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SCALAR CURVATURE RIGIDITY OF GEODESIC BALLS IN S^n

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Abstract

In this paper, we prove a scalar curvature rigidity result for geodesic balls in S^n. This result contrasts sharply with the counterexamples to Min-Oo's conjecture constructed in [**7**].

1. Introduction

This paper is concerned with rigidity phenomena involving the scalar curvature. These questions are motivated to a large extent by the positive mass theorem in general relativity, which was proved by Schoen and Yau [**18**] and Witten [**20**]. An important corollary of this theorem is that any Riemannian metric on $\mathbb{R}^n$ which has nonnegative scalar curvature and agrees with the Euclidean metric outside a compact set is necessarily flat.

It was observed by Miao [**16**] that the positive mass theorem implies the following rigidity result for metrics on the unit ball:

Theorem 1. *Suppose that g is a Riemannian metric on the unit ball $B^n \subset \mathbb{R}^n$ with the following properties:*

- *The scalar curvature of g is nonnegative.*
- *The induced metric on the boundary ∂B^n agrees with the standard metric on ∂B^n.*
- *The mean curvature of ∂B^n with respect to g is at least $n-1$.*

Then g is isometric to the standard metric on B^n.

An important generalization of Theorem 1 was proved by Shi and Tam [**19**].

Motivated by the positive mass theorem, Min-Oo [**17**] posed the following question:

Min-Oo's Conjecture. *Suppose that g is a Riemannian metric on the hemisphere S^n_+ with the following properties:*

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- *The scalar curvature of g is at least $n(n-1)$.*
- *The induced metric on the boundary ∂S^n_+ agrees with the standard metric on ∂S^n_+.*
- *The boundary ∂S^n_+ is totally geodesic with respect to g.*

Then g is isometric to the standard metric on S^n_+.

Min-Oo's conjecture has been verified in many special cases (see e.g. [**11**], [**13**], [**14**]). A related rigidity result for real projective space $\mathbb{RP}^3$ was established in [**3**] (see also [**4**]). A survey of these and other related results can be found in [**6**].

Very recently, counterexamples to Min-Oo's conjecture were constructed in [**7**].

Theorem 2 (S. Brendle, F.C. Marques, A. Neves [**7**]). *Given any integer $n \geq 3$, there exists a smooth Riemannian metric $\hat{g}$ on the hemisphere S^n_+ with the following properties:*

- *The scalar curvature of $\hat{g}$ is at least $n(n-1)$ at each point on S^n_+.*
- *The scalar curvature of $\hat{g}$ is strictly greater than $n(n-1)$ at some point on S^n_+.*
- *The metric $\hat{g}$ agrees with the standard metric in a neighborhood of ∂S^n_+.*

The proof of Theorem 2 relies on a perturbation analysis.

In this paper, we study the analogous rigidity question for geodesic balls in S^n of radius less than $\frac{\pi}{2}$. To fix notation, let $\overline{g}$ be the standard metric on S^n and let $f : S^n \to \mathbb{R}$ denotes the restriction of the coordinate function x_{n+1} to S^n. We will consider a domain of the form $\Omega = \{f \geq c\}$. If $c \geq \frac{2}{\sqrt{n+3}}$, we have the following rigidity result:

Theorem 3. *Consider the domain $\Omega = \{f \geq c\}$, where $c \geq \frac{2}{\sqrt{n+3}}$. Let g be a Riemannian metric on Ω with the following properties:*

- *$R_g \geq n(n-1)$ at each point in Ω.*
- *The metrics g and $\overline{g}$ induce the same metric on $\partial\Omega$.*
- *$H_g \geq H_{\overline{g}}$ at each point on $\partial\Omega$.*

If $g - \overline{g}$ is sufficiently small in the C^2-norm, then $\varphi^*(g) = \overline{g}$ for some diffeomorphism $\varphi : \Omega \to \Omega$ with $\varphi|_{\partial\Omega} = \mathrm{id}$.

We remark that the conclusion of Theorem 3 holds under the weaker assumption that g is close to $\overline{g}$ in $W^{2,p}$-norm for $p > n$.

Note that Theorem 3 is false for the hemisphere $\{f \geq 0\}$: by Theorem 4 in [**7**], there exist Riemannian metrics on the hemisphere which satisfy the assumptions of Theorem 3 and are arbitrary close to the standard metric $\overline{g}$ in the C^∞-topology, but which are not isometric to $\overline{g}$.

The proof of Theorem 3 relies on a perturbation analysis which is similar in spirit to Bartnik's work on the positive mass theorem (cf. [**1**], Section 5). Similar techniques have been employed in the study of the total scalar curvature functional (see e.g. [**2**], Section 4G) and the Yamabe flow (cf. [**5**]). Dai, Wang, and Wei [**8**],[**9**] have obtained interesting stability results for manifolds with parallel spinors, as well as for Kähler-Einstein manifolds.

2. The scalar curvature and boundary mean curvature of a perturbed metric

In this section, we consider a smooth manifold Ω with boundary $\partial\Omega$. Let $\overline{g}$ be a fixed Riemannian metric on Ω. Moreover, we consider another Riemannian metric $g = \overline{g} + h$, where $|h|_{\overline{g}} \leq \frac{1}{2}$ at each point in Ω. For abbreviation, we write $(h^2)_{ik} = \overline{g}^{jl}\, h_{ij}\, h_{kl}$.

Proposition 4. *The scalar curvature of g satisfies the pointwise estimate*

$$\begin{aligned}
&\Big| R_g - R_{\overline{g}} + \langle \mathrm{Ric}_{\overline{g}}, h\rangle - \langle \mathrm{Ric}_{\overline{g}}, h^2\rangle \\
&+ \frac{1}{4}\, \overline{g}^{ij}\, \overline{g}^{kl}\, \overline{g}^{pq}\, \overline{D}_i h_{kp}\, \overline{D}_j h_{lq} - \frac{1}{2}\, \overline{g}^{ij}\, \overline{g}^{kl}\, \overline{g}^{pq}\, \overline{D}_i h_{kp}\, \overline{D}_l h_{jq} \\
&+ \frac{1}{4}\, \overline{g}^{pq}\, \partial_p(\mathrm{tr}_{\overline{g}}(h))\, \partial_q(\mathrm{tr}_{\overline{g}}(h)) + \overline{D}_i\Big(g^{ik}\, g^{jl}\, (\overline{D}_k h_{jl} - \overline{D}_l h_{jk})\Big)\Big| \\
&\leq C\, |h|\, |\overline{D} h|^2 + C\, |h|^3.
\end{aligned}$$

Here, $\overline{D}$ denotes the Levi-Civita connection with respect to $\overline{g}$, and C is a positive constant which depends only on n.

Proof. The Levi-Civita connection with respect to g is given by

$$D_X Y = \overline{D}_X Y + \Gamma(X, Y),$$

where Γ is defined by

$$g(\Gamma(X, Y), Z) = \frac{1}{2}\, \big((\overline{D}_X h)(Y, Z) + (\overline{D}_Y h)(X, Z) - (\overline{D}_Z h)(X, Y)\big).$$

In local coordinates, the tensor Γ can be written in the form

$$\Gamma_{jk}^m = \frac{1}{2}\, g^{lm}\, (\overline{D}_j h_{kl} + \overline{D}_k h_{jl} - \overline{D}_l h_{jk}).$$

With this understood, the scalar curvature of g is given by

$$\begin{aligned}
R_g &= g^{ik}\, (\mathrm{Ric}_{\overline{g}})_{ik} + g^{ik}\, g^{jl}\, g_{pq}\, \Gamma_{il}^q\, \Gamma_{jk}^p - g^{ik}\, g^{jl}\, g_{pq}\, \Gamma_{jl}^q\, \Gamma_{ik}^p \\
&\quad - g^{ik}\, g^{jl}\, (\overline{D}_{i,k}^2 h_{jl} - \overline{D}_{i,l}^2 h_{jk})
\end{aligned}$$

(cf. [**7**], Proposition 16). This implies

$$\begin{aligned}
&\Big|R_g - R_{\overline{g}} + \langle \mathrm{Ric}_{\overline{g}}, h\rangle - \langle \mathrm{Ric}_{\overline{g}}, h^2\rangle \\
&- \frac{3}{4}\,\overline{g}^{ij}\,\overline{g}^{kl}\,\overline{g}^{pq}\,\overline{D}_i h_{kp}\,\overline{D}_j h_{lq} + \frac{1}{2}\,\overline{g}^{ij}\,\overline{g}^{kl}\,\overline{g}^{pq}\,\overline{D}_i h_{kp}\,\overline{D}_l h_{jq} \\
&+ \frac{1}{4}\,\overline{g}^{pq}\,\partial_p(\mathrm{tr}_{\overline{g}}(h))\,\partial_q(\mathrm{tr}_{\overline{g}}(h)) - \overline{g}^{ij}\,\overline{g}^{pq}\,\overline{D}_i h_{jp}\,\partial_q(\mathrm{tr}_{\overline{g}}(h)) \\
&+ \overline{g}^{ij}\,\overline{g}^{kl}\,\overline{g}^{pq}\,\overline{D}_i h_{jp}\,\overline{D}_k h_{lq} + g^{ik}\,g^{jl}\,(\overline{D}^2_{i,k} h_{jl} - \overline{D}^2_{i,l} h_{jk})\Big| \\
&\leq C\,|h|\,|\overline{D}h|^2 + C\,|h|^3.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\Big|R_g - R_{\overline{g}} + \langle \mathrm{Ric}_{\overline{g}}, h\rangle - \langle \mathrm{Ric}_{\overline{g}}, h^2\rangle \\
&+ \frac{1}{4}\,\overline{g}^{ij}\,\overline{g}^{kl}\,\overline{g}^{pq}\,\overline{D}_i h_{kp}\,\overline{D}_j h_{lq} - \frac{1}{2}\,\overline{g}^{ij}\,\overline{g}^{kl}\,\overline{g}^{pq}\,\overline{D}_i h_{kp}\,\overline{D}_l h_{jq} \\
&+ \frac{1}{4}\,\overline{g}^{pq}\,\partial_p(\mathrm{tr}_{\overline{g}}(h))\,\partial_q(\mathrm{tr}_{\overline{g}}(h)) + \overline{D}_i\Big(g^{ik}\,g^{jl}\,(\overline{D}_k h_{jl} - \overline{D}_l h_{jk})\Big)\Big| \\
&\leq C\,|h|\,|\overline{D}h|^2 + C\,|h|^3,
\end{aligned}$$

as claimed. q.e.d.

In the next step, we estimate the mean curvature of the boundary $\partial\Omega$ with respect to the metric $\overline{g}$. To that end, we assume that g and $\overline{g}$ induce the same metric on the boundary $\partial\Omega$; in other words, we assume that $h(X, Y) = 0$ whenever X and Y are tangent vectors to $\partial\Omega$.

Proposition 5. *Assume that g and $\overline{g}$ induce the same metric on the boundary $\partial\Omega$. Then the mean curvature of $\partial\Omega$ with respect to g satisfies*

$$\begin{aligned}
&\Big|2\,(H_g - H_{\overline{g}}) - \Big(h(\overline{\nu},\overline{\nu}) - \frac{1}{4}\,h(\overline{\nu},\overline{\nu})^2 + \sum_{a=1}^{n-1} h(e_a,\overline{\nu})^2\Big)\,H_{\overline{g}} \\
&+ \big(1 - \frac{1}{2}\,h(\overline{\nu},\overline{\nu})\big)\sum_{a=1}^{n-1}\big(2\,(\overline{D}_{e_a}h)(e_a,\overline{\nu}) - (\overline{D}_{\overline{\nu}}h)(e_a,e_a)\big)\Big| \\
&\leq C\,|h|^2\,|\overline{D}h| + C\,|h|^3.
\end{aligned}$$

Here, $\{e_a : 1 \leq a \leq n-1\}$ is a local orthonormal frame on $\partial\Omega$, and C is a positive constant that depends only on n.

Proof. Using the identity

$$H_g\,\nu - H_{\overline{g}}\,\overline{\nu} = -\sum_{a=1}^{n-1}(D_{e_a}e_a - \overline{D}_{e_a}e_a) = -\sum_{a=1}^{n-1}\Gamma(e_a,e_a),$$

we obtain

$$\begin{aligned}&2\left(H_g\, g(\nu,\overline{\nu}) - H_{\overline{g}}\, g(\overline{\nu},\overline{\nu})\right)\\&= -2\sum_{a=1}^{n-1} g(\Gamma(e_a,e_a),\overline{\nu}) = -\sum_{a=1}^{n-1}\left(2\,(\overline{D}_{e_a}h)(e_a,\overline{\nu}) - (\overline{D}_{\overline{\nu}}h)(e_a,e_a)\right).\end{aligned}$$

Clearly, $g(\overline{\nu},\overline{\nu}) = 1 + h(\overline{\nu},\overline{\nu})$. Moreover, it is easy to see that the vector $\overline{\nu} - \sum_{a=1}^{n-1} h(e_a,\overline{\nu})\, e_a$ is orthogonal to $\partial\Omega$ with respect to g. From this, we deduce that

$$\overline{\nu} - \sum_{a=1}^{n-1} h(e_a,\overline{\nu})\, e_a = \left(1 + h(\overline{\nu},\overline{\nu}) - \sum_{a=1}^{n-1} h(e_a,\overline{\nu})^2\right)^{\frac{1}{2}} \nu,$$

hence

$$g(\nu,\overline{\nu}) = \left(1 + h(\overline{\nu},\overline{\nu}) - \sum_{a=1}^{n-1} h(e_a,\overline{\nu})^2\right)^{\frac{1}{2}}.$$

Substituting these identities into the previous formula for H_g, the assertion follows. q.e.d.

3. Perturbations of the standard metric on S^n

We now consider perturbations of the standard metric $\overline{g}$ on S^n. To fix notation, let $f : S^n \to \mathbb{R}$ denote the restriction of the coordinate function x_{n+1} to S^n, and let $\Omega = \{f \geq c\}$ be a geodesic ball centered at the north pole. Here, c is a positive real number which will be specified later.

Let g be a Riemannian metric on Ω. We will assume throughout that g and $\overline{g}$ induce the same metric on the boundary $\partial\Omega$. Moreover, we assume that $g = \overline{g} + h$, where $|h|_{\overline{g}} \leq \frac{1}{2}$ at each point in Ω.

Our goal in this section is to estimate the integral

$$\int_\Omega (R_g - n(n-1))\, f\, d\mathrm{vol}_{\overline{g}}$$

(see also [**12**]).

Proposition 6. *We have*

$$\begin{aligned}
&\bigg| \int_\Omega (R_g - n(n-1) - (n-1)\,|h|_{\overline{g}}^2)\, f \, d\mathrm{vol}_{\overline{g}} + \frac{1}{4} \int_\Omega |\overline{D} h|^2 \, f \, d\mathrm{vol}_{\overline{g}} \\
&+ \frac{1}{4} \int_\Omega |\overline{\nabla}(\mathrm{tr}_{\overline{g}}(h))|^2 \, f \, d\mathrm{vol}_{\overline{g}} - \frac{1}{2} \int_\Omega \overline{g}^{ij}\, \overline{g}^{kl}\, \overline{g}^{pq}\, \overline{D}_i h_{kp}\, \overline{D}_l h_{jq}\, f \, d\mathrm{vol}_{\overline{g}} \\
&+ \int_\Omega \overline{g}^{ik}\, \overline{g}^{jp}\, \overline{g}^{lq}\, h_{pq}\, (\overline{D}_k h_{jl} - \overline{D}_l h_{jk})\, \partial_i f \, d\mathrm{vol}_{\overline{g}} \\
&+ \int_\Omega \overline{g}^{ip}\, \overline{g}^{kq}\, \overline{g}^{jl}\, h_{pq}\, (\overline{D}_k h_{jl} - \overline{D}_l h_{jk})\, \partial_i f \, d\mathrm{vol}_{\overline{g}} \\
&+ \int_{\partial\Omega} \overline{g}^{jl}\, (\overline{D}_k h_{jl} - \overline{D}_l h_{jk})\, \overline{\nu}^k \, f \, d\sigma_{\overline{g}} \\
&- \int_{\partial\Omega} \overline{g}^{jp}\, \overline{g}^{lq}\, h_{pq}\, (\overline{D}_k h_{jl} - \overline{D}_l h_{jk})\, \overline{\nu}^k \, f \, d\sigma_{\overline{g}} \\
&- \int_{\partial\Omega} \overline{g}^{kq}\, \overline{g}^{jl}\, h_{pq}\, (\overline{D}_k h_{jl} - \overline{D}_l h_{jk})\, \overline{\nu}^p \, f \, d\sigma_{\overline{g}} \bigg| \\
&\leq C \int_\Omega |h|\, |\overline{D} h|^2 \, d\mathrm{vol}_{\overline{g}} + C \int_\Omega |h|^3 \, d\mathrm{vol}_{\overline{g}} + C \int_{\partial\Omega} |h|^2\, |\overline{D} h| \, d\sigma_{\overline{g}},
\end{aligned}$$

where C is a positive constant that depends only on n and c.

Proof. Using Proposition 4 and the divergence theorem, we obtain

$$\begin{aligned}
&\bigg| \int_\Omega (R_g - n(n-1) + (n-1)\,\mathrm{tr}_{\overline{g}}(h) - (n-1)\,|h|_{\overline{g}}^2)\, f \, d\mathrm{vol}_{\overline{g}} \\
&+ \frac{1}{4} \int_\Omega |\overline{D} h|^2 \, f \, d\mathrm{vol}_{\overline{g}} - \frac{1}{2} \int_\Omega \overline{g}^{ij}\, \overline{g}^{kl}\, \overline{g}^{pq}\, \overline{D}_i h_{kp}\, \overline{D}_l h_{jq}\, f \, d\mathrm{vol}_{\overline{g}} \\
&+ \frac{1}{4} \int_\Omega |\overline{\nabla}(\mathrm{tr}_{\overline{g}}(h))|^2 \, f \, d\mathrm{vol}_{\overline{g}} - \int_\Omega g^{ik}\, g^{jl}\, (\overline{D}_k h_{jl} - \overline{D}_l h_{jk})\, \partial_i f \, d\mathrm{vol}_{\overline{g}} \\
&+ \int_{\partial\Omega} g^{ik}\, g^{jl}\, (\overline{D}_k h_{jl} - \overline{D}_l h_{jk})\, \overline{g}_{im}\, \overline{\nu}^m \, f \, d\sigma_{\overline{g}} \bigg| \\
&\leq C \int_\Omega |h|\, |\overline{D} h|^2 \, d\mathrm{vol}_{\overline{g}} + C \int_\Omega |h|^3 \, d\mathrm{vol}_{\overline{g}}.
\end{aligned}$$

Here, $\overline{\nu}$ denotes the outward-pointing unit normal vector to $\partial\Omega$ with respect to the metric $\overline{g}$. Using the identity $\overline{D}^2_{i,k} f = -f\, \overline{g}_{ik}$, we obtain

$$\begin{aligned}
&\int_\Omega (n-1)\,\mathrm{tr}_{\overline{g}}(h)\, f \, d\mathrm{vol}_{\overline{g}} - \int_\Omega \overline{g}^{ik}\, \overline{g}^{jl}\, (\overline{D}_k h_{jl} - \overline{D}_l h_{jk})\, \partial_i f \, d\mathrm{vol}_{\overline{g}} \\
&= -\int_{\partial\Omega} (\mathrm{tr}_{\overline{g}}(h)\, \partial_{\overline{\nu}} f - h(\overline{\nu}, \overline{\nabla} f))\, d\sigma_{\overline{g}} = 0.
\end{aligned}$$

Thus, we conclude that

$$\begin{aligned}
&\bigg| \int_\Omega (R_g - n(n-1) - (n-1)\,|h|_{\overline{g}}^2)\, f\, d\mathrm{vol}_{\overline{g}} + \frac{1}{4} \int_\Omega |\overline{D}h|^2\, f\, d\mathrm{vol}_{\overline{g}} \\
&+ \frac{1}{4} \int_\Omega |\overline{\nabla}(\mathrm{tr}_{\overline{g}}(h))|^2\, f\, d\mathrm{vol}_{\overline{g}} - \frac{1}{2} \int_\Omega \overline{g}^{ij}\, \overline{g}^{kl}\, \overline{g}^{pq}\, \overline{D}_i h_{kp}\, \overline{D}_l h_{jq}\, f\, d\mathrm{vol}_{\overline{g}} \\
&- \int_\Omega (g^{ik}\, g^{jl} - \overline{g}^{ik}\, \overline{g}^{jl})\, (\overline{D}_k h_{jl} - \overline{D}_l h_{jk})\, \partial_i f\, d\mathrm{vol}_{\overline{g}} \\
&+ \int_{\partial\Omega} g^{ik}\, g^{jl}\, (\overline{D}_k h_{jl} - \overline{D}_l h_{jk})\, \overline{g}_{im}\, \overline{\nu}^m\, f\, d\sigma_{\overline{g}} \bigg| \\
&\leq C \int_\Omega |h|\, |\overline{D}h|^2\, d\mathrm{vol}_{\overline{g}} + C \int_\Omega |h|^3\, d\mathrm{vol}_{\overline{g}}.
\end{aligned}$$

From this, the assertion follows easily. q.e.d.

In the remainder of this section, we will assume that h is divergence-free in the sense that $\overline{g}^{ik}\, \overline{D}_i h_{kl} = 0$.

Proposition 7. *Assume that h is divergence-free. Then*

$$\begin{aligned}
&\bigg| \int_\Omega (R_g - n(n-1))\, f\, d\mathrm{vol}_{\overline{g}} + \frac{1}{4} \int_\Omega |\overline{D}h|^2\, f\, d\mathrm{vol}_{\overline{g}} \\
&+ \frac{1}{4} \int_\Omega |\overline{\nabla}(\mathrm{tr}_{\overline{g}}(h))|^2\, f\, d\mathrm{vol}_{\overline{g}} + \frac{1}{2} \int_\Omega |h|_{\overline{g}}^2\, f\, d\mathrm{vol}_{\overline{g}} \\
&+ \frac{1}{2} \int_\Omega \mathrm{tr}_{\overline{g}}(h)^2\, f\, d\mathrm{vol}_{\overline{g}} + \frac{1}{4} \int_{\partial\Omega} (|h|_{\overline{g}}^2 + 3\, h(\overline{\nu}, \overline{\nu})^2)\, \partial_{\overline{\nu}} f\, d\sigma_{\overline{g}} \\
&+ \int_{\partial\Omega} \overline{g}^{jl}\, \overline{D}_k h_{jl}\, \overline{\nu}^k\, f\, d\sigma_{\overline{g}} - \frac{1}{2} \int_{\partial\Omega} \overline{g}^{kl}\, \overline{g}^{pq}\, h_{kp}\, \overline{D}_l h_{jq}\, \overline{\nu}^j\, f\, d\sigma_{\overline{g}} \\
&- \int_{\partial\Omega} \overline{g}^{jp}\, \overline{g}^{lq}\, h_{pq}\, (\overline{D}_k h_{jl} - \overline{D}_l h_{jk})\, \overline{\nu}^k\, f\, d\sigma_{\overline{g}} \\
&- \int_{\partial\Omega} \overline{g}^{kq}\, \overline{g}^{jl}\, h_{pq}\, \overline{D}_k h_{jl}\, \overline{\nu}^p\, f\, d\sigma_{\overline{g}} \bigg| \\
&\leq C \int_\Omega |h|\, |\overline{D}h|^2\, d\mathrm{vol}_{\overline{g}} + C \int_\Omega |h|^3\, d\mathrm{vol}_{\overline{g}} + C \int_{\partial\Omega} |h|^2\, |\overline{D}h|\, d\sigma_{\overline{g}},
\end{aligned}$$

where C is a positive constant that depends only on n and c.

Proof. Since $\overline{g}$ has constant sectional curvature 1, we have

$$\overline{D}^2_{i,l} h_{jq} = \overline{D}^2_{l,i} h_{jq} + h_{lq}\, \overline{g}_{ij} - h_{iq}\, \overline{g}_{jl} + h_{jl}\, \overline{g}_{iq} - h_{ij}\, \overline{g}_{lq}.$$

Since h is divergence-free, it follows that

$$\overline{g}^{ij}\, \overline{D}^2_{i,l} h_{jq} = n\, h_{lq} - \mathrm{tr}_{\overline{g}}(h)\, \overline{g}_{lq}.$$

This implies

$$\begin{aligned}
&-\int_\Omega \overline{g}^{ij}\,\overline{g}^{kl}\,\overline{g}^{pq}\,h_{kp}\,\overline{D}_l h_{jq}\,\partial_i f\,d\mathrm{vol}_{\overline{g}} - \int_\Omega \overline{g}^{ij}\,\overline{g}^{kl}\,\overline{g}^{pq}\,\overline{D}_i h_{kp}\,\overline{D}_l h_{jq}\,f\,d\mathrm{vol}_{\overline{g}} \\
&= \int_\Omega \overline{g}^{ij}\,\overline{g}^{kl}\,\overline{g}^{pq}\,h_{kp}\,\overline{D}^2_{i,l} h_{jq}\,f\,d\mathrm{vol}_{\overline{g}} - \int_{\partial\Omega} \overline{g}^{kl}\,\overline{g}^{pq}\,h_{kp}\,\overline{D}_l h_{jq}\,\overline{\nu}^j\,f\,d\sigma_{\overline{g}} \\
&= n\int_\Omega |h|^2_{\overline{g}}\,f\,d\mathrm{vol}_{\overline{g}} - \int_\Omega \mathrm{tr}_{\overline{g}}(h)^2\,f\,d\mathrm{vol}_{\overline{g}} - \int_{\partial\Omega} \overline{g}^{kl}\,\overline{g}^{pq}\,h_{kp}\,\overline{D}_l h_{jq}\,\overline{\nu}^j\,f\,d\sigma_{\overline{g}}.
\end{aligned}$$

From this, we deduce that

$$\begin{aligned}
&\int_\Omega \overline{g}^{ik}\,\overline{g}^{jp}\,\overline{g}^{lq}\,h_{pq}\,(\overline{D}_k h_{jl} - \overline{D}_l h_{jk})\,\partial_i f\,d\mathrm{vol}_{\overline{g}} \\
&- \frac{1}{2}\int_\Omega \overline{g}^{ij}\,\overline{g}^{kl}\,\overline{g}^{pq}\,\overline{D}_i h_{kp}\,\overline{D}_l h_{jq}\,f\,d\mathrm{vol}_{\overline{g}} \\
&= \frac{1}{2}\int_\Omega \overline{g}^{ik}\,\partial_k(|h|^2_{\overline{g}})\,\partial_i f\,d\mathrm{vol}_{\overline{g}} - \frac{1}{2}\int_\Omega \overline{g}^{ik}\,\overline{g}^{jp}\,\overline{g}^{lq}\,h_{pq}\,\overline{D}_l h_{jk}\,\partial_i f\,d\mathrm{vol}_{\overline{g}} \\
&+ \frac{n}{2}\int_\Omega |h|^2_{\overline{g}}\,f\,d\mathrm{vol}_{\overline{g}} - \frac{1}{2}\int_\Omega \mathrm{tr}_{\overline{g}}(h)^2\,f\,d\mathrm{vol}_{\overline{g}} \\
&- \frac{1}{2}\int_{\partial\Omega} \overline{g}^{kl}\,\overline{g}^{pq}\,h_{kp}\,\overline{D}_l h_{jq}\,\overline{\nu}^j\,f\,d\sigma_{\overline{g}}.
\end{aligned}$$

Integration by parts gives

$$\begin{aligned}
&\int_\Omega \overline{g}^{ik}\,\overline{g}^{jp}\,\overline{g}^{lq}\,h_{pq}\,(\overline{D}_k h_{jl} - \overline{D}_l h_{jk})\,\partial_i f\,d\mathrm{vol}_{\overline{g}} \\
&- \frac{1}{2}\int_\Omega \overline{g}^{ij}\,\overline{g}^{kl}\,\overline{g}^{pq}\,\overline{D}_i h_{kp}\,\overline{D}_l h_{jq}\,f\,d\mathrm{vol}_{\overline{g}} \\
&= -\frac{1}{2}\int_\Omega |h|^2_{\overline{g}}\,\Delta_{\overline{g}} f\,d\mathrm{vol}_{\overline{g}} + \frac{1}{2}\int_\Omega \overline{g}^{ik}\,\overline{g}^{jp}\,\overline{g}^{lq}\,h_{pq}\,h_{jk}\,\overline{D}^2_{i,l} f\,d\mathrm{vol}_{\overline{g}} \\
&+ \frac{1}{2}\int_{\partial\Omega} |h|^2_{\overline{g}}\,\partial_{\overline{\nu}} f\,d\sigma_{\overline{g}} - \frac{1}{2}\int_{\partial\Omega} \overline{g}^{ik}\,\overline{g}^{jp}\,h_{pq}\,h_{jk}\,\partial_i f\,\overline{\nu}^q\,d\sigma_{\overline{g}} \\
&+ \frac{n}{2}\int_\Omega |h|^2_{\overline{g}}\,f\,d\mathrm{vol}_{\overline{g}} - \frac{1}{2}\int_\Omega \mathrm{tr}_{\overline{g}}(h)^2\,f\,d\mathrm{vol}_{\overline{g}} \\
&- \frac{1}{2}\int_{\partial\Omega} \overline{g}^{kl}\,\overline{g}^{pq}\,h_{kp}\,\overline{D}_l h_{jq}\,\overline{\nu}^j\,f\,d\sigma_{\overline{g}} \\
&= \frac{2n-1}{2}\int_\Omega |h|^2_{\overline{g}}\,f\,d\mathrm{vol}_{\overline{g}} - \frac{1}{2}\int_\Omega \mathrm{tr}_{\overline{g}}(h)^2\,f\,d\mathrm{vol}_{\overline{g}} \\
&+ \frac{1}{2}\int_{\partial\Omega} |h|^2_{\overline{g}}\,\partial_{\overline{\nu}} f\,d\sigma_{\overline{g}} - \frac{1}{2}\int_{\partial\Omega} \overline{g}^{ik}\,\overline{g}^{jp}\,h_{pq}\,h_{jk}\,\partial_i f\,\overline{\nu}^q\,d\sigma_{\overline{g}} \\
&- \frac{1}{2}\int_{\partial\Omega} \overline{g}^{kl}\,\overline{g}^{pq}\,h_{kp}\,\overline{D}_l h_{jq}\,\overline{\nu}^j\,f\,d\sigma_{\overline{g}}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\int_\Omega \overline{g}^{ip}\,\overline{g}^{kq}\,\overline{g}^{jl}\,h_{pq}\,(\overline{D}_k h_{jl} - \overline{D}_l h_{jk})\,\partial_i f\,d\mathrm{vol}_{\overline{g}} \\
&= \int_\Omega \overline{g}^{ip}\,\overline{g}^{kq}\,h_{pq}\,\partial_k(\mathrm{tr}_{\overline{g}}(h))\,\partial_i f\,d\mathrm{vol}_{\overline{g}} \\
&= -\int_\Omega \overline{g}^{ip}\,\overline{g}^{kq}\,h_{pq}\,\mathrm{tr}_{\overline{g}}(h)\,\overline{D}^2_{i,k} f\,d\mathrm{vol}_{\overline{g}} + \int_{\partial\Omega} \overline{g}^{ip}\,h_{pq}\,\mathrm{tr}_{\overline{g}}(h)\,\partial_i f\,\overline{\nu}^q\,d\sigma_{\overline{g}} \\
&= \int_\Omega \mathrm{tr}_{\overline{g}}(h)^2\,f\,d\mathrm{vol}_{\overline{g}} + \int_{\partial\Omega} \overline{g}^{ip}\,h_{pq}\,\mathrm{tr}_{\overline{g}}(h)\,\partial_i f\,\overline{\nu}^q\,d\sigma_{\overline{g}}.
\end{aligned}$$

Putting these facts together, we obtain

$$\begin{aligned}
&\int_\Omega \overline{g}^{ip}\,\overline{g}^{kq}\,\overline{g}^{jl}\,h_{pq}\,(\overline{D}_k h_{jl} - \overline{D}_l h_{jk})\,\partial_i f\,d\mathrm{vol}_{\overline{g}} \\
&+ \int_\Omega \overline{g}^{ik}\,\overline{g}^{jp}\,\overline{g}^{lq}\,h_{pq}\,(\overline{D}_k h_{jl} - \overline{D}_l h_{jk})\,\partial_i f\,d\mathrm{vol}_{\overline{g}} \\
&- \frac{1}{2}\int_\Omega \overline{g}^{ij}\,\overline{g}^{kl}\,\overline{g}^{pq}\,\overline{D}_i h_{kp}\,\overline{D}_l h_{jq}\,f\,d\mathrm{vol}_{\overline{g}} \\
&= \frac{2n-1}{2}\int_\Omega |h|^2_{\overline{g}}\,f\,d\mathrm{vol}_{\overline{g}} + \frac{1}{2}\int_\Omega \mathrm{tr}_{\overline{g}}(h)^2\,f\,d\mathrm{vol}_{\overline{g}} \\
&+ \frac{1}{2}\int_{\partial\Omega} |h|^2_{\overline{g}}\,\partial_{\overline{\nu}} f\,d\sigma_{\overline{g}} - \frac{1}{2}\int_{\partial\Omega} \overline{g}^{ik}\,\overline{g}^{jp}\,h_{pq}\,h_{jk}\,\partial_i f\,\overline{\nu}^q\,d\sigma_{\overline{g}} \\
&+ \int_{\partial\Omega} \overline{g}^{ip}\,h_{pq}\,\mathrm{tr}_{\overline{g}}(h)\,\partial_i f\,\overline{\nu}^q\,d\sigma_{\overline{g}} - \frac{1}{2}\int_{\partial\Omega} \overline{g}^{kl}\,\overline{g}^{pq}\,h_{kp}\,\overline{D}_l h_{jq}\,\overline{\nu}^j\,f\,d\sigma_{\overline{g}} \\
&= \frac{2n-1}{2}\int_\Omega |h|^2_{\overline{g}}\,f\,d\mathrm{vol}_{\overline{g}} + \frac{1}{2}\int_\Omega \mathrm{tr}_{\overline{g}}(h)^2\,f\,d\mathrm{vol}_{\overline{g}} \\
&+ \frac{1}{4}\int_{\partial\Omega} (|h|^2_{\overline{g}} + 3\,h(\overline{\nu},\overline{\nu})^2)\,\partial_{\overline{\nu}} f\,d\sigma_{\overline{g}} - \frac{1}{2}\int_{\partial\Omega} \overline{g}^{kl}\,\overline{g}^{pq}\,h_{kp}\,\overline{D}_l h_{jq}\,\overline{\nu}^j\,f\,d\sigma_{\overline{g}}.
\end{aligned}$$

Hence, the assertion follows from Proposition 6. q.e.d.

4. Analysis of the boundary terms

In this section, we analyze the boundary terms in Proposition 7. As in the previous section, we assume that $\overline{g}$ is the standard metric on S^n, and $\Omega = \{f \geq c\}$ centered at the north pole. Moreover, we consider a Riemannian metric on Ω of the form $g = \overline{g} + h$, where $|h|_{\overline{g}} \leq \frac{1}{2}$ at each point in Ω.

Proposition 8. *Assume that h is divergence-free. Then*

$$\begin{aligned}
&\int_{\partial\Omega} \overline{g}^{jl}\, \overline{D}_k h_{jl}\, \overline{\nu}^k\, d\sigma_{\overline{g}} - \frac{1}{2} \int_{\partial\Omega} \overline{g}^{kl}\, \overline{g}^{pq}\, h_{kp}\, \overline{D}_l h_{jq}\, \overline{\nu}^j\, d\sigma_{\overline{g}} \\
&- \int_{\partial\Omega} \overline{g}^{jp}\, \overline{g}^{lq}\, h_{pq}\, (\overline{D}_k h_{jl} - \overline{D}_l h_{jk})\, \overline{\nu}^k\, d\sigma_{\overline{g}} \\
&- \int_{\partial\Omega} \overline{g}^{kq}\, \overline{g}^{jl}\, h_{pq}\, \overline{D}_k h_{jl}\, \overline{\nu}^p\, d\sigma_{\overline{g}} \\
&= - \int_{\partial\Omega} (1 - h(\overline{\nu}, \overline{\nu})) \sum_{a=1}^{n-1} \big(2\, (\overline{D}_{e_a} h)(e_a, \overline{\nu}) - (\overline{D}_{\overline{\nu}} h)(e_a, e_a)\big)\, d\sigma_{\overline{g}} \\
&+ \int_{\partial\Omega} \big(1 - \frac{1}{2}\, h(\overline{\nu}, \overline{\nu})\big)\, h(\overline{\nu}, \overline{\nu})\, H_{\overline{g}}\, d\sigma_{\overline{g}} \\
&+ \frac{3n-2}{2(n-1)} \int_{\partial\Omega} \sum_{a=1}^{n-1} h(e_a, \overline{\nu})^2\, H_{\overline{g}}\, d\sigma_{\overline{g}}.
\end{aligned}$$

Here, $\{e_a : 1 \leq a \leq n-1\}$ is a local orthonormal frame on $\partial\Omega$, and C is a positive constant that depends only on n and c.

Proof. Let $\{e_a : 1 \leq a \leq n-1\}$ be a local orthonormal frame on $\partial\Omega$. Since h is divergence-free, we have

$$\begin{aligned}
&\overline{g}^{jl}\, \overline{D}_k h_{jl}\, \overline{\nu}^k - \frac{1}{2}\, \overline{g}^{kl}\, \overline{g}^{pq}\, h_{kp}\, \overline{D}_l h_{jq}\, \overline{\nu}^j \\
&- \overline{g}^{jp}\, \overline{g}^{lq}\, h_{pq}\, (\overline{D}_k h_{jl} - \overline{D}_l h_{jk})\, \overline{\nu}^k - \overline{g}^{kq}\, \overline{g}^{jl}\, h_{pq}\, \overline{D}_k h_{jl}\, \overline{\nu}^p \\
&= -(1 - h(\overline{\nu}, \overline{\nu})) \sum_{a=1}^{n-1} \big(2\, (\overline{D}_{e_a} h)(e_a, \overline{\nu}) - (\overline{D}_{\overline{\nu}} h)(e_a, e_a)\big) \\
&+ \big(1 - \frac{1}{2}\, h(\overline{\nu}, \overline{\nu})\big) \sum_{a=1}^{n-1} (\overline{D}_{e_a} h)(e_a, \overline{\nu}) - \frac{1}{2} \sum_{a=1}^{n-1} (\overline{D}_{e_a} h)(\overline{\nu}, \overline{\nu})\, h(e_a, \overline{\nu}) \\
&+ \frac{3}{2} \sum_{a,b=1}^{n-1} h(e_a, \overline{\nu})\, (\overline{D}_{e_b} h)(e_a, e_b) - \sum_{a,b=1}^{n-1} h(e_a, \overline{\nu})\, (\overline{D}_{e_a} h)(e_b, e_b).
\end{aligned}$$

At this point, we define a one-form ω on $\partial\Omega$ by $\omega(e_a) = (1 - \frac{1}{2}\, h(\overline{\nu}, \overline{\nu}))\, h(e_a, \overline{\nu})$. Since $\partial\Omega$ is umbilic with respect to $\overline{g}$, we have

$$\overline{D}_{e_a} \overline{\nu} = \frac{1}{n-1}\, H_{\overline{g}}\, e_a,$$

where $H_{\overline{g}}$ denotes the mean curvature of $\partial\Omega$ with respect to the metric $\overline{g}$. Using this relation, we obtain the following formula for the divergence of ω:

$$\begin{aligned}\operatorname{div}_{\partial\Omega}(\omega) &= \big(1 - \frac{1}{2}\, h(\overline{\nu},\overline{\nu})\big) \sum_{a=1}^{n-1} (\overline{D}_{e_a} h)(e_a,\overline{\nu}) - \frac{1}{2} \sum_{a=1}^{n-1} (\overline{D}_{e_a} h)(\overline{\nu},\overline{\nu})\, h(e_a,\overline{\nu}) \\ &\quad - \big(1 - \frac{1}{2}\, h(\overline{\nu},\overline{\nu})\big)\, h(\overline{\nu},\overline{\nu})\, H_{\overline{g}} - \frac{1}{n-1} \sum_{a=1}^{n-1} h(e_a,\overline{\nu})^2\, H_{\overline{g}}.\end{aligned}$$

Moreover, we have the pointwise identities

$$\sum_{b=1}^{n-1} (\overline{D}_{e_b} h)(e_a,e_b) = \frac{n}{n-1}\, h(e_a,\overline{\nu})\, H_{\overline{g}}$$

and

$$\sum_{b=1}^{n-1} (\overline{D}_{e_a} h)(e_b,e_b) = \frac{2}{n-1}\, h(e_a,\overline{\nu})\, H_{\overline{g}}.$$

Putting these facts together, we obtain

$$\begin{aligned}&\overline{g}^{jl}\, \overline{D}_k h_{jl}\, \overline{\nu}^k - \frac{1}{2}\, \overline{g}^{kl}\, \overline{g}^{pq}\, h_{kp}\, \overline{D}_l h_{jq}\, \overline{\nu}^j \\ &- \overline{g}^{jp}\, \overline{g}^{lq}\, h_{pq}\, (\overline{D}_k h_{jl} - \overline{D}_l h_{jk})\, \overline{\nu}^k - \overline{g}^{kq}\, \overline{g}^{jl}\, h_{pq}\, \overline{D}_k h_{jl}\, \overline{\nu}^p \\ &= -(1 - h(\overline{\nu},\overline{\nu})) \sum_{a=1}^{n-1} \big(2\, (\overline{D}_{e_a} h)(e_a,\overline{\nu}) - (\overline{D}_{\overline{\nu}} h)(e_a,e_a)\big) \\ &+ \big(1 - \frac{1}{2}\, h(\overline{\nu},\overline{\nu})\big)\, h(\overline{\nu},\overline{\nu})\, H_{\overline{g}} + \frac{3n-2}{2(n-1)} \sum_{a=1}^{n-1} h(e_a,\overline{\nu})^2\, H_{\overline{g}} + \operatorname{div}_{\partial\Omega}(\omega).\end{aligned}$$

Therefore, the assertion follows from the divergence theorem. q.e.d.

Combining Proposition 8 and Proposition 5, we can draw the following conclusion:

Corollary 9. *If h is divergence-free, then we have*

$$\begin{aligned}&\bigg| \int_{\partial\Omega} (2 - h(\overline{\nu},\overline{\nu}))\, (H_g - H_{\overline{g}})\, d\sigma_{\overline{g}} \\ &- \int_{\partial\Omega} \overline{g}^{jl}\, \overline{D}_k h_{jl}\, \overline{\nu}^k\, d\sigma_{\overline{g}} + \frac{1}{2} \int_{\partial\Omega} \overline{g}^{kl}\, \overline{g}^{pq}\, h_{kp}\, \overline{D}_l h_{jq}\, \overline{\nu}^j\, d\sigma_{\overline{g}} \\ &+ \int_{\partial\Omega} \overline{g}^{jp}\, \overline{g}^{lq}\, h_{pq}\, (\overline{D}_k h_{jl} - \overline{D}_l h_{jk})\, \overline{\nu}^k\, d\sigma_{\overline{g}} \\ &+ \int_{\partial\Omega} \overline{g}^{kq}\, \overline{g}^{jl}\, h_{pq}\, \overline{D}_k h_{jl}\, \overline{\nu}^p\, d\sigma_{\overline{g}} \\ &+ \frac{1}{4} \int_{\partial\Omega} h(\overline{\nu},\overline{\nu})^2\, H_{\overline{g}}\, d\sigma_{\overline{g}} + \frac{n}{2(n-1)} \int_{\partial\Omega} \sum_{a=1}^{n-1} h(e_a,\overline{\nu})^2\, H_{\overline{g}}\, d\sigma_{\overline{g}} \bigg| \\ &\leq C \int_{\partial\Omega} |h|^2\, |\overline{D} h|\, d\sigma_{\overline{g}} + C \int_{\partial\Omega} |h|^3\, d\sigma_{\overline{g}},\end{aligned}$$

where C is a positive constant that depends only on n and c.

Proof. It follows from Proposition 5 that

$$\begin{aligned} &\bigg| \int_{\partial\Omega} (2 - h(\overline{\nu},\overline{\nu}))\,(H_g - H_{\overline{g}})\,d\sigma_{\overline{g}} \\ &- \int_{\partial\Omega} \Big(h(\overline{\nu},\overline{\nu}) - \frac{3}{4}\,h(\overline{\nu},\overline{\nu})^2 + \sum_{a=1}^{n-1} h(e_a,\overline{\nu})^2\Big)\,H_{\overline{g}}\,d\sigma_{\overline{g}} \\ &+ \int_{\partial\Omega} (1 - h(\overline{\nu},\overline{\nu})) \sum_{a=1}^{n-1} \big(2\,(\overline{D}_{e_a}h)(e_a,\overline{\nu}) - (\overline{D}_{\overline{\nu}}h)(e_a,e_a)\big)\,d\sigma_{\overline{g}} \bigg| \\ &\leq C \int_{\partial\Omega} |h|^2\,|\overline{D}h|\,d\sigma_{\overline{g}} + C \int_{\partial\Omega} |h|^3\,d\sigma_{\overline{g}}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} &\int_{\partial\Omega} \overline{g}^{jl}\,\overline{D}_k h_{jl}\,\overline{\nu}^k\,d\sigma_{\overline{g}} - \frac{1}{2} \int_{\partial\Omega} \overline{g}^{kl}\,\overline{g}^{pq}\,h_{kp}\,\overline{D}_l h_{jq}\,\overline{\nu}^j\,d\sigma_{\overline{g}} \\ &- \int_{\partial\Omega} \overline{g}^{jp}\,\overline{g}^{lq}\,h_{pq}\,(\overline{D}_k h_{jl} - \overline{D}_l h_{jk})\,\overline{\nu}^k\,d\sigma_{\overline{g}} \\ &- \int_{\partial\Omega} \overline{g}^{kq}\,\overline{g}^{jl}\,h_{pq}\,\overline{D}_k h_{jl}\,\overline{\nu}^p\,d\sigma_{\overline{g}} \\ &= - \int_{\partial\Omega} (1 - h(\overline{\nu},\overline{\nu})) \sum_{a=1}^{n-1} \big(2\,(\overline{D}_{e_a}h)(e_a,\overline{\nu}) - (\overline{D}_{\overline{\nu}}h)(e_a,e_a)\big)\,d\sigma_{\overline{g}} \\ &+ \int_{\partial\Omega} \big(1 - \frac{1}{2}\,h(\overline{\nu},\overline{\nu})\big)\,h(\overline{\nu},\overline{\nu})\,H_{\overline{g}}\,d\sigma_{\overline{g}} \\ &+ \frac{3n-2}{2(n-1)} \int_{\partial\Omega} \sum_{a=1}^{n-1} h(e_a,\overline{\nu})^2\,H_{\overline{g}}\,d\sigma_{\overline{g}} \end{aligned}$$

by Proposition 8. Putting these facts together, the assertion follows.

q.e.d.

Theorem 10. *Assume that h is divergence-free. Then*

$$\begin{aligned}
&\bigg| \int_\Omega (R_g - n(n-1))\, f \, d\text{vol}_{\overline{g}} + \int_{\partial\Omega} (2 - h(\overline{\nu},\overline{\nu}))\, (H_g - H_{\overline{g}})\, f \, d\sigma \\
&+ \frac{1}{4} \int_\Omega |\overline{D} h|^2 \, f \, d\text{vol}_{\overline{g}} + \frac{1}{4} \int_\Omega |\overline{\nabla}(\text{tr}_{\overline{g}}(h))|^2 \, f \, d\text{vol}_{\overline{g}} \\
&+ \frac{1}{2} \int_\Omega |h|_{\overline{g}}^2 \, f \, d\text{vol}_{\overline{g}} + \frac{1}{2} \int_\Omega \text{tr}_{\overline{g}}(h)^2 \, f \, d\text{vol}_{\overline{g}} \\
&+ \int_{\partial\Omega} h(\overline{\nu},\overline{\nu})^2 \, \partial_{\overline{\nu}} f \, d\sigma_{\overline{g}} + \frac{1}{2} \int_{\partial\Omega} \sum_{a=1}^{n-1} h(e_a,\overline{\nu})^2 \, \partial_{\overline{\nu}} f \, d\sigma_{\overline{g}} \\
&+ \frac{1}{4} \int_{\partial\Omega} h(\overline{\nu},\overline{\nu})^2 \, H_{\overline{g}} \, f \, d\sigma_{\overline{g}} + \frac{n}{2(n-1)} \int_{\partial\Omega} \sum_{a=1}^{n-1} h(e_a,\overline{\nu})^2 \, H_{\overline{g}} \, f \, d\sigma_{\overline{g}} \bigg| \\
&\leq C \int_\Omega |h| \, |\overline{D} h|^2 \, d\text{vol}_{\overline{g}} + C \int_\Omega |h|^3 \, d\text{vol}_{\overline{g}} \\
&+ C \int_{\partial\Omega} |h|^2 \, |\overline{D} h| \, d\sigma_{\overline{g}} + C \int_{\partial\Omega} |h|^3 \, d\sigma_{\overline{g}}.
\end{aligned}$$

Here, C is a positive constant that depends only on n and c.

Proof. Recall that f is constant along the boundary $\partial\Omega$. Hence, the assertion is a consequence of Proposition 7 and Corollary 9. q.e.d.

5. Proof of Theorem 3

To prove Theorem 3, we need an analogue of Ebin's slice theorem for manifolds with boundary [**10**] (see also [**12**]). The proof is standard, and works on any compact manifold with boundary.

Proposition 11. *Fix a real number $p > n$. If $\|g - \overline{g}\|_{W^{2,p}(\Omega,\overline{g})}$ is sufficiently small, we can find a diffeomorphism $\varphi : \Omega \to \Omega$ such that $\varphi|_{\partial\Omega} = \text{id}$ and $h = \varphi^*(g) - \overline{g}$ is divergence-free. Moreover,*

$$\|h\|_{W^{2,p}(\Omega,\overline{g})} \leq N \, \|g - \overline{g}\|_{W^{2,p}(\Omega,\overline{g})},$$

where N is a positive constant that depends only on Ω.

Proof. Let $\mathscr{S}$ denote the space of symmetric two-tensors on Ω of class $W^{2,p}$, and let $\mathscr{M}$ denote the space of Riemannian metrics on Ω of class $W^{2,p}$. Moreover, let $\mathscr{X}$ denote the space of vector fields of class $W^{3,p}$ that vanish along the boundary $\partial\Omega$, and let $\mathscr{D}$ denote the space of all diffeomorphisms $\varphi : \Omega \to \Omega$ of class $W^{3,p}$ satisfying $\varphi|_{\partial\Omega} = \text{id}$. Clearly, the tangent space to $\mathscr{M}$ at $\overline{g}$ can be identified with $\mathscr{S}$; similarly, the tangent space to $\mathscr{D}$ at the identity can be identified with $\mathscr{X}$.

There is a natural action

$$A : \mathscr{D} \times \mathscr{M} \to \mathscr{M}, \qquad (\varphi, g) \to \varphi^*(g).$$

Let us consider the linearization of A around the point $(\mathrm{id}, \overline{g})$. This gives a map $L : T_{\mathrm{id}}\mathscr{D} \to T_{\overline{g}}\mathscr{M}$. The map L sends a vector field $\xi \in \mathscr{X}$ to the Lie derivative $\mathscr{L}_\xi(\overline{g}) \in \mathscr{S}$. Standard elliptic regularity theory implies that

$$\mathscr{S} = \{\mathscr{L}_\xi(\overline{g}) : \xi \in \mathscr{X}\} \oplus \{h \in \mathscr{S} : h \text{ is divergence-free}\}$$

(compare [**12**], p. 523). Hence, the assertion follows from the implicit function theorem. q.e.d.

We now complete the proof of Theorem 3. Let g be a Riemannian metric on the domain $\Omega = \{f \geq c\}$ with the following properties:

- $R_g \geq n(n-1)$ at each point in Ω.
- The metrics g and $\overline{g}$ induce the same metric on $\partial\Omega$.
- $H_g \geq H_{\overline{g}}$ at each point on $\partial\Omega$.

If $\|g - \overline{g}\|_{W^{2,p}(\Omega,\overline{g})}$ is sufficiently small, Proposition 11 implies the existence of a diffeomorphism $\varphi : \Omega \to \Omega$ such that $\varphi|_{\partial\Omega} = \mathrm{id}$ and $h = \varphi^*(g) - \overline{g}$ is divergence-free.

Note that $R_{\varphi^*(g)} \geq n(n-1)$ at each point in Ω and $H_{\varphi^*(g)} \geq H_{\overline{g}}$ at each point on $\partial\Omega$. Applying Theorem 10 to the metric $\varphi^*(g) = \overline{g} + h$, we obtain

$$\begin{aligned}
&\frac{1}{4} \int_\Omega |\overline{D}h|^2 \, f \, d\mathrm{vol}_{\overline{g}} + \frac{1}{4} \int_\Omega |\overline{\nabla}(\mathrm{tr}_{\overline{g}}(h))|^2 \, f \, d\mathrm{vol}_{\overline{g}} \\
&+ \frac{1}{2} \int_\Omega |h|_{\overline{g}}^2 \, f \, d\mathrm{vol}_{\overline{g}} + \frac{1}{2} \int_\Omega \mathrm{tr}_{\overline{g}}(h)^2 \, f \, d\mathrm{vol}_{\overline{g}} \\
&+ \int_{\partial\Omega} h(\overline{\nu},\overline{\nu})^2 \, \partial_{\overline{\nu}} f \, d\sigma_{\overline{g}} + \frac{1}{2} \int_{\partial\Omega} \sum_{a=1}^{n-1} h(e_a,\overline{\nu})^2 \, \partial_\nu f \, d\sigma_{\overline{g}} \\
&+ \frac{1}{4} \int_{\partial\Omega} h(\overline{\nu},\overline{\nu})^2 \, H_{\overline{g}} \, f \, d\sigma_{\overline{g}} + \frac{n}{2(n-1)} \int_{\partial\Omega} \sum_{a=1}^{n-1} h(e_a,\overline{\nu})^2 \, H_{\overline{g}} \, f \, d\sigma_{\overline{g}} \\
&\leq C \int_\Omega |h| \, |\overline{D}h|^2 \, d\mathrm{vol}_{\overline{g}} + C \int_\Omega |h|^3 \, d\mathrm{vol}_{\overline{g}} \\
&+ C \int_{\partial\Omega} |h|^2 \, |\overline{D}h| \, d\sigma_{\overline{g}} + C \int_{\partial\Omega} |h|^3 \, d\sigma_{\overline{g}}.
\end{aligned}$$

If we choose $c \geq \frac{2}{\sqrt{n+3}}$, then

$$\frac{1}{4} \, H_{\overline{g}} \, f + \partial_{\overline{\nu}} f = \frac{n-1}{4} \, \frac{f^2}{|\overline{\nabla} f|} - |\overline{\nabla} f| = \frac{n-1}{4} \, \frac{c^2}{\sqrt{1-c^2}} - \sqrt{1-c^2} \geq 0$$

at each point on $\partial\Omega$. This implies

$$\begin{aligned}
&\frac{1}{4}\int_\Omega |\overline{D}h|^2 \, f \, d\mathrm{vol}_{\overline{g}} + \frac{1}{4}\int_\Omega |\overline{\nabla}(\mathrm{tr}_{\overline{g}}(h))|^2 \, f \, d\mathrm{vol}_{\overline{g}} \\
&+ \frac{1}{2}\int_\Omega |h|_{\overline{g}}^2 \, f \, d\mathrm{vol}_{\overline{g}} + \frac{1}{2}\int_\Omega \mathrm{tr}_{\overline{g}}(h)^2 \, f \, d\mathrm{vol}_{\overline{g}} \\
&\leq C\int_\Omega |h| \, |\overline{D}h|^2 \, d\mathrm{vol}_{\overline{g}} + C\int_\Omega |h|^3 \, d\mathrm{vol}_{\overline{g}} \\
&+ C\int_{\partial\Omega} |h|^2 \, |\overline{D}h| \, d\sigma_{\overline{g}} + C\int_{\partial\Omega} |h|^3 \, d\sigma_{\overline{g}}.
\end{aligned}$$

By the trace theorem, the error terms on the right hand side are bounded from above by $C\,\|h\|_{C^1(\Omega,\overline{g})}\,\|h\|^2_{W^{1,2}(\Omega,\overline{g})}$. Hence, if $\|h\|_{C^1(\Omega,\overline{g})}$ is sufficiently small, then h vanishes identically, and therefore $\varphi^*(g) = \overline{g}$. This completes the proof of Theorem 3.

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