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MINKOWSKI AREAS AND VALUATIONS

Monika Ludwig

Abstract

All continuous GL(n) covariant valuations that map convex bodies to convex bodies are completely classified. This establishes a characterization of moment and projection operators and shows that the Holmes-Thompson area is the unique Minkowski area that is also a bivaluation.

On finite dimensional Banach spaces, there are essentially different ways to choose a notion of surface area that is independent of the choice of a Euclidean coordinate system. In particular, *Minkowski areas* have this property. The precise definition of Minkowski area requires a set of conditions, which will be given in Section 2. Important examples of Minkowski areas are due to Busemann [**12**], Gromov [**19**], and Holmes and Thompson [**28**]. Central questions in the geometry of finite dimensional Banach spaces are those of finding the *right* notion of Minkowski area and, more generally, of *invariant area* (see Section 2 for the definition), and of finding distinctive properties of different such areas (see [**9**] and [**55**] for more information on area in Banach and Finsler spaces and see [**7**, **8**, **50**] for some recent results). As will be shown, if the critical *valuation* property is required and if invariant areas are defined for spaces with not necessarily symmetric unit balls, it turns out that Holmes-Thompson area is the only answer.

We need the following definitions. Let $\mathcal{K}^n$ be the space of convex bodies (that is, of compact convex sets) in $\mathbb{R}^n$ equipped with the Hausdorff metric, and let $\mathcal{K}_0^n$ be the subspace of convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors. A function z defined on a certain subset $\mathcal{C}$ of $\mathcal{K}^n$ and taking values in an abelian semigroup is called a *valuation* if

$$z(K) + z(L) = z(K \cup L) + z(K \cap L), \tag{1}$$

whenever $K, L, K \cup L, K \cap L \in \mathcal{C}$. A function $z : \mathcal{K}^n \times \mathcal{K}_0^n \to \langle \mathbb{R}, + \rangle$ is called a *bivaluation* if it is a valuation in both arguments. Valuations on convex bodies are a classical concept going back to Dehn's solution in 1900 of Hilbert's Third Problem. Starting with Hadwiger's celebrated classification of rigid motion invariant valuations and characterization of elementary mixed volumes, valuations have become a critical notion

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(see [**27**, **31**, **47**] and see [**1**, **2**, **3**, **4**, **5**, **6**, **11**, **15**, **29**, **30**, **49**, **51**, **52**, **53**] for some of the more recent contributions).

Invariant areas have strong invariance properties with respect to the general linear group, $\mathrm{GL}(n)$. In recent years, such *affine functions* on convex bodies have attracted increased interest (see, for example, [**10**, **14**, **18**, **21**, **25**, **26**, **41**, **42**, **43**, **45**, **46**, **56**, **57**]). Many of these affine functions on convex bodies can be completely characterized by their invariance and valuation properties (see, for example, [**22**, **23**, **24**, **33**, **34**, **35**, **36**, **37**, **38**, **40**]). The following result shows that the valuation property also completely characterizes the Holmes-Thompson area.

Theorem 1. *A functional $z : \mathcal{K}^n \times \mathcal{K}_0^n \to \mathbb{R}$ is a bivaluation and an invariant area if and only if there is a constant $c > 0$ such that*

$$z(K, B) = c\, V(K, \dots, K, \Pi\, B^*)$$

for every $K \in \mathcal{K}^n$ and $B \in \mathcal{K}_0^n$.

Here $V(K_1, \dots, K_n)$ denotes the mixed volume of $K_1, \dots, K_n \in \mathcal{K}^n$ and B^* denotes the polar body of $B \in \mathcal{K}_0^n$ (see Section 1). A convex body K is uniquely determined by its support function

$$h(K, v) = \max\{v \cdot x : x \in K\} \text{ for } v \in \mathbb{R}^n,$$

where $v \cdot x$ is the standard inner product of v and x. The *projection body*, $\Pi\, K$, of K is the convex body whose support function is given by

$$h(\Pi\, K, u) = V_{n-1}(K|u^\perp) \text{ for } u \in S^{n-1},$$

where V_{n-1} denotes $(n-1)$-dimensional volume, $K|u^\perp$ the image of the orthogonal projection of K onto the subspace orthogonal to u, and S^{n-1} the unit sphere in $\mathbb{R}^n$. Projection bodies are an important tool in geometric tomography (see [**16**]) and have found intriguing applications in recent years (see [**54**, **57**]).

The invariant area obtained in Theorem 1 is the Holmes-Thompson area, which is a Minkowski area. Thus Theorem 1 implies that every invariant area that is also a bivaluation is a Minkowski area and that the Holmes-Thompson area is (up to multiplication with a positive constant) the unique Minkowski area that is a bivaluation.

Theorem 1 is a consequence of a new classification of valuations. An operator $\mathrm{Z} : \mathcal{K}_0^n \to \langle \mathcal{K}^n, + \rangle$ is called a *Minkowski valuation* if (1) holds and addition on $\mathcal{K}^n$ is Minkowski addition (defined for $K, L \in \mathcal{K}^n$ by $K + L = \{x + y : x \in K,\, y \in L\}$). An operator $\mathrm{Z} : \mathcal{K}_0^n \to \langle \mathcal{K}^n, + \rangle$ is called $\mathrm{GL}(n)$ *covariant* if for some $q \in \mathbb{R}$,

$$\mathrm{Z}(\phi K) = |\det \phi|^q\, \phi\, \mathrm{Z}\, K \quad \text{for every } \phi \in \mathrm{GL}(n) \text{ and } K \in \mathcal{K}_0^n. \tag{2}$$

In [**36**], a classification of $\mathrm{GL}(n)$ covariant Minkowski valuations on the space of convex bodies containing the origin was obtained. Here we show that the space $\mathcal{K}_0^n$ allows additional important $\mathrm{GL}(n)$ covariant

Minkowski valuations and establish a complete classification of such valuations.

Theorem 2. *An operator* $\mathrm{Z} : \mathcal{K}_0^n \to \langle \mathcal{K}^n, + \rangle$, *where* $n \geq 3$, *is a continuous non-trivial* $\mathrm{GL}(n)$ *covariant valuation if and only if either there are constants* $c_0 \geq 0$ *and* $c_1 \in \mathbb{R}$ *such that*

$$\mathrm{Z}\, K = c_0\, \mathrm{M}\, K + c_1 m(K)$$

for every $K \in \mathcal{K}_0^n$ *or there is a constant* $c_0 \geq 0$ *such that*

$$\mathrm{Z}\, K = c_0\, \Pi\, K^*$$

for every $K \in \mathcal{K}_0^n$.

A valuation is called *trivial* if it is a linear combination of the identity and central reflection. The *moment body*, $\mathrm{M}\, K$, of K is defined by

$$h(\mathrm{M}\, K, u) = \int_K |u \cdot x| \, dx, \ \text{ for } \ u \in S^{n-1}.$$

When divided by the volume of K, the moment body of K becomes the *centroid body* of K and is a classical and important notion going back to at least Dupin (see [**16**]). The vector $m(K) = \int_K x \, dx$ is called the *moment vector* of K.

In the proof of Theorem 2, a classification of Minkowski valuations on the space of convex polytopes containing the origin in their interiors is established. These results are contained in Sections 4 and 5. The definition of invariant areas and Minkowski areas is given in Section 2. The proof of Theorem 2 is given in Section 6 and makes essential use of the new classification of Minkowski valuations.

1. Notation and background material

General references on convex bodies are the books by Gardner [**16**], Gruber [**20**], Schneider [**48**], and Thompson [**55**]. We work in Euclidean n-space, $\mathbb{R}^n$, and write $x = (x_1, \dots, x_n)$ for $x \in \mathbb{R}^n$. We denote by $e_1, \dots, e_n$ the vectors of the standard basis of $\mathbb{R}^n$.

For $K \in \mathcal{K}^n$, it follows immediately from the definition of the support function that for every $s > 0$ and $\phi \in \mathrm{GL}(n)$,

$$h(K, s\, x) = s\, h(K, x) \ \text{ and } \ h(\phi K, x) = h(K, \phi^t x), \tag{3}$$

where ϕ^t is the transpose of ϕ. Support functions are sublinear, that is, for all $x, y \in \mathbb{R}^n$,

$$h(K, x + y) \leq h(K, x) + h(K, y).$$

Let $x, y \in \mathbb{R}^n$ be given and let K_t for $t \geq 0$ be convex bodies. If $h(K_t, \pm y) = o(1)$ as $t \to 0$, then the sublinearity of support functions implies that

$$h(K_t, x) - h(K_t, -y) \leq h(K_t, x + y) \leq h(K_t, x) + h(K_t, y).$$

Hence

$$h(K_t, x+y) = h(K_t, x) + o(1) \quad \text{as } t \to 0. \tag{4}$$

If $h(K_t, \pm y) = o(1)$ uniformly (in some parameter) as $t \to 0$, then also (4) holds uniformly.

For $K, L \in \mathcal{K}^n$, the mixed volume $V(K, \dots, K, L) = V_1(K, L)$ is defined by

$$V_1(K, L) = \frac{1}{n} \lim_{\varepsilon \to 0+} \frac{V_n(K + \varepsilon L) - V_n(K)}{\varepsilon},$$

where V_n denotes n-dimensional volume. The following properties are immediate consequences of the definition (see [**48**]). For $K, L \in \mathcal{K}^n$,

$$V_1(\phi K, \phi L) = |\det \phi|\, V_1(K, L) \quad \text{for every } \phi \in \mathrm{GL}(n) \tag{5}$$

and

$$V_1(K, r\,L) = r\, V_1(K, L) \quad \text{for every } r \geq 0. \tag{6}$$

If $K, L_1, L_2 \in \mathcal{K}^n$, then

$$V_1(K, L_1 + L_2) = V_1(K, L_1) + V_1(K, L_2). \tag{7}$$

If $P \in \mathcal{K}^n$ is a polytope with facets $F_1, \dots, F_m$ lying in hyperplanes with outer normal unit vectors $u_1, \dots, u_m$ and $L \in \mathcal{K}^n$,

$$V_1(P, L) = \frac{1}{n} \sum_{i=1}^{m} h(L, u_i) V_{n-1}(F_i). \tag{8}$$

Let $\mathcal{K}_c^n$ denote the set of origin-symmetric convex bodies in $\mathbb{R}^n$. An immediate consequence of the equality case of Minkowski's inequality (see [**48**]) is the following result: If $L_1, L_2 \in \mathcal{K}_c^n$ and

$$V_1(K, L_1) = V_1(K, L_2) \quad \text{for every } K \in \mathcal{K}^n, \tag{9}$$

then $L_1 = L_2$.

Let $\mathcal{P}_0^n$ denote the set of convex polytopes in $\mathbb{R}^n$ that contain the origin in their interiors. A function defined on $\mathcal{P}_0^n$ is called *measurable* if it is Borel measurable, that is, the pre-image of any open set is a Borel set in the space $\mathcal{P}_0^n$ equipped with the Hausdorff metric. We require the following results on valuations on $\mathcal{P}_0^1$. Let $\nu : \mathcal{P}_0^1 \to \mathbb{R}$ be a measurable valuation that is homogeneous of degree p, that is, $\nu(t\,I) = t^p\, \nu(I)$ for all $t > 0$ and $I \in \mathcal{P}_0^1$. If $p = 0$, then there are $a, b \in \mathbb{R}$ such that

$$\nu([-s, t]) = a \log(\frac{t}{s}) + b \tag{10}$$

for every $s, t > 0$. If $p \neq 0$, then there are $a, b \in \mathbb{R}$ such that

$$\nu([-s, t]) = a\, s^p + b\, t^p \tag{11}$$

for every $s, t > 0$ (see [**34**], Equations (3) and (4)). These results follow from the fact that every measurable solution f of the Cauchy functional equation, $f(x + y) = f(x) + f(y)$, is linear.

An operator $\mathrm{Z}:\mathcal{P}_0^n\to\langle\mathcal{K}^n,+\rangle$ is called $\mathrm{GL}(n)$ *covariant of weight* q if

$$\mathrm{Z}\,\phi P=|\det\phi|^q\,\phi\,\mathrm{Z}\,P\quad\text{for every }\ \phi\in\mathrm{GL}(n)\text{ and }P\in\mathcal{P}_0^n.$$

An operator $\mathrm{Z}:\mathcal{P}_0^n\to\langle\mathcal{K}^n,+\rangle$ is called $\mathrm{GL}(n)$ *contravariant of weight* q if

$$\mathrm{Z}\,\phi P=|\det\phi|^q\,\phi^{-t}\,\mathrm{Z}\,P\quad\text{for every }\ \phi\in\mathrm{GL}(n)\text{ and }P\in\mathcal{P}_0^n,\tag{12}$$

where ϕ^{-t} is the inverse of the transpose of ϕ. Let $\mathrm{Z}:\mathcal{P}_0^n\to\langle\mathcal{K}^n,+\rangle$ be a valuation which is $\mathrm{GL}(n)$ contravariant of weight q. We associate with Z an operator Z^* in the following way. For $P\in\mathcal{P}_0^n$, the polar body, P^*, of P is defined by

$$P^*=\{x\in\mathbb{R}^n: x\cdot y\le 1\quad\text{for every }\ y\in P\}.$$

We define the operator $\mathrm{Z}^*:\mathcal{P}_0^n\to\langle\mathcal{K}^n,+\rangle$ by setting $\mathrm{Z}^*\,P=\mathrm{Z}\,P^*$ for $P\in\mathcal{P}_0^n$ and show that

(13) $\mathrm{Z}^*:\mathcal{P}_0^n\to\langle\mathcal{K}^n,+\rangle$ is a valuation $\mathrm{GL}(n)$ covariant of weight $-q$.

It follows immediately from the definition of polar body that for $P\in\mathcal{P}_0^n$,

$$(\phi P)^*=\phi^{-t}P^*\quad\text{for every }\ \phi\in\mathrm{GL}(n)\tag{14}$$

and that for $P,Q,P\cup Q\in\mathcal{P}_0^n$,

$$(P\cup Q)^*=P^*\cap Q^*\quad\text{and}\quad (P\cap Q)^*=P^*\cup Q^*.\tag{15}$$

Since Z is a valuation, for $P,Q\in\mathcal{P}_0^n$ with $P\cup Q\in\mathcal{P}_0^n$ it follows from (15) that

$$\begin{aligned}\mathrm{Z}^*\,P+\mathrm{Z}^*\,Q &= \mathrm{Z}\,P^*+\mathrm{Z}\,Q^*\\ &= \mathrm{Z}(P^*\cup Q^*)+\mathrm{Z}(P^*\cap Q^*)\\ &= \mathrm{Z}(P\cap Q)^*+\mathrm{Z}(P\cup Q)^*\\ &= \mathrm{Z}^*(P\cap Q)+\mathrm{Z}^*(P\cup Q).\end{aligned}$$

Thus Z^* is also a valuation. Since Z is $\mathrm{GL}(n)$ contravariant of weight q, it follows from (14) that

$$\mathrm{Z}^*(\phi P)=\mathrm{Z}(\phi P)^*=\mathrm{Z}(\phi^{-t}P^*)=|\det\phi|^{-q}\phi\,\mathrm{Z}^*\,P.$$

Thus (13) holds.

For $n=2$, we also associate with $\mathrm{Z}:\mathcal{P}_0^2\to\langle\mathcal{K}^2,+\rangle$ an operator $\mathrm{Z}^\perp$. Suppose that Z is a valuation which is $\mathrm{GL}(2)$ contravariant of weight q. Let $\rho_{\pi/2}$ denote the rotation by the angle $\pi/2$ and let $\rho_{-\pi/2}$ denote its inverse. Define $\mathrm{Z}^\perp:\mathcal{P}_0^2\to\langle\mathcal{K}^2,+\rangle$ by setting $\mathrm{Z}^\perp\,P=\rho_{-\pi/2}\,\mathrm{Z}\,P$ for $P\in\mathcal{P}_0^2$. We show that

(16) $\mathrm{Z}^\perp:\mathcal{P}_0^2\to\langle\mathcal{K}^2,+\rangle$ is a valuation $\mathrm{GL}(2)$ covariant of weight $q-1$.

Since Z is a valuation, we have for $P, Q \in \mathcal{P}_0^2$ with $P \cup Q \in \mathcal{P}_0^2$,

$$\begin{aligned} \mathrm{Z}^\perp P + \mathrm{Z}^\perp Q &= \rho_{-\pi/2}\, \mathrm{Z}\, P + \rho_{-\pi/2}\, \mathrm{Z}\, Q \\ &= \rho_{-\pi/2}(\mathrm{Z}(P) + \mathrm{Z}(Q)) \\ &= \rho_{-\pi/2}(\mathrm{Z}(P \cup Q) + \mathrm{Z}(P \cap Q)) \\ &= \mathrm{Z}^\perp(P \cup Q) + \mathrm{Z}^\perp(P \cap Q). \end{aligned}$$

Thus $\mathrm{Z}^\perp$ is also a valuation. For $\phi \in \mathrm{GL}(2)$ and $P \in \mathcal{P}_0^2$,

$$\mathrm{Z}^\perp(\phi P) = |\det \phi|^q\, \rho_{-\pi/2}\, \phi^{-t} \rho_{\pi/2}\, \mathrm{Z}^\perp P = |\det \phi|^{q-1} \phi\, \mathrm{Z}^\perp P.$$

Hence $\mathrm{Z}^\perp$ is $\mathrm{GL}(2)$ covariant of weight $q - 1$. Thus (16) holds.

In the proof of Theorem 2, we make use of the following results.

Theorem 1.1 ([**34**]). *A functional $z : \mathcal{P}_0^n \to \langle \mathbb{R}, + \rangle$, where $n \geq 2$, is a measurable valuation so that for some $q \in \mathbb{R}$,*

$$z(\phi P) = |\det \phi|^q\, z(P) \tag{17}$$

holds for every $\phi \in \mathrm{GL}(n)$ if and only if there is a constant $c \in \mathbb{R}$ such that

$$z(P) = c \quad \text{or} \quad z(P) = c\, V_n(P) \quad \text{or} \quad z(P) = c\, V_n(P^*)$$

for every $P \in \mathcal{P}_0^n$.

Theorem 1.2 ([**32**]). *Let $z : \mathcal{P}_0^2 \to \langle \mathbb{R}^2, + \rangle$ be a measurable valuation which is $\mathrm{GL}(2)$ covariant of weight q. If $q = 1$, then there is a constant $c \in \mathbb{R}$ such that*

$$z(P) = c\, m(P)$$

for every $P \in \mathcal{P}_0^2$. If $q = -2$, then there is a constant $c \in \mathbb{R}$ such that

$$z(P) = c\, \rho_{-\pi/2}\, m(P^*)$$

for every $P \in \mathcal{P}_0^2$. In all other cases, $z(P) = \{0\}$ for every $P \in \mathcal{P}_0^2$.

Theorem 1.3 ([**32**]). *Let $z : \mathcal{P}_0^n \to \langle \mathbb{R}^n, + \rangle$, where $n \geq 3$, be a measurable valuation which is $\mathrm{GL}(n)$ covariant of weight q. If $q = 1$, then there is a constant $c \in \mathbb{R}$ such that*

$$z(P) = c\, m(P)$$

for every $P \in \mathcal{P}_0^n$. In all other cases, $z(P) = \{0\}$ for every $P \in \mathcal{P}_0^n$.

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2. Background material on invariant areas

Let $(\mathbb{R}^n, \|\cdot\|)$ be a normed space and $B = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ its unit ball. An axiomatic approach to Minkowski areas following Busemann's ideas is presented in Thompson's book [**55**]. This approach has two important components. First, the notion of surface area is independent of the choice of a Euclidean coordinate system and has certain regularity properties. We call functionals with these properties *invariant areas*. Second, a Minkowski area assigns a notion of area in an *intrinsic* way. In [**55**], notions of volume for all Banach spaces in all dimensions are discussed. Here we restrict our attention to n-dimensional spaces and extend the definition of area to spaces with not necessarily origin-symmetric unit balls. For a hyperplane H, let $\mathcal{K}(H)$ denote the set of convex bodies contained in H.

Definition. A functional $z : \mathcal{K}^n \times \mathcal{K}^n_0 \to [0, \infty)$ is an *invariant area* if

(i) $z(K, B) = z(\phi K, \phi B)$ for all $\phi \in \mathrm{GL}(n)$;
(ii) $z(K, B) = z(K + x, B)$ for all $x \in \mathbb{R}^n$;
(iii) z is continuous in both variables;
(iv) for every hyperplane H, there exists a constant $c_H > 0$ such that $z(\cdot, B) = c_H\, V_{n-1}$ on $\mathcal{K}(H)$;
(v) for every polytope $P \in \mathcal{K}^n$ with facets $F_1, \ldots, F_m$, $z(F_1, B) \leq \sum_{i=2}^m z(F_i, B)$.

An invariant area $z : \mathcal{K}^n \times \mathcal{K}^n_0 \to [0, \infty)$ is a *Minkowski area* if it is *intrinsic*, that is, for every $(n-1)$-dimensional subspace H and every $B, B' \in \mathcal{K}^n_0$ satisfying $B' \cap H = B \cap H$, we have $z(K, B) = z(K, B')$ for every $K \in \mathcal{K}(H)$.

3. Extension

Let $\overline{\mathcal{P}}{}^n_0$ denote the set of convex polytopes P which are either in $\mathcal{P}^n_0$ or are the intersection of a polytope $P_0 \in \mathcal{P}^n_0$ and a polyhedral cone with apex at the origin. Here a polyhedral cone with apex at the origin is the intersection of finitely many closed halfspaces containing the origin in their boundaries. As a first step, we extend the valuation $\mathrm{Z} : \mathcal{P}^n_0 \to \langle \mathcal{K}^n, + \rangle$ to a valuation on $\overline{\mathcal{P}}{}^n_0$.

We need the following definitions. For $A_1, \ldots, A_k \subset \mathbb{R}^n$, we denote the convex hull of $A_1, \ldots, A_k$ by $[A_1, \ldots, A_k]$. For the convex hull of $A \subset \mathbb{R}^n$ and $u_1, \ldots, u_k \in \mathbb{R}^n$, we write $[A, u_1, \ldots, u_k]$. For $A \subset \mathbb{R}^n$, let

$$A^\perp = \{x \in \mathbb{R}^n : x \cdot y = 0 \quad \text{for every } \; y \in A\}.$$

For a hyperplane H containing the origin, let H^+ and H^- denote the complementary closed halfspaces bounded by H. Let $\mathcal{P}_0(H)$ denote the set of convex polytopes in H that contain the origin in their interiors relative to H. For $v \in \mathbb{R}^n$, let $\langle v \rangle$ denote the linear hull of v and $\mathcal{P}_0(v)$ the set of intervals in $\langle v \rangle$ that contain the origin in their interiors.

Let $C_1(\mathbb{R}^n)$ be the set of continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ that are positively homogeneous of degree 1, that is, $f(t\,x) = t\,f(x)$ for all $t \geq 0$ and $x \in \mathbb{R}^n$. Since $h(K_1 + K_2, \cdot) = h(K_1, \cdot) + h(K_2, \cdot)$ for $K_1, K_2 \in \mathcal{K}^n$, we see that if $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{K}^n, + \rangle$ is a valuation, then the operator Y on $\mathcal{P}_0^n$ defined by $\mathrm{Y}\,P = h(\mathrm{Z}\,P, \cdot)$ is a valuation taking values in $C_1(\mathbb{R}^n)$.

An operator $\mathrm{Y} : \mathcal{P}_0^n \to C_1(\mathbb{R}^n)$ is called $\mathrm{GL}(n)$ covariant of weight q if

$$\mathrm{Y}(\phi P) = |\det \phi|^q \ \mathrm{Y}\,P \circ \phi^t$$

for every $\phi \in \mathrm{GL}(n)$ and $P \in \mathcal{P}_0^n$. It is called $\mathrm{GL}(n)$ contravariant of weight q if

$$\mathrm{Y}(\phi P) = |\det \phi|^q \ \mathrm{Y}\,P \circ \phi^{-1}$$

for every $\phi \in \mathrm{GL}(n)$ and $P \in \mathcal{P}_0^n$.

Let H be a hyperplane containing the origin and $\mathrm{Y} : \mathcal{P}_0^n \to C_1(\mathbb{R}^n)$. We say that

$$\mathrm{Y} \text{ has the } \textit{Cauchy property} \text{ for } \mathcal{P}_0(H)$$

if for every $\varepsilon > 0$ and centered ball B, there exists $\delta > 0$ (depending only on ε and B) so that

$$\max_{x \in S^{n-1}} \big| \mathrm{Y}[P, u, v](x) - \mathrm{Y}[P, u', v'](x) \big| < \varepsilon \tag{18}$$

for every $u, u' \in H^- \backslash H$ and $v, v' \in H^+ \backslash H$ with $|u|, |u'|, |v|, |v'| < \delta$ and for every $P \in \mathcal{P}_0(H)$ with $P \subset B$. Let $w \in H^+ \backslash H$. We say that Y has the Cauchy property for $\mathcal{P}_0(H)$ with respect to $\langle w \rangle$ if (18) holds for $u, u', v, v' \in \langle w \rangle$.

Lemma 3.1. *If* $\mathrm{Y} : \mathcal{P}_0^n \to \langle C_1(\mathbb{R}^n), + \rangle$ *is a valuation so that for every hyperplane* H *containing the origin,*

$$\mathrm{Y} \textit{ has the Cauchy property for } \mathcal{P}_0(H), \tag{19}$$

then Y *can be extended to a valuation on* $\overline{\mathcal{P}}_0^n$. *Moreover, if* Y *is* $\mathrm{GL}(n)$ *covariant on* $\mathcal{P}_0^n$, *so is the extended valuation on* $\overline{\mathcal{P}}_0^n$. *If* Y *is* $\mathrm{GL}(n)$ *contravariant on* $\mathcal{P}_0^n$, *so is the extended valuation on* $\overline{\mathcal{P}}_0^n$.

Proof. For $j = 1, \ldots, n$, let $\mathcal{P}_j^n$ be the set of convex polytopes P containing the origin such that there exist $P_0 \in \mathcal{P}_0^n$ and hyperplanes $H_1, \ldots, H_i$, where $i \leq j$, containing the origin with linearly independent normal vectors and either

$$P = P_0 \cap H_1^+ \cap \cdots \cap H_i^+ \tag{20}$$

or

$$P = P_0 \cap H_1^+ \cap \cdots \cap H_{i-1}^+ \cap H_i. \tag{21}$$

Note that $H_1^+ \cap \cdots \cap H_i^+$ is the support cone at 0 of the polytope P defined by (20). For a hyperplane H and $j = 1, \ldots, n-1$, let $\mathcal{P}_j(H)$

be the set of convex polytopes P such that there exist $P_0 \in \mathcal{P}_0^n$ and hyperplanes $H_1, \dots, H_i$, where $i \le j$, containing the origin with linearly independent normal vectors such that

$$P = P_0 \cap H_1^+ \cap \cdots \cap H_i^+ \cap H. \tag{22}$$

Note that $\mathcal{P}_j(H) \subset \mathcal{P}_{j+1}^n$. If Υ is defined on $\mathcal{P}_j^n$ for $j \in \{1, \dots, n-1\}$ and H is a hyperplane containing the origin, we say that

$$\Upsilon \text{ has the Cauchy property for } \mathcal{P}_j(H)$$

if, for every $\varepsilon > 0$ and centered ball B, there exists $\delta > 0$ (depending only on ε and B) so that for every $P \in \mathcal{P}_j(H)$ defined by (22) with $P \subset B$, we have

$$\max_{x \in S^{n-1}} \big| \Upsilon[P, u, v](x) - \Upsilon[P, u', v'](x) \big| < \varepsilon$$

for every $u, u', v, v' \in H_1 \cap \cdots \cap H_i$ with $u, u' \in H^- \backslash H$ and $v, v' \in H^+ \backslash H$ such that $|u|, |u'|, |v|, |v'| < \delta$.

Define Υ on $\mathcal{P}_j^n$ for $j = 1, \dots, n$ inductively, starting with $j = 1$, in the following way. For P given by (20) or (21) with j hyperplanes, set

$$\Upsilon P = \lim_{u,v \to 0} \Upsilon[P, u, v], \tag{23}$$

where $u, v \in H_1 \cap \cdots \cap H_{j-1}$ and $u \in H_j^- \backslash H_j$ and $v \in H_j^+ \backslash H_j$.

We show that the limit in (23) exists uniformly on S^{n-1} and does not depend on the choice of H_j among $H_1, \dots, H_j$ and that for every hyperplane H containing the origin,

$$\Upsilon \text{ has the Cauchy property for } \mathcal{P}_j(H). \tag{24}$$

In addition, we show Υ has the following additivity properties:
If $P \in \mathcal{P}_{j-1}^n$ and H is a hyperplane such that $P \cap H^+, P \cap H^- \in \mathcal{P}_j^n$, then

$$\Upsilon P + \Upsilon(P \cap H) = \Upsilon(P \cap H^+) + \Upsilon(P \cap H^-). \tag{25}$$

If $P, Q, P \cap Q, P \cup Q \in \mathcal{P}_j^n$, where $j < n$, are defined by (20) by the same halfspaces $H_1^+, \dots, H_j^+$, then

$$\Upsilon P + \Upsilon Q = \Upsilon(P \cup Q) + \Upsilon(P \cap Q). \tag{26}$$

The operator Υ is well defined, and a valuation on $\mathcal{P}_0^n$ and properties (24) and (26) hold for $j = 0$ by assumption. Let $1 \le k \le n$. Suppose that Υ is well defined by (23) on $\mathcal{P}_{k-1}^n$. Further suppose that we have (24) and (26) for $0 \le j < k$ and that we have (25) for $1 \le j < k$.

First, we show that the limit in (23) exists uniformly on S^{n-1}. For $P \in \mathcal{P}_k^n$ given by (21), this follows from (24) for $j = k - 1$. So, let $P \in \mathcal{P}_k^n$ be given by (20) with $i = k$. Note that $[P, u, v] = [P, u]$ for $|v|$ suitably small. Suppose that $\bar{u} \in H_1 \cap \cdots \cap H_{k-1}$ with $\bar{u} \in H_k^- \backslash H_k$ is chosen such that $[P, u] \subseteq [P, \bar{u}]$ and $-\bar{u} \in P$. Then applying (26) with $j = k - 1$ gives

$$\Upsilon[P, u] + \Upsilon[P \cap H_k, \bar{u}, -\bar{u}] = \Upsilon[P, \bar{u}] + \Upsilon[P \cap H_k, u, -\bar{u}].$$

Consequently, it follows from (24) for $j = k-1$ that (23) exists uniformly on S^{n-1} and that ΥP is continuous on S^{n-1}. For $k > 1$ and $P \in \mathcal{P}_k^n$ given by (20) with $i = k$, we show that ΥP as defined by (23) does not depend on the choice of the hyperplane H_k among $H_1, \dots, H_k$. Let $u \in H_1 \cap \dots \cap H_{k-1}$ and $u \in H_k^- \backslash H_k$. Choose $w \in H_2 \cap \dots \cap H_k$ with $w \in H_1^- \backslash H_1$. Then applying (25) for $j = k - 1$ gives

$$\Upsilon[P, u, w] + \Upsilon[P \cap H_k, w] = \Upsilon[P, w] + \Upsilon[P \cap H_k, u, w].$$

Hence, by (24) for $j = k - 1$, there exists $\delta > 0$ such that on S^{n-1},

$$|\,\Upsilon[P, u, w] - \Upsilon[P, w]| = |\,\Upsilon[P \cap H_k, u, w] - \Upsilon[P \cap H_k, w]| < \varepsilon,$$

whenever $|u| < \delta$ and $[P \cap H_k, w]$ is contained in a suitable ball. Hence

$$\lim_{u,w \to 0} \Upsilon[P, u, w] = \lim_{w \to 0} \Upsilon[P, w].$$

Thus Υ is well defined on $\mathcal{P}_k^n$.

Next, we show that (24) holds for $j = k$. Let $\varepsilon > 0$ and a centered ball B be chosen. Suppose that $P \in \mathcal{P}_k(H)$ and that

$$P = P_0 \cap H_1^+ \cap \dots \cap H_k^+ \cap H$$

where $H, H_1, \dots, H_k$ have linearly independent normal vectors. Choose $z \in H \cap H_1 \cap \dots \cap H_{k-1}$ with $z \in H_k^- \backslash H_k$. Then $[P, z] \in \mathcal{P}_{k-1}(H)$ and by (24) for $j = k - 1$, there exists $\delta > 0$ so that for $u_i, v_i \in H_1 \cap \dots \cap H_{k-1}$ with $u_i \in H^- \backslash H$ and $v_i \in H^+ \backslash H$ for $i = 1, 2$, we have on S^{n-1},

$$|\,\Upsilon[P, z, u_1, v_1] - \Upsilon[P, z, u_2, v_2]| < \varepsilon, \tag{27}$$

whenever $|u_i|, |v_i| < \delta$ and $[P, z]$ is contained in B. By (23), letting $z \to 0$ in (27) shows that (24) holds for $j = k$.

Next, we show that (25) holds for $j = k$. Let $P \in \mathcal{P}_{k-1}^n$, that is, there exist $P_0 \in \mathcal{P}_0^n$ and $H_1, \dots, H_{k-1}$ such that $P = P_0 \cap H_1^+ \cap \dots \cap H_{k-1}^+$. Choose $u \in H_1 \cap \dots \cap H_{k-1}$, such that $u \in P \cap H^+ \backslash H$ and $-u \in P \cap H^-$. The four polytopes P, $[P \cap H, u, -u]$, $[P \cap H^+, -u]$, $[P \cap H^-, u]$ have the hyperplanes $H_1, \dots, H_{k-1}$ in common. Applying (26) for $j = k - 1$ gives

$$\Upsilon P + \Upsilon[P \cap H, u, -u] = \Upsilon[P \cap H^+, -u] + \Upsilon[P \cap H^-, u].$$

By (24) and definition (23), this implies that (25) holds for $j = k$.

Finally, we show that (26) holds for $j = k$. Choose $u \in H_1 \cap \dots \cap H_{k-1}$ with $u \notin H_k$ such that $-u \in P \cap Q$. Applying (26) for $j = k - 1$ shows that

$$\Upsilon[P, u] + \Upsilon[Q, u] = \Upsilon[P \cup Q, u] + \Upsilon[P \cap Q, u].$$

Because of definition (23), this implies that (26) holds for $j = k$.

The induction is now complete and Y is extended to $\mathcal{P}_n^n$. Let $P \in \mathcal{P}_n^n$ and let H be a hyperplane such that $P \cap H^+, P \cap H^- \in \mathcal{P}_n^n$. We show that

$$\mathrm{Y}\, P + \mathrm{Y}(P \cap H) = \mathrm{Y}(P \cap H^+) + \mathrm{Y}(P \cap H^-). \tag{28}$$

Since (25) holds, it suffices to show (28) for a polytope P whose support cone at 0 is bounded by n facets with linearly independent normal vectors.

First, let $n = 2$. Let $P = P_0 \cap H_1^+ \cap H_2^+$, where H_1 and H_2 are lines containing the origin and $P_0 \in \mathcal{P}_0^2$. Further, let $P \cap H^+ = H_1^+ \cap H^+ \cap P_0$ and $P \cap H^- = H_2^+ \cap H^- \cap P_0$. For $u \in H \cap (H_1^- \backslash H_1) \cap (H_2^- \backslash H_2)$, it follows from (25) that

$$\mathrm{Y}[P, u] + \mathrm{Y}[P \cap H, u] = \mathrm{Y}[P \cap H^+, u] + \mathrm{Y}[P \cap H^-, u].$$

By (23), this implies that

$$\lim_{u \to 0} \mathrm{Y}[P, u] + \mathrm{Y}[P \cap H] = \mathrm{Y}(P \cap H^+) + \mathrm{Y}(P \cap H^-). \tag{29}$$

On the other hand, it follows from (25) that

$$\mathrm{Y}[P, u] + \mathrm{Y}[P \cap H_1, w] = \mathrm{Y}[P, w] + \mathrm{Y}[P \cap H_1, u, w], \tag{30}$$

where $w \in H_1$ depends on u. Let B be a centered ball such that $[P \cap H_1, w] \subset B$ for $|u| < 1$ and let $\varepsilon > 0$. Since Y has the Cauchy property for $\mathcal{P}_0(H_1)$, it follows from (23) that on S^1,

$$|\,\mathrm{Y}[P \cap H_1, w] - \mathrm{Y}[P \cap H_1, u, w]| < \varepsilon$$

for $|u|$ sufficiently small. Thus, by (23), we obtain from (30) that $\lim_{u \to 0} \mathrm{Y}[P, u] = \mathrm{Y}\, P$. Combined with (29), this implies (28).

Second, let $n \geq 3$. Let $P = P_0 \cap H_1^+ \cap \cdots \cap H_n^+$, where $P_0 \in \mathcal{P}_0^n$. Since $P \cap H^+$, $P \cap H^- \in \mathcal{P}_n^n$, we can say that the support cone at 0 of $P \cap H^+$ is bounded by $H_1, H, H_3, \ldots, H_n$ and that the support cone at 0 of $P \cap H^-$ is bounded by $H, H_2, \ldots, H_n$, where $H_1 \cap H_2 \cap \cdots \cap H_{n-1} \subseteq H$. Therefore $\mathrm{Y}\, P = \lim_{u \to 0} \mathrm{Y}[P, u]$ and

$$\mathrm{Y}(P \cap H^+) = \lim_{u \to 0} \mathrm{Y}[P \cap H^+, u], \;\; \mathrm{Y}(P \cap H^-) = \lim_{u \to 0} \mathrm{Y}[P \cap H^-, u],$$

where $u \in H_1 \cap H_2 \cap \cdots \cap H_{n-1}$ and $u \in H_n^- \backslash H_n$. Applying (25) for $j = n$ shows that

$$\mathrm{Y}[P, u] + \mathrm{Y}[P \cap H, u] = \mathrm{Y}[P \cap H^+, u] + \mathrm{Y}[P \cap H^-, u].$$

Because of definition (23), this implies (28).

As a last step, we extend Y to a valuation on $\overline{\mathcal{P}}_0^n$. For $P \in \overline{\mathcal{P}}_0^n$, there are $P_1, \ldots, P_m \in \mathcal{P}_n^n$ such that $P = P_1 \cup \cdots \cup P_m$. It is proved in [**39**] that defining $h(\mathrm{Z}\, P, \cdot)$ by the inclusion-exclusion principle with $h(\mathrm{Z}\, P_i, \cdot)$, where $i = 1, \ldots, m$, leads to a well-defined extension of Z on $\overline{\mathcal{P}}_0^n$. Clearly, the extension is $\mathrm{GL}(n)$ covariant ($\mathrm{GL}(n)$ contravariant) if

Z is GL(n) covariant (GL(n) contravariant) on $\mathcal{P}^n_n$. This completes the proof of the lemma. q.e.d.

The following result is an immediate consequence of Lemma 3.1. Let $C_1^+(\mathbb{R}^n)$ denote the subset of non-negative functions in $C_1(\mathbb{R}^n)$.

Lemma 3.2. *If* $\mathrm{Y}: \mathcal{P}^n_0 \to \langle C_1^+(\mathbb{R}^n), +\rangle$ *is a valuation so that for every hyperplane H containing the origin and $w \in H^+\backslash H$, the operator* Y *has the Cauchy property for $\mathcal{P}_0(H)$ with respect to $\langle w\rangle$ and*

$$\lim_{u,v\to 0} \mathrm{Y}[P,u,v] = 0 \quad \textit{for every } P \in \mathcal{P}_0(H),$$

where $u, v \in \langle w\rangle$ with $u \in H^+\backslash H$ and $v \in H^-\backslash H$, then Y *can be extended to a valuation on $\overline{\mathcal{P}}^n_0$. Moreover, if* Y *is* GL(n) *covariant on $\mathcal{P}^n_0$, so is the extended valuation on $\overline{\mathcal{P}}^n_0$. If* Y *is* GL($n$) *contravariant on $\mathcal{P}^n_0$, so is the extended valuation on $\overline{\mathcal{P}}^n_0$.*

Proof. Let H be a hyperplane containing the origin and $P \in \mathcal{P}_0(H)$. Since Y is a non-negative valuation, for $u \in H^-\backslash H$ and $v \in H^+\backslash H$,

$$\begin{aligned} \mathrm{Y}[P,u,v] &= \mathrm{Y}[P,u,-t\,u] + \mathrm{Y}[P,-s\,v,v] - \mathrm{Y}[P,-s\,v,-t\,u] \\ &\le \mathrm{Y}[P,u,-t\,u] + \mathrm{Y}[P,-s\,v,v], \end{aligned}$$

when $t, s > 0$ are chosen suitably small. Thus (18) holds and Lemma 3.1 implies the existence of an extension. q.e.d.

4. The classification on $\mathcal{P}^2_0$

In Section 4.2, we prove the following result.

Proposition 4.1. *An operator* $\mathrm{Z}: \mathcal{P}^2_0 \to \langle \mathcal{K}^2, +\rangle$ *is a measurable valuation which is* GL(2) *covariant of weight $q \ge 0$ if and only if there are $c_0, c_1 \ge 0$, and $c_2 \in \mathbb{R}$ such that*

$$\mathrm{Z}\,P = \begin{cases} c_1\,\mathrm{M}\,P + c_2\,m(P) & \textit{for } q = 1 \\ c_0\,P + c_1\,(-P) & \textit{for } q = 0 \\ \{0\} & \textit{otherwise} \end{cases}$$

for every $P \in \mathcal{P}^2_0$.

By (16), Proposition 4.1 has the following immediate consequence.

Proposition 4.2. *An operator* $\mathrm{Z}: \mathcal{P}^2_0 \to \langle \mathcal{K}^2, +\rangle$ *is a measurable valuation which is* GL(2) *contravariant of weight $q \ge 1$ if and only if there are $c_0, c_1 \ge 0$, and $c_2 \in \mathbb{R}$ such that*

$$\mathrm{Z}\,P = \begin{cases} \rho_{\pi/2}(c_1\,\mathrm{M}\,P + c_2 m(P)) & \textit{for } q = 2 \\ \rho_{\pi/2}(c_0 P + c_1(-P)) & \textit{for } q = 1 \\ \{0\} & \textit{otherwise} \end{cases}$$

for every $P \in \mathcal{P}^2_0$.

By (13), Proposition 4.2 is equivalent to the classification of GL(2) covariant valuations of weight $q \le -1$. Hence Propositions 4.1 and 4.2 imply the following theorem. Note that the case $q \in (-1, 0)$ remains open.

Theorem 4.3. *An operator* $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{K}^2, + \rangle$ *is a measurable valuation which is* GL(2) *covariant of weight* $q \in (-\infty, -1] \cup [0, \infty)$ *if and only if there are* $c_0, c_1 \ge 0$, *and* $c_2 \in \mathbb{R}$ *such that*

$$\mathrm{Z}\, P = \begin{cases} c_1 \mathrm{M}\, P + c_2\, m(P) & \text{for } q = 1 \\ c_0 P + c_1(-P) & \text{for } q = 0 \\ \rho_{\pi/2}(c_0 P^* + c_1(-P^*)) & \text{for } q = -1 \\ \rho_{\pi/2}(c_1 \mathrm{M}\, P^* + c_2 m(P^*)) & \text{for } q = -2 \\ \{0\} & \text{otherwise} \end{cases}$$

for every $P \in \mathcal{P}_0^2$.

In the proof of Proposition 4.1, the following result is used.

Theorem 4.4 ([**36**]). *Let* $\mathrm{Z} : \overline{\mathcal{P}}_0^2 \to \langle \mathcal{K}^2, + \rangle$ *be a valuation which is* GL(n) *covariant of weight* q. *If* $q = 1$, *then there are constants* $c_1 \ge 0$ *and* $c_2 \in \mathbb{R}$ *such that*

$$\mathrm{Z}\, P = c_1 \mathrm{M}\, P + c_2\, m(P)$$

for every $P \in \overline{\mathcal{P}}_0^2$. *If* $q = 0$, *then there are constants* $a_0, b_0 \ge 0$ *and* $a_i, b_i \in \mathbb{R}$ *with* $a_i + b_0 + b_1 \ge 0$ *and* $a_0 + a_1 + b_i \ge 0$ *for* $i = 1, 2$ *such that*

$$\mathrm{Z}\, P = a_0\, P + b_0(-P) + \sum\nolimits_{i=1,2} (a_i\, \mathrm{E}_i(P) + b_i\, \mathrm{E}_i(-P))$$

for every $P \in \overline{\mathcal{P}}_0^2$. *In all other cases,* $\mathrm{Z}\, P = \{0\}$ *for every* $P \in \overline{\mathcal{P}}_0^2$.

Here E_1 and E_2 are certain operators that map each $P \in \mathcal{P}_0^2$ to $\{0\}$.

4.1. An auxiliary result. We derive a lemma that is also used for $n \ge 3$. For $v = (v', 1)$ with $v' \in \mathbb{R}^{n-1}$, define the map $\phi_v \in \mathrm{SL}(n)$ by $\phi_v e_n = v$ and $\phi_v e_i = e_i$ for $i = 1, \ldots, n-1$. For $r > 0$, define the map $\phi_r \in \mathrm{GL}(n)$ by $\phi_r e_n = r\, e_n$ and $\phi_r e_i$ to e_i for $i = 1, \ldots, n-1$. We say that a function $z : \mathcal{P}_0^n \to \langle \mathbb{R}, + \rangle$ is ϕ_r homogeneous of degree p if

$$z(\phi_r[P, s\, u, t\, v]) = r^p z([P, s\, u, t\, v])$$

for all $P \in \mathcal{P}_0(e_n^\perp)$, for all $r > 0$, and for all $u = (u', -1)$ and $v = (v', 1)$ with $u', v' \in \mathbb{R}^{n-1}$. Let B be a centered ball.

Lemma 4.5. *Suppose $z : \mathcal{P}_0^n \to \langle \mathbb{R}, + \rangle$ is a measurable valuation which is ϕ_r homogeneous of degree $p \neq 1$ and $v = (v', 1)$ with $v' \in \mathbb{R}^{n-1}$. If*

$$\lim_{s,t \to 0} z([P, -s\, e_n, t\, e_n])$$

exists uniformly for all $P \in \mathcal{P}_0(e_n^\perp)$ with $P \subset B$ and

$$\lim_{s,t \to 0} z(\phi_v [P, -s\, e_n, t\, e_n]) = z([P, -s\, e_n, t\, e_n])$$

uniformly for all $P \in \mathcal{P}_0(e_n^\perp)$ with $P \subset B$, then

$$\lim_{s,t \to 0} z([P, s\, u, t\, v]) = \lim_{s,t \to 0} z([P, -s\, e_n, t\, e_n])$$

uniformly for all $P \in \mathcal{P}_0(e_n^\perp)$ with $P \subset B$ and all $u = (u', -1)$ with $u' \in \mathbb{R}^{n-1}$.

Proof. First, we show that

$$\lim_{s,t \to 0} z([P, s\, u, t\, v]) \tag{31}$$

exists uniformly for $P \subset B$. Since z is a valuation and since u and v lie in complementary halfspaces, we have for $s, t > 0$ suitably small, $0 < t' < t$, and $t'' > 0$ suitably large with respect to s,

$$\begin{aligned} z([P, s\, u, t\, v]) &+ z([P, -t''\, v, t'\, v]) \\ &= z([P, s\, u, t'\, v]) + z([P, -t''\, v, t\, v]). \end{aligned} \tag{32}$$

Since $[P, -t''\, v, t\, v] = \phi_v [P, I]$ for $I = [-t''\, e_n, t\, e_n]$, we have by assumption

$$z([P, -t''\, v, t\, v]) = z(\phi_v [P, I]) = z([P, I]) + o(1) \tag{33}$$

as $t, t'' \to 0$ uniformly for $P \subset B$. Since $\lim_{s,t \to 0} z([P, I])$ exists uniformly for $P \subset B$, we obtain from (32) and (33) that for $t', t'' = O(t)$,

$$\begin{aligned} &z([P, s\, u, t\, v]) - z([P, s\, u, t'\, v]) \\ &\qquad = z([P, [-t''\, e_n, t\, e_n]]) - z([P, [-t''\, e_n, t'\, e_n]]) + o(1) = o(1) \end{aligned}$$

as $t \to 0$ uniformly for $P \subset B$. Similarly, for $s, t' > 0$ suitably small, $0 < s' < s$,

$$z([P, s\, u, t'\, v]) - z([P, s'\, u, t'\, v]) = o(1)$$

as $s \to 0$ uniformly for $P \subset B$. Thus the limit (31) exists uniformly for $P \subset B$.

For $P \in \mathcal{P}_0(e_n^\perp)$ fixed, we set

$$f(u', v') = \lim_{s,t \to 0} z([P, s\, u, t\, v]).$$

Note that $f(0,0) = \lim_{s,t \to 0} z([P, I])$. Since z is a valuation, we have for $r > 0$ suitably small,

$$z([P, s\, u, t\, v]) + z([P, -rs\, e_n, rt\, e_n]) = z([P, s\, u, rt\, e_n]) + z([P, -rs\, e_n, t\, v]).$$

Taking the limit as $s, t \to 0$ gives

$$f(u', v') + f(0,0) = f(u',0) + f(0,v'). \tag{34}$$

Since $\phi_v[P, s\,(u'+v', -1), t\,e_n] = [P, s\,u, t\,v]$, by assumption

$$z([P, s\,u, t\,v]) = z([P, s\,(u'+v', -1), t\,e_n]) + o(1).$$

Thus $f(u', v') = f(u'+v', 0)$. Setting $g(u') = f(u', 0) - f(0,0)$, it follows from (34) that

$$g(u' + v') = g(u') + g(v')$$

for $u', v' \in \mathbb{R}^{n-1}$. This is the Cauchy functional equation. Since z is measurable, so is g, and there is a vector $w(P) \in \mathbb{R}^{n-1}$ such that $g(u') = w(P) \cdot u'$. Thus

$$\lim_{s,t\to 0} z([P, s\,u, t\,v]) = w(P) \cdot (u' + v') + f(0,0). \tag{35}$$

Using ϕ_r, it follows from (35) and the assumption that

$$\frac{1}{r}\, w(P) \cdot (u' + v') + f(0,0) = r^p (w(P) \cdot (u' + v') + f(0,0)).$$

Since this holds for all $0 < r \le 1$, we obtain $w(P) = 0$ (and $f(0,0) = 0$ for $p \neq 0$). This implies the statement of the lemma. q.e.d.

4.2. Proof of Proposition 4.1. Let $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{K}^2, + \rangle$ be a measurable valuation which is GL(2) covariant of weight q.

Lemma 4.6. *For $q \neq -1$, there exist $a, b \in \mathbb{R}$ such that for all $s_i, t_i > 0$*

$$\begin{aligned} h(\mathrm{Z}[I_1, I_2], e_1) &= (a\, s_1^{q+1} + b\, t_1^{q+1})(s_2^q + t_2^q) \\ h(\mathrm{Z}[I_1, I_2], e_2) &= (a\, s_2^{q+1} + b\, t_2^{q+1})(s_1^q + t_1^q) \end{aligned}$$

where $I_i = [-s_i\, e_i, t_i\, e_i]$.

Proof. For $P \in \mathcal{P}_0^2$, set $z_i(P) = h(\mathrm{Z}\,P, e_i)$. If $\phi \in \mathrm{GL}(2)$ is the transformation that leaves e_1 fixed and multiplies e_2 by $r > 0$, then (2) implies $z_2([I_1, r\,I_2]) = r^{q+1} z_2([I_1, I_2])$. Thus $z_2([I_1, \cdot\,])$ is homogeneous of degree $q+1$ and it follows from (11) that there are $a(I_1), b(I_1) \in \mathbb{R}$ such that

$$z_2([I_1, I_2]) = a(I_1)\, s_2^{q+1} + b(I_1)\, t_2^{q+1}.$$

If $\phi \in \mathrm{GL}(2)$ is the transformation that leaves e_2 fixed and multiplies e_1 by $r > 0$, then (2) implies $z_2([r\,I_1, I_2]) = r^q z_2([I_1, I_2])$. Thus a and b are valuations that are homogeneous of degree q. For $q \neq 0$, it follows from (11) that there are constants $a, b, c, d \in \mathbb{R}$ such that

$$z_2([I_1, I_2]) = (a\, s_2^{q+1} + b\, t_2^{q+1})(c\, s_1^q + d\, t_1^q) \tag{36}$$

for all $s_i, t_i > 0$. For $q = 0$, it follows from (10) and (11) that there are constants $a, b, c, d \in \mathbb{R}$ such that

$$z_2([I_1, I_2]) = (a\, s_2 + b\, t_2)(c \log(\frac{t_1}{s_1}) + d) \tag{37}$$

for all $s_i, t_i > 0$. Define the transformation $\psi \in \mathrm{GL}(2)$ by $\psi e_1 = -e_1$ and $\psi e_2 = -e_2$. By (2) and (3), $z_2(\psi[I_1, I_2]) = z_2([-I_1, I_2]) = z_2([I_1, I_2])$. Combined with (36) and (37), this gives the formula for z_2. The formula for z_1 is obtained by applying $\rho_{\pi/2}$. q.e.d.

Lemma 4.7. *For $q \geq 0$, the operator $P \mapsto h(\mathrm{Z}\, P, \cdot)$ for $P \in \mathcal{P}_0^2$ has the Cauchy property for $\mathcal{P}_0(e_1)$.*

Proof. For $Q \in \mathcal{P}_0^2$, set $z_i(Q) = h(\mathrm{Z}\, Q, e_i)$ and let B be a centered ball. First, we show that there exist constants $a, b \in \mathbb{R}$ so that for all $I_1 = [-s_1\, e_1, t_1\, e_1]$, where $s_1, t_1 > 0$, for all $s, t > 0$ and for all $u = (u', -1)$ and $v = (v', 1)$, where $u', v' \in \mathbb{R}$, we have

$$z_2([I_1, s\, u, t\, v]) = (a\, s^{q+1} + b\, t^{q+1})(s_1^q + t_1^q) \tag{38}$$

whenever $[s\, u, t\, v]$ intersects I_1.

Let $I = [-s\, e_2, t\, e_2]$ and define $\phi_v, \phi_r \in \mathrm{GL}(2)$ as in Lemma 4.5. By Lemma 4.6, $\lim_{s,t\to 0} z_2([I_1, I])$ exists uniformly for $I_1 \subset B$. By (2), we have $z_2(\phi_r[I_1, I]) = r^{q+1} z_2([I_1, I])$ and

$$z_2([I_1, -s\, v, t\, v]) = z_2(\phi_v[I_1, I]) = z_2([I_1, I]).$$

Thus we apply Lemma 4.5 and obtain by Lemma 4.6 that

$$\lim_{s,t\to 0} z_2([I_1, s\, u, t\, v]) = 0. \tag{39}$$

Since z_2 is a valuation, we have for all $s', t' > 0$ sufficiently small,

$$z_2([I_1, s\, u, t\, v]) + z_2([I_1, -t'v, t'v]) = z_2([I_1, s\, u, t'v]) + z_2([I_1, -t'v, t\, v])$$
$$z_2([I_1, s\, u, t'v]) + z_2([I_1, s'u, -s'u]) = z_2([I_1, s\, u, -s'u]) + z_2([I_1, s'u, t'v]).$$

Hence we have for all $s', t' > 0$ sufficiently small,

$$\begin{aligned} z_2([I_1, s\, u, t\, v]) &= z_2([I_1, -t'\, v, t\, v]) - z_2([I_1, -t'\, v, t'\, v])) \\ &\quad + z_2([I_1, s\, u, -s'\, u] + z_2([I_1, s'\, u, t'\, v]) - z_2([I_1, s'\, u, -s'\, u]). \end{aligned}$$

Taking the limit as $s', t' \to 0$ and using (39) and Lemma 4.6 gives (38).

It follows from (2) that

$$h(\mathrm{Z}[I_1, -s\, v, t\, v], e_1) = h(\mathrm{Z}[I_1, I], \phi_v^t e_1) = h(\mathrm{Z}[I_1, I], e_1 + v'\, e_2).$$

Hence, Lemma 4.6, (38) and (4) imply

$$z_1([I_1, -s\, v, t\, v]) = (a\, s_1^{q+1} + b\, t_1^{q+1})(s^q + t^q) + o(1)$$

as $s,t\to 0$ uniformly for $I_1\subset B$. Note that $\lim_{s,t\to 0} z_1([I_1,I])$ exists uniformly for $I_1\subset B$ and that by (2), $z_1(\phi_r[I_1,u,v]) = r^q\, z_1([I_1,u,v])$. Thus we apply Lemma 4.5 and obtain by Lemma 4.6 that for $q>0$

$$\lim_{s,t\to 0} z_1([I_1, s\,u, t\,v]) = 0 \tag{40}$$

uniformly for $I_1\subset B$ and for $q=0$,

$$\lim_{s,t\to 0} z_1([I_1, s\,u, t\,v]) = 2(a\,s_1 + b\,t_1) \tag{41}$$

uniformly for $I_1\subset B$.

Let $x=(x_1,x_2)\in S^1$. From (4) and (39), we obtain that

$$h(\mathrm{Z}[I_1, s\,u, t\,v], x) = z_1([I_1, s\,u, t\,v])\, x_1 + o(1)$$

as $s,t\to 0$ uniformly for $I_1\subset B$. Combined with (40) and (41), this completes the proof of the lemma. q.e.d.

Set $\mathrm{Y}\,P = h(\mathrm{Z}\,P,\cdot)$ for $P\in\mathcal{P}_0^2$. Then $\mathrm{Y}:\mathcal{P}_0^2\to\langle C_1(\mathbb{R}^2),+\rangle$ is a valuation which is $\mathrm{GL}(2)$ covariant of weight q. For H a hyperplane containing the origin, Lemma 4.7 shows that Y has the Cauchy property for $\mathcal{P}_0(H)$. Hence Lemma 3.1 implies that we can extend Y and therefore Z to $\overline{\mathcal{P}}_0^2$. Thus Theorem 4.4 implies the statement of the proposition.

5. The classification on $\mathcal{P}_0^n$ for $n\ge 3$

The aim of this section is to establish the following result.

Theorem 5.1. *An operator $\mathrm{Z}:\mathcal{P}_0^n\to\langle\mathcal{K}^n,+\rangle$, where $n\ge 3$, is a measurable valuation which is $\mathrm{GL}(n)$ covariant of weight q if and only if there are $c_0,c_1\ge 0$ and $c_2\in\mathbb{R}$ such that*

$$\mathrm{Z}\,P = \begin{cases} c_1\,\mathrm{M}\,P + c_2\,m(P) & \text{for } q=1\\ c_0\,P + c_1(-P) & \text{for } q=0\\ c_0\,\Pi\,P^* & \text{for } q=-1\\ \{0\} & \text{otherwise}\end{cases}$$

for every $P\in\mathcal{P}_0^n$.

The proof of Theorem 5.1 is split into two cases. First, we derive in Section 5.1 the following classification of $\mathrm{GL}(n)$ covariant valuations.

Proposition 5.2. *An operator $\mathrm{Z}:\mathcal{P}_0^n\to\langle\mathcal{K}^n,+\rangle$, where $n\ge 3$, is a measurable valuation which is $\mathrm{GL}(n)$ covariant of weight $q>-1$ if and only if there are $c_0,c_1\ge 0$ and $c_2\in\mathbb{R}$ such that*

$$\mathrm{Z}\,P = \begin{cases} c_1\,\mathrm{M}\,P + c_2\,m(P) & \text{for } q=1\\ c_0\,P + c_1(-P) & \text{for } q=0\\ \{0\} & \text{otherwise}\end{cases}$$

for every $P\in\mathcal{P}_0^n$.

In Section 5.2, we derive the following classification of $\mathrm{GL}(n)$ contravariant valuations.

Proposition 5.3. *An operator* $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{K}^n, + \rangle$*, where* $n \geq 3$*, is a measurable valuation which is* $\mathrm{GL}(n)$ *contravariant of weight* $q \geq 1$ *if and only if there is a constant* $c \geq 0$ *such that*

$$\mathrm{Z}\, P = \begin{cases} c\, \Pi\, P & \text{for } q = 1 \\ \{0\} & \text{otherwise} \end{cases}$$

for every $P \in \mathcal{P}_0^n$.

By (13), Proposition 5.3 is equivalent to the classification of $\mathrm{GL}(n)$ covariant valuations of weight $q \leq -1$. Hence Propositions 5.2 and 5.3 imply Theorem 5.1, while Theorem 2 is an immediate consequence of Theorem 5.1.

In the proof of Theorem 5.1, the following results are used.

Theorem 5.4 ([**36**])**.** *Let* $\mathrm{Z} : \overline{\mathcal{P}}_0^n \to \langle \mathcal{K}^n, + \rangle$*, where* $n \geq 3$*, be a valuation which is* $\mathrm{GL}(n)$ *covariant of weight* q*. If* $q = 1$*, then there are* $c_1 \geq 0$ *and* $c_2 \in \mathbb{R}$ *such that*

$$\mathrm{Z}\, P = c_1\, \mathrm{M}\, P + c_2\, m(P)$$

for every $P \in \overline{\mathcal{P}}_0^n$*. If* $q = 0$*, then there are* $c_0, c_1 \geq 0$ *such that*

$$\mathrm{Z}\, P = c_0\, P + c_1(-P)$$

for every $P \in \overline{\mathcal{P}}_0^n$*. In all other cases,* $\mathrm{Z}\, P = \{0\}$ *for every* $P \in \overline{\mathcal{P}}_0^n$.

Theorem 5.5 ([**36**])**.** *Let* $\mathrm{Z} : \overline{\mathcal{P}}_0^n \to \langle \mathcal{K}^n, + \rangle$*, where* $n \geq 3$*, be a valuation which is* $\mathrm{GL}(n)$ *contravariant of weight* q*. If* $q = 1$*, then there are* $c_1, c_2, c_3 \in \mathbb{R}$ *with* $c_1 \geq 0$ *and* $c_1 + c_2 + c_3 \geq 0$ *such that*

$$\mathrm{Z}\, P = c_1\, \Pi\, P + c_2\, \Pi_0\, P + c_3(-\,\Pi_0\, P)$$

for every $P \in \overline{\mathcal{P}}_0^n$*. In all other cases,* $\mathrm{Z}\, P = \{0\}$ *for every* $P \in \overline{\mathcal{P}}_0^n$.

Here Π_0 is a certain operator with the property that $\Pi_0\, P = \{0\}$ for $P \in \mathcal{P}_0^n$.

5.1. Proof of Proposition 5.2. Let $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{K}^n, + \rangle$, where $n \geq 3$, be a measurable valuation which is $\mathrm{GL}(n)$ covariant of weight q. For $n > 3$, assume that Proposition 5.2 holds in dimension $(n-1)$.

Lemma 5.6. *For* $q > -1$*, there exist* $a, b \in \mathbb{R}$ *such that*

$$h(\mathrm{Z}[P, I], e_n) \quad = \quad \begin{cases} a\, s + b\, t & \text{for } q = 0 \\ (a\, s^2 + b\, t^2) V_{n-1}(P) & \text{for } q = 1 \\ 0 & \text{otherwise} \end{cases}$$

for every $I = [-s\, e_n, t\, e_n]$ *with* $s, t > 0$ *and* $P \in \mathcal{P}_0(e_n^\perp)$.

Proof. Set $z([P,I]) = h(\mathrm{Z}[P,I],e_n)$. If $\phi \in \mathrm{GL}(n)$ leaves $e_n^\perp$ fixed and multiplies e_n by $r>0$, then (2) implies $z([P,r\,I]) = r^{q+1} z([P,I])$. Thus $z([P,\cdot])$ is a valuation which is homogeneous of degree $q+1$. If $\phi \in \mathrm{GL}(n)$ is a transformation that leaves e_n fixed, then (2) implies $z(\phi[P,I]) = |\det\phi|^q z([P,I])$. Thus $z([\cdot,I])$ is a valuation for which (17) holds in dimension $(n-1)$. The statements follow from Theorem 1.1 and (11). q.e.d.

Lemma 5.7. *For $q>-1$ and $q\notin\{0,1\}$, we have $\mathrm{Z}[P,I]=\{0\}$ for every $I\in\mathcal{P}_0(e_n)$ and $P\in\mathcal{P}_0(e_n^\perp)$.*

Proof. By Lemma 5.6 and the $\mathrm{GL}(n)$ covariance of Z,

$$h(\mathrm{Z}[P,I],e_n) = h(\mathrm{Z}[P,I],-e_n) = 0. \tag{42}$$

For $I\in\mathcal{P}_0(e_n)$, define $\mathrm{Z}_I : \mathcal{P}_0^{n-1} \to \langle\mathcal{K}^{n-1},+\rangle$ by setting $h(\mathrm{Z}_I\,P,x) = h(\mathrm{Z}[P,I],x)$ for $x\in\mathbb{R}^{n-1}$ and $P\in\mathcal{P}_0^{n-1}$, where we identify $e_n^\perp$ and $\mathbb{R}^{n-1}$. Note that Z_I is a valuation on $\mathcal{P}_0^{n-1}$ that is $\mathrm{GL}(n-1)$ covariant of weight q.

Let $n=3$. Define $\mathrm{Z}_I^s : \mathcal{P}_0^2 \to \langle\mathcal{K}_c^2,+\rangle$ by $\mathrm{Z}_I^s\,P = \mathrm{Z}_I(P)+\mathrm{Z}_I(-P)$. Since Z_I^s is $\mathrm{GL}(2)$ covariant of weight q, Lemma 4.6 implies that there are $a(I)\in\mathbb{R}$ such that

$$\begin{aligned} h(\mathrm{Z}_I^s[I_1,I_2],e_1) &= a(I)(s_1^{q+1}+t_1^{q+1})(s_2^q+t_2^q)\\ h(\mathrm{Z}_I^s[I_1,I_2],e_2) &= a(I)(s_1^{q}+t_1^{q})(s_2^{q+1}+t_2^{q+1}) \end{aligned}$$

where $I_i = [-s_i\,e_i, t_i\,e_i]$ and $s_i,t_i>0$ for $i=1,2$. Define $Q\in\mathcal{P}_0^3$ as the convex hull of I_1, I_2, and I. Let ψ be the linear transformation so that $\psi\,e_1 = e_3$ and $\psi\,e_2=e_2$ and $\psi\,e_3=e_1$. Since Z is $\mathrm{GL}(3)$ covariant, it follows from (42) that

$$h(\mathrm{Z}(\psi Q),e_1) = h(\mathrm{Z}\,Q,e_3) = 0.$$

Let $I=[-s\,e_3,t\,e_3]$ with $s,t>0$ and set $s_i=s$ and $t_i=t$ for $i=1,2$. We conclude that

$$h(\mathrm{Z}_I^s[I_1,I_2],e_1)=0.$$

Hence $a(I)=0$. Consequently, Lemma 3.2 shows that we can extend Z_I^s to $\overline{\mathcal{P}}_0^2$. Theorem 4.4 shows that $\mathrm{Z}_I^s\,P=\{0\}$ for all $P\in\mathcal{P}_0^2$. Since $\mathrm{Z}_I(P)+\mathrm{Z}_I(-P)=\{0\}$ for all $P\in\mathcal{P}_0^2$, we see that Z_I is vector-valued. Consequently, Theorem 1.2 implies that $\mathrm{Z}_I\,P=\{0\}$ for all $P\in\mathcal{P}_0^2$. Hence the lemma holds true for $n=3$.

Let $n>3$. Since Proposition 5.2 holds in dimension $(n-1)$, we obtain that $\mathrm{Z}_I\,P=\{0\}$ for all $P\in\mathcal{P}_0(e_n^\perp)$. Hence the statement of the lemma holds true for every $n>3$. q.e.d.

Lemma 5.8. *If $q>-1$ and $q\notin\{0,1\}$, then $\mathrm{Z}\,P=\{0\}$ for all $P\in\mathcal{P}_0^n$.*

Proof. Set $\mathrm{Z}_s P = \mathrm{Z}\, P + \mathrm{Z}(-P)$ and $\mathrm{Y}\, P = h(\mathrm{Z}_s P, \cdot)$. Note that $\mathrm{Y} : \mathcal{P}_0 \to C_1^+(\mathbb{R}^n)$ is a $\mathrm{GL}(n)$ covariant valuation. By Lemma 5.7 and Lemma 3.2, we can extend Y and thus Z_s to $\overline{\mathcal{P}}_0^n$. Theorem 5.4 implies that $\mathrm{Z}_s P = \{0\}$ for all $P \in \mathcal{P}_0$. Hence Z is vector-valued and Theorem 1.3 implies the statement of the lemma. q.e.d.

Lemma 5.9. *There exist $a, b \in \mathbb{R}$ so that for all $P \in \mathcal{P}_0(e_n^\perp)$, for all $s, t > 0$, and for all $u = (u', -1)$ and $v = (v', 1)$ with $u', v' \in \mathbb{R}^{n-1}$,*

$$h(\mathrm{Z}[P, s\,u, t\,v], e_n) = \begin{cases} a\,s + b\,t & \text{for } q = 0 \\ (a\,s^2 + b\,t^2) V_{n-1}(P) & \text{for } q = 1 \end{cases}$$

whenever $[s\,u, t\,v]$ intersects P.

Proof. Set $z([P, s\,u, t\,v]) = h(\mathrm{Z}[P, s\,u, t\,v], e_n)$. Let B be a centered ball, let $I = [-s\,e_n, t\,e_n]$, and let $\phi_v, \phi_r \in \mathrm{GL}(n)$ be defined as in Lemma 4.5. By Lemma 5.6, $\lim_{s,t\to 0} z([P, I])$ exists uniformly for $I \subset B$. It follows from (2) that $z(\phi_r[P, u, v]) = r^{q+1} z([P, u, v])$ and

$$z([P, -s\,v, t\,v]) = z(\phi_v[P, I]) = z([P, I]).$$

Thus we apply Lemma 4.5 and obtain by Lemma 5.6 that

$$\lim_{s,t\to 0} z([P, s\,u, t\,v]) = 0. \tag{43}$$

Since for all $s', t' > 0$ sufficiently small,

$$\begin{aligned} z([P, s\,u, t\,v]) = {} & z([P, -t'\,v, t\,v]) - z([P, -t'\,v, t'\,v]) + \\ & + z([P, s\,u, -s'\,u]) + z([P, s'\,u, t'\,v]) - z([P, s'\,u, -s'\,u]), \end{aligned}$$

taking the limit as $s', t' \to 0$ and using (43) and Lemma 5.6 gives the statement of the lemma. q.e.d.

Lemma 5.10. *There exist $a, b \in \mathbb{R}$ such that for $x \in e_n^\perp$*

$$h(\mathrm{Z}[P, I], x) = \begin{cases} a\,h(P, x) + b\,h(-P, x) & \text{for } q = 0 \\ a(s+t)\,h(\mathrm{M}\,P, x) + b(s+t)\,h(m(P), x) & \text{for } q = 1 \end{cases}$$

for every $I = [-s\,e_n, t\,e_n]$ and $P \in \mathcal{P}_0(e_n^\perp)$.

Proof. For $I \in \mathcal{P}_0(e_n)$, define $\mathrm{Z}_I : \mathcal{P}_0^{n-1} \to \langle \mathcal{K}^{n-1}, + \rangle$ by setting $h(\mathrm{Z}_I\, P, x) = h(\mathrm{Z}[P, I], x)$ for $x \in \mathbb{R}^{n-1}$ and $P \in \mathcal{P}_0^{n-1}$, where we identify $e_n^\perp$ and $\mathbb{R}^{n-1}$. Note that Z_I is a valuation on $\mathcal{P}_0^{n-1}$ that is $\mathrm{GL}(n-1)$ covariant of weight q. By Theorem 4.3 for $n = 3$ and by Proposition 5.2 in dimension $(n-1)$ for $n > 3$, we obtain that there are $a(I), b(I) \in \mathbb{R}$ such that for $x \in e_n^\perp$,

$$h(\mathrm{Z}[P, I], x) = \begin{cases} a(I)\,h(P, x) + b(I)\,h(-P, x) & \text{for } q = 0 \\ a(I)\,h(\mathrm{M}\,P, x) + b(I)\,h(m(P), x) & \text{for } q = 1. \end{cases}$$

If $\phi_r \in \mathrm{GL}(n)$ is a transformation that leaves $e_n^\perp$ fixed and multiplies e_n by $r > 0$, then (2) implies $h(\mathrm{Z}[P, r\, I], x) = r^q h(\mathrm{Z}[P, I], x)$ for $x \in e_n^\perp$. Thus a and b are valuations on $\mathcal{P}_0^1$ which are homogeneous of degree q. By (2), we see that $a(I) = a(-I)$ and $b(I) = b(-I)$ for $I \in \mathcal{P}_0^1$. Thus (10) and (11) imply the statement of the lemma. q.e.d.

Lemma 5.11. *If $q = 0$ or $q = 1$, then the operator $P \mapsto h(\mathrm{Z}\, P, \cdot)$ for $P \in \mathcal{P}_0^n$ has the Cauchy property for $\mathcal{P}_0(e_n^\perp)$.*

Proof. Let B be a centered ball and $s, t > 0$. First, we show that for all $P \in \mathcal{P}_0(e_n^\perp)$ with $P \subset B$ and for all $u = (u', -1)$ and $v = (v', 1)$ with $u', v' \in \mathbb{R}^{n-1}$, we have for $x' \in e_n^\perp \cap B$,

$$(44) \quad h(\mathrm{Z}[P, s\, u, t\, v], x') = \begin{cases} a\, h(P, x') + b\, h(-P, x') + o(1) & \text{for } q = 0 \\ o(1) & \text{for } q = 1 \end{cases}$$

as $s, t \to 0$ uniformly whenever $[s\, u, t\, v]$ intersects P.

Let $I = [-s\, e_n, t\, e_n]$ and define $\phi_v, \phi_r \in \mathrm{GL}(n)$ as in Lemma 4.5. Since $[P, -s\, v, t\, v] = \phi_v[P, I]$, we obtain from (2) that

$$h(\mathrm{Z}[P, -s\, v, t\, v], x') = h(\mathrm{Z}(\phi_v[P, I]), x') = h(\mathrm{Z}[P, I], x' + (x' \cdot v)\, e_n).$$

By Lemmas 5.6 and 5.10 it follows from (4) that

$$h(\mathrm{Z}[P, -s\, v, t\, v], x') = \begin{cases} a\, h(P, x') + b\, h(-P, x') + o(1) & \text{for } q = 0 \\ o(1) & \text{for } q = 1 \end{cases}$$

as $s, t \to 0$ uniformly. In particular, $\lim_{s,t \to 0} h(\mathrm{Z}[P, I], x')$ exists uniformly. It follows from (2) that $h(\mathrm{Z}(\phi_r[P, I], x') = h(\mathrm{Z}[P, I], x')$. Thus we apply Lemma 4.5 for $z = h(\mathrm{Z}[P, u, v], x')$ and obtain by Lemma 5.10 that (44) holds.

Let $x = (x', x_n) \in S^{n-1}$, where $x' \in e_n^\perp$ and $x_n \in \mathbb{R}$. From (4) and Lemma 5.6, we obtain that

$$h(\mathrm{Z}[P, s\, u, t\, v], x) = h(\mathrm{Z}[P, s\, u, t\, v], x') + o(1)$$

as $s, t \to 0$. Combined with (44), this completes the proof of the lemma. q.e.d.

Lemma 5.12. *If $q = 0$, then there are constants $c_0, c_1 \geq 0$ such that*

$$\mathrm{Z}\, P = c_0\, P + c_1(-P)$$

for all $P \in \mathcal{P}_0^n$. If $q = 1$, then there are constants $c_1 \geq 0$ and $c_2 \in \mathbb{R}$ such that

$$\mathrm{Z}\, P = c_1\, \mathrm{M}\, P + c_2\, m(P)$$

for all $P \in \mathcal{P}_0^n$.

Proof. For $P \in \mathcal{P}_0^n$, set $\mathrm{Y}(P) = h(\mathrm{Z}\, P, \cdot)$. By Lemmas 5.9 and 5.11, we can apply Lemma 3.1 and extend Y and therefore Z to $\overline{\mathcal{P}}_0^n$. Hence Theorem 5.4 implies the statement of the lemma. q.e.d.

5.2. Proof of Proposition 5.3. Let $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{K}^n, + \rangle$, where $n \geq 3$, be a measurable valuation which is $\mathrm{GL}(n)$ contravariant of weight q. For $n > 3$, assume that Proposition 5.3 holds in dimension $(n-1)$.

Lemma 5.13. *For $q \geq 1$, there exist $a, b \in \mathbb{R}$ such that*

$$h(\mathrm{Z}[P,I], e_n) \quad = \quad \begin{cases} (a \log(t/s) + b) V_{n-1}(P) & \text{for } q = 1 \\ 0 & \text{otherwise} \end{cases}$$

for every $I = [-s\, e_n, t\, e_n]$ and $P \in \mathcal{P}_0(e_n^\perp)$.

Proof. Set $z([P,I]) = h(\mathrm{Z}[P,I], e_n)$. If $\phi \in \mathrm{GL}(n)$ leaves $e_n^\perp$ fixed and multiplies e_n by $r > 0$, then (12) implies $z([P, r\, I]) = r^{q-1} z([P,I])$. Thus $z([P, \cdot])$ is a valuation which is homogeneous of degree $q-1$. If $\phi \in \mathrm{GL}(n)$ is a transformation that leaves e_n fixed, then (12) implies $z(\phi [P, I]) = |\det \phi|^q z([P,I])$. Thus $z([\cdot, I])$ is a valuation for which (17) holds in dimension $(n-1)$. The statements follow from Theorem 1.1 and (10). q.e.d.

Lemma 5.14. *For $q > 1$, we have $\mathrm{Z}[P,I] = \{0\}$ for every $I \in \mathcal{P}_0(e_n)$ and $P \in \mathcal{P}_0(e_n^\perp)$.*

Proof. By Lemma 5.13 and the $\mathrm{GL}(n)$ contravariance of Z,

$$h(\mathrm{Z}[P,I], e_n) = h(\mathrm{Z}[P,I], -e_n) = 0. \tag{45}$$

For $I \in \mathcal{P}_0(e_n)$, define $\mathrm{Z}_I : \mathcal{P}_0^{n-1} \to \langle \mathcal{K}^{n-1}, + \rangle$ by setting $h(\mathrm{Z}_I\, P, x) = h(\mathrm{Z}[P,I], x)$ for $x \in \mathbb{R}^{n-1}$ and $P \in \mathcal{P}_0^{n-1}$, where we identify $e_n^\perp$ and $\mathbb{R}^{n-1}$. Note that Z_I is a $\mathrm{GL}(n-1)$ contravariant valuation of weight q on $\mathcal{P}_0^{n-1}$.

Let $n = 3$. Define $\mathrm{Z}_I^s : \mathcal{P}_0^2 \to \langle \mathcal{K}_c^2, + \rangle$ by $\mathrm{Z}_I^s\, P = \mathrm{Z}_I(P) + \mathrm{Z}_I(-P)$. Since Z_I^s is $\mathrm{GL}(2)$ contravariant of weight q, (16) implies that $\rho_{\pi/2}\, \mathrm{Z}_I$ is $\mathrm{GL}(2)$ covariant of weight $q-1$. It follows from Lemma 4.6 that there is $a(I) \in \mathbb{R}$ such that

$$\begin{aligned} h(\rho_{\pi/2}\, \mathrm{Z}_I^s[I_1, I_2], e_1) &= a(I)(s_1^q + t_1^q)(s_2^{q-1} + t_2^{q-1}) \\ h(\rho_{\pi/2}\, \mathrm{Z}_I^s[I_1, I_2], e_2) &= a(I)(s_1^{q-1} + t_1^{q-1})(s_2^q + t_2^q) \end{aligned}$$

where $I_i = [-s_i\, e_i, t_i\, e_i]$. Define $Q \in \mathcal{P}_0^3$ as the convex hull of I_1, I_2, and I. Let ψ be the linear transformation so that $\psi\, e_1 = e_3$ and $\psi\, e_2 = e_2$ and $\psi\, e_3 = e_1$. Since Z is $\mathrm{GL}(3)$ contravariant, it follows from (45) that

$$h(\mathrm{Z}(\psi Q), e_1) = h(\mathrm{Z}\, Q, e_3) = 0.$$

Let $I = [-s\, e_3, t\, e_3]$ with $s, t > 0$ and set $s_i = s$ and $t_i = t$ for $i = 1, 2$. We conclude that

$$h(\mathrm{Z}_I^s[I_1, I_2], e_1) = 0.$$

Hence $a(I) = 0$. Consequently, Lemma 3.2 shows that we can extend Z_I^s to $\overline{\mathcal{P}}_0^2$. By Theorem 4.4, we have $\rho_{-\pi/2}\, \mathrm{Z}_I^s\, P = \{0\}$ for all $P \in \mathcal{P}_0^2$. Since $\mathrm{Z}_I(P) + \mathrm{Z}_I(-P) = \{0\}$ for all $P \in \mathcal{P}_0^2$, Z_I is vector-valued. Theorem 1.2

implies that $\rho_{\pi/2}\,\mathrm{Z}_I\,P = \{0\}$ for all $P \in \mathcal{P}_0^2$. Hence the lemma holds true for $n = 3$.

Let $n > 3$. Since Proposition 5.3 holds in dimension $(n-1)$, we obtain that $\mathrm{Z}_I\,P = \{0\}$ for all $P \in \mathcal{P}_0(e_n^\perp)$. Hence the statement of the lemma holds true for every $n > 3$. q.e.d.

Lemma 5.15. *If $q > 1$, then $\mathrm{Z}\,P = \{0\}$ for all $P \in \mathcal{P}_0^n$.*

Proof. Set $\mathrm{Z}_s\,P = \mathrm{Z}\,P + \mathrm{Z}(-P)$ and $\mathrm{Y}\,P = h(\mathrm{Z}_s\,P, \cdot)$. Note that $\mathrm{Y} : \mathcal{P}_0 \to C_1^+(\mathbb{R}^n)$ is a $\mathrm{GL}(n)$ contravariant valuation. By Lemma 5.14 and Lemma 3.2, we can extend Y and thus Z_s to $\overline{\mathcal{P}}_0^n$. Theorem 5.5 implies that $\mathrm{Z}_s\,P = \{0\}$ for all $P \in \mathcal{P}_0$. Hence Z is vector-valued and Theorem 1.3 implies the statement of the lemma. q.e.d.

Lemma 5.16. *For $q = 1$, there exist $a, b \in \mathbb{R}$ such that for $x \in e_n^\perp$*

$$h(\mathrm{Z}[P,I],x) = \begin{cases} a(s+t)\,h(\rho_{\pi/2}P,x) + b(s+t)\,h(-\rho_{\pi/2}P,x) & \textit{for } n = 3 \\ a(s+t)\,h(\Pi\,P,x) & \textit{for } n > 3 \end{cases}$$

for every $I = [-s\,e_n, t\,e_n]$ and $P \in \mathcal{P}_0(e_n^\perp)$.

Proof. For $I \in \mathcal{P}_0(e_n)$, define $\mathrm{Z}_I : \mathcal{P}_0^{n-1} \to \langle \mathcal{K}^{n-1}, + \rangle$ by setting $h(\mathrm{Z}_I\,P,x) = h(\mathrm{Z}[P,I],x)$ for $x \in \mathbb{R}^{n-1}$ and $P \in \mathcal{P}^{n-1}$, where we identify $e_n^\perp$ and $\mathbb{R}^{n-1}$. Note that Z_I is a valuation on $\mathcal{P}_0^{n-1}$ that is $\mathrm{GL}(n-1)$ contravariant of weight $q = 1$.

By Proposition 4.2, we obtain that there are $a(I), b(I) \in \mathbb{R}$ such that for $x \in e_n^\perp$

$$h(\mathrm{Z}[P,I],x) = \begin{cases} a(I)\,h(\rho_{\pi/2}P,x) + b(I)\,h(-\rho_{\pi/2}P,x) & \text{for } n = 3 \\ a(I)\,h(\Pi\,P,x) & \text{for } n > 3. \end{cases}$$

If $\phi_r \in \mathrm{GL}(n)$ is a transformation that leaves $e_n^\perp$ fixed and multiplies e_n by $r > 0$, then (12) implies $h(\mathrm{Z}[P, r\,I],x) = r\,h(\mathrm{Z}[P,I],x)$. Thus a and b are valuations on $\mathcal{P}_0^1$ which are homogeneous of degree 1. By (12), we see that $a(I) = a(-I)$ and $b(I) = b(-I)$ for $I \in \mathcal{P}_0^1$. Thus (11) implies the statement of the lemma. q.e.d.

Lemma 5.17. *For $q = 1$, there exists $b \in \mathbb{R}$ such that*

$$h(\mathrm{Z}[P,I],e_n) = b\,V_{n-1}(P)$$

for every $I = [-s\,e_n, t\,e_n]$ and $P \in \mathcal{P}_0(e_n^\perp)$.

Proof. By Lemma 5.13, there are $a, b \in \mathbb{R}$ such that

$$h(\mathrm{Z}[P,I],e_n) = (a\log(\frac{t}{s}) + b)V_{n-1}(P). \tag{46}$$

Let $P = [I_1, \ldots, I_{n-1}]$ and let $I_i = [-s_i\,e_i, t_i\,e_i]$. Define $\psi \in \mathrm{GL}(n)$ by $\psi e_1 = e_n$ and $\psi e_n = e_1$ and $\psi e_i = e_i$ for $i = 2, \ldots, n-1$. Since Z is $\mathrm{GL}(n)$ contravariant, we have

$$h(\mathrm{Z}[P,I],e_n) = h(\mathrm{Z}(\psi[P,I]),e_1).$$

It follows from Lemma 5.16 that there are $c, d \in \mathbb{R}$ such that

$$h(\mathrm{Z}(\psi[P, I]), e_1) = c(s_1+t_1)h(\rho_{\pi/2}[\hat{I}, I_2], e_1)+d(s_1+t_1)h(-\rho_{\pi/2}[\hat{I}, I_2], e_1)$$

for $n = 3$ and

$$h(\mathrm{Z}(\psi[P, I]), e_1) = c(s_1 + t_1)\, h(\Pi[\hat{I}, I_2, \dots, I_{n-1}], e_1)$$

for $n > 3$, where $\hat{I} = [-s\, e_1, t\, e_1]$. Comparing coefficients in the above equations and in (46) gives that $a = 0$ and completes the proof of the lemma. q.e.d.

Lemma 5.18. *For $q = 1$, there exist constants $a, b \in \mathbb{R}$ so that for all $P \in \mathcal{P}_0(e_n^\perp)$, for all $u = (u', -1)$ and $v = (v', 1)$ with $u', v' \in \mathbb{R}^{n-1}$, and for all $s, t > 0$, we have for $x \in e_n^\perp$,*

$$h(\mathrm{Z}[P, s\, u, t\, v], x) = a(s + t)\, h(\rho_{\pi/2} P, x) + b(s + t)\, h(-\rho_{\pi/2} P, x)$$

for $n = 3$ and

$$h(\mathrm{Z}[P, s\, u, t\, v], x) = a(s + t)\, h(\Pi\, P, x)$$

for $n > 3$, whenever $[s\, u, t\, v]$ intersects P.

Proof. Let $I = [-s\, e_n, t\, e_n]$ with $s, t > 0$ and $x \in e_n^\perp$. Lemma 5.16 implies that $\lim_{s,t\to 0} h(\mathrm{Z}[P, I], x)$ exists uniformly for $P \in \mathcal{P}_0(e_n^\perp)$ contained in a centered ball. Let $\phi_v, \phi_r \in \mathrm{GL}(n)$ be defined as in Lemma 4.5. Since $[P, -s\, v, t\, v] = \phi_v[P, I]$, by (12)

$$h(\mathrm{Z}[P, -s\, v, t\, v], x) = h(\mathrm{Z}[P, I], \phi_v^{-1} x) = h(\mathrm{Z}[P, I], x).$$

By (12), we get

$$h(\mathrm{Z}(\phi_r[P, u, v], x) = r^{q-1} h(\mathrm{Z}[P, u, v], x).$$

Thus we apply Lemma 4.5 and obtain by Lemma 5.16 that

$$\lim_{s,t\to 0} h(\mathrm{Z}[P, s\, u, t\, v], x) = 0$$

for $x \in e_n^\perp$.

Since Z is a valuation, we have for $s', t' > 0$ sufficiently small,

$$\begin{aligned} h(\mathrm{Z}[P, s\, u, t\, v], x) = {} & h(\mathrm{Z}[P, -t'\, v, t\, v], x) - h(\mathrm{Z}[P, -t'\, v, t'\, v], x) \\ & + h(\mathrm{Z}[P, -s'\, u, s\, u], x) + h(\mathrm{Z}[P, s'\, u, t'\, v], x) \\ & - h(\mathrm{Z}[P, -s'\, u, s'\, u], x). \end{aligned}$$

Taking the limit as $s', t' \to 0$ and using Lemma 5.16 gives the statement of the lemma. q.e.d.

Lemma 5.19. *For $q = 1$, the operator $P \mapsto h(\mathrm{Z}\, P, \cdot)$ for $P \in \mathcal{P}_0^n$ has the Cauchy property for $\mathcal{P}_0(e_n^\perp)$.*

Proof. For $I = [-s\,e_n, t\,e_n]$ with $s, t > 0$, it follows from Lemma 5.17 that $\lim_{s,t\to 0} h(\mathrm{Z}[P, I], e_n)$ exists uniformly for $P \in \mathcal{P}_0(e_n^\perp)$ contained in a centered ball. Let $u = (u', -1)$ and $v = (v', 1)$ with $u', v' \in \mathbb{R}^{n-1}$, and let $\phi_v, \phi_r \in \mathrm{GL}(n)$ be defined as in Lemma 4.5. Since we have $[P, -s\,v, t\,v] = \phi_v[P, I]$, we obtain from (12) that

$$h(\mathrm{Z}[P, -s\,v, t\,v], e_n) = h(\mathrm{Z}(\phi_v[P, I]), e_n) = h(\mathrm{Z}[P, I], -v' + e_n).$$

Let B be a centered ball. By Lemmas 5.17 and 5.16, it follows from (4) that

$$h(\mathrm{Z}[P, -s\,v, t\,v], e_n) = b\,V_{n-1}(P) + o(1)$$

as $s, t \to 0$ uniformly for $P \subset B$. By (12), we have

$$h(\mathrm{Z}(\phi_r[P, u, v], e_n) = h(\mathrm{Z}[P, u, v], e_n).$$

Thus we apply Lemma 4.5 and obtain that

$$\lim_{s,t\to 0} h(\mathrm{Z}[P, s\,u, t\,v], e_n) = b\,V_{n-1}(P)$$

uniformly for $P \subset B$.

Let $x = (x', x_n) \in S^{n-1}$. From (4) and Lemma 5.16, we obtain that

$$h(\mathrm{Z}[P, s\,u, t\,v], x) = h(\mathrm{Z}[P, s\,u, t\,v], x') + o(1)$$

as $s, t \to 0$ uniformly for $x \in S^{n-1}$ and $P \subset B$. Combined with Lemma 5.16, this completes the proof of the lemma. q.e.d.

Lemma 5.20. *If $q = 1$, then there is a constant $c \geq 0$ such that*

$$\mathrm{Z}\,P = c\,\Pi\,P$$

for all $P \in \mathcal{P}_0^n$.

Proof. For $P \in \mathcal{P}_0^n$, set $\mathrm{Y}(P) = h(\mathrm{Z}\,P, \cdot)$. By Lemmas 5.18 and 5.19, we can apply Lemma 3.1 and extend Y and therefore Z to $\overline{\mathcal{P}}_0^n$. Hence Theorem 5.5 implies the statement of the lemma. q.e.d.

6. Proof of Theorem 1

Let H be a hyperplane containing the origin and let u be a unit normal vector to H. By Property (iv) in the definition of invariant area in Section 2, there exists $c_B(u) > 0$ such that

$$z(K, B) = c_B(u)\,V_{n-1}(K) \tag{47}$$

for all $K \in \mathcal{K}(H)$. Define $c_B(x)$ for $x \in \mathbb{R}^n$ by setting $c_B(t\,u) = t\,c_B(u)$ for $t > 0$. Note that $c_B : \mathbb{R}^n \to (0, \infty)$ is sublinear by Property (v) and hence a support function. Define the convex body $\mathbb{I}\,B$ by setting $h(\mathbb{I}\,B, x) = c_B(x)$ for $x \in \mathbb{R}^n$. Note that $\mathbb{I}\,B \in \mathcal{K}_0^n$ and that $\mathbb{I}\,B$ is origin symmetric.

By Groemer's Extension Theorem (see [**31**]), $z(\cdot, B)$ can be extended to a valuation on finite unions of convex polytopes in $\mathbb{R}^n$. Hence, for a polytope $P \in \mathcal{K}^n$ with facets $F_1, \ldots, F_m$ lying in hyperplanes with outer normal unit vectors $u_1, \dots, u_m$, we obtain by (47) and the definition of $\mathbb{I}\,B$ that

$$z(P,B) = \sum_{i=1}^{m} h(\mathbb{I}\,B, u_i) V_{n-1}(F_i) = n\, V_1(P, \mathbb{I}\,B),$$

where we used (8). By continuity,

$$z(K,B) = n\, V_1(K, \mathbb{I}\,B) \quad \text{for every } \; K \in \mathcal{K}^n.$$

The convex body $\mathbb{I}\,B$ is called *isoperimetrix* since it turns out to be the solution to the isoperimetric problem.

It follows from Property (i), (5), and (6) that

$$\begin{aligned} V_1(K, \mathbb{I}\,B) &= V_1(\phi K, \mathbb{I}(\phi B)) \\ &= |\det \phi|\, V_1(K, \phi^{-1}\, \mathbb{I}(\phi B)) \\ &= V_1(K, |\det \phi|\, \phi^{-1}\, \mathbb{I}(\phi B)) \end{aligned}$$

for all $K \in \mathcal{K}^n$ and $B \in \mathcal{K}_0^n$. Thus it follows from (9) that

$$\mathbb{I}(\phi B) = |\det \phi|^{-1}\, \phi\, \mathbb{I}\,B$$

for all $\phi \in \mathrm{GL}(n)$ and $B \in \mathcal{K}_0^n$. This shows that $\mathbb{I} : \mathcal{K}_0^n \to \mathcal{K}_0^n$ is $\mathrm{GL}(n)$ covariant of weight $q = -1$. Since we assume that z is a bivaluation,

$$z(K, B_1) + z(K, B_2) = z(K, B_1 \cup B_2) + z(K, B_1 \cap B_2)$$

and therefore by (7)

$$V_1(K, \mathbb{I}\,B_1 + \mathbb{I}\,B_2) = V_1(K, \mathbb{I}(B_1 \cup B_2) + \mathbb{I}(B_1 \cap B_2))$$

for $K \in \mathcal{K}^n$ and $B_1, B_2, B_1 \cup B_2 \in \mathcal{K}_0^n$. Thus it follows from (9) that $\mathbb{I}$ is a Minkowski valuation on $\mathcal{K}_0^n$. Since z is continuous, also $\mathbb{I}$ is continuous. Hence Theorem 4.3 and Theorem 5.1 imply that there is a constant $c \geq 0$ such that $\mathbb{I}\,B = c\, \Pi\, B^*$ for all $B \in \mathcal{K}_0^n$.

7. An open problem

Theorem 1 shows that the Holmes-Thompson area is the only bivaluation on $\mathcal{K}^n \times \mathcal{K}_0^n$ that is an invariant area. Is it also possible to obtain a complete classifications of bivaluations on $\mathcal{K}^n \times \mathcal{K}_c^n$ that are invariant areas? Is the Holmes-Thompson area again the unique such area?

References

[1] S. Alesker, *Continuous rotation invariant valuations on convex sets*, Ann. of Math. (2) **149** (1999), 977–1005, MR 1709308, Zbl 0941.52002.

[2] S. Alesker, *Description of translation invariant valuations on convex sets with solution of P. McMullen's conjecture*, Geom. Funct. Anal. **11** (2001), 244–272, MR 1837364, Zbl 0995.52001.

[3] S. Alesker, *Hard Lefschetz theorem for valuations, complex integral geometry, and unitarily invariant valuations*, J. Differential Geom. **63** (2003), 63–95, MR 2015260, Zbl 1073.52004.

[4] S. Alesker, *The multiplicative structure on continuous polynomial valuations*, Geom. Funct. Anal. **14** (2004), 1–26, MR 2053598, Zbl 1072.52011.

[5] S. Alesker, *Theory of valuations on manifolds, II*, Adv. Math. **207** (2006), 420–454, MR 2053598, Zbl 1072.52011.

[6] S. Alesker, *Theory of valuations on manifolds: a survey*, Geom. Funct. Anal. **17** (2007), 1321–1341, MR 2264077, Zbl 1117.52016.

[7] J.C. Álvarez Paiva, *Some problems on Finsler geometry*, Handbook of Differential Geometry. Vol. II, Elsevier/North-Holland, Amsterdam, 2006, 1–33, MR 2194667, Zbl 1147.53059.

[8] J.C. Álvarez Paiva & G. Berck, *What is wrong with the Hausdorff measure in Finsler spaces*, Adv. Math. **204** (2006), 647–663, MR 2249627, Zbl 1099.53018.

[9] J.C. Álvarez Paiva & A.C. Thompson, *Volumes on normed and Finsler spaces*, A Sampler of Riemann-Finsler geometry, Math. Sci. Res. Inst. Publ., vol. 50, Cambridge Univ. Press, Cambridge, 2004, 1–48, MR 2132656, Zbl 1078.53072.

[10] G. Berck, *Convexity of L_p-intersection bodies*, Adv. Math. **222** (2009), 920–936, MR 2553373, Zbl 1179.52005.

[11] A. Bernig, *Valuations with Crofton formula and Finsler geometry*, Adv. Math. **210** (2007), 733–753, MR 2303237, Zbl 1128.53047.

[12] H. Busemann, *Intrinsic area*, Ann. of Math. (2) **48** (1947), 234–267, MR 0020626, Zbl 0029.35301.

[13] H. Busemann, *The isoperimetric problem for Minkowski area*, Amer. J. Math. **71** (1949), 743–762, MR 0031762, Zbl 0038.10301.

[14] S. Campi & P. Gronchi, *The L^p-Busemann-Petty centroid inequality*, Adv. Math. **167** (2002), 128–141, MR 1901248, Zbl 1002.52005.

[15] J.H.G. Fu, *Structure of the unitary valuation algebra*, J. Differential Geom. **72** (2006), 509–533, MR 2219942, Zbl 1096.52003.

[16] R. Gardner, *Geometric Tomography*, second ed., Encyclopedia of Mathematics and its Applications, vol. 58, Cambridge University Press, Cambridge, 2006, MR 2251886, Zbl 1102.52002.

[17] R. Gardner & G. Zhang, *Affine inequalities and radial mean bodies*, Amer. J. Math. **120** (1998), 505–528, MR 1623396, Zbl 0908.52001.

[18] E. Grinberg & G. Zhang, *Convolutions, transforms, and convex bodies*, Proc. London Math. Soc. (3) **78** (1999), 77–115, MR 1658156, Zbl 0974.52001.

[19] M. Gromov, *Filling Riemannian manifolds*, J. Differential Geom. **18** (1983), 1–147, MR 0697984, Zbl 0515.53037.

[20] P.M. Gruber, *Convex and Discrete Geometry*, Grundlehren der Mathematischen Wissenschaften, vol. 336, Springer, Berlin, 2007, MR 2335496, Zbl 1139.52001.

[21] C. Haberl, *L_p intersection bodies*, Adv. Math. **217** (2008), 2599–2624, MR 2397461, Zbl 1140.52003.

[22] C. Haberl, *Star body valued valuations*, Indiana Univ. Math. J. **58** (2009), 2253–2276, MR 2583498, Zbl 1183.52003.

[23] C. Haberl, *Blaschke valuations*, Amer. J. Math., in press.

[24] C. Haberl & M. Ludwig, *A characterization of L_p intersection bodies*, Int. Math. Res. Not. **10548** (2006), 1–29, MR 2250020, Zbl 1115.52006.

[25] C. Haberl & F. Schuster, *Asymmetric affine L_p Sobolev inequalities*, J. Funct. Anal. **257** (2009), 641–658, MR 2530600, Zbl 1180.46023.

[26] C. Haberl & F. Schuster, *General L_p affine isoperimetric inequalities*, J. Differential Geom. **83** (2009), 1–26, MR 2545028, Zbl 1185.52005.

[27] H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer, Berlin, 1957, MR 0102775, Zbl 0078.35703.

[28] R.D. Holmes & A.C. Thompson, *n-dimensional area and content in Minkowski spaces*, Pacific J. Math. **85** (1979), 77–110, MR 0571628, Zbl 0467.51007.

[29] D.A. Klain, *Star valuations and dual mixed volumes*, Adv. Math. **121** (1996), 80–101, MR 1399604, Zbl 0858.52003.

[30] D.A. Klain, *Invariant valuations on star-shaped sets*, Adv. Math. **125** (1997), 95–113, MR 1427802, Zbl 0889.52007.

[31] D.A. Klain & G.-C. Rota, *Introduction to geometric probability*, Cambridge University Press, Cambridge, 1997, MR 1608265, Zbl 0896.60004.

[32] M. Ludwig, *Moment vectors of polytopes*, Rend. Circ. Mat. Palermo (2) Suppl. **70** (2002), part II, 123–138, MR 1962589, Zbl 1113.52031.

[33] M. Ludwig, *Projection bodies and valuations*, Adv. Math. **172** (2002), 158–168, MR 1942402, Zbl 1019.52003.

[34] M. Ludwig, *Valuations on polytopes containing the origin in their interiors*, Adv. Math. **170** (2002), 239–256, MR 1932331, Zbl 1015.52012.

[35] M. Ludwig, *Ellipsoids and matrix valued valuations*, Duke Math. J. **119** (2003), 159–188, MR 1991649, Zbl 1033.52012.

[36] M. Ludwig, *Minkowski valuations*, Trans. Amer. Math. Soc. **357** (2005), 4191–4213, MR 2159706, Zbl 1077.52005.

[37] M. Ludwig, *Intersection bodies and valuations*, Amer. J. Math. **128** (2006), 1409–1428, MR 2275906, Zbl 1115.52007.

[38] M. Ludwig & M. Reitzner, *A characterization of affine surface area*, Adv. Math. **147** (1999), 138–172, MR 1725817, Zbl 0947.52003.

[39] M. Ludwig & M. Reitzner, *Elementary moves on triangulations*, Discrete Comput. Geom. **35** (2006), 527–536, MR 2225672, Zbl 1105.52008.

[40] M. Ludwig & M. Reitzner, *A classification of $SL(n)$ invariant valuations*, Ann. of Math. **2** 172 (2010), 1219–1267.

[41] E. Lutwak, *On some affine isoperimetric inequalities*, J. Differential Geom. **23** (1986), 1–13, MR 0840399, Zbl 0592.52005.

[42] E. Lutwak, *The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem*, J. Differential Geom. **38** (1993), 131–150, MR 1231704, Zbl 0788.52007.

[43] E. Lutwak, D. Yang & G. Zhang, *L_p affine isoperimetric inequalities*, J. Differential Geom. **56** (2000), 111–132, MR 1863023, Zbl 1034.52009.

[44] E. Lutwak, D. Yang & G. Zhang, *Sharp affine L_p Sobolev inequalities*, J. Differential Geom. **62** (2002), 17–38, MR 1987375, Zbl 1073.46027.

[45] E. Lutwak, D. Yang & G. Zhang, *Moment-entropy inequalities*, Ann. Probab. **32** (2004), 757–774, MR 2039942, Zbl 1053.60004.

[46] E. Lutwak & G. Zhang, *Blaschke-Santaló inequalities*, J. Differential Geom. **47** (1997), 1–16, MR 1601426, Zbl 0906.52003.

[47] P. McMullen & R. Schneider, *Valuations on convex bodies*, Convexity and its Applications (P.M. Gruber and J.M. Wills, eds.), Birkhäuser, 1983, 170–247, MR 0731112, Zbl 0534.52001.

[48] R. Schneider, *Convex Bodies: the Brunn-Minkowski Theory*, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, 1993, MR 1216521, Zbl 0798.52001.

[49] R. Schneider & F. Schuster, *Rotation equivariant Minkowski valuations*, Int. Math. Res. Not. **72894** (2006), 1–20, MR 2272092, Zbl 1127.52017.

[50] R. Schneider & J.A. Wieacker, *Integral geometry in Minkowski spaces*, Adv. Math. **129** (1997), 222–260, MR 1462734, Zbl 0893.53028.

[51] F. Schuster, *Valuations and Busemann-Petty type problems*, Adv. Math. **219** (2008), 344–368, MR 2435426, Zbl 1146.52003.

[52] F. Schuster, *Crofton measures and Minkowski valuations*, Duke Math. J. **154** (2010), 1–30, MR 2668553, Zbl 05769087.

[53] F. Schuster & T. Wannerer, *GL(n) contravariant Minkowski valuations*, Trans. Amer. Math. Soc., in press.

[54] C. Steineder, *Subword complexity and projection bodies*, Adv. Math. **217** (2008), 2377–2400, MR 2388098, Zbl 05242999.

[55] A.C. Thompson, *Minkowski Geometry*, Cambridge University Press, Cambridge, 1996, MR 1406315, Zbl 0868.52001.

[56] D. Yang, *Affine integral geometry from a differential viewpoint*, in Handbook of Geometric Analysis, No. 2 (Lizhen Ji, ed.). Advanced Lectures in Mathematics **13**, (2010), 359–390, Zbl 05831742.

[57] G. Zhang, *The affine Sobolev inequality*, J. Differential Geom. **53** (1999), 183–202, MR 1776095, Zbl 1040.53089.

Institut für Diskrete Mathematik und Geometrie
Technische Universität Wien
Wiedner Hauptstr. 8-10/104
1040 Wien, Austria
and
Department of Mathematics
Polytechnic Institute of NYU
6 MetroTech Center
Brooklyn, NY 11201
E-mail address: monika.ludwig@tuwien.ac.at