

QUANTITATIVE EMBEDDED CONTACT HOMOLOGY

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Abstract

Define a “Liouville domain” to be a compact exact symplectic manifold with contact-type boundary. We use embedded contact homology to assign to each four-dimensional Liouville domain (or subset thereof) a sequence of real numbers, which we call “ECH capacities”. The ECH capacities of a Liouville domain are defined in terms of the “ECH spectrum” of its boundary, which measures the amount of symplectic action needed to represent certain classes in embedded contact homology. Using cobordism maps on embedded contact homology (defined in joint work with Taubes), we show that the ECH capacities are monotone with respect to symplectic embeddings. We compute the ECH capacities of ellipsoids, polydisks, certain subsets of the cotangent bundle of T^2 , and disjoint unions of examples for which the ECH capacities are known. The resulting symplectic embedding obstructions are sharp in some interesting cases, for example for the problem of embedding an ellipsoid into a ball (as shown by McDuff-Schlenk) or embedding a disjoint union of balls into a ball. We also state and present evidence for a conjecture under which the asymptotics of the ECH capacities of a Liouville domain recover its symplectic volume.

1. Introduction

Define a *Liouville domain* to be a compact symplectic manifold (X, ω) such that ω is exact, and there exists a contact form λ on ∂X with $d\lambda = \omega|_{\partial X}$. In this paper we introduce a new obstruction to symplectically embedding one four-dimensional Liouville domain into another, which turns out to be sharp in some interesting cases. For background on symplectic embedding questions more generally we refer the reader to [3] for an extensive discussion.

1.1. The main theorem. If (X, ω) is a four-dimensional Liouville domain, we use embedded contact homology to define a sequence of real numbers

$$0 = c_0(X, \omega) < c_1(X, \omega) \leq c_2(X, \omega) \leq \cdots \leq \infty$$

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which we call the (distinguished) *ECH capacities* of (X, ω) . The precise definition of these numbers is given in §4.1. Our main result is:

Theorem 1.1. *Let (X_0, ω_0) and (X_1, ω_1) be four-dimensional Liouville domains. Suppose there is a symplectic embedding of (X_0, ω_0) into the interior of (X_1, ω_1) . Then*

$$c_k(X_0, \omega_0) \leq c_k(X_1, \omega_1)$$

for each positive integer k , and the inequality is strict when $c_k(X_0, \omega_0) < \infty$.

Note that in Theorem 1.1, the four-manifolds X_0 and X_1 and their boundaries are not assumed to be connected. The proof of Theorem 1.1 uses cobordism maps on embedded contact homology induced by “weakly exact symplectic cobordisms”, which are defined using Seiberg-Witten theory by the construction in [12, 13].

1.2. Examples of ECH capacities. To see what Theorem 1.1 tells us, we now present some computations of ECH capacities. Given positive real numbers a, b , define the ellipsoid

$$(1.1) \quad E(a, b) := \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} \leq 1 \right\}.$$

In particular, define the ball

$$B(a) := E(a, a).$$

Also define the polydisk

$$(1.2) \quad P(a, b) := \{ (z_1, z_2) \in \mathbb{C}^2 \mid \pi|z_1|^2 \leq a, \pi|z_2|^2 \leq b \}.$$

All of these examples are given the restriction of the standard symplectic form $\omega = \sum_{i=1}^2 dx_i dy_i$ on $\mathbb{R}^4 = \mathbb{C}^2$. The first two are Liouville domains, because the 1-form

$$(1.3) \quad \lambda = \frac{1}{2} \sum_{i=1}^2 (x_i dy_i - y_i dx_i)$$

restricts to a contact form on the boundary of any smooth star-shaped domain. The polydisk is not quite a Liouville domain because its boundary is only piecewise smooth. However, as explained in §4.2, the definition of ECH capacities and Theorem 1.1 extend to arbitrary subsets of symplectic four-manifolds. (One expects to still get decent symplectic embedding obstructions for examples such as polydisks that can be approximated by Liouville domains.)

To describe the ECH capacities of the ellipsoid, let $(a, b)_k$ denote the k^{th} smallest entry in the matrix of real numbers $(am + bn)_{m, n \in \mathbb{N}}$. We then have:

Proposition 1.2. *The ECH capacities of an ellipsoid are given by*

$$c_k(E(a, b)) = (a, b)_{k+1}.$$

Note that in the definition of “ k^{th} smallest” we count with repetitions. For example:

Corollary 1.3. *The ECH capacities of a ball are given by*

$$c_k(B(a)) = da,$$

where d is the unique nonnegative integer such that

$$\frac{d^2 + d}{2} \leq k \leq \frac{d^2 + 3d}{2}.$$

Next we have:

Theorem 1.4. *The ECH capacities of a polydisk are given by*

$$c_k(P(a, b)) = \min \{am + bn \mid (m, n) \in \mathbb{N}^2, (m + 1)(n + 1) \geq k + 1\}.$$

Finally, to compute the ECH capacities of a disjoint union of examples whose ECH capacities are known, one can use:

Proposition 1.5. *Let (X_i, ω_i) be four-dimensional Liouville domains for $i = 1, \dots, n$. Then*

$$c_k \left(\prod_{i=1}^n (X_i, \omega_i) \right) = \max \left\{ \sum_{i=1}^n c_{k_i} (X_i, \omega_i) \mid \sum_{i=1}^n k_i = k \right\}.$$

1.3. Examples of symplectic embedding obstructions. One can now plug the above numbers into Theorem 1.1 to get explicit (but subtle, number-theoretic) obstructions to symplectic embeddings.

1.3.1. An ellipsoid into a ball (or ellipsoid). For example, consider the problem of symplectically embedding an ellipsoid into a ball. By scaling, we can encode this problem into a single function as follows: Given $a > 0$, define $f(a)$ to be the infimum over $c \in \mathbb{R}$ such that the ellipsoid $E(a, 1)$ symplectically embeds into the ball $B(c)$. By Theorem 1.1, Proposition 1.2, and Corollary 1.3, we have

$$(1.4) \quad f(a) \geq \sup_{k=2,3,\dots} \frac{(a, 1)_k}{(1, 1)_k} = \sup_{d=1,2,\dots} \frac{1}{d} (a, 1)_{(d^2+3d+2)/2}.$$

On the other hand, McDuff-Schlenk [19] computed the function f explicitly, obtaining a beautiful and complicated answer involving Fibonacci numbers. Using their result, they confirmed that the reverse inequality in (1.4) holds. Thus the ECH capacities give a sharp embedding obstruction in this case.

Update 1.6. More recently, McDuff [17] has shown that the ECH obstruction to symplectically embedding one ellipsoid into another is sharp: $\text{int}(E(a, b))$ symplectically embeds into $E(c, d)$ if and only if $(a, b)_k \leq (c, d)_k$ for all k .

1.3.2. A polydisk into a ball. Next let us consider the problem of symplectically embedding a polydisk into a ball. Given $a > 0$, define $g(a)$ to be the infimum over $c \in \mathbb{R}$ such that the polydisk $P(a, 1)$ symplectically embeds into the ball $B(c)$. By Theorems 1.1 and 1.4 and Corollary 1.3, we have

$$(1.5) \quad g(a) \geq \sup_{d=1,2,\dots} \min \left\{ \frac{am+n}{d} \mid (m,n) \in \mathbb{N}^2, \right. \\ \left. (m+1)(n+1) \geq \frac{(d+1)(d+2)}{2} \right\}.$$

Simple calculations in §7.2 then deduce:

Proposition 1.7. *The obstruction to symplectically embedding a polydisk into a ball satisfies*

$$(1.6) \quad g(a) \geq \begin{cases} 2, & 1 \leq a \leq 2, \\ 1 + \frac{a}{2}, & 2 \leq a \leq 3, \\ \frac{3}{2} + \frac{a}{3}, & 3 \leq a \leq 4. \end{cases}$$

Note that when $a \neq 2$ this is better than the lower bound $g(a) \geq \sqrt{2a}$ obtained by considering volumes. For a slightly larger than 4, a more complicated calculation which we omit shows that the best bound that can be obtained from (1.5) is

$$g(a) \geq \frac{19}{12} + \frac{5a}{16},$$

which comes from taking $d = 48$ in (1.5). We do not know much about the right hand side of (1.5) for larger a , although we do know that it is always at least $\sqrt{2a}$, see §1.5 below. By analogy with [19] one might guess that $g(a) = \sqrt{2a}$ when a is sufficiently large.

Remark 1.8. We do not know to what extent the bound (1.5) is sharp. In general, the obstruction from Theorem 1.1 to embedding a polydisk into an ellipsoid is not always sharp. For example, Proposition 1.2 and Theorem 1.4 imply that $P(1, 1)$ and $E(1, 2)$ have the same ECH capacities, namely

$$0, 1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, \dots$$

Thus the ECH capacities give no obstruction to symplectically embedding $P(1, 1)$ into $E(a, 2a)$ for any $a > 1$, and in particular tell us nothing more than volume comparison. However the Ekeland-Hofer capacities give an obstruction to symplectically embedding $P(1, 1)$ into $E(a, 2a)$ whenever $a < 3/2$. (The Ekeland-Hofer capacities of $P(1, 1)$ are $1, 2, 3, \dots$, while those of $E(a, 2a)$ are $a, 2a, 2a, 3a, 4a, 4a, \dots$, see [3].) Note that $P(1, 1)$ does symplectically embed into $E(a, 2a)$ whenever $a \geq 3/2$. Indeed, with the conventions of (1.1) and (1.2), $P(1, 1)$ is a subset of $E(3/2, 3)$.

1.3.3. A disjoint union of balls into a ball. The ECH capacities give the following obstruction to symplectically embedding a disjoint union of balls into a ball:

Proposition 1.9. *Suppose there is a symplectic embedding of $\coprod_{i=1}^n B(a_i)$ into the interior of $B(1)$. Then*

$$(1.7) \quad \sum_{i=1}^n d_i a_i < d$$

whenever (d_1, \dots, d_n, d) are nonnegative integers (not all zero) satisfying

$$\sum_{i=1}^n (d_i^2 + d_i) \leq d^2 + 3d.$$

Proof. Let $k_i := (d_i^2 + d_i)/2$ for $i = 1, \dots, n$, let $k := \sum_{i=1}^n k_i$, and let $k' := (d^2 + 3d)/2$. By Corollary 1.3 we have $c_{k_i}(B(a_i)) = d_i a_i$ and $c_{k'}(B(1)) = d$. Then

$$\sum_{i=1}^n d_i a_i = \sum_{i=1}^n c_{k_i}(B(a_i)) \leq c_k \left(\prod_{i=1}^n B(a_i) \right) < c_k(B(1)) \leq c_{k'}(B(1)) = d.$$

Here the first inequality holds by Proposition 1.5, the second inequality by Theorem 1.1, and the third inequality by our assumption that $k \leq k'$.

q.e.d.

Remark 1.10. Proposition 1.9 is not new and, as explained to me by Dusa McDuff, can also be deduced by applying Taubes’s “Seiberg-Witten = Gromov” theorem [20] to a symplectic blowup of $\mathbb{C}P^2$. The interesting point is that Proposition 1.9, and thus ECH, gives a sharp obstruction. Indeed, it follows from work of Biran [1, Thm. 3.2] that there exists a symplectic embedding of $\coprod_{i=1}^n B(a_i)$ into $B(1 + \varepsilon)$ for all $\varepsilon > 0$ if:

- (i) $\sum_{i=1}^n a_i^2 \leq 1$, i.e. the volume of $\coprod_i B(a_i)$ is less than or equal to that of $B(1)$, and
- (ii) the inequality $\sum_{i=1}^n d_i a_i \leq d$ holds for all tuples of nonnegative integers (d_1, \dots, d_n, d) satisfying $\sum_{i=1}^n d_i = 3d - 1$ and $\sum_{i=1}^n d_i^2 = d^2 + 1$.

(As explained in [19, §1.2], results of [15, 18] imply that one can replace the inequalities (ii) above by a certain subset thereof.) But Proposition 1.9 implies that conditions (i) and (ii) are also necessary for the existence of a symplectic embedding. Note here that by Proposition 8.4 below, the inequalities (1.7) imply the volume constraint (i).

1.4. More examples of ECH capacities. We can also compute the ECH capacities of certain subsets of the cotangent bundle of $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, such as the unit disk bundle, using results from [8]. Let $\|\cdot\|$ be a norm on \mathbb{R}^2 , regarded as a translation-invariant norm on TT^2 . Let

$\|\cdot\|^*$ denote the dual norm on $(\mathbb{R}^2)^*$, which we regard as a translation-invariant norm on T^*T^2 . That is, if $\zeta \in T_q^*T^2$, then

$$\|\zeta\|^* = \max \{ \langle \zeta, v \rangle \mid v \in T_qT^2, \|v\| \leq 1 \}.$$

Define

$$T_{\|\cdot\|^*} := \{ \zeta \in T^*T^2 \mid \|\zeta\|^* \leq 1 \},$$

with symplectic form obtained by restricting the standard symplectic form $\omega = \sum_{i=1}^2 dp_i dq_i$ on T^*T^2 . Here q_1, q_2 denote the standard coordinates on T^2 , and p_1, p_2 denote the corresponding coordinates on the cotangent fibers.

If $\|\cdot\|$ is smooth, then the unit ball in the dual norm $\|\cdot\|^*$ on \mathbb{R}^2 is smooth, and $T_{\|\cdot\|^*}$ is a Liouville domain, because $\lambda = \sum_{i=1}^2 p_i dq_i$ restricts to a contact form on the boundary. For example, if $\|\cdot\|$ is the Euclidean norm, then $T_{\|\cdot\|^*}$ is the unit disk bundle in the cotangent bundle of T^2 with the standard flat metric.

Theorem 1.11. *If $\|\cdot\|$ is a norm on \mathbb{R}^2 , then*

$$(1.8) \quad c_k(T_{\|\cdot\|^*}) = \min \{ \ell_{\|\cdot\|}(\Lambda) \mid |P_\Lambda \cap \mathbb{Z}^2| = k + 1 \}.$$

Here the minimum is over convex polygons Λ in \mathbb{R}^2 with vertices in \mathbb{Z}^2 , and P_Λ denotes the closed region bounded by Λ . Also $\ell_{\|\cdot\|}(\Lambda)$ denotes the length of Λ in the norm $\|\cdot\|$.

It is an interesting problem to understand the ECH capacities of the unit disk bundle in the cotangent bundle of more general surfaces than flat T^2 .

1.5. Volume conjecture. In all of the examples considered above, it turns out that the asymptotic behavior of the symplectic embedding obstruction given by Theorem 1.1 as $k \rightarrow \infty$ simply recovers the necessary condition that the volume of (X_0, ω_0) be less than or equal to that of (X_1, ω_1) . Here the volume of a four-dimensional Liouville domain (X, ω) is defined by

$$\text{vol}(X, \omega) = \frac{1}{2} \int_X \omega \wedge \omega.$$

The conjectural more general phenomenon is that the asymptotics of the ECH capacities are related to volume as follows:

Conjecture 1.12. *Let (X, ω) be a four-dimensional Liouville domain such that $c_k(X, \omega) < \infty$ for all k . Then*

$$\lim_{k \rightarrow \infty} \frac{c_k(X, \omega)^2}{k} = 4 \text{vol}(X, \omega).$$

It is not hard to check this for an ellipsoid, cf. Remark 3.13. It is also easy to check this for a polydisk (even though the conjecture is not applicable here since a polydisk is not quite a Liouville domain).

In §8 we further confirm that this conjecture holds for the examples in Theorem 1.11, as well as for any disjoint union or subset of examples for which the conjecture holds. Note that the hypothesis that $c_k(X, \omega) < \infty$ for all k holds only if the first Chern class (not the ECH capacity) $c_1(X, \omega) \in H^2(X; \mathbb{Z})$ restricts to a torsion class in $H^2(\partial X; \mathbb{Z})$, see Remark 4.4.

Conjecture 1.12 is related to the question of whether the Weinstein conjecture in three dimensions [21] can be refined to show that a closed contact 3-manifold has a Reeb orbit with an explicit upper bound on the symplectic action, see Remark 8.5.

1.6. Contents of the paper. There are in fact two basic ways to define ECH capacities of a four-dimensional Liouville domain (X, ω) : in addition to the “distinguished” ECH capacities $c_k(X, \omega)$ discussed above, there is also a more rudimentary notion which we call the “full ECH capacities” and which we denote by $\tilde{c}_k(X, \omega)$. The full ECH capacities satisfy an analogue of Theorem 1.1, but only under the additional assumption that if φ denotes the symplectic embedding in question, then $X_1 \setminus \varphi(\text{int}(X_0))$ is diffeomorphic to a product $[0, 1] \times Y^3$. The numbers $c_k(X, \omega)$ are a certain carefully selected subset of the numbers $\tilde{c}_k(X, \omega)$ for which the more general statement of Theorem 1.1 is true.

Both the full and distinguished ECH capacities of a four-dimensional Liouville domain (X, ω) with boundary Y are defined in terms of the embedded contact homology of (Y, λ) , where λ is a contact form on Y with $d\lambda = \omega|_Y$. In §2 we recall the necessary material about embedded contact homology.

In §3 we associate to a closed contact 3-manifold (Y, λ) a sequence of numbers $\tilde{c}_k(Y, \lambda)$, which we call its “full ECH spectrum”; these numbers measure the amount of symplectic action needed to represent certain classes in the embedded contact homology of (Y, λ) . The full ECH capacities of a four-dimensional Liouville domain are then defined to be the full ECH spectrum of its boundary. Proposition 1.2 above regarding the ECH capacities of ellipsoids is equivalent to Proposition 3.12 which is proved in this section.

In §4 we give the crucial definition of the “distinguished ECH spectrum” of a closed contact 3-manifold (Y, λ) with nonvanishing ECH contact invariant (e.g. the boundary of a Liouville domain). The distinguished ECH capacities of a four-dimensional Liouville domain are then defined to be the distinguished ECH spectrum of its boundary. This section also gives the proof of Theorem 1.1; once the correct definitions are in place, this is a simple application of the machinery of ECH cobordism maps from [13]. Finally, this section explains how to extend the definition of (distinguished) ECH capacities and Theorem 1.1 to arbitrary subsets of symplectic four-manifolds.

In §5 we compute the (distinguished) ECH spectrum of a disjoint union of contact 3-manifolds, which implies Proposition 1.5 above on the ECH capacities of a disjoint union of Liouville domains. In §6 we prove Theorem 1.11 regarding the ECH capacities of certain subsets of T^*T^2 . In §7 we prove Theorem 1.4 on the ECH capacities of a polydisk. Proposition 1.7 above on the obstruction to symplectically embedding a polydisk into a ball is also proved in this section. Finally, in §8 we discuss the volume conjecture 1.12 and two generalizations, and present some evidence for them.

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2. ECH preliminaries

We now review the necessary background on embedded contact homology.

2.1. Definition of ECH. Let Y be a closed oriented 3-manifold. A *contact form* on Y is a 1-form λ on Y with $\lambda \wedge d\lambda > 0$ everywhere. This determines a *contact structure*, namely the oriented 2-plane field $\xi = \text{Ker}(\lambda)$. We call the pair (Y, λ) a “contact 3-manifold”, although it is perhaps more usual to refer to the pair (Y, ξ) this way.

The contact form λ determines the *Reeb vector field* R characterized by $d\lambda(R, \cdot) = 0$ and $\lambda(R) = 1$. A *Reeb orbit* is a closed orbit of the Reeb vector field R , i.e. a map $\gamma : \mathbb{R}/T\mathbb{Z} \rightarrow Y$ for some $T > 0$ with $\gamma'(t) = R(\gamma(t))$, modulo reparametrization. A Reeb orbit is *nondegenerate* if its linearized return map, regarded as an endomorphism of the 2-dimensional symplectic vector space $(\xi_{\gamma(0)}, d\lambda)$, does not have 1 as an eigenvalue. A nondegenerate Reeb orbit is called *hyperbolic* if its linearized return map has real eigenvalues; otherwise it is called *elliptic*. We say that the contact form λ is nondegenerate if all Reeb orbits are nondegenerate.

If Y is a closed oriented 3-manifold with a nondegenerate contact form λ , and if $\Gamma \in H_1(Y)$, then the *embedded contact homology* with $\mathbb{Z}/2$ -coefficients, which we denote by $ECH(Y, \lambda, \Gamma)$, is defined. (ECH can also be defined over \mathbb{Z} , see [10, §9], but $\mathbb{Z}/2$ coefficients are sufficient for the applications in this paper.) This is the homology of a chain complex which is generated over $\mathbb{Z}/2$ by finite sets of pairs $\alpha = \{(\alpha_i, m_i)\}$ where the α_i 's are distinct embedded Reeb orbits, the m_i 's are positive integers, $m_i = 1$ whenever α_i is hyperbolic, and

$$\sum_i m_i [\alpha_i] = \Gamma \in H_1(Y).$$

We call such an α an *ECH generator*. We often use the multiplicative notation $\alpha = \prod_i \alpha_i^{m_i}$, even though the grading and differential on the chain complex do not behave simply with respect to this sort of multiplication.

To define the chain complex differential ∂ one chooses a generic almost complex structure J on $\mathbb{R} \times Y$ which is “admissible”, meaning that J is \mathbb{R} -invariant, $J(\partial_s) = R$ where s denotes the \mathbb{R} coordinate, and J sends ξ to itself, rotating positively with respect to the orientation $d\lambda$ on ξ . The coefficient $\langle \partial\alpha, \beta \rangle$ of the differential is then a count of J -holomorphic curves in $\mathbb{R} \times Y$ which have ECH index 1 and which as currents are asymptotic to $\mathbb{R} \times \alpha$ as $s \rightarrow \infty$ and asymptotic to $\mathbb{R} \times \beta$ as $s \rightarrow -\infty$. The detailed definition of the differential is given for example in [9, §7], using the ECH index defined in [5, 6]. We denote this chain complex by $ECC(Y, \lambda, \Gamma, J)$, and its homology by $ECH(Y, \lambda, \Gamma)$.

The $\mathbb{Z}/2$ -module $ECH(Y, \lambda, \Gamma)$ has a relative \mathbb{Z}/d -grading, where d denotes the divisibility of $c_1(\xi) + 2\text{PD}(\Gamma)$ in $H^2(Y; \mathbb{Z})/\text{Torsion}$. The detailed definition of the grading will not be needed here and can be found in [5, 6].

Although the differential on the chain complex $ECC(Y, \lambda, \Gamma, J)$ depends on J , the homology $ECH(Y, \lambda, \Gamma)$ does not. This follows from a much stronger theorem of Taubes [22, 23, 24, 25] asserting that there is a canonical isomorphism between embedded contact homology and a version of Seiberg-Witten Floer cohomology as defined by Kronheimer-Mrowka [14]. Namely, if Y is connected then there is a canonical isomorphism of relatively graded $\mathbb{Z}/2$ -modules

$$(2.1) \quad ECH_*(Y, \lambda, \Gamma) \xrightarrow{\simeq} \widehat{HM}^{-*}(Y, \mathfrak{s}_\xi + \text{PD}(\Gamma)),$$

where the right hand side denotes Seiberg-Witten Floer cohomology with $\mathbb{Z}/2$ -coefficients, and \mathfrak{s}_ξ is a spin-c structure determined by the contact structure. (This is also true with \mathbb{Z} coefficients.) As shown in [13], it follows from Taubes’s proof of (2.1) and the invariance properties of \widehat{HM} that the versions of $ECH(Y, \lambda, \Gamma)$ defined using different almost complex structures J are canonically isomorphic to each other.

In this paper we are almost exclusively concerned with the case $\Gamma = 0$.

2.2. Some additional structure on ECH. There is a canonical element

$$c(\xi) := [\emptyset] \in ECH(Y, \lambda, 0),$$

called the *ECH contact invariant*, represented by the ECH generator consisting of the empty set of Reeb orbits. This is a cycle in the ECH chain complex because any holomorphic curve counted by the differential must have at least one positive end, c.f. §2.3 below. The homology class $[\emptyset]$ depends only on the contact structure ξ (although not just on Y), and agrees with an analogous contact invariant in Seiberg-Witten Floer cohomology [26].

If Y is connected, then there is a degree -2 map

$$(2.2) \quad U : ECH(Y, \lambda, \Gamma) \longrightarrow ECH(Y, \lambda, \Gamma).$$

This is induced by a chain map which is defined similarly to the differential, but instead of counting holomorphic curves in $\mathbb{R} \times Y$ with ECH index one modulo translation, it counts holomorphic curves in $\mathbb{R} \times Y$ with ECH index two that pass through a chosen generic point $z \in \mathbb{R} \times Y$, see [11, §2.5]. Under the isomorphism (2.1), the U map (2.2) agrees with an analogous map on Seiberg-Witten Floer cohomology [26].

If (Y, λ) has connected components (Y_i, λ_i) for $i = 1, \dots, n$, then there are n different U maps U_1, \dots, U_n , where U_i is defined by taking $z \in \mathbb{R} \times Y_i$. The different maps U_i commute. Note also that in this case one has a canonical isomorphism of chain complexes

$$(2.3) \quad ECC(Y_1, \lambda_1, \Gamma_1, J_1) \otimes \cdots \otimes ECC(Y_n, \lambda_n, \Gamma_n, J_n) \xrightarrow{\cong} ECC(Y, \lambda, \Gamma, J),$$

which sends a tensor product of ECH generators on the left hand side to their union on the right, where $\Gamma = \sum_{i=1}^n \Gamma_i$ and J restricts to J_i on $\mathbb{R} \times Y_i$. Since we are working with field coefficients, this gives a canonical isomorphism on homology

$$(2.4) \quad ECH(Y, \lambda, \Gamma) = ECH(Y_1, \lambda_1, \Gamma_1) \otimes \cdots \otimes ECH(Y_n, \lambda_n, \Gamma_n).$$

Under this identification, U_i is the tensor product of the U map for (Y_i, λ_i) with the identity maps on the other factors.

2.3. Filtered ECH. If $\alpha = \{(\alpha_i, m_i)\}$ is a generator of the ECH chain complex, its *symplectic action* is defined by

$$\mathcal{A}(\alpha) := \sum_i m_i \int_{\alpha_i} \lambda.$$

The ECH differential (for any generic admissible J) decreases the action, i.e. if $\langle \partial\alpha, \beta \rangle \neq 0$ then $\mathcal{A}(\alpha) \geq \mathcal{A}(\beta)$. This is because if C is a J -holomorphic curve counted by $\langle \partial\alpha, \beta \rangle$, then $d\lambda|_C \geq 0$ everywhere. (In fact if $\langle \partial\alpha, \beta \rangle \neq 0$ then the strict inequality $\mathcal{A}(\alpha) > \mathcal{A}(\beta)$ holds, because $d\lambda$ vanishes identically on C if and only if the image of C is \mathbb{R} -invariant, in which case C has ECH index zero and so does not contribute to the differential.) Thus for any real number L , it makes sense to define the *filtered ECH*

$$ECH^L(Y, \lambda, \Gamma)$$

to be the homology of the subcomplex $ECC^L(Y, \lambda, \Gamma, J)$ of the ECH chain complex spanned by generators with action (strictly) less than L . It is shown in [13] that $ECH^L(Y, \lambda, \Gamma)$ does not depend on the choice of generic admissible J (although unlike the usual ECH it can change

when one deforms the contact form λ). For $L < L'$ the inclusion of chain complexes (for a given J) induces a map

$$\iota_* : ECH^L(Y, \lambda, \Gamma) \longrightarrow ECH^{L'}(Y, \lambda, \Gamma).$$

It is shown in [13] that this map does not depend on the choice of J . The usual ECH is recovered as the direct limit

$$ECH(Y, \lambda, \Gamma) = \lim_{\rightarrow} ECH^L(Y, \lambda, \Gamma).$$

Also, if c is a positive constant, then there is a canonical “scaling” isomorphism

$$(2.5) \quad s : ECH^L(Y, \lambda, \Gamma) \xrightarrow{\simeq} ECH^{cL}(Y, c\lambda, \Gamma).$$

The reason is that an admissible almost complex structure J for λ determines an admissible almost complex structure for $c\lambda$, such that the obvious identification of Reeb orbits gives an isomorphism of chain complexes. Again, it is shown in [13] that the resulting map (2.5) does not depend on the choice of J .

2.4. Weakly exact symplectic cobordisms. Let (Y_+, λ_+) and (Y_-, λ_-) be closed contact 3-manifolds.

Definition 2.1. An *exact symplectic cobordism* from (Y_+, λ_+) to (Y_-, λ_-) is a compact symplectic 4-manifold (X, ω) with $\partial X = Y_+ - Y_-$, such that there exists a 1-form λ on X with $d\lambda = \omega$ and $\lambda|_{Y_{\pm}} = \lambda_{\pm}$.

It is shown in [13] that if the contact forms λ_{\pm} are nondegenerate, then an exact symplectic cobordism as above induces maps of ungraded $\mathbb{Z}/2$ -modules

$$(2.6) \quad \bigoplus_{\Gamma_+ \in H_1(Y_+)} ECH^L(Y_+, \lambda_+, \Gamma_+) \longrightarrow \bigoplus_{\Gamma_- \in H_1(Y_-)} ECH^L(Y_-, \lambda_-, \Gamma_-)$$

satisfying various axioms. The idea of the construction is as follows. Consider the “symplectization completion” of X defined by

$$(2.7) \quad \overline{X} := ((-\infty, 0] \times Y_-) \cup_{Y_-} X \cup_{Y_+} ([0, \infty) \times Y_+).$$

As reviewed after Definition 2.2 below, the symplectic form ω on X extends over \overline{X} as $d(e^s \lambda_-)$ on $(-\infty, 0] \times Y_-$, where s denotes the $(-\infty, 0]$ coordinate, and as $d(e^s \lambda_+)$ on $[0, \infty) \times Y_+$. A suitable almost complex structure J on \overline{X} determines, via ω , a metric on \overline{X} . One then modifies ω and the metric on the ends to obtain a 2-form $\hat{\omega}$ and a metric which are \mathbb{R} -invariant on the ends. The map (2.6) is now induced by a chain map which is defined by counting solutions to the Seiberg-Witten equations on \overline{X} perturbed using a large multiple of the 2-form $\hat{\omega}$. In the limit as the perturbation gets large, the relevant Seiberg-Witten solutions give rise to (possibly broken) J -holomorphic curves in \overline{X} . The restriction of ω to any such J -holomorphic curve is pointwise nonnegative. The key fact needed to get a well-defined map on filtered ECH is then that if

α_{\pm} are smooth 1-chains in Y_{\pm} , and if Z is a smooth 2-chain in X with $\partial Z = \alpha_+ - \alpha_-$, then

$$(2.8) \quad \int_Z \omega = \int_{\alpha_+} \lambda_+ - \int_{\alpha_-} \lambda_-.$$

Of course this holds by the exactness assumption and Stokes's theorem.

We now show that the $\Gamma_{\pm} = 0$ component of the map (2.6) can still be defined under a slightly weaker assumption, in which we take d of the last equation in Definition 2.1:

Definition 2.2. A *weakly exact symplectic cobordism* from (Y_+, λ_+) to (Y_-, λ_-) is a compact symplectic 4-manifold (X, ω) with $\partial X = Y_+ - Y_-$, such that ω is exact and $\omega|_{Y_{\pm}} = d\lambda_{\pm}$.

For example, a four-dimensional Liouville domain as we have defined it is a weakly exact symplectic cobordism from a contact 3-manifold to the empty set. Note that for any weakly exact symplectic cobordism X as above, by a standard lemma there is an identification of a neighborhood of Y_+ in X with $(-\varepsilon, 0] \times Y_+$ such that on this neighborhood we have $\omega = d(e^s \lambda_+)$, where s denotes the $(-\varepsilon, 0]$ coordinate. Likewise a neighborhood of Y_- in X can be identified with $[0, \varepsilon) \times Y_-$ so that on this neighborhood $\omega = d(e^s \lambda_-)$. Thus one can still define the symplectization completion \overline{X} as in (2.7).

Theorem 2.3. *Let (X, ω) be a weakly exact symplectic cobordism from (Y_+, λ_+) to (Y_-, λ_-) , where Y_+ and Y_- are closed and the contact forms λ_{\pm} are nondegenerate. Then there exist maps*

$$(2.9) \quad \Phi^L(X, \omega) : ECH^L(Y_+, \lambda_+, 0) \longrightarrow ECH^L(Y_-, \lambda_-, 0)$$

of ungraded $\mathbb{Z}/2$ -modules, for each $L \in \mathbb{R}$, with the following properties:

(a) *If $L < L'$ then the following diagram commutes:*

$$\begin{array}{ccc} ECH^L(Y_+, \lambda_+, 0) & \xrightarrow{\Phi^L(X, \omega)} & ECH^L(Y_-, \lambda_-, 0) \\ \iota_* \downarrow & & \downarrow \iota_* \\ ECH^{L'}(Y_+, \lambda_+, 0) & \xrightarrow{\Phi^{L'}(X, \omega)} & ECH^{L'}(Y_-, \lambda_-, 0). \end{array}$$

In particular, it makes sense to define the direct limit

$$\Phi(X, \omega) := \lim_{\rightarrow} \Phi^L(X, \omega) : ECH(Y_+, \lambda_+, 0) \longrightarrow ECH(Y_-, \lambda_-, 0).$$

(b) $\Phi(X, \omega)[\emptyset] = [\emptyset]$.

(c) *If X is diffeomorphic to a product $[0, 1] \times Y$, then $\Phi(X, \omega)$ is an isomorphism.*

(d) *The diagram*

$$\begin{array}{ccc}
 ECH(Y_+, \lambda_+, 0) & \xrightarrow{\Phi(X, \omega)} & ECH(Y_-, \lambda_-, 0) \\
 \downarrow U_+ & & \downarrow U_- \\
 ECH(Y_+, \lambda_+, 0) & \xrightarrow{\Phi(X, \omega)} & ECH(Y_-, \lambda_-, 0)
 \end{array}$$

commutes, where U_{\pm} is the U map for any of the connected components of Y_{\pm} , as long as U_+ and U_- correspond to the same component of X .

Proof. Suppose first that Y_+ and Y_- are connected and that (X, ω) is exact as in Definition 2.1. In this case we define $\Phi^L(X, \omega)$ from the map (2.6) by restricting to the $\Gamma_+ = 0$ component and projecting to the $\Gamma_- = 0$ component. It follows from the main theorem in [13] that $\Phi^L(X, \omega)$ satisfies properties (a) and (b), and $\Phi(X, \omega)$ agrees with the $\Gamma_{\pm} = 0$ component of the induced map on Seiberg-Witten Floer cohomology via the isomorphisms (2.1) on both sides. Items (c) and (d) then follow from analogous results in Seiberg-Witten Floer theory [14].

If (X, ω) is only weakly exact, then one can no longer define a map (2.6), but one can still define a map on $\Gamma_{\pm} = 0$ components as in (2.9), again by perturbing the Seiberg-Witten equations on the symplectization completion \overline{X} using a large multiple of $\hat{\omega}$. One just needs to check that (2.8) holds when α_{\pm} is nullhomologous in Y_{\pm} . To do so, let λ be a 1-form on X with $d\lambda = \omega$. Then by Stokes’s theorem we have $\int_Z \omega = \int_{\alpha_+} \lambda - \int_{\alpha_-} \lambda$. On the other hand, since $\lambda|_{Y_{\pm}} - \lambda_{\pm}$ is a closed 1-form on Y_{\pm} and α_{\pm} is nullhomologous in Y_{\pm} , by Stokes’s theorem again we have $\int_{\alpha_{\pm}} (\lambda - \lambda_{\pm}) = 0$. Properties (a)–(d) hold as before.

When Y_+ and Y_- are not required to be connected, one can still construct the maps $\Phi^L(X, \omega)$ and prove properties (a) and (b) by deforming the Seiberg-Witten equations on \overline{X} using a large multiple of $\hat{\omega}$ as above (and we already know property (c) in this case). One can then prove property (d) by using the interpretation of the Seiberg-Witten U map in [26] (which counts index 2 Seiberg-Witten solutions in $\mathbb{R} \times Y$ satisfying a codimension 2 constraint at a chosen point) to construct a chain homotopy between the chain maps defining $U_+ \circ \Phi(X, \omega)$ and $\Phi(X, \omega) \circ U_-$ (by counting index 1 Seiberg-Witten solutions in the completed cobordism satisfying a codimension 2 constraint at any point along a suitable path). q.e.d.

3. Full ECH spectrum and capacities

We now introduce the full ECH spectrum and capacities, as a warmup for the distinguished ECH spectrum and capacities to be defined in §4.

3.1. The full ECH spectrum. Let Y be a closed oriented 3-manifold with a nondegenerate contact form λ .

Definition 3.1. For each positive integer k , define $\tilde{c}_k(Y, \lambda)$ to be the infimum over all $L \in \mathbb{R}$ such that the image of $ECH^L(Y, \lambda, 0)$ in $ECH(Y, \lambda, 0)$ has dimension at least k . The sequence $\{\tilde{c}_k(Y, \lambda)\}_{k=1,2,\dots}$ is called the *full ECH spectrum* of (Y, λ) .

Remark 3.2. (a) It follows from the definition that

$$0 \leq \tilde{c}_1(Y, \lambda) \leq \tilde{c}_2(Y, \lambda) \leq \dots \leq \infty.$$

Note that if Y is connected, then $\tilde{c}_k(Y, \lambda) < \infty$ for all k if and only if $c_1(\xi) \in H^2(Y; \mathbb{Z})$ is torsion. This is because Taubes's isomorphism (2.1), together with results of Kronheimer-Mrowka [14], imply that if Y is connected, then $ECH(Y, \lambda, \Gamma)$ is infinitely generated if and only if $c_1(\xi) + 2\text{PD}(\Gamma) \in H^2(Y; \mathbb{Z})$ is torsion.

(b) It follows immediately from the definition that

$$\tilde{c}_1(Y, \lambda) > 0 \iff c(\xi) = 0 \in ECH(Y, \lambda, 0).$$

(c) If c is a positive constant, then \tilde{c}_k satisfies the scaling property

$$(3.1) \quad \tilde{c}_k(Y, c\lambda) = c \cdot \tilde{c}_k(Y, \lambda).$$

This follows from the commutative diagram

$$\begin{array}{ccc} ECH^L(Y, \lambda, 0) & \longrightarrow & ECH(Y, \lambda, 0) \\ s \downarrow \simeq & & s \downarrow \simeq \\ ECH^{cL}(Y, c\lambda, 0) & \longrightarrow & ECH(Y, c\lambda, 0), \end{array}$$

where s is the scaling isomorphism (2.5). And commutativity of the above diagram is immediate from the definitions.

(d) One can also define analogues of the full ECH spectrum using $ECH(Y, \lambda, \Gamma)$ for $\Gamma \neq 0$. However restricting to Γ torsion is necessary to obtain well-defined capacities, see Lemma 3.9 below.

Lemma 3.3. *Let (X, ω) be a weakly exact symplectic cobordism from (Y_+, λ_+) to (Y_-, λ_-) . Assume that the contact forms λ_{\pm} are nondegenerate and that X is diffeomorphic to a product $[0, 1] \times Y$. Then for every positive integer k we have*

$$\tilde{c}_k(Y_-, \lambda_-) \leq \tilde{c}_k(Y_+, \lambda_+).$$

Proof. Fix $L \in \mathbb{R}$ and let I_{\pm} denote the image of $ECH^L(Y_{\pm}, \lambda_{\pm}, 0)$ in $ECH(Y_{\pm}, \lambda_{\pm}, 0)$. We need to show that $\dim(I_-) \geq \dim(I_+)$. By Theorem 2.3(a) we have a commutative diagram

$$(3.2) \quad \begin{array}{ccc} ECH^L(Y_+, \lambda_+, 0) & \longrightarrow & ECH(Y_+, \lambda_+, 0) \\ \downarrow \Phi^L(X, \omega) & & \downarrow \Phi(X, \omega) \\ ECH^L(Y_-, \lambda_-, 0) & \longrightarrow & ECH(Y_-, \lambda_-, 0). \end{array}$$

It follows from this diagram that $\Phi(X, \omega)(I_+) \subset I_-$. By Theorem 2.3(c) the map $\Phi(X, \omega)$ is an isomorphism, so $\dim(I_+) \leq \dim(I_-)$ as desired. q.e.d.

We now extend the definition of the full ECH spectrum to arbitrary (possibly degenerate) contact forms λ on Y .

Definition 3.4. Let (Y, λ) be any closed contact 3-manifold. Define

$$(3.3) \quad \tilde{c}_k(Y, \lambda) := \sup\{\tilde{c}_k(Y, f_- \lambda)\} = \inf\{\tilde{c}_k(Y, f_+ \lambda)\},$$

where the supremum is over smooth functions $f_- : Y \rightarrow (0, 1]$ such that the contact form $f_- \lambda$ is nondegenerate, and the infimum is over smooth functions $f_+ : Y \rightarrow [1, \infty)$ such that $f_+ \lambda$ is nondegenerate.

To confirm that this definition makes sense, we have:

Lemma 3.5. *The supremum and infimum in (3.3) are equal.*

Proof. We first show that $\sup\{\tilde{c}_k(Y, f_- \lambda)\} \leq \inf\{\tilde{c}_k(Y, f_+ \lambda)\}$. If f_-, f_+ are as in Definition 3.4, then

$$([0, 1] \times Y, d(((1 - s)f_- + sf_+) \lambda))$$

is an exact symplectic cobordism from $(Y, f_+ \lambda)$ to $(Y, f_- \lambda)$, where s denotes the $[0, 1]$ coordinate. Thus by Lemma 3.3 we have $\tilde{c}_k(Y, f_- \lambda) \leq \tilde{c}_k(Y, f_+ \lambda)$.

We now show that $\sup\{\tilde{c}_k(Y, f_- \lambda)\} \geq \inf\{\tilde{c}_k(Y, f_+ \lambda)\}$. Fix $\varepsilon > 0$. We can find a function $\phi : Y \rightarrow (0, \varepsilon)$ such that if $f_+ = e^\phi$, then the contact form $f_+ \lambda$ is nondegenerate. Define $f_- = e^{-\varepsilon} f_+$. Then by the scaling property (3.1) we have

$$\tilde{c}_k(Y, f_+ \lambda) = e^\varepsilon \tilde{c}_k(Y, f_- \lambda).$$

Thus $\inf\{\tilde{c}_k(Y, f_+ \lambda)\} \leq e^\varepsilon \sup\{\tilde{c}_k(Y, f_- \lambda)\}$. Now take $\varepsilon \rightarrow 0$. q.e.d.

Lemma 3.3 then extends to the possibly degenerate case:

Proposition 3.6. *Let (X, ω) be a weakly exact symplectic cobordism from (Y_+, λ_+) to (Y_-, λ_-) . Assume that X is diffeomorphic to a product $[0, 1] \times Y$. Then for every positive integer k we have*

$$\tilde{c}_k(Y_-, \lambda_-) \leq \tilde{c}_k(Y_+, \lambda_+).$$

Proof. If f_+ and f_- are functions as in Definition 3.4, then

$$(\{(s, y) \in \mathbb{R} \times Y_+ \mid 1 \leq e^s \leq f_+(y)\}, d(e^s \lambda_+))$$

is an exact symplectic cobordism from $(Y_+, f_+ \lambda_+)$ to (Y_+, λ_+) , and

$$(\{(s, y) \in \mathbb{R} \times Y_- \mid f_-(y) \leq e^s \leq 1\}, d(e^s \lambda_-))$$

is an exact symplectic cobordism from (Y_-, λ_-) to $(Y_-, f_- \lambda_-)$. Attaching these cobordisms to the positive and negative boundaries of X defines a subset of the symplectization completion (2.7) which is a

weakly exact symplectic cobordism, diffeomorphic to a product, from $(Y_+, f_+\lambda_+)$ to $(Y_-, f_-\lambda_-)$. By Lemma 3.3 we have

$$\tilde{c}_k(Y_-, f_-\lambda_-) \leq \tilde{c}_k(Y_+, f_+\lambda_+).$$

Taking the supremum over f_- on the left hand side and the infimum over f_+ on the right hand side completes the proof. q.e.d.

3.2. Full ECH capacities.

Definition 3.7. Let (X, ω) be a 4-dimensional Liouville domain with boundary Y . If k is a positive integer, define

$$\tilde{c}_k(X, \omega) := \tilde{c}_k(Y, \lambda),$$

where λ is a contact form on Y with $d\lambda = \omega|_Y$. We call the numbers $\{\tilde{c}_k(X, \omega)\}_{k=1,2,\dots}$ the *full ECH capacities* of (X, ω) .

Lemma 3.8. $\tilde{c}_k(X, \omega)$ does not depend on the choice of contact form λ .

Proof. Let λ' be another contact form on Y with $d\lambda' = \omega|_Y$. We need to show that

$$(3.4) \quad \tilde{c}_k(Y, \lambda) = \tilde{c}_k(Y, \lambda').$$

By modifying X slightly as in the proof of Proposition 3.6, we may assume that λ and λ' are nondegenerate. Equation (3.4) then follows immediately from Definition 3.1 and Lemma 3.9 below. q.e.d.

Lemma 3.9. Let Y be a closed oriented 3-manifold. Let λ, λ' be nondegenerate contact forms on Y with $d\lambda = d\lambda'$. Then there is an isomorphism $ECH(Y, \lambda, 0) \simeq ECH(Y, \lambda', 0)$, which is the direct limit of isomorphisms

$$ECH^L(Y, \lambda, 0) \simeq ECH^L(Y, \lambda', 0),$$

and which respects the U maps.

(The part about U maps is not needed here, but will be used in §4.1.)

Proof. Let R and R' denote the Reeb vector fields for λ and λ' respectively. Since $d\lambda = d\lambda'$, we have $R' = fR$ for some positive function $f : Y \rightarrow \mathbb{R}$. In particular there is a canonical bijection between the ECH generators of λ and those of λ' .

Now define a diffeomorphism

$$\begin{aligned} \phi : \mathbb{R} \times Y &\longrightarrow \mathbb{R} \times Y, \\ (s, y) &\longmapsto (f(y)s, y). \end{aligned}$$

If J is an almost complex structure on $\mathbb{R} \times Y$ as needed to define the ECH of λ , then $J' = \phi_*^{-1} \circ J \circ \phi_*$ is an almost complex structure as needed to

define the ECH of λ' . The canonical bijection on ECH generators then gives an isomorphism of chain complexes

$$(3.5) \quad ECC(Y, \lambda, \Gamma, J) \simeq ECC(Y, \lambda', \Gamma, J'),$$

because ϕ by definition induces a bijection on the relevant holomorphic curves. For the same reason, this isomorphism respects the U maps.

When $\Gamma = 0$, the isomorphism (3.5) further respects the symplectic action filtrations, because if $\alpha = \{(\alpha_i, m_i)\}$ is an ECH generator with $[\alpha] = 0$, then since $\lambda - \lambda'$ is a closed 1-form on Y , by Stokes's theorem we have $\sum_i m_i \int_{\alpha_i} \lambda = \sum_i m_i \int_{\alpha_i} \lambda'$. q.e.d.

Remark 3.10. We always have $\tilde{c}_1(X, \omega) = 0$, by Remark 3.2(b), because the ECH contact invariant $[\emptyset] \in ECH(Y, \lambda, 0)$ is nonzero by Theorem 2.3(b).

We can now prove a symplectic embedding obstruction, which is a warmup to Theorem 1.1:

Proposition 3.11. *Let (X_0, ω_0) and (X_1, ω_1) be four-dimensional Liouville domains. Suppose there is a symplectic embedding $\varphi : (X_0, \omega_0) \rightarrow (\text{int}(X_1), \omega_1)$ such that $X_1 \setminus \text{int}(\varphi(X_0))$ is diffeomorphic to a product $[0, 1] \times Y$. Then $\tilde{c}_k(X_0, \omega_0) \leq \tilde{c}_k(X_1, \omega_1)$ for all positive integers k .*

Proof. For $i = 0, 1$, write $Y_i = \partial X_i$, and let λ_i be a contact form on Y_i with $d\lambda_i = \omega_i|_{Y_i}$. Then $(X_1 \setminus \text{int}(\varphi(X_0)), \omega_1)$ is a weakly exact symplectic cobordism from (Y_1, λ_1) to (Y_0, λ_0) . Now apply Proposition 3.6. q.e.d.

3.3. The full ECH capacities of an ellipsoid. Recall the notation from Proposition 1.2.

Proposition 3.12. *The full ECH capacities of an ellipsoid are given by*

$$\tilde{c}_k(E(a, b)) = (a, b)_k.$$

Proof. For the contact form on $\partial E(a, b)$ obtained by restricting (1.3), the Reeb vector field is given by

$$R = 2\pi \left(a^{-1} \frac{\partial}{\partial \theta_1} + b^{-1} \frac{\partial}{\partial \theta_2} \right),$$

where $\partial/\partial \theta_j := x_j \partial/\partial y_j - y_j \partial/\partial x_j$.

Suppose that the ratio a/b is irrational. In this case there are just two embedded Reeb orbits $\gamma_1 = (z_2 = 0)$ and $\gamma_2 = (z_1 = 0)$. These are elliptic and nondegenerate and have action a and b respectively. In particular $\lambda|_{\partial E(a,b)}$ is nondegenerate, and the ECH generators have the form $\gamma_1^m \gamma_2^n$ where $m, n \in \mathbb{N}$. Of course these all correspond to $\Gamma = 0$ since $H_1(\partial E(a, b)) = 0$. The action of such a generator is given by

$$\mathcal{A}(\gamma_1^m \gamma_2^n) = am + bn.$$

Since all Reeb orbits are elliptic, all ECH generators have even grading (see [5, Prop. 1.6(c)]), so the differential on the ECH chain complex vanishes for any J . (The full calculation of the grading on the ECH chain complex in this example is given in [11, Ex. 4.2], but we do not need this here.) Thus the dimension of the image of $ECH^L(\partial E(a, b), \lambda, 0)$ in $ECH(\partial E(a, b), \lambda, 0)$ is

$$|\{(m, n) \in \mathbb{N}^2 \mid ma + nb < L\}|.$$

The proposition in this case follows immediately.

To prove the proposition when a/b is rational, choose real numbers $a_- < a < a_+$ and $b_- < b < b_+$ with a_-/b_- and a_+/b_+ irrational. By Proposition 3.11 we have

$$(a_-, b_-)_k = \tilde{c}_k(E(a_-, b_-)) \leq \tilde{c}_k(E(a, b)) \leq \tilde{c}_k(E(a_+, b_+)) = (a_+, b_+)_k.$$

For any given k , taking a limit as $a_{\pm} \rightarrow a$ and $b_{\pm} \rightarrow b$ proves that $\tilde{c}_k(E(a, b)) = (a, b)_k$ as claimed. q.e.d.

If $E(a, b)$ symplectically embeds into the interior of $E(c, d)$, then Propositions 3.11 and 3.12 tell us that

$$(3.6) \quad (a, b)_k \leq (c, d)_k$$

for all k . To understand this condition in examples, the following alternate description of $(a, b)_k$ is useful. Given $(m, n) \in \mathbb{N}^2$, let $T_{a/b}(m, n)$ denote the triangle in \mathbb{R}^2 whose edges are the coordinate axes together with the line through (m, n) of slope $-a/b$. Then

$$(a, b)_k = am + bn$$

where

$$k = |T_{a/b}(m, n) \cap \mathbb{N}^2|.$$

For example, we have $(a, b)_1 = 0$, as we already knew from Remark 3.10. Next, we have

$$(a, b)_2 = \begin{cases} a, & a/b \leq 1, \\ b, & a/b \geq 1. \end{cases}$$

Thus the condition (3.6) for $k = 2$ recovers the well-known fact that if $E(a, b)$ symplectically embeds into $E(c, d)$ then $\min(a, b) \leq \min(c, d)$. Next, assuming $a \geq b$, we have

$$(3.7) \quad (a, b)_3 = \begin{cases} 2b, & 2 \leq a/b, \\ a, & 1 \leq a/b \leq 2. \end{cases}$$

Another example is

$$(3.8) \quad (a, b)_6 = \begin{cases} 5b, & 5 \leq a/b, \\ a, & 4 \leq a/b \leq 5, \\ 4b, & 3 \leq a/b \leq 4, \\ a + b, & 2 \leq a/b \leq 3, \\ 3b, & 3/2 \leq a/b \leq 2, \\ 2a, & 1 \leq a/b \leq 3/2. \end{cases}$$

For example, return to the function f defined in §1.3.1 that measures the obstruction to symplectically embedding an ellipsoid into a ball. It is computed in [16] that $f(2) = 2$ and $f(5) = 5/2$. On the other hand, equation (3.7) implies that $(2, 1)_3/(1, 1)_3 = 2$, and equation (3.8) implies that $(5, 1)_6/(1, 1)_6 = 5/2$. This is how one confirms that the bound (1.4) (which we have already justified) is sharp for $a = 2, 5$.

Remark 3.13. If we write $L = am + bn$, then the triangle $T_{a/b}(m, n)$ has area $L^2/2ab$, so when L is large,

$$|T_{a/b}(m, n) \cap \mathbb{N}^2| = \frac{L^2}{2ab} + O(L).$$

Note also that $E(a, b)$ has volume $ab/2$. It follows that

$$(3.9) \quad \lim_{k \rightarrow \infty} \frac{\tilde{c}_k(E(a, b))^2}{k} = 4 \operatorname{vol}(E(a, b)).$$

In particular, the condition (3.6) for k large simply tells us that the volume of $E(a, b)$ is less than or equal to that of $E(c, d)$. (But the equality in (3.9) only holds in the limit, so that for given (a, b) and (c, d) , taking suitable small values of k often gives stronger conditions.)

4. Distinguished ECH spectrum and capacities

We now define modified versions of the full ECH spectrum and full ECH capacities which give obstructions to symplectic embeddings for non-product cobordisms.

4.1. Definitions and basic properties.

Definition 4.1. If λ is a nondegenerate contact form on a closed oriented three-manifold Y , and if $0 \neq \sigma \in ECH(Y, \lambda, \Gamma)$, define $c_\sigma(Y, \lambda)$ to be the infimum over $L \in \mathbb{R}$ such that σ is contained in the image of the map $ECH^L(Y, \lambda, \Gamma) \rightarrow ECH(Y, \lambda, \Gamma)$. As in §3.1, if λ is degenerate, define

$$c_\sigma(Y, \lambda) := \sup\{c_\sigma(Y, f\lambda)\},$$

where the supremum is over functions $f : Y \rightarrow (0, 1]$ such that $f\lambda$ is nondegenerate. This definition makes sense because $ECH(Y, f\lambda, \Gamma)$ does not depend on f . (The cobordism maps (2.6) for product cobordisms define a canonical isomorphism $ECH(Y, f\lambda, \Gamma) = ECH(Y, f'\lambda, \Gamma)$ whenever $f\lambda$ and $f'\lambda$ are nondegenerate.)

It follows from Lemma 5.2 below that if $\sigma \in ECH(Y, \lambda, 0)$, then $c_\sigma(Y, \lambda)$ is one of the numbers in the full ECH spectrum $\{\tilde{c}_k(Y, \lambda)\}$.

Lemma 4.2. *Let (X, ω) be a weakly exact symplectic cobordism from (Y_+, λ_+) to (Y_-, λ_-) , where λ_\pm are nondegenerate. Let $\sigma \in ECH(Y_+, \lambda_+, 0)$. Then*

$$c_\sigma(Y_+, \lambda_+) \geq c_{\Phi(X, \omega)(\sigma)}(Y_-, \lambda_-).$$

Proof. Let $L \in \mathbb{R}$. Suppose σ is in the image of the map

$$ECH^L(Y_+, \lambda_+, 0) \rightarrow ECH(Y_+, \lambda_+, 0).$$

Then it follows from the diagram (3.2) that $\Phi(X, \omega)(\sigma)$ is in the image of the map

$$ECH^L(Y_-, \lambda_-, 0) \rightarrow ECH(Y_-, \lambda_-, 0).$$

q.e.d.

Definition 4.3. If (Y, λ) is a closed connected contact three-manifold with $c(\xi) \neq 0$, and if k is a nonnegative integer, define

$$(4.1) \quad c_k(Y, \lambda) := \min \left\{ c_\sigma(Y, \lambda) \mid \sigma \in ECH(Y, \lambda, 0), U^k \sigma = [\emptyset] \right\}.$$

More generally, if (Y, λ) is a closed contact three-manifold with connected components Y_1, \dots, Y_n , and if $c(\xi) \neq 0$, define

$$c_k(Y, \lambda) := \min \left\{ c_\sigma(Y, \lambda) \mid \sigma \in ECH(Y, \lambda, 0), U_{i_1} \cdots U_{i_k} \sigma = [\emptyset] \quad \forall i_1, \dots, i_k \in \{1, \dots, n\} \right\}.$$

The sequence $\{c_k(Y, \lambda)\}_{k=0,1,\dots}$ is called the (*distinguished*) *ECH spectrum* of (Y, λ) .

Remark 4.4. (a) Any choice of chain map used to define the U map decreases the symplectic action, for the same reason that the differential does, see §2.3. It follows that

$$0 = c_0(Y, \lambda) < c_1(Y, \lambda) \leq c_2(Y, \lambda) \leq \cdots \leq \infty.$$

Here $c_k(Y, \lambda) = c_{k+1}(Y, \lambda) < \infty$ is possible when λ is degenerate.

(b) We have $c_k(Y, \lambda) < \infty$ for all k only if $c_1(\xi) \in H^2(Y; \mathbb{Z})$ is torsion. Proof: Without loss of generality Y is connected. Recall from Remark 3.2 that if $c_1(\xi)$ is not torsion then $ECH(Y, \lambda, 0)$ is finitely generated. But if $\sigma \in ECH(Y, \lambda, 0)$ and $U^k \sigma = [\emptyset]$ then $\dim(ECH(Y, \lambda, 0)) > k$, because it follows from $U[\emptyset] = 0$ that the classes $\sigma, U\sigma, \dots, U^k \sigma$ are linearly independent.

In simple examples the distinguished ECH spectrum is related to the full ECH spectrum defined previously as follows. Recall from [4] that there is a unique tight contact structure on S^3 , which is the one induced by a Liouville domain with boundary diffeomorphic to S^3 .

Proposition 4.5. *If Y is diffeomorphic to S^3 and if $\text{Ker}(\lambda)$ is the tight contact structure on Y , then*

$$c_k(Y, \lambda) = \tilde{c}_{k+1}(Y, \lambda).$$

We also have:

Proposition 4.6. *If (Y_i, λ_i) are closed contact 3-manifolds with non-vanishing ECH contact invariant for $i = 1, \dots, n$, then*

$$(4.2) \quad c_k \left(\prod_{i=1}^n (Y_i, \lambda_i) \right) = \max \left\{ \sum_{i=1}^n c_{k_i}(Y_i, \lambda_i) \mid \sum_{i=1}^n k_i = k \right\}.$$

The proofs of the above two propositions require an algebraic digression which is deferred to §5.

Proposition 4.7. *If (X, ω) is a weakly exact symplectic cobordism from (Y_+, λ_+) to (Y_-, λ_-) , then*

$$c_k(Y_+, \lambda_+) \geq c_k(Y_-, \lambda_-)$$

for each nonnegative integer k .

Proof. By the approximation argument in the proof of Proposition 3.6, we may assume that λ_+ and λ_- are nondegenerate.

Let $Y_{\pm}^1, \dots, Y_{\pm}^{n_{\pm}}$ denote the connected components of Y_{\pm} . Let $\sigma_+ \in \text{ECH}(Y_+, \lambda_+, 0)$ be a class with $U_{i_1} \cdots U_{i_k} \sigma_+ = [\emptyset]$ for all $i_1, \dots, i_k \in \{1, \dots, n_+\}$. Let $\sigma_- := \Phi(X, \omega)(\sigma_+) \in \text{ECH}(Y_-, \lambda_-, 0)$. Since each component of the cobordism X has at least one positive boundary component, it follows from Theorem 2.3(b),(d) that $U_{i_1} \cdots U_{i_k} \sigma_- = [\emptyset]$ for all $i_1, \dots, i_k \in \{1, \dots, n_-\}$. By Lemma 4.2 we have $c_{\sigma_+}(Y_+, \lambda_+) \geq c_{\sigma_-}(Y_-, \lambda_-)$. q.e.d.

Definition 4.8. By analogy with Definition 3.7, if (X, ω) is a 4-dimensional Liouville domain with boundary Y , and if k is a nonnegative integer, define

$$c_k(X, \omega) := c_k(Y, \lambda),$$

where λ is a contact form on Y with $d\lambda = \omega|_Y$. Lemma 3.9 shows that this does not depend on the choice of contact form λ , just like the full ECH capacities. The numbers $c_k(X, \omega)$ are called the (*distinguished*) ECH capacities of (X, ω) .

We can now prove the main symplectic embedding obstruction:

Proof of Theorem 1.1. For $i = 0, 1$, let $Y_i = \partial X_i$ and let λ_i be a contact form on Y_i with $d\lambda_i = \omega_i|_{X_i}$. Then X_1 minus the interior of the image of X_0 defines a weakly exact symplectic cobordism from (Y_1, λ_1) to (Y_0, λ_0) . By Proposition 4.7, $c_k(X_0, \omega_0) \leq c_k(X_1, \omega_1)$. But in fact the inequality is strict when $c_k(X_0, \omega_0) < \infty$, because the embedding sends X_0 into the interior of X_1 , so we can extend the embedding over $[0, \varepsilon] \times Y_0$ in the symplectization completion (2.7) of X_0 for some $\varepsilon > 0$. The above

argument together with the scaling isomorphism (2.5) then shows that $e^\varepsilon c_k(X_0, \omega_0) \leq c_k(X_1, \omega_1)$. q.e.d.

4.2. More general domains. We now explain how to extend the definition of the (distinguished) ECH capacities to some more general spaces.

Definition 4.9. Let (X, ω) be a subset of a symplectic four-manifold. If k is a positive integer, define

$$c_k(X, \omega) := \sup\{c_k(X_-, \omega)\},$$

where the supremum is over subsets $X_- \subset \text{int}(X)$ such that (X_-, ω) is a four-dimensional Liouville domain.

By definition, $c_k(X, \omega)$ depends only on the symplectic form on $\text{int}(X)$, and not on the symplectic four-manifold of which X is a subset. If (X, ω) is already a four-dimensional Liouville domain, then by Theorem 1.1 the above definition of $c_k(X, \omega)$ agrees with the previous one.

Remark 4.10. One could also try to define the full ECH capacities of a subset of a symplectic four-manifold as in Definition 4.9. However it is not clear if this would agree with the previous definition for Liouville domains, because of the extra assumption in Proposition 3.11. This is another way in which distinguished ECH capacities work better than full ECH capacities.

We now have the following extension of Theorem 1.1:

Proposition 4.11. *Suppose that (X_i, ω_i) is a subset of a symplectic four-manifold for $i = 0, 1$. If there is a symplectic embedding $\varphi : X_0 \rightarrow \text{int}(X_1)$, then $c_k(X_0, \omega_0) \leq c_k(X_1, \omega_1)$ for all k .*

Proof. This is a tautology. Let X_- be a subset of $\text{int}(X_0)$ such that (X_-, ω_0) is a four-dimensional Liouville domain. Then φ restricts to a symplectic embedding of X_- into $\text{int}(X_1)$, so by Definition 4.9,

$$c_k(X_-, \omega_0) \leq c_k(X_1, \omega_1).$$

Taking the supremum over X_- on the left hand side completes the proof. q.e.d.

Note also that Proposition 1.5 extends to the case when each (X_i, ω_i) is a subset of a symplectic four-manifold.

5. Algebraic interlude

The goal of this section is to prove Propositions 4.5 and 4.6. To simplify the notation, in this section write $H(Y, \lambda) := ECH(Y, \lambda, 0)$, and let $H^L(Y, \lambda)$ denote the image of $ECH^L(Y, \lambda, 0)$ in $ECH(Y, \lambda, 0)$. Also write $C_*(Y, \lambda, J) := ECC(Y, \lambda, 0, J)$, and let $C^*(Y, \lambda, J)$ denote the dual chain complex $\text{Hom}(C_*(Y, \lambda, J), \mathbb{Z}/2)$.

Definition 5.1. Let λ be a nondegenerate contact form on a closed 3-manifold Y . A basis $\{\sigma_k\}_{k=1,2,\dots}$ for $H(Y, \lambda)$ is *action-minimizing* if

$$(5.1) \quad c_{\sigma_k}(Y, \lambda) = \tilde{c}_k(Y, \lambda)$$

for all k .

Lemma 5.2. *Let λ be a nondegenerate contact form on a closed 3-manifold Y . Then:*

- (a) *There exists an action-minimizing basis for $H(Y, \lambda)$.*
- (b) *If $\{\sigma_k\}$ is an action-minimizing basis for $H(Y, \lambda)$, and if $0 \neq \sigma = \sum_j a_j \sigma_j \in H(Y, \lambda)$, then*

$$(5.2) \quad c_\sigma(Y, \lambda) = \tilde{c}_k(Y, \lambda)$$

where k is the largest integer such that $a_k \neq 0$.

Proof. (a) To construct an action-minimizing basis, increase L starting from 0, and whenever the dimension of $H^L(Y, \lambda)$ jumps, add new basis elements to span the rest of it. More precisely, there is a discrete set of nonnegative real numbers L such that

$$\dim(H^{L+\varepsilon}(Y, \lambda)) > \dim(H^L(Y, \lambda))$$

for all $\varepsilon > 0$. Denote these real numbers by $0 \leq L_1 < L_2 < \dots$. There are then integers $0 = k_0 < k_1 < k_2 < \dots$ such that

$$(5.3) \quad k_{i-1} < k \leq k_i \implies \tilde{c}_k(Y, \lambda) = L_i.$$

Now define a basis by taking $\{\sigma_k \mid k_{i-1} < k \leq k_i\}$ to be elements of $H^{L_{i+1}}(Y, \lambda)$ that project to a basis for $H^{L_{i+1}}(Y, \lambda)/H^{L_i}(Y, \lambda)$. Then equation (5.1) follows from the construction.

To prepare for the proof of (b), note also that conversely, by (5.3), any action-minimizing basis is obtained by the above construction.

(b) Continuing the notation from the proof of part (a), we have $\tilde{c}_k(Y, \lambda) = L_i$ for some i . By equation (5.1), $\sigma \in H^L(Y, \lambda)$ whenever $L > L_i$, so $c_\sigma(Y, \lambda) \leq L_i$. To prove the reverse inequality, suppose to get a contradiction that $\sigma \in H^{L_i}(Y, \lambda)$. Let σ' denote the contribution to σ from basis elements σ_j with $c_{\sigma_j}(Y, \lambda) < L_i$. Then $\sigma' \in H^{L_i}(Y, \lambda)$, so $\sigma - \sigma' \in H^{L_i}(Y, \lambda)$ as well. Now $\sigma - \sigma'$ is a linear combination of the basis elements $\{\sigma_k \mid k_{i-1} < k \leq k_i\}$. Since the latter are linearly independent in $H^{L_{i+1}}(Y, \lambda)/H^{L_i}(Y, \lambda)$, it follows that $\sigma - \sigma' = 0$, which is the desired contradiction. q.e.d.

Remark 5.3. One has to be careful in the proof of Lemma 5.2(b), because the equality

$$(5.4) \quad c_{\sigma_1+\dots+\sigma_n}(Y, \lambda) = \max\{c_{\sigma_i}(Y, \lambda) \mid i = 1, \dots, n\}$$

does not always hold for linearly independent elements $\sigma_1, \dots, \sigma_n$ of $H(Y, \lambda)$. However (5.4) does hold if the maximum on the right hand side

is realized by a unique $i \in \{1, \dots, n\}$, or if all of the classes $\sigma_1, \dots, \sigma_n$ have (definite and) distinct gradings.

Proof of Proposition 4.5. By the usual approximation arguments we may assume that λ is nondegenerate. Since Y is a homology sphere, the relative grading on ECH has a canonical refinement to an absolute \mathbb{Z} -grading in which the empty set of Reeb orbits has grading zero. With this grading convention, the ECH with $\mathbb{Z}/2$ -coefficients is given by

$$ECH_*(Y, \lambda, 0) = \begin{cases} \mathbb{Z}/2, & * = 0, 2, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

In addition, $U : ECH_*(Y, \lambda, 0) \rightarrow ECH_{*-2}(Y, \lambda, 0)$ is an isomorphism whenever $* \neq 0$. These facts follow from the isomorphism (2.1), together with the computation of the Seiberg-Witten Floer homology of S^3 in [14]. Finally, $[\emptyset]$ generates $ECH_0(Y, \lambda, 0)$. This follows from the above facts, or from direct computations for a standard tight contact form on S^3 , see [11, Ex. 4.2].

Now let σ_k denote the generator of $ECH_{2k}(Y, \lambda, 0)$. Since the U map decreases symplectic action we have

$$(5.5) \quad 0 = c_{\sigma_0}(Y, \lambda) < c_{\sigma_1}(Y, \lambda) < \dots < \infty.$$

It follows from (5.5) and Remark 5.3 that $c_{\sigma_k}(Y, \lambda) = \tilde{c}_{k+1}(Y, \lambda)$. Now a class $\sigma = \sum_j a_j \sigma_j$ satisfies $U^k \sigma = [\emptyset]$ if and only if $a_k = 1$ and $a_j = 0$ for $j > k$. By Lemma 5.2(b), each such class σ satisfies $c_\sigma(Y, \lambda) = \tilde{c}_{k+1}(Y, \lambda)$. q.e.d.

Before continuing, we need to recall the following elementary fact:

Lemma 5.4. *Let (C_*, ∂) be a chain complex over a field \mathbb{F} , and let $C'_* \subset C_*$ be a subcomplex. Suppose $\alpha_1, \dots, \alpha_n \in H_*(C_*)$ are linearly independent in $H_*(C_*)/H_*(C'_*)$, and let $y_1, \dots, y_n \in \mathbb{F}$. Then there exists a cocycle $\zeta \in \text{Hom}(C_*, \mathbb{F})$ which annihilates C'_* and sends $\alpha_i \mapsto y_i$ for each i .*

Proof. Let $x_i \in C_*$ be a cycle representing the homology class α_i . By hypothesis, x_1, \dots, x_n project to linearly independent elements of $C_*/(C'_* + \partial(C_*))$. Hence there is a linear map $\zeta : C_* \rightarrow \mathbb{F}$ sending $x_i \mapsto y_i$ for each i and annihilating the subspace $C'_* + \partial(C_*)$. This is the desired cocycle. q.e.d.

Proof of Proposition 4.6. By the usual approximation argument, we may assume that the contact forms λ_i are nondegenerate. We can also assume that each Y_i is connected. We now proceed in three steps.

Step 1. We first show that the left hand side of (4.2) is less than or equal to the right hand side. We can assume that the right hand side is finite. For each $i = 1, \dots, n$ and $j \geq 0$ with $c_j(Y_i, \lambda_i) < \infty$, choose

a class $\sigma_{i,j} \in H(Y_i, \lambda_i)$ with $U^j \sigma_{i,j} = [\emptyset]$, such that $\sigma_{i,j} \in H^L(Y_i, \lambda_i)$ whenever $L > c_j(Y_i, \lambda_i)$. Recalling the identification (2.4), define a class

$$\sigma := \sum_{j_1 + \dots + j_n = k} \sigma_{1,j_1} \otimes \dots \otimes \sigma_{n,j_n} \in H \left(\prod_{i=1}^n (Y_i, \lambda_i) \right).$$

Since symplectic action is additive under tensor product, $\sigma \in H^L(\prod_i (Y_i, \lambda_i))$ whenever L is greater than the right hand side of (4.2). So we just need to show that $U_{i_1} \dots U_{i_k} \sigma = [\emptyset]$ for all $i_1, \dots, i_k \in \{1, \dots, n\}$. Equivalently, since the different maps U_i commute, we need to show that if $\sum_{i=1}^n k_i = k$ then

$$U_1^{k_1} \dots U_n^{k_n} \sigma = [\emptyset].$$

To prove this last statement, observe that if $\sum_{i=1}^n j_i = k$ then

$$U_1^{k_1} \dots U_n^{k_n} (\sigma_{1,j_1} \otimes \dots \otimes \sigma_{n,j_n}) = \begin{cases} [\emptyset], & (j_1, \dots, j_n) = (k_1, \dots, k_n), \\ 0, & \text{otherwise.} \end{cases}$$

This is because if $(j_1, \dots, j_n) \neq (k_1, \dots, k_n)$ then $k_i > j_i$ for some i , so that

$$U_i^{k_i} \sigma_{i,j_i} = U_i^{k_i - j_i} [\emptyset] = 0,$$

where the last equality holds since U_i decreases symplectic action.

Step 2. We claim now that

$$(5.6) \quad H^L \left(\prod_{i=1}^n (Y_i, \lambda_i) \right) = \text{span} \left\{ \bigotimes_{i=1}^n H^{L_i}(Y_i, \lambda_i) \mid \sum_{i=1}^n L_i \leq L \right\}.$$

To prove this, for each $i = 1, \dots, n$, let $\{\sigma_{i,j}\}_{j=1,2,\dots}$ be an action-minimizing basis for $H(Y_i, \lambda_i)$. By Lemma 5.2(b), for each i and L_i we have

$$H^{L_i}(Y_i, \lambda_i) = \text{span}\{\sigma_{i,j} \mid c_{\sigma_{i,j}}(Y_i, \lambda_i) < L_i\}.$$

Thus equation (5.6) is equivalent to

$$(5.7) \quad H^L \left(\prod_{i=1}^n (Y_i, \lambda_i) \right) = \text{span} \left\{ \sigma_{1,j_1} \otimes \dots \otimes \sigma_{n,j_n} \mid \sum_{i=1}^n c_{\sigma_{i,j_i}}(Y_i, \lambda_i) < L \right\}.$$

The right hand side of (5.7) is a subset of the left, as in Step 1, because in the identification (2.3) the symplectic action is additive under tensor product. To prove the reverse inclusion, consider a class

$$(5.8) \quad \sigma = \sum_{j_1, \dots, j_n} a_{j_1, \dots, j_n} \sigma_{1,j_1} \otimes \dots \otimes \sigma_{n,j_n} \in H \left(\prod_{i=1}^n (Y_i, \lambda_i) \right).$$

Let

$$L' := \max \left\{ \sum_{i=1}^n c_{\sigma_{i,j_i}}(Y_i, \lambda_i) \mid a_{j_1, \dots, j_n} \neq 0 \right\}.$$

We need to show that $\sigma \notin H^{L'}(\prod_i (Y_i, \lambda_i))$.

To do so, choose (j_1, \dots, j_n) with $a_{j_1, \dots, j_n} \neq 0$ and $c_{\sigma_{i, j_i}}(Y_i, \lambda_i) = L_i$ where $\sum_{i=1}^n L_i = L'$. Choose an almost complex structure J_i on $\mathbb{R} \times Y_i$ as needed to define the ECH of λ_i . By Lemmas 5.2(b) and 5.4, there is a cocycle $\zeta_i \in C^*(Y_i, \lambda_i, J_i)$ sending $\sigma_{i, j_i} \mapsto 1$, annihilating all other basis elements $\sigma_{i, j}$ with $c_{\sigma_{i, j}}(Y_i, \lambda_i) = L_i$, and annihilating all ECH generators with action less than L_i . Then

$$\zeta_1 \otimes \dots \otimes \zeta_n \in C^* \left(\prod_{i=1}^n (Y_i, \lambda_i, J_i) \right)$$

sends $\sigma \mapsto 1$ and annihilates $H^{L'}(\prod_i(Y_i, \lambda_i))$. Therefore $\sigma \notin H^{L'}(\prod_i(Y_i, \lambda_i))$.

Step 3. We now show that the left hand side of (4.2) is greater than or equal to the right hand side. We need to show that if $\sum_{i=1}^n k_i = k$ then

$$c_k \left(\prod_{i=1}^n (Y_i, \lambda_i) \right) \geq \sum_{i=1}^n c_{k_i}(Y_i, \lambda_i).$$

To do so, let $L := \sum_{i=1}^n c_{k_i}(Y_i, \lambda_i)$. We will show that if $\sigma \in H^L(\prod_i(Y_i, \lambda_i))$, then $U_1^{k_1} \dots U_n^{k_n} \sigma \neq [\emptyset]$.

Expand σ as in (5.8). By Step 2,

$$(5.9) \quad a_{j_1, \dots, j_n} \neq 0 \implies \sum_{i=1}^n c_{\sigma_{j_i}}(Y_i, \lambda_i) < L.$$

Next, for each $i = 1, \dots, n$, we can choose $\zeta_i \in \text{Hom}(H(Y_i, \lambda_i), \mathbb{Z}/2)$ with the following two properties:

- (i) $\zeta_i([\emptyset]) = 1$.
- (ii) ζ_i annihilates $U^{k_i} \left(H^{c_{k_i}(Y_i, \lambda_i)}(Y_i, \lambda_i) \right)$.

Now let

$$\zeta = \zeta_1 \otimes \dots \otimes \zeta_n \in \text{Hom} \left(H \left(\prod_i (Y_i, \lambda_i) \right), \mathbb{Z}/2 \right).$$

By property (i) we have $\zeta([\emptyset]) = 1$. On the other hand,

$$\begin{aligned} \zeta \left(U_1^{k_1} \dots U_n^{k_n} \sigma \right) &= \left(\zeta_1 \circ U_1^{k_1} \right) \otimes \dots \otimes \left(\zeta_n \circ U_n^{k_n} \right) \sigma \\ &= \sum_{j_1, \dots, j_n} a_{j_1, \dots, j_n} \prod_{i=1}^n \zeta_i \left(U_i^{k_i} \sigma_{i, j_i} \right) \\ &= 0, \end{aligned}$$

where the last equality follows from (5.9) and (ii). Thus $U_1^{k_1} \dots U_n^{k_n} \sigma \neq [\emptyset]$ as desired. q.e.d.

6. The 3-torus

We now compute the distinguished ECH spectrum of the 3-torus with various contact forms.

6.1. Distinguished ECH spectrum of the standard 3-torus. Consider the 3-torus

$$(6.1) \quad Y = T^3 = (\mathbb{R}/2\pi\mathbb{Z})_\theta \times (\mathbb{R}^2/\mathbb{Z}^2)_{x,y}$$

with the standard contact form

$$(6.2) \quad \lambda = \cos \theta dx + \sin \theta dy.$$

The ECH of this example was studied in detail in [8]. Using these results, we can now compute the distinguished ECH spectrum:

Proposition 6.1. *If k is a nonnegative integer then*

$$(6.3) \quad c_k(T^3, \lambda) = \min \{ \ell(\Lambda) \mid |P_\Lambda \cap \mathbb{Z}^2| = k + 1 \}.$$

Here the minimum is over convex polygons Λ in \mathbb{R}^2 with vertices in \mathbb{Z}^2 , and P_Λ denotes the closed region bounded by Λ . Also $\ell(\Lambda)$ denotes the Euclidean length of Λ .

Proof. The proof has three steps.

Step 1. We first review what we need to know about the ECH of T^3 . The relative grading on $ECH_*(T^3, \lambda, 0)$ has a canonical refinement to an absolute \mathbb{Z} -grading in which the empty set has grading 0. With this convention, we have (by [8], or using the isomorphism (2.1) and [14, Prop. 3.10.1])

$$(6.4) \quad ECH_*(T^3, \lambda, 0) \simeq \begin{cases} (\mathbb{Z}/2)^3, & * \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

In addition, the map

$$U : ECH_*(T^3, \lambda, 0) \longrightarrow ECH_{*-2}(T^3, \lambda, 0)$$

is an isomorphism whenever $* \geq 2$. Finally, the contact invariant $[\emptyset]$ is nonzero (by [8], or because (T^3, λ) is the boundary of a Liouville domain).

We also need to know a bit about the ECH chain complex. The Reeb vector field is given by

$$R = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y}.$$

It follows that for every pair of relatively prime integers (m, n) there is a Morse-Bott circle of embedded Reeb orbits $\mathcal{O}_{m,n}$ sweeping out $\{\theta\} \times (\mathbb{R}^2/\mathbb{Z}^2)$ where

$$(6.5) \quad \cos \theta = \frac{m}{\sqrt{m^2 + n^2}}, \quad \sin \theta = \frac{n}{\sqrt{m^2 + n^2}}.$$

Each Reeb orbit $\gamma \in \mathcal{O}_{m,n}$ has symplectic action

$$(6.6) \quad \mathcal{A}(\gamma) = \sqrt{m^2 + n^2}.$$

There are no other embedded Reeb orbits.

Fix $L \in \mathbb{R}$. For any $\varepsilon > 0$, we can perturb the contact form λ to $f\lambda$ where $f : Y \rightarrow [1 - \varepsilon, 1]$, such that each Morse-Bott circle $\mathcal{O}_{m,n}$ with $\sqrt{m^2 + n^2} < L$ splits into an elliptic orbit $e_{m,n}$ and a hyperbolic orbit $h_{m,n}$, and these are the only embedded Reeb orbits with action less than L . As in [8, §11.3], a generator α of the ECH chain complex for $f\lambda$ with action less than L and with $\Gamma = 0$ then corresponds to a convex lattice polygon Λ_α , modulo translation, in which each edge is labeled ‘ e ’ or ‘ h ’. Note here that 2-gons and 0-gons are allowed, with the latter corresponding to the empty set of Reeb orbits.

By (6.6), the action of a generator α as above is given by

$$(6.7) \quad \mathcal{A}(\alpha) = \ell(\Lambda_\alpha) - O(\varepsilon).$$

Furthermore, it is shown in [8, §11.3] that with the above grading conventions, the grading of the generator α is given by

$$(6.8) \quad I(\alpha) = 2(|P_{\Lambda_\alpha} \cap \mathbb{Z}^2| - 1) - \#h(\alpha),$$

where $\#h(\alpha)$ denotes the number of edges of Λ_α that are labeled ‘ h ’.

Step 2. We now prove that the left hand side of (6.3) is less than or equal to the right hand side.

Fix a nonnegative integer k . Let Λ_0 be a length-minimizing convex polygon with $|P_{\Lambda_0} \cap \mathbb{Z}^2| = k + 1$. Let α_0 denote the ECH generator consisting of the polygon Λ_0 with all edges labeled ‘ e ’. (Assume that L above is chosen sufficiently large with respect to k so that this is defined.) The differential on the ECH chain complex in action less than L for suitable perturbation function f and almost complex structure J is computed in [8]: roughly speaking, the differential of a generator is the sum over all ways of “rounding a corner” and “locally losing one ‘ h ’”. Since the generator α_0 has no ‘ h ’ labels, it follows immediately that $\partial\alpha_0 = 0$. In addition, it follows from the computation of the U map in [8, §12.1.4] that the chain map U applied to a generator with all edges labeled ‘ e ’ is obtained by rounding a distinguished corner (depending on the choice of point $z \in Y$ used to define the chain map U) and leaving all edges labeled ‘ e ’. It follows that $U^k\alpha_0 = \emptyset$. Thus $[\alpha_0]$ is a class in ECH with $U^k[\alpha_0] = [\emptyset]$, so

$$c_k(T^3, f\lambda) \leq \mathcal{A}(\alpha_0) = \ell(\Lambda_0) - O(\varepsilon).$$

Taking $\varepsilon \rightarrow 0$ proves the desired inequality.

Step 3. We now prove that the left hand side of (6.3) is greater than or equal to the right hand side.

Let $\sigma \in ECH(T^3, f\lambda, 0)$ be a class with $U^k\sigma = [\emptyset]$. Since U is an isomorphism in grading ≥ 2 , it follows that $\sigma = [\alpha_0] + \sigma'$ where σ' is a

sum of classes of grading less than $2k$. Thus by Remark 5.3,

$$(6.9) \quad c_\sigma(T^3, f\lambda) = \max(c_{[\alpha_0]}(T^3, f\lambda), c_{\sigma'}(T^3, f\lambda)) \geq c_{[\alpha_0]}(T^3, f\lambda).$$

Next we observe that

(*) $\ell(\Lambda_0)$ is (up to $O(\varepsilon)$ error) the minimum of $\mathcal{A}(\alpha)$ where α is a generator with $\Gamma = 0$ and $I(\alpha) = 2k$.

This is because by (6.7), the above minimum of $\mathcal{A}(\alpha)$ is (up to $O(\varepsilon)$ error) the minimum of $\ell(\Lambda_\alpha)$ where α is a generator with $\Gamma = 0$ and $I(\alpha) = 2k$. But it follows immediately from (6.8) that the latter minimum is realized by a generator α in which all edges of Λ_α are labeled ‘ e ’ and $|P_{\Lambda_\alpha} \cap \mathbb{Z}^2| = k + 1$.

It follows from (*) that

$$c_{[\alpha_0]}(T^3, f\lambda) \geq \ell(\Lambda_0) - O(\varepsilon).$$

Combining with (6.9) and taking $\varepsilon \rightarrow 0$ proves the desired inequality.

q.e.d.

Remark 6.2. In principle one could compute the full ECH spectrum of T^3 from [8, Prop. 8.3], although this is not so simple. The latter proposition semi-explicitly describes a basis for the ECH consisting of elements p_k, u_k, v_k of grading $2k$ and s_k, t_k, w_k of grading $2k + 1$ for each nonnegative integer k . Here p_k is the unique class of grading $2k$ with $U^k p_k = [\emptyset]$. In particular, it follows from this description that in the notation of Definition 4.1,

$$c_{w_k} > c_{u_k} = c_{v_k} = c_{s_k} = c_{t_k} > c_{p_{k-1}}.$$

In addition it follows from the computation of the U map in [8, Lem. 8.4] that $c_{p_k} > c_{p_{k-1}}$, $c_{u_k} > c_{u_{k-1}}$, and so forth. The beginning of the full ECH spectrum is $c_{p_0} = 0$, $c_{p_1} = c_{u_0} = c_{v_0} = c_{s_0} = c_{t_0} = 2$, $c_{p_2} = c_{u_1} = c_{v_1} = c_{s_1} = c_{t_1} = c_{w_0} = 2 + \sqrt{2}$, $c_{p_3} = 4$, $c_{u_2} = c_{v_2} = c_{s_2} = c_{t_2} = c_{w_1} = 2 + 2\sqrt{2}$.

6.2. Distinguished ECH spectrum of some nonstandard 3-tori.

We now prove Theorem 1.11, computing the distinguished ECH capacities of the examples $T_{\|\cdot\|*}$ defined in §1.4. Note that this generalizes Proposition 6.1, because if $\|\cdot\|$ is the Euclidean norm on \mathbb{R}^2 , then λ restricts to $\partial T_{\|\cdot\|*}$ as the standard contact form (6.2) on T^3 .

Proof of Theorem 1.11. We may assume without loss of generality that the norm $\|\cdot\|$ is smooth. This follows from Proposition 4.11, because an arbitrary norm can be approximated from above and below by smooth norms, and for a given positive integer k the right hand side of (1.8) depends continuously on the norm.

Since the norm $\|\cdot\|$ is smooth, $T_{\|\cdot\|*}$ is a Liouville domain. We now follow the proof of Proposition 6.1 with appropriate modifications.

To start we compute the Reeb vector field of $\lambda = \sum_{i=1}^2 p_i dq_i$ on $\partial T_{\|\cdot\|}^*$. Let B denote the unit ball of the dual norm $\|\cdot\|^*$; observe that ∂B is a smooth convex curve in $(\mathbb{R}^2)^*$. Identify $(\mathbb{R}^2)^* = \mathbb{R}^2$ using the usual coordinates p_1, p_2 . Suppose $(p_1, p_2) \in \partial B$. There is a unique $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ such that the outward unit normal vector to ∂B at (p_1, p_2) (with respect to the Euclidean metric) is given by $(\cos \theta, \sin \theta)$. The Reeb vector field at (q_1, q_2, p_1, p_2) is then

$$R = (p_1 \cos \theta + p_2 \sin \theta)^{-1} \left(\cos \theta \frac{\partial}{\partial q_1} + \sin \theta \frac{\partial}{\partial q_2} \right).$$

It follows that for every pair of relatively prime integers (m, n) there is a Morse-Bott circle of embedded Reeb orbits $\mathcal{O}_{m,n}$, sweeping out $T^2 \times \{(p_1, p_2)\}$ where (p_1, p_2) corresponds as above to the unique θ satisfying (6.5). There are no other embedded Reeb orbits. Each Reeb orbit $\gamma \in \mathcal{O}_{m,n}$ has symplectic action

$$\mathcal{A}(\gamma) = p_1 m + p_2 n.$$

Now observe that since $\|\cdot\|$ is the dual norm of $\|\cdot\|^*$, we have

$$\|(m, n)\| = \max \{ \langle \zeta, (m, n) \rangle \mid \zeta \in B \}.$$

By the definition of θ , this maximum is realized by $\zeta = (p_1, p_2)$. In conclusion, each Reeb orbit $\gamma \in \mathcal{O}(m, n)$ has symplectic action

$$(6.10) \quad \mathcal{A}(\gamma) = \|(m, n)\|.$$

The rest of the proof is now the same as the proof of Proposition 6.1, with equation (6.6) replaced by (6.10), and ℓ replaced by $\ell_{\|\cdot\|}$. q.e.d.

7. The polydisk

7.1. The ECH capacities of a polydisk. We now prove Theorem 1.4 on the (distinguished) ECH capacities of a polydisk. One can calculate the ECH capacities of a polydisk by understanding the ECH chain complex of an appropriately smoothed polydisk, similarly to the calculations in [8] for T^3 as outlined in §6.1. However this is a long story, and we will instead take a shortcut using Theorems 1.1 and 1.11.

Proof of Theorem 1.4. The proof has two steps.

Step 1. Define a norm $\|\cdot\|$ on \mathbb{R}^2 by

$$(7.1) \quad \|(q_1, q_2)\| = \frac{a|q_1|}{2} + \frac{b|q_2|}{2}.$$

The dual norm is then

$$\|(p_1, p_2)\|^* = \max \left(\frac{2|p_1|}{a}, \frac{2|p_2|}{b} \right),$$

so that

$$T_{\|\cdot\|}^* = \{(q_1, q_2, p_1, p_2) \in T^*T^2 \mid |p_1| \leq a/2, |p_2| \leq b/2\}.$$

Denote this by $T(a, b)$.

Observe now that for any $\varepsilon > 0$, there is a symplectic embedding $P(a, b) \rightarrow T(a + \varepsilon, b + \varepsilon)$ defined by

$$(z_1, z_2) \mapsto (\phi_1(z_1), \phi_2(z_2)),$$

where $\phi_1 = (p_1, q_1)$ is an area-preserving embedding of the disc of area a into the cylinder $[-(a + \varepsilon)/2, (a + \varepsilon)/2] \times \mathbb{R}/\mathbb{Z}$, and $\phi_2 = (p_2, q_2)$ is an area-preserving embedding of the disc of area b into the cylinder $[-(b + \varepsilon)/2, (b + \varepsilon)/2] \times \mathbb{R}/\mathbb{Z}$. There is also a symplectic embedding $T(a - \varepsilon, b - \varepsilon) \rightarrow P(a, b)$ defined by

$$(q_1, q_2, p_1, p_2) \mapsto \pi^{-1/2} \left((a/2 + p_1)^{1/2} e^{2\pi i q_1}, (b/2 + p_2)^{1/2} e^{2\pi i q_2} \right).$$

Consequently, for any given k , applying Theorem 1.1 and taking $\varepsilon \rightarrow 0$ shows that

$$c_k(P(a, b)) = c_k(T(a, b)).$$

So by Theorem 1.11, we need to show that

$$(7.2) \quad \min \{ am + bn \mid (m + 1)(n + 1) \geq k + 1 \} \\ = \min \{ \ell_{\|\cdot\|}(\Lambda) \mid |P_\Lambda \cap \mathbb{Z}^2| = k + 1 \},$$

where in the first minimum $(m, n) \in \mathbb{N}^2$, and in the second minimum Λ is a convex polygon in \mathbb{R}^2 with vertices in \mathbb{Z}^2 .

Step 2. We now prove (7.2). Given a convex polygon Λ in \mathbb{R}^2 with vertices in \mathbb{Z}^2 , let m denote the horizontal displacement between the rightmost and leftmost vertices, and let n denote the vertical displacement between the top and bottom vertices. Then Λ is contained in a rectangle of side lengths m and n , so

$$|P_\Lambda \cap \mathbb{Z}^2| \leq (m + 1)(n + 1).$$

On the other hand it follows from (7.1) that

$$\ell_{\|\cdot\|}(\Lambda) = am + bn.$$

Hence the left hand side of (7.2) is less than or equal to the right hand side. But the reverse inequality also holds, because if one has $k + 1 \leq (m + 1)(n + 1)$, then inside a rectangle of side lengths m and n one can find a convex polygon Λ with $|P_\Lambda \cap \mathbb{Z}^2| = k + 1$. q.e.d.

7.2. Obstructions to embedding polydisks into balls. Let us now try to more explicitly understand the bound (1.5) (which we have now justified) for the function g defined in §1.3.2 that measures the obstruction to symplectically embedding a polydisk into a ball. The bound (1.5) can be written as $g(a) \geq \sup_{d=1,2,\dots} g_d(a)$, where

$$g_d(a) := \min \left\{ \frac{am + n}{d} \mid (m, n) \in \mathbb{N}^2, (m + 1)(n + 1) \geq \frac{(d + 1)(d + 2)}{2} \right\}.$$

Given d , one can compute the function g_d as follows. Let Λ_d denote the boundary of the convex hull of the set of lattice points $(m, n) \in \mathbb{N}^2$ with $(m+1)(n+1) \geq (d+1)(d+2)/2$. Then $g_d(a) = (am+n)/d$, where (m, n) is a (usually unique) vertex of the polygonal path Λ_d incident to edges of slope less than or equal to $-a$ and slope greater than or equal to $-a$. Using this observation, we can now give the:

Proof of Proposition 1.7. First consider $d = 1$. The path Λ_1 has vertices $(0, 2)$, $(1, 1)$, and $(2, 0)$. Since the vertex $(0, 2)$ is incident to edges of slope -1 and $-\infty$, the above discussion shows that

$$g_1(a) = 2, \quad a \geq 1.$$

This proves the first line of (1.6). To prove the rest of (1.6), take $d = 6$. The path Λ_6 has vertices $(0, 27)$, $(1, 13)$, $(2, 9)$, $(3, 6)$, $(4, 5)$, $(5, 4)$, $(6, 3)$, $(9, 2)$, $(13, 1)$, and $(27, 0)$. Since the vertex $(3, 6)$ is incident to edges of slope -1 and -3 , we get

$$g_6(a) = \frac{3a+6}{6}, \quad 1 \leq a \leq 3.$$

This implies the second line of (1.6). And since the vertex $(2, 9)$ is incident to edges of slope -3 and -4 , we obtain

$$g_6(a) = \frac{2a+9}{6}, \quad 3 \leq a \leq 4.$$

This gives the last line of (1.6).

q.e.d.

8. Volume and quantitative ECH

We now discuss and present evidence for Conjecture 1.12 and two generalizations, relating the asymptotics of quantitative ECH to volume.

8.1. Volume conjecture for the distinguished ECH spectrum.

If (Y, λ) is a closed contact 3-manifold, define

$$\text{vol}(Y, \lambda) := \int_Y \lambda \wedge d\lambda.$$

Conjecture 1.12 is then a special case of the following:

Conjecture 8.1. *Let (Y, λ) be a closed contact 3-manifold with non-vanishing ECH contact invariant. Suppose that $c_k(Y, \lambda) < \infty$ for all k . Then*

$$\lim_{k \rightarrow \infty} \frac{c_k(Y, \lambda)^2}{k} = 2 \text{vol}(Y, \lambda).$$

By Remark 3.13 and Proposition 4.5, this conjecture holds for ellipsoids. Here are some more examples:

Example 8.2. Consider T^3 as in (6.1) with the standard contact form λ in (6.2). Let Λ be a convex polygon as in (6.3). If $A(\Lambda)$ denotes the area enclosed by Λ , then

$$|P_\Lambda \cap \mathbb{Z}^2| = A(\Lambda) + O(\ell(\Lambda)).$$

It then follows from (6.3) and the isoperimetric inequality

$$\ell(\Lambda)^2 \geq 4\pi A(\Lambda)$$

that

$$\liminf_{k \rightarrow \infty} \frac{c_k(T^3, \lambda)^2}{k} \geq 4\pi.$$

On the other hand, approximating a circle with polygons shows that if k is large, then we can find a polygon Λ as in (6.3) with

$$\ell(\Lambda)^2 \leq 4\pi A(\Lambda) + O(\ell(\Lambda)),$$

so in fact

$$\lim_{k \rightarrow \infty} \frac{c_k(T^3, \lambda)^2}{k} = 4\pi.$$

Since $\text{vol}(T^3) = 2\pi$, Conjecture 8.1 is confirmed in this case.

Example 8.3. More generally, let $\|\cdot\|$ be a smooth norm on \mathbb{R}^2 , let B denote the unit ball in the dual norm $\|\cdot\|^*$, and consider the Liouville domain $T_{\|\cdot\|^*}$ from §6.2. We have $\text{vol}(T_{\|\cdot\|^*}) = A(B)$, where $A(B)$ denotes the area of B (with respect to the Euclidean metric). So it follows from Theorem 1.11 that Conjecture 1.12 in this case is equivalent to a sharp isoperimetric inequality

$$(8.1) \quad \ell_{\|\cdot\|}(\Lambda)^2 \geq 4A(B)A(\Lambda)$$

for a smooth convex curve Λ . Now (8.1) holds because if $A(\Lambda)$ is fixed, then $\ell_{\|\cdot\|}(\Lambda)$ is minimized when Λ is a scaling of a 90° rotation of ∂B , see [2, 27]; and one can check directly that in this case equality holds in (8.1).

Proposition 8.4. *If Conjecture 8.1 holds for closed contact three-manifolds (Y_i, λ_i) with nonvanishing contact invariant for $i = 1, \dots, n$, then it also holds for $(Y, \lambda) := \coprod_{i=1}^n (Y_i, \lambda_i)$.*

Proof. By Proposition 4.6, we can assume that $c_k(Y_i, \lambda_i) < \infty$ for all i and k , and we have

$$\lim_{k \rightarrow \infty} \frac{c_k(Y, \lambda)}{\sqrt{2k}} = \lim_{k \rightarrow \infty} \frac{1}{\sqrt{2k}} \max_{k_1 + \dots + k_n = k} \sum_{i=1}^n \sqrt{2k_i \text{vol}(Y_i, \lambda_i)},$$

provided that the limit on the right exists. If one drops the integrality requirement on k_i , then the maximum on the right is attained when

$$k_i = \frac{k \text{vol}(Y_i, \lambda_i)}{\text{vol}(Y, \lambda)}.$$

We then obtain

$$\lim_{k \rightarrow \infty} \frac{c_k(Y, \lambda)}{\sqrt{2k}} = \lim_{k \rightarrow \infty} \sum_{i=1}^n (\text{vol}(Y, \lambda))^{-1/2} \text{vol}(Y_i, \lambda_i) = \sqrt{\text{vol}(Y, \lambda)}$$

as required.

q.e.d.

Remark 8.5. Under the assumptions of Conjecture 8.1, in many (but not all) examples one in fact has $c_k(Y, \lambda) \leq \sqrt{2k \text{vol}(Y, \lambda)}$ for all k . Note that upper bounds on the numbers $c_k(Y, \lambda)$ imply quantitative refinements of the three-dimensional Weinstein conjecture, because by definition, if λ is nondegenerate, then (Y, λ) has at least k nonempty ECH generators of symplectic action at most $c_k(Y, \lambda)$. In particular, λ always has a Reeb orbit of symplectic action at most $c_1(Y, \lambda)$. If Y is a circle bundle over S^2 of euler class $e > 2$ with a prequantization contact form, then $\text{vol} = 4\pi^2 e$ but $c_1 = 2\pi e > \sqrt{2 \text{vol}}$, although there is still a Reeb orbit of action $2\pi < \sqrt{2 \text{vol}}$.

8.2. Volume conjecture for Liouville domains. We now confirm Conjecture 1.12 in some more cases.

Proposition 8.6. *Let (X_0, ω_0) be a 4-dimensional Liouville domain. Then:*

- (a) $\liminf_{k \rightarrow \infty} \frac{c_k(X_0, \omega_0)^2}{k} \geq 4 \text{vol}(X_0, \omega_0)$.
- (b) *Suppose that (X_0, ω_0) can be symplectically embedded into a 4-dimensional Liouville domain (X_1, ω_1) such that $c_k(X_1, \omega_1) < \infty$ for all k and Conjecture 1.12 holds for (X_1, ω_1) . Then Conjecture 1.12 holds for (X_0, ω_0) .*

Proof. (a) For any $\varepsilon > 0$, by using a finite cover of X_0 by Darboux charts, we can fill all but ε of the volume of (X_0, ω_0) with products of smoothed squares which are symplectomorphic to polydisks. Since Conjecture 1.12 is true for a polydisk, by Proposition 8.4 (applied to boundaries of smoothed polydisks) it is also true for a disjoint union of polydisks. Applying Theorem 1.1 then gives

$$\liminf_{k \rightarrow \infty} \frac{c_k(X_0, \omega_0)^2}{k} \geq 4(\text{vol}(X_0, \omega_0) - \varepsilon).$$

Since $\varepsilon > 0$ was arbitrary, this proves (a).

(b) Fill all but volume ε of the complement of X_0 in X_1 by polydisks and apply Theorem 1.1 again. q.e.d.

8.3. A more general volume conjecture. Conjecture 8.1 is a special case of the following more general conjecture. Let (Y, λ) be a closed contact 3-manifold. Recall that if $\Gamma \in H_1(Y)$ is such that $c_1(\xi) + 2\text{PD}(\Gamma) \in H^2(Y; \mathbb{Z})$ is torsion, then $ECH(Y, \lambda, \Gamma)$ has a relative \mathbb{Z} -grading, which can be arbitrarily normalized to an absolute \mathbb{Z} -grading. We then denote the grading of a generator x by $I(x) \in \mathbb{Z}$. Recall the notation c_σ from Definition 4.1.

Conjecture 8.7. *Let (Y, λ) be a closed connected contact 3-manifold, let $\Gamma \in H_1(Y)$, suppose that $c_1(\xi) + 2\text{PD}(\Gamma) \in H^2(Y; \mathbb{Z})$ is torsion, and choose an absolute \mathbb{Z} -grading as above on $ECH(Y, \lambda, \Gamma)$. Let $\{\sigma_k\}_{k=1,2,\dots}$ be a sequence of elements of $ECH(Y, \lambda, \Gamma)$ with definite gradings satisfying $\lim_{k \rightarrow \infty} I(\sigma_k) = \infty$. Then*

$$(8.2) \quad \lim_{k \rightarrow \infty} \frac{c_{\sigma_k}(Y, \lambda)^2}{I(\sigma_k)} = \text{vol}(Y, \lambda).$$

Note that the validity of (8.2) does not depend on the choice of absolute \mathbb{Z} -grading. Cliff Taubes has suggested to me that it may be possible to prove Conjecture 8.7 using the spectral flow estimates involved in the proof of (2.1).

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