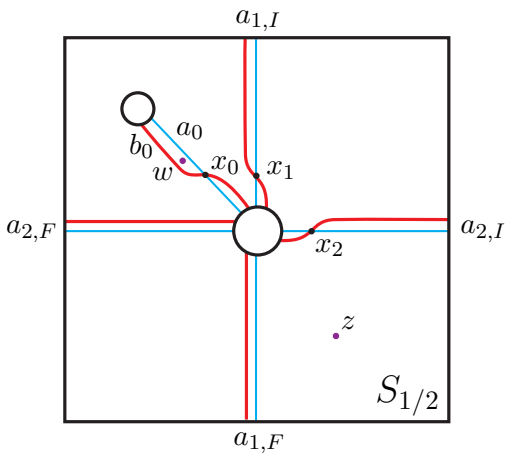


Figure 5

We have adopted the practice of Honda, Kazez and Matić of placing surrogate basepoints throughout the diagram whenever it is convenient. This signals that the local multiplicity of any domain contributing to the differential is zero in that region.

We have restricted our figures to the case where our page is a twice-punctured torus, and our monodromy map ϕ consists of two negative Dehn twists along the curve τ in Figure 1(b). The resulting doubly-pointed Heegaard diagram is shown in Figures 5 and 6.

Figure 5 shows the $S_{1/2}$ page of our open book, while Figure 6 shows the $-S_1$ page (note the reversed orientation). The invariant appears in Figure 5 as the intersection point $\mathbf{x} = (x_0, x_1, x_2)$.

Consider the small region southeast of x_0 in Figure 5. This region is equal to the region R in Figure 6. Let γ be the dashed arc connecting the region R to the z-pointed region. Denote by $(a_{i,*})$, the intersection point between $a_{i,*}$ and γ.

Lemma 3.2. *The doubly-pointed Heegaard diagram described above, and appearing in Figures* 5 *and* 6*, is weakly admissible.*

Proof. To show that the Heegaard diagram described above is *weakly admissible*, we must prove that any nontrivial periodic domain $P \subset \Sigma$ has both positive and negative multiplicities. Recall that a *domain* is a 2-chain consisting of a formal sum $\sum c_i D_i$ of components of the complement $\Sigma - (\alpha \cup \beta)$. A domain $P \subset \Sigma$ is said to be *periodic* if the boundary of P consists of a linear combination of α- and β-circles and if the local multiplicity of P at w is zero i.e $\partial P = \sum_{i=1}^{g} [n_i \alpha_i + m_i \beta_i]$, $n_i, m_i \in \mathbb{Z}$, $n_w(P) = 0$.

Let P be a nontrivial periodic domain for the pointed Heegaard diagram $(\Sigma, \beta, \alpha, w)$. Suppose P has nonzero local multiplicity in the

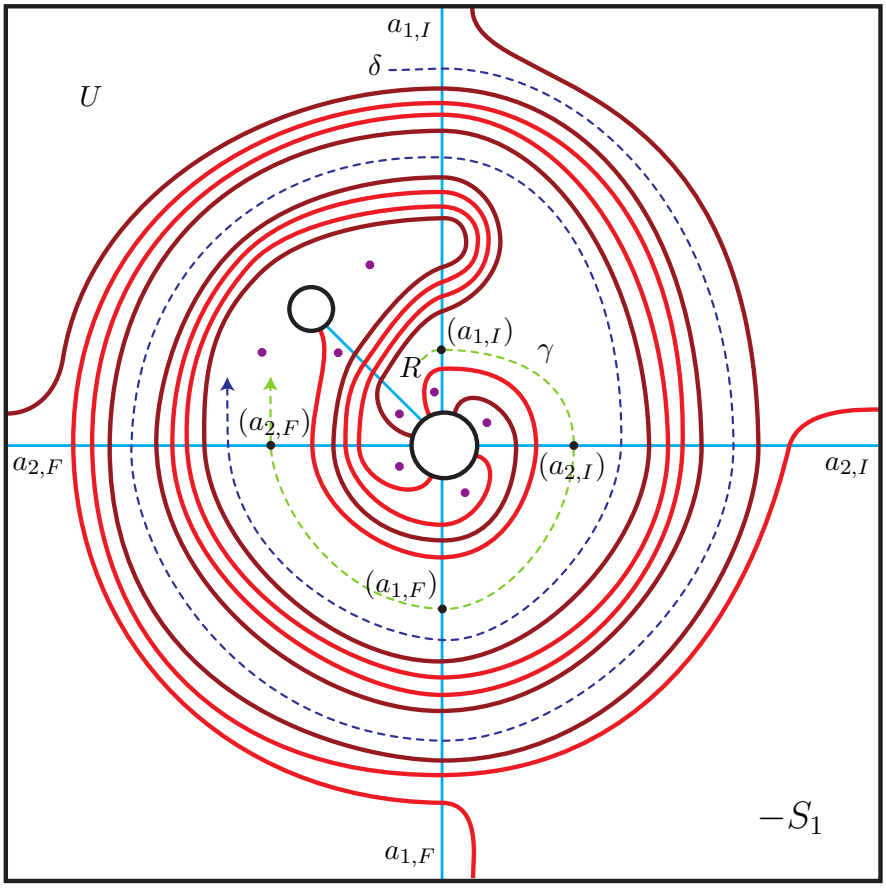


Figure 6

z-pointed region. Without loss of generality, we assume that this multiplicity is $+1$. In particular, the multiplicity of the region just above the point $(a_{2,F})$ in Figure 6 is $+1$. In order for the w-pointed region to have multiplicity zero, the α_0- and β_0-curves must be contained in the boundary of the periodic domain P and must appear with multiplicity ± 1 (depending on the chosen orientations of α_0 and β_0). This forces the small region southeast of x_0 in Figure 5 to have multiplicity $+2$. Since this region is the same as the region R in Figure 6, R must also have multiplicity $+2$.

Consider the dashed arc γ connecting R to the z-pointed region. In order for P to exist, the multiplicities of the regions intersected by γ must go from $+2$ in the region R to $+1$ in the z-pointed region.

However, the curve γ intersects each α-curve (other than α_0) in two points, each with opposite sign. Since the boundary of a periodic domain must be a sum of full α- and β-curves, if the local multiplicity of P increases (or decreases) by a factor of n as γ passes one of the intersection

points, it must decrease (or increase) by that same factor as γ passes the other intersection point. Therefore, the net change in the local multiplicity of the periodic domain P along γ between the region R and the z-pointed region is zero. We have seen that the multiplicity of the region R is $+2$, whereas the multiplicity of the z-pointed region was assumed to be $+1$. From this contradiction, we conclude that P must have local multiplicity zero in the z-pointed region.

Since each α- and each β-curve bound the z-pointed region on either side, and since P has local multiplicity zero in the z-pointed region, we conclude that if P is nontrivial, it must have both positive and negative multiplicities. q.e.d.

3.3. Computing $\widehat{\mathfrak{T}}(T)$. Let $\phi \in \pi_2(\mathbf{y}, \mathbf{y}')$ be a homotopy class with $n_z(\phi) = n_w(\phi) = 0$. Further, let $D_1, \ldots, D_n$ denote the closures of the components of the complement $\Sigma - (\alpha \cup \beta)$ and for $1 \leq i \leq n$, choose a point p_i in the interior of D_i. The *domain* of ϕ, denoted $D(\phi)$, is the 2-chain $\sum_{i=1}^{n} n_{p_i}(\phi) D_i$.

If the homotopy class ϕ is to have a holomorphic representative, then each of the $n_{p_i}(\phi)$ must be nonnegative by positivity of intersection. In other words, $D(\phi)$ is a positive domain and ϕ is a positive class.

Let $\mathbf{x} = (x_0, \ldots, x_{2g})$; we show that the transverse invariant $\widehat{\mathfrak{T}}(T) = [(x_0, \ldots, x_{2g})]$ is nonzero by proving that the generator $\mathbf{x}$ cannot appear in image of the Heegaard Floer differential. This is accomplished by showing that the set of positive classes $\phi \in \pi_2(\mathbf{y}, \mathbf{x})$ with $n_z(\phi) = n_w(\phi) = 0$ is empty for all $\mathbf{y} \neq \mathbf{x}$.

To draw a contradiction, we assume in what follows that D is the domain of a positive class $\phi \in \pi_2(\mathbf{y}, \mathbf{x})$ for some generator $\mathbf{y} \neq \mathbf{x}$ with $n_z(\phi) = n_w(\phi) = 0$.

Suppose R_1 and R_2 are two adjacent regions in a Heegaard diagram (i.e. R_1 and R_2 share an edge), and D is as above. In general, the multiplicities of R_1 and R_2 can differ arbitrarily. In our case, however, the multiplicities of any two adjacent regions can differ by at most one. To see this, observe that the boundary of D must consist of a collection of subarcs of the α- and β- curves joining the points in $\mathbf{y}$ to the points in $\mathbf{x}$. However, since each of the α- and β-curves in a Heegaard diagram coming from an open book decomposition bound the z-pointed region to either side, the boundary of D can not contain a full α or β curve. Therefore, the multiplicities of R_1 and R_2 can differ by at most one.

Consider the region R in Figure 6, and the curve γ connecting R to the z-pointed region.

Lemma 3.3. *Let $\phi \in \pi_2(\mathbf{y}, \mathbf{x})$ be a positive class satisfying $n_z(\phi) = n_w(\phi) = 0$ and let D be its corresponding positive domain. The net change in the local multiplicity of D between the region R and the z-pointed region along γ is nonnegative.*

Proof. The proof of Lemma 3.3 is similar to the proof of the admissibility lemma in Section 3.2. Recall that the curve γ intersects each α_i-curve in two points: $(a_{i,I})$ and $(a_{i,F})$. We show that if the local multiplicity of the regions intersected by γ decreases by a factor of -1 at the point $(a_{i,I})$, then there must be a corresponding increase in local multiplicity at the point $(a_{i,F})$. Similarly, we show that if the local multiplicity of the regions intersected by γ decrease by a factor of -1 at the point $(a_{i,F})$, then there must be a corresponding increase in local multiplicity at the point $(a_{i,I})$.

Observe that the local multiplicity along γ cannot decrease as γ passes over the point $(a_{1,I})$. Since $\phi \in \pi_2(\mathbf{y}, \mathbf{x})$ and D is a positive domain, if there is a decrease in local multiplicity at $(a_{1,I})$, a segment of the α_1-curve between the intersection point $(a_{1,I})$ and x_1 must be contained in the boundary of the D. Looking at the diagram in Figure 6, we see that any such arc has z-pointed region to the west, contradicting its existence.

In the genus one case, a similar argument shows that there can be no decrease in the local multiplicity of D at the point $(a_{2,I})$. So assume that either the genus of S is greater than one, or that we are considering an intersection point (a) beyond $(a_{2,I})$ along γ.

Suppose that $(a) = (a_{i,I})$, and that the local multiplicity along γ decreases by a factor -1 as it passes over at the point (a). Then, up to orientation, the segment of the α_i-curve beginning at the point (a) and traveling away from the center of the diagram to the point x_i is contained in the boundary of D. This implies that the region just past the intersection point $(a_{i,F})$ along γ gains a $+1$ boost in local multiplicity. Therefore, the increase in the local multiplicity at the point $(a_{i,F})$ balances the decrease in the local multiplicity at the point $(a_{i,I})$.

Similarly, if $(a) = (a_{i,F})$ and if the local multiplicity of D along γ decreases by a factor of -1 as γ passes over (a), then, again up to orientation, the segment of the α_i-curve beginning at the point $(a_{i,I})$ and traveling away from the center of the diagram to the point x_i is contained in the boundary of D. This implies that the region just past the intersection point $(a_{i,I})$ gains a $+1$ boost in local multiplicity. Thus, the decrease in local multiplicity at the point $(a_{i,F})$ is balanced by the increase in local multiplicity at the point $(a_{i,I})$.

Since each decrease in the local multiplicity of D along γ is balanced by a corresponding increase in local multiplicity somewhere else along γ, we have that the net change in the local multiplicity of D between the region R and the z-pointed region along the curve γ is nonnegative.

q.e.d.

Consider the region U in Figure 6, and the curve δ connecting U to the z-pointed region. By an argument similar to the proof of Lemma 3.3, we have the following:

Lemma 3.4. *Let $\phi \in \pi_2(\mathbf{y}, \mathbf{x})$ be a positive class satisfying $n_z(\phi) = n_w(\phi) = 0$ and let D be its corresponding positive domain. The net change in the local multiplicity of D between the region U and the z-pointed region along δ is nonnegative.*

Proof of Theorem 1. There are two main cases to consider.

Case 1: Assume that the positive domain D has nonzero local multiplicity in a region bordering the intersection point x_0. In this case, the region R in Figure 6 has multiplicity $+1$. By Lemma 3.3, this implies that the multiplicity of the z-pointed region must be at least $+1$, a contradiction.

Case 2: Now suppose that D has local multiplicity zero in all four of the regions bordering the point x_0. In particular, this means that the multiplicity of the region R is zero. We investigate the possible configurations of D near the center of Figure 6.

Suppose, for the moment, that all the regions bordered by the curve β_0 have zero multiplicity. Then, near the center of Figure 6, the regions of D with positive multiplicity are (locally) constrained to lie within the strip bounded by the darkened portions of the β-curves.

In order for this to be the case, the boundary of the domain must have veered off the α-curves while still contained within this strip. Therefore, all the α-curves are "used up" close to the center of the diagram (i.e. by the time they first intersect a darkened β-curve). This, in turn, forces the multiplicity of the region U in Figure 6 to be positive.

By Lemma 3.4, this implies that the multiplicity of the z-pointed region must be positive, a contradiction. Therefore, in order for such a nontrivial positive domain to exist, at least one of the regions bordered by β_0 must have nonzero multiplicity.

Recall that in Case 2 we are assuming that our domain D is constant near x_0. This means that the curve β_0 cannot be appear in the boundary of D with nonzero multiplicity, so at least one of the regions intersected by γ must have positive multiplicity. Let R' be the first region along γ with positive multiplicity, and let (a) be the $(a_{i,*})$ immediately preceding R'.

If $(a) = (a_{i,F})$, then by an argument similar to the proof of Lemma 3.3, it can be shown that the net change in multiplicity between the region R' and the z-pointed region must be nonnegative. The fact that (a) is a final point ensures that there can be no decrease in multiplicity at the point $(a_{i,I})$ since, by the definition of R', the regions to both sides of this point have multiplicity zero.

On the other hand, suppose $(a) = (a_{i,I})$. An argument similar to that in Lemma 3.3 demonstrates that for each decrease in multiplicity, there is a corresponding increase in multiplicity, except possibly at the point $(a_{i,F})$. If the multiplicity decreases at the point $(a_{i,F})$, then the segment of the α_i-curve from $(a_{i,F})$ to (a) must be contained in the boundary of

the domain. This then implies that the multiplicity of the region U is at least one.

Again, by Lemma 3.4, this forces the multiplicity of the z-pointed region to be positive, a contradiction. q.e.d.

This completes the proof of Theorem 1.

4. The vanishing theorem

In this section, we prove Theorem 2. The proof in this case is similar to the proof of Theorem 4.5 in [**HKM09b**]. The key differences are that we must be careful to incorporate the Legendrian knot L when choosing a Legendrian skeleton for the complement of the submanifold N, and that we must be cautious about the changes made to the diagram in the "spinning" process used to make the diagram strongly admissible.

Proof of Theorem 2. We begin by constructing a partial open book decomposition for the contact submanifold $(N, \xi|_N)$, which can be extended to an open book decomposition for all of (Y, ξ). Following [**HKM09b**], we must show that the basis $\{a_1, \dots, a_n\}$ for the partial open book decomposition of $(N, \xi|_N)$ can be extended to a basis $\{a_1, \dots, a_n, a'_0, a'_1, \dots, a'_s\}$ for an extended open book decomposition of (Y, ξ), where $L \cap (\bigcup a_i \cup \bigcup a'_j) = L \cap a'_0 = 1\text{pt}$.

Claim: We may assume without loss of generality that the complement of N is connected.

Proof of claim. Let (M, ξ) be a compact manifold with possibly nonempty boundary, and let $(M', \xi|_{M'})$ be a compact submanifold of (M, ξ) with convex boundary. In [**HKM08**], the authors show that the vanishing of the contact invariant for $(M', \xi|_{M'})$ implies the vanishing of the contact invariant for (M, ξ).

Suppose the complement of N is disconnected. Then, since the contact invariant $c(N, \xi|_N)$ vanishes, the contact manifold obtained by gluing the components of $Y - N$ not containing L to N must also have vanishing contact invariant. In particular, we may assume without loss of generality that $Y - N$ is connected. q.e.d.

Let K be a Legendrian skeleton for N, and let K' be an extension of the Legendrian knot L to a Legendrian skeleton for $N' = Y - N$ (see Figure 7). Assume that the univalent vertices of K and K' in ∂N do not intersect.

The Legendrian skeleton K gives us a partial open book decomposition for $(N, \xi|_N)$. Let $\nu(K)$ be a standard neighborhood of K inside of N, and let $\nu(K')$ be a standard neighborhood of K' inside of $Y - N$. We can build an open book decomposition for all of Y by considering

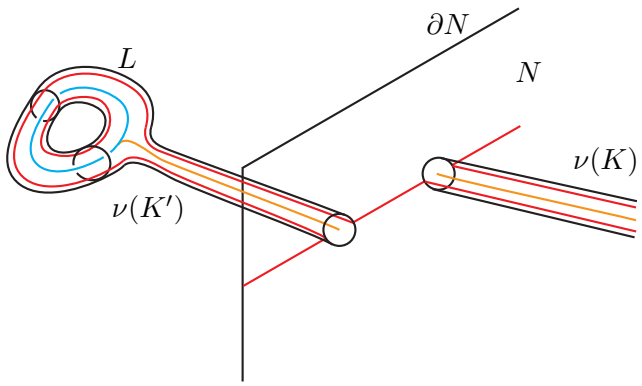


Figure 7

the handlebodies $(N' - \nu(K')) \cup \nu(K)$ and $\nu(K') \cup (N - \nu(K))$. By construction, each of these handlebodies are disk decomposable. A page S of the open book for (Y, ξ) is constructed from the page of the partial open book for $(N, \xi|_N)$ by repeatedly attaching 1-handles away from the portions of the open book coming from the boundary of $\nu(K)$. This construction is depicted in Figure 8.

In Figure 8, the portion of the page of the open book coming from the boundary of N is shown in black, and has its boundary lines thickened. The portion of the page coming from the boundary of $\nu(K)$ is lightly colored (orange), and appears in the lower right portion of the figure. Finally, the portion of the page coming from the extension of the open book to all of Y is also lightly colored (green), and appears in the lower left corner of the figure.

Let $\{a_1, \dots, a_n\}$ be a basis for the partial open book coming from $(N, \xi|_N)$, and let ϕ be the corresponding partially defined monodromy map for this open book. Consider a new partial open book, whose page is equal to S, and whose partially defined monodromy map is equal to ϕ. Because this new partial open book only differs from the partial open book coming from $(N, \xi|_N)$ by handle attachments away from $\partial\nu(K)$, the contact element for this new partial open book vanishes along with $c(N, \xi|_N)$.

Since $Y - N$ is connected, the basis $\{a_1, \dots, a_n\}$ can, after a suitable number of stabilizations to the Legendrian skeleton K', be extended to a basis for all of S.

By construction, the new monodromy map ϕ' extends ϕ, the monodromy map for N. Since L was included in the skeleton K', we now see L sitting on the page S. The local picture around $L \subset S$ (shown in blue) must look like that in Figure 8. In particular, we see the extended basis $\{a_1, \dots, a_n, a'_0, \dots, a'_s\}$ for S can be chosen so that L only intersects a'_0, and does so transversally in a single point.

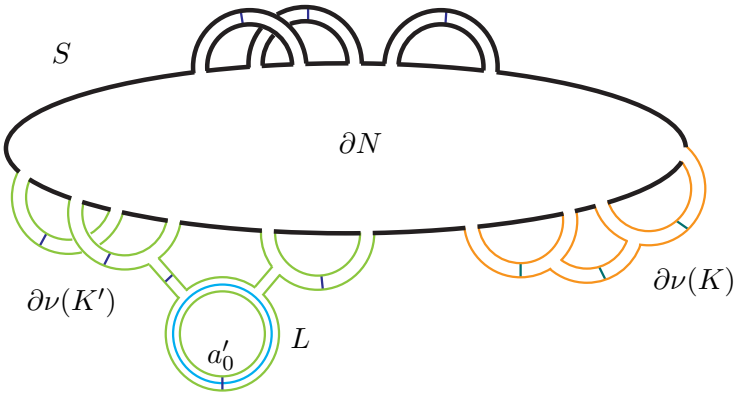


Figure 8

As was observed in [**LOSSz09**] (c.f. Proposition 2.13), the "spinning" isotopies needed to make this Heegaard diagram strongly admissible can be performed on the portion of the Heegaard diagram coming from the page S_1. This changes the monodromy map ϕ', but only within its isotopy class.

If we delete the α- and β-curves coming from $\{a'_0, a'_1, \dots, a'_s\}$, then we are left with a diagram which is essentially equivalent to that coming from the partial open book (S, ϕ), but whose monodromy has been changed by an isotopy. Since altering the monodromy map by an isotopy cannot change whether or not the contact element vanishes in sutured Floer homology, we know that the contact element corresponding to the partial open book (S, ϕ') vanishes. That is, if $\mathbf{x} = (x_1, \dots, x_n)$, then there exist c_i and $\mathbf{y}_i$ such that $\partial(\sum_i c_i \mathbf{y}_i) = \mathbf{x}$ in the sutured Floer homology of the manifold obtained from the partial open book (S, ϕ').

Let $\mathbf{x}' = (x'_0, x'_1, \dots, x'_s)$; we claim that $\partial(\sum_i c_i(\mathbf{y}_i, \mathbf{x}')) = (\mathbf{x}, \mathbf{x}')$ in $\mathrm{HFK}^-(-Y, L)$. The intersection points coming from $\mathbf{x}'$ must map to themselves via the constant map. This allows us to ignore the α- and β-curves corresponding to these intersection points, leaving us with a diagram which is essentially equivalent to the partial open book (S, ϕ'). q.e.d.

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