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ON A CHARACTERIZATION OF THE COMPLEX HYPERBOLIC SPACE

Ovidiu Munteanu

Abstract

Consider a compact Kähler manifold M^m with Ricci curvature lower bound $Ric_M \geq -2(m+1)$. Assume that its universal cover $\widetilde{M}$ has maximal bottom of spectrum $\lambda_1\left(\widetilde{M}\right) = m^2$. Then we prove that $\widetilde{M}$ is isometric to the complex hyperbolic space $\mathbb{CH}^m$.

1. Introduction

Complete Riemannian manifolds with negative Ricci curvature lower bound have been investigated by many authors. An important approach is to see how the spectrum of the Laplacian interacts with the geometry of the manifold. A classical result that we recall here is S.Y. Cheng's comparison theorem [**C**], which states that the hyperbolic space $\mathbb{H}^n$ has the greatest bottom of spectrum among all complete Riemannian manifolds with Ricci curvature at least the Ricci curvature of $\mathbb{H}^n$.

Therefore, if the Ricci curvature of a complete noncompact Riemannian manifold N^n of dimension n is bounded below by $Ric_N \geq -(n-1)$, then the bottom of the spectrum of the Laplacian has an upper bound $\lambda_1(N) \leq \frac{(n-1)^2}{4}$. This result is sharp, but we should point out that there are in fact many manifolds with maximal λ_1, and more examples can be found by considering hyperbolic manifolds $N = \mathbb{H}^n/\Gamma$ obtained by the quotient of $\mathbb{H}^n$ by a Kleinian group Γ (see [**S**]).

While in general we cannot determine the class of manifolds with λ_1 achieving its maximal value, recently there has been important progress in some directions.

P. Li and J. Wang have studied the structure at infinity of a complete noncompact Riemannian manifold that has $Ric_N \geq -(n-1)$ and maximal bottom of spectrum $\lambda_1(N) = \frac{(n-1)^2}{4}$. They proved that either the manifold is connected at infinity (i.e., it has one end) or it has two ends. In the case where it has two ends, it must split as a warped product of a compact manifold with the real line [**L-W2**]. Their result has

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since been extended in many other situations, e.g., Kähler manifolds, quaternionic Kähler manifolds, or locally symmetric spaces.

Recently X. Wang [**W**] has obtained an interesting result in a different setting. Suppose N^n is a compact Riemannian manifold with $Ric_N \geq -(n-1)$. Consider $\pi : \widetilde{N} \to N$ its universal cover and assume that $\lambda_1(\widetilde{N}) = \frac{(n-1)^2}{4}$. Then Wang proved that $\widetilde{N}$ is isometric to the hyperbolic space $\mathbb{H}^n$.

It should be pointed out that in Wang's theorem if the manifold N is assumed to have negative curvature (and removing the lower bound on Ricci curvature assumption) then stronger results are already known from the work of Ledrappier, Foulon and Labourie, and Besson, Courtois, and Gallot [**L**, **F-L**, **B-C-G**]. Indeed, let h denote the volume entropy of N, i.e.,

$$h = \lim_{R\to\infty} \frac{\log Vol\left(\widetilde{B}_p(R)\right)}{R},$$

where $\widetilde{B}_p(R)$ is the geodesic ball of radius R centered at p in $\widetilde{N}$.

Then, assuming N has negative curvature and $\lambda_1(\widetilde{N}) = \frac{1}{4}h^2$ from the works cited above, it results that N is locally symmetric.

However, Wang's theorem is quite powerful because it does not assume the metric has negative curvature.

It is a natural question to investigate these issues on Kähler manifolds.

A first question that one should ask is if Cheng's estimate can be improved in this case. The model space that we work with is now the complex hyperbolic space $\mathbb{CH}^m$. Recently, Li and Wang have proved [**L-W1**] that, for a complete noncompact Kähler manifold M^m of complex dimension m, if the bisectional curvature is bounded from below by $BK_M \geq -1$, then $\lambda_1(M) \leq m^2 = \lambda_1(\mathbb{CH}^m)$. Moreover, if the bottom of spectrum $\lambda_1(M)$ achieves its maximal value, then the manifold is either connected at infinity or it has two ends, and in this latter case it is diffeomorphic to the product of a compact manifold with the real line and the Kähler metric on M has a specialized form.

We recently improved Li and Wang's results for complete Kähler manifolds that have a Ricci curvature lower bound, $Ric_M \geq -2(m+1)$, which is a weaker assumption than $BK_M \geq -1$. We proved that the same estimate for λ_1 holds under the Ricci curvature lower bound, and, moreover, if the bottom of spectrum achieves its maximal value, the manifold has the same structure at infinity as in Li and Wang's theorem (see [**M**]). To obtain these results we developed a new argument, a sharp integral estimate for the gradient of a certain class of harmonic functions. We should point out that a sharp pointwise gradient estimate for harmonic functions is not known to be true for Kähler manifolds.

In this paper we will use our argument to prove the following result.

Theorem 1. *Let M^m be a compact Kähler manifold of complex dimension m and Ricci curvature bounded below by $Ric_M \geq -2(m+1)$. Assume its universal cover $\pi : \widetilde{M} \to M$ has maximal bottom of spectrum, $\lambda_1\left(\widetilde{M}\right) = m^2$. Then $\widetilde{M}$ is isometric to the complex hyperbolic space $\mathbb{CH}^m$.*

We want to comment now about the particular case when M has negative curvature.

For Kähler manifolds with bisectional curvature lower bound $BK_M \geq -1$, it follows from [**L-W1**] that the volume entropy verifies the sharp estimate $h \leq 2m$. So maximal bottom of spectrum in this case implies $\lambda_1 = \frac{1}{4}h^2$. However, for only Ricci curvature lower bound $Ric_M \geq -2(m+1)$, it is not known whether $h \leq 2m$, so it is not clear how to apply the Besson-Courtois-Gallot theorem in the negative curvature case.

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2. Proof of the Theorem

First, let us set the notation. We use the notations in [**L-W**, **M**]. Denote $ds^2 = h_{\alpha\bar{\beta}}dz^\alpha d\bar{z}^\beta$ the Kähler metric on $\widetilde{M}$ and let $Re\left(ds^2\right)$ be the Riemannian metric on $\widetilde{M}$.

Suppose that $\{e_1, e_2, \ldots, e_{2m}\}$ with $e_{2k} = Je_{2k-1}$ for $k \in \{1, 2, \ldots, m\}$ is an orthonormal frame with respect to the Riemannian metric on $\widetilde{M}$; then $\{v_1, \ldots, v_m\}$ is a unitary frame of $T_x^{1,0}\widetilde{M}$, where

$$v_k = \frac{1}{2}\left(e_{2k-1} - \sqrt{-1}e_{2k}\right).$$

With respect to any such unitary frame the following formulas hold:

$$\begin{aligned}\nabla f \cdot \nabla g &= 2\left(f_\alpha f_{\bar{\alpha}} + g_\alpha g_{\bar{\alpha}}\right), \\ \Delta f &= 4 f_{\alpha\bar{\alpha}}.\end{aligned}$$

In the statement of the theorem, the Ricci curvature lower bound refers to the Riemannian metric and it is equivalent to saying

$$Ric_{\alpha\bar{\beta}} \geq -(m+1)\,\delta_{\alpha\bar{\beta}}$$

with respect to any unitary frame.

To prove the theorem we follow the same approach as in [**W**], namely, we use the Kaimanovich entropy. This is defined using the minimal Martin boundary of $\widetilde{M}$, and the idea is that it can be bounded from below in terms of $\lambda_1\left(\widetilde{M}\right)$ and from above based on the Ricci curvature assumption. The fact that $\lambda_1\left(\widetilde{M}\right)$ is maximal then implies equality in these bounds, which will prove the theorem.

We first recall some facts about the Kaimanovich entropy.

There are a few equivalent formulations of this entropy. First, it can be defined as a limit of the heat kernel:

$$\beta(\widetilde{M}) = \lim_{t\to\infty}\left(-\frac{1}{t}\int_{\widetilde{M}} p\left(t,x,y\right)\log p\left(t,x,y\right)dy\right),$$

where p is the heat kernel on $\widetilde{M}$. This expression is useful because it can be shown that (a result of Ledrappier [**L**])

$$\beta(\widetilde{M}) \geq 4\lambda_1\left(\widetilde{M}\right).$$

There is another very useful formula for β, using the minimal Martin boundary of $\widetilde{M}$. Let us quickly recall some known facts (see e.g., [**A**]).

Let $H\left(\widetilde{M}\right)$ denote the space of harmonic functions on $\widetilde{M}$, with the topology of uniform convergence on compact sets. Observe that, for $O\in\widetilde{M}$ fixed,

$$K_O = \left\{u\in H\left(\widetilde{M}\right) : u\left(O\right) = 1,\ u > 0\right\}$$

is a compact and convex subset of $H\left(\widetilde{M}\right)$. Denote with $\partial^*\widetilde{M}$ the set of extremal points of K_O, i.e., points in K_O that do not lie in any open line segment in K_O. Note that a point of K_O is extremal if and only if it is a minimal positive harmonic function normalized at O; therefore $\partial^*\widetilde{M}$ is the minimal Martin boundary of $\widetilde{M}$. Since K_O is a metric space and it is compact and convex, by a theorem of Choquet it results that for any positive harmonic function h there is a unique Borel measure μ^h on the set of extremal points of K_O so that

$$h\left(x\right) = \int_{\partial^*\widetilde{M}}\xi\left(x\right)d\mu^h\left(\xi\right).$$

In particular, for $h=1$ there exists a unique measure ν on $\partial^*\widetilde{M}$ so that for any $x\in\widetilde{M}$,

$$\int_{\partial^*\widetilde{M}}\xi\left(x\right)d\nu\left(\xi\right) = 1.$$

Let Γ denote the group of deck transformations on $\widetilde{M}$; then there is a natural action of Γ on $\partial^*\widetilde{M}$, defined by

$$\left(\gamma\xi\right)\left(x\right) = \frac{\xi\left(\gamma^{-1}x\right)}{\xi\left(\gamma^{-1}O\right)},$$

for any $\xi\in\partial^*\widetilde{M}$ and for any $\gamma\in\Gamma$.

It is important to know how the measure ν is changed by the action of Γ on $\partial^*\widetilde{M}$. It can be seen, using the uniqueness property of ν, that if $\gamma_*\nu$ denotes the push forward measure, i.e., $(\gamma_*\nu)(B) = \nu\left(\gamma^{-1}B\right)$ for any Borel set $B \subset \partial^*\widetilde{M}$, then

$$\frac{d\gamma_*\nu(\xi)}{d\nu(\xi)} = \xi(\gamma O).$$

For $x \in \widetilde{M}$, define

$$\omega(x) = \int_{\partial^*\widetilde{M}} \xi^{-1}(x)\,|\nabla\xi|^2(x)\,d\nu(\xi),$$

and notice that ω descends on M. Indeed, for any $\gamma \in \Gamma$ we have that

$$|\nabla\xi|^2(\gamma x) = |\nabla(\gamma^*\xi)|^2(x),$$

where $\gamma^*\xi$ is the pull back of ξ, i.e., $\gamma^*\xi = \xi \circ \gamma$. Then it is easy to check using the Radon-Nikodym derivative that for $\eta = \gamma^{-1}\xi$ we have

$$\begin{aligned}
&\int_{\partial^*\widetilde{M}} \xi^{-1}(\gamma x)\,|\nabla\xi|^2(\gamma x)d\nu(\xi)\\
&\quad= \int_{\partial^*\widetilde{M}} \eta^{-1}(x)\,|\nabla\eta|^2(x)\frac{1}{\eta(\gamma^{-1}O)}d\gamma_*^{-1}\nu(\eta)\\
&\quad= \int_{\partial^*\widetilde{M}} \eta^{-1}(x)\,|\nabla\eta|^2(x)\,d\nu(\eta).
\end{aligned}$$

Hence it clearly follows that

$$\begin{aligned}
\omega(\gamma x) &= \int_{\partial^*\widetilde{M}} \xi^{-1}(\gamma x)\,|\nabla\xi|^2(\gamma x)\,d\nu(\xi)\\
&= \int_{\partial^*\widetilde{M}} \eta^{-1}(x)\,|\nabla\eta|^2(x)\,d\nu(\eta)\\
&= \omega(x).
\end{aligned}$$

We have shown that in fact ω is a well-defined function on M. This function can be used now to give another formula for the Kaimanovich entropy. Everywhere in this paper we will denote by dv_0 the normalized Riemannian volume form of M, i.e.,

$$dv_0 = \frac{1}{\int_M \sqrt{g}dx}\left(\sqrt{g}dx\right).$$

By a formula of Kaimanovich ([**K**]; see also [**L**, **W**]) the entropy can also be expressed as

$$\begin{aligned}
\beta(\widetilde{M}) &= \int_M \omega dv_0\\
&= \int_M \left(\int_{\partial^*\widetilde{M}} \xi^{-1}(x)\,|\nabla\xi|^2(x)\,d\nu(\xi)\right) dv_0.
\end{aligned}$$

Therefore, using Ledrappier's inequality for $\beta(\widetilde{M})$, we have the following:

$$4\lambda_1\left(\widetilde{M}\right) \leq \int_M \left(\int_{\partial^*\widetilde{M}} \xi^{-1}(x)\,|\nabla\xi|^2(x)\,d\nu(\xi)\right) dv_0. \tag{1}$$

For Riemannian manifolds, X. Wang has used this inequality together with the sharp Yau's gradient estimate ([**L-W2**]) to prove his result in the Riemannian setting.

For our problem, a sharp pointwise gradient estimate for Kähler manifolds is not known to be true. In [**M**] we developed an argument which makes possible to get sharp integral estimates for the gradient of a certain class of harmonic functions. This argument is based on successive integration by parts and the use of Ricci identities and it can be adapted to this setting.

Our goal is to show that

$$\int_M \left(\int_{\partial^*\widetilde{M}} \xi^{-1}(x)\,|\nabla\xi|^2(x)\,d\nu(\xi)\right) dv_0 \leq 4m^2.$$

Let $u = \log\xi$. Then a simple computation shows that

$$u_{\alpha\bar{\beta}} = \xi^{-1}\xi_{\alpha\bar{\beta}} - \xi^{-2}\xi_\alpha\xi_{\bar{\beta}}.$$

For a fixed $x \in \widetilde{M}$ consider

$$\int_{\partial^*\widetilde{M}} \xi(x)\left|u_{\alpha\bar{\beta}}\right|^2(x)\,d\nu(\xi).$$

We first claim that this integral is a finite number (depending on x). Indeed, since $\partial^*\widetilde{M}$ is compact and $d\nu$ is a finite measure, it suffices to show the integrand is bounded. But this is true because for fixed x we can bound $\left|\xi_{\alpha\bar{\beta}}\right|(x) \leq C(x)\,\xi(O) = C(x)$. This can be seen as follows. Consider $B_O(R)$ a geodesic ball of radius R big enough so that $x \in B_O(R)$. Note that there exists a constant $A > 0$ so that $\Delta\left|\xi_{\alpha\bar{\beta}}\right| \geq -A\left|\xi_{\alpha\bar{\beta}}\right|$ on $B_O(R)$. Such a constant A can be chosen to depend on the lower bound of the bisectional curvature on $B_O(R)$, using the Bochner formula. Using now the mean value inequality we get that there exists a constant C_1 depending on R and A so that

$$\left|\xi_{\alpha\bar{\beta}}\right|^2(x) \leq C_1 \int_{B_O(R)} \left|\xi_{\alpha\bar{\beta}}\right|^2.$$

It is known that by using integration by parts and suitable cut-off functions there exists a constant C_2 so that

$$\int_{B_O(R)} \left|\xi_{\alpha\bar{\beta}}\right|^2 \leq C_2 \int_{B_O(2R)} \xi^2.$$

The right side of this inequality can now be bounded by $C_3\xi^2(O)$, using the Harnack inequality. Obviously, these constants will depend on R; nevertheless, it follows that for x fixed $\left|\xi_{\alpha\bar{\beta}}\right|(x)$ will be bounded uniformly in ξ, which was our claim.

The second claim is that the function thus obtained actually descends on M. This claim can be showed as above, now using the fact that since M is Kähler, the deck transformations are holomorphic. Therefore for $\gamma \in \Gamma$ and $\gamma^*\xi$ the pull back of ξ we have

$$\left|(\log \xi)_{\alpha\bar{\beta}}\right|^2(\gamma x) = \left|(\log(\gamma^*\xi))_{\alpha\bar{\beta}}\right|^2(x).$$

The rest of the proof follows the same line as for the gradient of ξ (see above).

Therefore it makes sense to consider the following quantity:

$$\begin{aligned}
&\int_M \int_{\partial^*\widetilde{M}} \xi(x)\left|u_{\alpha\bar{\beta}}\right|^2(x) d\nu(\xi) dv_0 \\
&\quad = \int_M \int_{\partial^*\widetilde{M}} \xi^{-1}(x)\left|\xi_{\alpha\bar{\beta}}\right|^2(x) d\nu(\xi) dv_0 \\
&\quad\quad -2\int_M \int_{\partial^*\widetilde{M}} \xi^{-2}(x)\left(\xi_{\alpha\bar{\beta}}\xi_{\bar{\alpha}}\xi_{\beta}\right)(x)\, d\nu(\xi)\, dv_0 \\
&\quad\quad +\frac{1}{16}\int_M \int_{\partial^*\widetilde{M}} \xi^{-3}(x)\left|\nabla\xi\right|^4(x)\, d\nu(\xi)\, dv_0,
\end{aligned}$$

where each of the integrals on the right side are also well-defined by a similar discussion.

We now want to justify integration by parts to show that

$$\begin{aligned}
&\int_M \int_{\partial^*\widetilde{M}} \xi^{-1}(x)\left|\xi_{\alpha\bar{\beta}}\right|^2(x)\, d\nu(\xi)\, dv_0 \\
&\quad = \int_M \int_{\partial^*\widetilde{M}} \xi^{-2}(x)\left(\xi_{\alpha\bar{\beta}}\xi_{\bar{\alpha}}\xi_{\beta}\right)(x)\, d\nu(\xi)\, dv_0.
\end{aligned}$$

Consider (U_i) a covering of M with small open sets and let ρ_i be a partition of unity subordinated to this covering. We can choose (U_i) so that each U_i is diffeomorphic to an open set $\widetilde{U}_i \subset \widetilde{M}$ via π. We then have

$$\begin{aligned}
&\int_M \int_{\partial^*\widetilde{M}} \xi^{-1}(x)\left|\xi_{\alpha\bar{\beta}}\right|^2(x)\, d\nu(\xi)\, dv_0 \\
&\quad = \int_M \int_{\partial^*\widetilde{M}} \xi^{-1}(x)\,\xi_{\alpha\bar{\beta}}(x)\left(\xi_{\bar{\alpha}}(x)\sum_i \rho_i(\pi(x))\right)_{\beta} d\nu(\xi)\, dv_0 \\
&\quad = \sum_i \int_M \int_{\partial^*\widetilde{M}} \xi^{-1}(x)\,\xi_{\alpha\bar{\beta}}(x)\left(\xi_{\bar{\alpha}}(x)\rho_i(\pi(x))\right)_{\beta} d\nu(\xi)\, dv_0 \\
&\quad = \sum_i \int_{U_i} \int_{\partial^*\widetilde{M}} \xi^{-1}(x)\,\xi_{\alpha\bar{\beta}}(x)\left(\xi_{\bar{\alpha}}(x)\rho_i(\pi(x))\right)_{\beta} d\nu(\xi)\, dv_0
\end{aligned}$$

$$
\begin{aligned}
&= \sum_i \int_{\widetilde{U}_i} \int_{\partial^* \widetilde{M}} \xi^{-1}(x)\, \xi_{\alpha\bar{\beta}}(x) \left(\xi_{\bar{\alpha}}(x)\, \rho_i(\pi(x))\right)_\beta d\nu(\xi)\, dv_0 \\
&= -\sum_i \int_{\widetilde{U}_i} \int_{\partial^* \widetilde{M}} \left(\xi^{-1}(x)\, \xi_{\alpha\bar{\beta}}(x)\right)_\beta \left(\xi_{\bar{\alpha}}(x)\, \rho_i(\pi(x))\right) d\nu(\xi)\, dv_0 \\
&= \sum_i \int_{\widetilde{U}_i} \int_{\partial^* \widetilde{M}} \xi^{-2}(x) \left(\xi_{\alpha\bar{\beta}} \xi_{\bar{\alpha}} \xi_\beta\right)(x)\, \rho_i(\pi(x))\, d\nu(\xi)\, dv_0 \\
&= \sum_i \int_{U_i} \int_{\partial^* \widetilde{M}} \xi^{-2}(x) \left(\xi_{\alpha\bar{\beta}} \xi_{\bar{\alpha}} \xi_\beta\right)(x)\, \rho_i(\pi(x))\, d\nu(\xi)\, dv_0 \\
&= \sum_i \int_{M} \int_{\partial^* \widetilde{M}} \xi^{-2}(x) \left(\xi_{\alpha\bar{\beta}} \xi_{\bar{\alpha}} \xi_\beta\right)(x)\, \rho_i(\pi(x))\, d\nu(\xi)\, dv_0 \\
&= \int_{M} \int_{\partial^* \widetilde{M}} \xi^{-2}(x) \left(\xi_{\alpha\bar{\beta}} \xi_{\bar{\alpha}} \xi_\beta\right)(x)\, d\nu(\xi)\, dv_0.
\end{aligned}
$$

Let us point out that everywhere in these formulas (and in the paper) a priori the integrals on the minimal Martin boundary are taken for any (parameter) $x \in \widetilde{M}$. Then it can be justified that in fact these integrals on $\partial^* \widetilde{M}$ are invariant by the group of deck transformations, so they are well-defined functions on M. With this in mind, in the third line from the top one should also justify that for each i the functions on $\widetilde{M}$ defined by $x \to \int_{\partial^* \widetilde{M}} \xi^{-1}(x) \xi_{\alpha\bar{\beta}}(x) (\xi_{\bar{\alpha}}(x) \rho_i(\pi(x)))_\beta d\nu(\xi)$ descend on M. This can be done by the same argument, and using that $\gamma^*(\rho_i \circ \pi) = \rho_i \circ \pi$, for any $\gamma \in \Gamma$. It is also important that the function in ξ which is integrated on the minimal Martin boundary (for example $\xi \to \xi^{-1} \xi_{\alpha\bar{\beta}} \left(\xi_{\bar{\alpha}} \rho_i \circ \pi\right)_\beta$) be homogeneous of degree 1 in ξ. Thus we want to remark that not any integration by parts is allowed by this procedure of lifting the integrals on the universal covering. Finally, notice that we have also used that $\xi_{\alpha\bar{\beta}\beta} = 0$, which follows from the fact that ξ is harmonic.

This argument will be applied below every time we integrate by parts.

To simplify the writing, we will henceforth omit the argument x and the measure $d\nu$, but we always assume the integrals on $\partial^* \widetilde{M}$ are taken with respect to $d\nu$ and that all the functions integrated on $\partial^* \widetilde{M}$ depend on $x \in \widetilde{M}$. Each of these integrals on the minimal Martin boundary is invariant by the group of deck transformations so it legitimately defines a function on M.

We have thus proved that

$$
\begin{aligned}
\int_M \int_{\partial^* \widetilde{M}} \xi \left|u_{\alpha\bar{\beta}}\right|^2 \quad = \quad & -\int_M \int_{\partial^* \widetilde{M}} \xi^{-2} (\xi_{\alpha\bar{\beta}} \xi_{\bar{\alpha}} \xi_\beta) \\
& + \frac{1}{16} \int_M \int_{\partial^* \widetilde{M}} \xi^{-3} \left|\nabla \xi\right|^4.
\end{aligned}
$$

Let us use again integration by parts to see that

$$
\begin{aligned}
(2) \qquad & -\int_M \int_{\partial^* \widetilde{M}} \xi^{-2}(\xi_{\alpha\bar\beta}\xi_{\bar\alpha}\xi_\beta) = \int_M \int_{\partial^* \widetilde{M}} \xi_\alpha \left(\xi^{-2}\xi_{\bar\alpha}\xi_\beta\right)_{\bar\beta} \\
&= -\frac{1}{8}\int_M \int_{\partial^* \widetilde{M}} \xi^{-3} |\nabla\xi|^4 + \int_M \int_{\partial^* \widetilde{M}} \xi^{-2}\xi_{\bar\alpha\bar\beta}\xi_\alpha\xi_\beta .
\end{aligned}
$$

Note that the inequality

$$
\left|\xi_{\bar\alpha\bar\beta}\xi_\alpha\xi_\beta\right| \le \frac{1}{4}\left|\xi_{\alpha\beta}\right| |\nabla\xi|^2
$$

holds on $\widetilde{M}$, so that we get

$$
\begin{aligned}
(3) \qquad & 2\int_M \int_{\partial^* \widetilde{M}} \xi^{-2}\xi_{\bar\alpha\bar\beta}\xi_\alpha\xi_\beta \\
&\le \int_M \int_{\partial^* \widetilde{M}} 2\left(\xi^{-1/2}\left|\xi_{\alpha\beta}\right|\right)\left(\frac{1}{4}\xi^{-3/2}|\nabla\xi|^2\right) \\
&\le \frac{m}{m+1}\int_M \int_{\partial^* \widetilde{M}} \xi^{-1}\left|\xi_{\alpha\beta}\right|^2 + \frac{1}{16}\frac{m+1}{m}\int_M \int_{\partial^* \widetilde{M}} \xi^{-3}|\nabla\xi|^4 .
\end{aligned}
$$

Moreover, again integrating by parts we have

$$
\begin{aligned}
\int_M \int_{\partial^* \widetilde{M}} \xi^{-1}\left|\xi_{\alpha\beta}\right|^2 &= \int_M \int_{\partial^* \widetilde{M}} \xi^{-2}\xi_{\bar\alpha\bar\beta}\xi_\alpha\xi_\beta - \int_M \int_{\partial^* \widetilde{M}} \xi^{-1}\xi_\alpha\xi_{\bar\alpha\bar\beta\beta} \\
&\le \int_M \int_{\partial^* \widetilde{M}} \xi^{-2}\xi_{\bar\alpha\bar\beta}\xi_\alpha\xi_\beta + \frac{m+1}{4}\int_M \int_{\partial^* \widetilde{M}} \xi^{-1}|\nabla\xi|^2 .
\end{aligned}
$$

In the second line above, we have used the fact that ξ is harmonic, the Ricci identities, and the lower bound of the Ricci curvature:

$$
\begin{aligned}
-\xi_\alpha\xi_{\bar\alpha\bar\beta\beta} &= -\xi_\alpha\xi_{\bar\beta\bar\alpha\beta} \\
&= -\xi_\alpha\xi_{\bar\beta\beta\bar\alpha} - Ric_{\alpha\bar\beta}\xi_{\bar\alpha}\xi_\beta \\
&= -Ric_{\alpha\bar\beta}\xi_{\bar\alpha}\xi_\beta \\
&\le (m+1)\,\xi_\alpha\xi_{\bar\alpha} \\
&= \frac{m+1}{4}|\nabla\xi|^2 .
\end{aligned}
$$

Plug this inequality into (3) and it follows that

$$
\begin{aligned}
\frac{m+2}{m+1}\int_M \int_{\partial^* \widetilde{M}} \xi^{-2}\xi_{\bar\alpha\bar\beta}\xi_\alpha\xi_\beta &\le \frac{m}{4}\int_M \int_{\partial^* \widetilde{M}} \xi^{-1}|\nabla\xi|^2 \\
&+\frac{1}{16}\frac{m+1}{m}\int_M \int_{\partial^* \widetilde{M}} \xi^{-3}|\nabla\xi|^4 .
\end{aligned}
$$

Getting back to (2), we obtain

$$-\int_M\int_{\partial^*\widetilde{M}}\xi^{-2}\xi_{\alpha\bar{\beta}}\xi_{\bar{\alpha}}\xi_{\beta}\leq\left(-\frac{1}{8}+\frac{1}{16}\frac{(m+1)^2}{m\,(m+2)}\right)\int_M\int_{\partial^*\widetilde{M}}\xi^{-3}\left|\nabla\xi\right|^4$$
$$+\frac{m(m+1)}{4\,(m+2)}\int_M\int_{\partial^*\widetilde{M}}\xi^{-1}\left|\nabla\xi\right|^2.$$

We have thus proved that

$$\int_M\int_{\partial^*\widetilde{M}}\xi\left|u_{\alpha\bar{\beta}}\right|^2\leq\frac{1}{16}\frac{1}{m\,(m+2)}\int_M\int_{\partial^*\widetilde{M}}\xi^{-3}\left|\nabla\xi\right|^4$$
$$+\frac{m(m+1)}{4\,(m+2)}\int_M\int_{\partial^*\widetilde{M}}\xi^{-1}\left|\nabla\xi\right|^2.$$

The estimate from below is straightforward:

$$\left|u_{\alpha\bar{\beta}}\right|^2\geq\sum_{\alpha}\left|u_{\alpha\bar{\alpha}}\right|^2\geq\frac{1}{m}\left|\sum_{\alpha}u_{\alpha\bar{\alpha}}\right|^2=\frac{1}{16m}\xi^{-4}\left|\nabla\xi\right|^4.$$

Hence, this shows that

$$\int_M\int_{\partial^*\widetilde{M}}\xi^{-3}\left|\nabla\xi\right|^4\leq 4m^2\int_M\int_{\partial^*\widetilde{M}}\xi^{-1}\left|\nabla\xi\right|^2. \tag{4}$$

Finally, using the Schwarz inequality and the fact that $\int_{\partial^*\widetilde{M}}\xi=1$, we get

$$\begin{aligned}\int_M\int_{\partial^*\widetilde{M}}\xi^{-1}\left|\nabla\xi\right|^2 &\leq \left(\int_M\int_{\partial^*\widetilde{M}}\xi^{-3}\left|\nabla\xi\right|^4\right)^{\frac{1}{2}}\left(\int_M\int_{\partial^*\widetilde{M}}\xi\right)^{\frac{1}{2}}\\ &= \left(\int_M\int_{\partial^*\widetilde{M}}\xi^{-3}\left|\nabla\xi\right|^4\right)^{\frac{1}{2}}.\end{aligned}$$

Combined with (4) and (1) this gives indeed that

$$4\lambda_1\left(\widetilde{M}\right)\leq\int_M\left(\int_{\partial^*\widetilde{M}}\xi^{-1}(x)\left|\nabla\xi\right|^2(x)\,d\nu(\xi)\right)dv_0\leq 4m^2,$$

as claimed.

Since by hypothesis $\lambda_1\left(\widetilde{M}\right)=m^2$, it follows that all inequalities used in this proof will be (pointwise) equalities on $\widetilde{M}$ for almost all $\xi\in\partial^*\widetilde{M}$. Indeed, this is true because everywhere in our proof the inequalities were proved by integrating on $\partial^*\widetilde{M}$ some inequalities at $x\in\widetilde{M}$ that hold for each $\xi\in\partial^*\widetilde{M}$.

Tracing back our argument, in [**M**] we proved that for $B=\frac{1}{2m}\log\xi$ we have the following equalities on $\widetilde{M}$:

$$\begin{aligned}\left|\nabla B\right| &= 1\\ Hess_B(X,Y) &= -g(X,Y)+g(\nabla B,X)\,g(\nabla B,Y)\\ &\quad -g(J\nabla B,X)\,g(J\nabla B,Y),\end{aligned}$$

where $Hess_B$ denotes the real Hessian of B.

From the work of Li and Wang [**L-W**] we know that in this case, if the manifold $\widetilde{M}$ has bounded curvature then it is isometric to $\mathbb{CH}^m$. This is always the case for our setting, since $\widetilde{M}$ covers a compact manifold, so its curvature is bounded. q.e.d.

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Department of Mathematics
University of California
Irvine, CA 92697
E-mail address: omuntean@math.uci.edu