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A COMPACTNESS THEOREM FOR THE YAMABE PROBLEM

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Abstract

In this paper, we prove compactness for the full set of solutions to the Yamabe Problem if $n \leq 24$. After proving sharp pointwise estimates at a blowup point, we prove the Weyl Vanishing Theorem in those dimensions, and reduce the compactness question to showing positivity of a quadratic form. We also show that this quadratic form has negative eigenvalues if $n \geq 25$.

1. Introduction

Let (M^n, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$ without boundary. The Yamabe Problem consists of finding a constant scalar curvature metric $\widetilde{g}$, which is pointwise conformally related to g. This problem is equivalent to showing the existence of a positive solution to the equation

$$\Delta_g u - c(n) R_g u + K u^{\frac{n+2}{n-2}} = 0 \quad \text{on} \quad M, \tag{1.1}$$

where Δ_g is the Laplace-Beltrami operator associated with g, R_g is the scalar curvature of g, $c(n) = \frac{n-2}{4(n-1)}$, and K is a constant. More precisely, if $u > 0$ is a solution of (1.1) and we write $\widetilde{g} = u^{\frac{4}{n-2}} g$, then the scalar curvature of $\widetilde{g}$ is given by $c(n)^{-1} K$. As is well-known, the existence of a minimizing solution to the Yamabe problem was established through the combined works of Yamabe [**37**], Trudinger [**35**], Aubin [**1**], and Schoen [**26**].

The variational theory for nonminimal solutions of the Yamabe Problem is very rich. Examples such as $S^1 \times S^{n-1}$ (see [**28**]) show that there are generally a large number of high energy solutions with high Morse index. In fact, a theorem of Pollack [**25**] shows that on every compact manifold M^n with $n \geq 3$ and with positive scalar curvature, for any integer N there is a dense set (in a C^0 topology) of the positive conformal classes for which there are more than N inequivalent solutions of the Yamabe problem. These solutions generally have high energy and index. Thus, it is natural to ask what can be said about the full set of solutions to this problem.

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In a topics course at Stanford in 1988, the third author raised the question of compactness of the full set of solutions and proved some special cases of it. Notes for that course were written by D. Pollack, and although unpublished, they have been widely distributed. The cases which were covered in the Stanford notes are the locally conformally flat case, published in [**29**] and the three dimensional case, the argument for which is in the paper of Schoen and Zhang [**34**] (used there to establish a single simple point of blow-up for the prescribed scalar curvature problem on S^3). For a three manifold different from S^3, the positivity of the mass gives the full compactness statement by a virtually identical argument. The notes also give an outline of a proof that the Weyl tensor should vanish at a blow-up point. Over the past several years the results were improved by Druet [**9**] who did the cases $n \leq 5$ and by Marques [**19**] and Li and Zhang [**15**] who did $n \leq 7$ in general and arbitrary n under the assumption that the Weyl tensor vanishes nowhere to second order. We have recently become aware that Li and Zhang do $n \leq 11$ in [**16**]. In a surprising recent paper, Simon Brendle [**4**] has constructed examples of C^∞ metrics on spheres of dimension at least 52 for which the compactness statement fails. In a subsequent paper, Brendle and Marques [**6**] extend these examples to the dimensions $25 \leq n \leq 51$. In this paper, we give a proof of the statements of [**30**] for $n \leq 24$. We now describe these results in detail.

First, recall that if the principal eigenvalue of minus the conformal Laplacian, $-L_g = -\Delta_g + c(n)R_g$, is negative, then there is only one solution of (1.1). Moreover, if the principal eigenvalue is zero, then the problem becomes linear and solutions are unique up to multiplication by a positive constant. Therefore, the only interesting case is the positive one. If the underlying manifold is the round sphere (S^n, g_0), then Obata's theorem [**23**] classifies all solutions of (1.1), and this set is noncompact in the C^2 topology. On the other hand, the round sphere is the only compact manifold on which there can be a noncompact space of minimizing solutions, and in this paper, we extend this result to the space of arbitrary solutions for the Yamabe Problem, as well as its subcritical approximate problems provided $n \leq 24$. For any $p \in [1, \frac{n+2}{n-2}]$, set

$$\Phi_p = \{u > 0 \mid L_g u + K u^p = 0 \;\text{ on }\; M\},$$

then our main result is

Theorem 1.1. *Let (M^n, g) be a smooth compact Riemannian manifold of dimension $3 \leq n \leq 24$ without boundary. If (M^n, g) has positive Yamabe quotient and is not conformally diffeomorphic to (S^n, g_0), then for any $\varepsilon > 0$ there exists a constant $C > 0$ depending only on g and ε such that*

$$C^{-1} \leq u \leq C \quad \text{and} \quad \| u \|_{C^{2,\alpha}} \leq C,$$

for all $u \in \cup_{1+\varepsilon \le p \le \frac{n+2}{n-2}} \Phi_p$*, where* $0 < \alpha < 1$.

As in the construction of minimizers for the Yamabe problem, the proof of this theorem has both a local and a global aspect. The global aspect involves the Positive Mass Theorem (PMT). Recall that this theorem was established by Schoen and Yau [**31**] using a minimal hypersurface argument. This argument extends to the cases $n \le 7$ (see [**28**] for a discussion). A spinor proof of the theorem was provided by Witten [**36**] (see also [**13**]) in an argument which extends to arbitrary n assuming the manifold is spin. A special argument for the locally conformally flat case was given in [**32**] using the developing map. Our current theorem requires a more general PMT for high dimensional manifolds which are not locally conformally flat (and not necessarily spin). This theorem has been announced by Lohkamp [**18**] by an argument which extends the minimal hypersurface argument. In order to apply the PMT to get estimates for solutions of the Yamabe Problem, the Weyl tensor must vanish to a sufficiently high order so that the ADM energy can be defined. That is the main difficulty of the program, and the next result asserts that this is necessarily the case, if $n \le 24$. More precisely, we establish the following local result.

Theorem 1.2 (Weyl Vanishing Theorem)**.** *Let* g *be a smooth Riemannian metric defined in the unit* n*-ball* B_1, $6 \le n \le 24$*. Suppose that there is a sequence of solutions* $\{u_i\}$ *of*

$$L_g u_i + K u_i^{p_i} = 0 \quad \text{on} \quad B_1, \tag{1.2}$$

$p_i \in (1, \frac{n+2}{n-2}]$*, such that for any* $\varepsilon > 0$ *there is a constant* $C(\varepsilon)$ *such that* $\sup_{B_1 - B_\varepsilon} u_i \le C(\varepsilon)$ *and* $\lim_{i\to\infty}(\sup_{B_1} u_i) = \infty$*. Then the Weyl tensor* $W(g)$ *satisfies*

$$|W(g)|(x) \le C|x|^l$$

for some integer $l > \frac{n-6}{2}$*.*

We should also note that our compactness theorem holds without assuming the Positive Mass Theorem for the class of manifolds (M^n, g) satisfying

$$\sum_{k=0}^{\frac{n-6}{2}} |\nabla_g^k W_g(x)|^2 > 0$$

for all $x \in M$.

The proof of this theorem in the case of an isolated simple blow-up point relies on a new method to obtain local restrictions at a blowup point. The first step is to obtain sharp approximations of a blowing-up sequence of solutions in a neighborhood of the blowup point. This is achieved by establishing optimal pointwise estimates which generalize the ones obtained by the second author in [**19**]. The important point here is that in high dimensions one has to perform a refined blowup analysis, going beyond the standard bubble (rotationally symmetric solution). We should say that the approximate solutions introduced here were introduced by S. Brendle in [**3**] to generalize the results of Aubin [**1**] and of Hebey and Vaugon [**11**]. After we establish this kind of expansion for the blowing-up solutions, we can apply a Pohozaev-type identity in search of local obstructions. The relevant correction terms coming from the metric are then encoded in a quadratic form, whose positivity is sufficient to imply our stated results.

In the Appendix, we prove the positivity of this quadratic form if $n \leq 24$, noting also that this fails to be true if $n \geq 25$. This fact has been used by S. Brendle ($n \geq 52$, [**4**]), and Brendle and Marques ($25 \leq n \leq 51$, [**6**]) to give smooth counterexamples to compactness. We should say that the existence of a critical dimension for this problem comes as an interesting and surprising fact, which deserves to be better understood (see [**5**] for a survey). We note here that checking the positivity involves a calculation which is too lengthy to do by hand so we carried out this calculation in Maple. The Maple instructions and output of this calculation are posted at [**12**] for the interested reader to check.

The final step of the proof involves the reduction to the case of simple blow-up. This involves suitable rescalings to a situation in which the blow-up is simple for a sequence of metrics converging to the euclidean metric and for which the corresponding boundary term in the Pohozaev identity is positive. Combining the analysis described above to show that the interior terms are negligible, we rule out higher energy blow-up for $3 \leq n \leq 24$. We note that this is a local result, so that we can show that locally the blow-up set consists of a finite number of simple blow-up points. This generalizes results obtained by the third author in the locally conformally flat case.

One obvious consequence of Theorem 1.1 is to give an alternative proof of the solution to the Yamabe problem. This follows from the fact that standard variational methods can be used to give solutions to the subcritical equation (1.2) with $p_i \to \frac{n+2}{n-2}$ as $i \to \infty$. In fact, we can say more. More generally, the compactness theorem allows us to compute the total Leray-Schauder degree of all solutions to equation (1.1), and to obtain more refined existence theorems which we now discuss.

In this setting, it is convenient to recall that equation (1.1) arises from a variational problem. Namely, we consider the functional $\mathcal{R}(\widetilde{g})$ for $\widetilde{g} \in [g]$, the conformal class of the given metric g, where $\mathcal{R}(\widetilde{g})$ is the total scalar curvature

$$\mathcal{R}(\widetilde{g}) = \int_M R_{\widetilde{g}}\, d\omega_{\widetilde{g}}.$$

Here we have normalized $\widetilde{g}$ and g so that $\mathrm{Vol}(\widetilde{g}) = \mathrm{Vol}(g) = 1$. The critical points of $\mathcal{R}(\cdot)$ on the set of metrics $\widetilde{g} \in [g]$ with $\mathrm{Vol}(\widetilde{g}) = \mathrm{Vol}(g)$ then have constant scalar curvature, and by writing $\widetilde{g} = u^{\frac{4}{n-2}} g$, we get solutions u of equation (1.1) with $K = c(n)\mathcal{R}(u^{\frac{4}{n-2}}g)$. Note that by writing equation (1.2) as

$$L_g u + E(u)u^p = 0, \tag{1.3}$$

where $E(u)$ denotes the energy of u and is given by

$$E(u) = -\int_M uL_g u d\omega_g = \int_M (|\nabla_g u|^2 + c(n)R_g u^2) d\omega_g,$$

the volume constraint $\mathrm{Vol}(\widetilde{g}) = \mathrm{Vol}(g)$ is built into equation (1.3) (for $p = \frac{n+2}{n-2}$) since multiplying (1.3) by u and integrating by parts produces $\int_M u^{p+1} d\omega_g = 1$.

Since the questions which concern us here are only interesting in the case that the principal eigenvalue of minus the conformal Laplacian is positive, we may assume that it is invertible. We then define a map $F_p : \overline{\Omega}_\Lambda \to C^{2,\alpha}(M)$ by $F_p(u) = u + L_g^{-1}(E(u)u^p)$, where

$$\Omega_\Lambda = \{u \in C^{2,\alpha}(M) \mid \ \| u \|_{C^{2,\alpha}} < \Lambda, \quad u > \Lambda^{-1}\}.$$

From elliptic theory, we know that the map $u \mapsto L_g^{-1}(E(u)u^p)$ is a compact map from $\overline{\Omega}_\Lambda$ into $C^{2,\alpha}(M)$. Thus F_p is of the form I+compact, and we may define the Leray-Schauder degree (see [**21**]) of F_p in the region Ω_Λ with respect to $0 \in C^{2,\alpha}(M)$, denoted by $\deg(F_p, \Omega_\Lambda, 0)$, provided that $0 \neq F_p(\partial\Omega_\Lambda)$. The degree is an integer which counts with multiplicity the number of times that the value 0 is taken on by the map F_p. Notice that $F_p(u) = 0$ if and only if u is a solution of (1.3). Furthermore, the homotopy invariance of the degree tells us that $\deg(F_p, \Omega_\Lambda, 0)$ is constant for all $p \in [1, \frac{n+2}{n-2}]$ provided that $0 \neq F_p(\partial\Omega_\Lambda)$ for all $p \in [1, \frac{n+2}{n-2}]$. Moreover, in the linear case when $p = 1$, it is not difficult to calculate (as shown in [**29**]) that $\deg(F_1, \Omega_\Lambda, 0) = -1$ for all Λ sufficiently large. Therefore, Theorem 1.1 allows us to calculate the degree for all $p \in [1, \frac{n+2}{n-2}]$.

Theorem 1.3. *Let (M^n, g) satisfy the assumptions of Theorem* 1.1*, then for all Λ sufficiently large and all $p \in [1, \frac{n+2}{n-2}]$, we have*

$$\deg(F_p, \Omega_\Lambda, 0) = -1.$$

In the case that all solutions of the Yamabe problem are nondegenerate, as will be the case for a generic conformal class of Riemannian metrics, our previous results assert that there will be a finite number of solutions of the variational problem. Moreover, the strong Morse inequalities will hold for the Yamabe problem since these inequalities are well-known for subcritical equations, and Theorem 1.1 shows that all critical points converge as $p \to \frac{n+2}{n-2}$. It follows that

$$(-1)^\lambda \leq \sum_{\mu=0}^{\lambda} (-1)^{\lambda-\mu} C_\mu, \qquad \lambda = 0, 1, 2, \ldots,$$

where C_μ denotes the number of solutions of Morse index μ. Since there is a finite number of solutions, we then obtain

Theorem 1.4. *Let (M^n, g) satisfy the assumptions of Theorem 1.1, and suppose that all critical points in $[g]$ are nondegenerate. Then there is a finite number of critical points $g_1, \dots, g_k$, and we have*

$$1 = \sum_{j=1}^{k} (-1)^{I(g_j)},$$

where $I(g_j)$ denotes the Morse index of the variational problem with volume constraint.

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2. Basic Properties of Blowup

Let $D \subset \mathbb{R}^n$ be an open set endowed with a sequence of smooth Riemannian metrics $\{g_i\}_{i=1}^{\infty}$ converging in the C^N_{loc} topology to a smooth metric g, where N is large enough depending only on n. Consider a sequence of positive solutions $\{u_i\}_{i=1}^{\infty}$ of the equation

$$L_{g_i} u_i + K f_i^{-\delta_i} u_i^{p_i} = 0 \quad \text{in} \quad D, \tag{2.1}$$

where $1 < p_i \leq \frac{n+2}{n-2}$, $\delta_i = \frac{n+2}{n-2} - p_i$, $K = n(n-2)$, and $\{f_i\}_{i=1}^{\infty}$ is a sequence of smooth positive functions converging in the C^2_{loc} topology to a smooth positive function f. Later on it will be convenient to replace g_i by another member of its conformal class, $\hat{g}_i = \phi^{\frac{4}{n-2}} g_i$. In this case, we find

$$L_{\hat{g}_i}(\phi^{-1} u_i) = \phi^{-\frac{n+2}{n-2}} L_{g_i} u_i,$$

from which it follows that $\phi^{-1} u_i$ satisfies

$$L_{\hat{g}_i}(\phi^{-1} u_i) + K(\phi f_i)^{-\delta_i} (\phi^{-1} u_i)^{p_i} = 0,$$

which is an equation having the same form as (2.1). This is the reason for writing the subcritical equations as in (2.1).

A point $\overline{x} \in D$ is called a blow-up point for $\{u_i\}$ if $u_i(x_i) \to \infty$ for some $x_i \to \overline{x}$. The behavior of the sequence $\{u_i\}$ in a neighborhood of a blow-up point has been studied extensively; the following definitions are commonly used in this regard.

Definition 2.1. A point $\overline{x} \in D$ is an isolated blow-up point for $\{u_i\}$ if there exists a sequence $\{x_i\} \subset D$ where each x_i is a local maximum for u_i and $x_i \to \overline{x}$, such that:

1) $u_i(x_i) \to \infty$ as $i \to \infty$,
2) $u_i(x) \leq C d_{g_i}(x, x_i)^{-\frac{2}{p_i - 1}}$ for all $x \in B_R(x_i) \subset D$

for some constants R, $C > 0$, where $B_R(x_i)$ denotes the geodesic ball of radius R centered at x_i, and $d_{g_i}(x, x_i)$ represents the distance between x and x_i with respect to g_i.

Definition 2.2. Let $\{x_i\}$ and $\{u_i\}$ be as in Definition 2.1, and consider the spherical averages

$$\overline{u}_i(r) = |\partial B_r(x_i)|_{g_i}^{-1} \int_{\partial B_r(x_i)} u_i d\sigma_{g_i},$$

where $|\partial B_r(x_i)|_{g_i}$ denotes the area of $\partial B_r(x_i)$ with respect to g_i. Then $\overline{x} \in D$ is called an isolated simple blow-up point for $\{u_i\}$ if there exists $R > 0$ (independent of i) such that the functions $\widehat{u}_i(r) = r^{\frac{2}{p_i - 1}} \overline{u}_i(r)$ have exactly one critical point for $r \in (0, R)$.

We will now proceed to recall the basic properties of isolated and isolated simple blow-up. All the results of this section are originally due to Schoen [**27**] (but have appeared elsewhere, see [**14**], [**15**], [**19**]). We will not include the proofs here since they are by now standard.

The first result shows that near an isolated blow-up point, solutions of (2.1) satisfy a Harnack inequality. Unless otherwise noted, throughout this section we will be working in a normal coordinate system $x = (x^1, \ldots, x^n)$ centered at x_i, where $\{x_i\}$ is as in Definition 2.1. We will also denote $u_i(\exp_{x_i}(x))$ by $u_i(x)$ and $d_{g_i}(x, x_i)$ by $|x|$.

Proposition 2.3. *Let $x_i \to \overline{x}$ be an isolated blow-up point for the sequence $\{u_i\}$ of positive solutions to* (2.1)*. Then there exists $C > 0$ independent of i such that*

$$\max_{r \leq |x| \leq 2r} u_i(x) \leq C \min_{r \leq |x| \leq 2r} u_i(x)$$

for all $r \in (0, R/3)$.

We can now apply this Harnack inequality to obtain further information on the behavior of $\{u_i\}$ near an isolated blow-up point. Define $U(y) = (1 + |y|^2)^{\frac{2-n}{2}}$, and observe that if $\pi : \mathbb{S}^n - \{\infty\} \to \mathbb{R}^n$ is stereographic projection, then $(\pi^{-1})^* g_0 = 4U^{\frac{4}{n-2}} \delta$ where g_0 is the round metric on $\mathbb{S}^n$ and δ is the Euclidean metric on $\mathbb{R}^n$. It is for this reason that $U(y)$ is often referred to as the "standard bubble". The next lemma asserts that in the case of isolated blow-up, after suitable rescaling, the sequence $\{u_i\}$ approaches a standard bubble.

Proposition 2.4. *Let $x_i \to \overline{x}$ be an isolated blow-up point for the sequence $\{u_i\}$ of positive solutions to* (2.1)*. Then $p_i \to \frac{n+2}{n-2}$, and for any sequences $\epsilon_i \to 0$ and $R_i \to \infty$ as $i \to \infty$, we have*

$$|M_i^{-1} u_i(M_i^{-\frac{p_i-1}{2}} y) - U(y)|_{C^2(B_{R_i}(0))} \leq \epsilon_i$$

where $M_i = u_i(x_i)$ after possibly passing to a subsequence.

This result shows that isolated point blow-up gives good control in small balls of radius $R_i M_i^{-\frac{p_i-1}{2}}$. The next result says that in the case that the blow-up is isolated simple, control may be extended into a ball of fixed radius $R > 0$ independent of i. In particular, the blow-up is analogous to that of the Green's function for the conformal Laplacian centered at x_i. Recall that if R is sufficiently small, then there exists a unique Green's function $G_i(\cdot, x_i) \in C^2(\overline{B_R(x_i)} - \{x_i\})$ satisfying

$$L_{g_i} G_i = 0 \text{ in } B_R(x_i) - \{x_i\}, \quad G|_{\partial B_R(x_i)} = 0, \quad \lim_{x \to x_i} |x|^{n-2} G_i(x) = 1.$$

Proposition 2.5. *Let $x_i \to \overline{x}$ be an isolated simple blow-up point for the sequence $\{u_i\}$ of positive solutions to* (2.1)*. Then there exist constants C, $R > 0$ independent of i such that*

$$M_i u_i(x) \geq C^{-1} G_i(x, x_i), \quad M_i^{-\frac{p_i-1}{2}} \leq |x| \leq R, \tag{2.2}$$

$$M_i u_i(x) \leq C|x|^{2-n}, \quad |x| \leq R, \tag{2.3}$$

where $G_i(x, x_i)$ is the Green's function for L_{g_i} with Dirichlet boundary conditions in $B_R(x_i)$.

An immediate consequence of Proposition 2.5 is that $M_i u_i$ converges to a Green's function at an isolated simple blow-up point. To see this, observe that

$$L_{g_i}(M_i u_i(x)) + K M_i^{1-p_i} f_i^{-\delta_i} (M_i u_i(x))^{p_i} = 0,$$

and by (2.3) $M_i u_i$ is uniformly bounded on compact subsets of $B_R(\overline{x}) - \{\overline{x}\}$. It then follows from standard elliptic estimates that after passing to a subsequence $M_i u_i(x) \to G(x, \overline{x})$ in $C^2_{\text{loc}}(B_R(\overline{x}) - \{\overline{x}\})$, where $L_g G = 0$. By (2.2), $G(x, \overline{x})$ must have a nonremovable singularity at $\overline{x}$. Therefore, we have shown

Corollary 2.6. *Let $x_i \to \overline{x}$ be an isolated simple blow-up point for $\{u_i\}$. Then after passing to a subsequence*

$$M_i u_i(x) \to h(x) = G(x, \overline{x}) \quad \text{in} \quad C^2_{\text{loc}}(B_R(\overline{x}) - \{\overline{x}\}),$$

where $G(x, \overline{x})$ is a Green's function for L_g centered at $\overline{x}$.

3. Pohozaev Identity

In this section, we will establish the Pohozaev-type identity we will use in the subsequent blowup analysis.

Suppose $u : B_\sigma(0) \subset \mathbb{R}^n \to \mathbb{R}$ is a positive C^2 solution to the equation

$$\Delta u + K(x)u^p = -A(x), \tag{3.1}$$

where $p \neq -1$, Δ denotes the Euclidean Laplacian, and $K \in C^1$.

Define

$$P(r,u) = \int_{\partial B_r} \Big(\frac{n-2}{2} u \frac{\partial u}{\partial r} - \frac{r}{2}|\nabla u|^2 + r|\frac{\partial u}{\partial r}|^2 + \frac{1}{p+1} K(x) r u^{p+1} \Big) d\sigma(r), \tag{3.2}$$

whenever $0 < r < \sigma$.

The following lemma gives the radial Pohozaev-type identity:

Lemma 3.1. *Given* $0 < r < \sigma$*,*

$$\begin{aligned} P(r,u) = &- \int_{B_r} \left(x^k \partial_k u + \frac{n-2}{2} u \right) A(x) dx \\ &+ \left(\frac{n}{p+1} - \frac{n-2}{2} \right) \int_{B_r} K(x) u^{p+1} dx \\ &+ \frac{1}{p+1} \int_{B_r} (x^k \partial_k K(x)) u^{p+1} dx. \end{aligned}$$

Proof. Multiply the equation (3.1) by $x^k \partial_k u + \frac{n-2}{2} u$, and integrate by parts. The details may be found in [**17**] or [**19**]. q.e.d.

4. Linear Analysis and Scalar Curvature

In this section, we will establish important notation and define the functions we will need later to obtain sharp pointwise estimates around a blow-up point.

Since our problem is conformally invariant, we will sometimes work in conformal normal coordinates (see [**13**]) in order to simplify the analysis. Given an integer $N \geq 2$, there exists a positive function ϕ (which can be constructed explicitly) such that if $\tilde{g} = \phi^{\frac{4}{n-2}} g$, the volume element satisfies

$$\det(\tilde{g}_{ij}) = 1 + O(r^N)$$

in $\tilde{g}$-normal coordinates around p, where $r = d_{\tilde{g}}(p, \cdot)$. In such coordinates, it is more convenient to work with the Taylor expansion of the metric instead of dealing with derivatives of the Weyl tensor.

Remark. Throughout the paper, we will sometimes work as if $dv_g \equiv dx$ in conformal normal coordinates, ignoring the contributions from the volume element when they are negligible.

Let us introduce some notation. Throughout the rest of the paper $d = [\frac{n-2}{2}]$. In conformal normal coordinates, we will always write

$$g_{ij} = \exp(h_{ij}),$$

where $h_{ij}(x)x^j = 0$ and $\mathrm{tr}h_{ij}(x) = O(r^N)$. Here N is a large integer and $r = |x|$. In this case, $\det(g_{ij}) = 1 + O(|x|^N)$.

We will also define

$$H_{ij}(x) = \sum_{2 \le |\alpha| \le n-4} h_{ij\alpha} x^\alpha,$$

where $h_{ij}(x) = H_{ij}(x) + O(|x|^{n-3})$. Then $H_{ij}(x) = H_{ji}(x)$, $H_{ij}(x)x^j = 0$, and $\mathrm{tr}H_{ij}(x) = 0$. We will also use

$$H_{ij}^{(k)}(x) = \sum_{|\alpha|=k} h_{ij\alpha} x^\alpha,$$

and

$$|H^{(k)}|^2 = \sum_{|\alpha|=k} |h_{ij\alpha}|^2.$$

Let us now proceed to define our approximate solutions. The goal is to introduce $\tilde{z}_{\varepsilon_i}$ so that $U + \tilde{z}_{\varepsilon_i}$ gives a very good approximation, optimal in some sense, of the rescaled solutions around a blow-up point. Here $\varepsilon_i = M_i^{-\frac{p_i-1}{2}}$. The pointwise estimates will be derived later.

First, we will need to solve

$$\Delta\psi + n(n+2)U^{\frac{4}{n-2}}\psi = q_l U,$$

where q_l is a homogeneous polynomial of degree l, such that

$$\int_{S_1^{n-1}} q_l d\sigma_1 = 0, \qquad \int_{S_1^{n-1}} q_l x^i d\sigma_1 = 0$$

for $i = 1, \dots, n$. We say that given polynomials p and q are orthogonal if $\int_{S_1^{n-1}} pq d\sigma_1 = 0$.

Define $F(q_l) = \{\text{linear combinations of } |y|^{2j}\Delta^k q_l, 0 \le j \le k+2\}$. Note that $\deg(p) \le l + 4$ if $p \in F(q_l)$.

Proposition 4.1. *Let q_l be a homogeneous polynomial of degree l, orthogonal to $\langle 1, x^1, \dots, x^n\rangle$. Suppose $l < n - 4$. Then there exists a unique $\Gamma \in F(q_l)$ such that*

$$\Delta\psi + n(n+2)U^{\frac{4}{n-2}}\psi = q_l U \quad \textit{in} \quad \mathbb{R}^n, \tag{4.1}$$

where $\psi = \Gamma(1 + |y|^2)^{-\frac{n}{2}}$.

Proof. If we write $\psi = \Gamma(1+|y|^2)^{-\frac{n}{2}}$, then equation (4.1) is equivalent to

$$T(\Gamma) = p_l = (1 + |y|^2)^2 q_l,$$

where $T(\Gamma) = (1+|y|^2)\Delta\Gamma - 2ny\cdot\nabla\Gamma + 2n\Gamma$. It is easy to check that $T : F(q_l) \to F(q_l)$ and $p_l \in F(q_l)$.

Hence we just need to show T is injective. We also note that since $\Delta^{[\frac{l}{2}]}q_l = 0$, $\Gamma(0) = 0$ and $\nabla\Gamma(0) = 0$ for every $\Gamma \in F(q_l)$.

Suppose $T(\Gamma_0) = 0$, where $\Gamma_0 \in F(q_l)$.

If $\psi_0 = \Gamma_0(1+|y|^2)^{-\frac{n}{2}}$, then

$$\Delta\psi_0 + n(n+2)U^{\frac{4}{n-2}}\psi_0 = 0,$$

and since $\deg(\Gamma_0) \le l+4 < n$, we also have $\lim_{y\to\infty}\psi_0(y) = 0$.

We will need the following lemma proved in [**8**]:

Lemma 4.2 ([**8**]). *Suppose ψ is a solution to the equation*

$$\Delta\psi + n(n+2)U^{\frac{4}{n-2}}\psi = 0 \text{ in } \mathbb{R}^n. \tag{4.2}$$

If $\lim_{|y|\to\infty}\psi(y) = 0$, then there exist constants $c_0, c_1, \ldots, c_n$ such that

$$\psi(y) = c_0\left(\frac{n-2}{2}U + y\cdot\nabla U\right) + \sum_{j=1}^{n} c_j\frac{\partial U}{\partial y_j}.$$

The lemma implies $\Gamma_0 = 0$, due to the conditions $\Gamma_0(0) = 0$ and $\nabla\Gamma_0(0) = 0$. This finishes the proof of the proposition. q.e.d.

Given $\varepsilon > 0$, we define $\tilde{z}_\varepsilon$ to be the solution of

$$\Delta\tilde{z}_\varepsilon + n(n+2)U^{\frac{4}{n-2}}\tilde{z}_\varepsilon = c(n)\sum_{k=4}^{n-4}\partial_i\partial_j\tilde{H}_{ij}^{(k)}U, \tag{4.3}$$

given by Proposition 4.1, where

$$\tilde{H}_{ij}(y) = H_{ij}(\varepsilon y).$$

Since we are using conformal normal coordinates, integration by parts shows that

$$\int_{|x|=1}\partial_i\partial_j H_{ij} = 0,$$

and

$$\int_{|x|=1}x_l\partial_i\partial_j H_{ij} = 0,$$

where $1 \le l \le n$.

It follows from Proposition 4.1 that $\tilde{z}_\varepsilon(0) = 0$, $\nabla\tilde{z}_\varepsilon(0) = 0$, and

$$\int_{S_r}\tilde{z}_\varepsilon d\sigma_r = 0, \quad \int_{S_r}x^l\tilde{z}_\varepsilon d\sigma_r = 0, \quad 1 \le l \le n.$$

There is also a constant $C > 0$, independent of ε and H_{ij} such that

$$|\partial^\beta\tilde{z}_\varepsilon|(y) \le C\sum_{|\alpha|=4}^{n-4}\sum_{i,j}\varepsilon^{|\alpha|}|h_{ij\alpha}|(1+|y|)^{|\alpha|+2-n-|\beta|} \tag{4.4}$$

for $|\beta| = 0, 1, 2$.

Note that if $u_\varepsilon(x) = \varepsilon^{\frac{n-2}{2}}(\varepsilon^2 + |x|^2)^{\frac{2-n}{2}}$, and $\tilde{z}_\varepsilon(y) = \varepsilon^{\frac{n-2}{2}} z_\varepsilon(\varepsilon y)$, then equation (4.3) is equivalent to

$$\Delta z_\varepsilon + n(n+2)u_\varepsilon^{\frac{4}{n-2}} z_\varepsilon = c(n)\sum_{k=4}^{n-4} \partial_i\partial_j H_{ij}^{(k)} u_\varepsilon,$$

and the estimate (4.4) yields

$$|\partial^\beta z_\varepsilon|(x) \leq C\varepsilon^{\frac{n-2}{2}} \sum_{|\alpha|=4}^{n-4} \sum_{i,j} |h_{ij\alpha}|(\varepsilon + |x|)^{|\alpha|+2-n-|\beta|} \tag{4.5}$$

for $|\beta| = 0, 1, 2$.

In the rest of the paper, it will be important to have a good approximation of the scalar curvature in terms of h_{ij} in conformal normal coordinates. This is the content of the following proposition:

Proposition 4.3. *There exists a constant $C > 0$ such that*

$$\begin{aligned} &\left| R_g - \partial_i\partial_j h_{ij} + \partial_j(H_{ij}\partial_l H_{il}) - \frac{1}{2}\partial_j H_{ij}\partial_l H_{il} + \frac{1}{4}\partial_l H_{ij}\partial_l H_{ij} \right| \\ &\leq C\sum_{|\alpha|=2}^{d} \sum_{i,j} |h_{ij\alpha}|^2 |x|^{2|\alpha|} + C|x|^{n-3}, \end{aligned} \tag{4.6}$$

and

$$|R_g - \partial_i\partial_j h_{ij}| \leq C\sum_{|\alpha|=2}^{d} \sum_{i,j} |h_{ij\alpha}|^2 |x|^{2|\alpha|-2} + C|x|^{n-3} \tag{4.7}$$

if $|x| \leq \sigma \leq 1$, where C depends only on n and $|h|_{C^N(B_\sigma(0))}$.

Proof. It follows from the expression of R_g in local coordinates. q.e.d.

5. Refined Blowup Analysis

In this section, we will establish sharp pointwise estimates for the rescaled sequence of solutions when the blowup is isolated simple, generalizing to higher dimensions previous estimates of [**19**].

In what follows $\varepsilon_i = M_i^{-\frac{p_i-1}{2}}$, $v_i(y) = M_i^{-1} u_i(M_i^{-\frac{p_i-1}{2}} y)$, and H_{ij} comes from the Taylor expansion of the metric g_{ij} in conformal normal coordinates around x_i, as explained in the previous section. We will also use $\tilde{z}_i = \tilde{z}_{\varepsilon_i}$, constructed as in Section 4.

First, we prove the following pointwise estimate:

Proposition 5.1. *Suppose $n \geq 6$. Let $x_i \to \overline{x}$ be an isolated simple blow-up point for a sequence $\{u_i\}$ of positive solutions to* (2.1).

Then, after passing to conformal normal coordinates, there exist constants $\sigma, C > 0$ *such that*

$$|v_i - U - \tilde{z}_{\varepsilon_i}|(y) \le C \max_{2\le k\le d-1}\{\varepsilon_i^{2k}|H^{(k)}|^2(x_i), \varepsilon_i^{n-3}, \delta_i\} \tag{5.1}$$

for every $|y| \le \sigma\varepsilon_i^{-1}$.

Proof. Set $l_i = \sigma M_i^{\frac{p_i-1}{2}} = \sigma\varepsilon_i^{-1}$, and

$$\Lambda_i = \max_{|y|\le l_i} |v_i - U - \tilde{z}_i| = |v_i - U - \tilde{z}_i|(y_i)$$

for a certain $|y_i| \le l_i$.

We observe that if there exists a constant $c > 0$ such that $|y_i| \ge cl_i$ for every i, then the stronger inequality $\Lambda_i \le c\varepsilon_i^{n-2}$ holds. This follows from the estimates $v_i(y) \le cU(y) \le c|y|^{2-n}$ and

$$|\tilde{z}_i(y)| \le c\sum_{k=4}^{n-4}\varepsilon_i^k|y|^{k+2-n} \le c|y|^{2-n}$$

for $|y| \le \sigma\varepsilon_i^{-1}$, since

$$\Lambda_i = |(v_i - U - \tilde{z}_i)(y_i)| \le C|y_i|^{2-n} \le C\varepsilon_i^{n-2}.$$

Hence, we can assume $|y_i| \le \frac{l_i}{2}$.

Define

$$w_i(y) = \Lambda_i^{-1}(v_i - U - \tilde{z}_i)(y).$$

Then w_i satisfies

$$L_{\tilde{g}_i}w_i + b_iw_i = Q_i,$$

where

$$b_i(y) = K\tilde{f}_i^{-\delta_i}\frac{v_i^{p_i} - (U+\tilde{z}_i)^{p_i}}{v_i - U - \tilde{z}_i}(y),$$

and

$$\begin{aligned} Q_i(y) = \Lambda_i^{-1}\Bigg\{ & c(n)\varepsilon_i^2\Bigg(R_{g_i} - \sum_{l=2}^{n-6}(\partial_j\partial_kH_{jk})^{(l)}\Bigg)(\varepsilon_iy)U(y) \\ & + (\Delta - L_{\tilde{g}_i})(\tilde{z}_i) + O(|\tilde{z}_i|^2U^{\frac{6-n}{n-2}}) \\ & + K((U+\tilde{z}_i)^{\frac{n+2}{n-2}} - \tilde{f}_i^{-\delta_i}(U+\tilde{z}_i)^{p_i}) \\ & + M_i^{-(1+N)\frac{p_i-1}{2}}O(|y|^N)|y|(1+|y|^2)^{-\frac{n}{2}}\Bigg\}, \end{aligned} \tag{5.2}$$

where $\tilde{f}_i(y) = f_i(M_i^{-\frac{p_i-1}{2}}y)$, $(\tilde{g}_i)_{kl}(y) = (g_i)_{kl}(M_i^{-\frac{p_i-1}{2}}y)$ and $O(|y|^N)$ comes from the expansion of the volume element in conformal normal coordinates and N is as big as we want.

Since the blowup is isolated simple, from inequality $v_i \leq cU$, it is easy to check that

$$b_i(y) \leq c(1+|y|)^{-4} \tag{5.3}$$

for $|y| \leq l_i$.

The Green's representation formula gives

$$\begin{aligned} w_i(y) = &\int_{B_i} G_{i,L}(y,\eta)(b_i(\eta)w_i(\eta) - Q_i(\eta))d\eta \\ &- \int_{\partial B_i} \frac{\partial G_{i,L}}{\partial \nu}(y,\eta)w_i(\eta)ds, \end{aligned} \tag{5.4}$$

where B_i stands for $B_{l_i}(0)$, and $G_{i,L}$ is the Green function of $L_{\tilde{g}_i}$ in B_i with respect to the Dirichlet boundary condition.

The proof of Proposition 5.1 is by contradiction.

If the proposition is false, we necessarily have

$$\Lambda_i^{-1}\varepsilon_i^{2k}|H^{(k)}|^2(x_i) \to 0 \tag{5.5}$$

for every $2 \leq k \leq d-1$, as $i \to \infty$, and

$$\Lambda_i^{-1}\varepsilon_i^{n-3} \to 0, \quad \Lambda_i^{-1}\delta_i \to 0. \tag{5.6}$$

Let us proceed to estimate Q_i.

First,

$$\begin{aligned} &\left| R - \sum_{l=2}^{n-6}(\partial_j\partial_k H_{jk})^{(l)} \right|(\varepsilon y) \\ &\leq C\sum_{k=2}^{d-1}\varepsilon_i^{2k-2}|H^{(k)}|^2(x_i)|y|^{2k-2} + C\varepsilon_i^{n-5}|y|^{n-5}. \end{aligned} \tag{5.7}$$

It follows that

$$\begin{aligned} &\varepsilon_i^2\left| R - \sum_{l=2}^{n-6}(\partial_j\partial_k H_{jk})^{(l)} \right|(\varepsilon_i y)U(y) \\ &\leq C\sum_{k=2}^{d-1}\varepsilon_i^{2k}|H^{(k)}|^2(1+|y|)^{2k-n} + C\varepsilon_i^{n-3}(1+|y|)^{-3} \\ &\leq C\max_{2\leq k\leq d-1}\{\varepsilon_i^{2k}|H^{(k)}|^2(x_i)\}(1+|y|)^{2d-2-n} + C\varepsilon_i^{n-3}(1+|y|)^{-3}. \end{aligned}$$

Now

$$
\begin{aligned}
|(L_{\tilde{g}_i} - \Delta_y)(\tilde{z}_i)| &\le C \sum_{k=2}^{n-4} \varepsilon_i^{2k} |H^{(k)}|^2 (x_i)(1+|y|)^{2k-n} \\
&\quad + C \sum_{k=4}^{n-4} \varepsilon_i^{n+k-3} |H^{(k)}|(x_i)(1+|y|)^{k-3} \\
&\le C \max_{2\le k\le d-1} \{\varepsilon_i^{2k}|H^{(k)}|^2(x_i)\}(1+|y|)^{2d-2-n} \\
&\quad + C\varepsilon^{n-3}(1+|y|)^{-3}.
\end{aligned}
$$

Finally, since

$$
|\tilde{z}_\varepsilon|^2 \le C \sum_{k=4}^{n-4} \varepsilon^{2k} |H^{(k)}|^2 (1+|y|)^{2k+4-2n},
$$

we obtain

$$
|\tilde{z}_i|^2 U^{\frac{6-n}{n-2}} \le C \max_{2\le k\le d-1} \{\varepsilon_i^{2k}|H^{(k)}|^2(x_i)\}(1+|y|)^{2d-2-n} + C\varepsilon_i^{n-3}(1+|y|)^{-5}.
$$

Therefore,

$$
\begin{aligned}
|Q_i(y)| &\le C\Lambda_i^{-1}\{ \max_{2\le k\le d-1} \{\varepsilon_i^{2k}|H^{(k)}|^2(x_i)\}(1+|y|)^{2d-2-n} \\
&\quad + \varepsilon_i^{n-3}(1+|y|)^{-3} \\
&\quad + M_i^{-(1+N)\frac{p_i-1}{2}} O(|y|^N)|y|(1+|y|^2)^{-\frac{n}{2}} \\
&\quad + \delta_i(|\log(U+\tilde{z}_i)| + |\log \tilde{f}_i|)(1+|y|)^{-n-2}\}.
\end{aligned} \tag{5.8}
$$

Using the estimates (5.3) and (5.8), we get from the Green's representation formula (5.4) that w_i is bounded in C^2_{loc}, and

$$
|w_i(y)| \le C\left((1+|y|)^{-2} + \Lambda_i^{-1} \max_{2\le k\le d-1} \{\varepsilon_i^{2k}|H^{(k)}|^2(x_i), \varepsilon_i^{n-3}\} \right) \tag{5.9}
$$

for $|y| \le \frac{\sigma}{2}\varepsilon_i^{-1}$. We are using that $|w_i(y)| \le C\Lambda_i^{-1}\varepsilon_i^{n-2}$ when $|y| = \frac{\sigma}{2}\varepsilon_i^{-1}$, and also that $|G_{i,L}(y,\eta)| \le C|y-\eta|^{2-n}$ for $|y| \le \frac{l_i}{2}$.

Then, by standard elliptic estimates, there exists a subsequence, also denoted by w_i, converging to w satisfying

$$
\begin{cases}
\Delta w + n(n+2)U^{\frac{4}{n-2}} w = 0 & \text{in } \mathbb{R}^n, \\
|w(y)| \le C(1+|y|)^{-2}.
\end{cases}
$$

Hence, Lemma 4.2 implies that

$$
w(y) = c_0 \left(\frac{n-2}{2} U + y \cdot \nabla U \right) + \sum_{j=1}^n c_j \frac{\partial U}{\partial y_j}.
$$

The conditions $w(0) = \frac{\partial w}{\partial y_j}(0) = 0$ show that $c_j = 0$ for every j, in other words, $w(y) \equiv 0$. From here we conclude that $|y_i| \to \infty$ as $i \to \infty$.

This contradicts the estimate (5.9) and the limits (5.5) and (5.6) since $w_i(y_i) = 1$, and this finishes the proof. q.e.d.

In the next proposition, we estimate δ_i. This result and the Proposition 5.1 give us an estimate on $|v_i - U - \tilde{z}_i|$ independent of δ_i.

Proposition 5.2. *Under the same hypotheses of Proposition* 5.1*,*

$$\delta_i \leq C \max_{2 \leq k \leq d-1} \{\varepsilon_i^{2k} |H^{(k)}|^2(x_i), \varepsilon_i^{n-3}\}.$$

Proof. The proof will be again by contradiction. If the result is not true, then

$$\delta_i^{-1} \varepsilon_i^{2k} |H^{(k)}|^2(x_i) \to 0$$

for every $2 \leq k \leq d-1$, and

$$\delta_i^{-1} \varepsilon_i^{n-3} \to 0.$$

Hence Proposition 5.1 would imply that

$$|v_i - U - \tilde{z}_i|(y) \leq C\delta_i.$$

Define

$$w_i(y) = \delta_i^{-1}(v_i - U - \tilde{z}_i)(y),$$

so w_i is uniformly bounded. The equation satisfied by w_i is

$$L_{\tilde{g}_i} w_i + b_i w_i = \tilde{Q}_i(y),$$

where

$$b_i(y) = K \tilde{f}_i^{-\delta_i} \frac{v_i^{p_i} - (U + \tilde{z}_i)^{p_i}}{v_i - U - \tilde{z}_i}(y),$$

and

$$\begin{aligned} \tilde{Q}_i(y) = \delta_i^{-1} \Big\{ & c(n)\varepsilon_i^2 (R_{g_i} - \sum_{l=2}^{n-6} (\partial_j \partial_k H_{jk})^{(l)})(\varepsilon_i y) U(y) \\ & + (\Delta - L_{\tilde{g}_i})(\tilde{z}_i) + O(|\tilde{z}_i|^2 U^{\frac{6-n}{n-2}}) \\ & + K((U + \tilde{z}_i)^{\frac{n+2}{n-2}} - \tilde{f}_i^{-\delta_i} (U + \tilde{z}_i)^{p_i}) \\ & + M_i^{-(1+N)\frac{p_i - 1}{2}} O(|y|^N)|y|(1 + |y|^2)^{-\frac{n}{2}} \Big\}. \end{aligned} \tag{5.10}$$

We have, as in the previous proposition, that

$$\begin{aligned} |\tilde{Q}_i(y)| \leq C\delta_i^{-1} \Big\{ & \max_{2 \leq k \leq d-1} \{\varepsilon_i^{2k} |H^k|^2(x_i)\} (1 + |y|)^{2d-2-n} \\ & + \varepsilon_i^{n-3} (1 + |y|)^{-3} \\ & + M_i^{-(1+N)\frac{p_i - 1}{2}} O(|y|^N)|y|(1 + |y|^2)^{-\frac{n}{2}} \\ & + \delta_i (|\log(U + \tilde{z}_i)| + |\log \tilde{f}_i|)(1 + |y|)^{-n-2} \Big\}. \end{aligned} \tag{5.11}$$

By linear elliptic theory, we can suppose $w_i \to w$ in compact subsets.

If $\psi(y) = \frac{n-2}{2}U(y) + y \cdot \nabla U(y)$, then

$$\begin{aligned}\int_{|y|\le \frac{l_i}{2}} \psi(y)\delta_i^{-1}\Big(&\max_{2\le k\le d-1}\{\varepsilon_i^{2k}|H^{(k)}|^2(x_i)\}(1+|y|)^{2d-2-n} \\ &+ \varepsilon_i^{n-3}(1+|y|)^{-3} \\ &+ M_i^{-(1+N)\frac{p_i-1}{2}} O(|y|^N)|y|(1+|y|^2)^{-\frac{n}{2}}\Big) \to 0,\end{aligned} \tag{5.12}$$

since $d = [\frac{n-2}{2}]$. Note that when $i \to \infty$, we have:

$$\delta_i^{-1}K((U+\tilde{z}_i)^{\frac{n+2}{n-2}} - \tilde{f}_i^{-\delta_i}(U+\tilde{z}_i)^{p_i}) \to K(\log U(y) + \log f(\overline{x}))U^{\frac{n+2}{n-2}}$$

pointwise. Since

$$\int_{\mathbb{R}^n} \psi(y)U^{\frac{n+2}{n-2}}(y)dy = 0,$$

we can conclude

$$\lim_{i\to\infty}\int_{|y|\le \frac{l_i}{2}} \psi(y)\tilde{Q}_i(y)dy = n(n-2)\int_{\mathbb{R}^n} \psi(y)(\log U(y))U^{\frac{n+2}{n-2}}(y)dy.$$

On the other hand, integration by parts shows that

$$\begin{aligned}&\int_{|y|\le \frac{l_i}{2}} \psi(y)\tilde{Q}_i(y)dy \\ &= \int_{|y|\le \frac{l_i}{2}} \psi(y)(L_{\tilde{g}_i}w_i + b_i w_i)dy \\ &= \int_{|y|\le \frac{l_i}{2}} (L_{\tilde{g}_i}\psi(y) + b_i\psi)w_i dy + \int_{|y|=\frac{l_i}{2}} (\psi\frac{\partial w_i}{\partial r} - w_i\frac{\partial \psi}{\partial r})d\sigma.\end{aligned}$$

The integral on the boundary goes to zero when $i \to \infty$ because

$$\begin{cases} |\psi| = O(r^{2-n}), \quad |\nabla\psi| = O(r^{1-n}) \\ |w_i(\frac{l_i}{2})| \le c\delta_i^{-1}\varepsilon_i^{n-2}, \quad |\nabla w_i(\frac{l_i}{2})| \le c\delta_i^{-1}\varepsilon_i^{n-2}l_i^{-1}. \end{cases}$$

Taking the limit when $i \to \infty$, we would have

$$\lim_{i\to\infty}\int_{|y|\le \frac{l_i}{2}} \psi(y)\tilde{Q}_i(y)dy = \int_{\mathbb{R}^n} (\Delta\psi(y) + n(n+2)U^{\frac{4}{n-2}}\psi)w dy = 0$$

because $\Delta\psi(y) + n(n+2)U^{\frac{4}{n-2}}\psi = 0$, and since the limit should be independent of σ small.

This is a contradiction because a change of variables shows that

$$n(n-2)\int_{\mathbb{R}^n} \psi(y)(\log U(y))U^{\frac{n+2}{n-2}}(y)dy > 0.$$

That finishes the proof of the proposition. q.e.d.

The Propositions 5.1 and 5.2 imply

$$|v_i - U - \tilde{z}_i|(y) \le C \max_{2\le k\le d-1} \{\varepsilon_i^{2k} |H^{(k)}|^2(x_i), \varepsilon_i^{n-3}\} \tag{5.13}$$

for $|y| \le \sigma\varepsilon_i^{-1}$.

In the next result, we will apply the Green's representation formula again to get further decay.

Proposition 5.3. *Suppose that $n \ge 6$. Let $x_i \to \overline{x}$ be an isolated simple blow-up point for a sequence $\{u_i\}$ of positive solutions to (2.1). Then, after passing to conformal normal coordinates, there exist constants $\sigma, C > 0$ such that*

$$\begin{aligned}|\nabla^m (v_i - U - \tilde{z}_i)|(y) \le{}& C \sum_{k=2}^{d-1} \varepsilon_i^{2k} |H^{(k)}|^2(x_i)(1+|y|)^{2k+2-n-m} \\ &+ C\varepsilon_i^{n-3}(1+|y|)^{-1-m}\end{aligned} \tag{5.14}$$

for every $|y| \le \sigma\varepsilon_i^{-1}$, $0 \le m \le 2$.

Proof. Define

$$T_i = \max_{2\le k\le d-1} \{\varepsilon_i^{2k} |H^{(k)}|^2(x_i), \varepsilon_i^{n-3}\},$$

and

$$w_i(y) = (v_i - U - \tilde{z}_i)(y)$$

for $|y| \le \delta\varepsilon_i^{-1}$. Then our previous proposition implies w_i is uniformly bounded. As before, the equation satisfied is $L_{\tilde{g}_i} w_i + b_i w_i = \tilde{Q}_i(y)$, where

$$|b_i|(y) \le c(1+|y|)^{-4},$$

and one can check

$$\begin{aligned}|\tilde{Q}_i(y)| \le C\Bigg\{ & \sum_{k=2}^{d-1} \varepsilon_i^{2k}|H^{(k)}|^2(x_i)(1+|y|)^{2k-n} \\ &+ \varepsilon_i^{n-3}(1+|y|)^{-3} \\ &+ M_i^{-(1+N)\frac{p_i-1}{2}} O(|y|^N)|y|(1+|y|^2)^{-\frac{n}{2}} \\ &+ \delta_i(|\log(U+\tilde{z}_i)| + |\log \tilde{f}_i|)(1+|y|)^{-n-2}\Bigg\}.\end{aligned} \tag{5.15}$$

Since $|G_{i,L}(y,\eta)| \le C|y-\eta|^{2-n}$ for $|y| \le \frac{l_i}{2}$, the Green's representation formula

$$w_i(y) = \int_{B_i} G_{i,L}(y,\eta)(b_i(\eta)w_i(\eta) - \tilde{Q}_i(\eta))d\eta - \int_{\partial B_i} \frac{\partial G_{i,L}}{\partial\nu}(y,\eta)w_i(\eta)ds \tag{5.16}$$

implies

$$(5.17)\quad |w_i(y)| \le c\Big\{\sum_{k=2}^{d-1}\varepsilon_i^{2k}|H^{(k)}|^2(x_i)(1+|y|)^{2k+2-n} + T_i(1+|y|)^{-2} + \varepsilon_i^{n-3}(1+|y|)^{-1}\Big\}.$$

Now we plug the estimate (5.17) in the representation formula (5.16) repeatedly until we reach

$$|w_i(y)| \le C\sum_{k=2}^{d-1}\varepsilon_i^{2k}|H^{(k)}|^2(x_i)(1+|y|)^{2k+2-n} + C\varepsilon_i^{n-3}(1+|y|)^{-1}.$$

The derivative estimates follow from elliptic theory, finishing the proof. q.e.d.

Remark. The estimate (5.14) is equivalent, in the x-coordinates, to (5.18)

$$|u_i - U_\varepsilon - z_\varepsilon|(x) \le C\varepsilon^{\frac{n-2}{2}}\sum_{k=2}^{d-1}|H^{(k)}|^2(x_i)(\varepsilon+|x|)^{2k+2-n} + C\varepsilon^{\frac{n-2}{2}}(\varepsilon+|x|)^{-1}$$

for every $|x| \le \sigma$.

6. Weyl Vanishing Theorem

In this section, we prove the Weyl Vanishing Theorem at an isolated simple blow-up point if $n \le 24$.

In what follows $\theta_k = 1$ if $k = \frac{n-2}{2}$, and $\theta_k = 0$ otherwise.

Theorem 6.1. *Suppose that $6 \le n \le 24$. Let $x_i \to \overline{x}$ be an isolated simple blow-up point for a sequence $\{u_i\}$ of positive solutions to (2.1). Then*

$$(6.1)\qquad |\nabla_g^l W_g|^2(x_i) \le C\varepsilon_i^{n-6-2l}|\log\varepsilon_i|^{-\theta_{l+2}}$$

for every $0 \le l \le [\frac{n-6}{2}]$. In particular,

$$(6.2)\qquad |\nabla_g^l W_g|^2(\overline{x}) = 0$$

for $0 \le l \le [\frac{n-6}{2}]$.

Proof. If we define

$$P(r,u_i) = \int_{|x|=r}\left(\frac{n-2}{2}u_i\frac{\partial u_i}{\partial r} - \frac{r}{2}|\nabla u_i|^2 + r\left|\frac{\partial u_i}{\partial r}\right|^2 + \frac{1}{p_i+1}Kf_i^{-\delta_i}ru_i^{p_i+1}\right)d\sigma_r,$$

then the Pohozaev identity of Lemma 3.1 implies

$$
\begin{aligned}
P(r,u_i) = &-\int_{|x|\le r}\left(x^m\partial_m u_i + \frac{n-2}{2}u_i\right)(\Delta_g-\Delta_\delta)(u_i)dx \\
&- c(n)\int_{|x|\le r}\left(\frac{1}{2}x^k\partial_k R + R\right)u_i^2dx + c(n)\frac{r}{2}\int_{|x|=r} Ru_i^2 d\sigma_r \\
&+\left(\frac{n}{p_i+1}-\frac{n-2}{2}\right)\int_{|x|\le r} Kf_i^{-\delta_i}u_i^{p_i+1}dx \\
&- \frac{\delta_i}{p_i+1}\int_{|x|\le r} Kf_i^{-\delta_i-1}(x^m\partial_m f_i)u_i^{p_i+1}dx.
\end{aligned}
$$

Also observe that if we choose r sufficiently small, independently of i,

$$
\begin{aligned}
\left(\frac{n}{p_i+1}-\frac{n-2}{2}\right)&\int_{|x|\le r} Ku_i^{p_i+1}dx \\
&- \frac{\delta_i}{p_i+1}\int_{|x|\le r} Kf_i^{-\delta_i-1}(x^m\partial_m f_i)u_i^{p_i+1}dx \ge 0.
\end{aligned}
\tag{6.3}
$$

Hence, for small $r>0$

$$
\begin{aligned}
P(r,u_i) \ge &-\int_{|x|\le r}\left(x^m\partial_m u_i + \frac{n-2}{2}u_i\right)(\Delta_g-\Delta_\delta)(u_i)dx \\
&- c(n)\int_{|x|\le r}\left(\frac{1}{2}x^k\partial_k R + R\right)u_i^2dx + c(n)\frac{r}{2}\int_{|x|=r} Ru_i^2 d\sigma_r.
\end{aligned}
\tag{6.4}
$$

Since $M_iu_i\to h$ away from x_i, we have $A_i(r)\le C\varepsilon_i^{n-2}$, where

$$
\begin{aligned}
A_i(r) = &-c(n)\int_{|x|\le r}\left(\frac{1}{2}x^k\partial_k R + R\right)u_i^2dx \\
&-\int_{|x|\le r}\left(x^m\partial_m u_i + \frac{n-2}{2}u_i\right)((g^{kl}-\delta^{kl})\partial_{kl}u_i + \partial_k g^{kl}\partial_l u_i)dx.
\end{aligned}
$$

Define

$$
\begin{aligned}
\hat{A}_i(r) = &-c(n)\int_{|x|\le r}\left(\frac{1}{2}x^k\partial_k R + R\right)\phi_\varepsilon^2dx \\
&-\int_{|x|\le r}\left(x^m\partial_m \phi_\varepsilon + \frac{n-2}{2}\phi_\varepsilon\right)((g^{kl}-\delta^{kl})\partial_{kl}\phi_\varepsilon + \partial_k g^{kl}\partial_l \phi_\varepsilon)dx,
\end{aligned}
$$

where $\phi_\varepsilon = u_\varepsilon + z_\varepsilon$.

The pointwise estimate of Proposition 5.3 implies

$$
|u_i-\phi_{\varepsilon_i}| \le C\sum_{k=2}^{d-1}\varepsilon_i^{\frac{n-2}{2}}|H^{(k)}|^2(x_i)(\varepsilon_i+|x|)^{2k+2-n} + C\varepsilon_i^{\frac{n-2}{2}}(\varepsilon_i+|x|)^{-1},
$$

which together with the corresponding estimates on the derivatives imply

$$|A_i(r) - \hat{A}_i(r)| \leq C \sum_{k=2}^{d-1} \varepsilon_i^{2k+2} |H^{(k)}|^2(x_i) + Cr\varepsilon_i^{n-2}.$$

Hence,

$$\hat{A}_i(r) \leq C \sum_{k=2}^{d-1} \varepsilon_i^{2k+2} |H^{(k)}|^2(x_i) + C\varepsilon_i^{n-2}.$$

Now we will estimate the second integral in the definition of $\hat{A}_i(r)$.
First, since u_ε and $x^m \partial_m u_\varepsilon + \frac{n-2}{2} u_\varepsilon$ are rotationally symmetric,

$$\begin{aligned}
&\int_{|x|\leq r} \left(x^m \partial_m \phi_\varepsilon + \frac{n-2}{2} \phi_\varepsilon \right) ((g^{kl} - \delta^{kl}) \partial_{kl} \phi_\varepsilon + \partial_k g^{kl} \partial_l \phi_\varepsilon) \\
&= \int_{|x|\leq r} \left(x^m \partial_m \phi_\varepsilon + \frac{n-2}{2} \phi_\varepsilon \right) (\Delta_g - \Delta)(\phi_\varepsilon) \\
&= \int_{|x|\leq r} \left(x^m \partial_m \phi_\varepsilon + \frac{n-2}{2} \phi_\varepsilon \right) (\Delta_g - \Delta)(z_\varepsilon) \\
&= \int_{|x|\leq r} z_\varepsilon (\Delta_g - \Delta) \left(x^m \partial_m \phi_\varepsilon + \frac{n-2}{2} \phi_\varepsilon \right) \\
&= \int_{|x|\leq r} \left(x^m \partial_m z_\varepsilon + \frac{n-2}{2} z_\varepsilon \right) (\Delta_g - \Delta)(z_\varepsilon).
\end{aligned}$$

Here we are using that the metric Laplacian and the Euclidean Laplacian coincide, in conformal normal coordinates, when applied to rotationally symmetric functions.

But

$$\begin{aligned}
(6.5) \qquad &\left| \int_{|x|\leq r} \left(x^m \partial_m z_\varepsilon + \frac{n-2}{2} z_\varepsilon \right) (\Delta_g - \Delta)(z_\varepsilon) \right| \\
&\leq C \sum_{|\alpha|=2}^{[\frac{n-2}{3}]} \sum_{i,j} \varepsilon^{3|\alpha|} |h_{ij\alpha}|^3 |\log \varepsilon| + Cr\varepsilon^{n-2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
(6.6) \qquad &\left| \int_{|x|\leq r} \left(\frac{1}{2} x^k \partial_k R + R \right) z_\varepsilon^2 dx \right| \\
&\leq C \sum_{|\alpha|=2}^{[\frac{n-2}{3}]} \sum_{i,j} \varepsilon^{3|\alpha|} |h_{ij\alpha}|^3 |\log \varepsilon|^{\gamma_1} + Cr\varepsilon^{n-2}.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&- c(n) \int_{|x|\le r} \left(\frac{1}{2}x^k\partial_k R + R\right)(u_\varepsilon^2 + 2u_\varepsilon z_\varepsilon)dx \\
&\le C\sum_{k=2}^{d} \varepsilon_i^{2k+1}|H^{(k)}|^2(x_i) + C\varepsilon_i^{n-2}.
\end{aligned}
$$

Now the key estimate in the Appendix, Proposition A.4, implies that if $n \le 24$,

$$\sum_{|\alpha|=2}^{d}\sum_{i,j}|h_{ij\alpha}|^2\varepsilon^{2|\alpha|}|\log\varepsilon|^\theta \le C\varepsilon_i^{n-2}.$$

This finishes the proof. q.e.d.

As a corollary, we can improve the pointwise estimates of Proposition 5.3, if $n \le 24$.

Corollary 6.2. *Suppose that $6 \le n \le 24$. Let $x_i \to \overline{x}$ be an isolated simple blow-up point for a sequence $\{u_i\}$ of positive solutions to* (2.1)*. Then, after passing to conformal normal coordinates, there exist constants $\sigma, C > 0$ such that*

$$|\nabla^m(v_i - U - \tilde{z}_i)|(y) \le C\varepsilon_i^{n-3}(1+|y|)^{-1-m} \tag{6.7}$$

for every $|y| \le \sigma\varepsilon_i^{-1}$, $0 \le m \le 2$.

7. Local Sign Restriction

The next result concerns the local asymptotic analysis at a blowup point if $n \le 24$. It will be used together with the Positive Mass Theorem to exclude the possibility of blowup phenomenon on manifolds not conformally diffeomorphic to the sphere.

Define

$$P'(r,v) = \int_{|x|=r}\left(\frac{n-2}{2}v\frac{\partial v}{\partial\nu} - \frac{r}{2}|\nabla v|^2 + r\left|\frac{\partial v}{\partial \nu}\right|^2\right)d\sigma(r). \tag{7.1}$$

Theorem 7.1. *Suppose that $n \le 24$. Let $x_i \to \overline{x}$ be an isolated simple blow-up point for a sequence $\{u_i\}$ of positive solutions to* (2.1)*. If $u_i(x_i)u_i \to h$ away from the origin, then*

$$\liminf_{r\to 0} P'(r,h) \ge 0.$$

Proof. That is another application of the Pohozaev identity and the pointwise estimates. Recall inequality (6.4) for small $r > 0$:

$$\begin{aligned} P(r, u_i) \geq &- \int_{|x|\leq r} \left(x^m \partial_m u_i + \frac{n-2}{2} u_i \right) (\Delta_g - \Delta_\delta)(u_i) dx \\ &- c(n) \int_{|x|\leq r} \left(\frac{1}{2} x^k \partial_k R + R \right) u_i^2 dx + c(n) \frac{r}{2} \int_{|x|=r} R u_i^2 d\sigma_r. \end{aligned}$$

Note that

$$M_i^2 P(r, u_i) \to P'(r, h),$$

as $i \to \infty$.

The pointwise estimates established in Corollary 6.2 imply

$$\varepsilon_i^{2-n} |A_i(r) - \hat{A}_i(r)| \leq Cr,$$

where A_i and $\hat{A}_i$ were defined in the previous section.

Now

$$r M_i^2 \lim_{i\to\infty} \int_{|x|=r} R u_i^2 d\sigma_r = r \int_{|x|=r} R h^2 d\sigma_r.$$

The Weyl Vanishing Theorem (Theorem 6.1) implies $H_{ij} = O(|x|^{d+1})$ at the blowup point $\overline{x}$, so $\partial_i \partial_j H_{ij} = O(r^{d-1})$.

Therefore, $R_g = \partial_i \partial_j h_{ij} + O(|x|^{n-3})$.

Also, letting $\varepsilon \to 0$ in the pointwise estimates given by Corollary 6.2,

$$h = |x|^{2-n} + t(x) + O(|x|^{-1}),$$

where

$$t(x) = \lim_{\varepsilon\to 0} \varepsilon^{n-2} \tilde{z}_\varepsilon(\varepsilon^{-1} x) = O\left(\sum_{|\alpha|=4}^{n-4} |h_{ij\alpha}| |x|^{|\alpha|+2-n} \right).$$

The Weyl Vanishing Theorem implies $t(x) = O(|x|^{d+3-n})$. This can also be seen through the expansion of the Green's function of the conformal Laplacian.

Hence,

$$\begin{aligned} r h^2 R &= r(r^{2-n} + t + O(r^{-1}))^2 (\partial_i \partial_j h_{ij} + O(r^{n-3})) \\ &= r(r^{2-n} + t)^2 (\partial_i \partial_j h_{ij}) + O(r^{2-n}) \\ &= (\partial_i \partial_j h_{ij}) r^{5-2n} + O(r^{2-n}). \end{aligned}$$

Therefore,

$$\liminf_{r\to 0} \int_{|x|=r} r R h^2 d\sigma_r = 0.$$

By applying the Theorem 6.1 to the estimates (6.5) and (6.6), we get

$$\Big| - \int_{|x|\leq r} \left(x^m \partial_m \phi_\varepsilon + \frac{n-2}{2}\phi_\varepsilon\right)((g^{kl}-\delta^{kl})\partial_{kl}\phi_\varepsilon + \partial_k g^{kl}\partial_l \phi_\varepsilon) \\ - c(n)\int_{|x|\leq r}\left(\frac{1}{2}x^k \partial_k R + R\right) z_\varepsilon^2 dx\Big| \leq Cr\varepsilon^{n-2}$$

for $\varepsilon \leq r$.

The key estimate in the Appendix then implies, for $n \leq 24$,

$$\liminf_{r\to 0} P'(r,h) \geq 0. \tag{7.2}$$

q.e.d.

8. Blowup Set

We will show that the set of blowup points is finite and consists only of isolated simple blow-up points. This is going to be an application of the local sign restriction of Theorem 7.1.

First, we recall a well-known result:

Proposition 8.1. *Given small $\delta > 0$ and large $R > 0$, there exists a constant $C = C(\delta, R) > 0$ such that if u is a positive solution of (2.1) with $\max_M u > C$, then there exists $\{x_1, \dots, x_N\} \subset M$, $N = N(u) \geq 1$, where $\frac{n+2}{n-2} - p < \delta$ and each x_i is a local maximum of u such that:*

1) *$\{B_{r_i}(x_i)\}_{i=1}^N$ is a disjoint collection if $r_i = Ru(x_i)^{-\frac{p-1}{2}}$;*
2) *if $x = (x^1, \dots, x^n)$ is a normal coordinate system centered at x_i, then*
$$|u(x_i)^{-1}u(u(x_i)^{-\frac{p-1}{2}}y) - U(y)|_{C^2(B_R(0))} < \delta$$
where $y = u(x_i)^{\frac{p-1}{2}}x$;
3) *$u(x) \leq Cd_g(x, \{x_1, \dots, x_N\})^{-\frac{2}{p-1}}$ for all $x \in M$, and*
$$d_g(x_i, x_j)^{\frac{2}{p-1}}u(x_j) \geq C^{-1}$$
for $i \neq j$.

Let us start by proving that the local maxima $x_i = x_i(u)$ obtained in Proposition 8.1 cannot accumulate. That is, we will show that there exist constants $C_i(\delta, R) > 0$, $i = 1, 2$, such that for all $u > 0$ solving (2.1) with $\max_M u \geq C_1$, we must have

$$d_g(x_i(u), x_j(u)) \geq C_2.$$

In order to accomplish this, we will first prove that every isolated blow-up point must, in fact, be isolated simple.

Lemma 8.2. *Let $x_i \to \overline{x}$ be an isolated blow-up point for the sequence $\{u_i\}$ of positive solutions to (2.1). Then $\overline{x}$ is an isolated simple blow-up point for $\{u_i\}$.*

Proof. We proceed by contradiction, so assume that $\overline{x}$ is not an isolated simple blow-up point. Then there exist at least two critical points of $r^{\frac{2}{p_i-1}}\overline{u}_i(r)$ in the interval $(0,\overline{\tau}_i)$ for some $\overline{\tau}_i \to 0$. By Proposition 2.4, there can be at most one critical point in the interval $0 < r < r_i := R_i u_i(x_i)^{-\frac{p_i-1}{2}}$. Therefore, if τ_i is the second critical point then $\tau_i \geq r_i$ and $\tau_i \to 0$.

Let $x = (x^1, \ldots, x^n)$ be normal coordinates centered at x_i, and rescale $u_i(x)$ by

$$v_i(y) = \tau_i^{\frac{2}{p_i-1}} u_i(\tau_i y), \quad |y| < \tau_i^{-1},$$

where $x = \tau_i y$. Then $v_i(y)$ satisfies

$$L_h v_i(y) + K f_i^{-\delta_i}(\tau_i y) v_i^{p_i}(y) = 0$$

where $h_{\alpha\beta}(y) = g_{\alpha\beta}(\tau_i y)$. Furthermore, we have that $|y|^{\frac{2}{p_i-1}} v_i(y) \leq C$ for $|y| < \tau_i^{-1}$, $\lim_{i\to\infty} v_i(0) = \infty$, $r^{\frac{2}{p_i-1}}\overline{v}_i(r)$ has exactly one critical point on $0 < r < 1$, and

$$\frac{d}{dr}(r^{\frac{2}{p_i-1}}\overline{v}_i(r))|_{r=1} = 0. \tag{8.1}$$

It follows that the origin is an isolated simple blow-up point for $\{v_i\}$. As in the proof of Corollary 2.6 after passing to a subsequence, we have

$$v_i(0)v_i(y) \to h(y) := a|y|^{2-n} + b(y) \quad \text{in} \quad C^2_{loc}(\mathbb{R}^n - \{0\})$$

where $b(y)$ is harmonic on $\mathbb{R}^n$. Since $h(y)$ is positive $\liminf_{|y|\to\infty} b(y) \geq 0$, so that the maximum principle guarantees that $b(y) \geq 0$. By Liouville's theorem, $b(y) = b$ is constant. Moreover, using equality (8.1),

$$\frac{d}{dr}(r^{\frac{2}{p_i-1}}\overline{h}(r))|_{r=1} = 0,$$

which shows that $b = a > 0$. But this contradicts the local sign restriction of Theorem 7.1. q.e.d.

Having shown that isolated blow-up points are in fact isolated simple blow-up points, we can now rule out bubble accumulation.

Proposition 8.3. *Let δ, R, u, $C(\delta, R)$, and $\{x_1 \ldots, x_N\}$ be as in Proposition* 8.1. *If δ is sufficiently small and R is sufficiently large, then there exists a constant $\overline{C}(\delta, R) > 0$ such that if $\max_M u \geq C$ then $d_g(x_j, x_l) \geq \overline{C}$ for all $1 \leq j \neq l \leq N$.*

Proof. We proceed by contradiction. Suppose that such a constant $\overline{C}$ does not exist, then there exist sequences $p_i \to p \in (1, \frac{n+2}{n-2}]$ and $\{u_i\}$ with $\max_M u_i \geq C$ and

$$\lim_{i\to\infty} \min_{j\neq l} d_g(x_j(u_i), x_l(u_i)) = 0.$$

We can assume without loss of generality that

$$\sigma_i = d_g(x_1(u_i), x_2(u_i)) = \min_{j \neq l} d_g(x_j(u_i), x_l(u_i)) \to 0.$$

Then by item (3) of Proposition 8.1, we have $u_i(x_1), u_i(x_2) \to \infty$.

We now rescale by

$$v_i(y) = \sigma_i^{\frac{2}{p_i-1}} u_i(\exp_{x_1}(\sigma_i y)), \quad |y| < \sigma_i^{-1}.$$

Then $v_i(y)$ satisfies

$$L_h v_i(y) + K f_i^{-\delta_i}(\sigma_i y) v_i^{p_i}(y) = 0$$

where $h_{\alpha\beta}(y) = g_{\alpha\beta}(\sigma_i y)$. If $x_j(u_i) \in B_{\sqrt{\sigma_i}}(x_1)$ and we set $y_j = \sigma_i^{-1} x_j(u_i)$, then each y_j is a local maximum of $v_i(y)$ and by item (3) of Proposition 8.1,

$$|y - y_j|^{\frac{2}{p_i-1}} v_i(y) \leq C, \quad |y| < \sigma_i^{-1}.$$

Furthermore, $y_1 = 0$, $|y_2| = 1$, and $\min_{j \neq l} |y_j - y_l| \geq 1 + o(1)$, so in particular, $y_2(u_i) \to \overline{y}_2$ with $|\overline{y}_2| = 1$. It follows that $\{0, \overline{y}_2\}$ are isolated blow-up points for $v_i(y)$ as long as

$$v_i(0), v_i(y_2) \to \infty. \tag{8.2}$$

We now show (8.2). If $v_i(y_2)$ stays bounded but $v_i(0) \to \infty$, then $\{0\}$ is an isolated and hence isolated simple blow-up point, while $v_i(y)$ remains uniformly bounded near $\overline{y}_2$. Then by Proposition 2.5 $v_i(\overline{y}_2) \to 0$, but this cannot happen since $\sigma_i \geq \max\{R u_i(x_1)^{-\frac{p_i-1}{2}}, R u_i(x_2)^{-\frac{p_i-1}{2}}\}$ (here we use Proposition 8.1, item (1)), which implies that

$$v_i(0), v_i(\overline{y}_2) \geq R. \tag{8.3}$$

On the other hand, if both $v_i(0)$ and $v_i(y_2)$ remain bounded, then arguments similar to those of Proposition 2.4 show that $v_i \to v > 0$ in $C^2_{\text{loc}}(\mathbb{R}^n)$ where $v(y)$ satisfies

$$\Delta v + K v^{\frac{n+2}{n-2}} = 0, \quad \nabla v(0) = \nabla v(\overline{y}_2) = 0.$$

The results of Caffarelli, Gidas, and Spruck ([**7**]) then yield that $v \equiv 0$, which contradicts (8.3).

Now that (8.2) is established, we have that $\{0, \overline{y}_2\}$ are isolated simple blow-up points for $\{v_i\}$. Then according to Corollary 2.6

$$v_i(0) v_i(y) \to G(y) := a_1 |y|^{2-n} + a_2 |y - \overline{y}_2|^{2-n} + b(y) \quad \text{in} \quad C^2_{\text{loc}}(\mathbb{R}^n - S),$$

where S denotes the set of blow-up points for $\{v_i\}$, $b(y)$ is a harmonic function on $\mathbb{R}^n - (S - \{0, \overline{y}_2\})$, and $a_1, a_2 > 0$. By the maximum principle $b(y) \geq 0$, so that

$$G(y) = a_1 |y|^{2-n} + b + O(|y|) \quad \text{for } |y| \text{ near } 0,$$

for some constant $b > 0$. This, however, contradicts the sign condition of Theorem 7.1.

It is now clear from the above proposition that for any given sequence $\{u_i\}$, the integer $N = N(u_i)$ of Proposition 8.1 must remain uniformly bounded, otherwise there cannot exist a constant $\overline{C}(\delta, R) > 0$ such that $d_g(x_j, x_l) \geq \overline{C}$. We now have the stated goal of this section,

Corollary 8.4. *Let $\{u_i\}$ be a sequence of positive solutions of* (2.1) *on M with $\max_M u_i \to \infty$. Then $p_i \to \frac{n+2}{n-2}$, and the set of blow-up points is finite and consists only of isolated simple blow-up points.*

q.e.d.

9. The Compactness Result

The purpose of this section is to complete the proof of Theorems 1.1 and 1.2.

Proof of Theorem 1.2. Having shown that every blow-up point is isolated simple, Theorem 1.2 follows from Theorem 6.1. q.e.d.

In order to prove Theorem 1.1 we need the Positive Mass Theorem, which requires the introduction of asymptotically flat manifolds. A Riemannian n-manifold $(\mathcal{M}, \widehat{g})$ is called an asymptotically flat manifold of order τ if $\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_\infty$, where $\mathcal{M}_0$ is compact, and $\mathcal{M}_\infty$ is diffeomorphic to $\mathbb{R}^n - B_R$ $(R > 0)$ and the diffeomorphism provides a coordinate system $y = (y^1, \ldots, y^n)$ on $\mathcal{M}_\infty$ such that

$$\widehat{g}_{ij} = \delta_{ij} + O(|y|^{-\tau}), \quad \partial_{y^k} \widehat{g}_{ij} = O(|y|^{-\tau-1}), \quad \partial_{y^k y^m} \widehat{g}_{ij} = O(|y|^{-\tau-2}).$$

Such a coordinate system is called an asymptotic coordinate system. If $\tau > \frac{n-2}{2}$, we can define the mass of $(\mathcal{M}, \widehat{g})$ by the following limit:

$$m(g) = \lim_{R \to \infty} \int_{S_R} (\partial_i \widehat{g}_{ij} - \partial_j \widehat{g}_{ii}) \nu^j, \tag{9.1}$$

where ν is the Euclidean outward normal to S_R. The following is a generalization of the Positive Mass Theorem of Schoen and Yau ([**31**]) to arbitrary dimensions.

Theorem 9.1 ([**18**]). *Let $(\mathcal{M}, \widehat{g})$ be an n-dimensional asymptotically flat manifold of order $\tau > \frac{(n-2)}{2}$. Assume that the scalar curvature $R \geq 0$, and $R \in L^1(\mathcal{M}, \widehat{g})$. Then $m(\widehat{g}) \geq 0$, and $m(\widehat{g}) = 0$ if and only if $(\mathcal{M}, \widehat{g})$ is isometric to Euclidean space $\mathbb{R}^n$.*

Proof of Theorem 1.1. Standard elliptic estimates and the Harnack inequality show that it suffices to estimate $|u|_{C^0(M)}$ from above. We proceed by contradiction, and assume that $\cup_{1+\varepsilon \leq p \leq \frac{n+2}{n-2}} \Phi_p$ is not bounded in $C^0(M)$. Then there exists a sequence $u_i \in \Phi_{p_i}$, $1 + \varepsilon \leq p_i \leq \frac{n+2}{n-2}$, with

$$\max_M u_i \to \infty \quad \text{as} \quad i \to \infty.$$

By Corollary 8.4, we must have $p_i \to \frac{n+2}{n-2}$, and there exist a finite number $N > 0$ of isolated simple blowup points

$$x_i^{(1)} \to \overline{x}^{(1)}, \dots, x_i^{(N)} \to \overline{x}^{(N)}.$$

We can assume without loss of generality that

$$u_i(x_i^{(1)}) = \min\{u_i(x_i^{(1)}), \dots, u_i(x_i^{(N)})\}$$

for all i. Then set $w_i = u_i(x_i^{(1)})u_i$. Following the proof of Corollary 2.6 and using the Harnack inequality away from the blowup points, we have

$$w_i \to G := \sum_{j=1}^{N} a_j G_{\overline{x}^{(j)}} + b \quad \text{in} \quad C^2_{loc}(M - \{\overline{x}^{(1)}, \dots, \overline{x}^{(N)}\}),$$

where a_j are nonnegative constants, $a_1 > 0$, $G_{\overline{x}^{(j)}}$ is the Green's function for the conformal Laplacian with singularity at $\overline{x}^{(j)}$, and $b \in C^2(M)$. Note that since the first eigenvalue of minus the conformal Laplacian is positive, the fact that $L_g G = 0$ implies $b \equiv 0$.

Let $\widehat{g} = G_{\overline{x}^{(1)}}^{\frac{4}{n-2}} g$. We can assume that g is the metric conformally related to the given metric in Theorem 1.1, which produces conformal normal coordinates at $\overline{x}^{(1)}$; this follows from the fact that if g_0 is the given metric and $g = \phi^{\frac{4}{n-2}} g_0$ is the conformal normal metric, then $\phi^{-1} G_{\overline{x}^{(1)}}$ is the Green's function for g, where $G_{\overline{x}^{(1)}}$ is the Green's function for g_0. Then $(M - \{\overline{x}^{(1)}\}, \widehat{g})$ has scalar curvature $R_{\widehat{g}} \equiv 0$. Furthermore, a calculation shows that $(M - \{\overline{x}^{(1)}\}, \widehat{g})$ is asymptotically flat.

Indeed, it is well-known (see [**13**], [**33**]) that in a conformal normal coordinate system at $\overline{x}^{(1)} \in M$, the Green's function has the following asymptotic expansion:

$$G(x, \overline{x}^{(1)}) = |x|^{2-n}(1 + \chi_1(x) + \cdots + \chi_n(x)) + c \log|x| + \chi_{n+1}(x),$$

where χ_k is a homogeneous polynomial of degree k, $\chi_{n+1} = O(1)$, $\chi_1 = \chi_2 = \chi_3 \equiv 0$, and the log term only appears in even dimensions.

Since Theorem 6.1 implies $h_{ij\alpha}(\overline{x}^{(1)}) = 0$ for every $1 \le i, j \le n$ and $2 \le |\alpha| \le d$, it is not difficult to see (see proof in [**33**]) that

$$G(x, \overline{x}^{(1)}) = |x|^{2-n}\left(1 + \sum_{k=d+1}^{n-2} \chi_k(x)\right) + A + O(|x| \log|x|), \tag{9.2}$$

where

$$\int_{S_1^{n-1}} \chi_k = 0, \quad \int_{S_1^{n-1}} x_i \chi_k = 0 \tag{9.3}$$

for every $k \le n-2$, $1 \le i \le n$. This follows from the fact that in this case

$$R_g = \partial_i \partial_j h_{ij} + O(|x|^{n-3}),$$

and $\int_{S_1^{n-1}} \partial_i \partial_j h_{ij} = 0$.

If we introduce the asymptotic coordinates $y = |x|^{-2}x$, then the expansion (9.2) and the fact that $g_{ij} = \delta_{ij}+h_{ij}+O(|x|^{n-1})$, $h_{ij} = O(|x|^{d+1})$ will imply

$$\widehat{g}_{ij}(y) = \delta_{ij} + O(|y|^{-(d+1)})$$

for large $|y|$. More precisely, we can derive

$$\widehat{g}_{ij}(y) = G^{\frac{4}{n-2}}|y|^{-4}(\delta_{ij}+\overline{h}_{ij}-2y_iy_k|y|^{-2}\overline{h}_{kj}-2y_jy_l|y|^{-2}\overline{h}_{il}+O(|y|^{1-n})),$$

where $\overline{h}_{ij}(y) = h_{ij}(|y|^{-2}y)$. Recall $g_{ij} = \exp(h_{ij})$.

Since $d = [\frac{n-2}{2}]$, the Positive Mass Theorem (Theorem 9.1) can be applied, and therefore, $m(\widehat{g}) > 0$. Note that if $m(\widehat{g}) = 0$, then $(M - \{\overline{x}^{(1)}\}, \widehat{g})$ is isometric to $\mathbb{R}^n$, but $\mathbb{R}^n$ is conformal to $\mathbb{S}^n - \{\text{a point}\}$ which would then imply that (M, g) is conformal to $\mathbb{S}^n$.

We now observe that the terms involving χ_k or $\overline{h}$ do not contribute to the mass. Here we are using conformal normal coordinates and the equalities (9.3). A calculation then shows that the limit in (9.1) is a positive multiple of the coefficient A. Therefore, $A > 0$.

This contradicts the local sign restriction of Theorem 7.1, and we have finished the proof. q.e.d.

10. Nondegenerate Conformal Classes and Existence

Throughout this section, we will always assume $n \leq 24$.

In this section, we introduce a $C^{k,\alpha}$ topology on the set of conformal classes of metrics on a compact manifold M^n where k is a positive integer and $\alpha \in (0,1)$. We then show that for k sufficiently large (depending only on n), there is an open dense subset of the Yamabe-positive conformal classes in which there is a finite number of nondegenerate solutions of the equation $R = n(n-1)$. We then discuss the general existence theory (as outlined in the introduction) for these nondegenerate classes which includes Morse inequalities as well as a degree count.

We first fix a $C^{k,\alpha}$ norm on the open cone $\mathcal{M}$ of Riemannian metrics on M. This can be done by choosing a fixed background metric and measuring general metrics with respect to it. This norm then induces the same topology regardless of which smooth background metric is chosen. We denote by $\mathcal{C}$ the set of conformal classes of $C^{k,\alpha}$ metrics. If we fix a smooth volume form ω on M, we may identify $\mathcal{C}$ with the submanifold of $\mathcal{M}$ having volume form ω. We may then define a $C^{k,\alpha}$ Banach manifold topology on $\mathcal{C}$ by taking the induced topology from $\mathcal{M}$. It is easy to see that the topology induced on $\mathcal{C}$ does not depend on the smooth volume form ω which was chosen. The topology induced on $\mathcal{C}$ in this way will be referred to as the $C^{k,\alpha}$ topology.

We denote by $\mathcal{C}^+$ the (possibly empty) subset of $\mathcal{C}$ consisting of Yamabe-positive conformal classes. Note that if $k \geq 2$, $\mathcal{C}^+$ is an *open* subset of $\mathcal{C}$.

We will need the following compactness result which follows directly from the results of this paper:

Lemma 10.1. *Let $\mathcal{K}$ be a compact subset of $\mathcal{C}^+$ which does not contain a metric conformally diffeomorphic to a metric in the standard conformal class on S^n. If k is sufficiently large depending only on n, then the set of unit volume, constant scalar curvature metrics having conformal classes in $\mathcal{K}$ is a compact subset of $\mathcal{M}$.*

We now discuss nondegeneracy for constant scalar curvature metrics. Recall that we may either normalize the volume or the value of the scalar curvature to choose a unique representative of the solutions which are multiples of a given one. When we derived our estimate, we normalized volume but for the following discussion, we prefer to normalize the value of the scalar curvature to be the same as that of the standard unit n-sphere; that is, we consider solutions of the equation $R = n(n-1)$. It is easy to compute the linearization of this equation when the metric is varied within its conformal class; that is, if we consider a family of metrics $g_t = (1 + t\eta)^{4/(n-2)} g$ where η is a smooth function on M, and we compute the first t derivative of the equation $R(g_t) = n(n-1)$. This linearized equation is

$$\Delta\eta + n\eta = 0 \tag{10.1}$$

where Δ is the Laplace operator of the metric g. We say that a solution is *nondegenerate* if there is no nonzero solution of this equation; that is, if n is not an eigenvalue of Δ. It is easy to see that this condition is equivalent to the statement that the volume one rescaling of g is a nondegenerate critical point of the scalar curvature functional on its conformal class. We will say that a conformal class of metrics $[g]$ is *nondegenerate* if each metric in the class with constant scalar curvature $n(n-1)$ is nondegenerate. Since nondegenerate solutions are isolated within their conformal classes, it follows from Lemma 10.1 that there is only a finite number of solutions in any nondegenerate conformal class of metrics.

The Morse index $I(g)$ of a solution is defined to be the number (counted with multiplicity) of nonzero eigenvalues λ of Δ which are less than n. This agrees with the ordinary Morse index of the total scalar curvature functional at the unit volume rescaling of g on its conformal class.

We now consider the scalar curvature map R from $\mathcal{M}$ to the $C^{k-2,\alpha}$ functions on M, and we let $\mathcal{M}_1$ denote the subset of $\mathcal{M}$ consisting of metrics with scalar curvature equal to $n(n-1)$. Thus $\mathcal{M}_1$ is a closed subset of $\mathcal{M}$. A metric $g \in \mathcal{M}_1$ is a *regular point* of R if the differential of R at g is a surjective linear map from $C^{k,\alpha}$ symmetric $(0,2)$ tensors on M to $C^{k-2,\alpha}$ functions on M. It then follows from the inverse function theorem (see [**10**]) that the set of regular points of R is a smooth Banach submanifold of $\mathcal{M}$ which we denote $\mathcal{M}_1'$. In particular, $\mathcal{M}_1'$ is a relatively open subset of $\mathcal{M}_1$. We let $\mathcal{M}_1''$ denote the complement of $\mathcal{M}_1'$ in $\mathcal{M}_1$; that is, $\mathcal{M}_1'' = \mathcal{M}_1 \setminus \mathcal{M}_1'$. Thus we see that $\mathcal{M}_1''$ is a closed subset of $\mathcal{M}_1$. A direct calculation (see [**2**], p. 128) shows that a metric g is in $\mathcal{M}_1''$ if and only if there is a nonzero function f such that

$$f_{;ij} - fR_{ij} - \Delta f \ g_{ij} = 0. \tag{10.2}$$

(Note that the function f is an element of the cokernel of the linearized operator of R.) Taking the trace of equation (10.2), we see that the function f is a solution of (10.1), and hence g is necessarily a degenerate solution of $R = n(n-1)$.

We now consider the map T from $\mathcal{M}_1$ to $\mathcal{C}$ which assigns to a metric g its conformal class $[g]$. The next result shows that the set $\mathcal{M}_1''$ is negligible for our purposes.

Lemma 10.2. *The set $T(\mathcal{M}_1'')$ is a closed nowhere dense subset of $\mathcal{C}$.*

Proof. The fact that $T(\mathcal{M}_1'')$ is a closed set follows directly from Lemma 10.1 since a convergent sequence of points of $\mathcal{C}$ is a compact set, and suitably normalized solutions of (10.2) will converge to solutions of (10.2) for the limiting metric.

To prove that $T(\mathcal{M}_1'')$ is nowhere dense, we use the well-known fact (see for example [**20**]) that the equation (10.2) can be rewritten as a system for the pair (g, f) as follows

$$fR_{ij} - f_{;ij} - ng_{ij} = 0, \quad \Delta f + nf = 0,$$

and moreover, this system is elliptic on the region where f is nonzero. It follows that if we have a sequence of solutions (g_i, f_i) converging in $C^{k,\alpha}$ norm with f_i normalized to have L^2 norm one, then on any ball in which the limit f is strictly nonzero, we will have (possibly after a change of coordinates) the convergence of any number of derivatives of (g_i, f_i). Thus, to show that $T(\mathcal{M}_1'')$ is nowhere dense, it suffices to show that for any solution g of (10.2), we can find a sequence of conformal classes c_i converging to $[g]$ in $\mathcal{C}$ such that there are no choices of $g_i \in c_i$ such that g_i converges to g in C^{k+1} norm relative to any coordinate choice on a ball of fixed radius (with respect to g) in M. To find such a sequence, we observe that (for $n \geq 4$) it is possible to modify a metric near a point by an arbitrarily small amount in $C^{k,\alpha}$ while keeping the maximum value of the $(k-1)$-st derivative of the Weyl tensor large (strictly larger than the corresponding quantity for g' for all $g' \in [g]$ which are singular points of R). By doing such a deformation in small disjoint balls in (M, g), we can construct a sequence of metrics g_i converging to g in $C^{k,\alpha}$ norm with the property that in all balls of a fixed chosen radius the Weyl tensor $W(g_i)$ does **not** converge to $W(g)$ in the C^{k-1} norm. Setting $c_i = [g_i]$, we claim that for i sufficiently large we must have $c_i \notin T(\mathcal{M}_1'')$.

This follows because, as described above, if there were a subsequence of $c_i \in T(\mathcal{M}_1'')$, we could then choose a sequence of metrics $g_i' \in c_i$ with g_i' a singular point of R converging to a singular point $g' \in [g]$. It would then follow that for a ball contained away from the zero set of the limiting function, we would have convergence of all derivatives of g_i' to those of g'. In particular, the Weyl tensors would converge along with all derivatives. This contradicts the choice of g_i in light of the conformal invariance of the Weyl tensor. An analogous argument can be made for $n = 3$ using the three dimensional conformal tensor. This completes the proof that $T(\mathcal{M}_1'')$ is nowhere dense in $\mathcal{C}$. q.e.d.

The map T is a Fredholm map of index 0 whose linearization is an isomorphism if and only if g is a nondegenerate solution of $R = n(n-1)$. A conformal class is a reguar value of T if and only if the class is nondegenerate. It follows from Lemma 10.1 that the map T is a proper map, and in particular, that any regular value corresponds to a conformal class containing a finite number of nondegenerate solutions of $R = n(n-1)$. It also implies that the set of regular values of T is an *open* subset of $\mathcal{C}$. The Sard-Smale Theorem implies that the restriction of T to the Banach manifold $\mathcal{M}_1'$ has a dense set of regular values. Combining these remarks with Lemma 10.2, we obtain the following theorem:

Theorem 10.3. *There is an open dense subset of conformal classes in the natural $C^{k,\alpha}$ topology (for k large enough depending only on n) which are nondegenerate in the sense that each such conformal class contains a finite number of nondegenerate solutions of the equation $R = n(n-1)$.*

We now discuss the existence theory outlined in the introduction in more detail. It is known (see [**22**], p. 279) that the subcritical regularization of the Yamabe variational problem satisfies the Palais-Smale condition (on the manifold of nonnegative functions). If we are in a nondegenerate conformal class, then for $p < (n+2)/(n-2)$, but near $(n+2)/(n-2)$, all critical points must also be nondegenerate with the same Morse indices since they converge to Yamabe solutions in a C^2 topology. Since by the work of this paper all solutions converge, it is enough to prove the Morse inequalities, Theorem 1.3, and Theorem 1.4 for the subcritical (but nearly critical) case since these results refer only to the Morse indices of the solutions. These results follow from the Morse theory of Palais and Smale (see [**24**], p. 198).

Appendix A. Pohozaev's Quadratic Form

In what follows H_{ij}, where $1 \le i, j \le n$, will denote a matrix whose entries are polynomials in n variables. We will use repeated indices to denote summations. Given H_{ij} and W_{ij} matrices of polynomials, we define

$$(H_{ij}, W_{ij}) = \int_{|x|=1} H_{ij} W_{ij} d\sigma.$$

We say that $H_{ij} \in \mathcal{V}_k$, where k is a nonnegative integer, if each polynomial entry is homogeneous of degree k, and the following holds:

1) $H_{ij} = H_{ji}$;
2) $H_{ii} = 0$;
3) $x_i H_{ij} = 0$.

Note that

$$x_i \partial_j H_{ij} = \partial_j (x_i H_{ij}) - \delta_{ij} H_{ij} = 0.$$

We also define $\mathcal{V}_{\leq k} = \bigoplus_{j=2}^{k} \mathcal{V}_j$. Given $H_{ij} \in \mathcal{V}_{\leq k}$, we denote by $H_{ij}^{(l)}$ its component of degree l.

Let $(\delta H)_i = \partial_j H_{ij}$ and $\delta^2 h = \partial_i \partial_j H_{ij}$. We will refer to δH and $\delta^2 H$ as divergence and double divergence of H_{ij}, respectively.

Given $H_{ij} \in \mathcal{V}_k$, integration by parts shows that

$$\int_{|x|=1} \partial_i \partial_j H_{ij} = 0, \tag{A.1}$$

and

$$\int_{|x|=1} x_l \partial_i \partial_j H_{ij} = 0, \tag{A.2}$$

where $1 \leq l \leq n$.

Define

$$\begin{aligned} b_k &= \int_0^\infty \frac{s^{k+n-3}}{(1+s^2)^{n-1}} ds \quad \text{for} \quad k < n, \\ c_k &= -\int_0^\infty \frac{(1-s^2)s^{k+n-3}}{(1+s^2)^{n-1}} ds \quad \text{for} \quad k < n-2. \end{aligned}$$

The next lemma relates some of the above integrals.

Lemma A.1.

$$b_{2m} = \left(\prod_{j=1}^{m} \frac{n+2j-4}{n-2j} \right) b_0,$$

$$c_{2m} = \frac{4m}{n-2m-2} b_{2m}.$$

Proof. Integration by parts yields

$$b_{2m} = \frac{n+2m-4}{n-2m} b_{2m-2}.$$

The first equality follows by induction. The second equality follows from the first one since

$$c_{2m} = b_{2m+2} - b_{2m}.$$

q.e.d.

Recall $d = [\frac{n-2}{2}]$, and that $\theta_k = 1$ if $k = \frac{n-2}{2}$ while $\theta_k = 0$ otherwise. Let $H_{ij} \in \mathcal{V}_{\leq d}$.

If n is odd, let

$$\begin{aligned}&\frac{2}{n-2} I^{(n)}_{1,\varepsilon}(H,H)\\ &= \sum_{s,t=2}^{d} \varepsilon^{s+t} c_{s+t} \int_{S_1} \left(-\frac{1}{2} \partial_j H^{(s)}_{ij} \partial_l H^{(t)}_{il} + \frac{1}{4} \partial_l H^{(s)}_{ij} \partial_l H^{(t)}_{ij} \right),\end{aligned}$$

while, if n is even,

$$\begin{aligned}&\frac{2}{n-2} I^{(n)}_{1,\varepsilon}(H,H)\\ &= \sum_{s,t=2}^{d-1} \varepsilon^{s+t} c_{s+t} \int_{S_1} \left(-\frac{1}{2} \partial_j H^{(s)}_{ij} \partial_l H^{(t)}_{il} + \frac{1}{4} \partial_l H^{(s)}_{ij} \partial_l H^{(t)}_{ij} \right)\\ &\quad + \varepsilon^{n-2} |\log \varepsilon| \int_{S_1} \left(-\frac{1}{2} \partial_j H^{(d)}_{ij} \partial_l H^{(d)}_{il} + \frac{1}{4} \partial_l H^{(d)}_{ij} \partial_l H^{(d)}_{ij} \right).\end{aligned}$$

Throughout the rest of the appendix $g_{ij} = \exp(h_{ij})$, $\operatorname{tr} h_{ij}(x) = O(|x|^N)$, where N is large. We will also write $h_{ij}(x) = H_{ij}(x) + O(|x|^{d+1})$, and

$$H_{ij}(x) = \sum_{|\alpha|=2}^{d} h_{ij\alpha} x^\alpha.$$

In the following estimates, we will ignore the contributions coming from the volume element, since we can choose N sufficiently large.

Lemma A.2. *Given $\eta > 0$, there exists $C > 0$, depending only on n and $|g|_{C^N(B_\sigma(0))}$, such that*

$$\begin{aligned}&\left| -\int_{|x|\leq\sigma} \left(\frac{1}{2} x^k \partial_k R_g + R_g \right) u_\varepsilon^2 dx - I^{(n)}_{1,\varepsilon}(H,H) \right|\\ &\leq C\eta \sum_{|\alpha|=2}^{d} \sum_{i,j} |h_{ij\alpha}|^2 \varepsilon^{2|\alpha|} |\log \varepsilon|^{\theta_{|\alpha|}} + C\sigma\eta^{-1}\varepsilon^{n-2},\end{aligned}$$

if $\sigma \leq 1$ and ε is sufficiently small.

Proof. After integrating by parts,

$$-\int_{|x|\leq\sigma} \left(\frac{1}{2} x^k \partial_k R_g + R_g \right) u_\varepsilon^2 dx = \int_{|x|\leq\sigma} R_g u_\varepsilon \psi_\varepsilon dx - \frac{\sigma}{2} \int_{|x|=\sigma} R_g u_\varepsilon^2,$$

where $\psi_\varepsilon = \frac{n-2}{2} u_\varepsilon + x^k \partial_k u_\varepsilon$.

Using estimate (4.6), we conclude that there exists $C > 0$ such that for any given $\eta > 0$,

$$\left|R_g - \partial_i\partial_j h_{ij} + \partial_j(H_{ij}\partial_l H_{il}) - \frac{1}{2}\partial_j H_{ij}\partial_l H_{il} + \frac{1}{4}\partial_l H_{ij}\partial_l H_{ij}\right|$$
$$\leq C\sum_{|\alpha|=2}^{d}\sum_{i,j}|h_{ij\alpha}|^2|x|^{2|\alpha|} + \eta\sum_{|\alpha|=2}^{d}\sum_{i,j}|h_{ij\alpha}|^2|x|^{2|\alpha|-2} + C\eta^{-1}|x|^{n-3}.$$

Note that our expression differs from the one in (4.6) because here $\deg(H_{ij}) \leq d$.

Since

$$\int_{|x|\leq\sigma}(\partial_i\partial_j h_{ij} - \partial_j(H_{ij}\partial_l H_{il}))u_\varepsilon\psi_\varepsilon dx = 0,$$

we have

$$\int_{|x|\leq\sigma} R_g u_\varepsilon\psi_\varepsilon dx = \int_{|x|\leq\sigma}\left(\frac{1}{2}\partial_j H_{ij}\partial_l H_{il} - \frac{1}{4}\partial_l H_{ij}\partial_l H_{ij}\right)u_\varepsilon\psi_\varepsilon dx$$
$$+ O(\eta)\sum_{|\alpha|=2}^{d}\sum_{i,j}|h_{ij\alpha}|^2\varepsilon^{2|\alpha|}|\log\varepsilon|^{\theta_{|\alpha|}} + O(\sigma\eta^{-1}\varepsilon^{n-2}).$$

But

$$\int_{|x|\leq\sigma}\left(\frac{1}{2}\partial_j H_{ij}\partial_l H_{il} - \frac{1}{4}\partial_l H_{ij}\partial_l H_{ij}\right)u_\varepsilon\psi_\varepsilon dx$$
$$= \sum_{s,t=2}^{d}\left(\int_0^\sigma u_\varepsilon\psi_\varepsilon r^{s+t+n-3}dr\right)\int_{S_1}\left(\frac{1}{2}\partial_j H_{ij}^{(s)}\partial_l H_{il}^{(t)} - \frac{1}{4}\partial_l H_{ij}^{(s)}\partial_l H_{ij}^{(t)}\right),$$

where $S_1 = \{|x| = 1\}$.

Now

$$\int_0^\sigma u_\varepsilon\psi_\varepsilon r^{s+t+n-3}dr = \frac{n-2}{2}\varepsilon^{s+t}\int_0^{\frac{\sigma}{\varepsilon}}\frac{(1-r^2)r^{s+t+n-3}}{(1+r^2)^{n-1}}dr$$
$$= -\frac{n-2}{2}\varepsilon^{s+t}|\log\varepsilon|^{\theta_{\frac{s+t}{2}}}c_{s+t}^{1-\theta_{\frac{s+t}{2}}} + O(\varepsilon^{n-2}).$$

The result follows immediately. q.e.d.

If n is odd, define

$$I_{2,\varepsilon}^{(n)}(H,H) = -\sum_{k,l=4}^{d}k\varepsilon^{k+l}\int_{\mathbb{R}^n}\delta^2(H^{(k)})Z(H^{(l)})U dy,$$

while, if n is even,

$$I_{2,\varepsilon}^{(n)}(H,H) = -\sum_{k,l=4}^{d-1} k\varepsilon^{k+l}\int_{\mathbb{R}^n}\delta^2(H^{(k)})Z(H^{(l)})U\,dy$$
$$- d\varepsilon^{n-2}|\log\varepsilon|\int_{S_1}(\delta^2 H^{(d)})\Gamma^{(d+2)}(\delta^2 H^{(d)}).$$

Here $Z(H^{(k)}) = \Gamma(\delta^2 H^{(k)})(1+|y|^2)^{-\frac{n}{2}}$ denotes the solution of

$$\Delta Z(H^{(k)}) + n(n+2)U^{\frac{4}{n-2}}Z(H^{(k)}) = \frac{n-2}{4(n-1)}\delta^2(H^{(k)})U,$$

as in Section 4.

Lemma A.3. *Given $\eta > 0$, there exists $C > 0$, depending only on n and $|g|_{C^N(B_\sigma(0))}$, such that*

$$\left|-2\int_{|x|\le\sigma}\left(\frac{1}{2}x^k\partial_k R_g + R_g\right)u_\varepsilon z_\varepsilon dx - I_{2,\varepsilon}^{(n)}(H,H)\right|$$
$$\le C\eta\sum_{|\alpha|=2}^{d}\sum_{i,j}|h_{ij\alpha}|^2\varepsilon^{2|\alpha|}|\log\varepsilon|^{\theta_{|\alpha|}} + C\sigma\eta^{-1}\varepsilon^{n-2},$$

if $\sigma \le 1$ and ε is sufficiently small.

Proof. Recall

$$|z_\varepsilon|(x) \le C\varepsilon^{\frac{n-2}{2}}\sum_{|\alpha|=4}^{n-4}\sum_{i,j}|h_{ij\alpha}|(\varepsilon+|x|)^{|\alpha|+2-n}. \tag{A.3}$$

In particular,

$$|z_\varepsilon|(x) \le C\varepsilon^{\frac{n-2}{2}}(\varepsilon+|x|)^{6-n}.$$

Therefore, by (4.7)

$$-2\int_{|x|\le\sigma}\left(\frac{1}{2}x^k\partial_k R_g + R_g\right)u_\varepsilon z_\varepsilon dx$$
$$= -2\int_{|x|\le\sigma}\left(\frac{1}{2}x^k\partial_k(\delta^2 h) + \delta^2 h\right)u_\varepsilon z_\varepsilon dx$$
$$+\sum_{|\alpha|=2}^{d}\sum_{i,j}|h_{ij\alpha}|^2 o(\varepsilon^{2|\alpha|}|\log\varepsilon|^{\theta_{|\alpha|}}) + O(\sigma^5\varepsilon^{n-2}).$$

Moreover, using estimate (A.3),

$$
\begin{aligned}
&-2\int_{|x|\le\sigma}\left(\frac{1}{2}x^k\partial_k(\delta^2 h)+\delta^2 h\right)u_\varepsilon z_\varepsilon dx\\
&=-2\int_{|x|\le\sigma}\left(\frac{1}{2}x^k\partial_k(\delta^2 H)+\delta^2 H\right)u_\varepsilon z_\varepsilon dx\\
&\quad+O(\eta)\sum_{|\alpha|=2}^{d}\sum_{i,j}|h_{ij\alpha}|^2\varepsilon^{2|\alpha|}|\log\varepsilon|^{\theta_{|\alpha|}}+O(\sigma\eta^{-1}\varepsilon^{n-2})\\
&=-2\int_{|x|\le\sigma}\left(\frac{1}{2}x^k\partial_k(\delta^2 H)+\delta^2 H\right)u_\varepsilon z_\varepsilon^{\le d} dx\\
&\quad+O(\eta)\sum_{|\alpha|=2}^{d}\sum_{i,j}|h_{ij\alpha}|^2\varepsilon^{2|\alpha|}|\log\varepsilon|^{\theta_{|\alpha|}}+O(\sigma\eta^{-1}\varepsilon^{n-2}).
\end{aligned}
$$

Here

$$
\Delta z_\varepsilon^{\le d}+n(n+2)u_\varepsilon^{\frac{4}{n-2}}z_\varepsilon^{\le d}=c(n)\sum_{k=4}^{d}\delta^2 H^{(k)}u_\varepsilon,
$$

so that $z_\varepsilon^{\le d}$ depends linearly on $\delta^2 H$.

Now

$$
\begin{aligned}
&-2\int_{|x|\le\sigma}\left(\frac{1}{2}x^k\partial_k(\delta^2 H)+\delta^2 H\right)u_\varepsilon z_\varepsilon^{\le d} dx\\
&=-2\sum_{k,l=4}^{d}\int_{|x|\le\sigma}\left(\frac{1}{2}x^k\partial_k(\delta^2 H^{(k)})+\delta^2 H^{(k)}\right)u_\varepsilon z_\varepsilon^{(l)} dx\\
&=-\sum_{k,l=4}^{d}k\varepsilon^{k+l}\int_{|y|\le\sigma\varepsilon^{-1}}\delta^2(H^{(k)})Z(H^{(l)})U dy.
\end{aligned}
$$

If n is odd,

$$
\begin{aligned}
&-2\int_{|x|\le\sigma}\left(\frac{1}{2}x^k\partial_k(\delta^2 H)+\delta^2 H\right)u_\varepsilon z_\varepsilon^{\le d} dx\\
&=-\sum_{k,l=4}^{d}k\varepsilon^{k+l}\int_{\mathbb{R}^n}\delta^2(H^{(k)})Z(H^{(l)})U dy\\
&\quad+O\left(\sum_{|\alpha|=2}^{d}\sum_{i,j}|h_{ij\alpha}|^2\varepsilon^{n-2}\right).
\end{aligned}
$$

If n is even,

$$\begin{aligned}
&-2\int_{|x|\leq\sigma}\left(\frac{1}{2}x^k\partial_k(\delta^2 H)+\delta^2 H\right)u_\varepsilon z_\varepsilon^{\leq d}dx\\
&=-\sum_{k,l=4}^{d-1}k\varepsilon^{k+l}\int_{\mathbb{R}^n}\delta^2(H^{(k)})Z(H^{(l)})U dy\\
&\quad-d\varepsilon^{n-2}\int_{|y|\leq\sigma\varepsilon^{-1}}\delta^2(H^{(d)})Z(H^{(d)})U dy\\
&\quad+O\left(\sum_{|\alpha|=2}^{d-1}\sum_{i,j}|h_{ij\alpha}|^2\varepsilon^{n-2}\right)\\
&\quad+O(\eta)\sum_{|\alpha|=2}^{d-1}\sum_{i,j}|h_{ij\alpha}|^2\varepsilon^{2|\alpha|}+O\left(\eta^{-1}\sum_{|\alpha|=d}\sum_{i,j}|h_{ij\alpha}|^2\varepsilon^{n-2}\right).
\end{aligned}$$

But in this case

$$\begin{aligned}
&-d\varepsilon^{n-2}\int_{|y|\leq\sigma\varepsilon^{-1}}\delta^2(H^{(d)})Z(H^{(d)})U dy\\
&=-d\varepsilon^{n-2}\int_{|y|\leq\sigma\varepsilon^{-1}}(\delta^2 H^{(d)})\Gamma(\delta^2 H^{(d)})(1+|y|^2)^{1-n}dy\\
&=-d\varepsilon^{n-2}|\log\varepsilon|\int_{S_1}(\delta^2 H^{(d)})\Gamma^{(d+2)}(\delta^2 H^{(d)})\\
&\qquad+O\left(\sum_{|\alpha|=d}\sum_{i,j}|h_{ij\alpha}|^2\varepsilon^{n-2}\right).
\end{aligned}$$

q.e.d.

We want to study the positivity of the Pohozaev's quadratic form

$$I_\varepsilon^{(n)}=I_{1,\varepsilon}^{(n)}+I_{2,\varepsilon}^{(n)}$$

on the space $\mathcal{V}_{\leq d}$. The main result of this appendix is

Proposition A.4. *There exists $\beta>0$ such that, if $6\leq n\leq 24$,*

$$I_\varepsilon^{(n)}(H,H)\geq\beta\sum_{k=2}^{d}\varepsilon^{2k}|\log\varepsilon|^{\theta_k}(H_{ij}^{(k)},H_{ij}^{(k)})$$

for $H_{ij}\in\mathcal{V}_{\leq d}$.

Before proving Proposition A.4, we need to better understand the structure of $\mathcal{V}_{\leq d}$. We will begin by discussing a projection onto $\mathcal{V}_k$.

Let $\mathcal{P}_k$ denote the space of homogeneous polynomials of degree k.

Lemma A.5. *Let* $\tilde{H}_{ij}$ *be a symmetric matrix of homogeneous polynomials of degree* k*. Suppose there exist* $p, t \in \mathcal{P}_{k-2}$*,* $q_j \in \mathcal{P}_{k-1}$ *such that*

1) $\tilde{H}_{ii} = -p|x|^2$;
2) $x_i \tilde{H}_{ij} = -q_j |x|^2$;
3) $x_i x_j \tilde{H}_{ij} = -t|x|^4$.

If

$$b_j = q_j - \frac{p x_j}{2(n-1)} - \frac{(n-2) t x_j}{2(n-1)},$$

and

$$c = \frac{(p-t)|x|^2}{n-1},$$

then

$$H_{ij} = \tilde{H}_{ij} + b_i x_j + b_j x_i + c\delta_{ij} \in \mathcal{V}_k.$$

Remark. When this lemma applies, we will say $H_{ij} = \mathrm{Proj}(\tilde{H}_{ij})$.

Proof. It is straightforward to check that $H_{ij} = H_{ji}$, $H_{ii} = 0$ and $x_i H_{ij} = 0$. q.e.d.

We will say that

$$S_{ij} = \mathrm{mod}(x_i, x_j, \delta_{ij})$$

if there exist b_i, c such that $S_{ij} = b_i x_j + b_j x_i + c\delta_{ij}$.

Let us now define $\mathcal{L}_k : \mathcal{V}_k \to \mathcal{V}_k$ by

$$\mathcal{L}_k(H_{ij}) = \mathrm{Proj}(|x|^2 F(H_{ij})),$$

where

$$F(H_{ij}) = \frac{1}{4}\partial_j \partial_l H_{il} + \frac{1}{4}\partial_i \partial_l H_{jl} - \frac{1}{4}\Delta H_{ij}.$$

The Lemma A.5 implies

$$\begin{aligned}\mathcal{L}_k(H_{ij}) = {} & \frac{1}{4}|x|^2(\partial_j \partial_l H_{il} + \partial_i \partial_l H_{jl} - \Delta H_{ij}) \\ & - \frac{k}{4} x_j \partial_l H_{il} - \frac{k}{4} x_i \partial_l H_{jl} \\ & + \frac{1}{2(n-1)} \delta^2 H (x_i x_j - |x|^2 \delta_{ij}).\end{aligned} \tag{A.4}$$

Note that, given $H_{ij} \in \mathcal{V}_k$ and $W_{ij} \in \mathcal{V}_m$,

$$\begin{aligned} 4(\mathcal{L}_k(H_{ij}), W_{ij}) &= 4\int_{S_1} \mathrm{Proj}(|x|^2 F(H_{ij}))W_{ij} \\ &= 4\int_{S_1} F(H_{ij})W_{ij} \\ &= \int_{S_1} (\partial_j\partial_l H_{il} + \partial_i\partial_l H_{jl} - \Delta H_{ij})W_{ij} \\ &= -2\int_{S_1} \partial_l H_{il}\partial_j W_{ij} + \int_{S_1} \partial_l H_{ij}\partial_l W_{ij} \\ &\quad - k(n+k+m-2)\int_{S_1} H_{ij}W_{ij}. \end{aligned}$$

In particular, $\mathcal{L}_k : \mathcal{V}_k \to \mathcal{V}_k$ is symmetric with respect to the inner product $(\cdot,\cdot)$. Let us now proceed to the analysis of its eigenvalues.

Let $p_l \in \mathcal{P}_l$ such that $2 \le l \le k-2$ and $\Delta p_l = 0$. Define

$$\hat{H}_{ij} = \mathrm{Proj}(\partial_i\partial_j p_l|x|^{2m}),$$

where $k = l - 2 + 2m$. Note that $m \ge 2$.

Then Lemma A.5 implies

$$\begin{aligned} \hat{H}_{ij} = {} & \partial_i\partial_j p_l|x|^{2m} \\ & - (l-1)x_i\partial_j p_l|x|^{2m-2} - (l-1)x_j\partial_i p_l|x|^{2m-2} \\ & + \frac{n-2}{n-1}l(l-1)p_l x_i x_j|x|^{2m-4} + \frac{1}{n-1}l(l-1)p_l\delta_{ij}|x|^{2m-2}. \end{aligned}$$

A calculation shows that

$$\begin{aligned} \partial_i\hat{H}_{ij} = & -\frac{n-2}{n-1}(l-1)(n+l-1)\partial_j p_l|x|^{2m-2} \\ & + \frac{n-2}{n-1}l(l-1)(n+l-1)x_j p_l|x|^{2m-4}, \end{aligned} \tag{A.5}$$

and

$$\delta^2\hat{H} = \frac{n-2}{n-1}l(l-1)(n+l-1)(n+l-2)p_l|x|^{2m-4}. \tag{A.6}$$

From identity (A.5), and the fact that

$$\begin{aligned} \Delta\hat{H}_{ij} = {} & \left(2m(n+2m+2l-6) - 4(l-1)\right)\partial_i\partial_j p_l|x|^{2m-2} \\ & + \mathrm{mod}(x_i, x_j, \delta_{ij}), \end{aligned}$$

we obtain

$$|x|^2 F(\hat{H}_{ij}) = A_{l,m}\partial_i\partial_j p_l|x|^{2m} + \mathrm{mod}(x_i, x_j, \delta_{ij}).$$

Here

$$A_{l,m} = (l-1)\left(1 - \frac{n-2}{2(n-1)}(n+l-1)\right) - \frac{m}{2}(n+2m+2l-6). \tag{A.7}$$

Therefore,

$$\mathcal{L}_k(\hat{H}_{ij}) = A_{l,m}\hat{H}_{ij}.$$

Lemma A.6. *Let $H_{ij} \in \mathcal{V}_k$. Then there exist $p_{k-2q} \in \mathcal{P}_{k-2q}$, $q = 1, \dots, [\frac{k-2}{2}]$, $\Delta p_{k-2q} = 0$, such that, if*

$$(\hat{H}_q)_{ij} = \mathrm{Proj}(\partial_i\partial_j p_{k-2q}|x|^{2q+2}),$$

and

$$H_{ij} = W_{ij} + \sum_{q=1}^{[\frac{k-2}{2}]} (\hat{H}_q)_{ij},$$

then

$$\partial_i\partial_j W_{ij} = 0.$$

Moreover,

$$\begin{aligned}(W_{ij}, (\hat{H}_q)_{ij}) &= 0,\\ ((\hat{H}_q)_{ij}, (\hat{H}_s)_{ij}) &= 0 \quad \text{if } q \neq s,\\ \mathcal{L}_k((\hat{H}_q)_{ij}) &= A_{k-2q,q+1}(\hat{H}_q)_{ij}.\end{aligned}$$

Proof. The existence of the polynomials p_{k-2q} so that $\delta^2 W = 0$ follows from the decomposition of $\delta^2 H$ in spherical harmonics, noting equalities (A.1), (A.2), and using identity (A.6).

Moreover,

$$\begin{aligned}\int_{S_1} W_{ij}(\hat{H}_q)_{ij} &= \int_{S_1} W_{ij}\mathrm{Proj}(\partial_i\partial_j p_{k-2q}|x|^{2q+2})\\ &= \int_{S_1} W_{ij}\partial_i\partial_j p_{k-2q}\\ &= \int_{S_1} \partial_i\partial_j W_{ij} p_{k-2q}\\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\int_{S_1} (\hat{H}_s)_{ij}(\hat{H}_q)_{ij} &= \int_{S_1} (\hat{H}_s)_{ij}\mathrm{Proj}(\partial_i\partial_j p_{k-2q}|x|^{2q+2})\\ &= \int_{S_1} (\hat{H}_s)_{ij}\partial_i\partial_j p_{k-2q}\\ &= \int_{S_1} \partial_i\partial_j(\hat{H}_s)_{ij} p_{k-2q}\\ &= c\int_{S_1} p_{k-2s}p_{k-2q}\\ &= 0,\end{aligned}$$

if $s \neq q$. Here we are using identity (A.6). q.e.d.

Define $\mathcal{W}_k = \{W_{ij} \in \mathcal{V}_k : \partial_i\partial_j W_{ij} = 0\}$.

Let $W_{ij} \in \mathcal{W}_k$. A calculation then shows

$$\partial_i \mathcal{L}_k(W_{ij}) = -\frac{(n+k-2)k}{4}\partial_i W_{ij}. \tag{A.8}$$

In particular, $\partial_i\partial_j \mathcal{L}_k(W_{ij}) = 0$.

Hence $\mathcal{L}_k : \mathcal{W}_k \to \mathcal{W}_k$.

Suppose W_{ij} is an eigenvector of $\mathcal{L}_k$: $\mathcal{L}_k(W_{ij}) = \lambda W_{ij}$. The identity (A.8) implies that either

$$\lambda = -\frac{(n+k-2)k}{4},$$

or

$$\partial_i W_{ij} = 0.$$

Define $\mathcal{D}_k = \{D_{ij} \in \mathcal{V}_k : \partial_i D_{ij} = 0\}$. The identity (A.8) implies $\mathcal{L}_k : \mathcal{D}_k \to \mathcal{D}_k$.

Let $D_{ij} \in \mathcal{D}_k$. Then, from identity (A.4),

$$\mathcal{L}_k(D_{ij}) = -\frac{1}{4}|x|^2 \Delta D_{ij}.$$

Hence $|x|^2\Delta : \mathcal{D}_k \to \mathcal{D}_k$, and since

$$\begin{aligned}|x|^{2m+2}\Delta^{m+1} D_{ij} &= |x|^2\Delta(|x|^{2m}\Delta^m D_{ij}) \\ &\quad - 2m(n+2k-2m-2)|x|^{2m}\Delta^m D_{ij},\end{aligned}$$

an inductive argument shows $|x|^{2m}\Delta^m : \mathcal{D}_k \to \mathcal{D}_k$ for every $m \geq 1$.

Now let us consider the decomposition

$$D_{ij} = \sum_{q=0}^{[\frac{k}{2}]} |x|^{2q} M_{ij}^{(k-2q)},$$

where $\Delta M_{ij}^{(k-2q)} = 0$. Since $|x|^{2m}\Delta^m$ leaves $\mathcal{D}_k$ invariant, we get

$$|x|^{2q} M_{ij}^{(k-2q)} \in \mathcal{D}_k$$

for every $0 \leq q \leq [\frac{k}{2}]$. In other words, $M_{ij}^{(k-2q)} = M_{ji}^{(k-2q)}$, $M_{ii}^{(k-2q)} = 0$, $x_i M_{ij}^{(k-2q)} = 0$, and $\partial_i M_{ij}^{(k-2q)} = 0$.

In particular, $M_{ij}^{(0)} = 0$ or $M_{ij}^{(1)} = 0$, according to whether k is even or odd. In order to see this, just note that $M_{ij}^{(0)} x_i x_j = 0$ or $\partial_l M_{ij}^{(1)} x_i x_j = 0$.

Now

$$\begin{aligned}\mathcal{L}_k(|x|^{2q} M_{ij}^{(k-2q)}) &= -\frac{1}{4}|x|^2\Delta(|x|^{2q} M_{ij}^{(k-2q)}) \\ &= -\frac{1}{2}q(n-2q+2k-2)|x|^{2q} M_{ij}^{(k-2q)}.\end{aligned}$$

Therefore, we have proved

Lemma A.7. *Let $W_{ij} \in \mathcal{W}_k$. Then there exist $\hat{W}_{ij} \in \mathcal{W}_k$, $M_{ij}^{(k-2q)} \in \mathcal{D}_{k-2q}$, $q = 0, \ldots, [\frac{k-2}{2}]$, such that*

$$W_{ij} = \hat{W}_{ij} + \sum_{q=0}^{[\frac{k-2}{2}]} |x|^{2q} M_{ij}^{(k-2q)},$$

where

$$\mathcal{L}_k(\hat{W}_{ij}) = -\frac{(n+k-2)k}{4}\hat{W}_{ij},$$

and

$$\Delta M_{ij}^{(k-2q)} = 0.$$

Moreover,

$$\begin{aligned}(\hat{W}_{ij}, |x|^{2q} M_{ij}^{(k-2q)}) &= 0,\\ (|x|^{2q} M_{ij}^{(k-2q)}, |x|^{2s} M_{ij}^{(k-2s)}) &= 0 \quad \text{if } q \neq s,\\ \mathcal{L}_k(|x|^{2q} M_{ij}^{(k-2q)}) &= -\frac{1}{2} q(n-2q+2k-2)|x|^{2q} M_{ij}^{(k-2q)}.\end{aligned}$$

Remark. Note that

$$-\frac{(n+k-2)k}{4} \neq -\frac{1}{2}q(n-2q+2k-2) \tag{A.9}$$

for $q = 0, \ldots, [\frac{k-2}{2}]$.

Let us now go back to the study of the positivity of $I_\varepsilon^{(n)}$ on $\mathcal{V}_{\leq d}$.

Proof of Proposition A.4. Given $H_{ij}^{(m)} \in \mathcal{V}_m$, $\overline{H}_{ij}^{(k)} \in \mathcal{V}_k$, define

$$B(H_{ij}^{(m)}, \overline{H}_{ij}^{(k)}) = \int_{S_1} \left(-\frac{1}{2}\partial_j H_{ij}^{(m)} \partial_l \overline{H}_{il}^{(k)} + \frac{1}{4}\partial_l H_{ij}^{(m)} \partial_l \overline{H}_{ij}^{(k)}\right).$$

Hence,

$$\begin{aligned}B(H_{ij}^{(m)}, \overline{H}_{ij}^{(k)}) &= \int_{S_1} H_{ij}^{(m)} \left(\frac{1}{4}\partial_j \partial_l \overline{H}_{il}^{(k)} + \frac{1}{4}\partial_i \partial_l \overline{H}_{jl}^{(k)} - \frac{1}{4}\Delta \overline{H}_{ij}^{(k)}\right)\\ &\quad + \frac{(n+m+k-2)k}{4}\int_{S_1} H_{ij}^{(m)} \overline{H}_{ij}^{(k)}\\ &= \int_{S_1} H_{ij}^{(m)} \left(\mathcal{L}_k(\overline{H}_{ij}^{(k)}) + \frac{(n+m+k-2)k}{4}\overline{H}_{ij}^{(k)}\right).\end{aligned}$$

Let $H_{ij} \in \mathcal{V}_{\leq d}$. Then, using the notation of Lemma A.6,

$$H_{ij}^{(k)} = W_{ij}^{(k)} + \sum_{q=1}^{[\frac{k-2}{2}]} (\hat{H}_q)_{ij}^{(k)}$$

for $2 \leq k \leq d$. Let $W_{ij} = \sum_{k=2}^{d} W_{ij}^{(k)}$, and $\hat{H}_{ij} = \sum_{k=2}^{d} \sum_{q=1}^{[\frac{k-2}{2}]} (\hat{H}_q)_{ij}^{(k)}$.

Now

$$I_\varepsilon^{(n)}(W_{ij}^{(m)}, (\hat{H}_q)_{ij}^{(k)}) = I_{1,\varepsilon}^{(n)}(W_{ij}^{(m)}, (\hat{H}_q)_{ij}^{(k)}),$$

since $\delta^2 W^{(m)} = 0$.

But

$$\begin{aligned}
&I_{1,\varepsilon}^{(n)}(W_{ij}^{(m)}, (\hat{H}_q)_{ij}^{(k)}) \\
&= cB(W_{ij}^{(m)}, (\hat{H}_q)_{ij}^{(k)}) \\
&= c\int_{S_1} W_{ij}^{(m)} \left(\mathcal{L}_k((\hat{H}_q)_{ij}^{(k)}) + \frac{(n+m+k-2)k}{4}(\hat{H}_q)_{ij}^{(k)}\right) \\
&= c'\int_{S_1} W_{ij}^{(m)}(\hat{H}_q)_{ij}^{(k)} \\
&= c'\int_{S_1} W_{ij}^{(m)}\partial_i\partial_j p_{k-2q}^{(k)} \\
&= c'\int_{S_1} \partial_i\partial_j W_{ij}^{(m)} p_{k-2q}^{(k)} \\
&= 0.
\end{aligned}$$

Therefore,

$$I_\varepsilon^{(n)}(H_{ij}, H_{ij}) = I_\varepsilon^{(n)}(W_{ij}, W_{ij}) + I_\varepsilon^{(n)}(\hat{H}_{ij}, \hat{H}_{ij}),$$

and

$$(H_{ij}, H_{ij}) = (W_{ij}, W_{ij}) + (\hat{H}_{ij}, \hat{H}_{ij}).$$

Now, using Lemma A.7,

$$W_{ij}^{(k)} = \hat{W}_{ij}^{(k)} + \sum_{q=0}^{[\frac{k-2}{2}]} |x|^{2q}(M_k)_{ij}^{(k-2q)},$$

where $2 \le k \le d$. Let $\hat{W}_{ij} = \sum_{k=2}^d \hat{W}_{ij}^{(k)}$ and

$$D_{ij} = \sum_{k=2}^{d}\sum_{q=0}^{[\frac{k-2}{2}]} |x|^{2q}(M_k)_{ij}^{(k-2q)}.$$

Now

$$\begin{aligned}
&B(\hat{W}_{ij}^{(m)}, |x|^{2q}(M_k)_{ij}^{(k-2q)}) \\
&= \int_{S_1} \hat{W}_{ij}^{(m)} \Big\{\mathcal{L}_k(|x|^{2q}(M_k)_{ij}^{(k-2q)}) \\
&\qquad + \frac{(n+m+k-2)k}{4}|x|^{2q}(M_k)_{ij}^{(k-2q)}\Big\} \\
&= c''\int_{S_1} \hat{W}_{ij}^{(m)}(M_k)_{ij}^{(k-2q)},
\end{aligned}$$

where

$$c'' = \frac{(n+m+k-2)k}{4} - \frac{1}{2}q(n-2q+2k-2).$$

On the other hand,

$$\begin{aligned}
&B(\hat{W}_{ij}^{(m)}, |x|^{2q}(M_k)_{ij}^{(k-2q)}) \\
&= \int_{S_1} |x|^{2q}(M_k)_{ij}^{(k-2q)} \left(\mathcal{L}_m(\hat{W}_{ij}^{(m)}) + \frac{(n+m+k-2)m}{4}\hat{W}_{ij}^{(m)} \right) \\
&= \frac{km}{4}\int_{S_1} \hat{W}_{ij}^{(m)}(M_k)_{ij}^{(k-2q)}.
\end{aligned}$$

Therefore, the inequalities (A.9) imply

$$B(\hat{W}_{ij}^{(m)}, |x|^{2q}(M_k)_{ij}^{(k-2q)}) = 0,$$

and

$$\int_{S_1} \hat{W}_{ij}^{(m)}|x|^{2q}(M_k)_{ij}^{(k-2q)} = 0.$$

Hence,

$$I_\varepsilon^{(n)}(H_{ij}, H_{ij}) = I_\varepsilon^{(n)}(\hat{W}_{ij}, \hat{W}_{ij}) + I_\varepsilon^{(n)}(D_{ij}, D_{ij}) + I_\varepsilon^{(n)}(\hat{H}_{ij}, \hat{H}_{ij}),$$

and

$$(H_{ij}, H_{ij}) = (\hat{W}_{ij}, \hat{W}_{ij}) + (D_{ij}, D_{ij}) + (\hat{H}_{ij}, \hat{H}_{ij}).$$

Therefore, we divide the study of the positivity of the quadratic form into three cases.

When n is even, due to the log term, we should start by analyzing it on $\mathcal{V}_d$. Therefore we define

$$J(H_{ij}^{(d)}, H_{ij}^{(d)}) = B(H_{ij}^{(d)}, H_{ij}^{(d)}) - \int_{S_1} (\delta^2 H^{(d)})\Gamma^{(d+2)}(\delta^2 H^{(d)}),$$

and

$$\begin{aligned}
(I')_\varepsilon^{(n)}(H,H) = \frac{n-2}{2}\sum_{s,t=2}^{d'} \varepsilon^{s+t}c_{s+t}B(H_{ij}^{(s)}, H_{ij}^{(t)}) \\
- \sum_{s,t=4}^{d'} s\varepsilon^{s+t}\int_{\mathbb{R}^n} \delta^2(H^{(s)})Z(H^{(t)})U dy,
\end{aligned}$$

where $d' = [\frac{n-3}{2}]$.

Since $(I')_\varepsilon^{(n)}(H,H) = (I')_1^{(n)}(H_\varepsilon, H_\varepsilon)$, where $(H_\varepsilon)^{(s)} = \varepsilon^s H^{(s)}$, we will restrict our analysis to J and $(I')_1^{(n)}$.

Case 1. $H_{ij} = \hat{W}_{ij} = \sum_{k=2}^d \hat{W}_{ij}^{(k)}$, where $\mathcal{L}_k(\hat{W}_{ij}^{(k)}) = -\frac{(n+k-2)k}{4}\hat{W}_{ij}^{(k)}$. First note that

$$B(\hat{W}_{ij}^{(k)}, \hat{W}_{ij}^{(m)}) = \frac{km}{4}(\hat{W}_{ij}^{(k)}, \hat{W}_{ij}^{(m)}).$$

Therefore,

$$J(\hat{W}_{ij}^{(d)}, \hat{W}_{ij}^{(d)}) = \frac{d^2}{4}|\hat{W}_{ij}^{(d)}|^2.$$

Thus, J is always positive in this case.

On the other hand,

$$\begin{aligned}(I')_1^{(n)}(\hat{W}, \hat{W}) &= \frac{n-2}{8}\sum_{s,t=2}^{d'} stc_{s+t}(\hat{W}_{ij}^{(s)}, \hat{W}_{ij}^{(t)}) \\ &= \frac{n-2}{8}\sum_{2\leq s,t\leq d'}^{s,t \text{ even}} stc_{s+t}(\hat{W}_{ij}^{(s)}, \hat{W}_{ij}^{(t)}) \\ &\quad + \frac{n-2}{8}\sum_{2\leq s,t\leq d'}^{s,t \text{ odd}} stc_{s+t}(\hat{W}_{ij}^{(s)}, \hat{W}_{ij}^{(t)}).\end{aligned}$$

Since we are only interested in $n \leq 24$, we just have to consider the cases

$$\hat{W}_{ij} = \sum_{k=1}^{4} \hat{W}_{ij}^{(2k+2)},$$

$$\hat{W}_{ij} = \sum_{k=1}^{4} \hat{W}_{ij}^{(2k+1)}.$$

We are using the fact that $\hat{W}_{ij}^{(2)} = 0$, since $\mathcal{V}_2 \subset \mathcal{D}_2$.

Let

$$m_{kl}^{\text{even}} = (2k+2)(2l+2)c_{2k+2l+4},$$

and

$$m_{kl}^{\text{odd}} = (2k+1)(2l+1)c_{2k+2l+2}.$$

With the help of Lemma A.1, we can check (see [**12**]) that for each $1 \leq p \leq 4$, the matrix

$$(m_{kl}^{\text{even}})_{1\leq k,l\leq p}$$

is positive definite if $4p+6 < n \leq 24$. The same is true for

$$(m_{kl}^{\text{odd}})_{1\leq k,l\leq p},$$

if $4p+4 < n \leq 24$. That implies the positivity of $(I)_\varepsilon^{(n)}$ in Case 1, for $n \leq 24$.

Case 2. $H_{ij} = D_{ij} = \sum_{k=2}^{d} D_{ij}^{(k)}$, where

$$D_{ij}^{(k)} = \sum_{q=0}^{[\frac{k-2}{2}]} |x|^{2q}(M_k)_{ij}^{(k-2q)},$$

$(M_k)_{ij}^{(k-2q)} \in \mathcal{D}_{k-2q}$, and $\Delta(M_k)_{ij}^{(k-2q)} = 0$.

First,

$$B(|x|^{2q}(M_k)_{ij}^{(k-2q)}, |x|^{2q'}(M_m)_{ij}^{(m-2q')}) = c'' \int_{S_1} (M_k)_{ij}^{(k-2q)} (M_m)_{ij}^{(m-2q')}, \tag{A.10}$$

where

$$c'' = \frac{(n+m+k-2)k}{4} - \frac{1}{2}q(n-2q+2k-2).$$

This implies

$$I_\varepsilon^{(n)}(|x|^{2q}(M_k)_{ij}^{(k-2q)}, |x|^{2q'}(M_m)_{ij}^{(m-2q')}) = 0,$$

if $k-2q \neq m-2q'$.

Define $(E_s)_{ij} = \sum_{0\le 2q \le d-s} |x|^{2q}(M_{s+2q})_{ij}^{(s)}$, for $2 \le s \le d$.

Hence $D_{ij} = \sum_{s=2}^{d}(E_s)_{ij}$,

$$I_\varepsilon^{(n)}(D_{ij}, D_{ij}) = \sum_{s=2}^{d} I_\varepsilon^{(n)}((E_s)_{ij}, (E_s)_{ij}),$$

$$(D_{ij}, D_{ij}) = \sum_{s=2}^{d}((E_s)_{ij}, (E_s)_{ij}).$$

From the equality (A.10), we obtain

$$B(|x|^{2q}(M_{s+2q})_{ij}^{(s)}, |x|^{2q'}(M_{s+2q'})_{ij}^{(s)}) = \left(qq' + \frac{(n+2q+2q'+2s-2)s}{4}\right)\left((M_{s+2q})_{ij}^{(s)}, (M_{s+2q'})_{ij}^{(s)}\right).$$

Fix $2 \le s \le d$.

If $s+2q = s+2q' = d$, then

$$J(|x|^{d-s}(M_d)_{ij}^{(s)}, |x|^{d-s}(M_d)_{ij}^{(s)}) = \frac{1}{4}\left(d^2+s^2+s(n-2)\right)|(M_d)_{ij}^{(s)}|^2.$$

This implies J is always positive in Case 2.

Now let us turn to the analysis of $(I')_1^{(n)}$, so $s+2q \le d'$.

Since we only need to consider $n \le 24$, we can restrict ourselves to $s = 2, \dots, 10$. For each such s, the problem will be reduced to analyzing a matrix of size at most $[\frac{12-s}{2}] \times [\frac{12-s}{2}]$.

Now

$$(I')_1^{(n)}(E_s, E_s) = \frac{n-2}{2} \sum_{2q,2q'=0}^{d'-s} m_{q+1,q'+1}^{s}((M_{s+2q})_{ij}^{(s)}, (M_{s+2q'})_{ij}^{(s)}),$$

where

$$m_{q+1,q'+1}^{s} = c_{2s+2q+2q'}\left(qq' + \frac{(n+2q+2q'+2s-2)s}{4}\right).$$

After lengthy calculations (see [**12**]), and with the help of Lemma A.1, we can verify that for each $2 \leq s \leq 10$, $1 \leq p \leq [\frac{12-s}{2}]$, the matrix

$$(m^s_{k,l})_{1\leq k,l\leq p}$$

is positive definite if $4p + 2s - 2 < n \leq 24$.

That implies the positivity of $(I)^{(n)}_\varepsilon$ in Case 2, for $n \leq 24$.

Case 3. $H_{ij} = \hat{H}_{ij} = \sum_{k=4}^{d} \hat{H}^{(k)}_{ij}$, where $\hat{H}^{(k)}_{ij} = \sum_{q=1}^{[\frac{k-2}{2}]}(\hat{H}_q)^{(k)}_{ij}$ and $(\hat{H}_q)^{(k)}_{ij} = \mathrm{Proj}(\partial_i\partial_j(p_k)_{k-2q}|x|^{2q+2})$, $(p_k)_{k-2q} \in \mathcal{P}_{k-2q}$, $\Delta(p_k)_{k-2q} = 0$.

First note that by Lemma A.6

$$\begin{aligned}
&B((\hat{H}_q)^{(k)}_{ij}, (\hat{H}_{q'})^{(m)}_{ij})\\
&= \int_{S_1} (\hat{H}_{q'})^{(m)}_{ij}\left(\mathcal{L}_k((\hat{H}_q)^{(k)}_{ij}) + \frac{(n+m+k-2)k}{4}(\hat{H}_q)^{(k)}_{ij}\right)\\
&= \left(A_{k-2q,q+1} + \frac{(n+m+k-2)k}{4}\right)\int_{S_1}(\hat{H}_q)^{(k)}_{ij}(\hat{H}_{q'})^{(m)}_{ij}.
\end{aligned}$$

Now

$$\begin{aligned}
\int_{S_1}(\hat{H}_q)^{(k)}_{ij}(\hat{H}_{q'})^{(m)}_{ij} &= \int_{S_1}(\hat{H}_q)^{(k)}_{ij}\partial_i\partial_j(p_m)_{m-2q'}\\
&= \int_{S_1}\partial_i\partial_j(\hat{H}_q)^{(k)}_{ij}(p_m)_{m-2q'}\\
&= \alpha_{k-2q}\int_{S_1}(p_k)_{k-2q}(p_m)_{m-2q'},
\end{aligned}$$

where

$$\alpha_l = \frac{n-2}{n-1}l(l-1)(n+l-1)(n+l-2).$$

In particular,

$$B((\hat{H}_q)^{(k)}_{ij}, (\hat{H}_{q'})^{(m)}_{ij}) = 0, \quad ((\hat{H}_q)^{(k)}_{ij}, (\hat{H}_{q'})^{(m)}_{ij}) = 0,$$

if $k - 2q \neq m - 2q'$.

Recall

$$\begin{aligned}
&\Delta Z((\hat{H}_q)^{(k)}_{ij}) + n(n+2)U^{\frac{4}{n-2}}Z((\hat{H}_q)^{(k)}_{ij})\\
&= \frac{n-2}{4(n-1)}\alpha_{k-2q}(p_k)_{k-2q}|x|^{2q-2}U.
\end{aligned}$$

We can write

$$Z((\hat{H}_q)^{(k)}_{ij}) = \frac{n-2}{4(n-1)}\alpha_{k-2q}\Gamma_{k,q}(1+|x|^2)^{-\frac{n}{2}},$$

where

$$T(\Gamma_{k,q}) = (p_k)_{k-2q}(|x|^{2q-2} + 2|x|^{2q} + |x|^{2q+2}).$$

Recall $T(\Gamma) = (1+|y|^2)\Delta\Gamma - 2ny\cdot\nabla\Gamma + 2n\Gamma$.

Then

$$T(|x|^{2j}(p_k)_{k-2q}) = (2k+2j-4q-2)(2j-n)|x|^{2j}(p_k)_{k-2q} + 2j(n+2j+2k-4q-2)|x|^{2j-2}(p_k)_{k-2q},$$

and we can write

$$\Gamma_{k,q} = \sum_{j=0}^{q+1} \Gamma(k,q,j)|x|^{2j}(p_k)_{k-2q}.$$

The coefficients $\Gamma(k,q,j)$ can then be computed inductively in the following way:

$$\Gamma(k,q,q+1) = -\frac{1}{(2k-2q)(n-2q-2)},$$
$$\Gamma(k,q,q) = -\frac{2-2(q+1)(n+2k-2q)\Gamma(k,q,q+1)}{(2k-2q-2)(n-2q)},$$
$$\Gamma(k,q,q-1) = -\frac{1-2q(n+2k-2q-2)\Gamma(k,q,q)}{(2k-2q-4)(n-2q+2)},$$

and

$$\Gamma(k,q,j) = \frac{2(j+1)(n+2j+2k-4q)}{(2k+2j-4q-2)(n-2j)}\Gamma(k,q,j+1),$$

for $0 \le j \le q-2$.

Therefore,

$$\begin{aligned}
&\frac{4(n-1)}{n-2}\int_{\mathbb{R}^n} \delta^2((\hat{H}_{q'})_{ij}^{(m)})Z((\hat{H}_q)_{ij}^{(k)})U\,dy \\
&= \alpha_{k-2q}\int_{\mathbb{R}^n} \delta^2((\hat{H}_{q'})_{ij}^{(m)})\Gamma_{k,q}(1+|y|^2)^{1-n}dy \\
&= \alpha_{k-2q}\alpha_{m-2q'}\int_{\mathbb{R}^n} (p_m)_{m-2q'}\Gamma_{k,q}|y|^{2q'-2}(1+|y|^2)^{1-n}dy \\
&= \alpha_{k-2q}\alpha_{m-2q'}\sum_{j=0}^{q+1}\Gamma(k,q,j)\int_{\mathbb{R}^n} \frac{(p_m)_{m-2q'}(p_k)_{k-2q}|y|^{2j+2q'-2}}{(1+|y|^2)^{n-1}}dy \\
&= \alpha_{k-2q}\alpha_{m-2q'}\left(\sum_{j=0}^{q+1}\Gamma(k,q,j)b_{k+m-2q+2j}\right)\int_{S_1}(p_k)_{k-2q}(p_m)_{m-2q'}.
\end{aligned}$$

In particular,

$$\int_{\mathbb{R}^n} \delta^2((\hat{H}_{q'})_{ij}^{(m)})Z((\hat{H}_q)_{ij}^{(k)})U\,dy = 0,$$

if $k-2q \neq m-2q'$.

Define, for $2 \leq s \leq d-2$,

$$(\hat{E}_s)_{ij} = \sum_{q=1}^{[\frac{d-s}{2}]} \mathrm{Proj}(\partial_i \partial_j (p_{s+2q})_s |x|^{2q+2}).$$

Then $\hat{H}_{ij} = \sum_{s=2}^{d-2} (\hat{E}_s)_{ij}$, and

$$I_\varepsilon^{(n)}(\hat{H}_{ij}, \hat{H}_{ij}) = \sum_{s=2}^{d-2} I_\varepsilon^{(n)}((\hat{E}_s)_{ij}, (\hat{E}_s)_{ij}),$$

$$(\hat{H}_{ij}, \hat{H}_{ij}) = \sum_{s=2}^{d-2} ((\hat{E}_s)_{ij}, (\hat{E}_s)_{ij}).$$

When n is even, and if $s + 2q = s + 2q' = d$, then

$$\begin{aligned}
&J\left((\hat{H}_q)_{ij}^{(d)}, (\hat{H}_q)_{ij}^{(d)}\right) \\
&= B\left((\hat{H}_q)_{ij}^{(d)}, (\hat{H}_q)_{ij}^{(d)}\right) \\
&\quad + \frac{n-2}{4(n-1)} \frac{1}{(2d-2q)(n-2q-2)} (\alpha_s)^2 \int_{S_1} (p_{s+2q})_s^2 \\
&= t_{s,q} \int_{S_1} (p_{s+2q})_s^2.
\end{aligned}$$

Here

$$\begin{aligned}
t_{s,q} = \Bigg[&\left(A_{s,q+1} + \frac{(n-2)^2}{4} \right) \alpha_s \\
&\qquad + \frac{n-2}{4(n-1)} \frac{1}{(d+s)(n+s-d-2)} (\alpha_s)^2 \Bigg],
\end{aligned}$$

where

$$A_{s,q+1} = (s-1)\left(1 - \frac{n-2}{2(n-1)}(n+s-1)\right) - \frac{q+1}{2}(n+2q+2s-4).$$

It is possible to check that

$$t_{s,q} = \frac{1}{16} \frac{(4s^2 - 8s + 4 + 4ns - 8n + 5n^2 - n^3)^2}{(n-1)^2(n-2+2s)^2} \alpha_s,$$

if $s + 2q = \frac{n-2}{2}$. This implies J is always positive in this case.

Moreover,

$$
\begin{aligned}
&(I')_1^{(n)}((\hat{E}_s)_{ij},(\hat{E}_s)_{ij})\\
&= \frac{n-2}{2}\sum_{q,q'=1}^{[\frac{d'-s}{2}]} c_{2s+2q+2q'}B((\hat{E}_s)_{ij}^{(s+2q)},(\hat{E}_s)_{ij}^{(s+2q')})\\
&\quad - \sum_{q,q'=1}^{[\frac{d'-s}{2}]}(s+2q)\int_{\mathbb{R}^n}\delta^2((\hat{E}_s)_{ij}^{(s+2q)})Z((\hat{E}_s)_{ij}^{(s+2q')})U\,dy\\
&= \sum_{q,q'=1}^{[\frac{d'-s}{2}]} M(s,q,q')\int_{S_1}(p_{s+2q})_s(p_{s+2q'})_s,
\end{aligned}
$$

where

$$
\begin{aligned}
&M(s,q,q')\\
&= \frac{n-2}{2}c_{2s+2q+2q'}\alpha_s\left(A_{s,q+1}+\frac{(n+2s+2q+2q'-2)(s+2q)}{4}\right)\\
&\quad - \frac{n-2}{4(n-1)}\alpha_s^2(s+q+q')\left(\sum_{j=0}^{q+1}\Gamma(s+2q,q,j)b_{2s+2q'+2j}\right).
\end{aligned}
$$

Here we are using that

$$
\int_{\mathbb{R}^n}\delta^2(H^{(k)})Z(W^{(l)})U\,dy = \int_{\mathbb{R}^n}\delta^2(W^{(l)})Z(H^{(k)})U\,dy, \tag{A.11}
$$

since integration by parts implies

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\left(\Delta Z(H^{(k)})+n(n+2)U^{\frac{4}{n-2}}Z(H^{(k)})\right)Z(W^{(l)})dy\\
&= \int_{\mathbb{R}^n}\left(\Delta Z(W^{(l)})+n(n+2)U^{\frac{4}{n-2}}Z(W^{(l)})\right)Z(H^{(k)})dy.
\end{aligned}
$$

It is now possible to check that (see [**12**]), for every $2\le s\le 8$, and $1\le p\le[\frac{10-s}{2}]$, the matrix

$$
(M(s,q,q'))_{1\le q,q'\le p}
$$

is positive definite if $2s+4p+2<n\le 24$.

This finishes the proof of the positivity of $(I)_\varepsilon^{(n)}$ in Case 3, for $n\le 24$. q.e.d.

The following proposition states that $n=25$ is the critical dimension with respect to the positivity of the quadratic form.

Proposition A.8. *If $n\ge 25$, then the quadratic form $I_\varepsilon^{(n)}$ has negative eigenvalues.*

Proof. See [**12**] for details of the calculation. Let $(m^2)_{kl}$ be the matrix as in the proof of Case 2 above.

A calculation gives

$$\operatorname{discrim}\left(\sum_{k,l=1}^{2}(m^2)_{kl}a_k a_l, a_2\right)$$
$$= 16\frac{n^2(n+2)^2(n+4)(n^2-54n+152)}{(n-8)^2(n-6)^2(n-4)^2(n-10)}b_0^2.$$

This implies $((m^2)_{kl})_{1\le k,l\le 2}$ is not positive definite if $n \ge 52$.

On the other hand, it is possible to check that $((m^2)_{kl})_{1\le k,l\le 4}$ is not positive definite if $25 \le n \le 52$.

Given W_{ikjl} wih all the symmetries of the Weyl tensor, and such that $\sum |W_{ikjl}|^2 > 0$, define

$$D_{ij}^{(2)} = \sum_{k,l=1}^{n} W_{ikjl}x_k x_l,$$

and

$$D_{ij} = a_1 D_{ij}^{(2)} + a_2|x|^2 D_{ij}^{(2)} + a_3|x|^4 D_{ij}^{(2)} + a_4|x|^6 D_{ij}^{(2)}.$$

We conclude that, if $n \ge 25$, there are always a_1, a_2, a_3, a_4 so that

$$I_\varepsilon^{(n)}(D_{ij}, D_{ij}) < 0.$$

This finishes the proof. q.e.d.

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