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ON PARAMETERIZATIONS OF TEICHMÜLLER SPACES OF SURFACES WITH BOUNDARY

Ren Guo

Abstract

In [**5**], Luo introduced, for each real number λ, a new coordinate of the Teichmüller space of a surface with boundary: ψ_λ edge invariant. He proved that for $\lambda \geq 0$, the image of the Teichmüller space under ψ_λ edge invariant coordinate is an open cell. In this paper we verify his conjecture that for $\lambda < 0$, the image of the Teichmüller space is a bounded convex polytope.

1. Introduction

Suppose S is a compact connected surface of non-empty boundary and has negative Euler characteristic. It is well known that there are hyperbolic metrics with totally geodesic boundary on the surface S. Two such hyperbolic metrics are called *isotopic* if there is an isometry isotopic to the identity between them. The space of all isotopy classes of hyperbolic metrics on S, denoted by $T(S)$, is called the Teichmüller space of the surface S.

There are several known parameterizations of the Teichmüller spaces. In particular, using the 3-holed decomposition of a surface, Fenchel-Nielsen introduced the length-twist coordinate for $T(S)$. For more detail, see the book of Imayoshi & Taniguchi [**3**]. Using the Bonahon-Thurston shearing cocycles, Bonahon [**1**] produced a parametrization of the Teichmüller space $T(S)$. Analog to Penner's simplicial coordinate [**7**, **8**] of the decorated Teichmüller space, Ushijima [**9**], Luo [**4**] introduced the simplicial coordinate of the Teichmüller space $T(S)$. Recently, based on variational principles, Luo [**5**] introduced a family of coordinates of $T(S)$. To be more precise, for each real number λ, he introduced a ψ_λ edge invariant associated to a hyperbolic metric which turns out to be a coordinate of the Teichmüller space $T(S)$. When $\lambda \geq 0$, he proved that the image of the Teichmüller space under the coordinate is an open convex polytope independent of λ. Luo [**6**] conjectured that for $\lambda < 0$, the image of the Teichmüller space under ψ_λ

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edge invariant coordinate is a bounded convex polytope. The purpose of this paper is to verify this conjecture.

Let us begin by recalling the ψ_λ edge invariant coordinate introduced by Luo [**5**]. The coordinate depends on a fixed ideal triangulation of S. Recall that a colored hexagon is a hexagon with three non-pairwise adjacent edges labeled by red and the opposite edges labeled by black. Take a finite disjoint union of colored hexagons and identify all red edges in pairs by homeomorphisms. The quotient is a compact surface with non-empty boundary together with an ideal triangulation. The 2-cells in the ideal triangulation are quotients of the hexagons. The quotients of red edges (respectively black edges) are called the edges (respectively *B-arcs*) of the ideal triangulation. It it well known that every compact surface S of non-empty boundary and negative Euler characteristic admits an ideal triangulation.

In a hyperbolic metric, any hexagon in an ideal triangulation is isotopic (leaving the boundary of a surface invariant) to a hyperbolic right-angled hexagon. It is well known that a hyperbolic right-angled hexagon is determined up to isometry by the lengths of three red edges. Furthermore, for any $l_1, l_2, l_3 \in \mathbf{R}_{>0}$, there exists a unique colored hyperbolic right-angled hexagon whose three red edges have lengths l_1, l_2, l_3. For a proof, see Buser [**2**].

Given an ideally triangulated surface S with E the set of all edges, each hyperbolic metric d on S has a length coordinate $l_d : E \to \mathbf{R}_{>0}$ which assigns each edge e the length of the shortest geodesic arc homotopic to e relative to the boundary of S. On the other hand, given a function $l : E \to \mathbf{R}_{>0}$, we can produce a hyperbolic metric with totally geodesic boundary on S. This metric is constructed by making each 2-cell with red edges e_i, e_j, e_k a colored hyperbolic right-angled hexagon so that the lengths of the red edges are $l(e_i), l(e_j), l(e_k)$. Thus, the Teichmüller space $T(S)$ can be identified with the space $\mathbf{R}^E_{>0}$ by length coordinates.

In [**5**], Luo introduced the ψ_λ *edge invariant* of a hyperbolic metric as $\psi_\lambda : E \to \mathbf{R}$ defined by

$$\psi_\lambda(e) = \int_0^{\frac{a+b-c}{2}} \cosh^\lambda(t)dt + \int_0^{\frac{a'+b'-c'}{2}} \cosh^\lambda(t)dt$$

where e is an edge of an ideal triangulation shared by two hyperbolic right-angled hexagons and a, b, c, a', b', c' are lengths of the B-arcs labeled as in Figure 1. Now consider the map $\Psi_\lambda : T(S) \to \mathbf{R}^E$ sending a hyperbolic metric l to its ψ_λ edge invariant.

The following two theorems are proved in Luo [**5**]. The special case of $\lambda = 0$ was proved in Luo [**4**]. We use (S, T) to denote a surface S with an ideal triangulation T.

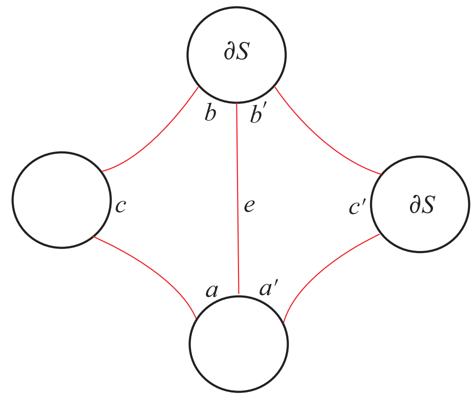

Figure 1. The lengths of B-arcs in the definition of ψ_λ edge invariant are labeled.

Theorem 1.1. *(Luo) Suppose (S, T) is an ideally triangulated surface. For any $\lambda \in \mathbf{R}$, the map $\Psi_\lambda : T(S) \to \mathbf{R}^E$ is a smooth embedding. In particular, each hyperbolic metric with geodesic boundary on (S, T) is determined up to triangulation preserving isometry by its ψ_λ edge invariant.*

An *edge path* $(H_0, e_1, H_1, ..., e_n, H_n)$ is a collection of hexagons and edges in an ideal triangulation so that two adjacent hexagons H_{i-1} and H_i sharing the edge e_i for $i = 1, ..., n$. An edge path $(H_0, e_1, H_1, ..., e_n, H_n)$ is an *edge cycle* if $H_0 = H_n$. For example see Figure 3. A *fundamental edge path (or fundament edge cycle)* is an edge path (or edge cycle) so that each edge in the ideal triangulation appears at most twice in the path (or cycle).

Theorem 1.2. *(Luo) Let $\lambda \geq 0$. For an ideal triangulated surface (S, T), $\Psi_\lambda(T(S)) = \{z \in \mathbf{R}^E |$ for each fundamental edge cycle $(H_0, e_1, H_1, ..., e_n, H_n = H_0)$, $\sum_{i=1}^n z(e_i) > 0\}$. Thus $\Psi_\lambda(T(S))$ is an open convex polytope independent of the parameter $\lambda \geq 0$.*

The main result of this paper is the verification of Luo's conjecture that, for $\lambda < 0$, the image of the Teichmüller space is a bounded open convex polytope.

Theorem 1.3. *Let $\lambda < 0$. For an ideal triangulated surface (S, T), $\Psi_\lambda(T(S))$ is the set of points $z \in \mathbf{R}^E$ satisfying*

1. $z(e) < 2 \int_0^\infty \cosh^\lambda(t) dt$ for each edge e;

2. $\sum_{i=1}^n z(e_i) > -2 \int_0^\infty \cosh^\lambda(t) dt$ for each fundamental edge path $(H_0, e_1, H_1, ..., e_n, H_n)$;

3. $\sum_{i=1}^n z(e_i) > 0$ for each fundamental edge cycle $(H_0, e_1, H_1, ..., e_n, H_n = H_0)$.

Thus $\Psi_\lambda(T(S))$ *is an open convex polytope. And*

$$\Psi_{\lambda_1}(T(S)) \subset \Psi_{\lambda_2}(T(S)) \subset \Psi_0(T(S))$$

for $\lambda_1 < \lambda_2 < 0$. *The intersection* $\cap_{\lambda=0}^{-\infty}\Psi_\lambda(T(S))$ *is empty.*

By definition, a single edge e_i together with two hexagons adjacent to e_i form an edge path. And a single edge e_i together with only one hexagon adjacent to e_i form an edge cycle. Thus the condition 2 requires that $z(e_i) > -2\int_0^\infty \cosh^\lambda(t)dt$ or the condition 3 requires that $z(e_i) > 0$. Therefor the image $\Psi_\lambda(\mathcal{T}(S))$ is contained in the box $(-c_\lambda, c_\lambda)^E \subset \mathbf{R}^E$, where $c_\lambda = 2\int_0^\infty \cosh^\lambda(t)dt$.

By the theorem above, we can see that the shape of the image of the Teichmüller space $T(S)$ is completely determined by the combinatorics the dual graph of the ideal triangulation T of the surface S.

The paper is organized as follows. In Section 2 we investigate degenerations of a hyperbolic right-angled hexagon. In Section 3 we prove the main result Theorem 1.3.

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2. Degenerations of a hyperbolic hexagon

In this section we always assume a hyperbolic right-angled hexagon has three non-pairwise adjacent edges of lengths l_1, l_2, l_3 and opposite B-arcs of lengths $\theta_1, \theta_2, \theta_3$ labeled in Figure 2. And recall that the r-coordinate is defined as $r_i = \frac{\theta_j+\theta_k-\theta_i}{2}$, where i, j, k are distinct.

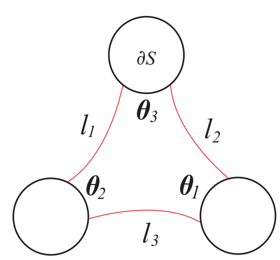

Figure 2. An hyperbolic right-angled hexagon with lengths of edges and B-arcs labeled.

We improve a lemma proved in Luo [**5**].

Lemma 2.1. *Consider* r_i *as a function of* (l_1, l_2, l_3). *We have* $\lim_{l_i\to 0} r_i = \infty$ *so that the convergence is uniform in* (l_1, l_2, l_3).

Proof. By the cosine law of a hyperbolic right-angled hexagon, we see that for $i \neq j \neq k \neq i$,

$$\begin{aligned}\cosh\theta_j &= \frac{\cosh l_j + \cosh l_i \cosh l_k}{\sinh l_i \sinh l_k} \\ &> \frac{\cosh l_i \cosh l_k}{\sinh l_i \sinh l_k} \\ &\geq \frac{\cosh l_i}{\sinh l_i}.\end{aligned}$$

Hence we have $\lim_{l_i \to 0} \theta_j = \infty$. Thus $\lim_{l_i \to 0} \frac{\cosh\theta_j}{\sinh\theta_j} = 1$. By symmetry we have $\lim_{l_i \to 0} \frac{\cosh\theta_k}{\sinh\theta_k} = 1$.

On the other hand, by the cosine law we see that for $i \neq j \neq k \neq i$,

$$\cosh l_i - \frac{\cosh\theta_j \cosh\theta_k}{\sinh\theta_j \sinh\theta_k} = \frac{\cosh\theta_i}{\sinh\theta_j \sinh\theta_k} > \frac{2e^{\theta_i}}{e^{\theta_j+\theta_k}} = \frac{2}{e^{2r_i}}.$$

Since the left hand side converges to 0 as $l_i \to 0$, we have $\lim_{l_i \to 0} r_i = \infty$.

To show the convergence is uniform, we consider the following formula called the *tangent law* derived in Luo [**5**]. For $i \neq j \neq k \neq i$,

$$\tanh^2 \frac{l_i}{2} = \frac{\cosh r_j \cosh r_k}{\cosh r_i \cosh(r_i + r_j + r_k)}.$$

By the formula,

$$\begin{aligned}&\tanh^2 \frac{l_i}{2} \\ =\ & \frac{1}{\cosh r_i} \cdot \frac{1}{(1 + \tanh r_j \tanh r_k)\cosh r_i + (\tanh r_j + \tanh r_k)\sinh r_i} \\ \geq\ & \frac{1}{\cosh r_i} \cdot \frac{1}{(1+1)\cosh r_i + (1+1)|\sinh r_i|} \\ \geq\ & \frac{1}{4\cosh^2 r_i}.\end{aligned}$$

It follows that

$$\cosh^2 r_i \geq \frac{1}{4\tanh^2 \frac{l_i}{2}}.$$

Thus r_i converges to ∞ uniformly. q.e.d.

Lemma 2.2. *(1) If (l_1, l_2, l_3) converges to (∞, f_1, f_2), where f_1 and f_2 are arbitrary positive numbers, then $(\theta_1, \theta_2, \theta_3)$ converges to (∞, f_3, f_4), where f_3 and f_4 are some positive numbers.*

(2) If (l_1, l_2, l_3) converges to (∞, ∞, f_5), where f_5 is an arbitrary positive number, then θ_3 converges to 0.

(3) If (l_1, l_2, l_3) converges to (∞, ∞, ∞), then we can chose a subsequence of (l_1, l_2, l_3) such that at least two of θ_1, θ_2 and θ_3 converge to 0.

Proof. (1) By the cosine law we have

$$\cosh\theta_1 = \frac{\cosh l_1 + \cosh l_2 \cosh l_3}{\sinh l_2 \sinh l_3},$$

if $\lim(l_1, l_2, l_3) = (\infty, f_1, f_2)$, we have $\lim\cosh\theta_1 = \infty$, or $\lim\theta_1 = \infty$. And since $\lim\frac{\cosh l_1}{\sinh l_1} = 1$,

$$\lim\cosh\theta_2 = \lim\frac{\cosh l_2 + \cosh l_1 \cosh l_3}{\sinh l_1 \sinh l_3} = \frac{\cosh f_2}{\sinh f_2} > 1.$$

Thus $\lim\theta_2$ is a positive finite number. By symmetry $\lim\theta_3$ is a positive finite number.

(2) If $\lim(l_1, l_2, l_3) = (\infty, \infty, f_5)$, we have

$$\lim\cosh\theta_3 = \lim\frac{\cosh l_3 + \cosh l_1 \cosh l_2}{\sinh l_1 \sinh l_2} = \lim\frac{\cosh l_3}{\sinh l_1 \sinh l_2} + 1 = 1.$$

Thus $\lim\theta_3 = 0$.

(3) If $\lim(l_1, l_2, l_3) = (\infty, \infty, \infty)$, we have

$$\lim\cosh\theta_i = \lim\frac{\cosh l_i + \cosh l_j \cosh l_k}{\sinh l_j \sinh l_k} = \lim\frac{\cosh l_i}{\sinh l_j \sinh l_k} + 1$$

$$= \lim\frac{2e^{l_i}}{e^{l_j+l_k}} + 1 = \lim 2e^{l_i-l_j-l_k} + 1.$$

Since $\lim e^{l_i-l_j-l_k}e^{l_j-l_i-l_k} = \lim e^{-2l_k} = 0$, by taking subsequence of (l_1, l_2, l_3), we may assume both $\lim e^{l_i-l_j-l_k}$ and $\lim e^{l_j-l_i-l_k}$ exist. Then one of $\lim e^{l_i-l_j-l_k}$ and $\lim e^{l_j-l_i-l_k}$ is 0. Hence at least two of $\lim\theta_1$, $\lim\theta_2$ and $\lim\theta_3$ are 0. q.e.d.

3. Proof of Theorem 1.3

Lemma 3.1. *If $a > 0$, then for any real number x, we have*

$$\int_0^{a+x}\cosh^\lambda(t)dt + \int_0^{a-x}\cosh^\lambda(t)dt > 0.$$

Proof. Let $f(a)$ be the function of the left hand side of the inequality. Since $f'(a) = \cosh^\lambda(a+x) + \cosh^\lambda(a-x) > 0$ and $f(0) = 0$, we have $f(a) > 0$ for $a > 0$. q.e.d.

Proof of Theorem 1.3. We denote the polytope defined by the inequalities in condition 1, 2, 3 by P_λ. First we claim $\Psi_\lambda(T(S)) \subset P_\lambda$. Indeed, fix a hyperbolic metric $l \in T(S)$. For any edge e, let r, r' be the r-coordinates of B-arcs facing e in the two hexagons adjacent to e, then

$$\psi_\lambda(e) = \int_0^r \cosh^\lambda(t)dt + \int_0^{r'}\cosh^\lambda(t)dt < 2\int_0^\infty \cosh^\lambda(t)dt.$$

Thus the condition 1 holds.

Given an edge path $(H_0, e_1, H_1, ..., e_n, H_n)$, for $i = 1, ..., n-1$, let a_i be the length of the B-arc in H_i adjacent to e_i and e_{i+1}. Denote the lengths of B-arcs in H_i facing e_i and e_{i+1} by b_i and c_i respectively as labelled in Figure 3 (a).

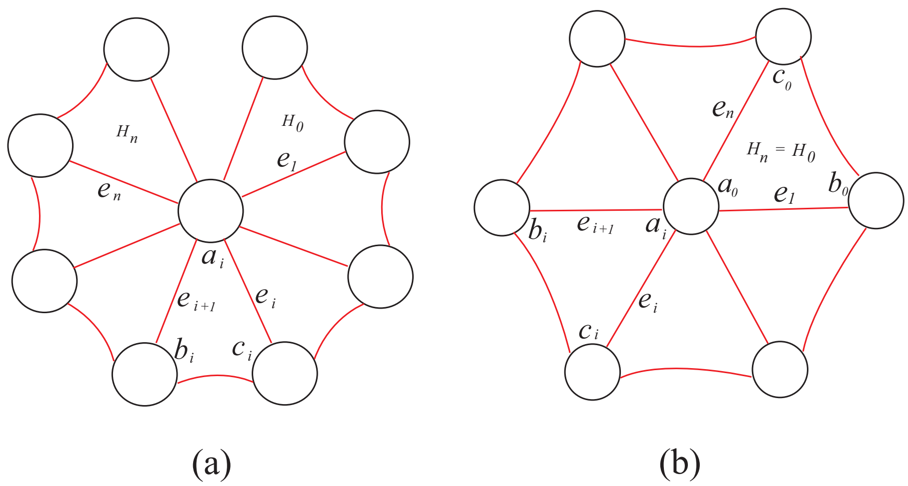

Figure 3. (a) An example of an edge path with lengths of B-arcs labeled. (b) An example of an edge cycle with lengths of B-arcs labeled.

Then by definition

$$\psi_\lambda(e_1) = \int_0^r \cosh^\lambda(t)dt + \int_0^{\frac{a_1+c_1-b_1}{2}} \cosh^\lambda(t)dt,$$

where r is the r-coordinate of the B-arc in H_0 facing e_1. For $i = 2, ..., n-1$,

$$\psi_\lambda(e_i) = \int_0^{\frac{a_{i-1}+b_{i-1}-c_{i-1}}{2}} \cosh^\lambda(t)dt + \int_0^{\frac{a_i+c_i-b_i}{2}} \cosh^\lambda(t)dt.$$

And

$$\psi_\lambda(e_n) = \int_0^{\frac{a_{n-1}+b_{n-1}-c_{n-1}}{2}} \cosh^\lambda(t)dt + \int_0^{r'} \cosh^\lambda(t)dt$$

where r' is the r-coordinate of the B-arc in H_n facing e_n.

Hence by Lemma 3.1,

$$\begin{aligned}\sum_{i=1}^{n}\psi_\lambda(e_i) &= \sum_{i=1}^{n-1}\Big(\int_0^{\frac{a_i+c_i-b_i}{2}}\cosh^\lambda(t)dt+\int_0^{\frac{a_i+b_i-c_i}{2}}\cosh^\lambda(t)dt\Big)\\ &+ \int_0^{r}\cosh^\lambda(t)dt+\int_0^{r'}\cosh^\lambda(t)dt\\ &> \int_0^{r}\cosh^\lambda(t)dt+\int_0^{r'}\cosh^\lambda(t)dt\\ &> 2\int_0^{-\infty}\cosh^\lambda(t)dt=-2\int_0^{\infty}\cosh^\lambda(t)dt.\end{aligned}$$

Thus the condition 2 holds.

Furthermore, if $(H_0, e_1, H_1, ..., e_n, H_n = H_0)$ is an edge cycle, H_0 contains both e_1 and e_n. Let a_0 be the length of B-arc in H_0 adjacent to e_1 and e_n, b_0, c_0 be the lengths of B-arcs facing e_n and e_0 respectively as labeled in Figure 3 (b). Thus the r-coordinates are $r = \frac{a_0+b_0-c_0}{2}$ and $r' = \frac{a_0+c_0-b_0}{2}$. Hence

$$\sum_{i=1}^{n}\psi_\lambda(e_i) > \int_0^{r}\cosh^\lambda(t)dt+\int_0^{r'}\cosh^\lambda(t)dt > 0$$

by Lemma 3.1. Thus the condition 3 holds.

Now by Theorem 1.1, $\Psi_\lambda : T(S) \to P_\lambda$ is an embedding. Therefore $\Psi_\lambda(T(S))$ is open in P_λ. We only need to show that $\Psi_\lambda(T(S))$ is closed in P_λ. This will finish the proof since P_λ is connected.

Take a sequence $l^{(m)} \in T(S)$ so that $\lim_{m\to\infty}\Psi_\lambda(l^{(m)}) = z \in P_\lambda$. By taking subsequence, we may assume that $\lim_{m\to\infty} l^{(m)} \in [0,\infty]^E$ exists and the length of each B-arc converges into $[0,\infty]$. We only need to show that $\lim_{m\to\infty} l^{(m)} \in (0,\infty)^E$. This will finish the proof since $z = \Psi_\lambda(\lim_{m\to\infty} l^{(m)})$.

Suppose otherwise that there is an edge $e \in E$ so that $\lim_{m\to\infty} l^{(m)}(e) \in \{0,\infty\}$. We will discuss two cases.

Case 1, $\lim_{m\to\infty} l^{(m)}(e) = 0$ for some $e \in E$. Let H, H' be the hexagons sharing e and $r^{(m)}, r'^{(m)}$ be the r-coordinates of the B-arcs in H, H' facing e. Then by Lemma 2.1, $\lim_{m\to\infty} r^{(m)} \to \infty$, $\lim_{m\to\infty} r'^{(m)} \to \infty$. Then

$$z(e) = \lim_{m\to\infty}\left(\int_0^{r^{(m)}}\cosh^\lambda(t)dt+\int_0^{r'^{(m)}}\cosh^\lambda(t)dt\right) = 2\int_0^{\infty}\cosh^\lambda(t)dt.$$

This is impossible since $z \in P_\lambda$ must satisfy the condition 1.

Due to case 1, we can assume $\lim_{m\to\infty} l^{(m)} \in (0,\infty]^E$.

Case 2, $\lim_{m\to\infty} l^{(m)}(e) = \infty$ for some $e \in E$. Define the subset

$$E_\infty = \{e \in E | \lim_{m\to\infty} l^{(m)}(e) = \infty\}.$$

We construct a graph G as follows. A vertex of G is a hexagon with at least one edge in E_∞. There is a *dual-edge* in G joining two vertexes if and only if the two hexagons corresponding to the vertexes share an edge in E_∞. The degree of a vertex of the graph G can only be 1, 2 or 3. Actually a vertex of degree 1, 2 or 3 is corresponding to the hexagon of type (1),(2) or (3) in Lemma 2.2 respectively.

We smooth the graph G at vertexes as follows. At a vertex of degree 1, we replace the small neighborhood of the vertex in G by a short smooth curve tangent to the unique dual-edge incident to the vertex as in Figure 4 (a). At a vertex v of degree of 2 or 3, every two dual-edges $\overline{e}_1, \overline{e}_2$ incident to v correspond to two edges e_1, e_2 in a hexagon. If the length of the B-arc adjacent to e_1, e_2 converges to 0, we replace the small neighborhood of the vertex v in G by a short smooth curve tangent to $\overline{e}_1, \overline{e}_2$. According to Lemma 2.2, every vertex of degree 2 can be smoothed as in Figure 4 (b) and there are two cases for a vertex of degree 3 according to the lengths of 2 or 3 B-arcs converge to 0 as in Figure 4 (c).

We denote by G' the graph smoothed at vertexes and the dual-edges of G' are the dual-edges of G. We claim that there exists a smooth closed curve in G' such that every dual-edge repeats at most twice in the closed curve. In fact, we give every dual-edge of G' an arbitrary orientation. Pick up any smooth closed curve in G' which may contains infinite number of dual-edges. If there exists an dual-edge $\overline{e}$ repeats with the same orientation in the closed curve, there is another smooth closed curve starting and ending at $\overline{e}$. By this procedure we can reduce the number of dual-edges of a closed curve. At last we obtain a smooth closed curve in G' such that every dual-edge repeats at most twice.

This smooth closed curve in G' corresponds a fundamental edge path or fundamental edge cycle in the ideal triangulation. First assume it is a fundamental edge path $(H_0, e_1, H_1, ..., e_n, H_n)$. Since the degree of the vertex corresponding to H_0 (or H_n) is 1, the lengths of other two edges other than e_1 (or e_n) converge to positive finite numbers in the sequence of metric $l^{(m)}$. By Lemma 2.2(1) the r-coordinate of the B-arc in H_0 (or H_n) facing e_1 (or e_n) converges to $-\infty$. By the construction of the edge path, the length of B-arc adjacent to e_i and e_{i+1} converges to 0 for $i = 1, ..., n-1$. And we denote b_i, c_i the limit of lengths of B-arcs in H_i facing e_i, e_{i+1} respectively, see Figure 3(a).

Hence

$$z(e_1) = \int_0^{-\infty} \cosh^\lambda(t)dt + \int_0^{\frac{c_1-b_1}{2}} \cosh^\lambda(t)dt.$$

For $i = 2, ..., n-1$,

$$z(e_i) = \int_0^{\frac{b_{i-1}-c_{i-1}}{2}} \cosh^\lambda(t)dt + \int_0^{\frac{c_i-b_i}{2}} \cosh^\lambda(t)dt.$$

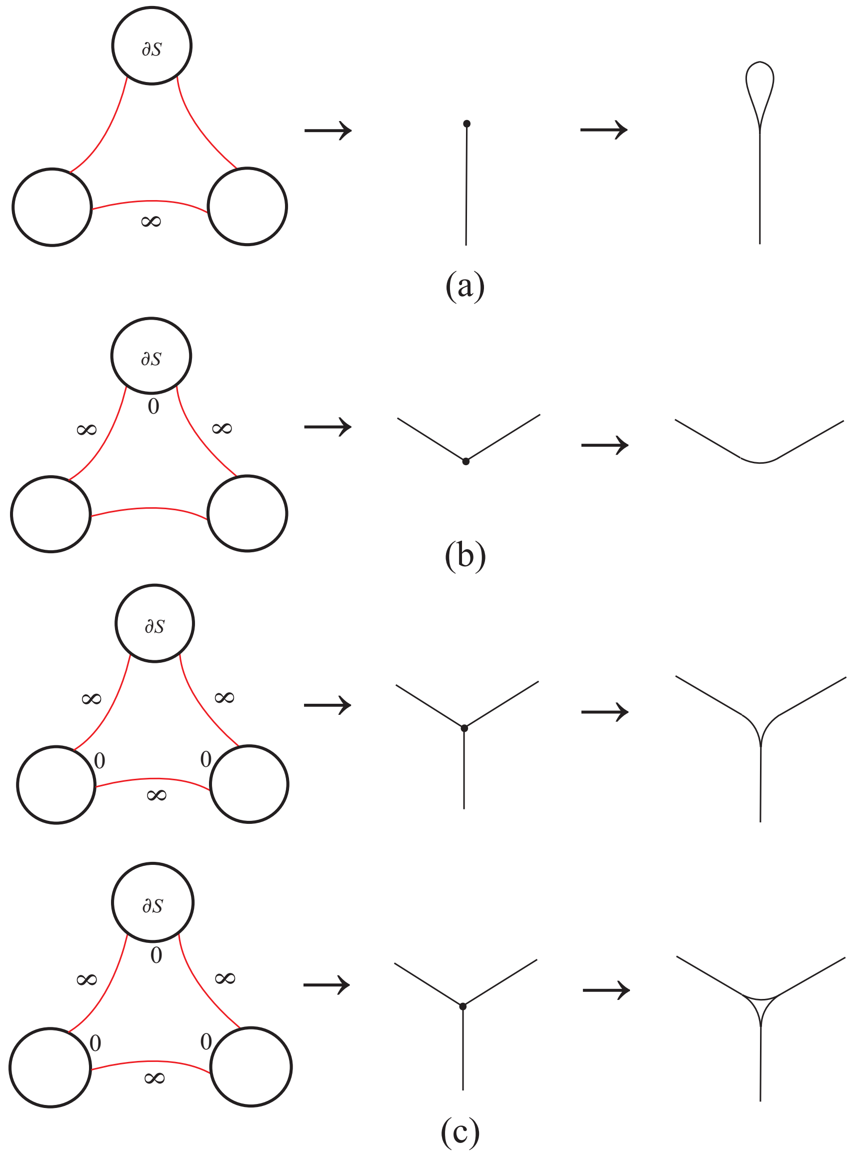

Figure 4. Smooth graph G at a vertex of degree 1 (a), degree 2 (b), degree 3 (c).

And

$$z(e_n) = \int_0^{\frac{b_{n-1}-c_{n-1}}{2}} \cosh^{\lambda}(t)dt + \int_0^{-\infty} \cosh^{\lambda}(t)dt.$$

Hence

$$
\begin{aligned}
&\sum_{i=1}^{n} z(e_i) \\
= \;& 2\int_0^{-\infty} \cosh^\lambda(t)dt + \sum_{i=1}^{n-1}\left(\int_0^{\frac{c_i-b_i}{2}} \cosh^\lambda(t)dt + \int_0^{\frac{b_i-c_i}{2}} \cosh^\lambda(t)dt\right) \\
= \;& -2\int_0^{\infty} \cosh^\lambda(t)dt,
\end{aligned}
$$

where the cancelation of the two terms in the bracket requires the condition $\lambda < 0$, since the two terms are finite when $\lambda < 0$. This is impossible since $z \in P_\lambda$ must satisfy the condition 2.

If the smooth closed curve in G' corresponds to a fundamental edge cycle $(H_0, e_1, H_1, ..., e_n, H_n = H_0)$, the length of B-arc in H_0 adjacent to e_1 and e_n is 0. Denote b_0, c_0 the lengths of B-arcs facing e_n and e_0, see Figure 3(b). Thus

$$z(e_1) = \int_0^{\frac{b_0-c_0}{2}} \cosh^\lambda(t)dt + \int_0^{\frac{c_1-b_1}{2}} \cosh^\lambda(t)dt,$$

$$z(e_n) = \int_0^{\frac{b_{n-1}-c_{n-1}}{2}} \cosh^\lambda(t)dt + \int_0^{\frac{c_0-b_0}{2}} \cosh^\lambda(t)dt.$$

And as in the case of fundamental edge path, for $i = 2, ..., n-1$,

$$z(e_i) = \int_0^{\frac{b_{i-1}-c_{i-1}}{2}} \cosh^\lambda(t)dt + \int_0^{\frac{c_i-b_i}{2}} \cosh^\lambda(t)dt.$$

Hence $\sum_{i=1}^n z(e_i) = 0$. This is impossible since $z \in P_\lambda$ must satisfy the condition 3.

We finish the proof of $\Psi_\lambda(T(S)) = P_\lambda$. Since there are only finite many fundament edge paths or fundament edge cycles in an ideal triangulation, $\Psi_\lambda(T(S))$ is defined by finite many inequalities in condition 1, 2, 3. Thus it is a open convex polytope.

The statement $\Psi_{\lambda_1}(T(S)) \subset \Psi_{\lambda_2}(T(S)) \subset \Psi_0(T(S))$ for $\lambda_1 < \lambda_2 < 0$ is obvious since the function $\int_0^\infty \cosh^\lambda(t)dt$ is increasing in λ and it is ∞ when $\lambda \geq 0$.

Since $0 < \cosh^\lambda(t) < \cosh^{-1}(t)$ for $\lambda < -1$ and $\int_0^\infty \cosh^{-1}(t)dt < \infty$, by Lebesgue's dominated convergence theorem, we have

$$\lim_{\lambda\to-\infty} \int_0^\infty \cosh^\lambda(t)dt = \int_0^\infty \lim_{\lambda\to-\infty} \cosh^\lambda(t)dt = 0.$$

Thus the intersection $\cap_{\lambda=0}^{-\infty}\Psi_\lambda(T(S))$ is the set of points $z \in \mathbf{R}^E$ satisfying $z(e) < 0$ for each edge e and $\sum_{i=1}^n z(e_i) > 0$ for each fundamental edge path $(H_0, e_1, H_1, ..., e_n, H_n)$. It is an empty set. q.e.d.

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Department of Mathematics
Rutgers University
Piscataway NJ 08854

Current address:
School of Mathematics
University of Minnesota
Minneapolis MN 55455

E-mail address: guoxx170@math.umn.edu