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MODULI SPACES OF POLARIZED SYMPLECTIC O'GRADY VARIETIES AND BORCHERDS PRODUCTS

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Abstract

We study moduli spaces of O'Grady's ten-dimensional irreducible symplectic manifolds. These moduli spaces are covers of modular varieties of dimension 21, namely quotients of hermitian symmetric domains by a suitable arithmetic group. The interesting and new aspect of this case is that the group in question is strictly bigger than the stable orthogonal group. This makes it different from both the K3 and the $\mathrm{K3}^{[n]}$ case, which are of dimension 19 and 20 respectively.

0. Introduction

Irreducible symplectic manifolds are simply connected compact Kähler manifolds which have a (up to scalar) unique 2-form, which is non-degenerate. In dimension two these are the K3 surfaces. In higher dimension there are, so far, four known classes of examples. These are deformations of degree n Hilbert schemes of K3 surfaces (the $\mathrm{K3}^{[n]}$ case), deformations of generalized Kummer varieties, and two examples of dimensions 6 and 10 due to O'Grady ([**OG2**], [**OG1**]).

From the point of view of the Beauville lattice these examples fall into two series. The first consists of K3 surfaces, the $\mathrm{K3}^{[n]}$ case, and O'Grady's example of dimension 10. The Beauville lattices are the unimodular K3-lattice $L_{\mathrm{K3}} = 3U \oplus 2E_8(-1)$, the lattice $L_{\mathrm{K3}} \oplus \langle -2(n-1)\rangle$, and $L_{\mathrm{K3}} \oplus A_2(-1)$. The moduli spaces of polarized irreducible symplectic manifolds of these classes are of dimensions 19, 20, and 21. The second series consists of generalized Kummer varieties and O'Grady's 6-dimensional varieties with Beauville lattices $3U \oplus \langle -2(n+1)\rangle$ and $3U \oplus \langle -2\rangle \oplus \langle -2\rangle$ respectively. Here the dimensions of the moduli spaces of polarized varieties are 4 and 5.

In order to describe moduli spaces of irreducible symplectic manifolds, one must first classify the possible types of the polarization. We do this in Section 3 for O'Grady's 10-dimensional example. As in the $\mathrm{K3}^{[n]}$ case, we find that we have a split and a non-split type. In this paper we

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shall mostly concentrate on the split case, when the modular group is maximal possible, but we shall also comment on the low degree non-split cases.

In the non-split case we expect Kodaira dimension $-\infty$ for the three cases of lowest Beauville degree, namely $2d = 12,\ 30,\ 48$. For the next case of Beauville degree $2d = 66$ we prove general type: see Corollary 4.3. The arguments used also suggest that $2d = 12$, 30, 48 might be the only degrees of non-split polarizations giving unirational moduli spaces.

We should like to comment that there is a natural series consisting of moduli of K3 surfaces of degree 2 (double planes branched along a sextic curve), the non-split $\mathrm{K3}^{[2]}$ case of Beauville degree $2d = 6$ (corresponding to cubic fourfolds and treated by Voisin in [**Vo**]), and O'Grady's example of dimension 10 with a non-split polarization of degree 12. The lattices which are orthogonal to the polarization vector in this series are $2U \oplus 2E_8(-1) \oplus A_n(-1)$ for $n = 1,\ 2,\ 3$. It would be very interesting to find a projective geometric realization of O'Grady's 10-dimensional irreducible symplectic manifolds with non-split Beauville degree 12.

In the split case we prove that the modular variety is of general type for most degrees using the method of constructing low weight cusp forms, as in the case of K3 surfaces. The existence of such a modular form proves that the modular variety is of general type, provided the form vanishes along the branch divisors. We construct these modular forms by using quasi pull-backs of Borcherds' form Φ_{12}. There is, however, one important difference between the split case for O'Grady varieties and the previous cases of K3 surfaces [**GHS1**] and the irreducible symplectic manifolds of $\mathrm{K3}^{[n]}$-type [**GHS5**]. The modular group is now no longer a subgroup of the stable orthogonal group: in fact, it is a degree 2 extension related to the root system G_2 (see Theorem 3.1 and (4) below). This fact changes considerably the geometry of the corresponding modular varieties. It makes the case of the O'Grady varieties with a split polarization very interesting. We modify the original method of [**GHS1**] and [**GHS5**] by considering involutions of the Dynkin diagrams and use this to prove results for the split polarization case (Sections 4–5). Here we make strong use of the classification of lattices of small rank and determinant (see Conway and Sloane [**CS**]).

The case of Beauville degree $2d = 2^n$ is exceptional because of very special relations between the root systems E_6 and F_4. We cannot obtain any results about the birational type of these modular varieties. However, if we take the double cover given by the stable orthogonal group, we can prove general type with the only exceptions the split polarizations $2d = 2,\ 4,\ 8$.

The geometry of roots is very special in this case and quite different from the K3 and the $K3^{[n]}$ case. Because of some very special coincidences we require no explicit Siegel type formulae for the representation of an integer by a lattice, nor do we have to enlist the help of a computer.

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1. Irreducible symplectic manifolds and moduli

We first recall the following.

Definition 1.1. *A complex manifold X is called an* irreducible symplectic manifold *or* hyperkähler manifold *if the following conditions are fulfilled:*

(i) *X is a compact Kähler manifold;*
(ii) *X is simply-connected;*
(iii) *$H^0(X, \Omega_X^2) \cong \mathbb{C}\sigma$ where σ is an everywhere non-degenerate holomorphic 2-form.*

It follows from the definition that X has even complex dimension, $\dim_{\mathbb{C}}(X) = 2n$, and that the canonical bundle ω_X is trivial (a trivializing section is given by σ^n). Moreover, the irregularity $q(X) = h^1(X, \mathcal{O}_X) = 0$. Irreducible symplectic manifolds are, together with Calabi-Yau manifolds and abelian varieties, one of the building blocks of compact Kähler manifolds with trivial canonical bundle (complex Ricci-flat manifolds). In dimension 2 the irreducible symplectic manifolds are the K3 surfaces. So far only four deformation types of such manifolds have been found. These are (deformations of) Hilbert schemes of points on K3 surfaces (also called irreducible symplectic manifolds of $K3^{[n]}$-type), (deformations of) generalized Kummer varieties, and two types of examples constructed by O'Grady (see [**OG1**], [**OG2**]).

For a K3 surface S the intersection form defines a non-degenerate, symmetric bilinear form on the second cohomology $H^2(S, \mathbb{Z})$, giving this cohomology group the structure of a lattice. More precisely,

$$H^2(S, \mathbb{Z}) \cong 3U \oplus 2E_8(-1) = L_{K3}$$

where U is the hyperbolic plane and $E_8(-1)$ is the unique even, negative definite unimodular lattice of rank 8. Similarly, one can also define a lattice structure on $H^2(X, \mathbb{Z})$ for all irreducible symplectic manifolds X, called the *Beauville lattice*. The easiest way to define this is the following. There exists a positive constant c, the *Fujiki constant*, such that the quadratic form q on $H^2(X,\mathbb{Z})$ defined by $(\alpha)^{2n} = cq(\alpha)^n$ determines a primitive integral bilinear form. This form has signature $(3, b_2(X) - 3)$.

Let L be an abstract lattice isomorphic to the Beauville lattice of an irreducible symplectic manifold. This defines a period domain

$$\Omega = \{[x] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (x,x) = 0,\ (x,\bar{x}) > 0\}.$$

Given a marking on an irreducible symplectic manifold, i.e. an isometry $\phi\colon H^2(X,\mathbb{Z}) \xrightarrow{\sim} L$, one can define the period point of X as the point in Ω defined by the line $\phi_{\mathbb{C}}(H^{2,0}(X))$. As in the K3 case, irreducible symplectic manifolds are unobstructed and local Torelli holds: that is, the period map of the Kuranishi family is a local isomorphism (see [**Be**]). Moreover, Huybrechts [**Huy**] proved surjectivity of the period map.

We are interested in moduli of polarized irreducible symplectic manifolds. By a polarization we mean a primitive ample line bundle $\mathcal{L}$ on X and we call $h = c_1(\mathcal{L}) \in H^2(X,\mathbb{Z})$ the polarization vector. Since $\mathcal{L}$ is ample, the Beauville degree $q(h)$ is strictly positive. Note that the geometric degree of the polarization is $cq(h)^n$.

In order to discuss moduli spaces of polarized irreducible symplectic varieties, one has to fix discrete data. These are firstly the Beauville lattice and the Fujiki invariant (which together determine the so-called numerical type of an irreducible symplectic manifold) and secondly the type of the polarization. Since the Beauville lattice L of an irreducible symplectic manifold is, in general, not unimodular, we cannot expect that any two polarization vectors of the same degree are equivalent under the orthogonal group $\mathrm{O}(L)$. (The case of K3 surfaces is an exception, since the K3 lattice is unimodular.) In general there will be several, but finitely many, $\mathrm{O}(L)$-orbits of such vectors. We call the choice of such an orbit the choice of a *polarization type*. Given a polarization type, we fix a representative $h \in L$ of it and consider the lattice $L_h = h^{\perp}_L$, which has signature $(2, b_2(X) - 3)$, and defines a homogeneous domain

$$\Omega_h = \Omega(L_h) = \{[x] \in \mathbb{P}(L_h \otimes \mathbb{C}) \mid (x,x) = 0,\ (x,\bar{x}) > 0\}.$$

This is a type IV bounded symmetric hermitian domain. It is of dimension $b_2(X) - 3$ and has two connected components

$$\Omega(L_h) = \mathcal{D}(L_h) \coprod \mathcal{D}(L_h)'.$$

The orthogonal group $\mathrm{O}(L_h)$ of the lattice L_h has an index 2 subgroup $\mathrm{O}^+(L_h)$ that fixes the components $\mathcal{D}(L_h)$ and $\mathcal{D}(L_h)'$. We also need the group

$$\mathrm{O}(L,h) = \{g \in \mathrm{O}(L) \mid g(h) = h\}. \tag{1}$$

Since this group maps the orthogonal complement L_h to itself, we can consider it as a subgroup of $\mathrm{O}(L_h)$. Let $\mathrm{O}^+(L,h) = \mathrm{O}(L,h) \cap \mathrm{O}^+(L_h)$.

Let $\mathcal{M}_h$ be the moduli space of polarized irreducible symplectic manifolds $(X,\mathcal{L})$ where X has numerical data as chosen above and where $\mathcal{L}$

is a primitive ample line bundle such that $c_1(\mathcal{L})$ is of the given polarization type. This moduli space exists by Viehweg's general theory as a quasi-projective variety. We do not know how many components $\mathcal{M}_h$ has, but Proposition 1.2 below allows us to work with each component separately.

Proposition 1.2. *Every component $\mathcal{M}_h^0$ of the moduli space $\mathcal{M}_h$ admits a dominant finite-to-one morphism*

$$\varphi\colon \mathcal{M}_h^0 \to \mathrm{O}^+(L,h)\backslash\mathcal{D}(L_h).$$

Proof. See [**GHS5**, Theorem 1.5]. q.e.d.

This is the starting point of our investigations. The importance of this result is that if the quotient $\mathrm{O}^+(L,h)\backslash\mathcal{D}(L_h)$ is of general type, then so is $\mathcal{M}_h^0$. We shall use this in Sections 4 and 5 to prove the main result of this paper.

For some irreducible symplectic manifolds, such as irreducible symplectic manifolds of $\mathrm{K3}^{[n]}$-type, the situation can be improved by introducing the group $\mathrm{Mon}^2(X) \subset \mathrm{O}(H^2(X,\mathbb{Z}))$, which is the group generated by the monodromy group operators acting on the second cohomology. This group was studied intensively by Markman ([**Mar1**], [**Mar2**], [**Mar3**]). If it is a normal subgroup, then it defines a subgroup $\mathrm{Mon}^2(L) \subset \mathrm{O}(L)$. One can then show (the proof of [**GHS5**, Theorem 2.3] for the $\mathrm{K3}^{[n]}$-type goes through unchanged) that one can factor the map φ from Proposition 1.2 as follows:

$$\begin{array}{ccc} \mathcal{M}_h^0 & \xrightarrow{\tilde{\varphi}} & (\mathrm{Mon}^2(L) \cap \mathrm{O}^+(L,h))\backslash\mathcal{D}(L_h) \\ & \searrow^{\varphi} & \downarrow \\ & & \mathrm{O}^+(L,h)\backslash\mathcal{D}(L_h). \end{array} \tag{2}$$

2. O'Grady's 10-dimensional example

O'Grady constructed his 10-dimensional irreducible symplectic manifolds using moduli spaces of sheaves on K3 surfaces. More precisely, let S be an algebraic K3 surface and consider the rank 2 sheaves $\mathcal{F}$ on S with trivial first Chern class $c_1(\mathcal{F}) = 0$ and second Chern class $c_2(\mathcal{F}) = 4$. Let H be a sufficiently general polarization, i.e. a polarization such that there is no non-trivial divisor class C with $C.H = 0$ and $C^2 \geq -4$. It is easy to find examples: in particular, every projective K3 surface with Picard number 1 has such a polarization. Let $\mathcal{M}_4$ be the moduli space of H-semistable sheaves. This is a singular variety whose smooth part carries a symplectic structure. The singularities occur at the semistable sheaves and these are sums of ideal sheaves $\mathcal{I}_Z \oplus \mathcal{I}_W$ where Z and W are 0-dimensional subschemes of S of length 2. O'Grady then considers Kirwan's desingularization $\widehat{\mathcal{M}}_4$ which has a canonical form vanishing on an irreducible divisor. He shows that this divisor is a $\mathbb{P}^2$-bundle whose normal bundle has degree -1 on each $\mathbb{P}^2$. Hence it can be contracted and the resulting 4-fold $\widetilde{\mathcal{M}}_4$ is O'Grady's irreducible symplectic manifold of dimension 10. It has second Betti number $b_2 = 24$. This also shows that these varieties have 22 deformation parameters and hence there are deformations of $\widetilde{\mathcal{M}}_4$ which do not arise from deformations of the underlying K3 surface.

In the case of O'Grady's 10-dimensional examples, the Beauville lattice is (as an abstract lattice) of the form:

$$L = 3U \oplus 2E_8(-1) \oplus A_2(-1)$$

where $A_2(-1)$ is the negative definite root lattice associated to A_2. The Fujiki invariant of O'Grady's 10-dimensional example is $c = 945$. This was shown by Rapagnetta [**Ra**]. Since the second cohomology of K3 surfaces is of the form $L = 3U \oplus 2E_8(-1)$ and the Beauville lattices of irreducible symplectic manifolds of $\mathrm{K3}^{[n]}$-type are of the form $L = 3U \oplus 2E_8(-1) \oplus \langle -2(n-1)\rangle$, one can see O'Grady's 10-dimensional example as the third type in a series. We previously treated the case of K3 surfaces in [**GHS1**] and the case of polarized varieties of $\mathrm{K3}^{[n]}$-type in [**GHS5**], where we restricted ourselves to the case of split polarizations (see [**GHS5**, Example 3.8] for a definition and details).

In the 10-dimensional case, the situation with respect to the monodromy group is as follows. Let $\mathrm{O}^{\mathrm{or}}(L)$ be the group of oriented orthogonal transformations of L (see [**Mar1**, Section 4.1] and in particular Remark 4.3 for a definition of oriented orthogonal transformations). A conjecture of Markman [**Mar3**, Conjecture 10.7] predicts that $\mathrm{Mon}^2(L) = \mathrm{O}^{\mathrm{or}}(L)$.

Since $\mathrm{O}(L,h) \cap \mathrm{O}^{\mathrm{or}}(L) = \mathrm{O}^+(L,h)$, the factorization (2) does not, unlike in some cases of $\mathrm{K3}^{[n]}$-type, improve the situation. In view of Verbitsky's results [**Ve**] we conjecture that the map $\varphi\colon \mathcal{M}_h^0 \to \mathrm{O}^+(L,h)\backslash\mathcal{D}(L_h)$ from Proposition 1.2 is indeed an open embedding.

There are two differences between the cases treated previously and this case. Firstly, the arithmetic group in question is no longer necessarily a subgroup of the stable orthogonal group (see Section 3). Secondly, the discriminant group of the lattices orthogonal to a polarization vector is no longer cyclic. This requires new considerations concerning the quasi pull-backs of the Borcherds form. We would also like to point out that the lattice theoretic part of this case is very different from the previous papers. The geometry of roots is very special here, and as a result we need neither arguments from analytic number theory nor any kind of Siegel formulae. The root geometry arguments in this paper are all elementary, but they are far from trivial.

3. The modular orthogonal group and the root system G_2

In this section we determine the modular group associated to the moduli spaces of polarized O'Grady varieties (see Theorem 3.1 below). A polarization corresponds to a primitive vector h with $h^2 = 2d > 0$ in

$$L_A = 3U \oplus 2E_8(-1) \oplus A_2(-1). \tag{3}$$

For any even lattice L, we denote the discriminant group of L by $D(L) = L^\vee/L$ where $L^\vee$ is the dual lattice of L. The discriminant group carries a discriminant quadratic form q_L (if L is even) with values in $\mathbb{Q}/2\mathbb{Z}$. The orthogonal group of the finite discriminant form is denoted by $\mathrm{O}(D(L))$. If $g \in \mathrm{O}(L)$, we denote by $\bar{g}$ its image in $\mathrm{O}(D(L))$. The *stable orthogonal group* $\widetilde{\mathrm{O}}(L)$ is defined by

$$\widetilde{\mathrm{O}}(L) = \ker(\mathrm{O}(L) \to \mathrm{O}(D(L))).$$

If $h \in L$, its *divisor* $\mathrm{div}(h)$ is the positive generator of the ideal $(h, L) \subset \mathbb{Z}$. Therefore $h^* = h/\mathrm{div}(h)$ is a primitive element of the dual lattice $L^\vee$ and $\mathrm{div}(h)$ is a divisor of $\det(L)$.

For the lattice L_A of (3), $D(L_A) \cong D(A_2(-1)) = \langle \bar{c} \rangle$ is the cyclic group of order 3 and $q_{L_A}(\bar{c}) = \frac{2}{3} \mod 2\mathbb{Z}$.

For any $h \in L_A$ with $h^2 > 0$ and $L_h = h^\perp_{L_A}$, we determine the structure of the modular group $\mathrm{O}^+(L_A, h) = \mathrm{O}(L_A, h) \cap \mathrm{O}^+(L_h)$ (see (1) and (2)). We have $\det(L_A) = 3$, so $\mathrm{div}(h)$ divides $(2d, 3)$.

Theorem 3.1. *Let $h \in L_A$ be a primitive vector of length $h^2 = 2d > 0$. The orthogonal complement $L_h = h^\perp_{L_A}$ is of signature $(2, 21)$. If $\mathrm{div}(h) = 3$, then*

$$L_h \cong L_Q = 2U \oplus 2E_8(-1) \oplus Q(-1),$$

where $Q(-1)$ is a negative definite even integral ternary quadratic form of determinant $-2d/3$. Its discriminant group $D(Q(-1)) \cong D(L_h)$ is cyclic of order $2d/3$ and

$$\mathrm{O}^+(L_A, h) \cong \widetilde{\mathrm{O}}^+(L_h).$$

If $\mathrm{div}(h) = 1$, then $L_h \cong L_{A,2d}$ where

$$L_{A,2d} = 2U \oplus 2E_8(-1) \oplus A_2(-1) \oplus \langle -2d \rangle,$$
$$D(L_h) \cong D(A_2(-1)) \oplus D(\langle -2d \rangle),$$

and

$$\mathrm{O}^+(L_A, h) \cong \mathrm{O}_G(L_{A,2d}) = \{g \in \mathrm{O}^+(L_{A,2d}) \mid \bar{g}|_{D(\langle -2d \rangle)} = \mathrm{id}\}.$$

Any totally isotropic subgroup of $D(A_2(-1)) \oplus D(\langle -2d \rangle)$ is cyclic.

The proof of Theorem 3.1, which is the main result of this section, will be given below, after the proof of Lemma 3.4. Before then, we make some comments and collect some preliminary results.

A polarization determined by a primitive vector h with $\operatorname{div}(h) = 1$ is called *split*. We note that if $(3, d) = 1$ (where $h^2 = 2d$), then the polarization is always split. If $3|d$, then the polarization is split if and only if the discriminant group of L_h is not cyclic. In the split case, the modular group $\mathrm{O}_G(L_{A,2d})$ is larger than the stable orthogonal group $\widetilde{\mathrm{O}}^+(L_h)$ because the elements of $\mathrm{O}_G(L_{A,2d})$ induce trivial action only on the second component of the discriminant group $D(L_h) \cong D(A_2(-1)) \oplus D(\langle -2d \rangle)$.

We recall that

$$[\mathrm{O}(A_2) : W(A_2)] = 2$$

where $\mathrm{O}(A_2)$ is the orthogonal group of the lattice A_2 and $W(A_2)$ is the Weyl group generated by reflections with respect to the roots of A_2. The group $\mathrm{O}(A_2)$ also contains reflections with respect to the vectors of square 6. The 2- and 6-roots of the lattice A_2 form together the root system G_2 and $\mathrm{O}(A_2) = W(G_2)$ (see [**Bou**]).

For any vector $l \in L_h$ with $l^2 < 0$, the reflection σ_l with respect to l belongs to $\mathrm{O}^+(L_h \otimes \mathbb{R})$. In particular, $\mathrm{O}(A_2(-1)) = W(G_2(-1))$ is a subgroup of $\mathrm{O}^+(L, h)$. Therefore

$$\mathrm{O}_G(L_{A,2d})/\widetilde{\mathrm{O}}^+(L_h) \cong W(G_2(-1))/W(A_2(-1)) \cong \mathbb{Z}/2\mathbb{Z}. \tag{4}$$

We note that in the case of polarized K3 surfaces or of polarized symplectic manifolds of $\mathrm{K3}^{[n]}$-type, the modular group of the corresponding modular varieties is identical to a stable orthogonal group (see [**GHS5**]). The degree 2 extension of the stable orthogonal group changes the geometry of the modular varieties considerably. This can be compared to the case of the moduli spaces of $(1, p)$-polarized abelian and Kummer surfaces (see [**GH**]).

Theorem 3.1 shows the difference between split and non-split polarizations. To prove it, we study the orbits of vectors in L. Using the standard discriminant group arguments (see [**Nik**] and the proof of Proposition 3.6 in [**GHS1**]), we get

Lemma 3.2. *Let L be any non-degenerate even integral lattice and let $h \in L_A$ be a primitive vector with $h^2 = 2d > 0$. If L_h is the orthogonal complement of h in L_A, then*

$$\det L_h = \frac{(2d) \cdot \det L_A}{\operatorname{div}(h)^2}.$$

A proof of the following classical result, known as the *Eichler criterion*, is given in [**GHS4**, Proposition 3.3].

Lemma 3.3. *Let L be a lattice containing two orthogonal isotropic planes. Then the $\widetilde{\mathrm{O}}(L)$-orbit of a primitive vector $l \in L$ is determined by two invariants: its length $l^2 = (l, l)$, and its image $l^* + L$ in the discriminant group $D(L)$.*

According to this Lemma 3.3, all primitive $2d$-vectors $l \in L_A$ with $\operatorname{div}(l) = 1$ belong to the same $\widetilde{\mathrm{O}}(L_A)$-orbit. If $\operatorname{div}(l) = 3$, then $l^* + L_A$ is a generator of $D(L_A) = D(A_2(-1))$. Therefore there are two $\widetilde{\mathrm{O}}(L_A)$-orbits of such vectors. A suitable element of $W(G_2(-1))$ makes these two $\widetilde{\mathrm{O}}(L_A)$-orbits into one $\mathrm{O}(L_A)$-orbit.

Lemma 3.4. *If h_{2d} is a vector of a non-split polarization with $h_{2d}^2 = 2d$, then $2d \equiv 12 \mod 18$. For any positive even integer $2d$ satisfying this congruence, there exists a primitive $h_{2d} \in L_A$ with* $\operatorname{div}(h_{2d}) = 3$.

Proof. We put $h_{2d} = u + xa + yb \in L_A$, where $u \in 3U \oplus 2E_8(-1)$ and $xa + yb \in A_2(-1) = \langle a, b\rangle$, where a, b are simple roots of $A_2(-1)$. Any primitive vector of a unimodular lattice has divisor 1. Therefore $u = 3v$ with $v \in 3U \oplus 2E_8(-1)$. A straightforward calculation shows that $\operatorname{div}(xa + yb)$ is divisible by 3 if and only if $x + y \equiv 0 \mod 3$. We have $x \equiv \pm 1 \mod 3$ and $y \equiv \mp 1 \mod 3$ since h_{2d} is primitive. Therefore

$$h_{2d}^2 = 9v^2 - 2(x+y)^2 + 6xy \equiv 12 \mod 18.$$

To construct a polarization vector of degree $18n - 6$, we take a vector $h = 3nu_1 + 3u_2 + (2a + b)$ where $U = \langle u_1, u_2\rangle$ is the first hyperbolic plane in L_A. q.e.d.

Proof of Theorem 3.1. Now we can calculate L_h. If the polarization is non-split, we take the vector $h_{2d} \in U \oplus A_2(-1)$ indicated above. We denote by $Q(-1)$ the orthogonal complement of h_{2d} in $U \oplus A_2(-1)$. According to Lemma 3.2, it is an even integral negative definite lattice of rank 3 and of determinant $-2d/3$, i.e.

$$L_h \cong 2U \oplus 2E_8(-1) \oplus Q(-1), \qquad \det Q(-1) = -\frac{2d}{3}.$$

To prove that $D(Q)$ is cyclic, we consider

$$\langle h\rangle \oplus L_h \subset L_A \subset L_A^\vee \subset \langle \frac{1}{2d} h\rangle \oplus L_h^\vee.$$

The lattice L_A defines the finite subgroup

$$H = L_A/(\langle h\rangle \oplus L_h) < D(\langle h\rangle) \oplus D(L_h).$$

We have $|H| = \det L_h = 2d/3$ because $H \cong (\langle \frac{1}{2d} h\rangle \oplus L_h^\vee)/L_A^\vee$. The projections

$$p_h \colon H \to D(\langle h\rangle), \qquad p_{L_h} \colon H \to D(L_h) \tag{5}$$

are injective because $\langle h\rangle$ and L_h are primitive in L_A (see [**Nik**, Prop. 1.5.1]). Therefore $H \cong D(L_h)$ and H is isomorphic to a subgroup of the cyclic group $D(\langle h\rangle)$.

To determine $\mathrm{O}(L_A, h)$, we consider the action of elements of this group on the discriminant group. Any $g \in \mathrm{O}(L_A, h)$ acts on $\langle h\rangle^\vee \oplus$

$L_h^\vee$ and induces an element $\bar{g} \in \mathrm{O}(D(L_A))$. Moreover, $\bar{g}$ acts on the subgroup H. For any $\bar{a} \in p_h(H)$, there exists a unique $\bar{b} \in p_{L_h}(H)$ such that $\bar{a}+\bar{b} \in H$. The action of $\bar{g}$ on $D(\langle h\rangle)$ is trivial. Therefore it is also trivial on the second component $\bar{b} \in p_{L_h}(H)$. But $p_{L_h}(H)$ is isomorphic to the whole group $D(L_h)$ if $\mathrm{div}(h) = 3$. Therefore $\mathrm{O}(L_A, h) \cong \widetilde{\mathrm{O}}(L_h)$. This proves the statement of Theorem 3.1 in the non-split case.

For a split polarization, we can take $h_{2d} = du_1 + u_2 \in U$. Then $(h_{2d})^\perp_U \cong \langle -2d\rangle$ and

$$L_h \cong 2U \oplus 2E_8(-1) \oplus A_2(-1) \oplus \langle -2d\rangle.$$

We may define H and the associated maps as in the non-split case: now $|H| = 2d$, and $p_{L_h}(H) \cong D(\langle -2d\rangle)$, and $\bar{g}$ acts trivially on $D(\langle -2d\rangle)$.

To finish the proof of Theorem 3.1, we analyse the isotropic elements of the discriminant group $D(A_2(-1)) \oplus D(\langle -2d\rangle)$ of the lattice L_h in the split case. If $(3, d) = 1$, then the latter group is cyclic. So we assume that $3|d$.

Let $\bar{l} = (\pm\bar{c}, \frac{x}{2d}\bar{h})$ where $\bar{c}$ is a generator of $D(A_2(-1))$ and x is taken modulo $2d$. We put $d = 3d_0 = 3ef^2$ where e is square free. It is easy to see that $\bar{l}$ is isotropic if and only if $x = 2yef$, where y is taken modulo $3f$, and $1 + ey^2 \equiv 0 \mod 3$. The element $\bar{l}$ is isotropic if and only if

$$\frac{2}{3} + \frac{x^2}{2d} \equiv 0 \mod 2d.$$

Then

$$4d + 3x^2 \equiv 0 \mod 12d \quad \text{or} \quad 12ef^2 + 3x^2 \equiv 0 \mod 36ef^2.$$

We see that $x = 2x_0$ and $ef^2 + x_0^2 \equiv 0 \mod 3ef^2$. Therefore $x_0 \equiv 0 \mod ef$ and $x = 2x_0 = 2efy$ where y is taken modulo $3f$ and

$$1 + ey^2 \equiv 0 \mod 3.$$

The last congruence is true if and only if

$$e \equiv 2 \mod 3 \quad \text{and} \quad y \not\equiv 0 \mod 3.$$

We proved that for $d = 3ef^2$ the isotropic elements with non-trivial first component are $(\pm\bar{c},\ \frac{y}{3f}\bar{h})$. All these elements belong to the union of two totally isotropic cyclic groups generated by $(\bar{c},\ (\bar{h}/3f))$ and by $(\bar{c},\ -(\bar{h}/3f))$. If a subgroup of the discriminant group contains two isotropic elements $(\bar{c}, y_i(\bar{h}/3f))$, where $y_1 \not\equiv y_2 \mod 3$, then $(\bar{0}, (y_1 - y_2)(\bar{h}/3f))$ is not isotropic because

$$\frac{6ef^2(y_1-y_2)^2}{9f^2} = \frac{2e(y_1-y_2)^2}{3} \not\equiv 0 \mod 2\mathbb{Z}.$$

Thus Theorem 3.1 is proved.

Example 1. *The smallest non-split polarizations* 12, 30, 48, 66. In the non-split case, the isomorphism class of the lattice L_h with $h^2 = 2d$ is uniquely defined by the genus of the ternary form Q of determinant $2d/3$. For the small polarizations of this example, the genus of Q contains only one class. The corresponding classes can be found in [**CS**, Table I]. We give a modified description of them using the language of root lattices, indicating the maximal root subsystem in the lattices Q and $Q^{\perp}_{E_8}$:

(6) $\det Q = 4,\quad Q = A_3,\qquad Q^{\perp}_{E_8} \cong D_5,$

(7) $\det Q = 10,\quad Q = (A_1)^{\perp}_{A_4},\qquad Q^{\perp}_{E_8} \cong A_1 \oplus A_4,$

(8) $\det Q = 16,\quad Q \supset A_2 \oplus \langle 48\rangle,\quad Q^{\perp}_{E_8} \supset A_4 \oplus \langle 48\rangle,$

(9) $\det Q = 22,\quad Q \supset A_2 \oplus \langle 66\rangle,\quad Q^{\perp}_{E_8} \supset A_3 \oplus A_1 \oplus \langle 44\rangle.$

4. Cusp forms of small weight and the Borcherds form Φ_{12}

Now we can formulate the main theorem of the paper.

Theorem 4.1. *Let d be a positive integer not equal to 2^n with $n \geq 0$. Then the modular variety*

$$M_{A,2d} = \mathrm{O}_G(L_{A,2d}) \setminus \mathcal{D}(L_{A,2d})$$

is of general type. Every component $\mathcal{M}_h^0$ of the moduli space $\mathcal{M}_h$ of ten-dimensional polarized O'Grady varieties with split polarization h of Beauville degree $h^2 = 2d \neq 2^{n+1}$ is of general type.

Remark. In Corollary 4.3 below, we prove general type of the moduli spaces $\mathcal{M}_h^0$ for the fourth non-split polarization, of Beauville degree 66 (see Example 1 of Section 3).

According to Proposition 1.2, it is enough to prove the main Theorem 4.1 for the modular varieties

$$M_{A,2d} = \mathrm{O}_G(L_{A,2d}) \setminus \mathcal{D}(L_{A,2d}) \quad \text{or} \quad M_Q^{(2d)} = \widetilde{\mathrm{O}}^+(L_Q) \setminus \mathcal{D}(L_Q)$$

(see notations of Theorem 3.1). The dimension of the modular variety $M_{A,2d}$ is 21, which is larger than 8. Therefore we can use the *low weight cusp form trick* from [**GHS1**].

Let L be an even integral lattice of signature $(2, n)$ with $n \geq 3$. A modular form of weight k and character det with respect to a subgroup $\Gamma < \mathrm{O}^+(L)$ of finite index is a holomorphic function $F\colon \mathcal{D}(L)^\bullet \to \mathbb{C}$ on the affine cone $\mathcal{D}(L)^\bullet$ over $\mathcal{D}(L)$ such that

$$F(tZ) = t^{-k}F(Z) \quad \forall\, t \in \mathbb{C}^* \qquad \text{and} \qquad F(gZ) = \det(g)F(Z) \quad \forall\, g \in \Gamma.$$

A modular form is a cusp form if it vanishes at every cusp. Cusp forms of character det vanish to integral order at any cusp (see [**GHS4**]). We denote the linear spaces of modular and cusp forms of weight k and character det for Γ by $M_k(\Gamma, \det)$ and $S_k(\Gamma, \det)$ respectively.

Theorem 4.2. *The modular variety $M_{A,2d}$ (or the modular variety $M_Q^{(2d)}$) is of general type if there exists a cusp form $F \in S_k(\mathrm{O}_G(L_{A,2d}),$*

det) *(or* $F \in S_k(\widetilde{\mathrm{O}}^+(L_Q), \det)$*) of weight* $k < 21$ *that vanishes of order at least one along the branch divisor of the modular projection*

$$\pi\colon \mathcal{D}(L_{A,2d}) \to \mathrm{O}_G(L_{A,2d}) \setminus \mathcal{D}(L_{A,2d})$$

(or the analogous projection for $\widetilde{\mathrm{O}}^+(L_Q)$*).*

This is a particular case of Theorem 1.1 in [**GHS1**].

The dimension of the modular variety is smaller than 26. Then we can use the *quasi pull-back* (see [**Bo**], [**BKPS**], [**Ko**], [**GHS1**] and equation (10) below) of the Borcherds modular form

$$\Phi_{12} \in M_{12}(\mathrm{O}^+(II_{2,26}), \det) \quad \text{where} \quad II_{2,26} \cong 2U \oplus 3E_8(-1).$$

We note that $\Phi_{12}(Z) = 0$ if and only if there exists $r \in II_{2,26}$ with $r^2 = -2$ such that $(r, Z) = 0$. Moreover, the multiplicity of the divisor of zeroes of Φ_{12} is 1 (see [**Bo**]). We used the quasi pull-back of Φ_{12} in order to construct cusp forms of small weight on the moduli spaces of polarized K3 surfaces (see [**GHS1**]) and on moduli spaces of split-polarized symplectic manifolds of $\mathrm{K3}^{[2]}$-type (see [**GHS5**]), which have dimension 19 and 20 respectively. The present case is of dimension 21. The non-split case is similar to the cases considered in [**GHS1**] and [**GHS5**] (see also the example at the end of this section) but the split case is different from the previous ones because we need a cusp form with respect to the modular group $\mathrm{O}_G(L_{A,2d})$, which is strictly larger than the stable orthogonal group $\widetilde{\mathrm{O}}^+(L_{A,2d})$. For this reason we will concentrate in this paper on the split case.

Let $S \subset E_8(-1)$ be a sublattice (primitive or not) of rank 3. For our present purpose we take the sublattice of polarizations $S = A_2(-1) \oplus \langle -2d \rangle$ or $S = Q(-1)$ from Theorem 3.1. The choice of S in $E_8(-1)$ determines an embedding of $L_S = 2U \oplus 2E_8(-1) \oplus S$ into $II_{2,26}$. The embedding of the lattice also gives us an embedding of the domain $\mathcal{D}(L_S) \subset \mathbb{P}(L_S \otimes \mathbb{C})$ into $\mathcal{D}(II_{2,26}) \subset \mathbb{P}(II_{2,26} \otimes \mathbb{C})$.

We put $R_S = \{r \in E_8(-1) \mid r^2 = -2,\ (r, S) = 0\}$, and $N_S = \# R_S$. Then the quasi pull-back of Φ_{12} is given by the following formula:

$$F_S(Z_S) = \left.\frac{\Phi_{12}(Z)}{\prod_{\{r \in R_S,\ r>0\}} (Z, r)}\right|_{\mathcal{D}(L_S)} \in M_{12+\frac{N_S}{2}}(\widetilde{\mathrm{O}}^+(L_S), \det), \tag{10}$$

where Z_S is the restriction of Z to $\mathcal{D}(L_S)$.

We fix a system of simple positive roots in $E_8(-1)$, and the notation $r > 0$ in the above formula means that we take the positive roots in R_S, i.e. we pick only one root in any $A_1 \subset R_S$. (The particular choice of a system of simple roots is not important.) The form F_S is a non-zero modular form of weight $12 + \frac{N_S}{2}$. By [**GHS1**, Theorems 6.2 and 4.2] it is a cusp form if $N_S \neq 0$, since any isotropic subgroup of the discriminant form of the lattice L_S is cyclic, by Theorem 3.1.

Example 2. *The smallest non-split polarizations.* We illustrate the method of Theorem 4.2 together with the quasi pull-back construction for the polarizations from Example 1 of Section 3. From the descriptions of $Q^\perp_{E_8}$ in (6) above, for the first three polarizations the cusp form F_Q is of weight 32, 23, and 22 respectively. But for the lattice Q of determinant 22, with $h^2 = 66$, there are 14 roots in $Q^\perp_{E_8}$ (two in A_1 and twelve in A_3) so we have a cusp form of small weight $12 + \frac{14}{2} = 19 < 21$:

$$F_Q^{(22)} \in S_{19}(\widetilde{\mathrm{O}}^+(L_Q), \det).$$

To apply Theorem 4.2 we need a cusp form of small weight with zero along the ramification divisor of the modular projection. According to [**GHS1**, Corollary 2.13] this divisor is determined by plus or minus reflections $\pm\sigma_r$ in the corresponding modular group. If σ_r is a reflection in this group, then $F_Q^{(22)}(\sigma_r(Z)) = -F_Q^{(22)}(Z)$ and $F^{(22)}(Z) = 0$ if $(Z, r) = 0$. If $-\sigma_r \in \widetilde{\mathrm{O}}^+(L_Q)$, then $\det(-\sigma_r) = 1$ because the dimension is odd. The weight of $F_Q^{(22)}$ is also odd, i.e. $F_Q^{(22)}(-Z) = -F_Q^{(22)}(Z)$. Therefore

$$-F_Q^{(22)}(\sigma_r(Z)) = F_Q^{(22)}(-\sigma_r(Z)) = \det(-\sigma_r)F_Q^{(22)}(Z) = F_Q^{(22)}(Z)$$

and $F_Q^{(22)}$ vanishes along the divisor defined by r. Applying Theorem 4.2 we obtain

Corollary 4.3. *The modular variety $M_Q^{(66)}$ is of general type. Every component $\mathcal{M}_h^0$ of the moduli space $\mathcal{M}_h$ of 10-dimensional polarized O'Grady varieties with non-split polarization h of Beauville degree $h^2 = 66$ is of general type.*

Any vector l of length 12, 30, or 48 with $\operatorname{div}(l) = 3$ is orthogonal to at least 20 roots in E_6. Hence we cannot apply the low weight cusp form trick. We conjecture that *for the three lowest non-split polarizations, of Beauville degrees $2d = 12$, 30, and 48, the corresponding moduli spaces are unirational.* Using the arithmetic and analytic methods developed in [**GHS1**] and [**GHS5**] we hope to prove that for other non-split polarizations the moduli spaces are of general type. In this paper we study the split polarization because this case is very different and has new phenomena appearing.

The Weyl group of E_8 acts transitively on the sublattices A_2. Let us fix a copy of $A_2(-1)$ in $E_8(-1)$. Then $(A_2(-1))^\perp_{E_8(-1)} \cong E_6(-1)$. Let $l \in E_6(-1)$ satisfy $l^2 = -2d$. We denote the quasi pull-back F_S for $S = A_2(-1) \oplus \langle l\rangle$ by F_l. The problem is to find such a vector l in $E_6(-1)$ that yields a modular form with respect to the larger group $\mathrm{O}_G(L_{A,2d})$.

Lemma 4.4. *Let us assume that $l \in E_6(-1)$, $l^2 = -2d$, is invariant with respect to the involution of the Dynkin diagram of $E_6(-1)$. Then the quasi pull-back F_l is modular with respect to $\mathrm{O}_G(L_{A,2d})$.*

Proof. We see that $\mathrm{O}_G(L_{A,2d}) = \langle \widetilde{\mathrm{O}}^+(L_{A,2d}), \sigma_6\rangle$ where σ_6 is a reflection with respect to any -6-vector in $A_2(-1)$ (see (4)). The involution $\sigma_6 \in W(G_2(-1))$ induces $-\,\mathrm{id}$ on the first component $D(A_2(-2))$ of the discriminant group $D(L_{A,2d})$. The Weyl group $W(E_6)$ is a subgroup of index 2 in $\mathrm{O}(E_6)$. The involution J of the Dynkin diagram of the fixed system of simple roots of $E_6(-1)$ induces $-\,\mathrm{id}$ on $D(E_6(-1))$, which is also cyclic of order 3. Using the fact that $(A_2)^{\perp}_{E_8} \cong E_6$, we can extend the element $J_6 = (\sigma_6, J)$ to an element in $\mathrm{O}(E_8) < \mathrm{O}^+(II_{2,26})$ where we consider σ_6 as an element in $\mathrm{O}^+(2U \oplus 2E_8(-1) \oplus A_2(-1))$. Let us introduce the coordinates $(Z_1, z_2, Z_3) \in \mathcal{D}(II_{2,26})$ corresponding to the sublattice

$$\big(2U \oplus 2E_8(-1) \oplus A_2(-1)\big) \oplus \langle l\rangle \oplus l^{\perp}_{E_6(-1)} \subset II_{2,26}$$

where $z_2 \in l \otimes \mathbb{C}$ and $Z_3 \in l^{\perp}_{E_6(-1)} \otimes \mathbb{C}$. We calculate the function

$$\frac{\Phi_{12}(J_6(Z_1, z_2, Z_3))}{\prod_{\{r \in R_l,\, r>0\}} (J_6(Z_1, z_2, Z_3), r)}\Bigg|_{\mathcal{D}(L_{A,2d})}$$

where $R_l = \{r \in E_6(-1) \mid r^2 = -2,\ (r, l) = 0\}$ is the set of roots in $E_8(-1)$ orthogonal to $S = A_2(-1) \oplus \langle l\rangle$. First, we find that it is equal to

$$\frac{\Phi_{12}((\sigma_6 Z_1, z_2, J(Z_3)))}{\prod_{\{r \in R_l,\, r>0\}} ((\sigma_6 Z_1, z_2, J(Z_3)), r)}\Bigg|_{\mathcal{D}(L_{A,2d})} = F_l(\sigma_6(Z_1, z_2))$$

because $J(l) = l$ and $J_6(z_2) = z_2$. Here we use the same symbols Z_1 and z_2 both for coordinates on the manifold $\mathcal{D}(II_{2,26})$ and their restriction to the submanifold $\mathcal{D}(L_{A,2d})$. Second, using the fact that Φ_{12} has character det, we find that the same function is equal to

$$\frac{(\det J_6)\Phi_{12}(Z_1, z_2, Z_3)}{\prod_{\{r \in R_l,\, r>0\}} ((Z_1, z_2, Z_3), J(r))}\Bigg|_{\mathcal{D}(L_{A,2d})} = -F_l((Z_1, z_2))$$

because $\det J = 1$, $\det \sigma_6 = -1$, $\det J_6 = -1$, and the involution J permutes the positive roots in $l^{\perp}_{E_6}$. We note also that $(\sigma_6 Z_1, z_2, J(Z_3), r) = (J(Z_3), r)_{E_6} = (Z_3, J(r))_{E_6}$. Therefore

$$F_l \in S_{12+\frac{N_l}{2}}(\mathrm{O}_G(L_{A,2d}), \det) \tag{11}$$

where $N_l = \#\{r \in E_6(-1) \mid r^2 = -2,\ (r, l) = 0\}$. q.e.d.

The weight of F_l is smaller than 21 if $N_l < 18$. In Section 4 we determine all d for which there exists a $(-2d)$-vector in $E_6(-1)$ invariant with respect to the automorphism of the Dynkin diagram. In the next lemma we study the ramification divisor of the modular projection of $\mathrm{O}_G(L_{A,2d})$. We studied this divisor for the modular groups $\widetilde{\mathrm{O}}^+(L)$ in [**GHS1**, Proposition 3.2] but the ramification divisor of $\mathrm{O}_G(L_{A,2d})$ is much larger.

Lemma 4.5. *If $-\sigma_r \in \mathrm{O}_G(L_{A,2d})$, then $r^2 = -2d$ and $\operatorname{div}(r) = 2d$, or $r^2 = -6d$ and $\operatorname{div}(r) = 3d$, or $r^2 = -2d$ and $\operatorname{div}(r) = d$.*

Proof. Let $r \in L_{A,2d}$ be a primitive vector and $r^2 = -2e$. If $\sigma_r : v \mapsto v - \frac{2(v,r)}{(r,r)} r \in \mathrm{O}^+(L_A)$, then

$$\operatorname{div}(r) \mid r^2 \mid 2\operatorname{div}(r) \quad \text{and} \quad \operatorname{div}(r) \mid \operatorname{lcm}(3, 2d).$$

We assume that $-\sigma_r \in \mathrm{O}_G(L_{A,2d})$. Then $\sigma_r|_{D(\langle -2d\rangle)} = -\operatorname{id}$ and for any $v \in L_{A,2d}^\vee$ we have

$$\sigma_r(v) + v = 2v - \frac{2(v,r)}{(r,r)} r = 2v - (v,r)\frac{r}{e} \in A_2(-1)^\vee + L_{A,2d}$$

where $(v,r) \in \mathbb{Z}$. This is true because we have no $D(\langle -2d\rangle)$-part in the sum $\sigma_r(v) + v$. In particular, there are the following relations between abelian groups

$$2 \cdot D(L_{A,2d}) \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z} \,<\, \mathbb{Z}/3\mathbb{Z} + \mathbb{Z}/e\mathbb{Z},$$

where the sum of the subgroup is taken in the discriminant group. Therefore $d|e$. We have

$$d \mid e \mid \operatorname{div}(r) \mid 2e \quad \text{and} \quad \operatorname{div}(r) \mid \operatorname{lcm}(3, 2d).$$

Our aim is to calculate the two lattices

$$L_{A,2d}^{(r)} = r^\perp_{L_{A,2d}} \text{ and } T_{r,d} = (L_{A,2d}^{(r)})^\perp_{II_{2,26}}. \tag{12}$$

According to Lemma 3.2 we have

$$\det T_{r,d} = \det L_{A,2d}^{(r)} = \frac{12de}{(\operatorname{div}(r))^2}.$$

Analysing all possible e and $\operatorname{div}(r)$ we see that $\det T_{r,d}$ is a divisor of 12. The possible cases are

$$\begin{array}{llll} e = d, & r^2 = 2d, & \operatorname{div}(r) = d, & \det T_{r,d} = 12; \\ e = d, & r^2 = 2d, & \operatorname{div}(r) = 2d, & \det T_{r,d} = 3; \\ e = 2d, & r^2 = 4d, & \operatorname{div}(r) = 2d, & \det T_{r,d} = 6; \\ e = 3d, & r^2 = 6d, & \operatorname{div}(r) = 3d, & \det T_{r,d} = 4; \\ e = 3d, & r^2 = 6d, & \operatorname{div}(r) = 6d, & \det T_{r,d} = 1; \\ e = 6d, & r^2 = 12d, & \operatorname{div}(r) = 6d, & \det T_{r,d} = 2. \end{array}$$

In [**CS**, Table I] one can find all indecomposable lattices of small rank and determinant. Analysing all lattices of determinant det | 12 and of rank $n \le 6$, we find the five classes

(13)
$$\det = 3,\ E_6; \quad \det = 4,\ D_6; \quad \det = 12,\ A_5 \oplus A_1,\ D_4 \oplus A_2,\ [D_5 \oplus \langle 12 \rangle]_2$$

where $[D_5 \oplus \langle 12 \rangle]_2$ denotes an overlattice of order 2 of $D_5 \oplus \langle 12 \rangle$. The root system of $[D_5 \oplus \langle 12 \rangle]$ is D_5. The formula for $\det T_{r,d}$ given above shows that only the cases mentioned in the lemma are possible. q.e.d.

Corollary 4.6. *Let l be as in Lemma 4.4. We assume that $N_l < 18$. Then the quasi pull-back F_l vanishes along the ramification divisor of the modular projection*

$$\pi \colon \mathcal{D}(L_{A,2d}) \to \mathrm{O}_G(L_{A,2d}) \setminus \mathcal{D}(L_{A,2d}).$$

Proof. The components of the branch divisor are

$$\mathcal{D}_r = \big\{[Z] \in \mathcal{D}(L_{A,2d}) \mid (r, Z) = 0\big\}$$

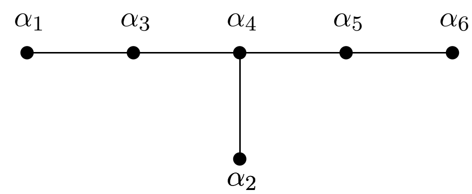
where $r \in L_{A,2d}$ and σ_r or $-\sigma_r$ is in $\mathrm{O}_G(L_{A,2d})$ (see [**GHS1**, Corollary 2.13]). If $\sigma_r \in \mathrm{O}_G(L_{A,2d})$, then F_l vanishes along $\mathcal{D}_r$ because F_l is modular with character det. Let $-\sigma_r \in \mathrm{O}_G(L_{A,2d})$. The divisor $\mathcal{D}_r$ coincides with the homogeneous domain $\mathcal{D}(L^{(r)}_{A,2d})$. The Borcherds modular form Φ_{12} vanishes of order $N/2$ where $N \ge |R(D_4 \oplus A_2)| = 30$ is the number of roots in the lattice $\det T_{r,d}$. Since $N_l < 18$, then the form F_l vanishes along $\mathcal{D}_r$ with order at least 7. q.e.d.

5. The $2d$-vectors in E_6 and the root system F_4

In this section we finish the proof of Theorem 4.1. To prove it we use Theorem 4.2, Lemma 4.4, and Lemma 4.6. We want to know for which $2d > 0$ there exists a vector $l \in E_6$ of length $l^2 = 2d$, invariant with respect to the involution J of the Dynkin diagram of E_6 and orthogonal to at least 2 and at most 16 roots in E_6. The answer is given in the next theorem.

Theorem 5.1. *A J-invariant vector l of length $l^2 = 2d$ that is orthogonal to at least 2 and at most 16 roots in E_6 exists if d is not equal to 2^n where $n \ge 0$.*

We give the proof of the theorem in Lemmas 5.2–5.5 below. We use the notation A_n, D_n, or E_n both for a lattice and for its root system because it is always clear from the context which is meant. We consider the Coxeter basis of simple roots in the lattice $E_6 = \langle \alpha_1, \dots, \alpha_6 \rangle$ (see [**Bou**, Table V]):



where

$$\alpha_1 = \frac{1}{2}(e_1 + e_8) - \frac{1}{2}(e_2 + e_3 + e_4 + e_5 + e_6 + e_7),$$
$$\alpha_2 = e_1 + e_2, \quad \alpha_k = e_{k-1} - e_{k-2} \ \ (3 \le k \le 6)$$

and $(e_1, \dots, e_8)$ is a Euclidean basis in $\mathbb{Z}^8$. To get the extended Dynkin diagram, one has to add the maximal root

$$\begin{aligned}\tilde{\alpha} &= \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 + e_8)\\ &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6.\end{aligned}$$

Then $(-\tilde{\alpha}, \alpha_2) = -1$ and $-\tilde{\alpha}$ is orthogonal to all other simple roots.

In the Euclidean basis (e_i) we have the following representation of E_6:

$$E_6 = \{l = x_1e_1 + \dots + x_5e_5 + x_6(e_6 + e_7 - e_8)\}, \qquad (14)$$
$$l^2 = x_1^2 + \dots + x_5^2 + 3x_6^3$$

where the x_i are either all integral or all half-integral, and in both cases $x_1 + \dots + x_6$ is an even integer. We recall that

$$\mathrm{Aut}(E_6) = W(E_6) \times \mathrm{Aut}(\text{Dynkin diagram of } E_6)$$

where the second factor is the cyclic group of order 2 generated by the involution J given by $J(\alpha_1) = \alpha_6$, $J(\alpha_3) = \alpha_5$, $J(\alpha_4) = \alpha_4$, $J(\alpha_2) = \alpha_2$.

The involution J defines sublattices $E_6^{J,+} = \{l \in E_6 \mid J(l) = l\}$ and $E_6^{J,-} = \{l \in E_6 \mid J(l) = -l\}$ of E_6.

Lemma 5.2. *The sublattice $E_6^{J,+} \oplus E_6^{J,-}$ is of index 4 in E_6. Moreover, $E_6^{J,+} \cong D_4$ and $E_6^{J,-} \cong A_2(2)$, where $A_2(2)$ is the lattice with the quadratic form $\left(\begin{smallmatrix} 4 & -2 \\ -2 & 4 \end{smallmatrix}\right)$ (the renormalization of the lattice A_2 by 2).*

Proof. From the definition of J we have $E_6^{J,+} = \langle \alpha_2, \alpha_4, \alpha_1 + \alpha_6, \alpha_3 + \alpha_5 \rangle$. This has another basis, namely

$$\begin{aligned}E_6^{J,+} &= \langle \alpha_2,\, \alpha_4,\, \alpha_3 + \alpha_4 + \alpha_5,\, (\alpha_1 + \alpha_6) + 2(\alpha_3 + \alpha_4 + \alpha_5) + 2\alpha_2 + \alpha_4 \rangle\\ &= \langle \alpha_2,\, \alpha_4,\, \alpha_3 + \alpha_4 + \alpha_5,\, -\tilde{\alpha} \rangle \cong D_4\end{aligned}$$

where α_2 is the central root of the Dynkin diagram of D_4. We denote $E_6^{J,+}$ by D_4^+.

If $J(u) = u$ and $J(v) = -v$, then $(u, v) = -(u, v) = 0$. Therefore

$$E_6^{J,-} = (D_4^+)^\perp_{E_6} \supseteq \langle \alpha_1 - \alpha_6,\, \alpha_3 - \alpha_5 \rangle \cong A_2(2) = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}.$$

A direct calculation shows that we have equality in the above inclusion of lattices. Then we have $\det D_4 = 4$ and $\det A_2(2) = 12$, so $[E_6, D_4^+ \oplus A_2(2)] = 4$. q.e.d.

In what follows we need some properties of the root systems D_4 and F_4. The lattice D_n is a sublattice of the Euclidean lattice $\mathbb{Z}^n$:

$$D_n = \{l = (x_1, \dots, x_n) \in \mathbb{Z}^n \mid x_1 + \dots + x_n \in 2\mathbb{Z}\}.$$

The lattice D_4 contains the twenty-four 2-roots

$$R_2(D_4) = \{\pm(e_i \pm e_j),\ 1 \le i < j \le 4\}$$

which form the root system D_4. But the lattice D_4 contains also the twenty-four 4-roots

$$R_4(D_4) = \{\pm e_1 \pm e_2 \pm e_3 \pm e_4,\ \pm 2e_i,\ 1 \le i \le 4\}.$$

By definition of the root system, F_4 equals

$$F_4 = R_2(D_4) \cup R_4(D_4).$$

The Weyl group of F_4 coincides with the orthogonal group of the lattice D_4:

$$\mathrm{O}(D_4) = W(F), \quad W(F_4)/W(D_4) \cong \mathrm{Aut}(\text{Dynkin diagram of } D_4) \cong S_3.$$

Lemma 5.3. *Let J be the involution of the Dynkin diagram of E_6.*
1) For any root $r \in R_2(E_6)$ we have

$$J(r) \neq r \Leftrightarrow (J(r), r) = 0.$$

2) For $D_4^+ = E_6^{J,+}$ we have

$$R_4(D_4^+) = \{r + J(r) \mid r \in R_2(E_6),\ r \neq J(r)\}.$$

3) Let $l \in D_4^+$ be orthogonal to a vector $l_4 \in R_4(D_4^+)$. Then l is orthogonal to the roots r and $J(r)$ from E_6 such that $l_4 = r + J(r)$ and $r \neq J(r)$.

Proof. 1) Lemma 5.2 gives us the following inclusion of lattices:

$$D_4^+ \oplus A_2(2) \subset E_6 \subset E_6^\vee \subset (D_4^+)^\vee \oplus A_2(2)^\vee. \tag{15}$$

We proved above that

$$[E_6 : (D_4^+ \oplus A_2(2))] = [D_4^\vee : D_4] = \det D_4 = 4.$$

It is easy to see that

$$D_4^\vee / D_4 = \{0,\ e_1 + D_4,\ \frac{1}{2}(e_1 + e_2 + e_3 \pm e_4) + D_4\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

where

$$q_{D_4}(e_1 + D_4) = q_{D_4}\big(\frac{1}{2}(e_1 + e_2 + e_3 \pm e_4) + D_4\big) \equiv 1 \mod 2\mathbb{Z}.$$

Analysing the discriminant form $A_2(2)^\vee / A_2(2)$, we see that it contains only three classes $\frac{1}{2}a$, $\frac{1}{2}b$, and $\frac{1}{2}(a+b)$ modulo A_2 (where a, b are simple roots in A_2) of square $1 \mod 2\mathbb{Z}$. Using (5), we see that the natural projection $E_6/(D_4^+ \oplus A_2(2))$ onto $D_4^\vee/D_4$ is surjective. It follows that if

$$l \in E_6, \quad l = l_+^* + l_-^*, \quad \text{where} \quad l_+^* \in (D_4^+)^\vee,\ l_-^* \in A_2(2)^\vee,\ l_+^* \notin D_4^+,$$

then $(l_+^*, l_+^*) \equiv 1 \mod 2\mathbb{Z}$.

Let us consider this representation $r_+^* + r_-^*$ for a root r in E_6. Then $r^2 = (r_+^*)^2 + (r_-^*)^2 = 2$ and the second component r_-^* is non-trivial if and only if $(r_+^*)^2 = (r_-^*)^2 = 1$ according to the argument above. Then $J(r) \neq r$ if and only if $(r, J(r)) = (r_+^*)^2 - (r_-^*)^2 = 0$.

2) We showed in Lemma 5.2 that E_6 contains exactly 24 J-invariant roots of D_4^+. Therefore there are $72 - 24 = 48$ non-invariant roots. For any non-invariant root r we proved in 1) that $(r, J(r)) = 0$. This gives us 24 pairs $(r, J(r))$ of non-invariant roots satisfying $(r + J(r))^2 = 4$ and $r + J(r) \in D_4^+$. To show that there is a bijection between the J-pairs and 4-roots in D_4^+, one can simply pick $\alpha_1 + J(\alpha_1)$ and take into account the fact that the Weyl group of D_4 acts transitively on the set of 4-vectors in D_4.

3) If $l \in D_4^+$, then $(l, r) = (l, J(r))$ for any root. Therefore $2(l, r) = (l, r + J(r)) = 0$. q.e.d.

Lemma 5.4. *For any positive integer d there exists a vector $l_{2d} \in D_4^+ = E_6^{J,+}$ of square $2d$ which is orthogonal to at least one root in E_6.*

Proof. We denote by $N_L(2d)$ the number of vectors of square $2d$ in a positive definite lattice L. We consider two cases: a vector l_{2d} is orthogonal to a J-invariant root r_J or to a non-J-invariant root r_n. In the first case, $l_{2d} \in (r_J)^\perp_{D_4^+} \cong 3A_1$. (See the fourth case in the proof of Lemma 5.5 below.) Then

$$N_{3A_1}(2d) = r_3(d)$$

where $r_3(d)$ is equal to the number of representations of d as a sum of three squares. It is classically known that

$$r_3(4^m d) = r_3(d) \quad \text{and} \quad r_3(d) > 0 \quad \text{if } d \neq 2^{2m}(8n + 7). \tag{16}$$

If $(l_{2d}, r_n) = 0$, then $(l_{2d}, r_n + J(r_n)) = 0$ where $r_n + J(r_n) = l_4 \in D_4^+$. But

$$(l_4)^\perp_{D_4^+} \cong A_3.$$

This follows from the form of the extended Dynkin diagram of D_4. For l_4 we can take the alternating sum of two orthogonal simple roots. Then the three other roots of the extended diagram form the orthogonal complement of l_4. We have $A_3 \cong D_3$. According to the definition of D_3, we have that $N_{A_3}(2d) = r_3(2d)$. The last number is not zero if $d \neq 2^{2m-1}(8n + 7)$. This and formula (16) show that for any d we have $N_{3A_1}(2d) + N_{A_3}(2d) > 0$. This proves the lemma. q.e.d.

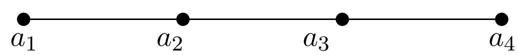
Lemma 5.5. *Let l_{2d} be a vector of square $2d$ in $E_6^{J,+}$. Then the number of roots in E_6 orthogonal to l_{2d} is smaller than 18 if and only if d is not equal to 2^n where $n \geq 0$.*

Proof. Let us assume that $|R_2((l_{2d})^{\perp}_{E_6})| \geq 18$. The root systems of rank at most 5 having at least 18 roots are

$$A_5,\ D_5,\ A_4 \oplus A_1,\ D_4 \oplus A_1,\ A_3 \oplus A_2,\ A_4,\ D_4.$$

1) The cases of $A_3 \oplus A_2$ and $D_4 \oplus A_1$ are not possible. $W(E_6)$ acts transitively on the roots and on the A_2-sublattices of E_6. We have $(A_1)^{\perp}_{E_6} \cong A_5$ and $(A_2)^{\perp}_{E_6} \cong A_2 \oplus A_2$. But A_5 does not contain D_4 and $A_2 \oplus A_2$ does not contain A_3.

2) Let us assume that $R_2((l_{2d})^{\perp}_{E_6}) = A_4$ or $A_4 \oplus A_1$. We show that neither case is possible. The vector l_{2d} is J-invariant. Therefore $J(A_4) = A_4$. The lattice A_4 is generated by its simple roots a_1, a_2, a_3, and a_4:



First we assume $a_1 \neq J(a_1)$ and $J(a_4) \neq a_4$. Then $(a_1, J(a_1)) = (a_4, J(a_4)) = 0$ according to Lemma 5.3. Therefore we have $J(a_4) \in \langle a_1, a_2 \rangle$ and $J(a_1) \in \langle a_3, a_4 \rangle$. If $J(a_1) \neq \pm a_4$, then A_4 contains two orthogonal sublattices $\langle a_1, J(a_4) \rangle$ and $\langle a_4, J(a_1) \rangle$ isomorphic to A_2, which is impossible.

If $J(a_1) = \pm a_4$, then $0 = (J(a_1), J(a_3)) = (\pm a_4, J(a_3))$ and $J(a_3) \in \langle a_1, a_2 \rangle$. But $J(a_3) \neq \pm a_1$ and we obtain that $J(a_3) \neq a_3$ and $(J(a_3), a_3) \neq 0$. This contradicts Lemma 5.3. Therefore we can assume that $a_1 = J(a_1)$ or $a_4 = J(a_4)$. If $a_1 = J(a_1)$, then $(a_1, J(a_4)) = 0$ and $J(a_4) \in \langle a_3, a_4 \rangle$. It follows that $J(a_4) = a_4$. An analogous argument shows that $J(a_3) = a_3$ and $J(a_2) = a_2$. Therefore J is the identity on A_4 and we obtain that A_4 is a sublattice of $D_4^+ = E_6^{J,+}$, which is impossible. If $R_2((l_{2d})^{\perp}_{E_6}) = A_4 \oplus A_1$, then again we have that $J(A_4) = A_4$ and $J|_{A_4} = \mathrm{id}$.

3) We have mentioned above that $(A_1)^{\perp}_{E_6} \cong A_5$ and that there is only one $W(E_6)$-orbit of A_1 in E_6. Therefore $(A_5)^{\perp}_{E_6} \cong A_1 = \langle 2 \rangle$. Any non-zero vector $l \in A_1$ ($l^2 = 2m^2$) will have the same orthogonal complement. Let us take a J-invariant vector $l \in 3A_1$ such that $l^2 = 2^{2n+1}k^2$ where k is odd. Then $N_{3A_1}(2) = r_3(1) = 6$ and

$$N_{3A_1}(2^{2n+1}k^2) = r_3(k^2) = \sum_{f|k} r_3^{pr}(k^2/f^2) = r_3^{pr}(1) + \cdots + r_3^{pr}(k^2),$$

which is > 6 if and only if $k > 1$. Here we denote by $r_3^{pr}(n)$ the number of primitive representation of n by three squares. According to Gauss, $r_3^{pr}(n) = 0$ if and only if $n \equiv 0 \mod 4$ or $n \equiv 7 \mod 8$. Therefore if

$2d = 2^{2n+1}$, then any $2d$-vector in $3A_1$ is a multiple of a root. If $l_{2d} \in A_3$, the situation is quite similar. We conclude that for $2d = 2^{2n+1}k^2$ there is a $2d$-vector which satisfies the conditions of the lemma if and only if $k > 1$.

4) We can compare the case when $l^\perp_{E_6} = D_5$ with the case of A_5. We have $(D_5)^\perp_{E_6} \cong \langle 12\rangle$. To see this we consider $(D_5)^\perp_{E_8} = A_3$ and $(A_2)^\perp_{E_8} = E_6$. There is only one $W(A_3)$-orbit of A_2 in A_3 and $(A_2)^\perp_{A_3} \cong \langle 12\rangle$. This gives us the sublattice $A_2 \oplus D_5 \oplus \langle 12\rangle$ in E_8. But we can find another orbit of 12-vectors in E_6 by taking a copy of A_2 in D_5. In fact, the 12-vector corresponding to the decomposition $\langle 12\rangle \oplus D_5 \subset E_6$ is not J-invariant. To get a J-invariant vector we take

$$l^+_{12} = 2\alpha_2 + \alpha_4 = 2e_1 + e_2 + e_3 \in E_6^{J,+}$$

(see the diagram of E_6 above). The roots of E_6 are the vectors

$$\pm e_i \pm e_j \ (1 \le i < j \le 5), \quad \pm\frac{1}{2}\big(e_8 - e_7 - e_6 \pm e_1 \pm \cdots \pm e_5\big)$$

where the number of minus signs in the last case is even. We see that there are six integral and eight half-integral roots orthogonal to l^+_{12}. Up to sign they are

$$e_3 - e_2,\ e_4 - e_5,\ e_4 + e_5;$$

$$\frac{1}{2}\big(e_8 - e_7 - e_6 + e_1 - e_2 - e_3 \pm (e_4 + e_5)\big),\ \frac{1}{2}\big(e_8 - e_7 - e_6 - e_1 + e_2 + e_3 \pm (e_4 - e_5)\big).$$

These roots form a root system $A_1 \oplus A_3$ where $A_1 = \langle \alpha_4\rangle = \langle e_3 - e_2\rangle$ and

$$A_3 = \langle e_4 - e_5,\ e_4 + e_5,\ \frac{1}{2}\big(e_8 - e_7 - e_6 + e_1 - e_2 - e_3 - e_4 - e_5\big)\rangle.$$

Therefore in the case $2d = 12$, a vector giving a low weight cusp form does exist.

5) Let us assume that $R_2((l_{2d})^\perp_{E_6}) = D_4$. Then $J(D_4) = D_4$. We can fix a system of simple roots (a_1, a_2, a_3, a_4) of D_4 (a_2 is the central root of the diagram).

First we prove that $J(a_2) = a_2$. Consideration of the extended Dynkin diagram of D_4 shows that $(A_1)^\perp_{D_4} \cong 3A_1$. The four pairwise orthogonal copies of A_1 in D_4 correspond to the vertices of the extended Dynkin diagram of D_4: a_1, a_3, a_4, and $-\tilde{a}$ where $\tilde{a} = a_1 + 2a_2 + a_3 + a_4$ is the maximal root of D_4 (see [**Bou**, Table IV]). If $J(b) \neq b$ for a root b, then $J(b)$ is orthogonal to b (Lemma 5.3). Therefore J permutes the roots a_1, a_3, a_4, and $-\tilde{a}$ with some possible changes of signs. Therefore

$$J(2a_2) = J(\tilde{a} - a_1 - a_3 - a_4) = \pm(a_1 + 2a_2 + a_3 + a_4 \pm a_1 \pm a_3 \pm a_4)$$

where all $\pm$ are independent. The maximal root $\tilde{a}$ is the only root represented by a linear combination of the simple roots having a coefficient greater than 1. That leaves only two possibilities: $J(2a_2) = \pm 2(a_1 + a_2 + a_3 + a_4)$ or $J(2a_2) = \pm 2a_2$. The first of those two does not occur because the root $a_1 + a_2 + a_3 + a_4$ is not orthogonal to a_2. Therefore $J(a_2) = a_2$.

Let us assume that J does not fix any of the four pairwise orthogonal copies A_1 in D_4. Let $J(a_1) \neq \pm a_3$ (the other cases are similar). Then the root $J(a_1 + a_2 + a_3) = a_2 + J(a_1) + J(a_3)$ is not equal to the root $a_1 + a_2 + a_3$ and it is not orthogonal to it. This contradicts Lemma 5.3-3. Therefore J fixes at least one A_1 among the four copies of A_1. So J fixes at least two copies, which form together with a_2 a root system A_3 on which J acts trivially. Therefore we have proved that if $l_{2d} \in E_6$, $J(l_{2d}) = l_{2d}$, and $R_2((l_{2d})^{\perp}_{E_6}) = D_4$, then the orthogonal complement of l_{2d} in $D_4^+ = E_6^{J,+}$ contains A_3. But $(A_3)^{\perp}_{D_4} \cong \langle 4 \rangle$. To see this one bears in mind two facts: $W(F_4) = \mathrm{O}(D_4)$ acts transitively on the set of 4-vectors in D_4 and

$$\langle a_3 - a_4 \rangle^{\perp}_{D_4} = \langle a_1,\, a_2,\, -\tilde{a} \rangle \cong A_3.$$

It follows that the vector l_{2d} is a multiple of a 4-vector l_4 in D_4^+

$$l_{2d} = m l_4, \quad l_4 \in 3A_1 \subset D_4^+ \text{ or } l_4 \in A_3 \subset D_4^+$$

(see Lemma 5.4).

If $2d = 4m^2$, then any $2d$-vector in $3A_1 \subset D_4^+$ or in $A_3 \subset D_4^+$ is a multiple of a corresponding 4-vector if and only if $2d = 4 \cdot 2^{2n}$. We use an argument similar to the case $d = 1$ (see part 3 of the proof above). If $2d = 4 \cdot 2^{2n} k^2$, with k odd, then

$$N_{3A_1}(4 \cdot 2^{2n} k^2) = r_3(2k^2) = \sum_{f|k} r_3^{pr}(2\frac{k^2}{f^2}) = r_3^{pr}(2) + \cdots + r_3^{pr}(k^2),$$

which is $> r_3(2) = 12$ if and only if $k > 1$. This finishes the proof of Lemma 5.5 and of Theorem 5.1. q.e.d.

We note that by a remark of Freitag [**Fr**, Hilfssatz 2.1, Kap. 3] one can calculate the geometric genus of a modular variety using cusp forms of canonical weight. In particular, we have

$$p_g(M_{A,2d}) = \dim S_{21}(\mathrm{O}_G(L_{A,2d}), \det).$$

In the cases of polarized K3 surfaces or polarized symplectic varieties of type $\mathrm{K3}^{[2]}$, we constructed canonical differential forms on the corresponding modular varieties using the quasi pull-back of Φ_{12}. In the case considered in this paper, this is not possible. From the proof of Lemma 5.5 we obtain

Corollary 5.6. *1. There are no J-invariant $2d$-vectors in E_6 which are orthogonal to exactly 18 roots in E_6.*
2. There are no $\mathrm{O}_G(L_{A,2d})$-modular quasi pull-backs of Φ_{12} of weight 21.

We think that cusp forms of canonical weight exist for $\mathrm{O}_G(L_{A,2d})$, but we expect the Beauville degree of the polarization to be rather large. To prove that the modular variety $M_{A,2d}$ with $d = 2^n$ is of general type for n large, we could use the method explained in [**GHS3**], which in turn uses the explicit formula for the Hirzebruch-Mumford volume found in [**GHS2**]. We conjecture that this variety is not of general type for small n; for example, for $n = 0, 1, 2$. An argument for this is given in Proposition 5.7 below.

The modular variety of symplectic 10-dimensional O'Grady varieties with a split polarization is a $2:1$ quotient of the modular variety

$$\widetilde{\mathrm{O}}^+(L_{A,2d}) \setminus \mathcal{D}(L_{A,2d}) \to \mathrm{O}_G(L_{A,2d}) \setminus \mathcal{D}(L_{A,2d}) = M_{A,2d}$$

because $[\mathrm{O}_G(L_{A,2d}) : \widetilde{\mathrm{O}}^+(L_{A,2d})] = 2$.

Proposition 5.7. *The modular variety $\widetilde{\mathrm{O}}^+(L_{A,2d}) \setminus \mathcal{D}(L_{A,2d})$ is of general type if $d \notin \{1, 2, 4\}$.*

Proof. We only have to consider the series $2d = 2^n$. If $2d = 2$, 4, or 8, then any vector l of length $l^2 = 2d$ is orthogonal to at least 20 roots. We have seen this for $2d = 2$ and $2d = 4$. The argument for $2d = 8$ is similar. Hence we cannot apply the low weight cusp form trick here.

The lattice $L_{A,2d}$ for $2d = 2^n$ with $n > 5$ can be considered as a sublattice of $L_{A,16}$, if n is even, or of $L_{A,32}$, if n is odd. Therefore the corresponding modular variety is a covering of finite order of one of the two varieties for $2d = 16$ or 32. Hence it is enough to prove that $\widetilde{\mathrm{O}}^+(L_{A,16}) \setminus \mathcal{D}(L_{A,16})$ and $\widetilde{\mathrm{O}}^+(L_{A,32}) \setminus \mathcal{D}(L_{A,32})$ are of general type.

1) Let $2d = 16$. Using the representation (14) of E_6, we put $l_{16} = 3e_1 + 2e_2 + e_3 + e_4 + e_5 \in E_6$. Inspection shows that there are 12 orthogonal roots (6 copies of A_1). Three "integral" copies are

$$e_3 - e_4,\ e_4 - e_5,\ e_3 - e_5.$$

Three "half-integral" copies are $\frac{1}{2}\big(-e_1 + e_2 \pm (e_3 - e_4) + e_5 - e_6 - e_7 + e_8\big)$ and $\frac{1}{2}\big(-e_1 + e_2 + e_3 + e_4 - e_5 - e_6 - e_7 + e_8\big)$. Then $(l_{16})^{\perp}_{E_6} \cong A_3$, where

$$A_3 = \langle \frac{1}{2}\big(-e_1 + e + 2 - e_3 + e_4 + e_5 - e_6 - e_7 + e_8\big),\ e_3 - e_4,\ e_4 - e_5\rangle.$$

2) Let $2d = 32$. We put $l_{32} = 4e_1 + 3e_2 + 2e_3 + e_6 + e_7 - e_8 \in E_6$. Then $(l_{32})^{\perp}_{E_6} \cong A_2 \oplus A_1$, where $A_1 = \langle e_4 + e_5\rangle$ and

$$A_2 = \langle \frac{1}{2}\big(e_1 - e + 2 + e_3 - e_4 + e_5 - e_6 - e_7 + e_8\big),\ e_4 - e_5\rangle.$$

The quasi pull-backs of Φ_{12} to $2U \oplus 2E_8(-1) \oplus A_2(-1) \oplus \langle -2d\rangle$ for the vectors l_{16} and l_{32} are cusp forms of weights 18 and 16 respectively, for the groups $\widetilde{\mathrm{O}}^+(L_{A,16})$ and $\widetilde{\mathrm{O}}^+(L_{A,32})$. The set of plus or minus reflections in $\widetilde{\mathrm{O}}^+(L_{A,2d})$ is a subset of the reflections considered in Lemma 4.5.

Therefore we can prove that $F_{l_{16}}$ and $F_{l_{36}}$ vanish on the branch divisor of the modular projection using the arguments of the proof of Corollary 4.6. To finish the proof, we apply Theorem 4.2. q.e.d.

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