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AN EXTENSION OF THE STABILITY THEOREM OF THE MINKOWSKI SPACE IN GENERAL RELATIVITY

Lydia Bieri

Abstract

In this paper, we sketch the proof of the extension of the stability theorem of the Minkowski space in General Relativity done explicitly in [**6**] and [**7**]. We discuss solutions of the Einstein vacuum (EV) equations (obtained in the author's Ph.D. thesis ([**6**]) in 2007). We solve the Cauchy problem for more general, asymptotically flat initial data than in the pioneering work of D. Christodoulou and S. Klainerman [**21**] or than in any other work. Moreover, we describe precisely the asymptotic behavior. Our relaxed assumptions on the initial data yield a spacetime curvature which is not bounded in $L^\infty(M)$. As a major result, we encounter in our work borderline cases, which we discuss in this paper as well. The fact that certain of our estimates are borderline in view of decay indicates that the conditions in our main theorem are sharp in so far as the assumptions on the decay at infinity on the initial data are concerned. Thus, the borderline cases are a consequence of our relaxed assumptions on the data, [**6**, **7**]. They are not present in the other works, as all of them place stronger assumptions on their data. We work with an invariant formulation of the EV equations. Our main proof is based on a bootstrap argument. To close the argument, we have to show that the spacetime curvature and the corresponding geometrical quantities have the required decay. In order to do so, the Einstein equations are decomposed with respect to specific foliations of the spacetime. This result generalizes the work of D. Christodoulou and S. Klainerman [**21**].

1. Introduction and Main Results

The laws of General Relativity (GR) are the Einstein equations linking the curvature of the spacetime to its matter content.

$$G_{\mu\nu} \;:=\; R_{\mu\nu} \;-\; \frac{1}{2}\, g_{\mu\nu}\, R \;=\; 2\, T_{\mu\nu}, \tag{1}$$

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(rationalized units $4\pi G = 1$), where, for $\mu, \nu = 0, 1, 2, 3$, $G_{\mu\nu}$ is called the Einstein tensor, $R_{\mu\nu}$ is the Ricci curvature tensor, R the scalar curvature tensor, g the metric tensor and $T_{\mu\nu}$ denotes the energy-momentum tensor.

This paper discusses the main results and steps of the proof of [**6**] and [**7**], dealing with the global, nonlinear stability of solutions of the Einstein vacuum equations in General Relativity. The case of (1) where $T_{\mu\nu} = 0$ are the Einstein vacuum (EV) equations. These read as follows:

$$R_{\mu\nu} \ = \ 0. \tag{2}$$

Solutions of the EV equations are spacetimes (M, g), where M is a four-dimensional, oriented, differentiable manifold and g is a Lorentzian metric obeying the EV equations. We study these equations for asymptotically flat systems. These are solutions where M looks like flat Minkowski space with diagonal metric $\eta = (-1, +1, +1, +1)$ outside of spatially compact regions. Many physical cases require studying the Einstein equations in vacuum. Isolated gravitating systems such as binary stars, clusters of stars, galaxies, etc., can be described in GR by asymptotically flat solutions of these equations, for they can be thought of as having an asymptotically flat region outside the support of the matter.

In view of the EV equations (2), what the sharp criteria are for nontrivial asymptotically flat initial data sets to yield a maximal development that is complete is an open problem. We generalize the results of D. Christodoulou and S. Klainerman [**21**]: Every strongly asymptotically flat, maximal, initial data which is globally close to the trivial data gives rise to a solution which is a complete spacetime tending to the Minkowski spacetime at infinity along any geodesic.'

We solve the Cauchy problem with more general, asymptotically flat initial data. In particular, we have one less power of r decay at spatial infinity and one less derivative than in [**21**]. We prove that also in this case, the initial data, under appropriate smallness conditions, yields a solution which is a complete spacetime, tending to the Minkowksi spacetime at infinity along any geodesic. In accordance with the initial data, the asymptotic flatness is correspondingly weaker. Contrary to the situation in [**21**], certain estimates in our proof are borderline in view of decay, indicating that the conditions in our main theorem on the decay at infinity on the initial data are sharp.

Our main results are stated in Theorems 1 and 3.

We construct global solutions (M, g) of the EV equations (2) for initial data specified in definition 2 below. We use two foliations given by a maximal time function t and an optical function u, respectively. The time function t foliates our four-dimensional spacetime into three-dimensional spacelike hypersurfaces H_t, being complete Riemannian manifolds. Whereas the optical function u induces a foliation of (M, g) into null hypersurfaces C_u, which we shall refer to as null cones. The intersections $H_t \cap C_u = S_{t,u}$ are two-dimensional compact Riemannian manifolds.

Definition 1. An initial data set is a triplet $(H, \bar{g}, k)$ with $(H, \bar{g})$ being a three-dimensional complete Riemannian manifold and k a two-covariant symmetric tensorfield on H, satisfying the constraint equations

$$\begin{aligned} \nabla^i \, k_{ij} \; - \; \nabla_j \; trk \quad &= \quad 0, \\ \bar{R} - \; \mid k \mid^2 \; + (trk)^2 \quad &= \quad 0. \end{aligned}$$

The constraint equations constrain the initial data. We recall that a development of an initial data set is an EV spacetime (M, g) together with an imbedding $i : H \to M$ such that g and k are the induced first and second fundamental forms of H in M. The barred quantities denote the metric and curvatures on H.

We work with a maximal time function t. That is, the level sets H_t of the time function t are required to be maximal spacelike hypersurfaces. Thus, they fulfill the equation $trk = 0$. (See below.) Also, the lapse function Φ is introduced after the definition of a time function (definition 13). For a time function t (that is, $dt \cdot X > 0$ for all future-directed timelike vectors X at all points $p \in M$) the corresponding lapse function Φ is given by $\Phi \; := \; (-g^{\mu\nu} \partial_\mu t \partial_\nu t)^{-\frac{1}{2}}$. From the structure equations with respect to the t-foliation, using the EV equations (2), we derive the constraint equations, the evolution equations, and the lapse equation below. The structure equations, consisting of the variation, Codazzi, and the trace of the Gauss equations, relate the spacetime curvature $R_{\alpha\beta\gamma\delta}$ to the Ricci curvature $\bar{R}_{ij}$ of H_t, the second fundamental form k, and the lapse function Φ. Note that in the three-dimensional leaf H_t, its Ricci curvature $\bar{R}_{ij}$ completely determines the induced Riemannian curvature tensor $\bar{R}_{ijkl}$ as follows ($\bar{R}$ is the scalar curvature $\bar{g}^{ij} \bar{R}_{ij}$):

$$\bar{R}_{ijkl} \; = \; \bar{g}_{ik} \, \bar{R}_{jl} \; + \; \bar{g}_{jl} \, \bar{R}_{ik} \; - \; \bar{g}_{il} \, \bar{R}_{jk} \; - \; \frac{1}{2} \, (\bar{g}_{ik} \, \bar{g}_{jl} \; - \; \bar{g}_{jk} \, \bar{g}_{il}) \, \bar{R}. \tag{3}$$

At this point, let us give the following formulas for the frame field (e_0, e_1, e_2, e_3) for M, where $e_0 = \frac{1}{\Phi} T$ denotes the future-directed unit normal to the H_t and (e_1, e_2, e_3) is an orthonormal frame tangent to the leaves of the foliation:

$$D_i e_0 \quad = \quad k_{ij} \; e_j, \tag{4}$$

$$D_i e_j \quad = \quad \nabla_i e_j \; + \; k_{ij} \; e_0, \tag{5}$$

$$D_0 e_0 \quad = \quad (\Phi^{-1} \nabla_i \Phi) \; e_i, \tag{6}$$

$$D_0 e_i \quad = \quad \bar{D}_0 e_i \; + \; (\Phi^{-1} \nabla_i \Phi) \; e_0, \tag{7}$$

$\bar{D}_0 e_i$ denoting the projection of $D_0 e_i$ to the tangent space of the foliation. Note that in a so-called Fermi propagated frame, it is $\bar{D}_0 e_i = 0$.

Here, note that $g_{00} = -1$, $g_{0i} = 0$, and $g_{ij} = \bar{g}_{ij} = g(e_i, e_j)$ for $i, j = 1, 2, 3$.

With respect to the foliation of the spacetime by a maximal time function t, the *constraint equations* take the following form:

$$trk = 0, \tag{8}$$

$$\nabla^i k_{ij} = 0, \tag{9}$$

$$\bar{R} = |k|^2. \tag{10}$$

The *evolution equations* for a maximal foliation are

$$\frac{\partial \bar{g}_{ij}}{\partial t} = 2\Phi k_{ij}, \tag{11}$$

$$\frac{\partial k_{ij}}{\partial t} = \nabla_i \nabla_j \Phi - (\bar{R}_{ij} - 2k_{im}k^m_j)\ \Phi. \tag{12}$$

Moreover, the *lapse equation* reads

$$\triangle\ \Phi = |\ k\ |^2\ \Phi. \tag{13}$$

In our work, we consider asymptotically flat initial data of the following form:

Definition 2 (AFB)**.** We define an asymptotically flat initial data set to be an AFB initial data set, if it is an asymptotically flat initial data set $(H_0, \bar{g}, k)$, where $\bar{g}$ and k are sufficiently smooth and for which there exists a coordinate system (x^1, x^2, x^3) in a neighbourhood of infinity such that with $r = (\sum_{i=1}^3 (x^i)^2)^{\frac{1}{2}} \to \infty$, it is

$$\bar{g}_{ij} = \delta_{ij} + o_3\ (r^{-\frac{1}{2}}), \tag{14}$$

$$k_{ij} = o_2\ (r^{-\frac{3}{2}}). \tag{15}$$

The initial data $(H_0, \bar{g}, k)$ has to satisfy the global smallness assumption below. We introduce $Q(a, x_{(0)})$ for the terms that have to be controlled by a small positive ϵ. At a later point in the proof, ϵ has to be taken suitably small, depending on other quantities.

$$\begin{aligned} Q(a, x_{(0)}) \quad = \quad & a^{-1}\ \Big(\int_{H_0} \Big(\ |\ k\ |^2\ + (a^2 + d_0^2)\ \ |\ \nabla k\ |^2 \\ & \qquad + (a^2 + d_0^2)^2\ \ |\ \nabla^2 k\ |^2\ \Big)\ d\mu_{\bar{g}} \\ & + \int_{H_0} \Big(\ (a^2 + d_0^2)\ \ |\ Ric\ |^2 \\ & \qquad + (a^2 + d_0^2)^2\ \ |\ \nabla Ric\ |^2\ \Big)\ d\mu_{\bar{g}}\ \Big), \end{aligned} \tag{16}$$

where a is a positive scale factor and d_0 denotes the distance function from an arbitrarily chosen origin $x_{(0)}$.

Let $\inf_{x_{(0)}, a} Q(x_{(0)}, a)$ denote the infimum over all choices of origin $x_{(0)}$ and all a of the quantity defined by (16).

We consider asymptotically flat initial data sets for which the metric $\bar{g}$ is complete and there exists a small positive ϵ such that

$$\inf_{x_{(0)},a} Q(x_{(0)}, a) \ < \ \epsilon. \tag{17}$$

One version of our main theorem is the following:

Theorem 1. *Any asymptotically flat, maximal initial data set, with complete metric $\bar{g}$, satisfying inequality (17), where the ϵ has to be taken sufficiently small, leads to a unique, globally hyperbolic, smooth, and geodesically complete solution of the EV equations, foliated by the level sets of a maximal time function. This development is globally asymptotically flat.*

For later reference, we state the global smallness assumption B as follows:

Global Smallness Assumption B:
An asymptotically flat initial data set satisfies the **global smallness assumption B** if the metric $\bar{g}$ is complete and there exists a sufficiently small positive ϵ such that

$$\inf_{x_{(0)},a} Q(x_{(0)}, a) \ < \ \epsilon. \tag{18}$$

The global smallness assumption B has to be considered together with the main theorem 3 in section 4. Then ϵ in (18) has to be taken suitably small such that the inequalities stated in main theorem 3 hold. This main theorem 3 is the most precise statement of our results.

To prove this result (theorem 1, respectively theorem 3), we do not need any preferred coordinate system, but we rely on the invariant formulation of the EV equations. Also, the *asymptotic behavior* is given in a *precise* way.

We remark that by *geodesically complete* is denoted what in GR is called *g-complete*, which means that every causal geodesic can be extended for all parameter values.

At this point, let us recall the result of D. Christodoulou and S. Klainerman [**21**]. They consider the following strongly asymptotically flat initial data set:

Definition 3 (SAFCK)**.** We define a strongly asymptotically flat initial data set in the sense of [**21**] and in the following denoted by SAFCK initial data set, to be an initial data set $(H, \bar{g}, k)$, where $\bar{g}$ and k are sufficiently smooth and there exists a coordinate system (x^1, x^2, x^3) defined in a neighbourhood of infinity such that,
as $r = (\sum_{i=1}^{3} (x^i)^2)^{\frac{1}{2}} \to \infty$,

$$\bar{g}_{ij} \ = \ (1 \ + \ \frac{2M}{r}) \ \delta_{ij} \ + \ o_4 \ (r^{-\frac{3}{2}}), \tag{19}$$

$$k_{ij} \ = \ o_3 \ (r^{-\frac{5}{2}}), \tag{20}$$

where M denotes the mass.

In order to state their global smallness assumption, Christodoulou and Klainerman introduce a quantity $Q_{CK}(x_{(0)}, b)$ that has to be controlled by a small positive ϵ. It is

$$\begin{aligned} Q_{CK}(x_{(0)}, b) \quad &= \quad \sup_H \ \big(\ b^{-2} \ (d_0^2 \ + \ b^2)^3 \ \mid Ric \mid^2 \ \big) \\ & \quad + b^{-3} \ \big(\ \int_H \ \sum_{l=0}^{3} \ (d_0^2 \ + \ b^2)^{l+1} \ \mid \nabla^l k \mid^2 \\ & \quad + \int_H \ \sum_{l=0}^{1} \ (d_0^2 \ + \ b^2)^{l+3} \ \mid \nabla^l B \mid^2 \ \big), \end{aligned} \tag{21}$$

with $d_0(x) = d(x_{(0)}, x)$ being the Riemannian geodesic distance between the point x and a given point $x_{(0)}$ on H; b is a positive constant, ∇^l denotes the l-covariant derivatives, and B (Bach tensor) is the following symmetric, traceless 2-tensor:

$$B_{ij} \ = \ \epsilon_j^{\ ab} \ \nabla_a \ (R_{ib} \ - \ \frac{1}{4} \ g_{ib} \ R).$$

It is used to formulate the following global smallness assumption in [**21**].

Global Smallness Assumption CK:
A strongly asymptotically flat initial data set is said to satisfy the **global smallness assumption CK** if the metric $\bar{g}$ is complete and there exists a sufficiently small positive ϵ such that

$$\inf_{x_{(0)} \in H, b \geq 0} \ Q_{CK}(x_{(0)}, b) \ < \ \epsilon. \tag{22}$$

Then one version of the main theorem in [**21**], is stated as follows:

Theorem 2 (D. Christodoulou and S. Klainerman, [**21**], p. 17, theorem 1.0.3)**.** *Any strongly asymptotically flat, maximal, initial data set that satisfies the global smallness assumption CK (*22*) leads to a unique, globally hyperbolic, smooth, and geodesically complete solution of the EV equations foliated by a normal, maximal time foliation. This development is globally asymptotically flat.*

The full version of their result is provided in [**21**], p.298, theorem 10.2.1.

There is no additional restriction on the data. The authors do not use a preferred coordinate system, but their proof relies on the invariant formulation of the EV equations. Moreover, they obtain a *precise description* of the *asymptotic behavior at null infinity.*

Our initial data ((14), (15)) is more general than that in ((19), (20)) in the sense that in (14) and in (15) we have one less derivative and less fall-off by one power of r than in ((19), (20)). We show **existence** and **uniqueness** of solutions of the EV equations under these **relaxed assumptions** on the initial data (AFB; see (18)). As we are assuming less on our initial data, the description of the **asymptotic behavior** of the curvature components is less precise than in [**21**] (with (19), (20)). However, it is as precise as it can be with these relaxed assumptions. The case we study does not tend as fast to Minkowski as the situation in [**21**]. In our proof, we use the main structure as in the proof of [**21**], namely, a bootstrap argument. However, the proof itself and the techniques differ considerably from the original one. Our more general case requires subtle and different treatment of the most delicate estimates. Another major difference to the situation studied in [**21**] by Christodoulou and Klainerman, and which arises from our relaxed assumptions, is the fact that we encounter **borderline cases in view of decay** in the power of r, indicating that the conditions in our main theorem on the decay at infinity of the initial data are **sharp**. Any further relaxation would make the corresponding integrals diverge and the argument would not close any more. As a consequence from imposing less conditions on our data, the **spacetime curvature is not in** $L^\infty(M)$. We only control one derivative of the curvature (Ricci) in $L^2(H)$. By the trace lemma, the Gauss curvature K in the leaves of the u-foliation S is only in $L^4(S)$. Contrary to that, in [**21**], the Ricci curvature is in $L^\infty(H)$, and in $L^\infty(S)$. The authors control two derivatives of the curvature (Ricci) in $L^2(H)$. Thus, this is a disadvantage and an advantage. First, as we do not have the curvature bounded in L^∞, certain steps of the proof become more subtle. On the other hand, we do not have to control the second derivatives of the curvature, which simplifies the proof considerably. A major simplification is the fact, that we do **not use any rotational vectorfields** in our proof. We gain control on the **angular derivatives** of the **curvature directly from the Bianchi equations**. Whereas in [**21**], a difficult construction of rotational vectorfields was necessary. Moreover, in our situation, **energy** and **linear momentum** are **well-defined and conserved**, whereas the **(ADM) angular momentum** is **not defined**. This is different to the situation investigated in [**21**], where all these quantities are well-defined and conserved.

The results of [**21**] yield the laws of gravitational radiation proposed by Bondi [**8**]. In particular, they explain the physical theory of the so-called memory effect (see [**12**]) in the framework of gravitational radiation. D. Christodoulou discusses this in his paper about nonlinear nature of gravitation and gravitational-wave experiments [**12**]. This memory effect is due to the nonlinear character of the asymptotic laws at future null infinity. The many well-known experiments to detect gravitational waves, going on and planned for the near future, build on this effect. In the same paper the formula for the power radiated to infinity at a given retarded time, in a given direction, per unit solid angle, is stated, as well as the formula for the total energy radiated to infinity in a given direction, per unit solid angle.

The full version of our main theorem is stated in theorem 3 in section 4. Crucial steps of the proof of the main theorem, in particular the bootstrap argument, are given in section 5. In section 6 we discuss the proof further and we investigate a borderline case.

For the reader who wishes to delve deeper into the area or the background material on which developments in this area depend, then among the vast interesting literature we suggest the following in addition to the references cited in this paper: [**1**], [**11**], [**13**], [**14**], [**15**], [**16**], [**19**], [**20**], [**23**], [**25**], [**26**], [**27**], [**28**], [**29**], [**30**], [**32**], [**35**], [**36**], [**37**], [**39**], [**40**], [**41**], [**42**], [**43**], [**44**].

2. Setting

The spacetime manifold (M, g) is defined above. For a Lorentzian metric g, there exists a vector V in T_pM such that $g_p(V, V) < 0$. Its g_p-orthogonal complement is defined as $\Sigma_V = \{X : \ g_p(X, V) = 0\}$ and g_p restricted to Σ_V is positive definite.

Then at each p in M we can choose a positive orthonormal frame $(e_0, e_1, e_2, e_3)_p$ continuously. We obtain the positive orthonormal frame field consisting of e_0, e_1, e_2, e_3 with

$$e_0 = \frac{V}{\sqrt{-g(V,V)}} \tag{23}$$

and e_1, e_2, e_3 being an orthonormal basis for Σ_V.

A given vector X in T_pM can be expanded as

$$\begin{aligned} X &= X^0 e_0 + X^1 e_1 + X^2 e_2 + X^3 e_3 \\ &= \sum_i X^i e_i \qquad\qquad (i = 0, 1, 2, 3). \end{aligned}$$

Consequently, it is

$$\begin{aligned} g(e_i, e_j) = \eta_{ij} &= diag(-1, +1, +1, +1). \\ g(X, X) &= -(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 \\ &= \sum_{ij} \eta_{ij} X^i X^j . \end{aligned}$$

At a point p in M, we distinguish three types of vectors. Namely, null, timelike, and spacelike vectors. The null vectors form a double cone at p, while the timelike vectors form an open set of two connected components—that is, the interior of this cone—and the spacelike vectors a connected open set being the exterior of the cone. They are defined as follows.

Definition 4. The null cone (or light cone) at p in M is

$$N_p \;=\; \{X \neq 0 \;\; \in \; T_pM : \; g_p(X,X) = 0\}.$$

The double cone consists of N_p^+ and N_p^-: $N_p \;=\; N_p^+ \cup N_p^-$.

Denote by I_p^+ the interior of N_p^+ and by I_p^- the interior of N_p^-.

Definition 5. The set of timelike vectors at p in M is given by

$$I_p \;:=\; I_p^+ \;\cup\; I_p^- \;=\; \{X \;\in\; T_pM : \; g_p(X,X) < 0\}.$$

Definition 6. The set of spacelike vectors at p in M is defined to be

$$S_p \;:=\; \{X \;\in\; T_pM : \; g_p(X,X) > 0\}.$$

Thus, S_p is the exterior of N_p.

Definition 7. A causal curve in M is a differentiable curve γ whose tangent vector $\dot{\gamma}$ at each point p in M belongs to $I_p \cup N_p$, i.e., is either timelike or null.

Definition 8. The causal future of a point p in M, denoted by $J^+(p)$, is the set of all points $q \in M$ for which there exists a future-directed causal curve initiating at p and ending at q.

Correspondingly, we can define $J^-(p)$, the causal past of p. We also need the causal future of a set S in M:

Definition 9. The causal future $J^+(S)$ of any set $S \;\subset\; M$, in particular in the case that S is a closed set, is

$$J^+(S) \;=\; \{q \;\in\; M : \; q \;\in\; J^+(p) \text{ for some } p \;\in\; S\}.$$

Similarly, the definition is given for $J^-(S)$. The boundaries $\partial J^+(S)$ and $\partial J^-(S)$ of $J^+(S)$ and $J^-(S)$, respectively, for closed sets S are null hypersurfaces. They are generated by null geodesic segments. The null geodesics generating $J^+(S)$ have past end points only on S. These null hypersurfaces $\partial J^+(S)$ and $\partial J^-(S)$ are realized as level sets of functions u satisfying the Eikonal equation $g^{\mu\nu}\partial_\mu u \partial_\nu u \;=\; 0$.

Definition 10. H is called a null hypersurface if at each point x in H the induced metric $g_x \mid T_xH$ is degenerate.

This means that there exists a $L \neq 0 \;\in\; T_xH$ such that

$$g_x(L,X) \;=\; 0 \quad \forall \; X \;\in\; T_xH.$$

Now, let u be a function for which each of its level sets is a null hypersurface. Then we can set

$$L^\mu \;=\; -g^{\mu\nu}\partial_\nu u$$

and we have

$$g(L,L) \;=\; 0.$$

The same reads in terms of du as follows: $g^{\mu\nu}\partial_\mu u \partial_\nu u = 0$, which is the *Eikonal equation.* In fact, L is a geodesic vectorfield, that is, the integral curves of L are null geodesics. A null hypersurface is generated by null geodesic segments.

Definition 11. A hypersurface H is called spacelike if at each x in H, the induced metric
$$g_x \mid_{T_x H} =: \bar{g}_x$$
is positive definite.

We observe that, $(H, \bar{g})$ is a proper Riemannian manifold. And the g-orthogonal complement of $T_x H$ is a one-dimensional subspace of $T_x M$ on which g_x is negative definite. Thus there exists a vector $N_x \in I_x^+$ of unit magnitude
$$g_x(N_x, N_x) = -1$$
whose *span* is this 1-dimensional subspace. We refer to N (the so-defined vectorfield along H) as the future-directed unit normal to H.

The second fundamental form of the hypersurface H is denoted as
$$k(X, Y) = g(D_X N, Y) \qquad \forall\ X,\ Y \in T_x H. \tag{24}$$

A very important notion in GR and in this work is a Cauchy hypersurface, being defined with the help of causal curves:

Definition 12. A Cauchy hypersurface is a complete spacelike hypersurface H in M (i.e., $(H, \bar{g})$ is a complete Riemannian manifold) such that if γ is any causal curve through any point $p \in M$, then γ intersects H at exactly one point.

A spacetime admitting a Cauchy hypersurface is called *globally hyperbolic.*

Assuming the spacetime to be globally hyperbolic, a *time function* t can be defined.

Definition 13. Let the spacetime (M, g) be globally hyperbolic. A time function is then a differentiable function t such that
$$dt \cdot X > 0 \tag{25}$$
for all $X \in I_p^+$ and for all $p \in M$.

The foliation given by the level surfaces H_t of t is called t*-foliation.* Denote by T the following future-directed normal to this foliation: $T^\mu = -\Phi^2 g^{\mu\nu}\partial_\nu t$. It is $Tt = T^\mu \partial_\mu t = 1$. Now, a spacetime foliated in this fashion is diffeomorphic to the product $\mathbb{R} \times \bar{M}$ where $\bar{M}$ is a 3-manifold, each level set H_t of t being diffeomorphic to $\bar{M}$. The integral curves of T are the orthogonal curves to the H_t-foliation. They are parametrized by t. Relative to this representation of M, the metric g reads
$$g = -\Phi^2 dt^2 + \bar{g} \tag{26}$$

with $\bar{g} = \bar{g}(t)$ denoting the induced metric on H_t. Note that $\bar{g}$ is positive definite. Here, Φ is the *lapse function* corresponding to the time function t. It is defined as follows:

$$\Phi \ := \ (-g^{\mu\nu}\partial_\mu t \partial_\nu t)^{-\frac{1}{2}}. \tag{27}$$

This lapse function measures the normal separation of the leaves H_t. By N (already given above) denote the unit normal $N = \Phi^{-1}T$. Its integral curves are the same as for T, but parametrized by arc length s.

In view of the first variational formula below, consider a frame field e_1, e_2, e_3 for H_t, Lie transported along the integral curves of T. That is, we have

$$[T, e_i] \ = \ 0$$

for $i = 1, 2, 3$. Denote $\bar{g}_{ij} = \bar{g}(e_i, e_j) = g(e_i, e_j)$. Then the *first variational formula* is

$$k_{ij} \ = \ k(e_i, e_j) \tag{28}$$

$$= \ \frac{1}{2\Phi}\frac{\partial \bar{g}_{ij}}{\partial t}. \tag{29}$$

One can choose a time function t, the level sets H_t of which are *maximal* spacelike hypersurfaces. This eliminates the indeterminacy of the evolution equations. The definition of a time function (definition 13) implies a freedom of choice. In fact, t being subject only to $dt \cdot X > 0$ for all $X \in I_p^+$ and for all $p \in M$ is arbitrary. We now fix our time function t by the condition to be *maximal*. This means we require the level sets H_t of the time function t to be *maximal spacelike hypersurfaces*. It describes the fact that any compact perturbation of H_t decreases the volume. Thus, H_t satisfies the maximal hypersurface equation

$$tr\ k \ = \ 0. \tag{30}$$

The existence of maximal surfaces in asymptotically flat spacetimes under slightly more general conditions, but for data with the same fall-off as ours, has been proven by R. Bartnik, P.T. Chruściel and N. O'Murchadha in [**4**]. It was first proven by R. Bartnik for stronger fall-off in [**2**].

Definition 14. A maximal time function is a time function t whose level sets are maximal spacelike hypersurfaces, being complete and tending to parallel spacelike coordinate hyperplanes at spatial infinity. We also require that the associated lapse function Φ tends to 1 at spatial infinity.

There is one such function up to an additive constant for each choice of family of parallel spacelike hyperplanes in Minkowski spacetime. These families are connected by the action of elements of the Lorentz group.

One can single out one family by choosing

$$P^i \ = \ 0. \tag{31}$$

Then the time function t is unique up to an additive constant.

The covariant differentiation on the spacetime M is denoted by D. For that on H write ∇. Whenever a different notation is used, it is indicated. In the sequel, denote by R the Riemannian curvature tensor of M, and by $\bar{R}$ the one of H. We shall work with the Weyl tensor W, not directly with the Riemannian curvature, for reasons explained below.

As motivated at the beginning, we are studying asymptotically flat solutions of the EV equations (2):

$$R_{\mu\nu} \ = \ 0.$$

Therefore, let us now explain what in general an asymptotically flat initial data set is. In view of the different types of asymptotic flatness, we first give the general definitions. Then, we can compare them with the definitions 2 and 3 from above describing the situations in [**21**] and [**6**], respectively.

Definition 15. A general asymptotically flat initial data set $(H, \bar{g}, k)$ is an initial data set such that

- the complement of a compact set in H is diffeomorphic to the complement of a closed ball in $\mathbb{R}^3$,
- and there exists a coordinate system (x^1, x^2, x^3) in this complement relative to which the metric components

$$\begin{array}{ccc} \bar{g}_{ij} & \rightarrow & \delta_{ij} \\ k_{ij} & \rightarrow & 0 \end{array}$$

  sufficiently rapidly as $r = (\sum_{i=1}^{3} (x^i)^2)^{\frac{1}{2}} \rightarrow \infty$.

Generally, one defines "strong" asymptotic flatness as follows:

Definition 16. A strongly asymptotically flat initial data set is an initial data set $(H, \bar{g}, k)$ with the following:

1) M is Euclidean at infinity.
2) There exists a chart on the neighbourhood of infinity in which the following holds:

$$\bar{g}_{ij} \ = \ (1 \ + \ \frac{2M}{r}) \ \delta_{ij} \ + \ o_2(r^{-1}). \tag{32}$$

3) It is

$$k_{ij} \ = \ o_1(r^{-2}). \tag{33}$$

M denotes the mass.

For certain asymptotically flat data sets, the ADM definitions of energy E, linear momentum P, and angular momentum J are well defined and finite. Let us write the ADM definitions in the following:

Definition 17 (Arnowitt, Deser, Misner (ADM)). Let $S_r = \{|x| = r\}$ be the coordinate sphere of radius r and dS_j the Euclidean oriented area element of S_r. Then we define the following

- Total Energy

$$E = \frac{1}{4} \lim_{r \to \infty} \int_{S_r} \sum_{i,j} (\partial_i \bar{g}_{ij} - \partial_j \bar{g}_{ii}) \, dS_j, \tag{34}$$

- Linear Momentum

$$P^i = -\frac{1}{2} \lim_{r \to \infty} \int_{S_r} (k_{ij} - \bar{g}_{ij} \, trk) \, dS_j, \tag{35}$$

- Angular Momentum

$$J^i = -\frac{1}{2} \lim_{r \to \infty} \int_{S_r} \epsilon_{ijm} x^j (k_{mn} - \bar{g}_{mn} \, trk) \, dS_n. \tag{36}$$

For strongly asymptotically flat initial data (definition 16), total energy, linear and angular momentum are well defined and conserved. Thus, also in the work of Christodoulou and Klainerman [[**21**], definition 3], all these quantities are well defined and conserved.

In our more general situation (see definition 2), the total energy and the linear momentum are shown to be well defined and conserved. We are still within the frame for which R. Bartnik's positive mass theorem applies [**3**].

Generally, total energy and linear momentum are well defined and conserved for asymptotically flat data sets such that there exists a coordinate system in the neighbourhood of infinity in which the following holds:

$$\bar{g}_{ij} = \delta_{ij} + o_2(r^{-\alpha}), \tag{37}$$

$$k_{ij} = o_1(r^{-1-\alpha}), \quad \alpha > \frac{1}{2}. \tag{38}$$

Let us now discuss the foliation of the spacetime given by u.

The *optical function* u is a solution of the *Eikonal equation*

$$g^{\alpha\beta} \frac{\partial u}{\partial x^\alpha} \frac{\partial u}{\partial x^\beta} = 0. \tag{39}$$

This equation tells us that the level sets C_u of u are null hypersurfaces.

The (t, u) *foliations* of the spacetime define a codimension 2 foliation by 2-surfaces

$$S_{t,u} = H_t \cap C_u, \tag{40}$$

the intersection between H_t (foliation by t) and a u-null-hypersurface C_u (foliation by u). The *area radius* $r(t, u)$ of $S_{t,u}$ is then defined as

$$r(t, u) = \sqrt{\frac{\text{Area } (S_{t,u})}{4\pi}}. \tag{41}$$

To construct this optical function u, we first choose a 2-surface $S_{0,0}$, diffeomorphic to S^2, in H_0. We assume the spacetime to have been constructed. Then the boundary $\partial J^+(S_{0,0})$ of the future of $S_{0,0}$ consists of an outer and an inner component. They are generated by the congruence of outgoing, respectively incoming, null geodesic normals to $S_{0,0}$. Now, the zero-level set C_0 of u is defined to be this outer component. In order to construct all the other level sets C_u for $u \neq 0$, we start on the last slice H_{t_*} of a spacetime slab. We solve on H_{t_*} an equation of motion of surfaces. We sketch it later in this paper. It forms a crucial part of our work. These level sets C_u are also outgoing null hypersurfaces. By construction, u is a solution of the Eikonal equation (39).

Important structures of the spacetime used in the proof are coming from a comparison with the Minkowski spacetime. Crucial are the canonical spacelike foliation, the null structure, and the conformal group structure. As the situation to be studied here is "close" to Minkowksi spacetime, we can use part of its conformal isometry group. In [**21**], the authors defined the action of the subgroup of the conformal group of Minkowski spacetime corresponding to the time translations, the scale transformations, the inverted time translations, and the spatial rotation group $O(3)$. For our present proof, we also define the actions for the first three of these, but not for $O(3)$. In contrast to [**21**], where the construction of the rotational vectorfields is a major part of the proof, we do not work with rotational vectorfields at all. Recalling the construction in [**21**], once the functions t and u have been fixed, the rotation group $O(3)$ takes any given hypersurface H_t onto itself. The orbit of $O(3)$ through a given point p is the corresponding surface $S_{t,u}$ through p. The surfaces (40) are the orbits of the rotation group $O(3)$ on H_t. In our situation, the vectorfields for the time and inverted time translations as well as for the scalings supply everything that is needed to obtain the estimates, as we shall see below. The group of time translations has already been defined. This corresponds to the choice of a canonical time function t. The integral curves of the generating vectorfield T are the timelike curves orthogonal to the hypersurfaces H_t, and are parametrized by t. For the corresponding group $\{f_\tau\}$, it holds that f_τ is a diffeomorphism of H_t onto $H_{t+\tau}$. Further, the vectorfields for the scaling and inverted time translations, that is, S and K, respectively, are also constructed with the help of the function u, as given below.

3. Important Structures and Former Results

We denote the deformation tensor of X by ${}^{(X)}\pi$. It is given as

$$ {}^{(X)}\pi_{\alpha\beta} = (\mathcal{L}_X g)_{\alpha\beta}, \tag{42} $$

$$ -\,{}^{(X)}\pi^{\alpha\beta} = (\mathcal{L}_X \, g^{-1})^{\alpha\beta}. \tag{43} $$

Moreover, a *Weyl tensor* W is defined to be a 4-tensor that satisfies all the symmetry properties of the curvature tensor and in addition is traceless.

Given a Weyl field W and a vectorfield X, the *Lie derivative* of W with respect to X is not, in general, a Weyl field, for it has trace. In fact, it is

$$g^{\alpha\gamma} \ (\mathcal{L}_X W_{\alpha\beta\gamma\delta}) \ = \ {}^{(X)}\pi^{\alpha\gamma} \ W_{\alpha\beta\gamma\delta}. \tag{44}$$

In view of this, we define the following *modified Lie derivative:*

$$\hat{\mathcal{L}}_X W \ := \ \mathcal{L}_X W \ - \ \frac{1}{2} \ {}^{(X)}[W] \ + \ \frac{3}{8} \ tr^{(X)}\pi \ W \tag{45}$$

with

$$\begin{aligned} {}^{(X)}[W]_{\alpha\beta\gamma\delta} \quad := \ & {}^{(X)}\pi^{\mu}{}_{\alpha} W_{\mu\beta\gamma\delta} \ + \ {}^{(X)}\pi^{\mu}{}_{\beta} W_{\alpha\mu\gamma\delta} \\ & + \ {}^{(X)}\pi^{\mu}{}_{\gamma} W_{\alpha\beta\mu\delta} \ + \ {}^{(X)}\pi^{\mu}{}_{\delta} W_{\alpha\beta\gamma\mu}. \end{aligned} \tag{46}$$

W is said to satisfy the *Bianchi equation* , if it is

$$D_{[\alpha} W_{\beta\gamma]\delta\epsilon} \ = \ 0.$$

To a Weyl field one can associate a tensorial quadratic form, a 4-covariant tensorfield which is fully symmetric and trace-free; this is a generalization of one found previously by Bel and Robinson [**5**]. As in [**21**], it is called the Bel-Robinson tensor:

$$Q_{\alpha\beta\gamma\delta} \ = \ \frac{1}{2} \ (W_{\alpha\rho\gamma\sigma} \ W_{\beta}{}^{\rho}{}_{\delta}{}^{\sigma} \ + \ {}^*W_{\alpha\rho\gamma\sigma} \ {}^*W_{\beta}{}^{\rho}{}_{\delta}{}^{\sigma}). \tag{47}$$

It satisfies the following positivity condition:

$$Q \ (X_1, \ X_2, \ X_3, \ X_4) \ \geq \ 0 \tag{48}$$

where X_1, X_2, X_3, and X_4 are future-directed timelike vectors. Moreover, if W satisfies the Bianchi equations then Q is divergence-free:

$$D^{\alpha} \ Q_{\alpha\beta\gamma\delta} \ = \ 0. \tag{49}$$

Equation (49) is a property of the Bianchi equations. In fact, they are covariant under conformal isometries. To be precise, let Ω be a positive function. Then, if $\Phi : M \to M$ is a conformal isometry of the spacetime, i.e.,

$$\Phi_* g \ = \ \Omega^2 g,$$

and if W is a solution, then $\Omega^{-1}\Phi_* W$ is also a solution.

The Bel-Robinson tensor Q is an important tool in our work. We shall come back to it.

For a long time, the burning question in the Cauchy problem for the EV equations (2) had been: Does there exist any non-trivial, asymptotically flat initial data with complete maximal development? In their pioneering work "The global nonlinear stability of the Minkowski space" [**21**], D. Christodoulou and S. Klainerman proved global existence and uniqueness of such solutions under certain smallness conditions on the initial data.

But this question has its roots back in the 1920s, 1930s and 1950s. Important preliminary work was done by de Donder, Friedrichs, Schauder, Sobolev, Petrovsky, Leray, and more people. A nice exposition about these early developments and further references can be found in [**22**]. Finally, in 1952, Y. Choquet-Bruhat focussed the question of local existence and uniqueness of solutions, in GR. In [**9**] she treated the Cauchy problem for the Einstein equations, locally in time, she showed existence and uniqueness of solutions, reducing the Einstein equations to wave equations and introducing harmonic coordinates. She proved the well-posedeness of the local Cauchy problem in these coordinates. The local result led to a global theorem proved by Y. Choquet-Bruhat and R. Geroch in [**10**], stating the existence of a unique maximal future development for each given initial data set.

Regarding the question of completeness or incompleteness of this maximal future development, R. Penrose gave an answer in his incompleteness theorem [**38**], stating that, if in the initial data set $(H, \bar{g}, k)$ H is non-compact (but complete), if the positivity condition on the energy holds, and, H contains a closed trapped surface S, the boundary of a compact domain in H, then the corresponding maximal future development is incomplete.

Definition 18. A closed trapped surface S in a non-compact Cauchy hypersurface H is a two-dimensional surface in H, bounding a compact domain such that

$$tr\chi \ < \ 0 \quad \text{on } S.$$

The second fundamental form χ is given below in (57). The Penrose theorem and its extensions by S. Hawking and R. Penrose led directly to the question formulated above: Is there any non-trivial asymptotically flat initial data whose maximal development is complete? The answer was given in the joint work of D. Christodoulou and S. Klainerman [**21**]. A rough version of the theorem is stated at the end of subsection 2, whereas a more precise version is given in theorem 2. The problem studied by Christodoulou and Klainerman in [**21**] was suggested by S.-T. Yau to Klainerman in 1978. Christodoulou elaborated the method of [**21**] and combined it with new ideas in his remarkable work "The Formation of Black Holes in General Relativity" [**18**].

N. Zipser in [**45**] studied the Einstein equations with the energy-momentum tensor being equal to the stress-energy tensor of an electromagnetic field satisfying the Maxwell equations. As in [**21**], she considered strongly asymptotically flat initial data on a maximal hypersurface. She generalized the result of [**21**] by proving the global nonlinear stability of the trivial solution of the Einstein-Maxwell equations. Lately, a proof under stronger conditions for the global stability of Minkowski space for the EV equations and asymptotically flat Schwarzschild initial data was given by H. Lindblad and I. Rodnianski [**33**], [**34**], the latter for EV (scalar field) equations. They worked with a wave coordinate gauge, showing the wave coordinates to be stable globally. Concerning the asymptotic behavior, the results are less precise than the ones of Christodoulou and Klainerman in [**21**]. Moreover, there are more conditions to be imposed on the data than in [**21**]. There is a variant for the exterior part of the proof from [**21**] using a double-null foliation by S. Klainerman and F. Nicolò in [**31**]. Also a semiglobal result was given by H. Friedrich [**24**] with initial data on a spacelike hyperboloid.

A still open question is: What is the sharp critera for non-trivial asymptotically flat initial data sets to give rise to a maximal development that is complete? Or, to what extent can the result of [**21**] be generalized?

The results of [**21**, **45**], and our new result [**6**], [**7**] are much more general than the others cited above, as all the other works place stronger conditions on the data.

4. Detailed Statement of the Main Results

In this section, we provide the most precise version of our main results. In order to state our main theorem in full detail, we have to introduce the corresponding norms. The definitions of these norms can be found in subsection 5.2.

We recall (16) and the global smallness assumption B (18) from the Introduction. The small positive ϵ on the right-hand side of (18) has to be chosen suitably small later in the proof, depending on other quantities, such that the inequalities in the main theorem 3 hold.

Theorem 3. (Main Theorem) *Any asymptotically flat, maximal initial data set (AFB) of the form given in definition* 2 *in the Introduction satisfying the global smallness assumption B stated in the 'Introduction', inequality (*18*), leads to a unique, globally hyperbolic, smooth, and geodesically complete solution of the EV equations, foliated by the level sets of a maximal time function t, defined for all $t \geq -1$. Moreover, there exists a global, smooth optical function u that is a solution of the Eikonal equation defined everywhere in the exterior region $r \geq \frac{r_0}{2}$, with $r_0(t)$ denoting the radius of the* 2*-surface $S_{t,0}$ of intersection between the hypersurfaces H_t and a fixed null cone C_0 with vertex at a point on H_{-1}. With respect to this foliation the following holds:*

$$ {}^e\mathcal{R}_{[1]}, \quad {}^e\mathcal{K}_{[2]}, \quad {}^e\mathcal{O}_{[2]}, \quad {}^e\mathcal{L}_{[2]} \quad \leq \quad \epsilon_0, \tag{50} $$

$$ {}^e\mathcal{K}_0^\infty, \quad {}^e\mathcal{O}_0^\infty, \quad {}^e\mathcal{L}_0^\infty \quad \leq \quad \epsilon_0. \tag{51} $$

Moreover, in the complement of the exterior region, the following holds:

$$ {}^{i}\mathcal{R}_{[1]}, \quad {}^{i}\mathcal{K}_{[2]}, \quad {}^{i}\mathcal{L}_{[2]} \quad \leq \quad \epsilon_0, \tag{52} $$

$$ {}^{i}\mathcal{K}_{0}^{\infty}, \quad {}^{i}\mathcal{L}_{0}^{\infty} \quad \leq \quad \epsilon_0. \tag{53} $$

The strict inequalities hold for $t = 0$ *with* ϵ *on the right-hand sides. The norms are given in subsection* 5.2.

In the next section, we are going to state the main steps of the proof and to explain the bootstrap argument in detail.

We recall that in the global smallness assumption B (18) the initial data has to be smaller than a sufficiently small positive ϵ. Later in the proof, this ϵ has to be taken suitably small, depending on other quantities. In the bootstrap assumptions BA0–BA2 ((135)–(137)) and in (138), the considered quantities have to be smaller than a small positive ϵ_0. We estimate the main quantities at times t by their values at $t = 0$, which are controlled by inequalities with ϵ on their right-hand sides. Then, choosing ϵ sufficiently small, the right-hand sides of these inequalities can be made strictly smaller than ϵ_0 from the bootstrap assumptions. The bootstrap argument is explained in detail in the next section.

Our main theorem 3 provides existence and uniqueness of solutions under the relaxed assumptions (AFB, see (18)) as well as describes the asymptotic behavior as precisely as possible under these relaxed assumptions. Compared to the result [**21**] main theorem 10.2.1, p. 298 (we give one version of their main theorem in theorem 2), we impose less on our initial data. That is, we assume one less power of r decay of the data at infinity and one less derivative to be controlled.

5. Crucial Steps of the Proof of the Main Theorem—Bootstrap Argument

Our proof consists of one large *bootstrap argument*, containing other arguments of the same type but at different levels.

First, we give the **main steps of the proof of our main result.** They can be summarized as follows:

1. *Energy*. Estimate a *quantity* $\mathcal{Q}_1(W)$, which is an integral over H_t involving the Bel-Robinson tensor Q of the spacetime curvature W and of the Lie derivatives of W as below. At time t, this quantity $\mathcal{Q}_1(W)$ can be calculated by its value at $t = 0$ and an integral from 0 to t, which both are controlled. We use the vectorfields T time translations, S scaling, K inverted time translations, and $\bar{K} = K + T$. (See below). $\mathcal{Q}_1(W)$ is given by

$$ \mathcal{Q}_1(W) \quad = \quad Q_0 \; + \; Q_1 \tag{54} $$

with Q_0 and Q_1 being the following integrals:

$$Q_0(t) = \int_{H_t} Q\ (W)\ (\bar{K}, T, T, T), \tag{55}$$

$$\begin{aligned} Q_1(t) &= \int_{H_t} Q\ (\hat{\mathcal{L}}_S W)\ (\bar{K}, T, T, T) \\ &\quad + \int_{H_t} Q\ (\hat{\mathcal{L}}_T W)\ (\bar{K}, \bar{K}, T, T). \end{aligned} \tag{56}$$

2. The components of the *Weyl tensor* W are estimated by a *comparison argument* with $\mathcal{Q}_1(W)$. The estimates for the Weyl tensor W rely heavily on the fact that W satisfies the Bianchi equations.
3. The *geometric* quantities are estimated from *curvature assumptions* using the optical structure equations, elliptic estimates, evolution equations, and tools like Sobolev inequalities.

The bootstrapping allows us to go from local to global. The whole procedure can mainly be split into three parts.

- *Bootstrap assumptions:* initial assumptions on the main geometric quantities of the two foliations, i.e., $\{H_t\}$ and $\{C_u\}$.
- *Local existence theorem:* It guarantees the local existence of a unique solution and the preservation of the asymptotic behavior in space of the metric, the second fundamental form, and the curvature.
- *Bootstrap argument:* Together with the evolution equations, it yields the global existence of a unique solution as well as shows the asymptotic behavior as above to be preserved.

Now we are going to state and to explain the crucial steps of the proof of the main theorem, that is, of the *bootstrap argument*.

Before that, let us say a few words about the geometric quantities in point three. In order to estimate the geometric quantities of point three above in the framework of a bootstrap argument, one assumes that the curvature components satisfy suitable bounds. To see how this is done, let us consider the affine foliation $\{S_s\}$ of C. Here, the generating vectorfield L of C is geodesic, and s is the corresponding affine parameter function. We shall now give the idea for the case of the second fundamental form χ of S_s relative to C,

$$\chi(X, Y) = g(D_X L, Y), \tag{57}$$

for any pair of vectors $X, Y \in T_p S_s$. To estimate χ, we first split it into its trace $tr\chi$ and traceless part $\hat{\chi}$. Then, one estimates χ from the propagation equation

$$\frac{\partial tr\chi}{\partial s} + \frac{1}{2}\ (tr\chi)^2 + |\ \hat{\chi}\ |^2 = 0 \tag{58}$$

and the elliptic system on each section S_s of C (the Codazzi equations)

$$d\not{i}v\ \hat{\chi}_a = \frac{1}{2}\, d_a tr\chi + f_a \tag{59}$$

where f_a involves curvature.

Assuming estimates for the spacetime curvature on the right-hand side of (59) yields estimates for the quantities controlling the geometry of C as described by its foliation $\{S_s\}$. This is discussed in more detail below.

5.1. Local Existence Theorem. Relying on the *local existence theorem*, [[**21**], statement theorem 10.2.2: p. 299/300, proof 10.2.2: p. 304–310], we show global existence of a unique, globally hyperbolic, smooth, and geodesically complete solution of the Einstein-vacuum equations, coming from initial data stated in definition 2 (AFB) and inequality 18 (global smallness assumption B). The local existence theorem is stated and proven in [**21**] for their problem, and it also holds in our case. It requires the second fundamental form k to be in L^∞, which is satisfied in our situation. The proof of the local existence theorem in [**21**] mainly uses the ideas developed in the proof of the well-known existence result of Choquet-Bruhat [**9**] and modifies them simply. In [**21**], the authors formulate the conditions in the local existence theorem according to their situation imposing $Ric(\bar{g}_0) \in H_{2,1}(H, \bar{g}_0)$ and $k \in H_{3,1}(H, \bar{g}_0)$, while we impose $Ric(\bar{g}_0) \in H_{1,1}(H, \bar{g}_0)$ and $k \in H_{2,0}(H, \bar{g}_0)$. The proof still holds in the same way, as it only requires k in L^∞, which in our situation is true.

5.2. Norms. We define the norms as they appear in the main theorem 3 and as we use them in subsection 5.3.

We consider a spacetime slab $\bigcup_{t\in[0,t_*]} H_t$. In this section, we assume that this slab is foliated by a maximal time function t and by an optical function u. In what follows, we shall introduce the basic norms for the curvature R, the second fundamental form k, the lapse function ϕ, and the components χ, ζ, ω of the Hessian of u. In most of the definitions below, we follow the notation of [**21**].

Let V be a vectorfield tangent to S. Then we define the norms on S:

$$\| V \|_{p,S}\ (t,u) = \Big(\int_{S_{t,u}} | V |^p \ d\mu_\gamma \Big)^{\frac{1}{p}}, \quad \text{for } 1 \le p < \infty, \tag{60}$$

$$= \sup_{S_{t,u}} | V | \ , \quad \text{for } p = \infty. \tag{61}$$

Sometimes, we also denote these norms by $| V |_{p,S}\ (t,u)$.

The following norms are stated for the interior and the exterior regions of each hypersurface H_t.

The *interior region* I, also denoted by H_t^i, is all the points in H_t for which

$$r \le \frac{r_0(t)}{2}.$$

The *exterior region* U, also denoted by H_t^e, is all the points in H_t for which

$$r \ \geq \ \frac{r_0(t)}{2}.$$

Here, $r_0(t)$ is the value of r corresponding to the area of $S_{t,0}$, the surface of intersection between C_0 and H_t. And $u_1(t)$ is the value of u corresponding to $r_0(t)/2$.

Now we introduce

$$\| V \|_{p,i} \ = \ \Big(\int_{H_t^i} | V |^p \Big)^{\frac{1}{p}}, \quad \text{for } 1 \ \leq \ p \ < \infty, \tag{62}$$

$$\| V \|_{\infty,i} \ = \ \sup_{H_t^i} | V |, \tag{63}$$

$$\| V \|_{p,e} \ (t) \ = \ \Big(\int_{H_t^e} | V |^p \Big)^{\frac{1}{p}}, \quad \text{for } 1 \ \leq \ p \ < \infty, \tag{64}$$

$$\| V \|_{\infty,e} \ (t) \ = \ \sup_{H_t^e} | V | \, . \tag{65}$$

Sometimes, we denote the norms ((62)–(65)) also by $|V|_{p,I}$, $|V|_{p,U}(t)$, or $\|V\|_{p,I}$, $\|V\|_{p,U}(t)$ or $|V|_{p,i}$, $|V|_{p,e}(t)$, respectively.

5.2.1. Norms for the Curvature Tensor R**.** We are now going to define the norms $\overline{\mathcal{R}_0} \ (W)$ and $\overline{\mathcal{R}_1} \ (W)$ as well as $\mathcal{R}_0(W)$ and $\mathcal{R}_1(W)$ for the null curvature components.

The norms are stated for the interior and exterior regions of the hypersurface H_t.

Norms $\overline{\mathcal{R}_0}$ (W) and $\overline{\mathcal{R}_1}$ (W):

On each slice H_t, we denote by $\overline{\mathcal{R}_q} \ (W)$ the following maximum:

$$\overline{\mathcal{R}_q} \ (W) \ = \ \max \big(\overline{{}^e\mathcal{R}_q}(W), \ \overline{{}^i\mathcal{R}_q}(W) \big), \qquad q \ = \ 0, \ 1. \tag{66}$$

By $\overline{{}^e\mathcal{R}_q}(W)$ we denote the exterior and by $\overline{{}^i\mathcal{R}_q}(W)$ the interior L^2-norms of the curvature given as follows: In the interior region I, we define for $q = 0, 1$,

$$\overline{{}^i\mathcal{R}_q} \ = \ r_0^{1+q} \ \| D^q \ W \|_{2,I} \, . \tag{67}$$

Then, one sets

$$\overline{{}^i\mathcal{R}_{[0]}} \ = \ \overline{{}^i\mathcal{R}_0}, \tag{68}$$

$$\overline{{}^i\mathcal{R}_{[1]}} \ = \ \overline{{}^i\mathcal{R}_{[0]}} \ + \ \overline{{}^i\mathcal{R}_1}. \tag{69}$$

In the following, the weight function τ_- and the curvature components $\alpha, \underline{\alpha}, \beta, \underline{\beta}, \rho, \sigma$ are defined in ((151)–(156)) and right before (151).

We define the exterior norms $\overline{{}^e\mathcal{R}_0}(W)$ and $\overline{{}^e\mathcal{R}_1}(W)$ to be

$$\begin{aligned}\overline{{}^e\mathcal{R}_0}\,(W)^2 \;&=\; \int_U \tau_-^2 \mid \underline{\alpha} \mid^2 + \int_U r^2 \mid \underline{\beta} \mid^2 + \int_U r^2 \mid \rho \mid^2 + \int_U r^2 \mid \sigma \mid^2 \\ &\quad + \int_U r^2 \mid \beta \mid^2 + \int_U r^2 \mid \alpha \mid^2 \end{aligned} \tag{70}$$

and

$$\begin{aligned}\overline{{}^e\mathcal{R}_1}\,(W)^2 \;&=\; \int_U \tau_-^2 r^2 \mid \nabla\!\!\!/\, \underline{\alpha} \mid^2 + \int_U r^4 \mid \nabla\!\!\!/\, \underline{\beta} \mid^2 + \int_U r^4 \mid \nabla\!\!\!/\, \rho \mid^2 \\ &\quad + \int_U r^4 \mid \nabla\!\!\!/\, \sigma \mid^2 + \int_U r^4 \mid \nabla\!\!\!/\, \beta \mid^2 + \int_U r^4 \mid \nabla\!\!\!/\, \alpha \mid^2 \\ &\quad + \int_U \tau_-^4 \mid \nabla\!\!\!/_N \underline{\alpha} \mid^2 + \int_U \tau_-^2 r^2 \mid \nabla\!\!\!/_N \underline{\beta} \mid^2 \\ &\quad + \int_U r^4 \mid \nabla\!\!\!/_N \rho \mid^2 + \int_U r^4 \mid \nabla\!\!\!/_N \sigma \mid^2 + \int_U r^4 \mid \nabla\!\!\!/_N \beta \mid^2 \\ &\quad + \int_U r^4 \mid \nabla\!\!\!/_N \alpha \mid^2 . \end{aligned} \tag{71}$$

We refer to the norms of the components of R by the formulas for $q = 0, 1$,

$$\begin{aligned}\overline{{}^e\mathcal{R}_q}(\underline{\alpha}) \;&=\; \parallel \tau_- r^q \nabla\!\!\!/^{\,q} \underline{\alpha} \parallel_{2,e}, \\ \overline{{}^e\mathcal{R}_q}(\alpha) \;&=\; \parallel r^{q+1} \nabla\!\!\!/^{\,q} \alpha \parallel_{2,e}, \\ &\cdots, \end{aligned}$$

and correspondingly for the remaining components.

Denote $\underline{\alpha}_N = \nabla\!\!\!/_N \underline{\alpha}$, $\alpha_N = \nabla\!\!\!/_N \alpha$ and correspondingly for the other curvature components. Then, we set

$$\begin{aligned}\overline{{}^e\mathcal{R}_0}[\underline{\alpha}] \;&=\; \overline{{}^e\mathcal{R}_0}(\underline{\alpha}), \\ \overline{{}^e\mathcal{R}_1}[\underline{\alpha}] \;&=\; \Big(\overline{{}^e\mathcal{R}_1}(\underline{\alpha})^2 \;+\; \overline{{}^e\mathcal{R}_0}(\underline{\alpha}_N)^2 \Big)^{\frac{1}{2}}. \end{aligned} \tag{72}$$

Similarly as in (72), we proceed with all the other null components of the curvature. This allows us now to define for $q = 0, 1$ the following:

$$\overline{{}^e\mathcal{R}_q} \;=\; \Big(\overline{{}^e\mathcal{R}_q}[\underline{\alpha}]^2 \;+\; \overline{{}^e\mathcal{R}_q}[\underline{\beta}]^2 \;+\; \cdots \;+\; \overline{{}^e\mathcal{R}_q}[\alpha]^2 \Big)^{\frac{1}{2}}. \tag{73}$$

Then, one sets

$$\overline{{}^e\mathcal{R}_{[0]}} \;=\; \overline{{}^e\mathcal{R}_0}, \tag{74}$$

$$\overline{{}^e\mathcal{R}_{[1]}} \;=\; \overline{{}^e\mathcal{R}_{[0]}} \;+\; \overline{{}^e\mathcal{R}_1}. \tag{75}$$

Norms $\mathcal{R}_{\mathbf{0}}(\mathbf{W})$ and $\mathcal{R}_{\mathbf{1}}(\mathbf{W})$:

First, we introduce some notation. (See also [**21**], [**6**], [**7**].) The *signature* s is defined to be the difference of the number of contractions with L minus the number of contractions with $\underline{L}$. See also subsection 6.1. Then we introduce the following.

Definition 19. Let W be an arbitrary Weyl tensor and let ξ be any of its null components. Let $\not{D}_3 \xi$ and $\not{D}_4 \xi$ denote the projections to $S_{t,u}$ of $D_3 \xi$ and $D_4 \xi$, respectively. Define the following $S_{t,u}$-tangent tensors:

$$\begin{aligned} \xi_3 &= \not{D}_3 \xi + \frac{3-s}{2} tr \underline{\chi} \xi, \\ \xi_4 &= \not{D}_4 \xi + \frac{3+s}{2} tr \chi \xi. \end{aligned}$$

Similarly as above, we define on each slice H_t the quantity $\mathcal{R}_q \ (W)$ as the maximum

$$\mathcal{R}_q \ (W) = \max \big({}^e\mathcal{R}_q(W), \ {}^i\mathcal{R}_q(W) \big), \qquad q = 0, \ 1. \tag{76}$$

By ${}^e\mathcal{R}_q(W)$ we denote the exterior and by ${}^i\mathcal{R}_q(W)$ the interior L^2-norms of the curvature given as follows: In the interior region I, we define for $q = 0, 1$,

$$ {}^i\mathcal{R}_q = r_0^{1+q} \ \| \ D^q \ W \ \|_{2,I} \ . \tag{77}$$

Then, one sets

$$ {}^i\mathcal{R}_{[0]} = {}^i\mathcal{R}_0, \tag{78}$$

$$ {}^i\mathcal{R}_{[1]} = {}^i\mathcal{R}_{[0]} + {}^i\mathcal{R}_1. \tag{79}$$

We define the exterior norms ${}^e\mathcal{R}_0$ and ${}^e\mathcal{R}_1$ as follows:

$$\begin{aligned} {}^e\mathcal{R}_0 \ (W)^2 = & \int_U \tau_-^2 \mid \underline{\alpha} \mid^2 + \int_U r^2 \mid \underline{\beta} \mid^2 + \int_U r^2 \mid \rho \mid^2 + \int_U r^2 \mid \sigma \mid^2 \\ & + \int_U r^2 \mid \beta \mid^2 + \int_U r^2 \mid \alpha \mid^2 \end{aligned} \tag{80}$$

and

$$\begin{aligned} {}^e\mathcal{R}_1 \ (W)^2 = & \int_U \tau_-^2 r^2 \mid \nabla\!\!\!/ \, \underline{\alpha} \mid^2 + \int_U r^4 \mid \nabla\!\!\!/ \, \underline{\beta} \mid^2 + \int_U r^4 \ \mid \nabla\!\!\!/ \, \rho \mid^2 \\ & + \int_U r^4 \ \mid \nabla\!\!\!/ \, \sigma \mid^2 + \int_U r^4 \mid \nabla\!\!\!/ \, \beta \mid^2 + \int_U r^4 \mid \nabla\!\!\!/ \, \alpha \mid^2 \\ & + \int_U \tau_-^4 \mid \underline{\alpha}_3 \mid^2 + \int_U r^4 \mid \underline{\alpha}_4 \mid^2 + \int_U \tau_-^2 r^2 \mid \underline{\beta}_3 \mid^2 \\ & + \int_U r^4 \mid \underline{\beta}_4 \mid^2 + \int_U r^4 \mid \rho_3 \mid^2 + \int_U r^4 \mid \rho_4 \mid^2 \\ & + \int_U r^4 \mid \sigma_3 \mid^2 + \int_U r^4 \mid \sigma_4 \mid^2 + \int_U r^4 \mid \beta_3 \mid^2 \\ & + \int_U r^4 \mid \beta_4 \mid^2 + \int_U r^4 \mid \alpha_3 \mid^2 + \int_U r^4 \mid \alpha_4 \mid^2 \ . \end{aligned} \tag{81}$$

We refer to the norms of the components of R by the formulas, for $q = 0, 1$,

$$\begin{aligned} {}^e\mathcal{R}_q(\underline{\alpha}) &= \| \tau_- r^q \nabla\!\!\!/^{\,q} \underline{\alpha} \|_{2,e} \\ {}^e\mathcal{R}_q(\alpha) &= \| r^{q+1} \nabla\!\!\!/^{\,q} \alpha \|_{2,e}, \\ \cdots, & \end{aligned}$$

and correspondingly for the remaining components.

Then, we set

$$\begin{aligned} {}^e\mathcal{R}_0[\underline{\alpha}] &= {}^e\mathcal{R}_0(\underline{\alpha}), \\ {}^e\mathcal{R}_1[\underline{\alpha}] &= \Big({}^e\mathcal{R}_1(\underline{\alpha})^2 + {}^e\mathcal{R}_0(\underline{\alpha}_3)^2 + {}^e\mathcal{R}_0(\underline{\alpha}_4)^2 \Big)^{\frac{1}{2}}. \end{aligned} \tag{82}$$

Similarly, this is done for all the other null components of the curvature. Thus, we define the following for $q = 0, 1$:

$$ {}^e\mathcal{R}_q = \Big({}^e\mathcal{R}_q[\underline{\alpha}]^2 + {}^e\mathcal{R}_q[\underline{\beta}]^2 + \cdots + {}^e\mathcal{R}_q[\alpha]^2 \Big)^{\frac{1}{2}}. \tag{83}$$

Then, one sets

$$ {}^e\mathcal{R}_{[0]} = {}^e\mathcal{R}_0, \tag{84}$$

$$ {}^e\mathcal{R}_{[1]} = {}^e\mathcal{R}_{[0]} + {}^e\mathcal{R}_1. \tag{85}$$

5.2.2. Norms for the Second Fundamental Form k of the t-Foliation. First, let us define ${}^i\mathcal{K}_q^p$ and ${}^e\mathcal{K}_q^p$ to be the interior and exterior weighted L^p-norms of the q-covariant derivatives of the components of the second fundamental form k. The quantity $\mathcal{K}_q^p$ we then define as follows:

$$\mathcal{K}_q^p = \max\big({}^i\mathcal{K}_q^p, \ {}^e\mathcal{K}_q^p \big), \quad q = 0, 1, 2 \quad \text{and} \quad 1 \leq p < \infty. \tag{86}$$

Correspondingly, we have

$$\mathcal{K}_0^\infty = \max\big({}^i\mathcal{K}_0^\infty, \ {}^e\mathcal{K}_0^\infty \big). \tag{87}$$

The interior norms ${}^i\mathcal{K}_q^p$ in (86) are given by

$$ {}^i\mathcal{K}_q^p = r_0^{1+q-\frac{2}{p}} \| D^q k \|_{p,i} . \tag{88}$$

In view of the exterior norms ${}^e\mathcal{K}_q^p$, let us remind ourselves that the second fundamental form k relative to the radial foliation of u on H_t decomposes into

$$\begin{aligned} k_{NN} &= \delta, \\ k_{AN} &= \epsilon_A, \\ k_{AB} &= \eta_{AB}. \end{aligned} \tag{89}$$

In addition, η decomposes into its trace $tr\eta = -\delta$ and its traceless part $\hat{\eta}$. Let us also introduce the following notation:

$$
\begin{aligned}
\delta_4 &= \not{D}_4\, \delta, \\
\epsilon_4 &= \not{D}_4\, \epsilon, \\
\hat{\eta}_4 &= \not{D}_4\, \hat{\eta} + \frac{1}{2}\, tr\chi\, \hat{\eta}, \\
\delta_3 &= \not{D}_3\, \delta, \\
\epsilon_3 &= \not{D}_3\, \epsilon, \\
\hat{\eta}_3 &= \not{D}_3\, \hat{\eta} + \frac{1}{2}\, tr\underline{\chi}\, \hat{\eta}.
\end{aligned}
\tag{90}
$$

Then, we set

$$
\begin{aligned}
{}^e\mathcal{K}^p_q(\delta) &= \| r^{(\frac{3}{2}-\frac{3}{p}+q)}\, \not{\nabla}^q \delta \|_{p,e}, \\
{}^e\mathcal{K}^p_q(\epsilon) &= \| r^{(\frac{3}{2}-\frac{3}{p}+q)}\, \not{\nabla}^q \epsilon \|_{p,e}, \\
{}^e\mathcal{K}^p_q(\hat{\eta}) &= \| r^{(\frac{3}{2}-\frac{3}{p}+q)}\, \not{\nabla}^q \hat{\eta} \|_{p,e},
\end{aligned}
\tag{91}
$$

$$
\begin{aligned}
{}^e\mathcal{K}^p_{q+1}(\delta_4) &= \| r^{(\frac{5}{2}-\frac{3}{p}+q)}\, \not{\nabla}^q \delta_4 \|_{p,e}, \\
{}^e\mathcal{K}^p_{q+1}(\delta_3) &= \| r^{(\frac{3}{2}-\frac{3}{p}+q)} \tau_-^{(\frac{3}{2}-\frac{1}{p})}\, \not{\nabla}^q \delta_3 \|_{p,e}, \\
{}^e\mathcal{K}^p_{q+1}(\epsilon_4) &= \| r^{(\frac{5}{2}-\frac{3}{p}+q)}\, \not{\nabla}^q \epsilon_4 \|_{p,e}, \\
{}^e\mathcal{K}^p_{q+1}(\epsilon_3) &= \| r^{(\frac{3}{2}-\frac{3}{p}+q)} \tau_-^{(\frac{3}{2}-\frac{1}{p})}\, \not{\nabla}^q \epsilon_3 \|_{p,e}, \\
{}^e\mathcal{K}^p_{q+1}(\hat{\eta}_4) &= \| r^{(\frac{5}{2}-\frac{3}{p}+q)}\, \not{\nabla}^q \hat{\eta}_4 \|_{p,e}, \\
{}^e\mathcal{K}^p_{q+1}(\hat{\eta}_3) &= \| r^{(1-\frac{2}{p}+q)} \tau_-^{(\frac{3}{2}-\frac{1}{p})} \not{\nabla}^q \hat{\eta}_3 \|_{p,e}\ .
\end{aligned}
\tag{92}
$$

Next, one sets

$$
\begin{aligned}
{}^e\mathcal{K}^p_0[\delta] &= {}^e\mathcal{K}^p_0(\delta), \\
{}^e\mathcal{K}^p_1[\delta] &= {}^e\mathcal{K}^p_1(\delta) + {}^e\mathcal{K}^p_1(\delta_3) + {}^e\mathcal{K}^p_1(\delta_4), \\
{}^e\mathcal{K}^p_2[\delta] &= {}^e\mathcal{K}^p_2(\delta) + {}^e\mathcal{K}^p_2(\delta_3) + {}^e\mathcal{K}^p_2(\delta_4),
\end{aligned}
\tag{93}
$$

and correspondingly we do this for ϵ and $\hat{\eta}$. Then for all $q = 0, 1, 2$:

$$
{}^e\mathcal{K}^p_q = {}^e\mathcal{K}^p_q[\delta] + {}^e\mathcal{K}^p_q[\epsilon] + {}^e\mathcal{K}^p_q[\hat{\eta}]. \tag{94}
$$

For the case $p = 2$, which will be used later on, we write simply

$$
\mathcal{K}_q = \mathcal{K}^2_q.
$$

Thus, we define the basic spacetime norms for k as follows:

$$ {}^{i}\mathcal{K}_{[0]} = {}^{i}\mathcal{K}_{0}, \tag{95} $$

$$ {}^{i}\mathcal{K}_{[1]} = {}^{i}\mathcal{K}_{[0]} + {}^{i}\mathcal{K}_{1}, \tag{96} $$

$$ {}^{i}\mathcal{K}_{[2]} = {}^{i}\mathcal{K}_{[1]} + {}^{i}\mathcal{K}_{2}, \tag{97} $$

$$ {}^{e}\mathcal{K}_{[0]} = {}^{e}\mathcal{K}_{0}, \tag{98} $$

$$ {}^{e}\mathcal{K}_{[1]} = {}^{e}\mathcal{K}_{[0]} + {}^{e}\mathcal{K}_{1}, \tag{99} $$

$$ {}^{e}\mathcal{K}_{[2]} = {}^{e}\mathcal{K}_{[1]} + {}^{e}\mathcal{K}_{2}, \tag{100} $$

$$ \mathcal{K}_{[q]} = {}^{i}\mathcal{K}_{[q]} + {}^{e}\mathcal{K}_{[q]}, \tag{101} $$

Correspondingly, we define $\overline{{}^{i}\mathcal{K}_{q}^{p}}$ and $\overline{{}^{e}\mathcal{K}_{q}^{p}}$ to be the interior and exterior weighted L^p-norms as given below. Then we define $\overline{\mathcal{K}_{q}^{p}}$ as follows:

$$ \overline{\mathcal{K}_{q}^{p}} = \max\left(\overline{{}^{i}\mathcal{K}_{q}^{p}},\ \overline{{}^{e}\mathcal{K}_{q}^{p}}\right), \qquad q = 0,\ 1,\ 2 \ \text{ and } \ 1 \leq p < \infty. \tag{102} $$

Correspondingly, we have

$$ \overline{\mathcal{K}_{0}^{\infty}} = \max\left(\overline{{}^{i}\mathcal{K}_{0}^{\infty}},\ \overline{{}^{e}\mathcal{K}_{0}^{\infty}}\right), \tag{103} $$

Now, in the interior region I we define the following norms for k:

$$ \overline{{}^{i}\mathcal{K}_{q}^{p}} = r_0^{1+q-\frac{2}{p}} \parallel D^q k \parallel_{p,i} . \tag{104} $$

On the other hand, in the exterior region U, we define the following norms:

$$ \begin{aligned} \overline{{}^{e}\mathcal{K}_{q}^{p}}(\delta) &= \Big\{ \parallel r^{(\frac{3}{2}-\frac{3}{p}+q)}\ \nabla\!\!\!\!/^{\,q}\delta \parallel_{p,e}^{p} \\ &\quad + \sum_{i+j=q,\ j\geq 1} \parallel r^{(2-\frac{2}{p}+i)}\ (\tau_-)^{1-\frac{2}{p}+j-1}\ \nabla\!\!\!\!/^{\,i}\nabla_N^{j}\delta \parallel_{p,e}^{p} \Big\}^{\frac{1}{p}}, \\ \overline{{}^{e}\mathcal{K}_{q}^{p}}(\epsilon) &= \Big\{ \parallel r^{(\frac{3}{2}-\frac{3}{p}+q)}\ \nabla\!\!\!\!/^{\,q}\epsilon \parallel_{p,e}^{p} \\ &\quad + \sum_{i+j=q,\ j\geq 1} \parallel r^{(2-\frac{2}{p}+i)}\ (\tau_-)^{1-\frac{2}{p}+j-1}\ \nabla\!\!\!\!/^{\,i}\nabla\!\!\!\!/_N^{\,j}\epsilon \parallel_{p,e}^{p} \Big\}^{\frac{1}{p}}, \\ \overline{{}^{e}\mathcal{K}_{q}^{p}}(\hat{\eta}) &= \Big\{ \parallel r^{(\frac{3}{2}-\frac{3}{p}+q)}\ \nabla\!\!\!\!/^{\,q}\hat{\eta} \parallel_{p,e}^{p} \\ &\quad + \sum_{i+j=q,\ j\geq 1} \parallel r^{(1-\frac{2}{p}+i)}\ (\tau_-)^{1-\frac{2}{p}+j}\ \nabla\!\!\!\!/^{\,i}\nabla\!\!\!\!/_N^{\,j}\hat{\eta} \parallel_{p,e}^{p} \Big\}^{\frac{1}{p}}, \end{aligned} \tag{105} $$

Then we set

$$ \overline{{}^{e}\mathcal{K}_{q}^{p}} = \overline{{}^{e}\mathcal{K}_{q}^{p}}(\delta) + \overline{{}^{e}\mathcal{K}_{q}^{p}}(\epsilon) + \overline{{}^{e}\mathcal{K}_{q}^{p}}(\hat{\eta}) \tag{106} $$

similarly as above. For the case $p = 2$, we write simply

$$ \overline{\mathcal{K}_{q}} = \overline{\mathcal{K}_{q}^{2}}. $$

Putting this together into the basic norms of the second fundamental form,

$$\overline{\mathcal{K}_{[0]}} = \overline{\mathcal{K}_0}, \tag{107}$$
$$\overline{\mathcal{K}_{[1]}} = \overline{\mathcal{K}_{[0]}} + \overline{\mathcal{K}_1}, \tag{108}$$
$$\overline{\mathcal{K}_{[2]}} = \overline{\mathcal{K}_{[1]}} + \overline{\mathcal{K}_2}. \tag{109}$$

5.2.3. Norms for the Lapse Function Φ. Analogously to the previous cases, we first define ${}^{i}\mathcal{L}_q^p$ and ${}^{e}\mathcal{L}_q^p$ to be the interior and exterior weighted L^p-norms of the $(q+1)$-covariant derivatives of the logarithm φ of the lapse function ϕ. Then we also define

$$\mathcal{L}_q^p = \max\left({}^{i}\mathcal{L}_q^p, \ {}^{e}\mathcal{L}_q^p \right), \tag{110}$$

for $q = 0, 1, 2$. Correspondingly, we define

$$\mathcal{L}_0^\infty = \max\left({}^{i}\mathcal{L}_0^\infty, \ {}^{e}\mathcal{L}_0^\infty \right). \tag{111}$$

The interior norms ${}^{i}\mathcal{L}_q^p$ in (110) are given by

$${}^{i}\mathcal{L}_q^p = r_0^{1+q-\frac{2}{p}} \parallel D^{q+1}\varphi \parallel_{p,i} . \tag{112}$$

In order to state the norms for ${}^{e}\mathcal{L}_q^p$, we first decompose $\nabla\varphi$ as follows:

$$\not{\varphi}_A = \not{\nabla}_A \varphi, \tag{113}$$
$$\varphi_N = \nabla_N \varphi. \tag{114}$$

That is, we have

$$\not{\varphi}_A = \not{\nabla}_A\varphi = \frac{1}{\phi}\not{\nabla}_A\phi,$$
$$\varphi_N = \nabla_N\varphi = \frac{1}{\phi}\nabla_N\phi.$$

Next, we set

$$
\begin{aligned}
{}^{e}\mathcal{L}^{p}_{q}(\not\phi\,) &= \| \, r^{(\frac{3}{2}-\frac{3}{p}+q)} \; \not\nabla^{q} \not\phi \; \|_{p,e}, \\
{}^{e}\mathcal{L}^{p}_{q}(\varphi_N) &= \| \, r^{(\frac{3}{2}-\frac{3}{p}+q)} \; \not\nabla^{q} \varphi_N \, \|_{p,e}, \qquad (115)\\
{}^{e}\mathcal{L}^{p}_{q+1}(\not D_{4} \not\phi\,) &= \| \, r^{(\frac{5}{2}-\frac{3}{p}+q)} \; \not\nabla^{q} \not D_{4} \not\phi \; \|_{p,e}, \\
{}^{e}\mathcal{L}^{p}_{q+1}(\not D_{3} \not\phi\,) &= \| \, r^{(\frac{5}{2}-\frac{3}{p}+q)} \; \not\nabla^{q} \not D_{3} \not\phi \; \|_{p,e}, \\
{}^{e}\mathcal{L}^{p}_{q+1}(D_4 \; \varphi_N) &= \| \, r^{(\frac{5}{2}-\frac{3}{p}+q)} \; \not\nabla^{q} D_4 \; \varphi_N \, \|_{p,e}, \\
{}^{e}\mathcal{L}^{p}_{q+1}(D_3 \; \varphi_N) &= \| \, r^{(\frac{3}{2}-\frac{3}{p}+q)} \tau_{-}^{\frac{3}{2}-\frac{1}{p}} \; \not\nabla^{q} D_3 \; \varphi_N \, \|_{p,e}, \qquad (116)\\
{}^{e}\mathcal{L}^{p}_{q+1}(\not D_{S} \not\phi\,) &= \| \, r^{(\frac{3}{2}-\frac{3}{p}+q)} \; \not\nabla^{q} \not\phi \; \|_{p,e}, \\
{}^{e}\mathcal{L}^{p}_{q+1}(\not D_{S} \; \varphi_N) &= \| \, r^{(\frac{3}{2}-\frac{3}{p}+q)} \; \not\nabla^{q} \varphi_N \, \|_{p,e} \qquad (117)\\
{}^{e}\mathcal{L}^{p}_{q+2}(\not D_{S} \not D_{4} \not\phi\,) &= \| \, r^{(\frac{5}{2}-\frac{3}{p}+q)} \; \not\nabla^{q} \not D_{4} \not\phi \; \|_{p,e}, \\
{}^{e}\mathcal{L}^{p}_{q+2}(\not D_{S} \not D_{3} \not\phi\,) &= \| \, r^{(\frac{5}{2}-\frac{3}{p}+q)} \; \not\nabla^{q} \not D_{3} \not\phi \; \|_{p,e}, \\
{}^{e}\mathcal{L}^{p}_{q+2}(D_S D_4 \; \varphi_N) &= \| \, r^{(\frac{5}{2}-\frac{3}{p}+q)} \; \not\nabla^{q} D_4 \varphi_N \, \|_{p,e}, \\
{}^{e}\mathcal{L}^{p}_{q+2}(D_S D_3 \; \varphi_N) &= \| \, r^{(\frac{3}{2}-\frac{3}{p}+q)} \tau_{-}^{\frac{3}{2}-\frac{1}{p}} \; \not\nabla^{q} D_3 \varphi_N \, \|_{p,e} \, . \qquad (118)
\end{aligned}
$$

Then, we set

$$
\begin{aligned}
{}^{e}\mathcal{L}^{p}_{0}[\not\phi\,] &= {}^{e}\mathcal{L}^{p}_{0}(\not\phi\,), \\
{}^{e}\mathcal{L}^{p}_{1}[\not\phi\,] &= {}^{e}\mathcal{L}^{p}_{1}(\not\phi\,) + {}^{e}\mathcal{L}^{p}_{1}(\not D_{3} \not\phi\,) + {}^{e}\mathcal{L}^{p}_{1}(\not D_{4} \not\phi\,), \\
{}^{e}\mathcal{L}^{p}_{2}[\not\phi\,] &= {}^{e}\mathcal{L}^{p}_{2}(\not\phi\,) + {}^{e}\mathcal{L}^{p}_{2}(\not D_{3} \not\phi\,) + {}^{e}\mathcal{L}^{p}_{2}(\not D_{4} \not\phi\,), \qquad (119)
\end{aligned}
$$

and correspondingly for φ_N.

Then, we define for $q = 0, 1, 2$,

$$
{}^{e}\mathcal{L}^{p}_{q} = {}^{e}\mathcal{L}^{p}_{q}[\not\phi\,] + {}^{e}\mathcal{L}^{p}_{q}[\varphi_N]. \qquad (120)
$$

These norms we use for $p = 2$ and we write

$$
{}^{e}\mathcal{L}_{q} = {}^{e}\mathcal{L}^{2}_{q}.
$$

Finally, we define

$$
\begin{aligned}
{}^{i}\mathcal{L}_{[0]} &= {}^{i}\mathcal{L}_{0}, \\
{}^{i}\mathcal{L}_{[1]} &= {}^{i}\mathcal{L}_{[0]} + {}^{i}\mathcal{L}_{1}, \\
{}^{i}\mathcal{L}_{[2]} &= {}^{i}\mathcal{L}_{[1]} + {}^{i}\mathcal{L}_{2}, \\
{}^{e}\mathcal{L}_{[0]} &= {}^{e}\mathcal{L}_{0}, \\
{}^{e}\mathcal{L}_{[1]} &= {}^{e}\mathcal{L}_{[0]} + {}^{e}\mathcal{L}_{1}, \\
{}^{e}\mathcal{L}_{[2]} &= {}^{e}\mathcal{L}_{[1]} + {}^{e}\mathcal{L}_{2}, \qquad (121)
\end{aligned}
$$

and

$$\mathcal{L}_{[q]} \;=\; {}^{i}\mathcal{L}_{[q]} \;+\; {}^{e}\mathcal{L}_{[q]}. \tag{122}$$

5.2.4. Norms for the Hessian of the Optical Function u. Here, we state the L^2-norms for the Hessian D^2u of the optical function u. Let ${}^{i}\mathcal{O}_q$ and ${}^{e}\mathcal{O}_q$ denote respectively the interior and exterior norms. We set

$$\mathcal{O}_q \;=\; \max\big(\, {}^{i}\mathcal{O}_q, \;\; {}^{e}\mathcal{O}_q \,\big), \tag{123}$$

for $q = 0, 1, 2$. Correspondingly, we define

$$\mathcal{O}_0^{\infty} \;=\; \max\big(\, {}^{i}\mathcal{O}_0^{\infty}, \;\; {}^{e}\mathcal{O}_0^{\infty} \,\big). \tag{124}$$

The Hessian of u decomposes relative to a null frame into $\chi_{AB}, \zeta_A, \omega$. See also subsection 6.3, before formula (184). Denote by $a = \mid \nabla u \mid^{-1}$ the lapse function of the foliation induced by u on each H_t.

In the interior region, we express the components of D^2u with respect to the standard null frame in terms of $\chi' = a\chi$, $\zeta' = \zeta$, $\omega' = a^{-1}\omega$. In the interior let us define

$$\begin{aligned}
{}^{i}\,\mathcal{O}_q\,(tr\chi' - \overline{tr\chi'}) \;&=\; r_0^q \sum_{i+j+k=q} \|\, \nabla\!\!\!/^{\,i} D_3^j D_4^k\, (tr\chi' - \overline{tr\chi'})\, \|_{2,i}, \\
{}^{i}\,\mathcal{O}_q\,(\hat{\chi}') \;&=\; r_0^q \sum_{i+j+k=q} \|\, \nabla\!\!\!/^{\,i} D\!\!\!\!/\,_3^{\,j} D\!\!\!\!/\,_4^{\,k}\, \hat{\chi}'\, \|_{2,i}, \\
{}^{i}\,\mathcal{O}_q\,(\zeta') \;&=\; r_0^q \sum_{i+j+k=q} \|\, \nabla\!\!\!/^{\,i} D_3^j D_4^k\, \zeta'\, \|_{2,i}, \\
{}^{i}\,\mathcal{O}_q\,(\omega') \;&=\; r_0^q \sum_{i+j+k=q} \|\, \nabla\!\!\!/^{\,i} D_3^j D_4^k\, \omega'\, \|_{2,i} \quad .
\end{aligned} \tag{125}$$

Then, we set

$${}^{i}\,\mathcal{O}_q \;=\; {}^{i}\,\mathcal{O}_q\,(tr\chi' - \overline{tr\chi'}) + {}^{i}\,\mathcal{O}_q\,(\hat{\chi}') + {}^{i}\,\mathcal{O}_q\,(\zeta') + {}^{i}\,\mathcal{O}_q\,(\omega'). \tag{126}$$

In the exterior region, we work with the l-pair and the null frame related to it. The components of the Hessian of u in the exterior region with respect to the l-null frame, that is $tr\chi, \hat{\chi}, \zeta, \omega$, behave differently.

Let us also introduce

$$
\begin{aligned}
tr\chi_4 &= D_4 tr\chi \ + \ \frac{1}{2}\, tr\chi \ tr\chi, \\
tr\chi_3 &= D_3 tr\chi \ + \ \frac{1}{2}\, tr\underline{\chi} \ tr\chi, \\
\hat{\chi}_4 &= \not{D}_4 \, \hat{\chi} \ + \ tr\chi \ \hat{\chi}, \\
\hat{\chi}_3 &= \not{D}_3 \, \hat{\chi} \ + \ tr\underline{\chi} \ \hat{\chi}, \\
\zeta_4 &= \not{D}_4 \, \zeta \ + \ \frac{1}{2}\, tr\chi \ \zeta, \\
\zeta_3 &= \not{D}_3 \, \zeta \ + \ \frac{1}{2}\, tr\underline{\chi} \ \zeta, \\
\omega_4 &= D_4 \omega \\
\omega_3 &= D_3 \omega. \qquad (127)
\end{aligned}
$$

Then, we define

$$
\begin{aligned}
{}^e\mathcal{O}_q \ (tr\chi - \overline{tr\chi}) &= \| \, r^q \ \not{\nabla}^q (tr\chi - \overline{tr\chi}) \, \|_{2,e}, \\
{}^e\mathcal{O}_q \ (\hat{\chi}) &= \| \, r^q \ \not{\nabla}^q \hat{\chi} \, \|_{2,e}, \\
{}^e\mathcal{O}_q \ (\zeta) &= \| \, r^q \ \not{\nabla}^q \zeta \, \|_{2,e}, \\
{}^e\mathcal{O}_q \ (\omega) &= \| \, r^{-\frac{1}{2}+q} \tau_-^{\frac{1}{2}} \ \not{\nabla}^q \omega \, \|_{2,e}, \qquad (128) \\
{}^e\mathcal{O}_q \ (\chi) &= \max \big\{ \ {}^e\mathcal{O}_q \ (tr\chi - \overline{tr\chi}), \ {}^e\mathcal{O}_q \ (\hat{\chi}) \big\}, \\
{}^e\mathcal{O}_{q+1} \ (\hat{\chi}_4) &= \| \, r^{1+q} \ \not{\nabla}^q \hat{\chi}_4 \, \|_{2,e}, \\
{}^e\mathcal{O}_{q+1} \ (\hat{\chi}_3) &= \| \, r^{1+q} \ \not{\nabla}^q \hat{\chi}_3 \, \|_{2,e}, \\
{}^e\mathcal{O}_{q+1} \ (tr\chi_4) &= \| \, r^{1+q} \ \not{\nabla}^q tr\chi_4 \, \|_{2,e}, \\
{}^e\mathcal{O}_{q+1} \ (tr\chi_3) &= \| \, r^q \tau_- \ \not{\nabla}^q tr\chi_3 \, \|_{2,e}, \\
{}^e\mathcal{O}_{q+1} \ (\zeta_4) &= \| \, r^{1+q} \ \not{\nabla}^q \zeta_4 \, \|_{2,e}, \\
{}^e\mathcal{O}_{q+1} \ (\zeta_3) &= \| \, r^{1+q} \ \not{\nabla}^q \zeta_3 \, \|_{2,e}, \\
{}^e\mathcal{O}_{q+1} \ (\omega_4) &= \| \, r^{\frac{1}{2}+q} \tau_-^{\frac{1}{2}} \ \not{\nabla}^q \omega_4 \, \|_{2,e}, \\
{}^e\mathcal{O}_{q+1} \ (\omega_3) &= \| \, r^{-\frac{1}{2}+q} \tau_-^{\frac{3}{2}} \ \not{\nabla}^q \omega_3 \, \|_{2,e} \, . \qquad (129)
\end{aligned}
$$

Next, we set

$$
\begin{aligned}
{}^e\mathcal{O}_0[tr\chi] &= {}^e\mathcal{O}_0(tr\chi), \\
{}^e\mathcal{O}_1[tr\chi] &= {}^e\mathcal{O}_1(tr\chi) \ + \ {}^e\mathcal{O}_1(tr\chi_3) \ + \ {}^e\mathcal{O}_1(tr\chi_4), \\
{}^e\mathcal{O}_2[tr\chi] &= {}^e\mathcal{O}_2(tr\chi) \ + \ {}^e\mathcal{O}_2(tr\chi_3) \ + \ {}^e\mathcal{O}_2(tr\chi_4). \qquad (130)
\end{aligned}
$$

Correspondingly, we proceed for $\hat{\chi}, \zeta, \omega$.
Now, for $q = 0, 1, 2$ one sets

$$
{}^e\mathcal{O}_q = {}^e\mathcal{O}_q^2 = {}^e\mathcal{O}_q[tr\chi] + {}^e\mathcal{O}_q[\hat{\chi}] + {}^e\mathcal{O}_q[\zeta] + {}^e\mathcal{O}_q[\omega]. \qquad (131)
$$

Then we set

$$
\begin{aligned}
{}^{i}\,\mathcal{O}^{\infty}_{[0]} \quad &= \quad {}^{i}\,\mathcal{O}^{\infty}_{0} \;+\; r_0^{\frac{1}{2}}\,\sup_{I} \mid \overline{tr\chi} - \frac{2}{r} \mid \;+\; \sup_{I} \mid a \;-\; 1 \mid, \\
{}^{e}\,\mathcal{O}^{\infty}_{[0]} \quad &= \quad {}^{e}\,\mathcal{O}^{\infty}_{0} \;+\; \sup_{r \geq \frac{r_0}{2}} r^{\frac{1}{2}} \mid \overline{tr\chi} - \frac{2}{r} \mid \;+\; \sup_{r \geq \frac{r_0}{2}} \mid a \;-\; 1 \mid . \qquad (132)
\end{aligned}
$$

Finally, we define

$$
\begin{aligned}
{}^{i}\,\mathcal{O}_{[0]} \quad &= \quad {}^{i}\,\mathcal{O}_{0} \;+\; {}^{i}\,\mathcal{O}^{\infty}_{[0]}, \\
{}^{i}\,\mathcal{O}_{[1]} \quad &= \quad {}^{i}\,\mathcal{O}_{[0]} \;+\; {}^{i}\,\mathcal{O}_{1}, \\
{}^{i}\,\mathcal{O}_{[2]} \quad &= \quad {}^{i}\,\mathcal{O}_{[1]} \;+\; {}^{i}\,\mathcal{O}_{2}, \\
{}^{e}\,\mathcal{O}_{[0]} \quad &= \quad {}^{e}\,\mathcal{O}_{0} \;+\; {}^{e}\,\mathcal{O}^{\infty}_{[0]}, \\
{}^{e}\,\mathcal{O}_{[1]} \quad &= \quad {}^{e}\,\mathcal{O}_{[0]} \;+\; {}^{e}\,\mathcal{O}_{1}, \\
{}^{e}\,\mathcal{O}_{[2]} \quad &= \quad {}^{e}\,\mathcal{O}_{[1]} \;+\; {}^{e}\,\mathcal{O}_{2}, \qquad (133)
\end{aligned}
$$

and

$$
\mathcal{O}_{[q]} \quad = \quad {}^{i}\,\mathcal{O}_{[q]} \;+\; {}^{e}\,\mathcal{O}_{[q]}. \qquad (134)
$$

5.3. Bootstrap Argument. The method of bootstrapping constitutes the core of the proof of the main theorem. Several steps are needed to close the bootstrap loop.

We define $\mathcal{S}$ to be the set of all $t \geq 0$ such that there exists a spacetime slab $\bigcup_{t' \in [0,t]} H_{t'}$ endowed with a canonical optical function with respect to which the following *bootstrap assumptions* hold:

- **BA0:** For all $t' \in [0, t]$, assume that

$$
\frac{1}{2}\,(1 \;+\; t') \;\leq\; r_0(t') \;\leq\; \frac{3}{2}\,(1 \;+\; t'). \qquad (135)
$$

- **BA1:** For all $t' \in [0, t]$, assume that

$$
{}^{e}\mathcal{O}^{\infty}_{0},\; \mathcal{K}^{\infty}_{0},\; \mathcal{L}^{\infty}_{0} \quad \leq \quad \epsilon_0. \qquad (136)
$$

- **BA2:** For all $t' \in [0, t]$, assume that

$$
\mathcal{R}_{[1]},\;\; {}^{e}\mathcal{O}_{[2]},\; \mathcal{K}_{[2]},\; \mathcal{L}_{[2]} \quad \leq \quad \epsilon_0. \qquad (137)
$$

Additional Assumption on the last slice H_t:

$$
\sup_{S_u} \left\{ r^{\frac{3}{2}} \mid \nabla\!\!\!/ \log a \right\} \quad \leq \quad \epsilon_0. \qquad (138)
$$

First, we show that the set $\mathcal{S}$ is not empty. Of course, the bootstrap assumptions hold at $t = 0$. Afterwards, we define the supremum of $\mathcal{S}$ as t_*. If $t_* = \infty$, then the global existence is proved. We use a continuity argument to derive that indeed $t_* = \infty$. This goes by contradiction, assuming that $t_* < \infty$. Then it is $t_* \in \mathcal{S}$, and the inequalities must be saturated at t_*. It will be shown that this cannot happen. This yields that $t_* = \infty$.

Step 1: We show that the set $\mathcal{S}$ is not empty. That is, it contains at least $t = 0$. In view of the local existence theorem we can construct the past slab $\bigcup_{t' \in [-1,0]} H_{t'}$ and the initial cone C_0 with vertex at a point on H_{-1}. Then an exterior optical function u on $t = 0$ is constructed by solving the inverse lapse problem, starting on the 2-surface $S_{0,0} = C_0 \cap H_0$, according to the procedure explained in Step 4 for "The last slice." We proceed analogously to Step 4 for a background radial function u' on H_0. One then shows that the norms $\mathcal{R}_{[1]}, \mathcal{K}_{[2]}, \mathcal{L}_{[2]}, {}^e\mathcal{O}_{[2]}$, as well as ${}^e\mathcal{O}_0^\infty, \mathcal{K}_0^\infty, \mathcal{L}_0^\infty$ can be made arbitrarily small, that is, the assumptions **BA1** and **BA2** are fulfilled. The additional assumption (138) follows from the main result in Step 4.

Step 2: Let $t_* = \sup \mathcal{S}$. If $t_* = \infty$, then the global existence is proved. Now, we assume $t_* < \infty$. Then, it is $t_* \in \mathcal{S}$. The hypersurface H_{t_*} is called the "last slice" of the t-foliation of the bootstrap argument. One defines an interior optical function u in the interior region of the spacetime slab $\bigcup_{t \in [0,t_*]} H_t$, for which one derives the following inequalities:

$$\begin{aligned} {}^i\mathcal{O}_{[2]} &\leq c\,\big(\mathcal{R}_{[1]} + \mathcal{K}_{[2]} + \mathcal{L}_{[2]}\big), \\ {}^i\mathcal{O}_0^\infty &\leq c\,\big(\mathcal{R}_{[1]} + \mathcal{K}_0^\infty + \mathcal{L}_0^\infty\big). \end{aligned} \tag{139}$$

Then one obtains

$$ {}^i\mathcal{O}_{[2]},\ {}^i\mathcal{O}_0^\infty \quad \leq \quad c\,\epsilon_0. \tag{140}$$

The global function u is obtained by the matching of two optical functions, defined in the exterior and in the interior region respectively. The exterior optical function is crucial, as it describes the structure of null infinity. And one constructs it by solving the Eikonal equation with initial conditions on the last slice. For the interior optical function one prescribes initial conditions on a "central line" given by an integral curve of the vectorfield T. The quantities related to the foliation given in the exterior region are more subtle to estimate than the ones in the interior. Thus, proving the exterior estimates takes the largest part of the work. Further information is given in [**6**], [**7**], and [**21**].

Step 3: This is the core part of the proof. It splits into three sections, namely, obtaining estimates for

a) $\mathcal{R}_{[1]}$,
b) $\mathcal{K}_{[2]}, \mathcal{L}_{[2]}$, respectively, $\mathcal{K}_0^\infty, \mathcal{L}_0^\infty$,
c) ${}^e\mathcal{O}_{[2]}$, respectively, ${}^e\mathcal{O}_0^\infty$.

Using step 2 and the bootstrap assumptions **BA0**, **BA1**, **BA2**, we show that the size of the norms ${}^e\mathcal{O}_0^\infty, \mathcal{K}_0^\infty, \mathcal{L}_0^\infty$ and $\mathcal{R}_{[1]}, \mathcal{K}_{[2]}, \mathcal{L}_{[2]}, {}^e\mathcal{O}_{[2]}$ cannot exceed a constant multiple of the respective size of the data at $t = 0$. Therefore, given any $\epsilon_0 > 0$ we can choose $\epsilon > 0$ sufficiently small such that

$$
\begin{aligned}
{}^{e}\mathcal{O}_0^{\infty},\ \mathcal{K}_0^{\infty},\ \mathcal{L}_0^{\infty} \quad &\leq \quad \frac{1}{2}\,\epsilon_0, \\
{}^{e}\mathcal{O}_{[2]},\ \mathcal{R}_{[1]},\ \mathcal{K}_{[2]},\ \mathcal{L}_{[2]} \quad &\leq \quad \frac{1}{2}\,\epsilon_0.
\end{aligned} \tag{141}
$$

This is achieved as follows:

a) In this part, we use the bootstrap assumptions **BA0**, **BA1**, **BA2** and inequality (140) to check all the assumptions of the comparison theorem, which in item 2 of the main steps of the proof of our main result laid out at the beginning of section 5 allows us to estimate the curvature by $Q_1(W)$, as well as of the main theorem concerning the error estimates, where we estimate the three terms of $Q_1(W)$, as given in (54)–(56).

According to the main theorem for the error estimates, which is from item 1 of the main steps of the proof of our main result laid out at the beginning of section 5, we have

$$
Q_1 * \ + Q_0 * \ \leq \ c\ (Q_1(0)\ +\ Q_0(0)). \tag{142}
$$

Thus by the comparison theorem of item 2 at the beginning of section 5, we conclude that, for all $t \in [0, t_*]$,

$$
\mathcal{R}_{[1]}(t)\ \leq\ c\,\mathcal{R}_{[1]}(0). \tag{143}
$$

We recall that in our work we do not need rotational vectorfields at all to derive the estimates for $\mathcal{R}_{[1]}(t)$. Contrary to [**21**], where rotational vectorfields were used in a crucial way, we only work with the vectorfields T, S and $\bar{K}$ in conjunction with the Bianchi equations to deduce the required estimates for $\mathcal{R}_{[1]}(t)$. That is, we define the quantities Q_0 and Q_1 with help of the vectorfields T, S, and $\bar{K}$ as given also above. Then we estimate $\mathcal{R}_{[1]}(t)$ in terms of Q_0 and Q_1 by the comparison argument.

From (143) and from the fact that we can bound $\mathcal{R}_{[1]}(0)$ by $c \cdot \epsilon$, we decuce

$$
\mathcal{R}_{[1]}(t)\ \leq\ c\,\epsilon. \tag{144}
$$

Choosing ϵ sufficiently small yields

$$
\mathcal{R}_{[1]}(t)\ \leq\ \frac{1}{2}\,\epsilon_0. \tag{145}
$$

b) We show that the bootstrap assumptions **BA0**, **BA1**, **BA2** and the inequality (140) imply the following:

$$
\mathcal{K}_{[2]},\ \mathcal{L}_{[2]}\ \leq\ c\,\mathcal{R}_{[1]}.
$$

Then, by the Sobolev inequalities, we deduce that

$$
\mathcal{K}_0^{\infty},\ \mathcal{L}_0^{\infty}\ \leq\ c\,\mathcal{R}_{[1]},
$$

and therefore, choosing ϵ sufficiently small, we conclude

$$\mathcal{K}_0^\infty,\ \mathcal{L}_0^\infty \leq \frac{1}{2}\,\epsilon_0, \tag{146}$$

$$\mathcal{K}_{[2]},\ \mathcal{L}_{[2]} \leq \frac{1}{2}\,\epsilon_0. \tag{147}$$

c) Here, we show that the bootstrap assumptions **BA0**, **BA1**, **BA2** imply the following:

$$\begin{aligned} {}^e\mathcal{O}_0^\infty &\leq c\,\epsilon, \\ {}^e\mathcal{O}_{[2]} &\leq c\,\epsilon. \end{aligned}$$

Therefore, if ϵ is sufficiently small, this yields

$${}^e\mathcal{O}_0^\infty \leq \frac{1}{2}\,\epsilon_0, \tag{148}$$

$${}^e\mathcal{O}_{[2]} \leq \frac{1}{2}\,\epsilon_0. \tag{149}$$

Step 4: This step is to be considered together with the previous one. However, as it is a crucial point within the whole procedure of the bootstrap argument, we formulate it separately.

We show that we can extend our spacetime beyond the time t_*. In particular, we use the result of the previous step together with the local existence theorem, with initial data at t_*, to extend the spacetime up from t_* to $t_* + \delta$. Also, the optical function u' of the spacetime slab $\bigcup_{t\in[0,t_*]} H_t$ is extended by continuing the null geodesic generators of the hypersurfaces $C_{u'}$ into the future up to $t_* + \delta$. We choose δ to be sufficiently small, such that the size of the norms $\mathcal{R}'_{[1]}, \mathcal{K}'_{[2]}, \mathcal{L}'_{[2]},\ {}^e\mathcal{O}'_{[2]}$ and $\mathcal{K}'^\infty_0, \mathcal{L}'^\infty_0,\ {}^e\mathcal{O}'^\infty_0$ remains strictly smaller than ϵ_0. Moreover, $\sup_{S'_{u'}} \{ r'^{\frac{3}{2}} \mid \nabla\!\!\!/\,' \log a' \}$ in (138) is strictly smaller than ϵ_0.

Now, we start with $H_{t_*+\delta}$ as last slice. The cut $S_{t_*+\delta,\ 0} = H_{t_*+\delta} \cap C_0$ of this last slice with C_0 is the initial 2-surface, from which we start, to solve the appropriate equation of motion of surfaces and construct a new optical function u on $H_{t_*+\delta}$. We recall that the function u' gives the background foliation here. As the level surfaces of u in $H_{t_*+\delta}$ do not exist yet, but have to be constructed, we use the bootstrap assumptions on the quantities of the background foliation and the comparison between the two foliations induced by u' and u to control the curvature and geometric quantities of the foliation by u. Thus, using a bootstrap argument, we construct the new optical function u on $H_{t_*+\delta}$ by solving an equation of motion of surfaces on $H_{t_*+\delta}$ starting from $S_{t_*+\delta,\ 0}$. This is then extended to the past.

In view of the continuity properties of the equations, we deduce that the new norms $\mathcal{R}_{[1]}$, $\mathcal{K}_{[2]}$, $\mathcal{L}_{[2]}$, ${}^e\mathcal{O}_{[2]}$ and $\mathcal{K}_0^\infty$, $\mathcal{L}_0^\infty$, ${}^e\mathcal{O}_0^\infty$ can be made arbitrarily close to the previous ones by choosing δ suitably small. One therefore checks that the bootstrap assumptions BA1 and BA2 as well as BA0 and inequality (138) still hold. Thus, we obtain that $t_* + \delta \in \mathcal{S}$, which contradicts the assumption that $t_* < \infty$.

Step 5: To complete the proof of the main theorem 3, one shows that the optical function ${}^{(t)}u$ defined on the slab $\bigcup_{t' \in [0,t]} H_{t'}$, starting from the last slice H_t in the exterior approaches a global exterior optical function u as $t \to \infty$.

6. Discussion and Outline of the Proof

6.1. General Ideas and Concepts, Curvature, and Energy. There are several important properties of our spacetime which play a crucial role in our proof. We are going to discuss them now and also give an outline of the proof of our main theorem (theorem 1, respectively, of the full version of our main theorem 3).

In order to obtain the precise estimates, it is necessary to work with appropriate foliations of the spacetime (M, g). These are the foliation into hypersurfaces H_t given by the time function t and the foliation into u-null-hypersurfaces C_u by the optical function u, as they were introduced earlier. The slices H_t of our spacetime are three-dimensional, complete, Riemannian manifolds, diffeomorphic to $\mathbb{R}^3$ and Euclidean at infinity. Each slice carries a structure induced by the level hypersurfaces of the optical function u. Thus, the (t, u) foliations of the spacetime (M, g) define a codimension-2-foliation by 2-surfaces $S_{t,u} = H_t \cap C_u$.

The *asymptotic behavior* of the *curvature tensor* R and the *Hessian* of t and u can only be *fully described* by decomposing them into *components tangent and normal to* $S_{t,u}$.

One achieves this by introducing *null pairs* consisting of two future-directed null vectors e_4 and e_3 orthogonal to $S_{t,u}$ with e_4 tangent to C_u and

$$\langle e_4, e_3 \rangle = -2. \tag{150}$$

Note that the null pair (e_3, e_4) is only determined up to a transformation of the form

$$(e_3, e_4) \mapsto (a^{-1} e_3, a e_4), \quad a > 0.$$

It is uniquely determined if we also impose the condition that $e_4 - e_3$ is tangential to the H_t.

The null pair is completed with an orthonormal frame e_1, e_2 on $S_{t,u}$ to form a *null frame.* The *null decomposition* of a tensor relative to a null frame e_4, e_3, e_2, e_1 is obtained by taking *contractions* with the vectorfields e_4, e_3.

Also define

$$\tau_-^2 := 1 + u^2.$$

With respect to this frame, we obtain the following *null decomposition of the Riemann curvature tensor of an EV spacetime,* where the capital indices take the values $1, 2$:

$$R_{A3B3} = \underline{\alpha}_{AB}, \tag{151}$$

$$R_{A334} = 2\ \underline{\beta}_A, \tag{152}$$

$$R_{3434} = 4\ \rho, \tag{153}$$

$$^*R_{3434} = 4\ \sigma, \tag{154}$$

$$R_{A434} = 2\ \beta_A, \tag{155}$$

$$R_{A4B4} = \alpha_{AB}, \tag{156}$$

with
α, $\underline{\alpha}$: S-tangent, symmetric, traceless tensors,
β, $\underline{\beta}$: S-tangent 1-forms,
ρ, σ : scalars .

We show, as a part of our main result, that these components are controlled in the sense of the main theorem. And our estimates yield the decay behavior:

$$\begin{aligned} \underline{\alpha} &= O\ (r^{-1}\ \tau_-^{-\frac{3}{2}}), \\ \underline{\beta} &= O\ (r^{-2}\ \tau_-^{-\frac{1}{2}}), \\ \rho,\ \sigma,\ \alpha,\ \beta &= o\ (r^{-\frac{5}{2}}) \end{aligned}$$

At this point, let us recall that in [**21**] the null components have the following decay properties:

$$\begin{aligned} \underline{\alpha} &= O\ (r^{-1}\ \tau_-^{-\frac{5}{2}}), \\ \underline{\beta} &= O\ (r^{-2}\ \tau_-^{-\frac{3}{2}}), \\ \rho &= O\ (r^{-3}), \\ \sigma &= O\ (r^{-3}\ \tau_-^{-\frac{1}{2}}), \\ \alpha,\ \beta &= o\ (r^{-\frac{7}{2}}). \end{aligned}$$

The fact that in [**6**, **7**] only one derivative of the curvature (Ricci) in H is controlled means that the curvature is not pointwise bounded. What one has is only the following, where *Ric* includes corresponding weights according to (16):

$$Ric\ \in\ W^{1,2}(H).$$

For the Gauss curvature K in the leaves of the u-foliation S the trace lemma gives,

$$K\ \in\ L^4(S).$$

Whereas in [**21**], the authors control two derivatives of the curvature in $L^2(H)$, giving $L^\infty(H)$ bounds: *Ric* including weights as in [**21**]:

$$Ric \in L^\infty(H).$$

This yields for the Gauss curvature K in the surfaces S that $K \in L^\infty(S)$.

In our case, we also control two derivatives of the second fundamental form k. Therefore, by Sobolev inequalities, it is

$$k \in L^\infty(H).$$

Then, also in the surfaces S, the second fundamental form k lies in $L^\infty(S)$.

Working with this approach, there are two main difficulties to be discussed. They have been stated and solved by Christodoulou and Klainerman in [**21**]. As these concepts are also crucial in our work, let us now say what they are. However, employing these concepts in our setting requires fundamentally new ideas in our proofs. It shall be explained below. Now, the said difficulties are:

1) "Energy estimates"

2) A general spacetime has no symmetries. Thus the conformal isometry group is trivial. Hence, the vectorfields needed to construct conserved quantities do not exist.

1) As the goal now is to find estimates for the spacetime curvature to give control on regularity, one way to attack the problem could be to focus on the definition of the energy-momentum tensor appropriate to a geometric Lagrangian, namely, considering the variation of the action $\mathcal{A}$ with respect to the underlying metric. Generally, for a domain D with compact closure in M and Lagrangian L the action $\mathcal{A}$ is defined as

$$\mathcal{A}[D] = \int_D L \, d\mu_g. \tag{157}$$

Variations supported in D of the action, with respect to the underlying metric, yield the energy-momentum tensor as follows:

$$\dot{\mathcal{A}}[D] = -\frac{1}{2} \int_D T^{\mu\nu} \, \dot{g}_{\mu\nu} \, d\mu_g. \tag{158}$$

But this approach would not work here (or in [**21**]), because the variation (158) vanishes for the gravitational Lagrangian $L = -\frac{1}{4} R d\mu_g$, that is for the Einstein-Hilbert action:

$$\mathcal{A}[D] = -\frac{1}{4} \int_D R \, d\mu_g. \tag{159}$$

This vanishing is stated in the Euler-Lagrange equations for gravitation, which is in the EV equations.

An alternative, one could think, could be Noether's theorem after subtracting an appropriate divergence relative to a background metric.

But the energy could give control on the solutions only after the isoperimetric constant is controlled. Therefore, the energy alone could not help to prove regularity. For a discussion of this problem, see also [**17**].

However, the right way to resolve the first difficulty is the following: Consider the Bianchi identities

$$D_{[\alpha}R_{\beta\gamma]\delta\epsilon} \ = \ 0 \tag{160}$$

as differential equations for the curvature and the Einstein equations $R_{\mu\nu} = 0$ as algebraic conditions on the curvature. At this point, breaking the connection between the metric and the curvature, one introduces the *Weyl tensorfield* W of a given spacetime (M, g), which has all the symmetry properties of the curvature tensor and in addition is traceless

$$g^{\alpha\beta} \ W_{\alpha\mu\beta\nu} \ = \ 0, \tag{161}$$

which is the analogue of the Einstein equations, and satisfies the Bianchi equations,

$$D_{[\epsilon}W_{\alpha\beta]\gamma\delta} \ = \ 0. \tag{162}$$

We remark that the Bianchi equations are linear.

Note that the Riemann curvature tensor has 20 independent components, whereas the conformal curvature and Ricci tensors have 10 components each.

The Bel-Robinson tensor Q is defined out of the Weyl tensor W and given above in (47). The main properties of Q are stated after its definition (47). This quantity Q can then be thought of as the "energy-momentum tensor" in our setting.

The Bel-Robinson tensor Q, in fact, plays the same role for solutions of the Bianchi equations as the energy-momentum tensor of an electromagnetic field plays for the solutions of the Maxwell equations.

Assume to be given three vectorfields X, Y, Z, each of which generating a 1-parameter group of conformal isometries of the spacetime (M, g). Then the 1-form

$$P \ = \ - \ Q(\cdot, X, Y, Z)$$

is divergence-free. It follows thus that the integral on a Cauchy hypersurface H

$$\int_H {}^*P$$

is conserved (and is positive definite, if all of the vectorfields X, Y, Z are timelike future-directed), recalling that ${}^*P$ is the dual 3-form ${}^*P_{\mu\alpha\beta} = P^{\nu}\epsilon_{\nu\mu\alpha\beta}$.

To investigate the Einstein equations, being hyperbolic and nonlinear, we use energy estimates of the type in [**21**]. Aiming at global results, the classical energy estimates do not suffice, their usefulness being limited to obtaining results which are local in time. Instead, we introduce energies $Q_0(t)$ and $Q_1(t)$ (see definition above), being integrals over H_t involving the Bel-Robinson tensor Q of the spacetime curvature W and of the Lie derivatives of W, which serve to estimate the curvature components by a comparison argument. This is one of the core parts of our work, and it is different from the work of D. Christodoulou and S. Klainerman in a fundamental way, which will be explained below.

The quantities $Q_0(t)$ and $Q_1(t)$ themselves are estimated by a continuity argument to be bounded by a multiple of the initial value $Q_1(0)$. More precisely, we show the error terms, which are generated while estimating the growth of Q_0 and Q_1, to be controlled. In this procedure it is important to assess the structure of these nonlinear terms. It turns out that the most troublesome terms cancel by identities that are consequences of the covariance and algebraic properties of the Einstein equations.

As a major result emerges the fact that the estimates for the most delicate of these error terms are borderline. This means that any further relaxation of the assumptions would lead to divergence and the argument would not close anymore. This needs further explanation. Contrary to many problems in analysis, where the principal terms, that is, the terms containing the highest derivatives, are the most sensitive ones to estimate whereas the non-principal terms (containing fewer or no derivatives) are usually easier to handle, here the most difficult terms to be estimated are of higher order with respect to asymptotic behavior (that is they have less decay), but they are non-principal from the point of view of differentiability. On the other hand the expressions, which are principal with respect to derivatives behave better asymptotically, and therefore can be controlled easier. Thus, by "borderline" we always mean borderline from the point of view of decay (asymptotic behavior). It is an essential difference between the situation investigated by D. Christodoulou and S. Klainerman in [**21**] and ours that their worst terms still being of lower order in asymptotic behavior than ours, the borderline case does not appear, whereas in our setting the estimates for the highest order terms in view of asymptotic behavior are really borderline.

2) The second difficulty is that a general spacetime has no symmetries, that is, the conformal isometry group is trivial. Then one could not construct integral conserved quantities by using vectorfields in conjunction with energy-momentum tensors.

The solution is as follows: As a spacetime arising from arbitrary asymptotically flat initial data is itself supposed to be asymptotically flat at spacelike infinity in general, and also, with corresponding smallness assumptions of the initial data, as the time tends to infinity, one could expect the spacetime to approach Minkowski spacetime. Now, Minkowski spacetime has a large conformal isometry group. The idea is to use part of it in the following way. One defines in the limit an action of a subgroup. Next, one extends this action backwards in time up to the initial hypersurface in a manner as to obtain an action of the said subgroup globally. This has to be done in a way such that the deviation from conformal isometry is globally small and goes to zero at infinity sufficiently rapidly. It is described by the circumstance that the trace-free part ${}^{(X)}\hat{\pi}$ of the deformation tensor ${}^{(X)}\pi := \mathcal{L}_X g$ of the generating vectorfield X is globally small and approaches zero sufficiently fast at infinity. In order to derive a complete system of estimates, we define the action of the subgroup of the conformal group of Minkowski spacetime corresponding to the time translations, the scaling, and the inverted time translations. We recall that, contrary to the work of S. Christodoulou and S. Klainerman [**21**], our proof does not involve any rotational vectorfields.

The action of the group of time translations is the easiest to define. Having chosen a canonical maximal time function t, the corresponding time translation vectorfield T generates the action, taking the maximal hypersurfaces into each other, as described above. This and the optical function u, from which the action of the other groups are defined, are introduced above, where we discuss the (t, u)-foliation of the spacetime.

Let us define the vectorfields S for the scaling and K for the inverted time translation. First, we introduce the function $\underline{u}$ to be

$$\underline{u} \; = \; u \; + \; 2 \, r. \tag{163}$$

We define also

$$\tau_+^2 \; := \; 1 \; + \; \underline{u}^2.$$

The time translation vectorfield T has already been defined. We only remark here that it can be written as in the subsequent formula. Let L and $\underline{L}$ be respectively the outgoing and incoming null normals to the surface $S_{t,u}$ given by (40), for which the component along T is equal to T. Also, the integral curves of L are the null geodesic generators of the null hypersurfaces C_u parametrized by t.

Then T is expressed as

$$T \; = \; \frac{1}{2} \, \big(L \; + \; \underline{L} \big). \tag{164}$$

The generator S of scalings is defined to be

$$S \; = \; \frac{1}{2} \, \big(\underline{u} \, L \; + \; u \, \underline{L} \big). \tag{165}$$

And the generator K of inverted time translations is defined as

$$K \; = \; \frac{1}{2} \, \big(\underline{u}^2 \, L \; + \; u^2 \, \underline{L} \big). \tag{166}$$

Then the vectorfield $\bar{K} = K + T$ reads as

$$\bar{K} \; = \; \frac{1}{2} \, \big(\tau_+^2 \, L \; + \; \tau_-^2 \, \underline{L} \big). \tag{167}$$

These vectorfields are used to construct quantities whose growth can be controlled in terms of the quantities themselves. This procedure is in the spirit Noether's theorem. We observe that it is crucial to work with a characteristic foliation of the spacetime in order to obtain the required quasi-conformal isometries.

In the subsequent paragraphs, we are going to outline how the energies Q_0 and Q_1 are constructed and estimated.

Before we do this, however, let us say a few words about the following. As pointed out in the main step 3 above, we have to estimate the geometric quantities by curvature assumptions and using mainly elliptic estimates on the surfaces $S_{t,u}$ and evolution equations in H_t and C_u. The elliptic tools are used in many situations. In particular, they are crucial in deriving inequalities for the components of the second fundamental form. Note that in the estimates we have to make a difference between angular and normal components and derivatives relative to the radial foliation. This is a consequence of the decay behavior of the spacetime curvature (in the wave zone). Following the notation of [**21**], we call such estimates degenerate, and the usual type non-degenerate. At several points, we work with L^p estimates on the surfaces S. In order to obtain them, we need the uniformization theorem so that we can use the Calderon-Zygmund theory for the corresponding Hodge systems on the standard sphere. We prove the uniformization theorem in [**7**] and in [**6**] in chapter 9, theorem 13 for our setting, where the Gauss curvature K is in $L^4(S)$.

As a central part of the proof, we state and prove the comparison theorem from the main step 2 above. It estimates the components of the Weyl curvature by the quantity $\mathcal{Q}_1(W)$ introduced above. This quantity $\mathcal{Q}_1(W)$ is shown to be bounded. For the vectorfields T, S, $\bar{K}$, the corresponding deformation tensors are calculated. Here, the Bianchi equations play a crucial role. In fact, the Bianchi equations allow us to obtain the estimates of the angular derivatives of our curvature components directly. We re-emphasize that no rotational vectorfields are needed in the present proof. In [**21**], the authors introduced rotational vectorfields to obtain the corresponding angular derivatives. While this is different in our work, another fact is used similarly, that is, the principle of conservation of signature.

In the error estimates we show the quantity $Q_1(W)$, that is, $Q_0(t)$ and $Q_1(t)$, to be bounded. In view of estimating $Q_1(W)$ from (56), we continue as follows. $Q_0(t)$ is controlled directly, once $Q_1(t)$ is estimated. The integral $Q_1(t)$ for t_* can be split into

$$\begin{aligned} Q_1(t_*) \quad = \quad & \int_{H_0} Q\ (\hat{\mathcal{L}}_S W)\ (\bar{K}, T, T, T) \\ & \quad + \int_{H_0} Q\ (\hat{\mathcal{L}}_T W)\ (\bar{K}, \bar{K}, T, T) + \mathcal{E}_1(W, t_*), \end{aligned}$$

where $\mathcal{E}_1(W, t_*)$ is the integral on V_{t_*} of the absolute values of the error terms. V_{t_*} denotes the spacetime slab $\bigcup_{t \in [0,t_*]} H_t$, for which the bootstrap assumptions hold. That is, the corresponding quantities are bounded by a small positive constant ϵ_0. These error terms are generated because of the fact that the integral (56) on H_t differs from an integral over H_0, as the vectorfields T, S, K are not exact conformal Killing fields (only quasi-conformal). The expressions in $\mathcal{E}_1(W, t_*)$ to be integrated are quadratic in the Weyl fields W and linear in the deformation tensors $\hat{\pi}$ of the vectorfields.

Generally, for the integral on H_t, we have the following formula, (see [**7**] and [**6**] chapter 5, proposition 13), for three arbitrary vectorfields X, Y, Z and T denoting the unit normal to the foliation by a time function t:

$$\begin{aligned}
&\int_{H_t} Q(W)(X,Y,Z,T)\ d\mu_g \\
&\quad = \int_{H_0} Q(W)(X,Y,Z,T)\ d\mu_g \\
&\qquad + \int_0^t \Big(\int_{H_{t'}} (divQ)_{\beta\gamma\delta}\ X^\beta\ Y^\gamma\ Z^\delta \\
&\qquad\qquad + \frac{1}{2} \int_{H_{t'}} Q_{\alpha\beta\gamma\delta}\ \big(\ {}^{(X)}\pi^{\alpha\beta}\ Y^\gamma\ Z^\delta\ +\ {}^{(Y)}\pi^{\alpha\beta}\ Z^\gamma\ X^\delta \\
&\qquad\qquad\qquad + {}^{(Z)}\pi^{\alpha\beta}\ X^\gamma\ Y^\delta \big)\ \Phi\ d\mu_g \Big)\ dt'.
\end{aligned}$$

We also need the following integral on the cones C_u:

$$\begin{aligned}
\tilde{Q}_1(W,u,t) \quad &= \quad \int_{C_u(t_0,t)} Q(\hat{\mathcal{L}}_S W)(\bar{K}, T, T, e_4) \\
&\qquad + \int_{C_u(t_0,t)} Q(\hat{\mathcal{L}}_T W)(\bar{K}, \bar{K}, T, e_4)
\end{aligned} \tag{168}$$

with $C_u(t_0, t)$ denoting the part of the cone C_u between H_{t_0} and H_t for $0 \leq t_0 \leq t$.

And what we have just said for (56) also holds for (168). In fact, we estimate

$$Q_1^* \ = \ \max\ \Big(\sup_{t\in[0,t_*]} Q_1(W,t), \quad \sup_{t\in[0,t_*]} \sup_{u_* \geq u_0(t)} \tilde{Q}_1(W,u,t) \Big). \tag{169}$$

Thus, we have the inequality

$$Q_1^* \ \leq\ Q_1(0)\ +\ \mathcal{E}_1(t_*).$$

Therefore, we have to estimate the error terms $\mathcal{E}_1(t_*)$. We prove by a bootstrap argument (see [**7**], [**6**], chapter 6, theorem 7) that

$$\mathcal{E}_1(t_*)\ \leq\ C\epsilon_0 Q_1^*, \tag{170}$$

which for ϵ_0 sufficiently small implies

$$Q_1^* \leq 2\, Q_1(0). \tag{171}$$

In deriving (170) we encounter borderline estimates for the most delicate terms.

6.2. Borderline Estimates. We now give an example of a borderline estimate. To do so, let us first give the formula for the error terms.

Let V_t denote the spacetime slab $\bigcup_{t'\in[0,t]} H_{t'}$.

The first term in $Q_1(t)$ from formula (56) is

$$\begin{aligned}
\int_{H_t} Q\ (\hat{\mathcal{L}}_S W)\ (\bar{K}, T, T, T)\ &=\ \int_{H_0} Q\ (\hat{\mathcal{L}}_S W)\ (\bar{K}, T, T, T) \\
&+ \int_{V_t} \Phi\ (div\ Q(\hat{\mathcal{L}}_S W))_{\beta\gamma\delta}\ \bar{K}^\beta\ T^\gamma\ T^\delta \\
&+ \frac{1}{2} \int_{V_t} \Phi\ Q(\hat{\mathcal{L}}_S W)_{\alpha\beta\gamma\delta}\ \ {}^{(\bar{K})}\pi^{\alpha\beta} T^\gamma T^\delta \\
&+ \int_{V_t} \Phi\ Q(\hat{\mathcal{L}}_S W)_{\alpha\beta\gamma\delta}\ {}^{(T)}\pi^{\alpha\beta} \bar{K}^\gamma T^\delta.
\end{aligned}$$

Similarly, we obtain for the remaining terms

$$\begin{aligned}
Q_1(W,t) &\leq Q_1(W,0)\ +\ \mathcal{E}_1(W,t), \\
Q_0(W,t) &\leq Q_0(W,0)\ +\ \mathcal{E}_0(W,t)
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{E}_1(W,t)\ &=\ \int_{V_t} \Phi\ \mid (div\ Q(\hat{\mathcal{L}}_S W))_{\beta\gamma\delta}\ \bar{K}^\beta\ T^\gamma\ T^\delta \mid \\
&+ \int_{V_t} \Phi\ \mid (div\ Q(\hat{\mathcal{L}}_T W))_{\beta\gamma\delta}\ \bar{K}^\beta\ \bar{K}^\gamma\ T^\delta \mid \\
&+ \frac{1}{2} \int_{V_t} \Phi\ \mid Q(\hat{\mathcal{L}}_S W)_{\alpha\beta\gamma\delta}\ \ {}^{(\bar{K})}\pi^{\alpha\beta} T^\gamma T^\delta \mid \\
&+ \int_{V_t} \Phi\ \mid Q(\hat{\mathcal{L}}_S W)_{\alpha\beta\gamma\delta}\ {}^{(T)}\pi^{\alpha\beta} \bar{K}^\gamma T^\delta \mid \\
&+ \int_{V_t} \Phi\ \mid Q(\hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}\ \ {}^{(\bar{K})}\pi^{\alpha\beta} \bar{K}^\gamma T^\delta \mid \\
&+ \frac{1}{2} \int_{V_t} \Phi\ \mid Q(\hat{\mathcal{L}}_T W)_{\alpha\beta\gamma\delta}\ {}^{(T)}\pi^{\alpha\beta} \bar{K}^\gamma \bar{K}^\delta \mid,
\end{aligned}$$

$$\begin{aligned}\mathcal{E}_0(W,t) \ = \ & \int_{V_t} \Phi \ \mid \underbrace{(div \ Q(W))_{\beta\gamma\delta}}_{=0} \ \bar{K}^{\beta} \ T^{\gamma} \ T^{\delta} \mid \\ + \ & \frac{1}{2} \int_{V_t} \Phi \ \mid Q(W)_{\alpha\beta\gamma\delta} \ \ ^{(\bar{K})}\pi^{\alpha\beta} T^{\gamma} T^{\delta} \mid \\ + \ & \int_{V_t} \Phi \ \mid Q(W)_{\alpha\beta\gamma\delta} \ ^{(T)}\pi^{\alpha\beta} \bar{K}^{\gamma} T^{\delta} \mid .\end{aligned}$$

There are several borderline cases appearing in $\mathcal{E}_1(W,t_*)$. One of them we encounter in the integral

$$\int_{V^e_{t_*}} \tau_+^2 \ \Phi \ \mid (\rho,\sigma) \ (\hat{\mathcal{L}}_S W) \mid \mid tr\chi \mid \mid \ ^{(S)}\hat{i} \mid \mid \underline{\alpha} \mid , \tag{172}$$

which arises in the most delicate term

$$\int_{V^e_{t_*}} \Phi \tau_+^2 \ \mid (div \ Q(\hat{\mathcal{L}}_S W))_{334} \mid$$

when estimating $\int_{V^e_{t_*}} \Phi \ \mid (div \ Q(\hat{\mathcal{L}}_S W))_{\beta\gamma\delta} \ \bar{K}^{\beta} \ T^{\gamma} \ T^{\delta} \mid$ from above. We stress the fact that $(div \ Q(\hat{\mathcal{L}}_S W))_{334}$ being multiplied by τ_+^2 involves parts with the worst decay properties, which require more subtle estimates, as we shall see in the treatment of the borderline case (172). Note that $V^e_{t_*}$ is the exterior region as introduced earlier. We can split the integrals in $\mathcal{E}_1(W,t_*)$ and $\mathcal{E}_0(W,t_*)$ into the interior region $V^i_{t_*}$ where $r \leq \frac{r_0}{2}$, and the exterior region $V^e_{t_*}$ where $r \geq \frac{r_0}{2}$. The interior integrals are estimated in a straightforward way, as in the interior all the components of the tensors $DW, \hat{\mathcal{L}}_T W, \hat{\mathcal{L}}_S W$ and all the components of the deformation tensors of the vectorfields $T, S, \bar{K}$ behave in the same manner. Whereas in the exterior, the components of the said tensors behave differently. Therefore, the exterior estimates are more complicated, as they depend on the structure of the nonlinear terms.

Writing out the terms of $\int_{V^e_{t_*}} \Phi\tau_+^2 \ \mid (div \ Q(\hat{\mathcal{L}}_S W))_{334} \mid$ in detail, we find that the most delicate, namely, the borderline terms, are of the form (172). The quantities we have to pay special attention to are $\underline{\alpha}$ and $^{(S)}i$. Here, $\underline{\alpha}$ is the null curvature component (151) and $^{(S)}i$ is the null component $^{(S)}\hat{\pi}_{AB}$ of the deformation tensor (see (42)) tangent to the surface. In view of the coarea formula, we write the integral (172) over $V^e_{t_*}$ as an integral on C_u and an integral with respect to u:

$$\begin{aligned}& \int_{V^e_{t_*}} \tau_+^2 \ \Phi \ \mid (\rho,\sigma) \ (\hat{\mathcal{L}}_S W) \mid \mid tr\chi \mid \mid \ ^{(S)}\hat{i} \mid \mid \underline{\alpha} \mid \\ & \quad = \int_{-\infty}^{u_*} du \int_{C_u} \tau_+^2 \ \Phi \ \mid (\rho,\sigma) \ (\hat{\mathcal{L}}_S W) \mid \mid tr\chi \mid \mid \ ^{(S)}\hat{i} \mid \mid \underline{\alpha} \mid\end{aligned}$$

We calculate on C_u:

$$\int_{C_u} \tau_+^2 \, \Phi \ | (\rho, \sigma) \ (\hat{\mathcal{L}}_S W) | \, | tr\chi | \, | \ {}^{(S)}\hat{i} \ | \, | \underline{\alpha} |$$

$$\leq c' \sup_{C_u} \{ \tau_+ \, | tr\chi | \} \int_{C_u} \tau_+ \, | (\rho, \sigma) \ (\hat{\mathcal{L}}_S W) | \, | \ {}^{(S)}\hat{i} \ | \, | \underline{\alpha} |$$

$$\leq \ c \Big(\int_{C_u} \tau_+^2 \, | (\rho, \sigma) \ (\hat{\mathcal{L}}_S W) |^2 \ \Big)^{\frac{1}{2}} \Big(\int_{C_u} | \ {}^{(S)}\hat{i} \ |^2 \, | \underline{\alpha} |^2 \Big)^{\frac{1}{2}}.$$

Whereas the first integral on the right-hand side is bounded by curvature assumptions, the second integral has to be investigated further. Thus, one obtains

$$\Big(\int_{C_u} | \ {}^{(S)}\hat{i} \ |^2 \, | \underline{\alpha} |^2 \Big)^{\frac{1}{2}} \ = \ \Big(\int_0^{t_*} \Big\{ \int_{S_{t,u}} | \ {}^{(S)}\hat{i} \ |^2 \, | \underline{\alpha} |^2 \Big\} dt \Big)^{\frac{1}{2}}$$

$$= \ \Big(\int_0^{t_*} \ \| \ | \ {}^{(S)}\hat{i} \ | \, | \underline{\alpha} | \ \|^2_{L^2(S_{t,u})} \ dt \ \Big)^{\frac{1}{2}}$$

$$\leq \ \Big(\int_0^{t_*} \ \| \ {}^{(S)}\hat{i} \ \|^2_{L^4(S_{t,u})} \, \| \, \underline{\alpha} \, \|^2_{L^4(S_{t,u})} \ dt \ \Big)^{\frac{1}{2}}, \tag{173}$$

The problem reduces to estimating the last integral. We have

$$\Big(\int_0^{t_*} \ \| \ {}^{(S)}\hat{i} \ \|^2_{L^4(S_{t,u})} \, \| \, \underline{\alpha} \, \|^2_{L^4(S_{t,u})} \ dt \ \Big)^{\frac{1}{2}}$$

$$\leq \ c \ \tau_-^{-\frac{3}{2}} \ \Big(\int_0^{t_*} (1+t)^{-1} \ \| \ {}^{(S)}\hat{i} \ \|^2_{L^4(S_{t,u})} \ dt \ \Big)^{\frac{1}{2}}. \tag{174}$$

The curvature assumptions give

$$\sup_{t, \ \mathrm{on} C_u} \big\{ r^{\frac{1}{2}} \ \| \, \underline{\alpha} \, \|_{L^4(S_{t,u})} \ \big\} \ \ \leq \ \ c \ \tau_-^{-\frac{3}{2}},$$

and the assumption on ${}^{(S)}i$ is

$$\| \ r^{\frac{1}{2}} \ {}^{(S)}i \ \|_{\infty, e} \leq \ \epsilon_0,$$

which gives

$$\| \ {}^{(S)}\hat{i} \ \|_{L^4(S_{t,u})} \leq \ C\epsilon_0.$$

If these bounds are substituted in (174), a logarithmic divergence would result. Thus the integral in (174) is borderline. We show that in fact it is bounded. Any relaxation of our assumptions on the data involved would make this integral diverge. Now, in view of the definition of ${}^{(S)}\hat{i}$ and writing out the Lie derivative, we obtain

$${}^{(S)}\hat{i} \ = \ \frac{1}{2} \, \underline{u} \, \hat{\chi} \ + \ \frac{1}{2} \, u \, \underline{\hat{\chi}}.$$

Thus, the ${}^{(S)}\hat{i}$ involves the terms $r\hat{\chi}$ and $u\underline{\hat{\chi}}$. Whereas the latter has order of decay $O(r^{-1}\tau_-^{\frac{1}{2}})$, the first term is only of order $o(r^{-\frac{1}{2}})$. Therefore, the difficulty lies in estimating $r\hat{\chi}$. Using the Codazzi equations and the assumption that $\beta \in L^4(S_{t,u})$, by elliptic estimates from the results to control $r\hat{\chi}$ in terms of β, we obtain

$$\| \, r\hat{\chi} \, \|_{L^4(S_{t,u})} \; + \; \| \, r^2 \nabla\!\!\!/ \, \hat{\chi} \, \|_{L^4(S_{t,u})} \quad \leq \quad c \; \| \, r^2 \beta \, \|_{L^4(S_{t,u})} \, . \tag{175}$$

As we want to integrate $(1+t)^{-1} \| \; {}^{(S)}\hat{i} \; \|^2_{L^4(S_{t,u})}$ in (174), we study the integral for $\| \, r^{\frac{3}{2}} \beta \, \|^2_{L^4(S_{t,u})}$:

$$\begin{aligned}
& \int_0^{t_*} \| \, r^{\frac{3}{2}} \beta \, \|^2_{L^4(S_{t,u})} \; dt \\
& \quad \leq \; c \int_0^{t_*} \| \, r\beta \, \|_{L^2(S_{t,u})} \cdot \| \, r\beta \, \|_{L^2(S_{t,u})} \; dt \\
& \qquad + \; c \int_0^{t_*} \| \, r\beta \, \|_{L^2(S_{t,u})} \cdot \| \, r^2 \nabla\!\!\!/ \, \beta \, \|_{L^2(S_{t,u})} \; dt \qquad (176) \\
& \quad \leq \; c \left(\int_0^{t_*} \| \, r\beta \, \|^2_{L^2(S_{t,u})} \, dt \right)^{\frac{1}{2}} \left(\int_0^{t_*} \| \, r\beta \, \|^2_{L^2(S_{t,u})} \, dt \right)^{\frac{1}{2}} \\
& \qquad + \; c \left(\int_0^{t_*} \| \, r\beta \, \|^2_{L^2(S_{t,u})} \, dt \right)^{\frac{1}{2}} \left(\int_0^{t_*} \| \, r^2 \nabla\!\!\!/ \, \beta \, \|^2_{L^2(S_{t,u})} \, dt \right)^{\frac{1}{2}} \qquad (177) \\
& \quad \leq \; c \left(\int_{C_u} r^2 \mid \beta \mid^2 \right)^{\frac{1}{2}} \left(\int_{C_u} r^2 \mid \beta \mid^2 \right)^{\frac{1}{2}} \\
& \qquad + \; c \left(\int_{C_u} r^2 \mid \beta \mid^2 \right)^{\frac{1}{2}} \left(\int_{C_u} r^4 \mid \nabla\!\!\!/ \, \beta \mid^2 \right)^{\frac{1}{2}} . \qquad (178)
\end{aligned}$$

The right-hand side of the last inequality is bounded by the curvature assumptions on β. In order to prove the first inequality, we apply the isoperimetric inequality on $S_{t,u}$ to $\mid \beta \mid^2$. Using the fact that for a function f we have $\int_{S_{t,u}} (f-\bar{f})^2 = \int_{S_{t,u}} f^2 - \int_{S_{t,u}} f\bar{f} = \int_{S_{t,u}} f^2 - \int_{S_{t,u}} \bar{f}^2$, we obtain by first applying the isoperimetric and then Hölder inequality

$$\begin{aligned}
\int_{S_{t,u}} \mid \beta \mid^4 \quad \leq \quad & c \, r^{-2} \left(\int_{S_{t,u}} \mid \beta \mid^2 d\mu_\gamma \right) \\
& \times \left\{ \left(\int_{S_{t,u}} \mid \beta \mid^2 d\mu_\gamma \right) \; + \; \left(\int_{S_{t,u}} r^2 \mid \nabla\!\!\!/ \, \beta \mid^2 d\mu_\gamma \right) \right\}
\end{aligned}$$

Multiplying by r^6, we derive

$$\| \, r^{\frac{3}{2}} \beta \, \|^2_{L^4(S_{t,u})} \leq \; C \left\{ \; \| \, r\beta \, \|^2_{L^2(S_{t,u})} \; + \; \| \, r\beta \, \|_{L^2(S_{t,u})} \| \, r^2 \nabla\!\!\!/ \, \beta \, \|_{L^2(S_{t,u})} \; \right\}. \tag{179}$$

Finally, estimate (176) for the integral over t is deduced, and the Cauchy-Schwarz inequality then gives the right-hand side of (177) respectively (178) which is bounded. What we have shown is

$$\begin{aligned} \| r^{-\frac{1}{2}} \, {}^{(S)}\hat{i} \|_{L^2([0,t_*], L^4(S_{t,u}))} &\leq c \left(\int_{C_u} r^2 \mid \beta \mid^2 \right)^{\frac{1}{2}} \left(\int_{C_u} r^2 \mid \beta \mid^2 \right)^{\frac{1}{2}} \\ &\quad + c \left(\int_{C_u} r^2 \mid \beta \mid^2 \right)^{\frac{1}{2}} \left(\int_{C_u} r^4 \mid \nabla\!\!\!/ \, \beta \mid^2 \right)^{\frac{1}{2}} \\ &\leq c \, \epsilon_0 \, Q_1^*. \end{aligned}$$

From this it directly follows that

$$\int_{V_{t_*}^e} \tau_+^2 \, \Phi \; \mid (\rho, \sigma) \, (\hat{\mathcal{L}}_S W) \mid \mid tr\chi \mid \mid \, {}^{(S)}\hat{i} \mid \mid \underline{\alpha} \mid \; \leq \; c \, \epsilon_0 \, Q_1^*.$$

We observe that only the estimates of ${}^{(S)}i$ in terms of β yield the required bounds for this integral. Any further relaxation of our assumptions would lead to divergence of this integral. Therefore, this is indeed a borderline case.

The estimates for many quantities in our work are borderline. Let us point out here that this is true also for χ. Most of these borderline cases require a more careful treatment.

6.3. Controlling Geometric Quantities and the Construction of the Optical Function. The main step 3 above consists of estimating the geometric quantities with respect to the foliations $\{H_t\}$ and $\{C_u\}$. The core part in here is the treatment of the components of the second fundamental form k and the Hessian of u. In view of the results mentioned in the previous paragraph, the estimates derived here have to be appropriate. Then this makes it possible to close the proof of the main theorem and to derive the main results. To make this more precise, let us discuss in the following the procedure for how it is done, and give the different quantities that are estimated.

We introduce the basic norms of our geometric quantities in subsection 5.2, that is, in particular, of the components of k in 5.2.2 and of the components of the Hessian of u in 5.2.4. With the definitive version of our main theorem 3 comes a concrete description of the asymptotic behavior of the geometric quantities.

The second fundamental form k decomposes into the scalar $k_{NN} = \delta$, the S-tangent 1-form $k_{AN} = \epsilon_A$, and the S-tangent symmetric 2-tensor $k_{AB} = \eta_{AB}$. The worst decay properties have $\hat{\eta}_{AB}$, that is, the traceless part of η_{AB}. Having good estimates for the curvature, we can then use standard tools. Here, elliptic estimates are applied to derive the desired results. The main part is to prove them in the wave zone, where the behavior of the components and of their derivatives depends on the direction. The elliptic system on H_t for k is given by

$$trk = 0, \tag{180}$$

$$curl\ k = H, \tag{181}$$

$$div\ k = 0. \tag{182}$$

H is the magnetic part of the spacetime curvature relative to the time foliation. We also have

$$\bar{R}_{ij} = k_{ia}\, k^a{}_j + E_{ij}, \tag{183}$$

where E denotes the electric part of the spacetime curvature relative to the time foliation. This elliptic system on H_t for k is decomposed relative to the radial foliation, that is, the foliation of each H_t by the surfaces $S_{t,u}$. From the fact that the nonlinear terms in these equations behave better in view of decay than the worst linear ones follows that the estimates are essentially linear but nevertheless yield control of the full nonlinear problem. The estimates, being essentially linear in the present situation in contrast to [**21**], simplify the proof considerably.

In order to study the components of the Hessian of u, we apply the basic method from the original proof in [**21**], namely, the method of treating propagation equations along the cones C_u coupled to elliptic systems on the surfaces $S_{t,u}$. However, our estimates differ fundamentally from the ones in [**21**]. The reason for that is the fact that we do not have any L^∞ bounds on the curvature, but we only control one derivative of the curvature in the hypersurfaces H_t and the Gauss curvature K in the surfaces $S_{t,u}$ lies in L^4, as explained above. Our situation yields borderline estimates for χ (see [**7**], chapter 7), whereas in [**21**], there are no borderline estimates. In fact, our assumptions on the fall-off cannot be relaxed. Thus, they are sharp with respect to decay.

Relative to a null frame (introduced before) the Hessian of u decomposes into χ_{AB}, ζ_A, ω, satisfying the following equations:

$$\begin{aligned} \frac{d\chi_{AB}}{ds} &= -\chi_{AC}\,\chi_{CB} + \alpha_{AB}, \\ \frac{d\zeta_A}{ds} &= -\chi_{AC}\,\zeta_B + \chi_{AB}\,\underline{\zeta}_B - \beta_A, \\ \frac{d\omega}{ds} &= 2\,\underline{\zeta}\cdot\zeta - |\,\zeta\,|^2 - \rho. \end{aligned}$$

As α is traceless, the trace of χ fulfills an equation without curvature terms, namely, (58):

$$\frac{dtr\chi}{ds} = -\frac{1}{2}\,(tr\chi)^2 - |\,\hat{\chi}\,|^2$$

with $\hat{\chi}$ denoting the traceless part of χ. For $\hat{\chi}$ the null-Codazzi equations form an elliptic system on the surfaces $S_{t,u}$. Recall (59), which in detail read

$$\text{(184)} \qquad \not{d}\!iv\ \hat{\chi}_a \ = \ -\beta_a \ + \ \frac{1}{2}\nabla\!\!\!/_a tr\chi \ - \ \hat{\chi}_a{}^b\ \zeta_b \ + \ \frac{1}{2}\ tr\chi\ \zeta_a.$$

$\nabla\!\!\!/$ is the induced covariant differentiation on the surfaces $S_{t,u}$. As a main goal here, we show that χ is one degree of differentiability smoother than the curvature. This is achieved by a bootstrap argument with a certain assumption on the curvature term β on the right- hand side of (184), as sketched above. The divergence equation (184) is a Hodge system. Thus, the bootstrap argument together with (58), (184), the method of treating such coupled systems of propagation and elliptic equations together with the Hodge theory yield the estimates. In the same line, we obtain the results for ζ and ω. For ζ, we have the Hodge system

$$\begin{aligned} \not{d}\!iv\,\zeta \quad &= \quad -\,\mu \ - \ \rho \ + \ \frac{1}{2}\ \hat{\chi}\ \cdot\ \underline{\hat{\chi}}, \\ c\!\!\!/url\,\zeta \quad &= \quad \sigma \ - \ \frac{1}{2}\ \hat{\chi}\ \wedge\ \underline{\hat{\chi}}, \end{aligned}$$

with $\underline{\chi}$ being the second fundamental form

$$\underline{\chi}(X,Y) \ = \ g(D_X\underline{L},Y)$$

for $X,Y \in T_pS$ and $\underline{L}$ being the inward null normal and μ denoting the *mass aspect function*

$$\mu \ = \ K \ + \ \frac{1}{4}\ tr\chi\ tr\underline{\chi} \ - \ \not{d}\!iv\,\zeta.$$

The propagation equation for μ is derived to be

$$\text{(185)} \qquad \begin{aligned} \frac{d\mu}{ds} \ + \ \frac{3}{2}\ tr\chi\ \mu \quad &= \quad -\ \frac{1}{4}\ tr\underline{\chi}\ \mid\hat{\chi}\mid^2 \ + \frac{1}{2}\ tr\chi\ \mid\zeta\mid^2 \\ &\quad\ + 2\ \not{d}\!iv\,\chi\cdot\zeta \ + \ \hat{\chi}\cdot\nabla\!\!\!/\,\hat{\otimes}\zeta, \end{aligned}$$

with $(\nabla\!\!\!/\,\hat{\otimes}\zeta)_{ab} = \nabla\!\!\!/_a\zeta_b + \nabla\!\!\!/_b\zeta_a - \gamma_{ab}\ \not{d}\!iv\,\zeta$.

Within this framework, to obtain our estimates, the uniformization theorem for $K \in L^4(S)$ is needed, as it is mentioned above and proven in [**6**, **7**].

A further crucial part of our main proof deals with the last slice H_{t_*} of the spacetime. Here, we study the construction of the optical function u on this last slice. It is done by solving an inverse lapse problem for the function u on H_{t_*}. That is, starting from $S_{t_*,0} = H_{t_*} \cap C_0$, the intersection of the last slice with the initial cone C_0, the function u is defined to be the solution of

$$\mid\nabla u\mid^{-1} = \ a, \qquad u\ \mid_{S_{t_*,0}} = \ 0,$$

where a on each $S_{t_*,u}$ fulfills

$$\triangle\!\!\!\!/\ \log a \ = \ f \ - \ \bar{f} \ - \ \not{d}\!iv\,\epsilon, \qquad \overline{\log a} \ = \ 0$$

with

$$f = K - \frac{1}{4} (tr\chi)^2 .$$

This method of solving an inverse lapse problem for u is the same as in [**21**], whereas the proof itself differs fundamentally from the one in [**21**]. Again the fact that we do not have any L^∞ bounds on the curvature is crucial, but the curvature only lies in H^1 of the hypersurface H_{t_*}.

Constructing u as a solution of the inverse lapse problem, that is, solving an equation of motion of surfaces, was first done by D. Christodoulou and S. Klainerman in [**21**]. One might first think of applying other, easier methods in order to construct u. But, as we are going to explain now, they would not match our requirements.

What we refer to as the "last slice" is the maximal hypersurface $\mathcal{H}_{t_*}$, which bounds in the future the spacetime slab that we are constructing in the continuity argument. The obvious choice of u on $\mathcal{H}_{t_*}$, namely, minus the signed distance function from $S_{t_*,0}$, is inappropriate because this distance function is only as smooth as the induced metric $\bar{g}_{t_*}$. It is not one order better, which would be the maximal possible. Thus, there would be a loss of one order of differentiability. But the problem does not allow us to lose derivatives. That is, with this loss of one order of differentiability, the estimates would fail to close.

To overcome this difficulty, we define u on $\mathcal{H}_{t_*}$ in a different way, namely, by solving an equation of motion of surfaces on $\mathcal{H}_{t_*}$, as described above.

Why would the use of other methods like the inverse mean curvature flow (IMCF) not work here? Using IMCF, the problem could be solved in the outward direction only, whereas the equation of motion of surfaces can be solved in both directions, which is what we need.

The equation we use here has to have the smoothing property described above as well as has to be solvable in both directions. This excludes the IMCF and similar methods. It turns out that the equation of motion of surfaces yields exactly what we need.

Let us explain now the principal ideas in the proof of the main theorem in the last slice and also show in which sense our relaxed assumptions on the curvature require a special treatment.

The main results of this part are again obtained by a bootstrap argument. We have to assume estimates for the spacetime curvature on the last slice H_{t_*}. To be precise, these assumptions have to be made with respect to the background foliation, as the level surfaces of u on the last slice do not yet exist, but have to be constructed. Note that, at the beginning, only $S_{t_*,0}$ is given. We have to use the estimates on the background foliation to control the curvature and geometric components of the foliation by u. We estimate $tr\chi$, $\hat{\chi}$, ζ, and their first angular derivatives, as well as K in dimensionless L^4-norms on the surfaces $S_{t_*,u}$. We show this by bootstrapping. A crucial role in the proof is played by the trace lemma, yielding L^4-bounds on the curvature components in $S_{t_*,u}$. Relying on them, we then apply elliptic theory to obtain the estimates of the theorem. By a straightforward argument, the said quantities are shown to be controlled correspondingly in H_{t_*}. Next, integrating in the last slice over $u \in [0, \infty)$ yields the estimates for the second angular derivatives of $tr\chi$, $\hat{\chi}$, and ζ in L^2-norms in H_{t_*}.

Yet in order to apply the trace lemma, as said above, one has to work more. We estimate on H_{t_*} the components of the above quantities of the foliation by the function u by corresponding quantities with respect to the background foliation given by u'. Let u' be a smooth function without critical points, defined in a tubular neighborhood U' of S_0, and let $S'_{u'}$ be the level sets of u' on H_{t_*} of our background foliation. Denote

$$U' \;=\; \bigcup_{u' \in (u'_0, \infty)} S'_{u'}.$$

The function u' is introduced and specified in detail in [**7**] and in [**6**] in chapter 11, theorem 6. The curvature components with respect to its foliation are small. However, this function will not in general satisfy the equations of the inverse lapse problem above. Therefore, we use the estimates on the background foliation to control the curvature and geometric components of the foliation by u. For the foliation in the bootstrap argument, with respect to the function u, we denote

$$U \;=\; \bigcup_{u \in [0, u_1)} S_u.$$

We have to overcome the following difficulty: By assumption, the curvature components of the background foliation lie in $H^1(U')$. How can they be bounded in $H^1(U)$? There is no straightforward procedure to bound the curvature components in the surfaces of the foliation given by u directly, as is done in [**21**], where the corresponding curvature components of the background foliation are in L^∞. Thus the situation here is different. We solve the problem as follows: The transformation coefficients between quantities referring to the two foliations being bounded, we derive that the curvature components relative to the u-foliation lie in $H^1(U)$. Only now, the trace lemma can be applied to obtain the curvature components with respect to the u-foliation to lie in $L^4(S_u)$. We re-emphasize that, as a consequence from controlling one less derivative, the derivatives of these components are not bounded in the surfaces S_u.

The main results of the last slice are formulated in L^4-norms. Actually, they hold for L^p-norms with $2 < p \leq 4$. The upper bound 4 is given by the trace lemma, the lower bound 2 by the fact that at certain levels of the proof, in the surfaces $S_{t_*,u'}$, we have to bound the L^∞-norms of the quantities we estimate, and we only have them in L^p up to their first derivatives, we have to require $p > 2$. This is necessary in view of the fact that for the said surfaces it is $W^p_m \hookrightarrow L^\infty$ for $mp > 2$.

Finally, all the bootstrap arguments close and so does the overall bootstrap argument, which finishes the proof of the main theorem.

Conlcuding, we find that the estimates for some of our main quantities are borderline from the point of view of decay, which means that it is not possible to relax further our assumptions. This indicates that the conditions in our theorem are sharp in so far as the assumptions on the decay at infinity on the initial data are concerned.

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Department of Mathematics
Harvard University
Cambridge, MA 02138
E-mail address: lbieri@math.harvard.edu