

**GEOMETRY OF FOLIATIONS AND FLOWS I:
ALMOST TRANSVERSE PSEUDO-ANOSOV FLOWS
AND ASYMPTOTIC BEHAVIOR OF FOLIATIONS**

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Abstract

Let \mathcal{F} be a foliation in a closed 3-manifold with negatively curved fundamental group and suppose that \mathcal{F} is almost transverse to a quasigeodesic pseudo-Anosov flow. We show that the leaves of the foliation in the universal cover extend continuously to the sphere at infinity; therefore the limit sets of the leaves are continuous images of the circle. One important corollary is that if \mathcal{F} is a Reebless, finite depth foliation in a hyperbolic 3-manifold, then it has the continuous extension property. Such finite depth foliations exist whenever the second Betti number is non zero. The result also applies to other classes of foliations, including a large class of foliations where all leaves are dense, and infinitely many examples with one sided branching. One extremely useful tool is a detailed understanding of the topological structure and asymptotic properties of the 1-dimensional foliations in the leaves of $\tilde{\mathcal{F}}$ induced by the stable and unstable foliations of the flow.

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1. Introduction

A 2-dimensional foliation in a 3-manifold is called Reebless if it does not have a Reeb component: a foliation of the solid torus so that the boundary is a leaf and the interior is foliated by plane leaves spiralling towards the boundary. The boundary leaf does not inject in the fundamental group level and is compressible. Novikov [No] showed that transversely orientable, Reebless foliations and the underlying manifolds have excellent topological properties. This result was extended by Rosenberg [Ros], Palmeira [Pa] and many others. Reebless foliations exist for instance whenever M is irreducible, orientable and the second homology of M is not finite [Ga1, Ga3]. They also exist in much more generality by work of Roberts [Ro1, Ro2, Ro3], Thurston [Th5] and many others.

The goal of this article is to analyse geometric properties of foliations. Let \mathcal{F} be a Reebless foliation in M^3 with negatively curved fundamental group. Reebless implies that M is irreducible [Ros]. In this article we will not make use of Perelman's fantastic results [Pe1, Pe2, Pe3, Mo-Ti], which imply that the manifold is hyperbolic.

Since M^3 is closed, irreducible with negatively curved fundamental group, then the universal cover is canonically compactified with a sphere at infinity (denoted by S_∞^2), with compactification a closed ball [Be-Me]. The covering transformations act by homeomorphisms in the compactified space. Let $\tilde{\mathcal{F}}$ be the lifted foliation to the universal cover \tilde{M} . The leaves of $\tilde{\mathcal{F}}$ are topological planes and they are properly embedded in \tilde{M} [No], hence they only limit in the sphere at infinity. For hyperbolic manifolds, the relationship between objects in hyperbolic 3-space (isometric to \tilde{M}) and their limit sets in the sphere at infinity is central to the theory of such manifolds [Th1, Th2, Mar]. The same is true if $\pi_1(M)$ is negatively curved [Gr, Gh-Ha]. Candel proved that there is a metric in M so that all leaves of \mathcal{F} are hyperbolic (that is constant curvature -1) [Ca] and so the universal cover of each leaf of \mathcal{F} is isometric to the hyperbolic plane (\mathbf{H}^2). The *continuous extension question* asks whether these leaves extend continuously to the sphere at infinity, that is: given the inclusion map from a leaf F of $\tilde{\mathcal{F}}$ to \tilde{M} , is there a continuous extension to a map $F \cup \partial_\infty F$ to $\tilde{M} \cup S_\infty^2$? Here $\partial_\infty F$ is the ideal boundary of F which is homeomorphic to a circle. If this is true for every F , we say that \mathcal{F} has the *continuous extension property*. In that case the restriction of the map to $\partial_\infty F$ expresses the limit set of F in S_∞^2 as the continuous image of a circle, showing it is locally connected.

In this article we prove the continuous extension property for a very large class of foliations. First we need two definitions. A pseudo-Anosov flow is *almost transverse* to a foliation if an appropriate blow up of the

flow is transverse to the foliation. A blow up is obtained by replacing a (possibly empty) collection of singular orbits by a union of annuli. Another property that is important is a metric property: A flow is *quasigeodesic* if it is uniformly efficient up to a bounded multiplicative distortion in measuring distances in the universal cover. This is extremely important for manifolds with negatively curved fundamental group [Th1, Th3, Gr]. Our main result is the following:

Main theorem. *Let \mathcal{F} be a foliation in M^3 closed, atoroidal. Suppose that \mathcal{F} is almost transverse to a quasigeodesic pseudo-Anosov flow Φ , which has some prong singularity (that is, not a topological Anosov flow). This implies that M has negatively curved fundamental group. Then \mathcal{F} has the continuous extension property. Therefore the limit sets of leaves of $\tilde{\mathcal{F}}$ are locally connected. The set of foliations almost transverse to a flow is open in the Hirsch topology of foliations.*

The hypothesis implies that \mathcal{F} is transversely orientable. Since M has a singular pseudo-Anosov flow then M is irreducible [Ga-Oe] and the stable/unstable foliations of Φ split to genuine laminations in M [Ga-Ka]. A fundamental result of Gabai and Kazez [Ga-Ka] then implies that M has negatively curved fundamental group. For simplicity of statements we frequently use the group negative curvature hypothesis, but in most places that could be substituted by atoroidal. We remark again that we are not using Perelman's results, which imply further that M is hyperbolic.

We also stress that the quasigeodesic hypothesis for Φ is essential for the main theorem.

Notice that it is not necessary to assume that \mathcal{F} is Reebless – we prove that the condition of being almost transverse to a pseudo-Anosov flow implies that \mathcal{F} is Reebless.

As a first consequence, we prove the continuous extension property for all Reebless finite depth foliations in hyperbolic 3-manifolds. Roughly, a foliation is *finite depth* if all leaves are proper and the leaves are perfectly fitted along the cutting surfaces of a hierarchy of the manifold [Ga1, Ga3]. In particular, there are compact leaves. These foliations exist whenever the second homology is not finite [Ga1].

Corollary A. *Let \mathcal{F} be a Reebless finite depth foliation in M^3 closed hyperbolic. Then \mathcal{F} has the continuous extension property. In particular, the limit sets of the leaves are all locally connected.*

Hence, any orientable, hyperbolic 3-manifold with non finite second homology has such a foliation with the continuous extension property. Conjecturally any closed, hyperbolic 3-manifold has a finite cover with positive first Betti number. This would imply there is always a foliation with the continuous extension property in a finite cover. The proof of Corollary A is simple given previous results: If necessary, take a double

cover and assume that \mathcal{F} is transversely oriented. Then Mosher and Gabai proved that such \mathcal{F} is almost transverse to a pseudo-Anosov flow Φ [Mo5]. We proved, jointly with Mosher that these flows are quasigeodesic [Fe-Mo]. The main theorem then implies corollary A. The result in [Fe-Mo] is only for finite depth foliations: the proof depends heavily on the existence of a compact leaf, M being hyperbolic and the direct association with a hierarchy. By Thurston's geometrization theorem [Th1, Th2, Mor] it suffices to assume that M is atoroidal.

Corollary B. *There are infinitely many foliations with all leaves dense which have the continuous extension property. Many of these have one sided branching.*

Foliations with all leaves dense can be obtained, for example, starting with finite depth foliations and doing small perturbations – keeping it still almost transverse to the same quasigeodesic pseudo-Anosov flow. A construction is carefully explained by Gabai [Ga3], providing infinitely many examples with dense leaves to which corollary B applies. The examples occur whenever the second Betti number of M is non zero. In fact, whenever a foliation \mathcal{F} satisfies the hypothesis of the main theorem, then any \mathcal{F}' sufficiently close to \mathcal{F} will also be transverse to the same flow. By the main theorem again, \mathcal{F}' will have the continuous extension property. This perturbation feature of the main theorem is not shared by any previous result concerning the continuous extension property.

A foliation \mathcal{F} is **R-covered** if the leaf space of $\tilde{\mathcal{F}}$ is homeomorphic to the real numbers. Equivalently, this leaf space is Hausdorff. A foliation which is not **R-covered** has *branching*; that is, there are non separated points in the leaf space of $\tilde{\mathcal{F}}$. This leaf space is oriented (being a simply connected, perhaps non Hausdorff 1-manifold) and there is a notion of branching in the positive or negative directions. If it branches only in one direction, the foliation is said to have *one sided branching*. Foliations with one sided branching, where all leaves are dense and the foliation is transverse to a suspension pseudo-Anosov flow (which is quasigeodesic), were constructed by Meigniez [Me]. This provides infinitely many examples with one sided branching to which corollary B applies.

After this article was written, we used the results of this article to prove the continuous extension property for foliations which are **R-covered** or with one sided branching [Fe11]. We showed that a certain transverse pseudo-Anosov flow is quasigeodesic. The result then follows from a direct application of the main theorem here. This shows that the main theorem is a very useful result.

A very important tool in the proof of the main theorem is an analysis of the topological structure of the pseudo-Anosov flow. Let Φ_1 be the original pseudo-Anosov flow almost transverse to \mathcal{F} . To make the flow transverse to \mathcal{F} one needs in general to blow up a collection of singular

orbits into a collection of flow saturated annuli so that each boundary is a closed orbit of the new flow Φ . The blown up flow Φ is called an *almost pseudo-Anosov flow* (see Section 3). If $\tilde{\Phi}$ is the lifted flow to the universal cover \tilde{M} and \mathcal{O} is its orbit space, then \mathcal{O} is homeomorphic to the plane \mathbf{R}^2 [Fe-Mo] – this is true for pseudo-Anosov and almost pseudo-Anosov flows. When one blows up some singular orbits into a collection of joined annuli, the stable/unstable singular foliations also blow up. The two new singular foliations Λ^s, Λ^u are everywhere transverse to each other except at the singularities and the blown up annuli. The blown up annuli are part of both singular foliations, hence Λ^s, Λ^u are tangent to each other there. Since \mathcal{F} is now transverse to the blown up foliations Λ^s, Λ^u , then the stable/unstable blown up foliations Λ^s, Λ^u induce singular 1-dimensional foliations in the leaves of \mathcal{F} and hence also in the leaves of $\tilde{\mathcal{F}}$. The behavior of this is described in the following result, which is of independent interest also:

Theorem C. *Let \mathcal{F} be a foliation with hyperbolic leaves in M^3 closed. Let Φ_1 be a pseudo-Anosov flow almost transverse to \mathcal{F} and let Φ be a corresponding almost pseudo-Anosov flow transverse to \mathcal{F} . Let Λ^s, Λ^u be the stable/unstable 2-dimensional foliations of Φ and $\tilde{\Lambda}^s, \tilde{\Lambda}^u$ the lifts to \tilde{M} . Given F leaf of $\tilde{\mathcal{F}}$, let $\tilde{\Lambda}_F^s, \tilde{\Lambda}_F^u$ be the induced singular 1-dimensional foliations in F . Then*

- *For every ray l in a leaf of $\tilde{\Lambda}_F^s$ or $\tilde{\Lambda}_F^u$, l limits in a single point of $\partial_\infty F$.*
- *If the stable/unstable foliations $\tilde{\Lambda}^s, \tilde{\Lambda}^u$ of Φ have Hausdorff leaf space, then the leaves of $\tilde{\Lambda}_F^s, \tilde{\Lambda}_F^u$ are uniform quasigeodesics in F , the bound is independent of the leaf. In general, the leaves of $\tilde{\Lambda}_F^s, \tilde{\Lambda}_F^u$ are not quasigeodesic.*
- *Any non Hausdorffness (of say $\tilde{\Lambda}_F^s$) is associated to a Reeb annulus of $\Lambda^s \cap \mathcal{F}$ in a leaf of \mathcal{F} : There is an associated set in \tilde{M} , so that when projected to M it is a Reeb annulus or spirals towards a Reeb annulus.*

For the next two items we assume that one cannot collapse a countable number of foliated I -bundles of \mathcal{F} so that the resulting foliation is topologically conjugate to the stable foliation of a suspension Anosov flow.

- *The set of ideal points of leaves of $\tilde{\Lambda}_F^s$ is dense in $\partial_\infty F$ and similarly for $\tilde{\Lambda}_F^u$.*
- *If two rays of the same leaf of $\tilde{\Lambda}_F^s$ limit to the same ideal point in $\partial_\infty F$ then this leaf of $\tilde{\Lambda}_F^s$ does not contain a singularity and the region in F bounded by this leaf projects in M to a set in a leaf of \mathcal{F} which is either contained in or asymptotic to a Reeb annulus.*

This is a completely general result: one does not need negatively curved fundamental group of M or any metric properties of Φ . Theorem C is a key tool used in the proof of the main theorem.

The article is organized as follows: In the next section we discuss previous results on the continuous extension property and the strategy of the proof of the main theorem and Theorem C. In Section 3 we present basic definitions and results concerning pseudo-Anosov flows and almost pseudo-Anosov flows. In Section 4 we analyse projections to the orbit space. In Sections 5 and 6 we analyse the singular foliations $\tilde{\Lambda}_F^s, \tilde{\Lambda}_F^u$ and asymptotic properties of their leaves, proving Theorem C. In Section 7 we prove the continuous extension property – the main theorem. In the final section we comment on general relationships between foliations and Kleinian groups.

In a subsequent article we analyse other important consequences of quasigeodesic behavior of Φ for flows and foliations [Fe10].

2. Historical remarks and strategy of proofs

Here we review what is known about the continuous extension property. In a seminal work, Cannon and Thurston [Ca-Th] proved this property when \mathcal{F} is a fibration over the circle. Previously Thurston showed that a fibering manifold is hyperbolic when the monodromy of the fibration is pseudo-Anosov [Th1, Th3, Th4, Bl-Ca]. Since the fundamental group of a leaf of \mathcal{F} is a normal subgroup of the fundamental group of M , then the limit set of every leaf of $\tilde{\mathcal{F}}$ is the whole sphere. In this way Cannon and Thurston produced many examples of group invariant Peano curves.

Another extremely important class of foliations is the following: A foliation is *proper* if the leaves never limit on themselves – this is in the foliation sense and it means that a sufficiently small transversal to a given leaf only meets the leaf in a single point. In particular, leaves are not dense. In a proper foliation there are compact leaves which are said to have *depth* 0. The depth of a leaf is inductively defined to be i (for finite i) if $i - 1$ is the maximum of the depths of leaves in the (foliation) limit set of the leaf. A foliation has *finite depth* if it is proper and there is a finite upper bound to the depths of all leaves.

Gabai proved that whenever a compact 3-manifold M is irreducible, orientable and the second homology group $H_2(M, \partial M, \mathbf{Z})$ is not finite, then there is a Reebless finite depth foliation associated to each non torsion homology class [Ga1, Ga3]. The foliation is directly associated to a hierarchy of the manifold and as such is strongly connected with the topological structure of the manifold. These results had several fundamental consequences for the topology of 3-manifolds [Ga1, Ga2, Ga3]. If M is hyperbolic (or atoroidal), then one important question

is whether these finite depth foliations have the continuous extension property.

Subsequently, Gabai and Mosher showed [Mo5] that any Reebless, transversely oriented, finite depth foliation in a closed, atoroidal 3-manifold admits a pseudo-Anosov flow Φ which is almost transverse to it. Roughly, a flow is *pseudo-Anosov* if it has transverse hyperbolic dynamics – even though it may have finitely many singularities. It has stable and unstable two dimensional foliations which in general are singular. The term *almost transverse* means that one may need to blow up one singular orbit (or more) into a finite collection of joined annuli to make the flow transverse to the foliation. See detailed definitions and comments in Section 3. Since \mathcal{F} has a compact leaf and M is atoroidal, Thurston [Th1, Th3] proved that M is in fact hyperbolic.

We proved, jointly with Mosher, that these pseudo-Anosov flows almost transverse to finite depth foliations in hyperbolic 3-manifolds are *quasigeodesic* [Fe-Mo]. This means that flow lines are uniformly efficient in measuring distance in relative homotopy classes, or equivalently, uniformly efficient in measuring distance in the universal cover. This was first proved by Mosher [Mo4] for a class of flows transverse to some examples of depth one foliations obtained by handle constructions. Another concept is that of quasi-isometric behavior: a foliation (perhaps singular) is *quasi-isometric* if its leaves are uniformly efficient in measuring distance in the universal cover. There are no non singular 2 dimensional quasi-isometric foliations in closed 3-manifolds with negatively curved fundamental group [Fe1]. As for *singular* foliations, the situation is quite different and there are examples. The stable/unstable singular foliations of the quasigeodesic flows above may be quasi-isometric [Fe6] and may not [Mo5, Fe6]. In general, quasi-isometric behavior of Λ^s (or Λ^u) implies that Φ is quasigeodesic.

If both Λ^s and Λ^u are quasi-isometric *and* the flow is transverse (as opposed to being only almost transverse) to the finite depth foliation, then as we proved in [Fe6], \mathcal{F} has the continuous extension property. We stress that this result only applies to finite depth foliations – the proof depends, amongst other things, on induction in the depth. To be able to apply this result we needed to check the quasi-isometric behavior of Λ^s, Λ^u and transversality between \mathcal{F} and Φ . This was very tricky and we could only do that for some *depth one* foliations. More to the point, it is known that these conditions do not always hold for general finite depth foliations. Corollary A proves the continuous extension property for all finite depth foliations: there are no restrictions on the depth of the foliation, or about transversality of the flow with the foliation or quasi-isometric behavior of Λ^s, Λ^u .

The continuous extension property was also proved for another class of foliations: A foliation \mathcal{F} is *uniform* if any two leaves in the universal

cover are a bounded distance apart; the bound depends on the individual leaves. Hence, \mathcal{F} is \mathbf{R} -covered. Thurston [Th5] proved that there is a large class of uniform foliations. If in addition $\pi_1(M)$ is negatively curved, then, Thurston [Th5] proved, there is a pseudo-Anosov flow transverse to \mathcal{F} . From this it is easy to prove that the flow has quasi-isometric stable/unstable foliations. In this case it also easily implies that the foliation \mathcal{F} has the continuous extension property. The arguments are a very clever generalization of the fibering situation.

Notice that in all the previous results, there is a pseudo-Anosov flow Φ transverse to \mathcal{F} and so that the stable/unstable foliations of Φ are quasi-isometric singular foliations. Both of these properties were crucial in all proofs. The main theorem implies all the previous results about the continuous extension property, and it has the unique feature that it applies to an open set of foliations.

The main theorem can potentially be widely applicable because of the abundance of pseudo-Anosov flows almost transverse to foliations: Thurston proved this for fibrations [Th4]. It is also true for all \mathbf{R} -covered foliations [Fe7, Cal1], and Calegari proved it for all foliations with one sided branching [Cal2]. Calegari also has done extensive analysis for minimal foliations [Cal3] and many other foliations [Cal3]. The missing ingredient is the quasigeodesic property of these pseudo-Anosov flows which is needed to apply the main theorem. In general, the quasigeodesic property for a pseudo-Anosov flow (or an arbitrary flow) is very hard to obtain. This property was only known when the pseudo-Anosov flow is almost transverse to a foliation of one of the following types: finite depth or uniform. One main goal in the study of a pseudo-Anosov flow Φ in M with Gromov hyperbolic fundamental group is to decide whether Φ is quasigeodesic. There are many examples where it is not quasigeodesic [Fe2].

After this article was written, we proved that for every foliation which is \mathbf{R} -covered or with one sided branching, the transverse pseudo-Anosov flows constructed in [Fe7, Cal1, Cal2] are always quasigeodesic. This proves the continuous extension property for these foliations.

Why almost transversality and not just transversality? In many cases (\mathbf{R} -covered, one sided branching), the pseudo-Anosov flow is actually transverse to \mathcal{F} . But for finite depth foliations (which have two sided branching), there are many examples where it is impossible to make the pseudo-Anosov flow transverse to the foliation [Mo5] and one can only get almost transversality. We will have more comments about this in Section 3. Finite depth foliations are extremely important as they are strongly connected to the topology of the underlying manifold. Also, in some sense, foliations with two sided branching are probably more common than foliations which are either \mathbf{R} -covered or with one sided branching.

We also remark that in all previous results concerning the continuous extension property, Theorem C was a crucial property on which the whole analysis hinged. In the previous situations, the analysis of leaves of $\tilde{\Lambda}_F^s, \tilde{\Lambda}_F^u$ was either trivial or substantially easier. In particular, in these situations the leaves of $\tilde{\Lambda}_F^s, \tilde{\Lambda}_F^u$ were always uniform quasigeodesics in F , which simplified subsequent proofs considerably. Such is not the case in general. In particular, if $\tilde{\Lambda}^s, \tilde{\Lambda}^u$ do not have Hausdorff leaf space, then a priori the leaves of $\tilde{\Lambda}_F^s, \tilde{\Lambda}_F^u$ do not have to be quasigeodesic. The proof of theorem C works for any pseudo-Anosov flow almost transverse to a foliation with hyperbolic leaves. The proof uses the denseness of contracting directions for foliations as proved by Thurston [Th6, Th7] when he introduced the universal circle for foliations – even though we do not directly use the universal circle here. The basic idea is: if any ray of (say) $\tilde{\Lambda}_F^s$ does not limit to a single point in $\partial_\infty F$, then it limits in a non trivial interval of $\partial_\infty F$. Zoom into this interval and analyse the situation in the limit. This is actually the easiest statement to prove in Theorem C. The facts about rays with the same ideal point and non Hausdorffness are much trickier, but they will be essential in the analysis of the main theorem. The results of Theorem C are also used in other contexts, for example, to analyse rigidity of pseudo-Anosov flows almost transverse to a given foliation. This will be explored in a future article.

The proof of the main theorem has 2 parts: given a leaf F of $\tilde{\mathcal{F}}$, one first constructs an extension to the ideal boundary and then one shows that it is continuous. To define the extension, one uses the foliations $\tilde{\Lambda}_F^s, \tilde{\Lambda}_F^u$ as they hopefully define a basis neighborhood of an ideal point p of F . The best situation is that the leaves of $\tilde{\Lambda}^s, \tilde{\Lambda}^u$ which contain these leaves of $\tilde{\Lambda}_F^s, \tilde{\Lambda}_F^u$ define basis neighborhoods of unique points in S_∞^2 , hence defining the image of p in S_∞^2 . There are several difficulties here: first, the leaves of $\tilde{\Lambda}_F^s, \tilde{\Lambda}_F^u$ are not quasigeodesics in F , so much more care is needed. Another problem is that the foliations $\tilde{\Lambda}^s, \tilde{\Lambda}^u$ in general do not have Hausdorff leaf space. This keeps recurring throughout the proof. A further difficulty is that if intersections of leaves L_i of say $\tilde{\Lambda}^s$ with a leaf F of $\tilde{\mathcal{F}}$ escape in F as i converges to infinity, it does not follow that the L_i themselves escape compact sets in \tilde{M} . Consequently, there are several cases to be analysed.

Another important feature in the proofs of the main theorem and Theorem C is the following: Let Θ be the projection map from \tilde{M} to \mathcal{O} – \mathcal{O} is the orbit space of $\tilde{\Phi}$, Θ projects flow lines to points. A leaf of $\tilde{\mathcal{F}}$ intersects an orbit of $\tilde{\Phi}$ at most once defining an injective projection of F to $\Theta(F)$. Here $\Theta(F)$ equal to \mathcal{O} implies that the foliation is \mathbf{R} -covered. An important question is to determine the boundary $\Theta(F)$ as a subset of \mathcal{O} . This turns out to be a collection of subsets of stable/unstable leaves

in \mathcal{O} . This result is different than what happens for pseudo-Anosov flows transverse to foliations and its proof is much more delicate. This is analysed in Section 4.

3. Preliminaries: Pseudo-Anosov flows and almost pseudo-Anosov flows

Let Φ be a flow on a closed, oriented 3-manifold M . We say that Φ is a *pseudo-Anosov flow* if the following are satisfied:

- For each $x \in M$, the flow line $t \rightarrow \Phi(x, t)$ is C^1 , it is not a single point, and the tangent vector bundle $D_t\Phi$ is C^0 .

- There is a finite number of periodic orbits $\{\gamma_i\}$, called *singular orbits*, such that the flow is “topologically” smooth off of the singular orbits (see below).

- The flowlines are contained in two singular 2-dim transverse foliations Λ^s, Λ^u which are non singular topological foliations outside of $\{\gamma_i\}$ and whose local intersections are flow lines of Φ . A singular leaf is homeomorphic to $P \times I/f$ where P is a p -prong in the plane and f is a homeomorphism from $P \times \{1\}$ to $P \times \{0\}$. In a stable leaf, f contracts towards towards the prongs and in an unstable leaf it expands away from the prongs. We restrict to p at least 3, that is, we do not allow 1-prongs.

- In an arbitrary stable leaf all orbits are forward asymptotic, in an unstable leaf all orbits are backwards asymptotic. They are like Anosov foliations off of the singular orbits.

Basic references for pseudo-Anosov flows are [Mo3, Mo5] and for 3-manifolds [He].

Notation/definition. The singular foliations lifted to \widetilde{M} are denoted by $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$. If $x \in M$ let $W^s(x)$ denote the leaf of Λ^s containing x . Similarly one defines $W^u(x)$ and in the universal cover $\widetilde{W}^s(x), \widetilde{W}^u(x)$. Similarly, if α is an orbit of Φ define $W^s(\alpha)$, etc. Also, let $\widetilde{\Phi}$ be the lifted flow to \widetilde{M} .

Now we review the results about the topology of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ that will be needed later. We refer to [Fe4, Fe6] for detailed definitions, explanations and proofs. The orbit space of $\widetilde{\Phi}$ in \widetilde{M} is homeomorphic to the plane \mathbf{R}^2 [Fe-Mo] and is denoted by $\mathcal{O} \cong \widetilde{M}/\widetilde{\Phi}$. Let $\Theta : \widetilde{M} \rightarrow \mathcal{O} \cong \mathbf{R}^2$ be the projection map. If L is a leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$, then $\Theta(L) \subset \mathcal{O}$ is a tree which is either homeomorphic to \mathbf{R} if L is regular, or is a union of p -rays all with the same starting point if L has a singular p -prong orbit. The foliations $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ induce 1-dimensional foliations $\mathcal{O}^s, \mathcal{O}^u$ in \mathcal{O} . Its leaves are $\Theta(L)$ as above. If L is a leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$, then a *sector* is a component of $\widetilde{M} - L$; similarly for $\mathcal{O}^s, \mathcal{O}^u$. If B is any subset of \mathcal{O} , we denote by $B \times \mathbf{R}$ the set $\Theta^{-1}(B)$. The same notation $B \times \mathbf{R}$ will be

used for any subset B of \widetilde{M} : it will just be the union of all flow lines through points of B .

Definition 3.1. Let L be a leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$. A slice (or slice leaf) of L is $l \times \mathbf{R}$ where l is a properly embedded copy of the reals in $\Theta(L)$. For instance, if L is regular then L is its only slice. If a slice is the boundary of a sector of L then it is called a line leaf of L . If a is a ray in $\Theta(L)$ then $A = a \times \mathbf{R}$ is called a half leaf of L . If ζ is an open (or closed or half closed) segment in $\Theta(L)$ it defines a *flow band* L_1 of L by $L_1 = \zeta \times \mathbf{R}$. Same notation for the foliations $\mathcal{O}^s, \mathcal{O}^u$ of \mathcal{O} .

If $F \in \widetilde{\Lambda}^s$ and $G \in \widetilde{\Lambda}^u$ then F and G intersect in at most one orbit. Also, suppose that a leaf $F \in \widetilde{\Lambda}^s$ intersects two leaves $G, H \in \widetilde{\Lambda}^u$ and so does $L \in \widetilde{\Lambda}^s$. Then F, L, G, H form a *rectangle* in \widetilde{M} and there is no singularity in the interior of the rectangle [Fe6]. There will be two generalizations of rectangles: 1) perfect fits = rectangle with one corner removed and 2) lozenges = rectangle with two opposite corners removed.

We stress that we will also denote by rectangles, perfect fits, lozenges and product regions the projection of these regions to $\mathcal{O} \cong \mathbf{R}^2$.

Definition 3.2 ([Fe2, Fe4, Fe6], Perfect fits). Two leaves $F \in \widetilde{\Lambda}^s$ and $G \in \widetilde{\Lambda}^u$ form a perfect fit if $F \cap G = \emptyset$ and there are half leaves F_1 of F and G_1 of G and also flow bands $L_1 \subset L \in \widetilde{\Lambda}^s$ and $H_1 \subset H \in \widetilde{\Lambda}^u$ so that the set

$$\overline{F_1} \cup \overline{H_1} \cup \overline{L_1} \cup \overline{G_1}$$

separates \widetilde{M} , and the joint structure of $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ in this region R is that of a rectangle with a corner orbit removed. The removed corner corresponds to the perfect fit of F and G which do not intersect.

We refer to Fig. 1, a for perfect fits. There is a product structure in the interior of R : there are two stable boundary sides and two unstable ones. An unstable leaf intersects one stable boundary side (not in the corner) if and only if it intersects the other stable boundary side (not in the corner). We also say that the leaves F, G are *asymptotic*.

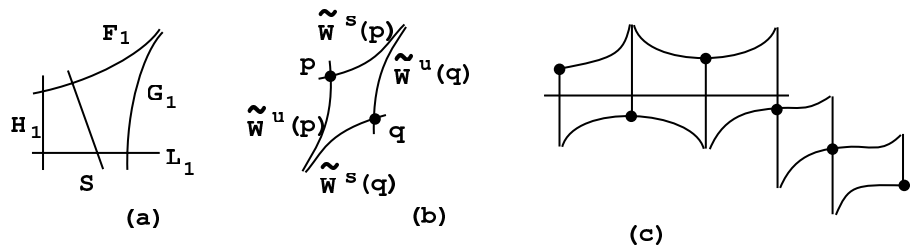


Figure 1. a. Perfect fits in \widetilde{M} . b. A lozenge. c. A chain of lozenges.

Definition 3.3 ([Fe2, Fe4, Fe6], Lozenges). A lozenge is a region of \widetilde{M} whose closure is homeomorphic to a rectangle with two corner orbits removed. More specifically, two points p, q form the corners of a lozenge if there are half leaves A, B of $\widetilde{W}^s(p), \widetilde{W}^u(p)$ defined by p and C, D half leaves of $\widetilde{W}^s(q), \widetilde{W}^u(q)$ so that A and D form a perfect fit and so do B and C . The region bounded by the lozenge is R and it does not have any singularities in the interior. See Fig. 1, b.

There may be singular orbits on the sides of the lozenge and the corner orbits.

Two lozenges are *adjacent* if they share a corner and there is a stable or unstable leaf intersecting both of them, see Fig. 1, c. Therefore they share a side. A *chain of lozenges* is a collection $\{\mathcal{C}_i\}, i \in I$, where I is an interval (finite or not) in \mathbf{Z} ; so that if $i, i+1 \in I$, then \mathcal{C}_i and \mathcal{C}_{i+1} share a corner, see Fig. 1, c. Consecutive lozenges may be adjacent or not. The chain is finite if I is finite.

Definition 3.4. Suppose A is a flow band in a leaf of $\widetilde{\Lambda}^s$. Suppose that for each orbit γ of $\widetilde{\Phi}$ in A there is a half leaf B_γ of $\widetilde{W}^u(\gamma)$ defined by γ so that: for any two orbits γ, β in A , a stable leaf intersects B_β if and only if it intersects B_γ . This defines a stable product region which is the union of the B_γ . Similarly define unstable product regions.

There are no singular orbits of $\widetilde{\Phi}$ in the interior of A .

We abuse convention and call a leaf L of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$ *periodic* if there is a non trivial covering translation g of \widetilde{M} with $g(L) = L$. This is equivalent to $\pi(L)$ containing a periodic orbit of Φ . In the same way an orbit γ of $\widetilde{\Phi}$ is *periodic* if $\pi(\gamma)$ is a periodic orbit of Φ .

We say that two orbits γ, α of $\widetilde{\Phi}$ (or the leaves $\widetilde{W}^s(\gamma), \widetilde{W}^s(\alpha)$) are connected by a chain of lozenges $\{\mathcal{C}_i\}, 1 \leq i \leq n$, if γ is a corner of \mathcal{C}_1 and α is a corner of \mathcal{C}_n .

If \mathcal{C} is a lozenge with corners β, γ and g is a non trivial covering translation leaving β, γ invariant (and so also the lozenge), then $\pi(\beta), \pi(\gamma)$ are closed orbits of Φ which are freely homotopic to the inverse of each other [Fe4].

Theorem 3.5 ([Fe4, Fe6]). *Let Φ be a pseudo-Anosov flow in M^3 closed and let $F_0 \neq F_1 \in \widetilde{\Lambda}^s$, so that there is a non trivial covering translation g with $g(F_i) = F_i, i = 0, 1$. Let $\alpha_i, i = 0, 1$ be the periodic orbits of $\widetilde{\Phi}$ in F_i , so $g(\alpha_i) = \alpha_i$. Then α_0 and α_1 are connected by a finite chain of lozenges $\{\mathcal{C}_i\}, 1 \leq i \leq n$, and g leaves invariant each lozenge \mathcal{C}_i as well as their corners. By the remark above, there are closed orbits β, γ of Φ which are freely homotopic to the inverse of each other.*

A chain from α_0 to α_1 is called *minimal* if all lozenges in the chain are distinct. Exactly as proved in [Fe4] for Anosov flows, it follows that there is a unique minimal chain from α_0 to α_1 and also all other chains have to contain all the lozenges in the minimal chain.

The main result concerning non Hausdorff behavior in the leaf spaces of $\tilde{\Lambda}^s, \tilde{\Lambda}^u$ is the following:

Theorem 3.6 ([Fe4, Fe6]). *Let Φ be a pseudo-Anosov flow in M^3 closed. Suppose that $F \neq L$ are not separated in the leaf space of $\tilde{\Lambda}^s$. Then the following facts happen:*

- F is periodic and so is L .
- Let F_0, L_0 be the line leaves of F, L which are not separated from each other. Let V_0 be the sector of F bounded by F_0 and containing L . Let α be the periodic orbit in F_0 and H_0 be the component of $(\tilde{W}^u(\alpha) - \alpha)$ contained in V_0 . Let g be a non trivial covering translation with $g(F_0) = F_0, g(H_0) = H_0$ and g leaves invariant the components of $(F_0 - \alpha)$. Then $g(L_0) = L_0$.
- Let β be the periodic orbit in L_0 . As $g(\beta) = \beta, g(\alpha) = \alpha$, then $\pi(\alpha), \pi(\beta)$ are closed orbits of Φ which are freely homotopic in M . By Theorem 3.5 F_0 and L_0 are connected by a finite chain of lozenges $\{A_i\}, 1 \leq i \leq n$. Consecutive lozenges are adjacent. They all intersect a common stable leaf C . There is an even number of lozenges in the chain, see Fig. 2. In particular there are closed orbits of Φ which are freely homotopic to the inverse of each other.
- In addition let $\mathcal{B}_{F,L}$ be the set of leaves non separated from both F and L . Put an order in $\mathcal{B}_{F,L}$ as follows: It is easy to see that the stable leaf C as above intersects all the lozenges associated to $\mathcal{B}_{F,L}$. Put an orientation in the set of orbits of C contained in the union of the lozenges and their sides. If $R_1, R_2 \in \mathcal{B}_{F,L}$ let α_1, α_2 be the respective periodic orbits in R_1, R_2 . Then $\tilde{W}^u(\alpha_i) \cap C \neq \emptyset$ and let $a_i = \tilde{W}^u(\alpha_i) \cap C$. We define $R_1 < R_2$ in $\mathcal{B}_{F,L}$ if a_1 precedes a_2 in the orientation of the set of orbits of C . There are two options: $\mathcal{B}_{F,L}$ is either order isomorphic to $\{1, \dots, n\}$ for some $n \in \mathbf{N}$; or $\mathcal{B}_{F,L}$ is order isomorphic to the integers \mathbf{Z} .
- Also, if there are $Z, S \in \tilde{\Lambda}^s$ so that $\mathcal{B}_{Z,S}$ is infinite, then there is an incompressible torus in M transverse to Φ . In particular, M cannot be atoroidal.
- Up to covering translations, there are only finitely many non Hausdorff leaves in $\tilde{\Lambda}^s$.

Notice that $\mathcal{B}_{F,L}$ is a discrete set in this order. For detailed explanations and proofs, see [Fe4, Fe6].

Proposition 3.7. *Let φ be a (topological) Anosov flow so that every leaf of its stable foliation $\tilde{\Lambda}^s$ intersects every leaf of its unstable foliation*

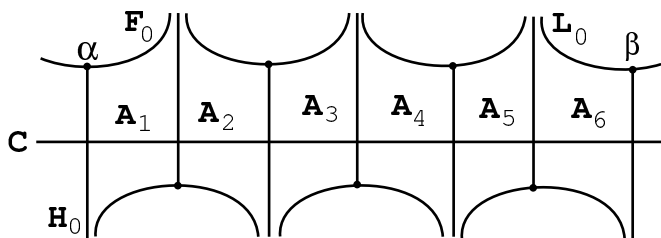


Figure 2. The correct picture between non separated leaves of $\tilde{\Lambda}^s$.

$\tilde{\Lambda}^u$. Then φ is topologically conjugate to a suspension Anosov flow. In particular, M fibers over the circle with fiber a torus and Anosov monodromy.

Proof. This result is proved by Barbot [Ba1] when φ is a smooth Anosov flow. That means it is C^1 and it has also strong stable/unstable foliations and contraction on the level of tangent vectors along the flow. Here we only have the weak foliations and orbits being asymptotic in their leaves. With the correct interpretation all the steps carry through to the general situation.

Lift to a finite cover where Λ^s, Λ^u are transversely orientable. A cross section to the flow in the finite cover projects to a cross section in the manifold (after cut and paste following Fried [Fr]) and so we can prove the result in the cover.

First, the flow φ is expanding: there is $\epsilon > 0$ so that no distinct orbits are always less than ϵ away from each other. Brunella [Br] then proved that the flow admits a surface of section. From this it follows that the flow has a Markov partition. The important consequence for us here is the following: If F is a leaf of $\tilde{\Lambda}^s$ which is left invariant by g , then there is a closed orbit of φ in $\pi(F)$ and all orbits are asymptotic to this closed orbit. Similarly for $\tilde{\Lambda}^u$.

What this means is the following: consider the action of $\pi_1(M)$ in the leaf space of $\tilde{\Lambda}^s$ which is the reals (\mathbf{R}) . Let g in $\pi_1(M)$, which fixes a point in \mathbf{R} . There is L in $\tilde{\Lambda}^s$ with $g(L) = L$. So there is orbit γ of $\tilde{\varphi}$ with $g(\gamma) = \gamma$. Let U be the unstable leaf of $\tilde{\varphi}$ with γ contained in U . Then $g(U) = U$. If g is associated to the positive direction of γ , then g acts as a contraction in the set of orbits of U with γ as the only fixed point. Since every leaf of $\tilde{\Lambda}^u$ intersects every leaf of $\tilde{\Lambda}^s$ then the set of orbits in U is equivalent to the set of leaves of $\tilde{\Lambda}^s$. This implies the important fact:

Conclusion. If g is in $\pi_1(M)$ and has a fixed point in the leaf space of $\tilde{\Lambda}^s$ then it is of hyperbolic type and has a single fixed point.

Using this topological characterization Barbot [Ba1] showed that $G = \pi_1(M)$ is metabelian; in fact, he showed that the commutator subgroup $[G, G]$ is abelian. In particular $\pi_1(M)$ is solvable. This used only an action by homeomorphisms in \mathbf{R} satisfying the conclusion above. Barbot [Ba1] also proved that the leaves of Λ^s, Λ^u are dense in M .

Plante [P11], showed that if \mathcal{F} a minimal foliation in $\pi_1(M)$ solvable then \mathcal{F} is transversely affine: there is a collection of charts $f_i : U_i \rightarrow \mathbf{R}^2 \times \mathbf{R}$, so that the transition functions are affine in the second coordinate. Using this Plante [P11, P12] constructs a homomorphism

$$C : \pi_1(M) \rightarrow \mathbf{R},$$

which measures the logarithm of how much distortion there is along an element of $\pi_1(M)$. This is a cohomology class. Every closed orbit γ of φ traversed in the positive direction has a transversal fence which is expanding - this implies that $C(\gamma)$ is positive. Plante then refers to a criterion of Fried [Fr] to conclude that φ has a cross section and therefore it is easily seen that φ is topologically conjugate to a suspension Anosov flow. This finishes the proof of the proposition. q.e.d.

Theorem 3.8 ([Fe6]). *Let Φ be a pseudo-Anosov flow. If there is a stable or unstable product region, then Φ is topologically conjugate to a suspension Anosov flow. In particular, Φ is non singular.*

Blown up orbits and almost pseudo-Anosov flows.

We now describe almost pseudo-Anosov flows. This is taken from [Mo5]. First we describe the blow up of a singular orbit. Let γ be a singular orbit of a pseudo-Anosov flow φ and let C be a small disk transverse to φ through a point p of γ . The set $W^s(\gamma) \cup W^u(\gamma)$ intersects C in a tree T , which is made of a single common vertex and $2n$ prongs, if the stable/unstable leaves $W^s(\gamma), W^u(\gamma)$ have n prongs. The stable and unstable prongs alternate in C . Orient the prongs so that stable prongs point towards p and unstable prongs point away from p . Hence T is an oriented tree. Up to topological conjugacy we may assume in polar coordinates centered at p that $T = \{(r, \theta) \mid \theta = k\pi/n, k = 0, 1, \dots, 2n-1\}$.

Choose a Poincaré section for γ , that is, a smaller disk C' so that every point of C' flows forward to a point in C . This defines a continuous map $f : C' \rightarrow C$ which fixes p and is a homeomorphism onto its image. This f sends the corresponding pieces of T into T , contracting stable directions and expanding unstable directions. There is a rotation or reflection R in C so that R prescribes where the sectors of C defined by T go to under f . For instance, R may be a rotation by $2\pi/n$.

We first blow up f . Let D be a small subdisk of C' containing p in its interior. Let T^* be an oriented tree in C which agrees with T outside D and such that each vertex v of T^* is “pseudo-hyperbolic”, meaning as you go around the edges incident to v , the orientations of the edges of

T^* alternate pointing toward and away from v . Each vertex of T^* must have $2i$ edges for some $i \geq 2$. The point here is that T^* is created from T by replacing p with a finite subtree. There are finitely many ways to choose T^* , up to compactly supported isotopy. The new edges of T^* created by the blow up are called the finite edges of T^* .

Given this data there is a C^0 perturbation f^* of f , and a continuous map $h : C \rightarrow C$ so that

- f^* leaves T^* invariant.
- For each edge E of T^* , the first return map of f^* acts as a translation on $\text{int}(E)$ moving points in the direction of the orientation on E .
- h collapses the finite edges of T^* to the point p and h is otherwise 1 to 1.
- h is a semiconjugacy from f^* to f , that is, $f \circ h = h \circ f^*$.
- h is close to the identity map in the sup norm and h equals the identity on $C - D$.

We say that f^* is obtained from f by *dynamically blowing up* the pseudo-hyperbolic fixed point p . Each choice of T^* determines a unique f^* up to conjugacy by isotopy. There are therefore finitely many ways to dynamically blow up a pseudo-hyperbolic fixed point, up to conjugation by isotopy.

Now define a dynamic blow up of γ by altering the flow φ near γ as follows. Let D' so that $f(D') \subset D$. Alter φ so that the first return map $f : D' \rightarrow D$ is replaced by $f^* : D' \rightarrow D$. This has the effect of altering the generating vector field of φ inside the “mapping torus” $T_g = \{\varphi(x, s) \mid x \in D', 0 \leq s \leq t(x)\}$ and leaving the vector field unaltered outside of T_g . Let φ^* be the new flow.

The orbit γ gets blown up into a collection of flow invariant annuli. In each annulus A , the boundary components are closed orbits of φ^* which are isotopic to γ as oriented orbits. In the interior of A orbits move from one boundary to the other, as given by the orientation of the corresponding edge of T^* . There is a blow down map $\xi : M \rightarrow M$ which is homotopic to the identity and isotopic to the identity in the complement of the annuli. It sends the collection of annuli into γ . The map ξ sends orbits of φ^* to orbits of φ and preserves orientation.

Definition 3.9. Let φ be a pseudo-Anosov flow in M^3 closed. Then φ^* is an almost pseudo-Anosov flow associated to φ if φ^* is obtained from φ by dynamically blowing some singular orbits of φ .

The reason for considering almost pseudo-Anosov flows is as follows. All the constructions of pseudo-Anosov flows transverse to foliations are in fact constructions of a pair of laminations – stable and unstable – which are transverse to each other and to \mathcal{F} [Th4, Mo5, Fe7, Cal1, Cal2]. The intersection of the laminations is oriented producing a flow

in this intersection. One then collapses the complementary regions to the laminations to produce transverse singular foliations and a pseudo-Anosov flow. The transversality problem occurs in the last step, the blow down of complementary regions. In certain situations, for example for finite depth foliations, one cannot guarantee transversality of the flow and foliation after the blow down, see Fig. 3. In the annulus in Fig. 3 the orbits flow from β towards α . See extended explanations by Mosher in [Mo5].

The necessity of almost transversality as opposed to transversality was first discovered by Mosher in [Mo1] for positioning surfaces with respect to certain \mathbf{Z} invariant flows (in non compact manifolds). This was further explored in [Mo2] where it was analysed when it is necessary to blow up a pseudo-Anosov flow before it became transverse to a given surface.

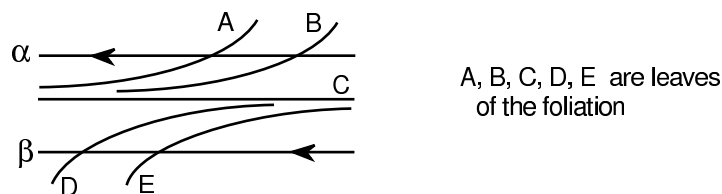


Figure 3. Obstruction to transversality. Here α, β are isotopic closed orbits. If the flow is to be pseudo-Anosov, then one has to collapse α, β together. But then the collapsed flow will not be transverse to the foliation.

Given a dynamic blow up φ^* of φ with corresponding blow down map ξ , the inverse images ξ^{-1} of Λ^s, Λ^u are singular foliations, which have some flow invariant annuli. These are the annuli that come from blowing up some singular orbits – they are called the *blown up annuli*. We still denote by Λ^s, Λ^u the blown up stable/unstable foliations of an almost pseudo-Anosov flow. They are transverse to each other except at the blown up annuli. The same notation is used for $\tilde{\Lambda}^s, \tilde{\Lambda}^u$, etc..

Because φ^* is transverse to the foliation \mathcal{F} ; $\tilde{\Lambda}^s, \tilde{\Lambda}^u$ are transverse to $\tilde{\mathcal{F}}$ and induce singular 1-dimensional foliations which are denoted by $\tilde{\Lambda}_F^s, \tilde{\Lambda}_F^u$ in any leaf F of $\tilde{\mathcal{F}}$.

The objects perfect fits, lozenges, product regions, etc. all make sense in the setting of almost pseudo-Anosov flows: they are just the blow ups of the same objects for the corresponding pseudo-Anosov flows. Since the interior of these objects does not have singularities, the blow up operation does not affect these interiors. There may be singular orbits in the boundary which get blown into a collection of annuli. All the results in this section still hold for almost pseudo-Anosov flows. For example, if F, L in $\tilde{\Lambda}^s$ are not separated from each other, then they are connected

by an even number of lozenges all intersecting a common stable leaf. Since parts of the boundary of these may be in blown annuli, there is not a product structure in the closure of the union of the lozenges, but there is still a product structure of $\tilde{\Lambda}^s, \tilde{\Lambda}^u$ in the interior. We stress that there are no new leaves of $\tilde{\Lambda}^s, \tilde{\Lambda}^u$ created in the blow up and the non Hausdorff structure of $\tilde{\Lambda}^s, \tilde{\Lambda}^u$ is preserved.

4. Projections of leaves of $\tilde{\mathcal{F}}$ to the orbit space

Notation. In some of the proofs and arguments that follow we will be working with an almost pseudo-Anosov flow transverse to a foliation \mathcal{F} . In those cases, for notational simplicity we denote the almost pseudo-Anosov flow by Φ and a corresponding blow down pseudo-Anosov flow by Φ_1 . Hence Φ is some blow up of Φ_1 . This is different than the notation φ, φ^* from the last section.

Then let Φ be an almost pseudo-Anosov flow which is transverse to a foliation \mathcal{F} . This implies that \mathcal{F} is Reebless – we provide a proof of this at the end of this section. An orbit of $\tilde{\Phi}$ intersects a leaf of $\tilde{\mathcal{F}}$ at most once – because the leaves of $\tilde{\mathcal{F}}$ are properly embedded and $\tilde{\Phi}$ is transverse to $\tilde{\mathcal{F}}$. Hence the projection $\Theta : F \rightarrow \Theta(F)$ is injective. We want to determine the set of orbits a leaf of $\tilde{\mathcal{F}}$ intersects – in particular we want to determine the boundary $\partial\Theta(F)$. As it turns out, $\partial\Theta(F)$ is composed of a disjoint union of slice leaves in $\mathcal{O}^s, \mathcal{O}^u$.

We assume throughout this section that Φ is *blow down minimal* with respect to being transverse to \mathcal{F} . This means that no blow down of some flow annuli of Φ produces a flow transverse to \mathcal{F} .

Since Φ is transverse to \mathcal{F} , there is $\epsilon > 0$ so that if a leaf F of $\tilde{\mathcal{F}}$ intersects an orbit of $\tilde{\Phi}$ at p , then it intersects every orbit of $\tilde{\Phi}$ which passes ϵ near p and the intersection is also very near p . To understand $\partial\Theta(F)$ one main ingredient is that when considering pseudo-Anosov flows, flow lines in the same stable leaf are forward asymptotic. So if F intersects a given orbit in a very future time then it also intersects a lot of other orbits in the same stable leaf since in future time they converge. In the limit this produces a stable boundary leaf of $\Theta(F)$. The blow up operation disturbs this: it is not true that orbits in the same stable leaf of an almost pseudo-Anosov flow are forward asymptotic: when they pass very near a blow up annulus the orbits are distorted and their distance can increase enormously: a priori they can go from being arbitrarily small to a number which is, say, close to 1. This is the key difficulty in this section. Hence, we first analyse the blow up operation more carefully.

Convention. Given Φ an almost pseudo-Anosov flow, let Φ_1 be a corresponding pseudo-Anosov flow associated to Φ . The notation $\tilde{W}^s(x)$

($\widetilde{W}^u(x)$, etc..) will denote the stable leaf of $\widetilde{\Phi}$ or $\widetilde{\Phi}_1$, where the context will make clear which one it is.

Recall that $\pi : \widetilde{M} \rightarrow M$ denotes the universal covering map.

We will start with Φ_1 and understand the blow up procedure. The blown up annuli come from singular orbits. The *lift annuli* are the lifts of blown up annuli to \widetilde{M} . Their projections to \mathcal{O} are called *blown segments*. If L is a blown up leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$ the components of L minus the lift annuli are called the *prongs*. A *quarter* associated to an orbit γ of $\widetilde{\Phi}_1$ is the closure of a connected component of $\widetilde{M} - (\widetilde{W}^u(\gamma) \cup \widetilde{W}^s(\gamma))$. Its boundary is a union of γ and half leaves in the stable and unstable leaves of γ . We analyse the case that γ is periodic. We will be interested in a neighborhood V of γ in this quarter which projects to M near the closed orbit $\pi(\gamma)$. We will understand the blow up in the projection of a quarter. Glueing up different quarters gives the overall picture of the blow up operation. In the projected quarter $\pi(V)$ in M there is a cross section to the flow Φ_1 . The orbits across the cross section are determined by which stable and unstable leaf they are in. The return map on the stable direction is a contraction and an expansion in the unstable direction. Any contraction is topologically conjugate to, say, $x \rightarrow x/2$, and an expansion is conjugate to $x \rightarrow 2x$. Hence the local return map is topologically conjugate to

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix},$$

which is a linear map. The whole discussion here is one of *topological* conjugacy. Locally this flow is topologically conjugate to $Z_t = (x, y, 0) \rightarrow (2^{-t}x, 2^t y, t)$. In this model the flow is now smooth. The next step is the blow up: Think of the blow up annulus as the set of unit tangent vectors to γ associated to the quarter region. The flow in the annulus is given by the action of DZ_t on the tangent vectors. It has 2 closed orbits (the boundary ones corresponding to the stable and unstable leaves). The other orbits are asymptotic to the stable closed orbit in negative time and to the unstable closed orbit in positive time. This makes it into a continuous flow in this blown up part. For future reference, recall this fact that in a blow up annulus the boundary components are orbits of the flow and in the interior the flow lines go from one boundary to the other without a Reeb annulus picture (there is a cross section to the flow in the annulus). Do this for each quarter region that is blown up. One can then glue up the 2 sides of the appropriate annuli because they are all of the same topological picture (using the standard model above). This describes the blown up operation in a quarter. There is clearly a blow down map which sends orbits of the blown up flow Φ to orbits of Φ_1 and collapses connected unions of annuli into a single p -prong singular orbit.

We quantify these: let ϵ very small so that any two orbits of Φ_1 which are always less than ϵ apart in forward time, then they are in the same stable leaf. Let \mathcal{Z} the union of the singular orbits of Φ_1 which are blown up. Let $\epsilon' \ll \epsilon$ and let U be the ϵ' tubular neighborhood of \mathcal{Z} . Let U' be the $\epsilon'/2$ tubular neighborhood of \mathcal{Z} . Choose the blow up map to be the identity in the complement of U' , that is the blown up annuli are also contained in U' . The blow down map is then an isometry of the Riemannian metric outside U' . Choose the blow down to move points very little in U' . Isotope \mathcal{F} so that it is transverse to the flow Φ . We are now ready to analyse $\partial\Theta(F)$.

Theorem 4.1. *Let F in $\widetilde{\mathcal{F}}$. Then $\Theta(F)$ is an open subset of \mathcal{O} . Any boundary component of $\Theta(F)$ is a slice of a leaf of \mathcal{O}^s or \mathcal{O}^u . If it is a slice of \mathcal{O}^s , then as $\Theta(F)$ approaches l , the corresponding points of F escape in the positive direction. The same result holds for unstable boundary slices.*

Proof. Here $\Theta : \widetilde{M} \rightarrow \mathcal{O}$ is associated with the almost pseudo-Anosov flow Φ . First, notice that since F is transverse to $\widetilde{\Phi}$, $\Theta(F)$ is an open set. Hence $\partial\Theta(F)$ is disjoint from $\Theta(F)$. The important thing to notice is that the two metrics (blown and original) are the same outside the small neighborhood U' of the blown up annuli. If two points are in the same stable leaf of Φ , then their orbits under the blow down flow Φ_1 are asymptotic in forward time. The same is true for Φ , for big enough time if the point is outside U . This is because the points of the corresponding orbits of Φ_1 will be *both* outside U' – this is the reason for the construction of two neighborhoods U', U . The following setup will be used in *all* cases.

Setup. Let v in $\partial\Theta(F)$ and v_i in $\Theta(F)$ with v_i converging to v . Let p_i in F with $\Theta(p_i) = v_i$ and let w in \widetilde{M} with $\Theta(w) = v$. Let D be any small disk in \widetilde{M} transverse to $\widetilde{\Phi}$ with w in the interior of D . For i big enough v_i is in $\Theta(D)$, so there are t_i real numbers with $p_i = \widetilde{\Phi}_{t_i}(w_i)$ and w_i are in D , so w_i converges to w . As v is not in $\Theta(F)$, $|t_i|$ grows without bound. Without loss of generality assume up to subsequence that $t_i \rightarrow +\infty$. Same proof if $t_i \rightarrow -\infty$. We will prove that there is a slice leaf L of $\widetilde{W}^s(w)$ so that $\Theta(L) \subset \partial\Theta(F)$ and F goes up as it “approaches” L . The stable/unstable leaves here are those of the almost pseudo-Anosov flow and they may have blown up annuli.

One transversality fact used here is the following: for each δ sufficiently small, there is $\delta' > 0$ so that if x, y are δ close in M then the \mathcal{F} leaf through x intersects the Φ -orbit through y in a point δ' close to x . Then δ' goes to 0 as δ does.

Case 1. Suppose that w is not in a blown up leaf.

First, we show that we can assume that *no* w_i is in $\widetilde{W}^s(w)$. Otherwise, up to subsequence assume that *all* w_i are in $\widetilde{W}^s(w)$. The orbits through w_i and w start out very close and aside from the time they stay in $\pi^{-1}(U)$ they are always very close. Let B be the component of the intersection of F with the flow band from $\widetilde{\Phi}_{\mathbf{R}}(w_i)$ to $\widetilde{\Phi}_{\mathbf{R}}(w)$ in the stable leaf $\widetilde{W}^s(w)$, which contains p_i . Then B does not intersect $\widetilde{\Phi}_{\mathbf{R}}(w)$ so it has to either escape up or down. If it escapes down it will have to intersect a small segment from w_i to w and hence so does F . For i big enough w_i is arbitrarily near w , so transversality of \mathcal{F} and Φ then implies that F will intersect $\widetilde{\Phi}_{\mathbf{R}}(w)$ near w , contradiction see Fig. 4, a.

We now consider the case that B escapes up. If the forward orbit through w is not always in $\pi^{-1}(U)$ then at those times outside of $\pi^{-1}(U)$ it will be arbitrarily close to $\widetilde{\Phi}_{\mathbf{R}}(w_i)$ and transversality implies again that F intersects $\widetilde{\Phi}_{\mathbf{R}}(w)$. If the forward orbit of w always stays in $\pi^{-1}(U)$, the same happens after the blow down so the blow down orbit is in the stable leaf of the singular orbit which is being blown up. This does not happen in Case 1.

We can now assume that all v_i are in a sector of $\mathcal{O}^s(v)$ with l the boundary of this sector and $L = l \times \mathbf{R}$, the line leaf of $\widetilde{W}^s(w)$ which is the boundary of this sector.

Now let q in l . We will show that q is in $\partial\Theta(F)$ so $l \subset \partial\Theta(F)$. There is a segment $[q, v]$ contained in l . Choose x in L with $\Theta(x) = q$. Let α be a segment in $\widetilde{W}^s(w)$ transverse to the flow lines and going from x to w . Notice that x, w are in \widetilde{M} and q, v are in the orbit space \mathcal{O} . Let x_i converging to x and x_i in $\widetilde{W}^s(w_i)$. We can do that since all w_i are in the same sector of $\widetilde{W}^s(w)$. Choose segments α_i from x_i to w_i in $\widetilde{W}^s(w_i)$ and transverse to the flowlines of $\widetilde{\Phi}$ in $\widetilde{W}^s(w_i)$.

Claim. For every orbit γ of $\widetilde{\Phi}$ intersecting α_i in y , γ intersects F in $\widetilde{\Phi}_s(y)$ where s converges to ∞ as $i \rightarrow \infty$.

Suppose there are $a_0 > 0$ and i_0 in \mathbf{N} so that

$$\widetilde{\Phi}_{[a_0, t_i]}(w_i) \subset \pi^{-1}(U) \quad \text{for all } i \geq i_0.$$

Then $\widetilde{\Phi}_{[a_0, \infty)}(w)$ is contained in the closure of $\pi^{-1}(U)$. As seen before, this implies that w is in a blown up stable leaf, which is not the hypothesis of Case 1. This also cannot happen in any subsequence of (w_i) . Therefore, up to subsequence, there are arbitrarily big times s_i between 0 and t_i so that $\widetilde{\Phi}_{s_i}(w_i)$ is not in $\pi^{-1}(U)$. Hence $\widetilde{\Phi}_{\mathbf{R}}(x_i)$ is very close to $\widetilde{\Phi}_{s_i}(w_i)$ and the argument above implies that F cannot escape up or down, so F has to intersect $\widetilde{\Phi}_{\mathbf{R}}(x_i)$. Hence the segment $[\Theta(x_i), v_i]$ of $\mathcal{O}^s(v)$ is contained in $\Theta(F)$ and so $[x, v]$ is contained in the closure of $\Theta(F)$. Also the time s so that $\widetilde{\Phi}_s(y)$ hits F goes to infinity; hence

$[x, v]$ cannot intersect $\Theta(F)$. This is because there would be bounded times of intersection of $[x, v] \times \mathbf{R}$ with F , and so similarly in a small neighborhood, by transversality. This proves the claim.

We conclude that $[x, v] \subset \partial\Theta(F)$, and hence $l \subset \partial\Theta(F)$ as desired. If there is a sequence z_i in F escaping down with $\Theta(z_i)$ converging to a point in l , then by connectedness there is one intersecting a compact middle region – this would force an intersection of F with $l \times \mathbf{R}$ which is impossible.

This finishes the proof of Case 1. In this case we proved there is a *line leaf* l of $\Theta(L)$ with $l \subset \partial\Theta(F)$ and F escapes up as $\Theta(F)$ approaches l .

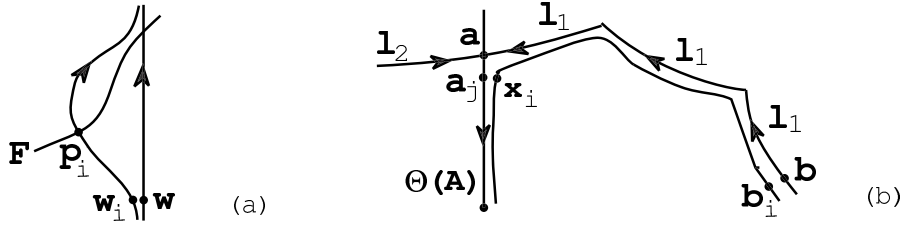


Figure 4. a. A strangling neck is being forced.
b. A slice in a leaf of \mathcal{O}^s or \mathcal{O}^u . Here $x_i = \Theta(z_i)$.

Case 2. w is in a blown up leaf, but F does not intersect a lift annulus in $\widetilde{W}^s(w)$.

Refer to the setup above. As before, we first show we can assume w_i are not in $\widetilde{W}^s(w)$. Otherwise, up to subsequence assume all w_i are in $\widetilde{W}^s(w)$. Since F does not intersect lift annuli in $\widetilde{W}^s(w)$, then w_i are all in prongs of $\widetilde{W}^s(w)$. Up to subsequence we can assume they are all in the same prong C of $\widetilde{W}^s(w)$ which has boundary an orbit γ of $\widetilde{\Phi}$. It follows that w is in γ . All the orbits in C are forward asymptotic to γ , even in the blown up situation. The strangling necks analysis of Case 1 shows that F will be forced to intersect $\widetilde{\Phi}_{\mathbf{R}}(w)$. This cannot occur.

Hence, assume all v_i are in a sector of $\mathcal{O}^s(v)$ bounded by a line leaf l . Let L be $l \times \mathbf{R}$. Let q be a point in l and choose x, α, x_i and α_i as in the proof of case 1. Choose a small disc D which is transverse to $\widetilde{\Phi}$ and has α in its interior. For i big enough, D intersects lift annuli only in $\widetilde{W}^s(x)$. This is because the union of the blown annuli forms a compact set in M , so either α intersects a lift annulus, in which case there is no other lift annulus nearby, or D is entirely disjoint from lift annuli. Now apply the arguments of Case 1. This shows that $\Theta(L)$ is contained in $\overline{\Theta(F)}$ and is disjoint from $\Theta(F)$, so contained in $\partial\Theta(F)$ and F escapes up as it approaches L . This finishes the proof of Case 2.

Now we need to understand what happens when F intersects a lift annulus in general. We treat that in a special case. We need the following facts before addressing this case. A lift annulus W through b is contained in $\widetilde{W}^s(b)$ and $\widetilde{W}^u(b)$ so there is not stable/unstable flow directions in W . However, there are still such directions in ∂W , because one attracts nearby orbits of $\widetilde{\Phi}$ in W and the other one repels nearby orbits in W . In this generalized sense the first one is stable and the second one is unstable. If a is in an endpoint of a blown segment, then all local components of $\mathcal{O}^s(a) - a$, $\mathcal{O}^u(a) - a$ near a are either generalized stable or unstable. With this understanding there is an even number of such components and they alternate between generalized stable and unstable. Some local components of $\mathcal{O}^s(a) - a$ are also local components of $\mathcal{O}^u(a) - a$ if they are blown segments. One key thing to remember is that generalized stable and unstable alternate.

Case 3. Suppose that F intersects some lift annulus A contained in $\widetilde{W}^s(w)$.

Then F does not intersect both boundary orbits of A . Otherwise, in the annulus $\pi(A)$ in M , the foliation $\Lambda^s \cap \mathcal{F}$ has an arc leaf connecting the two boundary components. Hence all leaves of $\Lambda^s \cap \mathcal{F}$ in $\pi(A)$ are of this type and one can collapse $\pi(A)$ to a single orbit, still keeping Φ transverse to \mathcal{F} . This contradicts that Φ is blow down minimal with respect to \mathcal{F} . Hence either $F \cap A$ is contained in the interior of A or it intersects only one boundary leaf.

Assume without loss of generality that F escapes *up* in one direction. This defines an orbit γ of $\widetilde{\Phi}$ with $a = \Theta(\gamma)$ in $\partial\Theta(F)$. The orbit γ has to be in the boundary of the lift annulus A . This is because an interior orbit is asymptotic to both boundary orbits and hence would intersect F . We now look at the picture in \mathcal{O} . Consider the stable leaf $\mathcal{O}^s(a)$. Notice that $\Theta(F)$ intersects $\Theta(A)$. From the point of view of γ , orbits in A move away from γ in future time, that is A is an unstable direction from γ . This means that $\Theta(A)$ is generalized unstable as seen from a . It follows that there are two generalized stable sides of $\mathcal{O}^s(a)$, one on each side of $\Theta(A)$, which are the closest to $\Theta(A)$. Choose one side, start at a and follow along $\mathcal{O}^s(a)$ if needed through blown segments and eventually into a prong in $\mathcal{O}^s(a)$ so as to produce a piece of a line leaf of $\mathcal{O}^s(a)$ in that direction. This path is regular on the side associated to $\Theta(A)$ and defines a half leaf l_1 of $\mathcal{O}^s(a)$. Similarly define l_2 in the other direction, see Fig. 4, b. Let l be the union of l_1 and l_2 . Then l is a slice leaf of $\mathcal{O}^s(a)$ but is not a line leaf since $\Theta(A)$ is in $\mathcal{O}^s(a)$ and is not in l .

Claim. l is contained in $\partial\Theta(F)$ and F escapes positively as $\Theta(F)$ approaches l .

Let b in l_1 with b not in blown segment, that is, b in a prong. Choose b_i in $\mathcal{O}^u(b)$, with $b_i \rightarrow b$ and in the component of $\mathcal{O} - l$ containing $\Theta(A) - a$. Let D be an embedded disc in \widetilde{M} which is transverse to $\widetilde{\Phi}$ and projects to \mathcal{O} to a neighborhood of the arc ξ in l_1 from a to b . Let y_i in D with $\Theta(y_i) = b_i$, $y_i \rightarrow y$ with $\Theta(y) = b$. Assume that y is not in $\pi^{-1}(U)$. Choose b so that it is not in the unstable leaf of one singular orbit, hence $\widetilde{W}^u(y)$ does not contain lift annuli. In addition, choose y_i so that $\widetilde{W}^s(y_i)$ does not contain lift annuli either.

Choose points u_j in $F \cap A$ so that $\Theta(u_j) = a_j$ converges to a . For each j the set $\Theta(F)$ contains a small neighborhood V_j of $\Theta(u_j)$ with V_j converging to a when j converges to infinity. Notice that a is not in V_j as a is not in $\Theta(F)$. The leaves $\mathcal{O}^s(b_i)$ are getting closer and closer to l_1 and $\Theta(A)$. For j fixed there is i big enough so that $\mathcal{O}^s(b_i)$ intersects V_j . Let

$$z_i \in F \cap \widetilde{W}^s(y_i) \text{ with } \Theta(z_i) \in V_j;$$

here i depends on j . Let $z_i = \widetilde{\Phi}_{t_i}(r_i)$ with r_i in D . By choosing j and i converging to infinity we get that $\Theta(z_i)$ converges to a and we can ensure that the arc of $D \cap \widetilde{W}^s(y_i)$ between r_i and y_i is converging to the arc η of $\widetilde{W}^s(\gamma) \cap D$ with $\Theta(\eta) = \xi$. We can also choose V_j small enough so that t_i converges to $+\infty$.

The orbits $\widetilde{\Phi}_{\mathbf{R}}(y_i), \widetilde{\Phi}_{\mathbf{R}}(r_i)$ are very close in the forward direction as long as they are outside $\pi^{-1}(U)$. Since $\widetilde{W}^s(y_i)$ does not contain lift annuli then for times s converging to infinity $\widetilde{\Phi}_s(y_i)$ is not in $\pi^{-1}(U)$. Consider the flow band C in $\widetilde{W}^s(y_i)$ between $\widetilde{\Phi}_{\mathbf{R}}(r_i)$ and $\widetilde{\Phi}_{\mathbf{R}}(y_i)$. The leaf F intersects $\widetilde{\Phi}_{\mathbf{R}}(r_i)$ in $\widetilde{\Phi}_{t_i}(r_i)$ with t_i converging to infinity. Then an analysis exactly as in Case 1 considering strangling necks and the arcs B in that proof shows that $F \cap \widetilde{W}^s(y_i)$ cannot escape up before intersecting $\widetilde{\Phi}_{\mathbf{R}}(y_i)$.

Suppose that F escapes down before intersecting $\widetilde{\Phi}_{\mathbf{R}}(y_i)$. We show that this is impossible. Since $F \cap \widetilde{W}^s(y_i)$ has points z_i in the forward direction from D and points in the backwards direction from D it follows that $F \cap \widetilde{W}^s(y_i)$ must intersect D in at least a point q_i . Up to subsequence we may assume that q_i converges to q in $\widetilde{W}^s(y)$. This will be an iterative process. Let $u_1 = q$. It is crucial to notice that in the flow band of $\widetilde{W}^s(y)$ between $\widetilde{\Phi}_{\mathbf{R}}(y)$ and γ the flow lines tend to get closer to γ , that is, either they project to closed orbits freely homotopic to $\pi(\gamma)$ or they are asymptotic to one of these orbits moving closer to γ . We now consider the component of $F \cap \widetilde{W}^s(y)$ containing u_1 and follow it towards γ . This component does not intersect γ and by the above it can only escape down in $\widetilde{W}^s(y)$. As it escapes down it produces points c_i in $\widetilde{W}^s(r_i)$ and as before produces points c'_i in D , which up to

subsequence converge to c in $D \cap F$. By construction, c is not u_1 and its orbit is closer to γ . Let $u_2 = c$. We can iterate this process. Notice the u_i cannot accumulate in D , or else all the corresponding points of F are in a compact set of \widetilde{M} . On the other hand, the process does not terminate. This produces a contradiction.

The contradiction shows that in fact the arc $\Theta(C)$ is in $\Theta(F)$, which implies that $\xi = \Theta(\eta)$ is contained in $\overline{\Theta(F)}$. As the time to hit F from D grows with i , this shows that $\Theta(F)$ does not intersect ξ and hence ξ is contained in $\partial\Theta(F)$. As b is arbitrary this shows that $l \subset \partial\Theta(F)$ and F escapes up as $\Theta(F)$ approaches l . This finishes the analysis of Case 3.

Case 4. w is in a blown up leaf and F intersects some lift annulus A in $\widetilde{W}^s(w)$ – final part.

In Case 3 we obtained a slice boundary component l of $\Theta(F)$. But we do not yet know if l contains $v = \Theta(w)$ and whether l is a stable or unstable boundary. Now we prove that there is l a stable slice which contains $v = \Theta(w)$. This l may be different than the one produced in Case 3.

Recall the setup: $v = \Theta(w)$ is in $\partial\Theta(F)$ and there are v_i in $\Theta(F)$ with v_i converging to v and with p_i in $(v_i \times \mathbf{R}) \cap F$. Also $p_i = \widetilde{\Phi}_{t_i}(w_i)$ with w_i converging to w in \widetilde{M} and t_i converging to infinity. Let ξ be the blown segment $\Theta(A)$.

The analysis of Case 3 shows that $\Theta(F)$ contains the interior of $\Theta(A)$. Suppose first that v is in ξ . Then v is in the boundary of ξ and by Case 3 again F escapes up or down when $\Theta(F)$ approaches a slice which contains v . If it escapes up, then the slice is a stable slice and we obtain the desired result in this case. We now show that F does not escape down. Let l be the unstable slice in $\partial\Theta(F)$ associated to this. Then l cuts in half a small disk neighborhood of v in \mathcal{O} . The set $\Theta(F)$ intersects only one component of the complement, the one which intersects ξ . As F escapes down when $\Theta(F)$ approaches l , then for all points in $\Theta(F)$ near enough v the corresponding point in F is flow backwards from D . This contradicts the fact that t_i is converging to infinity. Therefore, F cannot escape down as it approaches l .

We can now assume that v is not in ξ . By changing ξ if necessary assume that ξ is the blown segment in $\mathcal{O}^s(v)$ intersected by $\Theta(F)$ which is closest to v . Let z be the endpoint of ξ separating the rest of ξ from v in $\mathcal{O}^s(v)$.

We first show that z is not in $\Theta(F)$. Suppose that is not the case and let b the intersection point of $z \times \mathbf{R}$ and F . Since ξ is the last blown segment of $\mathcal{O}^s(v)$ between ξ and v intersected by $\Theta(F)$ and $\Theta(F)$ contains an open neighborhood of z , it follows that v is in a prong B of $\mathcal{O}^s(v)$ with endpoint z . Let τ be the component of $F \cap \widetilde{W}^s(b)$ containing b . Since F does not intersect $v \times \mathbf{R}$, it escapes. As the region between

$\tilde{\Phi}_{\mathbf{R}}(b)$ and $v \times \mathbf{R}$ is a prong, F cannot escape up. As seen in the arguments for Case 3, F cannot escape down either. This shows that z cannot be in $\Theta(F)$.

It follows that F escapes either up or down as $\Theta(F)$ approaches z . Suppose first that it escapes up. Then we are in the situation of Case 3 and we produce a stable slice l in $\partial\Theta(F)$ with F going up as $\Theta(F)$ approaches l . If v is not in l then l separates v from $\Theta(F)$. This contradicts v_i in $\Theta(F)$ with v_i converging to v . Hence v is in l with F escaping up as $\Theta(F)$ approaches l . This is exactly what we want finishing the analysis in this situation.

The last situation is F escaping down in A as $\Theta(F)$ approaches z . By Case 3 there is a slice leaf l in $\mathcal{O}^u(z)$ with l contained in $\partial\Theta(F)$ and F escaping down as $\Theta(F)$ approaches l . We want to show that this case cannot happen. Notice that the blown segments of $\mathcal{O}^s(z)$ are exactly the same as the blown segments of $\mathcal{O}^u(z)$. The sets $\mathcal{O}^s(z), \mathcal{O}^u(z)$ differ exactly in the prongs and as they go around the collection of blown segments. The collection of all prongs in $\mathcal{O}^s(z), \mathcal{O}^u(z)$ also alternates between stable and unstable as it goes around the union of the blown segments. Here l is the union of l_1, l_2 , each contained in a line leaf of $\mathcal{O}^u(z)$

Suppose first that v is in l . This contradicts F escaping down and $t_i \rightarrow \infty$. Finally suppose that v is not in l . We claim that in this case l separates v from $\Theta(F)$. Let β be the path in $\mathcal{O}^s(v)$ from z to v . If β only intersects l in z , then the separation property follows because l_1 and l_2 contain the local components of $\mathcal{O}^s(z) \cup \mathcal{O}^u(z) - z$ which are closest to $\Theta(A)$. This was part of the construction of l in Case 3. The path β cannot start in l_1 or l_2 , hence l separates $\Theta(F)$ from v .

If on the other hand $\beta \cap l = \tau$ is not a single point, then it is a union of blown segments. Let u be the other endpoint of τ . By regularity of l_1 and l_2 on the $\Theta(F)$ side it follows that each blown up segment in τ has flow direction away from z . Hence τ is generalized stable at u . Therefore, the next component of $(\mathcal{O}^s(u) \cup \mathcal{O}^u(u) - u)$ on the $\Theta(F)$ side is generalized unstable and that is contained in l . In this case it also follows that l separates v from $\Theta(F)$. As seen before, this is a contradiction.

This finishes the proof of Theorem 4.1.

q.e.d.

This has an important consequence that will be used extensively in this article.

Proposition 4.2. *Let F in $\tilde{\mathcal{F}}$ and L in $\tilde{\Lambda}^s$ or $\tilde{\Lambda}^u$. Then the intersection $F \cap L$ is connected.*

Proof. By transversality of \mathcal{F} and Φ , the intersection $C = \Theta(F) \cap \Theta(L)$ is open in $\Theta(L)$. Suppose there are 2 disjoint components A and B of C . Then there is v in ∂A with v separating A from B . There

are v_i in A with v_i converging to v . By the previous proposition F escapes up or down in $A \times \mathbf{R}$ as $\Theta(F)$ approaches v . Assume wlog that F escapes up. Then there is a slice leaf l of $\mathcal{O}^s(v)$ with $l \subset \partial\Theta(F)$ and F escapes up as $\Theta(F)$ approaches l . Since l and $\Theta(F)$ are disjoint then B is disjoint from l . In addition, v separates B from A in $\Theta(L)$. From the construction in the previous theorem it follows that the slice l is the closest to A , so l separates A from B . Hence $\Theta(F)$ cannot intersect B , contrary to assumption. This finishes the proof. q.e.d.

As promised, we prove that \mathcal{F} almost transverse to a pseudo-Anosov flow implies \mathcal{F} is Reebless:

Proposition 4.3. *Let \mathcal{F} be a foliation almost transverse to a pseudo-Anosov flow Φ_1 and transverse to a corresponding almost pseudo-Anosov flow Φ . Then \mathcal{F} is Reebless.*

Proof. Suppose that \mathcal{F} is not Reebless and consider a Reeb component which is a solid torus V bounded by a torus T . Assume that the flow Φ is incoming along T .

Recall that there are some singular orbits of Φ_1 which blow up into a collection of flow annuli of Φ . Suppose that V intersects one of these annuli A . Then since Φ is incoming along T , the torus T cannot intersect the closed orbits in ∂A . Hence it intersects the interior of A , say in a point p , and the forward orbit of p will limit in a closed orbit which is contained in the interior of V .

If on the other hand V does not intersect these blown annuli, then the blow down operation does not affect the flow in V . Since orbits of Φ_1 trapped inside V cannot escape V , the shadow lemma for pseudo-Anosov flows [Han, Man, Mo1] shows that there is also a periodic orbit of Φ_1 (and hence also of Φ) in $V - T$. Notice that the shadow lemma is for pseudo-Anosov flows and not for almost pseudo-Anosov flows, and that is why we split the analysis into 2 cases.

In any case there is a closed orbit γ of Φ contained in the interior of V . Consider the generalized stable/unstable local leaves at γ . Since Φ is incoming along T , the generalized unstable leaves have to be contained in V . We eventually obtain that a whole half leaf of $W^u(\gamma)$ is contained in V . A lift \tilde{V} to \tilde{M} is homeomorphic to $D^2 \times \mathbf{R}$, because closed orbits of Φ are not null homotopic. The procedure above produces a half leaf of $\tilde{W}^u(\tilde{\gamma})$ contained in \tilde{V} . This contradicts the fact that $\tilde{W}^u(\tilde{\gamma})$ is properly embedded [Ga-Oe]. This shows that \mathcal{F} is Reebless. q.e.d.

5. Asymptotic properties in leaves of the foliation

Let Φ be an almost pseudo-Anosov flow transverse to a foliation \mathcal{F} with hyperbolic leaves. Given a leaf F of $\tilde{\mathcal{F}}$, let $\tilde{\Lambda}_F^s, \tilde{\Lambda}_F^u$ be the induced one dimensional (singular) foliations in F . In this section we

study asymptotic properties of rays in $\tilde{\Lambda}_F^s$. First we mention a result of Thurston [Th5, Th6, Th7] concerning contracting directions, which for convenience we state for 3-manifolds:

Theorem 5.1 (Thurston). *Let \mathcal{F} be a codimension one foliation with hyperbolic leaves in M^3 closed. Then for every x in any leaf F of $\tilde{\mathcal{F}}$ and every $\epsilon > 0$ there is a dense set of geodesic rays of F starting at x such that: for any such ray r there is a transversal β to $\tilde{\mathcal{F}}$ starting at x so that any leaf L intersecting β and any y in r , the distance between y and L is less than ϵ . If there is not a holonomy invariant transverse measure in M whose support contains $\pi(F)$, then one can show that the directions are actually contracting: if y escapes in r then the distance between y and L converges to 0. Finally, if $\pi(F)$ is not a compact leaf, then one can choose the β to have x in the interior.*

There is a carefully written, published version of this result in [Ca-Du]. The directions above where distance to nearby L goes to 0 are called *contracting directions*. The other ones where distance is bounded by ϵ are called ϵ *non expanding directions*.

The goal of this section is to show that given a leaf L of $\tilde{\mathcal{F}}$ and a ray l of $\tilde{\Lambda}_L^s$, l converges to a single point in $\partial_\infty L$. We first analyse the non \mathbf{R} -covered case. The proof is very involved and is done by way of contradiction. Later we deal with the \mathbf{R} -covered situation. This result is a natural extension of a result by Levitt [Le] who proved that if \mathcal{G} is a foliation with prong singularities in a closed hyperbolic surface, then in the universal cover, an arbitrary ray converges to a single point in the circle at infinity. The situation for non compact leaves of foliations is much more delicate.

Proposition 5.2. *Let Φ be an almost pseudo-Anosov flow transverse to a foliation \mathcal{F} in M^3 closed and \mathcal{F} with hyperbolic leaves. Suppose that \mathcal{F} is not \mathbf{R} -covered. Given a leaf L of $\tilde{\mathcal{F}}$ and an arbitrary ray l in a leaf of $\tilde{\Lambda}_L^s$ or $\tilde{\Lambda}_L^u$, l limits to a single point in $\partial_\infty L$. The limit depends on the ray l .*

Proof. We do the proof for $\tilde{\Lambda}_L^s$. Since M is compact and \mathcal{F} is transverse to Φ , then there is $\nu_0 > 0$ so that for ϵ positive sufficiently small: if p in \tilde{M} is less than ϵ distant from a leaf F of $\tilde{\mathcal{F}}$, then the flow line through p intersects F less than $\nu_0\epsilon$ away from p . Let l be a ray in $\tilde{\Lambda}_L^s$. Because \mathcal{F} and Φ are transverse, L is properly embedded in \tilde{M} and the leaves of $\tilde{\Lambda}^s$ are properly embedded, it follows that l is a properly embedded ray in L . Therefore it can only limit in $\partial_\infty L$.

Suppose by way of contradiction that l limits on 2 distinct points a_0, b_0 in $\partial_\infty L$. Fix p a basepoint in L . Since l limits in a_0, b_0 , there are compact arcs l_i of l with endpoints which converge to a_0, b_0 respectively in $L \cup \partial_\infty L$ and so that the distance from l_i to p in L converges to infinity.

Also, we can assume that the l_i converges to a segment v in $\partial_\infty L$, where v connects a_0, b_0 . This is in the Hausdorff topology of closed sets in $L \cup \partial_\infty L$, which is a closed disk.

The key idea is to bring this situation to a compact part of \widetilde{M} . Choose a sequence p_i at bounded distance from points in l_{k_i} so that that p_i converges in $L \cup \partial_\infty L$ to a point a in the interior of v . The bound depends on the sequence. Up to subsequence assume that there are converging translations g_i in $\pi_1(M)$ and a point p_0 in \widetilde{M} so that $g_i(p_i)$ converges to p_0 in \widetilde{M} .

We claim that the set of possible limits p_0 obtained as above projects to a sublamination of \mathcal{F} . Clearly if $g_i(p_i)$ converges to p_0 and q is in the same leaf L_0 of $\widetilde{\mathcal{F}}$ as p_0 , then the distance from p_0 to q is finite and there are q_i in L with $d_L(q_i, p_i)$ bounded and $g_i(q_i)$ converging to q . Also, q_i converges to a in $\partial_\infty L$. In addition, if a sequence of such limits c_j converges to c_0 then a diagonal process shows that c_0 is also obtained as a single limit. This proves the claim. Choose a minimal sublamination \mathcal{L} .

A leaf F of $\widetilde{\mathcal{F}}$ is isometric to the hyperbolic plane. A *wedge* W in F with corner b and ideal set an interval $B \subset \partial_\infty F$ is the union of the rays in F from b with ideal point in B . The angle of the wedge is the angle that the boundary rays of W make at b . For any such sequence p_i as above, then the visual angle at p_i subtended by the arc v in $\partial_\infty L$ grows to 2π . Therefore the angle of wedge with corner p_i and ideal set $\partial_\infty L - v$ converges to 0. This is called the *bad wedge*.

Assume up to subsequence that $g_i(p_i)$ is converging to p_0 in a leaf L_0 of $\widetilde{\mathcal{F}}$ and that the directions of the bad wedges with corners $g_i(p_i)$ in $g_i(L)$ are converging to the direction r_0 of L_0 . Let c be the ideal point of r_0 in $\partial_\infty L_0$.

Suppose first that $\pi(L_0)$ is not compact – we shall see later that this is in fact always the case. Thurston's theorem shows that the set of two sided contracting directions (or ϵ non expanding directions) in L_0 is dense in $\partial_\infty L_0$. We will use these contracting directions to transport a lot of the structure of $\widetilde{\Lambda}_{L_0}^s$ to nearby leaves. Choose s_0, s_1 to be rays in L_0 on opposite sides of r_0 defining contracting directions (or ϵ non expanding directions) very near r_0 so that together they form a small wedge W in L_0 with corner p_0 . There is an interval of leaves near L_0 so that any such leaf V is less than ϵ away from s_0, s_1 . Then a flow line of $\widetilde{\Phi}$ through any point in s_0 or s_1 intersects V less than $\nu_0\epsilon$ away. So s_0 flows to a curve in V , where we can assume it has geodesic curvature very close to 0, if ϵ is sufficiently small. It is therefore a quasigeodesic with a well defined ideal point. The same happens for s_1 and the flow images u_0, u_1 of s_0, s_1 in V define a generalized wedge W' in V . The

ideal points e_0, e_1 of u_0, u_1 are close and bound an interval I which is almost all of $\partial_\infty g_i(L)$.

By construction, $g_i(l)$ is a ray which accumulates in an interval of $\partial_\infty g_i(L)$ which contains I in its interior if i is big enough. There are then subarcs τ_j of $g_i(l)$ with endpoints a_j, b_j in u_0, u_1 respectively so that a_j converges to e_0 and b_j converges to e_1 and τ_j converges to I , see Fig. 5. Here i is fixed and j varies. Since a_j, b_j are in u_0, u_1 , they flow (by $\tilde{\Phi}$) to points in L_0 . The images in L_0 are in the same leaf of $\tilde{\Lambda}^s$. By Proposition 4.2 these images are in the same leaf of $\tilde{\Lambda}_{L_0}^s$. Hence the whole segment τ_j flows into L_0 .

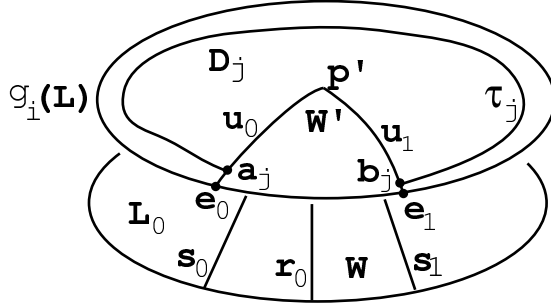


Figure 5. Transporting the structure between leaves $g_i(L)$ and L_0 .

The point p_0 flows into p' in $g_i(L)$ under the flow. The arc τ_j together with subarcs of u_0, u_1 from a_j, b_j to p' bound a disc D_j in $g_i(L)$. The arguments above show that the boundary of D_j flows into L_0 producing a curve in L_0 bounding a disc B_j . The segments of $\tilde{\Phi}$ connecting points in ∂D_j to points in ∂B_j produce an annulus C_j . Then $D_j \cup C_j \cup B_j$ is an embedded sphere in \tilde{M} and hence bounds an embedded ball. Since orbits of $\tilde{\Phi}$ are properly embedded in \tilde{M} , it follows that all orbits of $\tilde{\Phi}$ intersecting D_j will also intersect B_j . Hence there is a product flow in this ball. Since this is true for all j then the union of the D_j flows into L_0 . The union of the D_j is the closure of $g_i(L) - W'$. The image is contained in the closure of $L_0 - W$ in L_0 - call the closure J .

We claim that the image is in fact J . All the τ_j are in the same leaf of $\tilde{\Lambda}^s$ and hence all their flow images in L_0 also are. Since rays of $\tilde{\Lambda}_{L_0}^s$ are properly embedded in L_0 , when j converges to infinity the images of τ_j in L_0 escape compact sets. This shows the claim. Therefore, the flow produces a homeomorphism between the closure of $L_0 - W$ and the closure of $g_i(L) - W'$. Clearly the same is true for any leaf in the interval associated to the contracting (non expanding) directions s_0, s_1 .

We can now show that $\pi(L_0)$ is not compact. Notice that the structure of $\tilde{\Lambda}_{g_i(L)}^s$ in $g_i(L) - W'$ flows over to L_0 . In particular, there are

many rays of $\tilde{\Lambda}_{L_0}^s$ which do not have a single limit in $\partial_\infty L_0$. This implies that $\pi(L_0)$ is not compact. This is because Levitt [Le] proved that given any singular foliation with prong singularities in a closed hyperbolic surface R , the rays of the lift to \tilde{R} all have unique limit points in the ideal boundary. This shows that there cannot be any limit L_0 as above so that $\pi(L_0)$ is a compact leaf. In particular, the minimal lamination \mathcal{L} is not a compact leaf and hence it has no compact leaves.

By Thurston's theorem, one consequence of $\pi(L_0)$ not compact is that there is a dense set of contracting directions on both sides of L_0 . Therefore, we can expand this conclusion to the other side of L_0 , so L_0 is in the interior of this interval. In particular, we have the following conclusion:

Conclusion. In any limit leaf L_0 with a limit direction r_0 of bad wedges the following happens: Let c be the ideal point of r_0 and A a closed interval of $\partial_\infty L_0 - \{c\}$. Then there is a leaf l of $\tilde{\Lambda}_{L_0}^s$ with compact subsegments l_i so that the endpoints of l_i converge to the endpoints a, b of A and l_i converges to A . In particular, l_i escapes compact sets. There are also subsegments v_i with both endpoints converging to c and so that v_i converges to sets in $\partial_\infty L_0$ which contain A . Finally, for sufficiently near leaves there is a wedge in L_0 which forms a product flow region with these nearby leaves.

To get the first assertion above just follow l beyond the endpoint of l_i near b until it returns near a again. As a preliminary step to obtain Proposition 5.2 we prove the following:

Lemma 5.3. *Under the assumption that \mathcal{F} is not \mathbf{R} -covered the following happens: For any limit $g_i(p_i)$ converging to p_0 , the distinguished direction of the bad wedge associated to $g_i(p_i)$ converges to a single direction at p_0 . In addition this direction varies continuously with the leaves in $\tilde{\mathcal{L}}$.*

Proof. Suppose there are subsequences q_i, p_i converging to points in the interior of the segment v of $\partial_\infty L$ with $g_i(p_i), h_i(q_i) \rightarrow p_0 \in L_0 \in \tilde{\mathcal{F}}$, but the directions of the wedges converge to r_0, r_1 distinct geodesic rays in L_0 . We will first show that there is an interval of leaves of $\tilde{\mathcal{F}}$ so that the flow $\tilde{\Phi}$ is a product flow in this region.

Using the limit direction r_0 we produce a wedge W in L_0 so that the closure of $L_0 - W$ is part of a product flow region with nearby leaves of $\tilde{\mathcal{F}}$. Using the other limit direction r_1 we produce a flow product region associated to another wedge region W_* disjoint from $W - p_0$. Together they produce a global product structure of the flow in a neighborhood of L_0 .

This shows that there is a neighborhood N of L_0 in the leaf space of $\tilde{\mathcal{F}}$ so that the flow is a product flow in N . In particular there is no non

Hausdorffness of $\tilde{\mathcal{F}}$ in this neighborhood. This is a very strong property as we shall see below. It implies a global product structure of the flow.

Consider the neighborhood N as above which is an open subset of \tilde{M} . Consider the translates $g(N)$ where g runs through all elements of the fundamental group. Let P be the component containing N of the union of these translates. It is easy to see that the set P is precisely invariant: if g is in $\pi_1(M)$ and $g(P)$ intersects P then $g(P)$ is equal to P . In addition \mathcal{F} restricted to P has leaf space homeomorphic to \mathbf{R} because of the product flow property.

Suppose first that P is not all of \tilde{M} ; hence ∂P is a non empty collection of leaves of $\tilde{\mathcal{F}}$. Let C be the projection of P to M . Then C is open, saturated by leaves of \mathcal{F} . Notice that $g(P)$ does not intersect ∂P for any g in $\pi_1(M)$ for otherwise $g(P)$ intersects P and so $g(P) = P$. It follows that $\pi(\partial P)$ is disjoint from C hence C is a proper open, foliated subset of M .

Dippolito [Di] developed a theory of such open, saturated subsets. Let \bar{C} be the metric completion of C . There is an induced foliation in \bar{C} , which we will also denote by \mathcal{F} . Then

$$\bar{C} = V \cup \bigcup_1^n V_i$$

where V is compact and may be all of \bar{C} . Each nonempty V_i is an I -bundle over a non compact surface with boundary, so that \mathcal{F} is a foliation transverse to the I -fibers. Each component of the intersection $\partial V_i \cap V$ is an annulus (or Möebius band) with induced foliation transverse to the I fibers. In our situation with \mathcal{F} transverse to the flow, if V is not \bar{C} , we can choose V big enough so that the flow induces an I -fibration in each V_i .

Let \tilde{R} be a component of ∂P and $R = \pi(\tilde{R})$, so R is a component of ∂C . Parametrize the leaves of $\tilde{\mathcal{F}}$ in P as $F_t, 0 < t < 1$ with t increasing with flow direction. A leaf in the boundary of P which is the limit of leaves in P that are limiting from the positive side above has to be the limit of F_t as t goes to 0: Suppose that S is in the boundary of P and there are x_i in F_{t_i} with t_i converging to $t_0 > 0$ and x_i converging to x in S . Then S and F_{t_0} are not separated from each other. For i big enough the flow line through x_i will intersect S and therefore this flow line will not intersect F_{t_0} . This contradicts the fact that F_{t_0} and F_{t_i} have a flow product structure.

Suppose then that \tilde{R} is a limit of F_t where t converges to 0. Suppose first that R is compact. Suppose there are t_i converging to 0 so that F_{t_i} are in $\tilde{\mathcal{L}}$. Then since \mathcal{L} is a closed subset of M it follows that \tilde{R} is in $\tilde{\mathcal{L}}$ and so R is in \mathcal{L} . But R is compact, contradicting the fact that \mathcal{L} has no compact leaves. There is then $a > 0$ which is the smallest a so that

F_a is in $\tilde{\mathcal{L}}$ – notice that $\tilde{\mathcal{L}}$ has leaves in P . For any g in $\pi_1(R)$, $g(N) \cap N$ is not empty; hence $g(N) = N$. It follows that $g(F_a) = F_b$ for some b . If b is not a then by taking g^{-1} if necessary we may assume that $b < a$. But as F_b is in $\tilde{\mathcal{L}}$, this contradicts the definition of a . Hence $g(F_a) = F_a$ for any g in $\pi_1(R)$. This implies that $\pi(F_a)$ is a closed surface, again contradiction.

We conclude that R is not compact, hence it eventually enters some V_i (the point here is that V is not \overline{C}). The flow restricted to any component of $\partial V_i \cap \overline{C}$ goes from one component to the other in the annulus. This implies that all $\pi(F_t)$ intersect this annulus. There is then a leaf B of \mathcal{L} which enters V_i . Going deeper and deeper in this non compact I -bundle will produce a limit point which is not in C . This shows the very important fact that \mathcal{L} is not contained in C and therefore

$$\mathcal{E} = \mathcal{L} \cap (M - C) \neq \emptyset.$$

In addition \mathcal{E} is not equal to \mathcal{L} since \mathcal{L} has leaves in C and $(M - C)$ is closed. Hence \mathcal{E} is a non trivial, proper sublamination of \mathcal{L} . This contradicts the fact that \mathcal{L} is a minimal lamination.

This shows that the assumption $P \neq \tilde{M}$ is impossible. Hence $P = \tilde{M}$, which implies the flow $\tilde{\Phi}$ produces a global product picture of $\tilde{\mathcal{F}}$ and in particular \mathcal{F} is \mathbf{R} -covered, contrary to assumption.

This shows the limits of the bad wedges are unique directions in the limit leaves and they vary continuously from leaf to leaf: otherwise one obtains bad wedges in very near leaves with definitely separated directions. The same proof above then applies. This finishes the proof of Lemma 5.3. q.e.d.

Continuation of the proof of Proposition 5.2.

By the previous lemma we know that limit directions of bad wedges are unique and they vary continuously in leaves of $\tilde{\mathcal{L}}$. These unique directions are distinguished in their respective leaves.

We first show that any complementary region of \mathcal{L} (if any) is an I -bundle with a product flow.

Lift to a double cover if necessary to assume that M is orientable. Assume this is the original foliation \mathcal{F} , flow Φ , etc. Let Z be a leaf of $\tilde{\mathcal{L}}$. Since Z has a distinguished ideal point, the fundamental group of $\pi(Z)$ can be at most \mathbf{Z} . Since there is a transverse flow and M is orientable this implies that $\pi(Z)$ is either a plane or an annulus.

Let U be a complementary region of \mathcal{L} with boundary leaves R_1, R_2, R_3 , etc. As explained before, the completion of U has a compact thick part and the non compact arms which are in thin, I -bundle regions. Suppose first that R_1 is a plane. There is a big disk D so that $R_1 - D$ is contained in the thin arms and flows across U to another boundary components of U . By connectedness it flows into a single boundary

component R_2 of U . Then ∂D flows into a curve γ in R_2 which is null homotopic in M . The flow segments in M produce an annulus C in the completion of U . Since \mathcal{F} is Reebless then γ bounds a disk D' in R_2 and so R_2 is a plane. The union $D \cup C \cup D'$ is an embedded sphere in \widetilde{M} which bounds a ball B . Since orbits of $\widetilde{\Phi}$ are properly embedded in \widetilde{M} , it follows that the flow has to be a product flow in B as well. This shows that flow is a product in the completion of U .

Suppose now that each R_i is an annulus. Let F be a lift of R_1 to \widetilde{M} with F in the boundary of a component \widetilde{U} of $\pi^{-1}(U)$. In R_1 there are two disjoint open annuli A_1, A_2 contained in the thin arms so that $B = R_1 - (A_1 \cup A_2)$ is a closed annulus in the core. Then A_1, A_2 flow across U in the arms to two annuli C_1, C_2 contained in leaves R_2, R_3 in the boundary of U . Lifting to $F = \widetilde{R}_1$ we see leaves of $\widetilde{\Lambda}_F^s$ limiting in an interval of $\partial_\infty F$ with very small complement (near the distinguished ideal point of F). This implies that these leaves of $\widetilde{\Lambda}_F^s$ will have points in the lifts $\widetilde{A}_1, \widetilde{A}_2$ of A_1, A_2 to F . By Proposition 4.2, it follows that an entire ray of such a leaf of $\widetilde{\Lambda}_F^s$ flows across \widetilde{U} to the boundary leaves of \widetilde{U} . This shows that the lifts $\widetilde{C}_1, \widetilde{C}_2$ of C_1, C_2 are in the same leaf of $\widetilde{\mathcal{F}}$. This implies that $R_2 = R_3$. In the same way a half of the infinite strip \widetilde{B} flows into \widetilde{R}_2 . Since B is compact, then all of B flows into R_2 . This implies that the region U is an I -bundle. It is also easy to show that the flow is a product in this I -bundle.

This implies that we can collapse this complementary region along flow lines to completely eliminate it. This is because even in the universal cover we are eliminating product regions of the flow and the asymptotic behavior is still preserved in the remaining regions. This can be done to all complementary regions, and therefore we can assume there are no complementary regions, that is $\mathcal{L} = \mathcal{F}$ or that \mathcal{F} is minimal.

Let F_1, F_2 be leaves of $\widetilde{\mathcal{F}}$ which are not separated from each other. Consider leaves F of $\widetilde{\mathcal{F}}$ which are very close to points in both F_1 and F_2 . As stated in the conclusion in the beginning of the proof of this proposition, there is a wedge of F which flows into F_1 and similarly for F_2 . Hence there are half planes E_1, E_2 of F which flow into F_1, F_2 . As F_1, F_2 are not separated, this implies that E_1, E_2 are disjoint. Fix a point w in F and a big enough radius r so that the disk D of radius r around w intersects both E_1, E_2 . Again as seen in the conclusion above there is an arc l in a leaf of $\widetilde{\Lambda}_F^s$ so that both endpoints of l are outside D and in E_1 . In addition, we can assume that l is entirely outside D , and as seen from p the visual measure of l is almost 2π . This implies that l intersects E_2 . Since the endpoints of l are in E_1 , which flows to F_1 , Proposition 4.2 implies that the whole arc l flows into F_1 . The points of l in E_2 will also flow to F_2 . This is a contradiction to the fact that F_1, F_2 are not separated from each other.

This contradiction shows that \mathcal{F} has to be \mathbf{R} -covered and finishes the proof of Proposition 5.2 q.e.d.

Next we analyse the \mathbf{R} -covered situation which is of interest on its own:

Theorem 5.4. *Let \mathcal{F} be an \mathbf{R} -covered foliation with hyperbolic leaves and let Φ be a pseudo-Anosov flow almost transverse to \mathcal{F} . Then Φ is actually transverse to \mathcal{F} . In addition, for any leaf F of $\tilde{\mathcal{F}}$ and for any ray l in $\tilde{\Lambda}_F^s$ it converges to a unique ideal point in $\partial_\infty F$. The limit may depend on l .*

Proof. If Φ is not transverse to \mathcal{F} , let Φ^* be an almost pseudo-Anosov flow which is transverse to \mathcal{F} and is a blow up of Φ . Notice this is not the same notation as in Proposition 5.2 – here we prove Φ is equal to Φ^* . We assume that Φ^* is minimal: no blow down of Φ^* can be transverse to \mathcal{F} . Then there is a flow annulus A of Φ^* with closed orbits γ_1, γ_2 in the boundary, so that A blows down to a single orbit of Φ .

By minimality the foliation induced by \mathcal{F} in A has leaves which spiral to at least one boundary component – which they do not intersect. Lifting this picture to the universal cover, one obtains an orbit of $\tilde{\Phi}^*$ which does not intersect every leaf of $\tilde{\mathcal{F}}$. This means that the flow $\tilde{\Phi}^*$ is not *regulating* for $\tilde{\mathcal{F}}$ [Th6, Th7]. We also say that Φ^* does not regulate \mathcal{F} . In [Fe9] we analysed a similar situation and proved the following: if Υ is a pseudo-Anosov flow transverse to an \mathbf{R} -covered foliation and Υ is not regulating, then Υ is an \mathbf{R} -covered Anosov flow. The same arguments work with an almost pseudo-Anosov flow transverse to an \mathbf{R} -covered foliation. This shows that Φ^* is an \mathbf{R} -covered Anosov flow and has no (topological) singularities. In particular, Φ^* is equal to Φ , that is the original flow is already transverse to \mathcal{F} . This proves the first assertion of the theorem.

Assume by way of contradiction that there is L' in $\tilde{\mathcal{F}}$ and l in $\tilde{\Lambda}_{L'}^s$ which does not converge to a single point in $\partial_\infty L'$. As in the proof of Proposition 5.2, we construct a minimal sublamination \mathcal{L} of \mathcal{F} such that: for every L in $\tilde{\mathcal{L}}$ there is an ideal point u in $\partial_\infty L$ so that for every closed segment J in $\partial_\infty L - \{u\}$ there is a ray l of $\tilde{\Lambda}_L^s$ which has subsegments limiting to J . As shown in the proof of Proposition 5.2, \mathcal{L} cannot be a compact leaf.

Suppose first that every leaf of \mathcal{F} is a plane. Rosenberg [Ros] proved that M is the 3-dimensional torus T^3 . This manifold is a Seifert fibered space. In this case Brittenham [Bri] proved that an essential lamination is isotopic to one which is either vertical (a union of Seifert fibers) or horizontal (transverse to the fibers). So after isotopy assume \mathcal{L} has one of these types. If \mathcal{L} has a vertical leaf B , then geometrically it is a product of the reals with the circle. Hence it is a Euclidean leaf and in the

universal cover it has polynomial growth of area. If \mathcal{L} has a horizontal leaf B , then because the fibration is a product, there is a projection to a T^2 fiber, which distorts distances by a bounded amount. Again the same growth properties hold. But the leaves of \mathcal{F} are hyperbolic, which is a contradiction. We conclude that M cannot be T^3 .

Let then F in $\tilde{\mathcal{L}}$ with $\pi(F)$ not simply connected. Let g in $\pi_1(M)$ non trivial with $g(F) = F$ and ξ be the axis of g in F . At least one ideal point of ξ , call it u , is not the direction of a fixed limit of bad wedges. Let ξ' be a ray of ξ with ideal point u . Then as explained before there is a ray l of $\tilde{\Lambda}_F^s$ and segments l_i of l , bounded by a_i, b_i both points in ξ' , so that l_i escapes compact sets and converges to a non trivial segment in $\partial_\infty F$ and a_i, b_i converge to u . We may assume that $l_i \cap \xi' = \{a_i, b_i\}$ and also that all l_i are in the same side of ξ' . Let e_0 be the translation length of g in F .

If the distance from a_i to b_i along ξ is bigger than e_0 then this produces a contradiction as follows: There is an integer n so that $g^n(a_i)$ is in the open segment (a_i, b_i) of ξ and $g^n(b_i)$ is outside of the closed segment $[a_i, b_i]$. Since the arc l_i only intersects ξ' in a_i, b_i , then l_i , together with $[a_i, b_i]$, bounds a closed disk in F and $g^n(a_i)$ is in (a_i, b_i) . But $g^n(b_i)$ is outside and $g^n(l_i)$ is also on this side of ξ' , so this produces a transverse self intersection of $\tilde{\Lambda}_F^s$. If $g^n(l_i)$ is contained in the leaf v which contains l_i , then $g^n(v) = v$ and this produces infinitely many singularities in v , which is impossible. Hence $g^n(l_i)$ is not in v and the transverse intersection is impossible.

We conclude that the distance in ξ from a_i to b_i is bounded. Up to subsequence we may assume there are integers n_i so that $g^{n_i}(a_i)$ converges to a_0 and $g^{n_i}(b_i)$ converges to b_0 , both limits in ξ of course. Since the lengths of $g^{n_i}(l_i)$ are converging to infinity, it follows that a_0, b_0 are not in the same leaf of $\tilde{\Lambda}_F^s$. By Proposition 4.2 it follows that a_0, b_0 are not in the same leaf of $\tilde{\Lambda}^s$. But for each i , the pair of points $g^{n_i}(a_i), g^{n_i}(b_i)$ is in the same leaf of $\tilde{\Lambda}^s$. This implies that the leaf space of $\tilde{\Lambda}^s$ is not Hausdorff. Next we prove that this is not possible.

First of all, this implies that Φ is regulating for \mathcal{F} , for otherwise the aforementioned result from [Fe9] shows that Φ is an \mathbf{R} -covered Anosov flow – in particular $\tilde{\Lambda}^s$ has Hausdorff leaf space. Also, by Theorem 3.6 the fact that $\tilde{\Lambda}^s$ has non Hausdorff leaf space implies that there are closed orbits α, β of Φ so that α is freely homotopic to the inverse of β . Let h be a covering translation associated to α and $\tilde{\alpha}, \tilde{\beta}$ lifts of α, β to \tilde{M} which are left invariant by h . Without loss of generality assume that h acts on $\tilde{\alpha}$ sending points forwards. As $\alpha \cong \beta^{-1}$ this implies that h acts on $\tilde{\beta}$ taking points backwards. But since both of them intersect all leaves of $\tilde{\mathcal{F}}$ (by the regulating property) then as seen from $\tilde{\alpha}$ the

translation h acts increasingly in the leaf space of $\tilde{\mathcal{F}}$, with opposite behavior when considering $\tilde{\beta}$. This is a contradiction and finishes the proof of Theorem 5.4. q.e.d.

Proposition 5.2 and Theorem 5.4 imply the following:

Corollary 5.5. *Let Φ be an almost pseudo-Anosov flow transverse to \mathcal{F} with hyperbolic leaves in M^3 closed. For any F of $\tilde{\mathcal{F}}$ and any ray l of $\tilde{\Lambda}_F^s$ or $\tilde{\Lambda}_F^u$, l converges to a single point in $\partial_\infty F$.*

Remark. Group invariance and compactness of M are both essential here. For example, start with a nicely behaved singular foliation of \mathbf{H}^2 , so that all rays converge. It could be a foliation by geodesics or for instance the lift of the stable singular foliation associated to a suspension. Fix a base point p . Now rotate the leaves at a distance d of p by an angle d . In this situation all rays limit in all points of $\partial_\infty L$; in fact, they spiral indefinitely into it. Another operation is to fix a ray through p and then distort the rest more and more one way and the other way. Here we have the leaves getting closer and closer to segments in $\partial_\infty F$ which are complementary to the ideal point associated to the ray.

6. Properties of leaves of $\tilde{\Lambda}_F^s, \tilde{\Lambda}_F^u$ and their ideal points

In this section Φ is an almost pseudo-Anosov flow transverse to \mathcal{F} with hyperbolic leaves. As in the previous section there is no restriction on M or Φ here. In the previous section we proved that for any ray r of a leaf of $\tilde{\Lambda}_F^s$ or $\tilde{\Lambda}_F^u$ (F leaf of $\tilde{\mathcal{F}}$), the ray has a unique ideal point in $\partial_\infty F$, which will be denoted by r_∞ . We analyse further properties of leaves of $\tilde{\Lambda}_F^s, \tilde{\Lambda}_F^u$ and their ideal points.

We first put a topology on the union of ideal boundaries of an interval of leaves. Let p in F leaf of $\tilde{\mathcal{F}}$ and τ a transversal to $\tilde{\mathcal{F}}$ with p in the interior. For any L in $\tilde{\mathcal{F}}$ intersecting τ , the ideal boundary is in 1-1 homeomorphic correspondence with the unit tangent bundle to L at $\tau \cap L$: ideal points correspond to rays in L starting at $L \cap \tau$. This puts a topology in

$$\mathcal{A} = \cup \{ \partial_\infty L \mid L \text{ leaf of } \tilde{\mathcal{F}}, \text{ with } L \cap \tau \neq \emptyset \}$$

making it into an annulus homeomorphic to $\cup \{ T_q^1 \tilde{\mathcal{F}}, q \in \tau \}$. This topology in \mathcal{A} is independent of the choice of transversal τ . The following definition/result is proved in [Fe7] or [Cal1].

Definition 6.1 (markers). Given a foliation \mathcal{F} by hyperbolic leaves of M^3 closed, there is $\epsilon > 0$ such that: Let v be a geodesic ray in a leaf F so that it is associated to a contracting (or ϵ non expanding) direction of F . For any leaf L sufficiently near F , all the points of v have near points in L and define a curve denoted by v_L , which one can assume is

geodesic in L ; hence v_L has a unique ideal point denoted by a_L . The union m of the a_L is called a marker and is a subset of $\mathcal{A} = \cup \{\partial_\infty L\}$. Then m is an embedded curve in \mathcal{A} in the topology defined above.

In addition, the markers are continuous in \mathcal{A} in the following sense: Let z in $\partial_\infty F$ and a_i, b_i in $\partial_\infty F$ which are in markers associated to contracting (non expanding) directions on a fixed side of F . Suppose that the sequence of open intervals (a_i, b_i) in $\partial_\infty F$ contains z and converges to z as i converges to infinity. Let α_i, β_i be the markers in that side of $\partial_\infty F$ containing a_i, b_i respectively. Let L_i in $\tilde{\mathcal{F}}$ be a sequence of leaves converging to F and on that side of F so that $\partial_\infty L_i$ intersects both α_i and β_i . In the annulus \mathcal{A} of circles at infinity, consider the rectangle R_i bounded by (a_i, b_i) in $\partial_\infty F$, the parts of α_i, β_i between $\partial_\infty F$ and $\partial_\infty L_i$ and the small segment in $\partial_\infty L_i$ bounded by $\partial_\infty L_i \cap \alpha_i$ and $\partial_\infty L_i \cap \beta_i$. Then the sets R_i converge to z as i converges to infinity. This is proved in [Fe7].

From now on the ϵ is chosen small enough to also satisfy the conclusions of the definition above and also that any set in \tilde{M} of diameter less than 10ϵ is in a product box of $\tilde{\mathcal{F}}$ and $\tilde{\Phi}$. Given a curve ζ in a leaf F with starting point p and limiting on a unique point q in $\partial_\infty F$, let ζ^* denote the geodesic ray of F with same starting and ideal points.

Lemma 6.2. *Let E be a leaf of $\tilde{\Lambda}^s$ and p the starting point of the ray r of $E \cap F$. Assume that r does not have any singularity. For any leaf L of $\tilde{\mathcal{F}}$ near F , $E \cap L$ has a ray r_L which is near r . The ideal points of r_L in $\partial_\infty L$ vary continuously with L in the topology of \mathcal{A} defined above.*

Proof. We do the proof for say the positive side of F . We consider r without singularity or else we would have to check the various exterior rays in $\tilde{\Lambda}_F^s$ emanating from p . We can always get a subray of r which has no singularities.

Let $u = r_\infty$. Choose contracting (or ϵ non expanding) directions in both sides of u , with ideal points very close to u . Let them be defined by geodesic rays r_0, r_1 starting at p . There is τ a small flow segment of $\tilde{\Phi}$ starting at p and in that side of F so that for any L intersecting τ , L is asymptotic to F along the r_0, r_1 rays, or at least always $\leq \epsilon$ from F . Hence r_0, r_1 flow along $\tilde{\Phi}$ to L . Let s_0, s_1 be the flow images in L . The ϵ is also chosen small enough so that s_0, s_1 have geodesic curvature very small (this ϵ depends only on M and \mathcal{F}). In particular the curves s_0, s_1 are a small bounded distance (depending only on ϵ) from the corresponding geodesic arcs s_0^*, s_1^* in L . Let the ideal points of s_0, s_1 in $\partial_\infty L$ be denoted by v_0, v_1 and let J_L be the small closed interval in $\partial_\infty L$ bounded by v_0, v_1 . Then v_0, v_1 are in the markers associated to r_0, r_1 respectively and so they vary continuously with L when seen in \mathcal{A} .

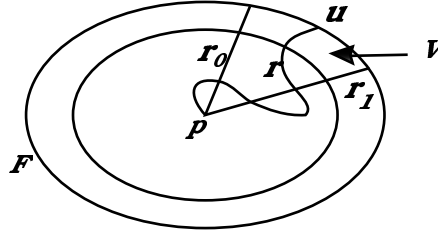


Figure 6. Leaf in wedge defined by markers.

Consider $\xi = E \cap L$ and the rays l of ξ starting at $\tau \cap L$ and containing some points which flow back to points in r . It may be that ξ has singularities – even if r does not – but there are only finitely many such rays. We want to prove that the ideal point of any such ray is in J_L . As the rectangles R_i defined above converge to u in \mathcal{A} this will prove the continuity property of the lemma.

Choose $d > 0$ so that outside of a disk D of radius d in F , r is in the small wedge W of F defined by r_0, r_1 , see Fig. 6. Choose τ small enough so that if L intersects τ , then the entire disk D is ϵ near L . Let V be the closure in F of $W - D$. The boundary ∂V in F consists of subrays of r_0, r_1 and an arc in ∂D . Therefore all points in ∂V are less than ϵ from L and flow to L under $\tilde{\Phi}$ with image a curve γ in L . This curve contains subrays of s_0, s_1 and it is properly embedded in L . Points of F near ∂V also flow to L so there is a unique component U of $L - \gamma$ which has some points flowing back to points in V . We want to show that the ray l is eventually contained in U .

Let r_{init} be the subarc of r between p and the last point c_0 of r in D . As p and c_0 flow into L , Proposition 4.2 shows that the entire arc r_{init} flows into L ; let δ be its image in L . As r is singularity free, then so is δ and hence δ is contained in any ray l of $E \cap L$ in that direction. After c_0 the curve r enters V and so l must enter U after δ . If after that the ray l exits U then it must cross $\partial U = \gamma$ in a point c_1 . But c_1 flows back to F and one can apply Proposition 4.2 again in the backwards direction to show that c_1 has to flow to a point in r . This contradicts the choice of c_0 .

This shows that l is eventually entirely contained in U and therefore l_∞ is a point in J_L . This shows the continuity property as desired and finishes the proof of the lemma. q.e.d.

Now we have a property which will be crucial to a lot of our analysis.

Proposition 6.3. *Suppose \mathcal{F} is not a blow up of a foliation topologically conjugate to the stable foliation of a suspension Anosov flow. Then the set of ideal points of rays of $\tilde{\Lambda}_F^s$ is dense in $\partial_\infty F$.*

Proof. Recall that a blow up replaces a leaf Z of \mathcal{F} by a foliated I -bundle with base Z . Suppose that there is F in $\tilde{\mathcal{F}}$ so that the set of ideal points in $\tilde{\Lambda}_F^s$ is not dense in $\partial_\infty F$. Let J be an open interval in $\partial_\infty F$ free of such ideal points. Choose p_i in F , p_i converging to a point in J . The visual angle of J as seen from p_i converges to 2π , so the complementary wedge W_i with corner p_i has angle which converges to zero. Up to subsequence assume that $g_i(p_i)$ converges to p_0 in a leaf L of $\tilde{\mathcal{F}}$ and the small wedges $g_i(W_i)$ converge to a geodesic ray s in L with ideal point z .

Claim. In L all the rays of $\tilde{\Lambda}_L^s$ converge to z .

Suppose there is x different from z so that x is an ideal point of a ray r in $\tilde{\Lambda}_L^s$. Then r is contained in $\tilde{W}^s(c_0)$ for some c_0 in \tilde{M} and for $g_i(F)$ sufficiently near L then $\tilde{W}^s(c_0)$ intersects $g_i(F)$. Any ray of $\tilde{W}^s(c_0) \cap g_i(F)$ which is near r will have all ideal points near x in the topology of the corresponding annulus \mathcal{A} of ideal circles near $\partial_\infty L$. This is a consequence of the previous lemma. But $(g_i(W_i))$ converges to the geodesic ray s , so the sets $g_i(\partial_\infty F - J)$ converge to z in \mathcal{A} . There are no ideal points of leaves of $\tilde{\Lambda}_{g_i(F)}^s$ in $g_i(J)$. This contradicts the fact that the ideal points above are very near x and proves the claim.

The proof of Proposition 6.3 is similar to that of Proposition 5.2. As in that proposition consider the set of possible limits $g_i(p_i)$ as above. This projects to a lamination in M and let \mathcal{L} be a minimal sublamination. The claim shows that each leaf of $\tilde{\mathcal{L}}$ has a distinguished ideal point towards which all rays of $\tilde{\Lambda}_L^s$ converge. The arguments in the claim also prove that if τ is a transversal to $\tilde{\mathcal{F}}$, then the distinguished ideal points of leaves of $\tilde{\mathcal{L}}$ intersecting τ vary continuously in the corresponding ideal annulus. Because of the distinguished ideal point property, each leaf of \mathcal{L} has fundamental group at most \mathbf{Z} . If needed lift to a double cover so that all leaves of \mathcal{F} are orientable. Hence a leaf of \mathcal{L} is either a plane or an annulus.

Consider a complementary component U of \mathcal{L} and a boundary leaf A of U . If A is a plane then as in the proof of Proposition 5.2, the component U is an I -bundle over A and the flow Φ is a product in U . This region can be collapsed away.

Suppose now that A is an annulus. Assume that flow lines through A flow into U . Again we want to show that U is a product region. As in the proof of Proposition 5.2 let A_1, A_2 be two noncompact, disjoint annuli in A with $A - (A_1 \cup A_2)$ a compact annulus and A_1, A_2 contained in the thin, I -bundle region. Then A_1, A_2 flow entirely into leaves B and C in ∂U . Suppose first that B, C are different. Lift to the universal cover to produce lifts $\tilde{U}, \tilde{A}, \tilde{A}_1, \tilde{A}_2, \tilde{B}, \tilde{C}$. Then \tilde{A}_1, \tilde{A}_2 are disjoint half planes of \tilde{A} which flow, say positively, respectively into \tilde{B} and \tilde{C} . Let

g be the generator of the isotropy group of \tilde{A} , which has fixed points in z, y where z is the distinguished ideal point in \tilde{A} . The argument will show there is a leaf in $\tilde{\Lambda}_A^s$ which also has ideal point in y , contradiction.

From a point in \tilde{A}_1 draw a geodesic segment of \tilde{A} to a point in \tilde{A}_2 . Let p be the first point of this segment which does not flow positively into \tilde{B} . Then $\Theta(p)$ is in the boundary of $\Theta(\tilde{B})$. Also points in the segment near p flow to \tilde{B} in positive time, hence by Theorem 4.1, there is a slice leaf l of $\mathcal{O}^s(\Theta(p))$ which is in the boundary of $\Theta(\tilde{B})$. Notice that every point in l is a limit of points in $\Theta(B)$ on that side. The set $(l \times \mathbf{R})$ intersects \tilde{A} in at least p : if l is contained in $\Theta(\tilde{A})$ then it generates a properly embedded copy of the reals in a leaf s of $\tilde{\Lambda}_A^s$; otherwise, the part that is contained in $\Theta(\tilde{A})$ also does. Every point of s is a limit of points that flow positively into \tilde{B} . Therefore, no point in s can flow positively in \tilde{C} or else we would have points flowing both in \tilde{B} and \tilde{C} .

This shows that $\Theta(s)$ separates $\Theta(\tilde{B})$ from $\Theta(\tilde{C})$. Since \tilde{A}_1 flows into \tilde{B} and \tilde{A}_2 flows into C , it now follows that the leaf s of $\tilde{\Lambda}_A^s$ is a bounded distance from the axis r of g . In particular, it has ideal points z and y , contradiction. This shows that $B = C$. In fact, the same arguments show that all of the points in A flow into B , since that happens for the complement of a compact annulus in A and then the arguments above apply here. Hence U is a product region. Therefore we can collapse \mathcal{F} to a minimal foliation.

As in Theorem 5.2 we can then show that \mathcal{F} is \mathbf{R} -covered. Suppose this is not the case and let F_1, F_2 be non separated leaves. Let L_i in $\tilde{\mathcal{F}}$ be leaves converging to both F_1, F_2 . Let u_1, u_2 be the distinguished ideal points in $\partial_\infty F_1, \partial_\infty F_2$ respectively. Let a_1, b_1 be points in $\partial_\infty F_1$ very near u_1 and on opposite sides of u_1 and which are in markers associated to contracting or ϵ non expanding directions in F_1 associated to the L_i side. Let r_1 be the geodesic in F_1 with ideal points a_1, b_1 . Similarly for F_2 producing a_2, b_2, r_2 . For i big enough L_i is at most ϵ far from all points in r_1, r_2 . Therefore r_1 flows (by $\tilde{\Phi}$) into a curve s_1 in L_i and r_2 flows into s_2 . This implies that s_1, s_2 are disjoint in L_i . Also, s_1 has ideal points a'_1, b'_1 which are in markers containing a_1, b_1 respectively (this is using a transversal to $\tilde{\mathcal{F}}$ through a point in F_1). Similarly, s_2 has ideal points a'_2, b'_2 in markers containing a_2, b_2 (using transversal to $\tilde{\mathcal{F}}$ through a point in F_2). As s_1, s_2 are disjoint, a'_1, b'_1 do not link a'_2, b'_2 in $\partial_\infty L_i$, see Fig. 7.

The ideal point a'_1 cannot be in a marker to $\partial_\infty F_1$ and to $\partial_\infty F_2$ at the same time since they are non separated leaves. Hence the points a'_1, b'_1, a'_2, b'_2 are all distinct. Let J_1 be the interval of $\partial_\infty L_i$ bounded by a'_1, b'_1 and not containing the other points, and similarly define J_2 . For simplicity we are omitting the dependence of J_1, J_2 on L_i (or on i).

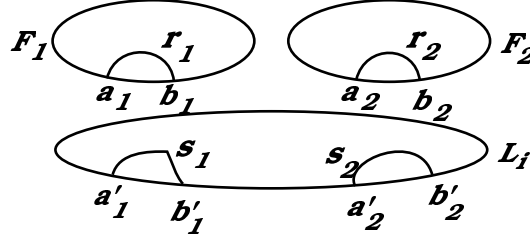


Figure 7. Pushing ideal points near.

Now consider E a leaf of $\tilde{\Lambda}^s$ intersecting F_1 . Then $E \cap F_1$ has a ray with ideal point u_1 , which is in the interval (a_1, b_1) of $\partial_\infty F_1$. The proof of Lemma 6.2 shows that if L_i is close enough to F_1 then the ideal points of the corresponding rays of $(E \cap L_i)$ have to be in J_1 . In the same way using F_2 one shows that the distinguished ideal point has to be in J_2 . Since J_1, J_2 are disjoint, this is a contradiction. This shows that \mathcal{F} is \mathbf{R} -covered.

Since \mathcal{F} is \mathbf{R} -covered Theorem 5.4 implies that Φ can be chosen to be a pseudo-Anosov flow.

Also as \mathcal{F} is \mathbf{R} -covered we can choose a transversal τ intersecting all the leaves of $\tilde{\mathcal{F}}$. This shows that the union of all the circles at infinity has a natural topology making it into a cylinder \mathcal{A} . This situation of \mathbf{R} -covered foliations is carefully analysed in [Fe7]. The fundamental group of M acts on \mathcal{A} by homeomorphisms. The union of the distinguished ideal points of leaves of the distinct leaves of $\tilde{\mathcal{F}}$ is a continuous curve ζ in \mathcal{A} which is group invariant.

Suppose first that \mathcal{F} admits a holonomy invariant transverse measure. We want to show that this case cannot happen. Then \mathcal{F} is *uniform*: any two leaves of $\tilde{\mathcal{F}}$ are a bounded distance from each other [Th5] and there is a quasiisometry between any two such leaves F, E . This induces a homeomorphism between $\partial_\infty F, \partial_\infty E$ and this is group equivariant. The quotient of \mathcal{A} by these identifications is the universal circle \mathcal{U} for \mathcal{F} in this situation [Th5] and $\pi_1(M)$ acts on \mathcal{U} . Since the distinguished direction is invariant there is a global fixed point in \mathcal{U} for this action. Recall that \mathcal{F} is minimal. If Φ is not regulating for \mathcal{F} , then as seen in the proof of Theorem 5.4, \mathcal{F} is topologically conjugate to the stable foliation of an \mathbf{R} -covered Anosov flow φ . Since \mathcal{F} cannot be topologically conjugate to the stable foliation of a suspension Anosov flow, it follows that φ has the *skewed* type [Fe2]. But then as seen in [Fe2, Th5, Fe7] the action of $\pi_1(M)$ on \mathcal{U} does not have a global fixed point. We conclude that Φ is regulating for \mathcal{F} . As seen in [Fe9] this implies that M is atoroidal. It is proved in [Fe7, Section 4] that the action in \mathcal{U} is not uniformly quasymmetric. Then this action cannot have a global fixed point. This shows that this case cannot happen.

We conclude that there is no holonomy invariant transverse measure. Therefore Thurston's theorem shows the existence of contracting directions and not just ϵ non expanding directions. Consider only the markers which are associated to contracting directions. If ζ intersects a marker m , that corresponds to a direction in a leaf of $\tilde{\mathcal{F}}$ which is contracting. Under the flow $\tilde{\Phi}$ this gets reflected in the contracted leaves nearby, that is the marker is contained in ζ . Since \mathcal{F} is minimal and ζ is $\pi_1(M)$ invariant, this shows that the entire curve ζ is a marker associated to contracting directions. The results from [Fe7] apply here, in particular Lemma 3.17 through Proposition 3.21 of [Fe7]: they show that no other direction in $\tilde{\mathcal{F}}$ (outside of ζ) is a contracting direction. By Thurston's theorem again, there would be a holonomy invariant transverse measure, contradiction.

Therefore ζ has no contracting directions. The same analysis of [Fe7] now shows that for any leaf F in $\tilde{\mathcal{F}}$ and every direction other than the distinguished direction, it is a contracting direction. In fact, it is a contracting direction with any other leaf of the foliation.

This is a very interesting situation. Let a_F be the distinguished ideal point of F leaf of $\tilde{\mathcal{F}}$. Consider a one dimensional foliation in \tilde{M} whose leaves are geodesics in leaves F of $\tilde{\mathcal{F}}$ and which have one ideal point a_F . Let $\tilde{\xi}$ be the flow which is unit speed tangent to this foliation and moves towards the ideal point a_F .

This is a flow in \tilde{M} . Clearly in each leaf of $\tilde{\mathcal{F}}$, it is a smooth flow. If q_i in L_i of $\tilde{\mathcal{F}}$ converge to q in L , then the geodesics of L_i with ideal point a_{L_i} converge to the geodesic through q in L with ideal point a_L . This is because the ideal points a_F vary continuously with F and q_i converges to q – this is the local trivialization of the union of the circles at infinity using the tangent bundles to a transversal. Hence, $\tilde{\xi}$ varies continuously.

Since ζ is group invariant, this induces a flow in M , which is tangent to the foliation \mathcal{F} . Clearly it is smooth along the leaves of \mathcal{F} and usually just continuous in the transverse direction.

This flow is a topological Anosov flow: the stable foliation is just the original foliation \mathcal{F} . The unstable foliation: Let p in leaf L of $\tilde{\mathcal{F}}$, and let γ be the flow line of $\tilde{\xi}$ through p . Then γ has positive ideal point a_L and negative ideal point v . As explained above, v is in a marker m which is associated to a contracting direction and so that m intersects all ideal circles. For each F in $\tilde{\mathcal{F}}$, let m_F be the intersection of m and $\partial_\infty F$. Let γ_F be the geodesic in F with ideal points a_F and m_F . Let E_p be the union of these γ_F . Then all orbits of $\tilde{\xi}$ in E_p are backwards asymptotic by construction. By construction the E_p are either disjoint or equal as p varies in \tilde{M} and they form a group invariant foliation in \tilde{M} . This is the

unstable foliation. Hence ξ is a topological Anosov flow. Notice that in the universal cover every stable leaf intersects every unstable leaf and vice versa.

By Proposition 3.7 it follows that ξ is topologically conjugate to a suspension Anosov flow. The foliation \mathcal{F} is then obtained as a blow up of at most countable many leaves of a foliation which is topologically conjugate to the stable foliation of this flow. This is disallowed by hypothesis and finishes the proof of Proposition 6.3. q.e.d.

Remarks.

- 1) The hypothesis is necessary. Suppose that \mathcal{F} is the stable foliation of a suspension Anosov flow, ξ so that it is transversely orientable. Perturb the flow slightly so that flow lines are still tangent to the original *unstable* foliation of ξ . The new flow, call it Φ , is transverse to \mathcal{F} ; it has the same unstable foliation as ξ but different stable foliation. The flow Φ is not regulating for \mathcal{F} . The intersections of leaves of $\tilde{\Lambda}^s$ with leaves F of $\tilde{\mathcal{F}}$ are all horocycles with the same ideal point which is the positive ideal point of flow lines in F . So the ideal points of rays of leaves of $\tilde{\Lambda}_F^s$ are not dense in $\partial_\infty F$. Notice that these leaves of $\tilde{\Lambda}_F^s$ are not quasigeodesics in F either. This example is studied in detail in Section 7 of [Fe9].
- 2) The construction of the Anosov flow at the end of the proof above is a special case of a more general construction in [Cal3].

Now we want to study metric properties of slices of leaves of $\tilde{\Lambda}_F^s$. The best metric property such leaves could have is that they are *quasi-geodesic*: this means that length along the curve is at most a bounded multiplicative distortion of length in the leaf F of $\tilde{\mathcal{F}}$ [Th1, Gr, Gh-Ha, CDP]. If the bound is k then we say the curve is a k -quasigeodesic. Since F is hyperbolic this would imply that such leaves (the non singular ones) are a bounded distance from true geodesics. Very unfortunately for us, this is not true in general, even modulo Remark 1) above. But there are still some good properties.

Let $\mathcal{H}^s, \mathcal{H}^u$ be the leaf spaces of $\tilde{\Lambda}^s, \tilde{\Lambda}^u$ respectively. Clearly, since \mathcal{H}^s may be non Hausdorff, it could be that some $\tilde{\Lambda}_F^s$ does not have Hausdorff leaf space. This easily would imply that the slices of $\tilde{\Lambda}_F^s$ are not uniformly quasigeodesic [Fe1]. This in fact occurs, see Mosher [Mo1, Mo5]. But it could be that individual rays $\tilde{\Lambda}_F^s$ are quasigeodesic – with the quasigeodesic constant depending on the particular ray. We are not able to prove this and we do not conjecture what happens in generality. But we are able to prove a weaker property, which will be enough for our purposes. If r is a ray in a leaf of $\tilde{\Lambda}_F^s$, recall that r^* is the unique geodesic ray in F with same starting point as r and same ideal point. We would like to prove that r, r^* are a bounded distance

apart, but we do not know if that is true, even modulo Remark 1). Still we prove some useful properties.

Running hypothesis for the remainder of the article.

We assume that one cannot collapse countably many complementary regions of \mathcal{F} so that the resulting foliation is topologically conjugate to the stable foliation of a suspension Anosov flow.

The main reason for this hypothesis is that we can then apply the previous proposition. The hypothesis is not restrictive for the applications to geometric properties of foliations when $\pi_1(M)$ is Gromov hyperbolic, because the manifolds discarded are for instance toroidal.

Lemma 6.4. *There is $\delta_0 > 0$ so that for any F in $\tilde{\mathcal{F}}$ and any ray r in a leaf of $\tilde{\Lambda}_F^s$, given any segment of length δ_0 in r^* , there is a point in this segment which is less than δ_0 from r in F . That implies that r^* is in the neighborhood of radius $2\delta_0$ of r in F .*

Proof. This means that $r^* \subset N_{2\delta_0}(r)$ in F . We do not know if the converse holds. Suppose the lemma is not true. Then there are F_i leaves of $\tilde{\mathcal{F}}$, r_i rays of $\tilde{\Lambda}_{F_i}^s$ and p_i in r_i^* so that $B_{2i}(p_i)$ (in F_i) does not intersect r_i . There is one side of r_i^* in F_i so that r_i goes around that side, see Fig. 8, a. Let q_i be inside a half disk of $B_{2i}(p_i)$ with $B_i(q_i)$ tangent to r_i^* and $\partial B_{2i}(p_i)$, see Fig. 8, a.

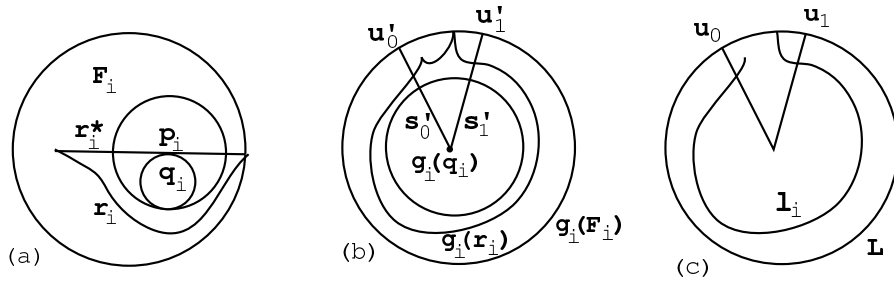


Figure 8. a. Limits of points. b. Going around disks in F_i . c. The picture in L .

As usual, up to subsequence there are g_i in $\pi_1(M)$ with $g_i(q_i)$ converging to q_0 in L leaf of $\tilde{\mathcal{F}}$ and so that the geodesic segments ζ_i from $g_i(q_i)$ to $g_i(p_i)$ in F_i converge to a geodesic ray s in L . Choose two markers with points u_0, u_1 in $\partial_\infty L$ very close to s_∞ and on opposite sides of it. The markers are associated to the side of L where the $g_i(F_i)$ are limiting to. Let s_0, s_1 be the geodesic rays of L starting at q_0 and with ideal points u_0, u_1 . For i big enough, $g_i(F_i)$ is ϵ close to both s_0 and s_1 and so these two rays flow (under $\tilde{\Phi}$) to curves s'_0, s'_1 in $g_i(F_i)$. The ideal points u'_0, u'_1 of s'_0, s'_1 are in the markers above.

For i big enough, the ray $g_i(r_i)$ has a subray which goes around $g_i(B_i(q_i))$ in $g_i(F_i)$ and has ideal point in the small segment of $\partial_\infty g_i(F_i)$ defined by u'_0, u'_1 , see Fig. 8, b. Since s'_0, s'_1 flows back to L this figure flows back to L producing a ray l_i of $\tilde{\Lambda}_L^s$ which goes around a big disk in L centered at q_0 and has ideal point in the small segment bounded by u_0, u_1 , see Fig. 8, c. As i goes to infinity, these l_i escape to infinity in L because bigger and bigger disks in $g_i(F_i)$ flow to L . This implies that there is no ideal point of a ray of $\tilde{\Lambda}_L^s$ outside the small segment of $\partial_\infty L$ bounded by u_0, u_1 . This contradicts the previous proposition that such ideal points are dense in $\partial_\infty L$.

This finishes the proof of Lemma 6.4.

q.e.d.

Lemma 6.5. *Fix the leaf F of $\tilde{\mathcal{F}}$. The limit points of rays of $\tilde{\Lambda}_F^s$ vary continuously in $\partial_\infty F$ except for the non Hausdorffness in the leaf space of $\tilde{\Lambda}_F^s$.*

Proof. Suppose that p_i converges to p in F , with respective rays r_i converging to the ray r of $\tilde{\Lambda}_F^s$. Let l be the leaf of $\tilde{\Lambda}_F^s$ through p . Up to subsequence assume the r_i are all in the same sector of l defined by p and that they form a nested sequence of rays. Then the ideal points $(r_i)_\infty$ form a monotone sequence in $\partial_\infty F$. Perhaps some ideal points are the same. If $(r_i)_\infty$ does not converge to r_∞ there is an interval v in $\partial_\infty F$, between the limit and r_∞ . Since the ideal points are dense in $\partial_\infty F$, there is w leaf of $\tilde{\Lambda}_F^s$ with w_∞ in v . Therefore, there is l' not separated from l with r_i converging to l' as well. In this fashion we can go from l to l' . This shows that if there is no leaf of $\tilde{\Lambda}_F^s$ non separated from l in that side and in the direction the rays r_i go, then the limit points vary continuously.

We analyse a bit further the non Hausdorffness. In the setup above there are subrays of r_i with points converging to a point in l' and we can restart the analysis with l' instead of l . If there are finitely many leaves non separated from l and l' we can assume that l, l' are consecutive. Then they have subrays which share an ideal point. If m is the last leaf non separated from l, l' in the direction the rays r_i go to, then there is a ray ζ of m so that there are subrays of r_i with points converging to a point in ζ and $(r_i)_\infty$ converges to ζ_∞ . If there are infinitely many such leaves non separated from l , then we can order them as $\{l_j\}, j \in \mathbf{N}$ all in the direction the rays r_i go to. The ideal points of l_j form a monotone sequence in $\partial_\infty F$ which converge to a point u in $\partial_\infty F$. The arguments above show that $(r_i)_\infty$ converges to u .

q.e.d.

Our next goal is to analyse the non Hausdorffness in the leaf space of $\tilde{\Lambda}_F^s$. We also want to understand when can the ideal points of two different rays of the same leaf of $\tilde{\Lambda}_F^s$ be the same. A *Reeb annulus* is an annulus A with a foliation so that the boundary components are leaves

and every leaf in the interior is a topological line which spirals towards the two boundary components in the same direction. In the universal cover the lifted foliation does not have Hausdorff leaf space. The lifted foliation to the universal cover is called a *Reeb band*.

Definition 6.6 (spike region). A stable spike region in a leaf F of $\tilde{\mathcal{F}}$ is a closed $\tilde{\Lambda}_F^s$ saturated set \mathcal{E} satisfying:

- There are finitely many boundary leaves of \mathcal{E} which are line leaves of $\tilde{\Lambda}_F^s$. The ideal points of consecutive rays in the boundary of \mathcal{E} are the same; otherwise, they are distinct (like an ideal polygon in F).
- The region \mathcal{E} is a bounded distance from the ideal polygon of F with these vertices. The bound is not universal in $\tilde{\mathcal{F}}$.
- There is an ideal point z of \mathcal{E} so that every leaf in the interior of \mathcal{E} has both ideal points equal to z . In addition, the leaves in the interior are nested. The finitely many leaves in the boundary are all non separated from each other and they are limits of nested sequences of interior leaves.
- There is no singularity of $\tilde{\Lambda}_F^s$ in the interior of \mathcal{E} . If there is a singularity of $\tilde{\Lambda}_F^s$ in a boundary leaf τ of \mathcal{E} then the interior of \mathcal{E} is contained in the sector defined by the line leaf τ .

Similarly define an unstable spike region. A spike region is either stable or unstable.

Proposition 6.7. *Let E be a leaf in $\tilde{\mathcal{F}}$ and v a slice leaf of $\tilde{\Lambda}_E^s$. Suppose that both ideal points of v are the same. Then v is contained in the interior of a stable spike region B of E . In addition, either B projects to a Reeb annulus in a leaf of \mathcal{F} or for any two consecutive rays in ∂B , the region between them projects to a set asymptotic to a Reeb annulus in a leaf of \mathcal{F} . Similarly for $\tilde{\Lambda}_E^u$.*

Proof. We do the proof for $\tilde{\Lambda}_E^s$. Let v be a slice as above with ideal point x in $\partial_\infty E$. Let C be the region bounded by v in E which only limits in x . First we assume that v is a line leaf of some leaf of $\tilde{\Lambda}_E^s$. We will show that the region C as it approaches x , projects to a set in M which limits to a Reeb annulus in a leaf of \mathcal{F} . The process will be done in a series of steps. The reader should be warned that the proof of this proposition is very long with several intermediate results and lemmas.

Choose z_0 in v and let e_1, e_2 be the rays of v defined by z_0 . Let ζ^* be the geodesic ray of E starting at z_0 and with ideal point x . Then ζ^* is contained in the $2\delta_0$ neighborhood of e_1 or e_2 , where δ_0 is the constant of Lemma 6.4. It follows that we can choose p_i, q_i in e_1, e_2 respectively with p_i, q_i converging to x in $E \cup \partial_\infty E$ and also $d_E(p_i, q_i) < 4\delta_0$. Let e_1^i be the subray of e_1 starting at p_i and e_2^i the subray of e_2 starting at q_i .

Up to subsequence there are p_0, q_0 in \widetilde{M} and g_i in $\pi_1(M)$ with $g_i(p_i), g_i(q_i)$ converging to p_0, q_0 respectively. The distance condition implies p_0, q_0 are in the same leaf of $\widetilde{\mathcal{F}}$, let F be this leaf. Then $g_i(E)$ converges to F and perhaps other leaves as well.

For i big enough the flowlines of $\widetilde{\Phi}$ through $g_i(p_i), g_i(q_i)$ go through to u_i and v_i in F . Also $u_i \rightarrow p_0, v_i \rightarrow q_0$, convergence in F . If the leaf of $\widetilde{\Lambda}_F^s$ through p_0 contains q_0 then for i big enough the arcs in leaves of $\widetilde{\Lambda}_F^s$ from u_i to v_i will have bounded length and bounded diameter. The same will happen for the arcs of $g_i(v)$ between $g_i(p_i)$ and $g_i(q_i)$, contradiction. Hence p_0, q_0 are not in the same leaf of $\widetilde{\Lambda}_F^s$. Let l be the leaf of $\widetilde{\Lambda}_F^s$ through p_0 and r be the one through q_0 . Let

$$L = \widetilde{W}^s(p_0), \quad R = \widetilde{W}^s(q_0), \quad \text{so } l \subset L, \quad r \subset R.$$

As $L \cap F$ is connected, then L, R are distinct and also are not separated from each other in the leaf space of $\widetilde{\Lambda}^s$. We assume that the leaves of $\widetilde{\Lambda}^s$ through u_i form a nested collection with i .

The first goal is to show that there are l, r line leaves of $\widetilde{\Lambda}_F^s$ as above so that l, r have a common ideal point. Let α_i be the leaf of $\widetilde{\Lambda}_F^s$ containing u_i . Let β_i be the ray of α_i starting at u_i and containing points in flowlines which go to the ray $g_i(e_1^i)$. Similarly let γ_i be the subray of α_i starting at v_i and associated with the ray $g_i(e_2^i)$. Let \mathcal{C}_1 (resp. \mathcal{C}_2) be the collection of line leaves of $\widetilde{\Lambda}_F^s$ that β_i (resp. γ_i) converges to, including the ray of l (resp. r). Let \mathcal{C} be the collection of all line leaves of $\widetilde{\Lambda}_F^s$ which are non separated from l, r . Then \mathcal{C} contains \mathcal{C}_1 and \mathcal{C}_2 . For any element τ in \mathcal{C} , let $B(\tau)$ be the leaf of $\widetilde{\Lambda}^s$ containing it. All of the $B(\tau)$ are not separated from each other, and they are in the set of leaves \mathcal{B} of $\widetilde{\Lambda}^s$ non separated from both L, R . By Theorem 3.6, the set \mathcal{B} has a linear order, making it order isomorphic to either \mathbf{Z} or a finite set. This induces an order in \mathcal{C} which can be chosen so that an arbitrary element of \mathcal{C}_1 is bigger than any element in \mathcal{C}_2 .

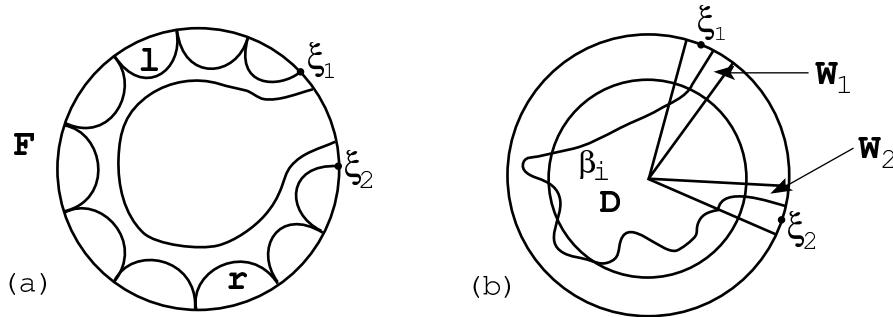


Figure 9. a. Non Hausdorffness in the limit.
b. Showing $\xi_1 = \xi_2$.

If there are finitely many elements in \mathcal{C}_1 let l' be the last one and let ξ_1 be the ideal point of the ray of l' corresponding to the direction of the rays β_i . Otherwise, the ideal points of the leaves in \mathcal{C}_1 form a weakly monotone sequence in $\partial_\infty F$ and let ξ_1 be the limit of this sequence. Similarly, define ξ_2 associated to r , see Fig. 9, a. Fix a basepoint x_0 in F . The first thing to prove is the following:

Lemma 6.8. $\xi_1 = \xi_2$.

Proof. Suppose this is not true. Choose 2 markers very near ξ_1 on either side of ξ_1 bounding an interval J_1 in $\partial_\infty F$ and similarly choose markers near ξ_2 and interval J_2 so that J_1, J_2 are disjoint. Let W_1 be the wedge of F centered at the point x_0 with ideal set J_1 and W_2 the wedge of F centered also at x_0 with ideal set J_2 . For i big enough, both boundaries of W_1 and W_2 flow into $g_i(E)$.

Suppose first that there is a last leaf l' in \mathcal{C}_1 . Then l' has a ray which is eventually contained in a strictly smaller wedge W'_1 , since the ideal point of l' is ξ_1 . Now choose a big disk D of F centered in x_0 . Let N_1 be the closure of $W_1 - D$. Choose D big enough so that l' enters N_1 through ∂D and is then entirely in W'_1 . For i big enough β_i will be close to l' for a long distance. By Lemma 6.5 the ideal points of β_i converge to ξ_1 as i converges to infinity, since l' is the last leaf non separated from l in that side. The ideal point is in the limit set of the subwedge W'_1 . If the rays β_i keep exiting W_1 , then since they are trapped by l' and β_{i_0} (for some i_0), it follows that the sequence β_i has additional limits besides the leaves in \mathcal{C}_1 —a contradiction. Therefore for big enough i , the β_i enters N_1 through ∂D and stays in N_1 from then on.

We want to get the same result when \mathcal{C}_1 is infinite. In that case let $\{\nu_j, j \in \mathbf{N}\}$ be the leaves in \mathcal{C}_1 ordered with same ordering as in \mathcal{C}_1 and $\nu_1 = l$. Since the leaves ν_i are non separated from each other then they cannot accumulate anywhere in F as $i \rightarrow \infty$ and ν_j escapes compact sets as j grows. The ideal points of ν_j are also converging to ξ_1 . By density of ideal points of $\tilde{\Lambda}_F^s$ in $\partial_\infty F$ the leaves ν_j cannot be getting closer to non trivial intervals in $\partial_\infty F$. Hence there is j_0 so that for

$$j \geq j_0, \nu_j \text{ is very close to } \xi_1 \text{ in } F \cup \partial_\infty F$$

and so contained in W_1 . Now, an argument entirely similar as in the case \mathcal{C}_1 finite implies that for i big enough, β_i has subrays contained in N_1 . The same holds for γ_i producing subrays contained in the corresponding set N_2 — perhaps the disk D needs to be bigger to satisfy all these conditions.

There is $a_1 > 0$ and i_0 so that for $i \geq i_0$, except for the initial segment of length a_1 , β_i is entirely contained in N_1 and similarly for γ_i and N_2 . Choose k_0 big enough so that D is ϵ close to $g_k(E)$ for any $k \geq k_0$. Then D flows in $g_k(E)$ under $\tilde{\Phi}$ and so do $\partial W_1, \partial W_2$. For i bigger than both i_0, k_0 the ray β_i flows into the ray $g_i(e_1^i)$ (notice these do not have

singularities). The ray $g^i(e_1^i)$ has to be in the generalized wedge which is bounded by the image of ∂W_1 in $g_i(E)$. Similarly for γ_i . This argument is made in Lemma 6.2. These two generalized wedges have disjoint ideal sets in $\partial_\infty g_i(E)$. Therefore $g_i(e_1^i)$ and $g_i(e_2^i)$ do not have the same ideal points. This is a contradiction because e_1, e_2 have the same ideal point in $\partial_\infty E$. This proves that $\xi_1 = \xi_2$ and proves Lemma 6.8. q.e.d.

Continuation of the proof of Proposition 6.7.

Recall that v is a slice leaf of $\tilde{\Lambda}_E^s$ with both ideal points x . The first part of the proof was this: in E zoom in towards the ideal point x of $\partial_\infty E$ and use covering translations g_i of \tilde{M} to map back these points near a point in \tilde{M} which is in a leaf F . We will now do this process again, but starting with F instead of E . By taking translates of F we will limit to a leaf F^* . As opposed to the analysis with E , now there is the advantage of having leaves of $\tilde{\Lambda}_F^s$ which are non separated from each other. These non separated leaves are much better suited to perturbation arguments as seen below.

Recall the sequences $(\beta_i), (\gamma_i)$ of rays in leaves of $\tilde{\Lambda}_F^s$ and contained in F . The previous lemma shows that $\xi_1 = \xi_2$ and this implies that the ideal points of β_i, γ_i are all the same and equal to ξ_1 . Let $\xi = \xi_1$. Let μ be the geodesic ray in F starting at x_0 (the basepoint in F) with ideal point ξ . Since $(\beta_i)_\infty = (\gamma_i)_\infty = \xi$, then Lemma 6.4 implies that for z in μ far enough from p_0 , we can choose a point in β_i which is less than $2\delta_0$ away from z in F . Call this point $b_i(z)$. Similarly define $c_i(z)$ in γ_i . This is for any i in \mathbf{N} . For each z in μ we may take a subsequence of the $b_i(z)$ which converges in F and the limit is denoted by $b(z)$. Similarly define $c(z)$. By definition of \mathcal{C}_1 the point $b(z)$ has to be in one of the leaves of \mathcal{C}_1 and similarly $c(z)$ is in \mathcal{C}_2 . The $b(z), c(z)$ are not uniquely defined and most likely do not vary continuously with z .

Lemma 6.9. *There is at least one element ζ of \mathcal{C}_1 which has ideal point ξ . Similarly for \mathcal{C}_2 .*

Proof. If there are finitely elements in \mathcal{C}_1 then the last one satisfies this property. Suppose then that \mathcal{C}_1 is infinite. As z varies in μ , then so does $b(z)$. If there are z escaping in μ so that $b(z)$ is in the same element ζ of \mathcal{C}_1 then ζ has an appropriate ray with ideal point ξ . In this case we are done.

Otherwise, we can find z_k in μ converging to ξ so that $b(z_k)$ are in leaves $\nu_{m(k)}$ of \mathcal{C}_1 which are all distinct. We can choose z_k so that the $m(k)$ increases with k . In the same way we may assume that $c(z_k)$ are in distinct elements of \mathcal{C}_2 . Let

$$B_k = \tilde{W}^s(b(z_k)), \quad C_k = \tilde{W}^s(c(z_k)), \quad \text{both in } \mathcal{B}.$$

Recall that \mathcal{B} is the set of leaves of $\widetilde{\Lambda}^s$ non separated from both L, R . As the length from $b(z_k)$ to $c(z_k)$ in F is bounded by $4\delta_0$, then up to subsequence assume $\pi(b(z_k)), \pi(c(z_k))$ converge in M . For n, k big enough there is $h_{n,k}$ covering translation of \widetilde{M} so that $h_{n,k}(b(z_n))$ is very close to $b(z_k)$ and $h_{n,k}(c(z_n))$ is very close to $c(z_k)$. Suppose $n \gg k$, and let $h = h_{n,k}$ for simplicity. Then B_k has a point $b(z_k)$ very close to $h(b(z_n)) \in h(B_n)$ and similarly $c(z_k)$ in C_k very close to $h(c(z_n)) \in h(C_n)$. But B_k is non separated from C_k and similarly for $h(B_n), h(C_n)$, so the only way this can happen is if

$$h(B_n) = B_k, \quad h(C_n) = C_k.$$

It is exactly here that we use in an essential way the fact that B_i and C_i are not separated from each other. This perturbation argument will be used again in this section and could not be done using the leaf E . The fact above implies that h sends the set of leaves non separated from B_i, C_i to itself, that is h acts on \mathcal{B} . Notice that $B_k < B_n$ in the order of \mathcal{B} because $n > k$ and $C_k \geq C_n$ (the C_k could be all the same, but if they are not then they decrease in the order). Since $h(B_n) = B_k$ then h acts as a decreasing translation in the ordered set \mathcal{B} . But since $h(C_n) = C_k$, h acts as a non decreasing translation. These two facts are incompatible. This in fact shows that \mathcal{C}_1 is finite.

This implies that we have to have at least one element in \mathcal{C}_1 with ideal point ξ . The same happens for \mathcal{C}_2 . This finishes the proof of the lemma. q.e.d.

Continuation of the proof of Proposition 6.7.

Since the sequence $\{\beta_i\}$ also converges to the leaf ζ of $\widetilde{\Lambda}_F^s$ we can rename the objects and assume that $l = \zeta$ and p_0 is a point in l . This can be accomplished by choosing different points p_i in the ray e_1 . Similarly do the same thing in the other direction. We state this conclusion:

Conclusion. There are p_i, q_i in e_1, e_2 respectively, escaping these rays, so that $d_E(p_i, q_i) < 4\delta_0$ and there are covering translations g_i so that: $g_i(p_i)$ converges to p_0 , $g_i(q_i)$ converges to q_0 . Both p_0, q_0 are in F and are (respectively) in rays l, r of $\widetilde{\Lambda}_F^s$. Also l, r converge to the same ideal point ξ in $\partial_\infty F$ and l, r are not separated in the leaf space of $\widetilde{\Lambda}_F^s$.

We will continue this perturbation approach. First we prove the following:

Claim 1. The region in F “between” l and r projects to a strip in M which is asymptotic or spirals towards a Reeb annulus of $\mathcal{F} \cap \Lambda^s$ in M .

Let z_i be points in l converging to ξ and w_i in r converging to ξ , so that $d_F(z_i, w_i)$ is always less than $4\delta_0$. Up to subsequence assume there

are h_i covering translations with

$$h_i(z_i) \rightarrow z_0, \quad h_i(w_i) \rightarrow w_0.$$

Notice that $h_i(L), h_i(R)$ are non separated from each other and $h_i(L) \rightarrow \widetilde{W}^s(z_0), h_i(R) \rightarrow \widetilde{W}^s(w_0)$. The argument in the previous lemma then implies that

$$h_i(L) = h_j(L), \quad h_i(R) = h_j(R), \quad \forall i, j \geq i_0.$$

Discard the first i_0 terms and postcompose h_i with $(h_{i_0})^{-1}$ (that is consider $(h_{i_0}^{-1} \circ h_i)$ instead of h_i) and assume that this is the original sequence h_i . This implies that $h_i(L) = L, h_i(R) = R$ for all i . So the h_i are all in the intersection of the isotropy groups of L and R . This group is generated by a covering translation h . Therefore, there are n_i with $h_i = h^{n_i}$. Since $h_i(z_i) \rightarrow z_0$ and the $\{z_i, | i \in \mathbf{N}\}$ do not accumulate in \widetilde{M} then $|n_i| \rightarrow \infty$. In addition since L, R are not separated from each other, then h preserves orientation in \mathcal{O} and h also preserves each individual line leaf, slice and possible lift annulus of L .

Let F^* be the leaf of $\widetilde{\mathcal{F}}$ containing z_0, w_0 . Then $h_i(F)$ converges to F^* . Since z_i is in L and $h_i(L) = L$ then $h_i(z_i)$ is in L and so z_0 is in L . It follows that L intersects F^* . Up to subsequence and perhaps taking the inverse of h , assume that n_i converges to $+\infty$.

The first possibility is that $h(F) = F$. Recall that $F \cap L$ is connected and is equal to l , similarly $r = F \cap R$. As $h(L) = L$ then

$$h(l) = h(F \cap L) = F \cap L = l.$$

So $l = F \cap L$ projects to a closed curve in $\pi(F)$. In particular l has one endpoint x and the other endpoint y which is the other fixed point of h acting on $F \cup \partial_\infty F$. Similarly for $r = F \cap R$. They project to two closed curves in $\pi(F)$ which bound an annulus in $\pi(F)$. By Euler characteristic reasons, there can be no singularities inside the annulus. This annulus has a sequence of leaves $(\pi(\alpha_i))$ which converge to both $\pi(l)$ and $\pi(r)$. This implies that the annulus in $\pi(F)$ has a Reeb foliation. In particular there are only two leaves l, r in \mathcal{C} and $\mathcal{C}_1 = \{l\}$ and $\mathcal{C}_2 = \{r\}$.

The second possibility is that that $h(F)$ is not equal to F . Let \mathcal{H} be the leaf space of $\widetilde{\mathcal{F}}$. This is a one dimensional manifold, which is simply connected, but usually not Hausdorff [Ba2] and h acts on \mathcal{H} . An analysis of group actions on simply connected non Hausdorff spaces was iniatilly done by Barbot in [Ba2] and subsequently in [Ro-St, Fe8]. One possibility is that h acts freely on \mathcal{H} . Then h has an axis τ in \mathcal{H} which is invariant under h and h acts by a translation on τ . In general τ is not properly embedded in \mathcal{H} , see [Fe8]. As all the $h^{n_i}(F)$ intersect a common transversal, then the analysis in [Ba2] shows that F has to be in the axis of h and $h^n(F)$ converges to a collection of non separated leaves of $\widetilde{\mathcal{F}}$. Notice that $(h^n(F))$ cannot escape \widetilde{M} , as z_i are all in F

and $h_i(z_i)$ converges to z_0 . It follows that in this case F^* and $h(F^*)$ are non separated from each other and are distinct.

The other situation is that h has fixed points in \mathcal{H} . In general the set of fixed points of \mathcal{H} is not a closed set, but the set of points z in \mathcal{H} so that z and $h(z)$ are not separated in \mathcal{H} is a closed subset Z of \mathcal{H} [Ba2, Ro-St]. None of the images of F under h can be in Z , so F is in a component of $\mathcal{H} - Z$. Since $h^{n_i}(F)$ all intersect a common transversal then they all have to be in the same component U of $\mathcal{H} - Z$. Let i_0 be the smallest positive integer so that $h^{i_0}(U) = U$. It follows that all n_i are multiples of i_0 . Since $h^{n_i}(F)$ converges to F^* then the leaf F^* is in the boundary of the component U . In order to analyse this case let $f = h^{i_0}$. Then $f(U) = U$ and $f^n(F)$ converges to F^* and perhaps other leaves as well. In either case we want to prove the following:

Claim 2. $f(F^*) = F^*$.

Suppose this is not true. The leaves $f(F^*), F^*$ are not separated in \mathcal{H} . This implies that $\Theta(F^*)$ and $\Theta(f(F^*))$ are disjoint subsets of \mathcal{O} , see fig. 10. Therefore there are boundary leaves separating them. But L intersects both F^* and $f(F^*)$ as L intersects F^* and is invariant under $f = h^{i_0}$. Therefore both $\Theta(F^*)$ and $\Theta(f(F^*))$ intersect the same stable leaf $\Theta(L)$.

Suppose that there is a stable boundary component of $\Theta(F^*)$ separating it from $\Theta(f(F^*))$. Then it has to be a slice of $\Theta(L)$ as this set intersects both of them. It would not be a line leaf of $\Theta(L)$. But as remarked before, $f = h^{i_0}$ leaves invariant all the slices, line leaves and lift annuli of L and this contradicts $\Theta(f(F^*))$ being disjoint from $\Theta(F^*)$. This implies there is an unstable boundary component of $\Theta(F^*)$ separating it from $\Theta(f(F^*))$, see Fig. 10.

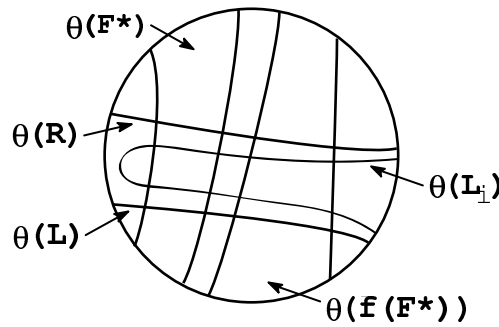


Figure 10. Contradiction in the orbit space \mathcal{O} .

In the same way $\Theta(R)$ intersects both $\Theta(F^*)$ and $\Theta(f(F^*))$. Let $L_i = \widetilde{W}^s(u_i)$. Recall from the beginning of the proof of Proposition 6.7 that u_i, v_i are points in F with u_i converging in F to the point p_0 which

is also in L and v_i converging in F to the point q_0 which is also in R . Then $\Theta(L_i)$ converges to $\Theta(L) \cup \Theta(R)$ (maybe other leaves as well). So

$$\Theta(L_i) \text{ intersects both } \Theta(F^*) \text{ and } \Theta(f(F^*))$$

for i big enough. The intersection of $\Theta(L_i)$ with at least one of $\Theta(F^*)$ or $\Theta(f(F^*))$ cannot be connected, see Fig. 10. This contradicts Proposition 4.2. This contradiction implies that $h(F^*) = F^*$ and proves Claim 2.

So far we have proved the following: in any case there is i_0 a positive integer so that if $f = h^{i_0}$ then $f(F^*) = F^*$. As $f(L) = L$ then $f(F^* \cap L) = F^* \cap L$ and similarly $f(F^* \cap R) = F^* \cap R$. This produces an annulus B in $\pi(F^*)$. The sequence of leaves (α_i) , $i \in \mathbf{N}$ limits to l and r in F . We then use $f^n(\alpha_i)$ and flow to F^* to obtain a sequence of leaves converging to both $F^* \cap L$ and $F^* \cap R$. This shows that the region of F^* bounded by $F^* \cap R$ and $F^* \cap L$ bounds a band \tilde{B} which is a bounded distance from a geodesic in F^* and projects to a Reeb annulus of $\Lambda^s \cap \mathcal{F}$ in a leaf of \mathcal{F} .

But to prove Proposition 6.7, we really want these facts for F and not just F^* . That is, we want a region in $\pi(E)$ which spirals towards a Reeb annulus. This turns out to be true: $\pi(E)$ has a band converging to $\pi(F)$ and $\pi(F)$ has a band converging to a Reeb annulus in $\pi(F^*)$. Since the annulus is compact, it turns out the second step is unnecessary. This depends on an analysis of holonomy of the foliation \mathcal{F} near the annulus in $\pi(F^*)$ as explained below.

Claim 3. The point $\pi(p_0)$ of $\pi(F)$ is in the boundary of a Reeb annulus of $\mathcal{F} \cap \Lambda^s$ contained in $\pi(F)$. This implies that $F = F^*$.

Recall that z_0 is in $F^* \cap L$. Then $\pi(z_0)$ is in $\pi(F^* \cap L) = \eta$, which is a closed curve since h^{i_0} leaves invariant both F^* and L and their intersection is connected. Removing some z_i if necessary, we may assume that all $\pi(z_i)$ are very close and up to a covering translation, also assume that F is close to F^* , where $f(F^*) = F^*$. Previous arguments in the proof imply that for i big enough $h_i(z_i)$ is in the same local sheet of $\tilde{\Lambda}^s$ as z_0 . Hence the points $\pi(z_i)$ are in $W^s(\pi(z_0)) = \pi(L)$ and converge to $\pi(z_0)$. This shows that $\pi(F \cap L)$ is asymptotic to η in the direction corresponding to the projection of the direction of escaping z_i in the ray of $F \cap L$. Namely η has contracting holonomy (of \mathcal{F}) in the side the $\pi(z_i)$ are converging to and eventually $\pi(z_i)$ is in the domain of contraction of η .

This means that the direction of F associated to the ideal point ξ is a contracting direction towards F^* . The rays in the leaves $F^* \cap L$, $F^* \cap R$ in F^* are a bounded distance from a geodesic ray in F^* with the same ideal point. The contraction above implies that the corresponding rays

$F \cap L$, $F \cap R$ of F are also a bounded distance from a ray in F with ideal point ξ .

Now recall the points p_i in E . We have $g_i(p_i)$ very close to p_0 (in \widetilde{M}) in the leaf l of $\widetilde{\Lambda}_F^s$. Also $\pi(l)$ is eventually in a region contracting towards a Reeb annulus of \mathcal{F} . Hence if i is big enough the $g_i(p_i)$ will also be in this region. Therefore the leaf of $\Lambda^s \cap \mathcal{F}$ through $\pi(p_i)$ will be contracted towards the Reeb annulus in $\pi(F^*)$ in that direction. This implies that the limit of the $\pi(p_i)$ is already in a Reeb annulus, and consequently the limit of the $g_i(p_i)$ is already in a Reeb band.

It now follows that $\pi(F) = \pi(F^*)$. Hence the second perturbation procedure (from points in F to points in F^*) in fact does not produce any new leaf. Therefore up to covering translations the leaf E is asymptotic to F in the direction of the ideal point x in $\partial_\infty E$. This proves Claim 3.

Let V be the region of E bounded by v with ideal point x . Then outside of a compact part it projects very near a Reeb annulus in $\pi(F)$ and so this tail of V has no singularity of the foliation $\widetilde{\Lambda}_E^s$. In fact, the arguments show that the tail of V flows into the interior of a Reeb band in a nearby leaf U of $\widetilde{\mathcal{F}}$. Then leaves of $\widetilde{\Lambda}_E^s$ near v on the outside of V will also flow to interior of the Reeb band in U . Therefore, there are appropriate 2 rays on the outside so that they will be in the same leaf of $\widetilde{\Lambda}_U^s$ and hence in the same leaf of $\widetilde{\Lambda}^s$. It follows that in E the leaf v is also approximated on the outside by a leaf which has a line leaf with both ideal points the same. Hence v has no singularities.

So far we proved the following:

Conclusion. Let v be a slice of $\widetilde{\Lambda}_E^s$ with two rays converging to the same ideal point x of $\partial_\infty E$ and V the region of E bounded by v . Then V has no singularity of $\widetilde{\Lambda}_E^s$. Also $\pi(V)$ is either contained in or asymptotic to a Reeb annulus in a leaf of \mathcal{F} and so E is asymptotic to a Reeb band in a leaf F in the direction x .

Continuation of the proof of Proposition 6.7.

What we want to prove is that in E itself the region V is contained in the interior of a spike region. Notice it is not true in general that $\pi(V)$ is contained in a Reeb annulus, only that it is asymptotic to a Reeb annulus. For instance, start with a leaf of \mathcal{F} having a Reeb annulus and blow that into an I-bundle. Then produce non trivial holonomy associated to the core of the Reeb annulus. Then one produces Reeb bands which are asymptotic to but not contained in Reeb annuli.

Since V is asymptotic to the Reeb band in F , it turns out that (after rearranging by covering translations) that E intersects both leaves L and R of $\widetilde{\Lambda}^s$. Their intersection produces two leaves e_L, e_R of $\widetilde{\Lambda}_E^s$ which are not separated from each other in $\widetilde{\Lambda}_F^s$ and which have the same ideal point x . There are then leaves l_i of $\widetilde{\Lambda}_E^s$ all with ideal point x and which

converge to $e_L \cup e_R$. This follows from the fact that in F the same is true and E is asymptotic to F in the direction of x , plus the connectivity of the intersection of E with leaves of $\tilde{\Lambda}^s$.

The sequence l_i can converge to other leaves as well, all of which will be non separated from e_L, e_R . The set of limits is an ordered set and any other leaf is between e_L and e_R . By Theorem 3.6 there are only finitely many leaves. We refer to Fig. 11, a, where for simplicity we consider 4 leaves in the limit: e_L, e_Y, e_Z, e_R contained in leaves L, Z, Y, R of $\tilde{\Lambda}^s$. These leaves of $\tilde{\Lambda}^s$ are non separated from each other. Let ξ be the region of E which is the union of the region bounded by all the l_i plus the boundary leaves, which are non separated from e_L, e_R . Clearly every leaf in the interior has ideal point x and has no singularity. We want to show that ξ is a spike region.

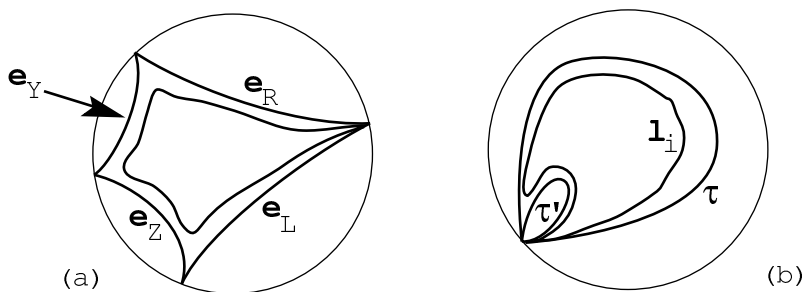


Figure 11. a. l_i converging to non separated leaves e_L, e_Z, e_Y, e_R of $\tilde{\Lambda}_E^s$. b. Nested families and identifications of ideal points.

Any two consecutive leaves of $\partial\xi$ in this ordering will have rays with same ideal point and a sequence of leaves (l_i) of $\tilde{\Lambda}_E^s$ converging to them. This situation is important on its own and is analysed in the following proposition:

Proposition 6.10. *Suppose v_1, v_2 are non separated leaves in $\tilde{\Lambda}_G^s$ for some G leaf of $\tilde{\mathcal{F}}$. Suppose that there are no leaves non separated from v_1, v_2 in between them. Then the corresponding rays of v_1, v_2 are a bounded distance from a geodesic ray of G and so these rays have the same ideal point. In M this region either projects to or is asymptotic to a Reeb annulus.*

Proof. We do essentially the same proof as in the case of leaves of $\tilde{\Lambda}_G^s$ with same ideal points, except that we go in the direction of the non Hausdorffness. Because there are no non separated leaves in between v_1, v_2 , the corresponding rays have the same ideal point. Choose w_i, y_i in these rays of v_1, v_2 and escaping towards the ideal point and so that $d_G(w_i, y_i)$ is less than $4\delta_0$. We do the limit analysis using $f_i(w_i), f_i(y_i)$

converging in \widetilde{M} . Because v_1, v_2 are non separated from each other, it follows that $f_i(w_i), f_j(w_j)$ are in the same stable leaf (of $\widetilde{\Lambda}^s$) for i, j big enough. Hence we can readjust so that they are all in the same stable leaf and similarly for $f_i(y_i)$. The same arguments as before show that that region of G between v_1, v_2 projects in M to set in a leaf of \mathcal{F} which is either contained in or asymptotic to a Reeb annulus. The results follow. In general nothing can be said about the other direction in the leaves v_1, v_2 : in particular it does not follow at all that the other rays of v_1, v_2 have to have the same ideal point. q.e.d.

Conclusion of the proof of Proposition 6.7.

Given this last proposition then for any two consecutive rays in $\partial\mathcal{E}$ it follows that they are a bounded distance from a geodesic ray in E . All that is needed to show that \mathcal{E} is a spike region is to prove that the ideal points of the rays in the boundary are distinct except for consecutive rays.

Suppose that τ is a line leaf in $\partial\mathcal{E}$ so that τ has identified ideal points. Our analysis so far shows that τ is in the interior of another region similar to the one constructed above so that all leaves have just one common ideal point. Since the l_i limit on τ , then the ideal point of τ has to be x . In addition, the leaves in this new region have to be nested. But if the l_i together with τ are a nested family of leaves of $\widetilde{\Lambda}_F^s$, then τ is outside the l_i hence the region in E bounded by τ enclosed the whole region \mathcal{E} , see Fig. 11, b. There is at least one other leaf τ' in $\partial\mathcal{E}$. The same arguments we used for τ can be applied to τ' . But it is impossible that the l_i are also nested with the τ' .

This shows that the ideal points of \mathcal{E} are distinct except as mandated by consecutive rays. In addition any line leaf in the boundary of \mathcal{E} has distinct ideal points and rays which are a bounded distance from geodesic rays. It follows that the whole leaf is a bounded distance from a geodesic in E . This shows that \mathcal{E} is a spike region. This finishes the proof of Proposition 6.7. q.e.d.

If $\widetilde{\Lambda}^s$ has Hausdorff leaf space one can say much more about metric properties of leaves of $\widetilde{\Lambda}_F^s$:

Proposition 6.11. *Suppose that Φ is an almost pseudo-Anosov flow transverse to a foliation \mathcal{F} with hyperbolic leaves. Suppose that $\widetilde{\Lambda}^s$ has Hausdorff leaf space. Then there is $k_0 > 0$ so that for any F leaf of $\widetilde{\Lambda}^s$, then the slice leaves of $\widetilde{\Lambda}_F^s$ are uniform k_0 -quasigeodesics.*

Proof. If there is a leaf F of $\widetilde{\mathcal{F}}$ and a slice leaf of $\widetilde{\Lambda}_F^s$ with only one ideal point, then the proof of Proposition 6.7 shows that there are leaves of $\widetilde{\Lambda}^s$ non separated from each other. This is impossible.

Suppose now that for any integer i , there are x_i in \widetilde{M} , x_i in leaves F_i of $\widetilde{\mathcal{F}}$ with x_i in line leaves l_i of $\widetilde{\Lambda}_{F_i}^s$ with distance from x_i to l_i^* in F_i going to infinity. Here l_i^* is the geodesic in F_i with same ideal points as l_i . Up to covering translations assume x_i converges to x . Also assume all x_i are in the same sector of $\widetilde{\Lambda}^s$ defined by x . Since l_i converges to l , the arguments in Lemmas 6.5 and 6.2 would show that the ideal points of l are the same. This was just disproved above.

Given that, the line leaves are within some global distance a_0 of the respective geodesics in their leaves. It is well known that these facts imply that the slice leaves of $\widetilde{\Lambda}_F^s$ are uniform quasigeodesics. For a proof of this well known fact see for example [Fe-Mo]. q.e.d.

Remark. The proofs in Sections 5 and 6 only used $\widetilde{\Lambda}^s, \widetilde{\Lambda}_F^s, \widetilde{\Lambda}_E^s, \dots$ and did *not* really use the unstable foliation. Hence it was not an issue here that $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ are not transverse in the lift annuli. But this non transversality feature will be very relevant in the next section.

7. Continuous extension of leaves

The purpose of this section is to prove the main theorem: the continuous extension property for leaves of foliations which are almost transverse to quasigeodesic singular pseudo-Anosov flows in atoroidal 3-manifolds. As seen before, this implies that M has negatively curved fundamental group.

Suppose first that Φ is an almost pseudo-Anosov flow which is transverse to a foliation \mathcal{F} with hyperbolic leaves in a general closed 3-manifold M . Given a leaf F of $\widetilde{\mathcal{F}}$ we introduce geodesic “laminations” in F coming from $\widetilde{\Lambda}_F^s, \widetilde{\Lambda}_F^u$. We only work with the stable foliation, similar results hold for the unstable foliation. Assume that a leaf l of $\widetilde{\Lambda}_F^s$ is not singular. If both ideal points are the same let l^* be empty. Otherwise let l^* be the geodesic with same ideal points as l . If l is singular, then no line leaves of l have the same ideal point by Proposition 6.7. For each line leaf e of l let e^* be the corresponding geodesic and l^* their union. Let now τ_F^s be the union of these geodesics of F . Line leaves of $\widetilde{\Lambda}_F^s$ do not have transverse intersections and therefore the same happens for leaves of τ_F^s . By the running hypothesis (stated prior to Lemma 6.4), Lemma 6.3 applies. Otherwise τ_F^s could be empty as examples in Section 6 show.

Suppose that $\widetilde{\Lambda}_F^s$ has non separated leaves l, v which are not in the boundary of a spike region. Then there are l_i converging to $l \cup v$ (and maybe other leaves as well), but l_i^* does not converge to l^* or v^* . In that case τ_F^s may fail to be a geodesic lamination as it may fail to be a closed subset of F . Notice none of the limit leaves can have identified ideal points, because then they would be in the interior of a spike region

(Proposition 6.7) and have a neighborhood which is product foliated. Let $\bar{\tau}_F^s$ be the closure of τ_F^s . Then $\bar{\tau}_F^s$ is a geodesic lamination in F . Similarly define $\tau_F^u, \bar{\tau}_F^u$. In a complementary region U of $\bar{\tau}_F^s$ associated to non Hausdorffness, there is one boundary component which is added (a leaf of $\bar{\tau}_F^s - \tau_F^s$) and which is the limit of the l_i^* as above. All of the other boundary leaves of the region are associated to the non separated leaves of $\tilde{\Lambda}_F^s$ and are in τ_F^s .

Lemma 7.1. *The new leaves in $\bar{\tau}_F^s$ (that is those in $\bar{\tau}_F^s - \tau_F^s$) come from non Hausdorff pairs (l, v) of $\tilde{\Lambda}_F^s$ as in the description above.*

Proof. Let e_i in τ_F^s converging to e not in τ_F^s . Then choose l_i line leaves in $\tilde{\Lambda}_F^s$ with $e_i = l_i^*$. Given u a point in e , there is u_i in l_i^* very close to u . Then there is a global constant δ_1 (for \mathcal{F}) so that there are p_i in l_i which are δ_1 close to u_i . The proof of this is similar to that of Lemma 6.4. Up to subsequence assume that p_i converges to p_0 and let l be the line leaf of $\tilde{\Lambda}_F^s$ containing p_0 , hence the sequence l_i converges to l and maybe other leaves. Since the l_i^* converges to e which is not in τ_F^s and l^* is in τ_F^s , it follows that l_i^* does not converge to l^* . Hence the ideal points of l_i do not converge to the ideal points of l . By Lemma 6.5 this is associated to a non Hausdorff situation: l_i converging to l and other leaves as well and l^* is the added leaf associated to this non Hausdorffness. This finishes the proof of the lemma. q.e.d.

Lemma 7.2. *The complementary regions of $\bar{\tau}_F^s$ in F are ideal polygons associated to either singular leaves or to non Hausdorff behavior of $\tilde{\Lambda}^s$. If M is atoroidal then these polygons are finite sided.*

Proof. Let t be in a complementary region U of $\bar{\tau}_F^s$. Let e be a leaf in the boundary ∂U . Let I be the interval of $\partial_\infty F - \partial e$ containing other ideal points of U . Suppose first that e is an actual leaf of τ_F^s , which comes from a line leaf l of $\tilde{\Lambda}_F^s$. It may be that l is contained in a singular leaf z of $\tilde{\Lambda}_F^s$ which is singular on the “ t ” side. This means that z has ideal points in I . In that case t is in the complementary region obtained by splitting z . This region must be U . If on the other hand l is not singular on the side containing t , then we may assume there are l_i leaves of $\tilde{\Lambda}_F^s$ with ideal points in the closure of I in $\partial_\infty F$, with l_i converging to l . If the ideal points of l_i converge to that of l then eventually l_i^* separates t from e and t is not in the complementary region U – impossible. Hence the ideal points of l_i do not converge to ∂e and there is non Hausdorffness and a complementary region in that side of l . Then t needs to be in this complementary region (which is U) and e is a boundary leaf of U which comes from a line leaf of τ_F^s .

Suppose now that e is an added leaf. There are l_i leaves of $\tilde{\Lambda}_F^s$ with $e_i = l_i^*$ converging to e on the side opposite to t , otherwise t is not in U . Then l_i converges to more than one leaf of $\tilde{\Lambda}_F^s$ producing non

Hausdorff behavior and a complementary region with e in its boundary. The t is in the region associated to this non Hausdorff behavior, so the complementary region must be U .

If there is a complementary region of $\bar{\tau}_F^s$ with infinitely many sides then it is associated to non Hausdorff behavior and so there are leaves l_i of $\tilde{\Lambda}_F^s$ converging to infinitely many distinct leaves of $\tilde{\Lambda}_F^s$. Then there is L leaf of $\tilde{\Lambda}^s$ which is non separated from infinitely many other leaves. Theorem 3.6 implies that there is a $\mathbf{Z} \oplus \mathbf{Z}$ subgroup of $\pi_1(M)$, contradiction to hypothesis. q.e.d.

We now turn to the continuous extension property.

Theorem 7.3 (Main theorem). *Let \mathcal{F} be a foliation in M^3 closed, atoroidal. Suppose that \mathcal{F} is almost transverse to a quasigeodesic, singular pseudo-Anosov flow Φ_1 and transverse to an associated almost pseudo-Anosov flow Φ . Singular means Φ_1 is not a topological Anosov flow. Then for any leaf F of $\tilde{\mathcal{F}}$, the inclusion map $\Psi : F \rightarrow \tilde{M}$ extends to a continuous map*

$$\Psi : F \cup \partial_\infty F \rightarrow \tilde{M} \cup S_\infty^2.$$

The map Ψ restricted to $\partial_\infty F$, gives a continuous parametrization of the limit set of F , which is then locally connected.

Proof. Since Φ_1 is singular then one can blow the singular stable leaves into a union of non singular leaves to produce a lamination in M which is essential and genuine [Ga-Ka]. As M is atoroidal, Gabai and Kazez proved that $\pi_1(M)$ is negatively curved [Ga-Ka].

To prove the main theorem some difficulties are that $\tilde{\Lambda}^s, \tilde{\Lambda}^u$ may have non Hausdorff leaf space [Mo5, Fe6] and so given F , $\tilde{\Lambda}_F^s, \tilde{\Lambda}_F^u$ may have non Hausdorff leaf space. This implies that the leaves of $\tilde{\Lambda}_F^s, \tilde{\Lambda}_F^u$ may not be uniform quasigeodesics in F . In addition the leaves of $\tilde{\Lambda}^s, \tilde{\Lambda}^u$ are not quasi-isometrically embedded in \tilde{M} . The proof of the theorem is done in two steps: first we define an extension and then we show that it is continuous.

The proof will fundamentally use the fact that Φ_1 is a quasigeodesic pseudo-Anosov flow. It was proved in [Fe-Mo] that this implies that Φ is a quasigeodesic flow as well. From now on we use the stable/unstable foliations $\tilde{\Lambda}^s, \tilde{\Lambda}^u$ of $\tilde{\Phi}$. First we need to review some facts about quasigeodesic almost pseudo-Anosov flows. If γ is an orbit of $\tilde{\Phi}$ then it is a quasigeodesic and hence has unique distinct ideal points γ_+ and γ_- in S_∞^2 corresponding to the positive and negative flow directions respectively [Th1, Gr, Gh-Ha, CDP]. Hence given x in \tilde{M} define

$$\eta_+(x) = \gamma_+, \quad \eta_-(x) = \gamma_-, \quad \eta_+(x) \neq \eta_-(x),$$

where γ is the $\tilde{\Phi}$ flowline through x .

One important point is that η_+, η_- are continuous functions from \widetilde{M} to S_∞^2 [Fe2, Fe6]. This occurs because Φ is uniformly quasigeodesic [Mo3, Fe6]. In addition if L is a leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$ and a is a limit point of L in S_∞^2 , then there is an orbit γ of $\widetilde{\Phi}$ contained in L with either $\gamma_- = a$ or $\gamma_+ = a$, that is, any limit point of L is an ideal point of one of its flow lines [Fe2, Fe6].

Also any such L in $\widetilde{\Lambda}^s$ is Gromov negatively curved in its path metric [Gr, Gh-Ha, Fe2, Fe6] and hence has an intrinsic ideal boundary ∂L . The ideal boundary of L consists of a single forward ideal point and distinct negative ideal points for each flow line if L is not a blown up leaf [Fe2, Fe6]. When L is a blown up leaf then given any lift annulus A in L then all orbits in A have the same *negative* limit point as well, since they are a bounded distance from each other in the negative direction. The set $L \cup \partial_\infty L$ is a natural compactification of L in the Gromov sense. For instance if L is a non singular leaf, then $L \cup \partial_\infty L$ is a closed disk. In this case the foliation by flow lines in L is equivalent to the foliation in \mathbf{H}^2 by geodesics sharing a fixed point in S_∞^1 . Similarly for leaves G of $\widetilde{\Lambda}^u$.

A very important fact for us is that the inclusion

$$\kappa : L \rightarrow \widetilde{M} \quad \text{extends to a continuous map} \quad \kappa : L \cup \partial L \rightarrow \widetilde{M} \cup S_\infty^2,$$

[Fe2, Fe6]. This all follows from the fact that Φ is quasigeodesic. This works for any L in $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$. If L is in $\widetilde{\Lambda}^s$ there is a unique distinguished ideal point in S_∞^2 denoted by

$$L_+ := \text{the forward limit point in } S_\infty^2 \text{ of any flow line in } L \subset \widetilde{M}.$$

If in addition Λ^s is a quasi-isometric foliation, then the extension κ is always a homeomorphism to its image, but this is not true if Λ^s is not quasi-isometric [Fe6]. Likewise for L in $\widetilde{\Lambda}^u$.

Throughout the proof we fix a homeomorphism of $\widetilde{M} \cup S_\infty^2$ with the closed unit ball in \mathbf{R}^3 . The Euclidean metric in this ball induces the visual distance in $\widetilde{M} \cup S_\infty^2$. Then $\text{diam}(B)$ denotes the diameter in this distance for any subset B of $\widetilde{M} \cup S_\infty^2$. A notation used throughout here is the following: if A is a subset of a leaf F of $\widetilde{\mathcal{F}}$, then \overline{A} is its closure in $F \cup \partial_\infty F$.

We now produce an extension $\Psi : \partial_\infty F \rightarrow S_\infty^2$.

Case 1. Suppose that v in $\partial_\infty F$ is not an ideal point of a ray in $\widetilde{\Lambda}_F^s$ or in $\widetilde{\Lambda}_F^u$.

Since $\pi_1(M)$ is negatively curved, then complementary regions of $\overline{\tau}_F^s$ are finite sided ideal polygons. Hence there are e_i in $\overline{\tau}_F^s$ so that $\{e_i \cup \partial e_i\}$, $i \in \mathbf{N}$ define a neighborhood basis of v (in $F \cup \partial_\infty F$) and $\{e_i\}$ forms a nested sequence of geodesics in F . Here ∂e_i are the ideal points of e_i in $\partial_\infty F$. We say that the $\{e_i\}$ define a neighborhood basis

at v . Assume that no two e_i share an ideal point – possible because of hypothesis. If e_i is in $\bar{\tau}_F^s - \tau_F^s$ then it is the limit of leaves in τ_F^s and by adjusting the sequence above we can assume that e_i is always in τ_F^s . Let l_i in $\tilde{\Lambda}_F^s$ with $l_i^* = e_i$ and let L_i be leaves of $\tilde{\Lambda}^s$ with $l_i \subset L_i$. Similarly there are c_i in $\bar{\tau}_F^u$ defining a neighborhood basis of v . Up to subsequence we may assume that $e_1, c_1, e_2, c_2, \dots$ are nested and none of them have any common ideal points (in $F \cup \partial_\infty F$) and c_i is in τ_F^u . Let b_i in $\tilde{\Lambda}_F^u$ with $b_i^* = c_i$ and B_i leaves of $\tilde{\Lambda}^u$ with $b_i \subset B_i$.

At this point we need the following result:

Lemma 7.4. *Let L leaf of $\tilde{\Lambda}^s$, B leaf of $\tilde{\Lambda}^u$ and F leaf of $\tilde{\mathcal{F}}$ so that F intersects both L and B : $l = L \cap F$, $b = F \cap B$. Suppose that b and l are disjoint in F . Then L does not intersect B in \tilde{M} .*

Proof. Suppose not. Recall that $\Theta(L), \Theta(B)$ are finite pronged, non compact trees and they intersect in a compact subtree. The union is also a finite pronged tree. In addition $\Theta(L \cap B)$ is connected. The sets $\Theta(l), \Theta(b)$ are disjoint in this union. Let x be a boundary point of $\Theta(l)$ which is either in $\Theta(L \cap B)$ or separates $\Theta(L \cap B)$ from $\Theta(l)$ in this union, see Fig. 12, a.

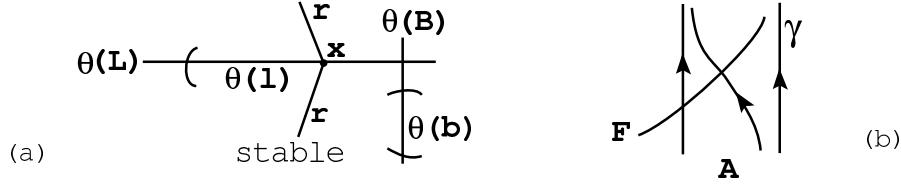


Figure 12. a. Obstruction to intersections of leaves.
b. The case of F escaping up.

Let $\gamma = x \times \mathbf{R}$, an orbit of $\tilde{\Phi}$. The first possibility is that F escapes up as $\Theta(F)$ approaches x . Then γ is a repelling orbit with respect to the $\Theta(l)$ side, see Fig. 12, b. By Theorem 4.1 γ is in the boundary of a lift annulus A . This means that $\Theta(l)$ is a generalized unstable prong from the point of view of x . By Theorem 4.1 there is a stable slice r of $\mathcal{O}^s(x)$ with r contained in $\partial\Theta(F)$ and F escapes up as $\Theta(F)$ approaches r , see Fig. 12, a. The two sides of r are the closest generalized prongs to $\Theta(l)$ on either side of $\Theta(l)$. This implies that r separates $\Theta(b)$ from $\Theta(F)$ see fig. 12, a. Then $\Theta(b)$ cannot be contained in $\Theta(F)$, contradiction.

The second option is that F escapes down as $\Theta(F)$ approaches x along $\Theta(l)$. Here there is a slice r of $\mathcal{O}^u(x)$ with r contained in $\partial\Theta(F)$ and the closest to $\Theta(l)$ on both sides of $\Theta(l)$. Either $\Theta(b) \subset r$ or r separates $\Theta(b)$ from $\Theta(F)$. In any case $\Theta(b)$ does not intersect $\Theta(F)$, again a contradiction. This finishes the proof of the lemma. q.e.d.

Claim 1. In Case 1, both L_i and B_i escape in \tilde{M} .

Notice $e_i \cap c_j = \emptyset$ for any i, j . This implies that l_i, b_j have empty intersection for any i, j .

The lemma above shows that $L_i \cap B_j$ is empty for any i, j , and they form nested sequences of leaves in \widetilde{M} . Suppose that the sequence $\{L_i\}$ does not escape compact sets. Then there is L in $\widetilde{\Lambda}^s$ which is a limit of L_i (and possibly other leaves as well). Let α be an orbit in L which is *not in a lift annulus*. Then $\widetilde{W}^u(\alpha)$ is transverse to L in α and hence intersects L_i for i big enough. Since the L_i, B_j are nested this would force $\widetilde{W}^u(\alpha)$ to intersect B_j for j big enough, contradiction. It follows that both L_i and B_j escape compact sets as $i, j \rightarrow \infty$. This proves Claim 1.

Let r be a geodesic ray in F with ideal point v . For each i , there is a subray of r contained in the component of $F - l_i$ which is in a small neighborhood of v . Hence $\Psi(r)$ has a subray which is contained in the corresponding component V_i of $\widetilde{M} - L_i$. These components V_i form a nested sequence. The ray $\Psi(r)$ can only limit in the limit set of V_i in S_∞^2 . We need the following lemma which will be a key tool throughout the proof.

Lemma 7.5 (basic lemma). *Let $\{Z_i\}$ be a sequence of leaves or line leaves or slices or any non empty flow saturated sets in leaves of either in $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$ (not all leaves Z_i need to be in the same singular foliation). If the sets Z_i escape compact sets in \widetilde{M} , then up to taking a subsequence it follows that \overline{Z}_i converges to a point in S_∞^2 as closed subsets of $\widetilde{M} \cup S_\infty^2$.*

Proof. Let Y_i be the leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$ which contains Z_i . Up to subsequence assume $Y_i \in \widetilde{\Lambda}^s$. The statement is equivalent to $\text{diam}(Z_i)$ converges to 0. Otherwise up to subsequence we can assume $\text{diam}(Z_i) > a_0$ for some a_0 and all i and hence no subsequence can converge to a single point in S_∞^2 . Then there is p_i in Z_i with visual distance from p_i to $(Y_i)_+$ is bigger than $a_0/2$. Notice that $(Y_i)_+$ is a point in \overline{Z}_i since Z_i is flow saturated, so it contains at least one orbit of $\widetilde{\Phi}$ in Y_i . Let γ_i the orbit of $\widetilde{\Phi}$ through p_i . If $(\gamma_i)_-$ is very close to $(\gamma_i)_+ = (Y_i)_+$ then the geodesic with these ideal points has very small visual diameter. Since γ_i is a global bounded distance from this geodesic [Gr, Gh-Ha, CDP], the same is true for γ_i contradiction to the choice of p_i . Hence the geodesic above intersects a fixed compact set in \widetilde{M} and so does γ_i . This contradicts the fact that Z_i escape compact sets in \widetilde{M} and finishes the proof. q.e.d.

To finish the analysis of Case 1, we claim that the limit sets of V_i above shrink to a single point in S_∞^2 . The limit sets of the V_i form a weakly monotone decreasing sequence, because the L_i are nested and

V_i are decreasing. If the limit set does not have diameter going to zero, then there are points in the limit set of L_i which are at least $2\delta_2$ apart for some fixed $\delta_2 > 0$. By the previous lemma the L_i cannot escape compact sets in \widetilde{M} , contradiction. Since the limit sets of V_i shrinks to a point in S_∞^2 , let $\Psi(v)$ be this point. Clearly $\Psi(r)$ limits to this point and so does $\Psi(r')$ for any other geodesic ray r' in F with ideal point v . This settles Case 1.

Case 2. Suppose that v is an ideal point of a leaf of $\widetilde{\Lambda}_F^s$ or $\widetilde{\Lambda}_F^u$.

Let l be a ray in say $\widetilde{\Lambda}_F^s$ which limits on v and r a geodesic ray on F with ideal point v . Then l is contained in L leaf of $\widetilde{\Lambda}^s$. Either $\Theta(l)$ escapes in $\Theta(L)$ or limits to a point x in $\Theta(L)$. Consider the first case. Then in the intrinsic geometry of L , the ray l converges to the positive ideal point of L , hence in $\widetilde{M} \cup S_\infty^2$, the image $\Psi(l)$ converges to L_+ . In the second possibility let $\beta = x \times \mathbf{R}$, an orbit of $\widetilde{\Phi}$. As l escapes in F then in L it either escapes up or down. If it escapes down then it converges to the negative ideal point of β in $L \cup \partial_\infty L$ and hence $\Psi(l)$ converges to β_- . Otherwise l escapes up in L as $\Theta(l)$ approaches x . In this case β is in the boundary of a lift annulus and l converges to the positive ideal point in $L \cup \partial_\infty L$ and so $\Psi(l)$ converges to L_+ again.

In either case let $\Psi(v)$ be the limit point of $\Psi(l)$. Similarly if l is a ray of $\widetilde{\Lambda}_F^u$.

Every point in the geodesic ray r it is $2\delta_0$ close to a point in l in F , hence the limit of $\Psi(r)$ in $\widetilde{M} \cup S_\infty^2$ is the same as that of $\Psi(l)$. If l' is another ray of $\widetilde{\Lambda}_F^s$ or $\widetilde{\Lambda}_F^u$ converging to v , then it will have points boundedly close to r which escape in l' and therefore $\Psi(l')$ has the same ideal point in S_∞^2 . Therefore $\Psi(v)$ is well defined.

This finishes the construction of the extension of Ψ to $\partial_\infty F$.

Proof of continuity of the extension.

$$\Psi : F \cup \partial_\infty F \rightarrow \widetilde{M} \cup S_\infty^2.$$

Case 1. v is not an ideal point of a ray in $\widetilde{\Lambda}_F^s$ or $\widetilde{\Lambda}_F^u$.

Let r be a geodesic ray in F with ideal point v . Recall the extension construction. There are l_i in $\widetilde{\Lambda}_F^s$ shrinking to v in $F \cup \partial_\infty F$ and similarly b_i in $\widetilde{\Lambda}_F^u$, assumed to be nested with the l_i . Let $\{l_i^*\}$ define a neighborhood basis of v in $F \cup \partial_\infty F$. Let L_i in $\widetilde{\Lambda}^s$ with $l_i \subset L_i$, and $b_i \subset B_i \in \widetilde{\Lambda}^u$ as in the construction Case 1. Then the L_i, B_i escape in \widetilde{M} . Let U_i be the component of $F - l_i$ containing a subray of r and V_i the component of $\widetilde{M} - L_i$ containing U_i . Notice that $\Psi(U_i) \subset V_i$. Let now z in \overline{U}_i with the closure taken in $F \cup \partial_\infty F$ and \overline{V}_i the closure of V_i in $\widetilde{M} \cup S_\infty^2$. Then \overline{U}_i is a neighborhood of v in $F \cup \partial_\infty F$. If z is in $\Psi(\overline{U}_i)$ then using either Case 1 or Case 2 constructions in the extension part shows that z is a limit of points in $\Psi(U_i) \subset V_i$. As seen in the

construction arguments $\text{diam}(\overline{V}_i)$ converges to 0, showing continuity of Ψ at v . This finishes the proof in this case.

Case 2. v is an ideal point of a ray of $\tilde{\Lambda}_F^s$ or $\tilde{\Lambda}_F^u$.

This case is considerably more complicated, with several possibilities.

Case 2.1. v is an ideal point of $\tilde{\Lambda}_F^s$ but not of $\tilde{\Lambda}_F^u$ (or vice versa).

Suppose the first option occurs. There is l ray in $\tilde{\Lambda}_F^s$ with ideal point v . We may assume that l is not in a leaf of $\tilde{\Lambda}_F^s$ with same ideal points. Otherwise we can choose l to be one of the boundary leaves of the corresponding spike region. Since v is not an ideal point of $\tilde{\Lambda}_F^u$, Lemma 7.2 implies that there are g_i line leaves in $\tilde{\Lambda}_F^u$ defining a basis neighborhood system at v . Let g_i be contained in G_i leaves of $\tilde{\Lambda}^u$. Let L in $\tilde{\Lambda}^s$ containing l . If G_i escapes in \tilde{M} as $i \rightarrow \infty$, then as seen in Case 1, we are done. Let then G_i converge to the finite set of leaves

$$\mathcal{V} = H_1 \cup H_2 \dots \cup H_m \quad \text{leaves of } \tilde{\Lambda}^u.$$

Notice that this has to be a finite union by Theorem 3.6 as M is atoroidal. We can assume that G_i is non singular and the sequence $\{G_i\}, i \in \mathbf{N}$ is nested.

Case 2.1.1. Suppose that L intersects \mathcal{V} , say $L \cap H_1 \neq \emptyset$.

Then $\Psi(l)$ escapes down in \tilde{M} as $\Theta(l)$ approaches $\Theta(L \cap H_1)$. Otherwise $L \cap H_1$ is in the boundary of a lift annulus A and l has a subray contained in this lift annulus. But then the subray is also in $\tilde{\Lambda}_F^u$, so v is an ideal point of a leaf in $\tilde{\Lambda}_F^u$ – disallowed in this case. As $\Psi(l)$ escapes down in L , then in the intrinsic compactification $L \cup \partial_\infty L$ we have that $\Psi(l)$ converges to $(L \cap H_1)_-$ which is the intrinsic negative ideal point of $(L \cap H_1)$. Therefore the ideal point of $\Psi(l)$ in $\tilde{M} \cup S_\infty^2$ is $(L \cap H_1)_-$ which is equal to $(H_1)_-$, the negative ideal point of H_1 in S_∞^2 . Since the values of $\Psi(p)$ for p in $\partial_\infty F$ are obtained as limits of values in $\Psi(F)$, then we only need to show that if z_k is in F and z_k converges to p as $k \rightarrow \infty$, then $\Psi(z_k)$ converges to $\Psi(p)$. Suppose this is not the case.

Except for a finite initial segment, then $\Psi(l)$ does not intersect a lift annulus and it is transverse to $\tilde{\Lambda}_F^u$ in F . Parametrize the leaves of $\tilde{\Lambda}_F^u$ intersected by l as $\{g_t, t \in \mathbf{R}_+\}$, contained in $G_t \in \tilde{\Lambda}^u$ (by an abuse of notation think of the G_i as a discrete subcollection of the $G_t, t \in \mathbf{R}_+$). Let

$$\mathcal{U} = \bigcup_{t>0} G_t.$$

No g_t (or leaf of $\tilde{\Lambda}_F^u$) has ideal point v in $\partial_\infty F$. As $\{g_i\}, i \in \mathbf{N}$ converges to v in $F \cup \partial_\infty F$, then g_t escapes compact sets in F as $t \rightarrow \infty$ and the ideal points of g_t converge to v on either side of v . Up to subsequence assume either every z_k is contained in \mathcal{U} or every z_k is disjoint from \mathcal{U} .

Situation 1. Suppose that $\Psi(z_k)$ is not in \mathcal{U} for any k .

Since z_k is very close to v in the compactification $F \cup \partial_\infty F$ and g_t converges to v in $F \cup \partial_\infty F$ when $t \rightarrow \infty$, then there are t, s with z_k between g_t and g_s (in F). Notice z_k is not in any of them. Now there is a unique time t_k so that exactly at that time $\Psi(z_k)$ switches from being in one side of G_t in \widetilde{M} to the other (equivalently compare the z and g_t in F). In particular, either there is a line leaf L_{t_k} of G_{t_k} which separates $\Psi(z_k)$ from all the other G_t , see Fig. 13, b, or there is a leaf L_{t_k} non separated from G_{t_k} with $\Psi(z_k)$ either in L_{t_k} or L_{t_k} separates $\Psi(z_k)$ from all G_t , see Fig. 13, a. This can be seen in the leaf space of $\widetilde{\Lambda}^u$, which is a non Hausdorff tree [Fe8, Ga-Ka, Ro-St].

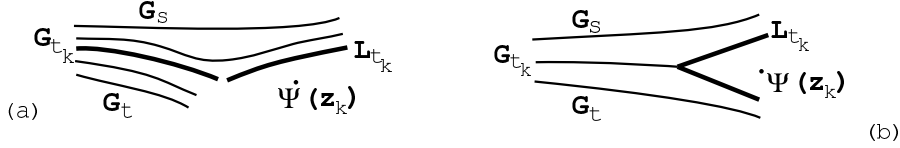


Figure 13. a. Non separated leaf separating points.
b. Line leaf separating points.

Claim 2. In the Gromov-Hausdorff topology of closed sets of $\widetilde{M} \cup S_\infty^2$, the sets \overline{L}_{t_k} converge to $(H_1)_-$ as $k \rightarrow \infty$.

If L_{t_k} is a line leaf of G_{t_k} , then $(L_{t_k})_- = (G_{t_k})_-$. If L_{t_k} is not separated from G_{t_k} then also $(L_{t_k})_- = (G_{t_k})_-$. This is because there are E_i leaves of $\widetilde{\Lambda}^u$ with E_i converging to $L_{t_k} \cup G_{t_k}$. So there are x_i, y_i in E_i with $x_i \rightarrow x, y_i \rightarrow y$ and $x \in L_{t_k}, y \in G_{t_k}$. Since η_- is continuous we have

$$\begin{aligned} \eta_-(x_i) &\rightarrow \eta_-(x) = \eta_-(L_{t_k}), & \eta_-(y_i) &\rightarrow \eta_-(y) = \eta_-(G_{t_k}) \\ &\text{and } \eta_-(x_i) = \eta_-(y_i). \end{aligned}$$

The last equality occurs because x_i, y_i are in the same unstable leaf E_i . Therefore $(L_{t_k})_-$ converges to $(H_1)_-$ when $k \rightarrow \infty$. Suppose that \overline{L}_{t_k} does not converge to $(H_1)_-$ in $\widetilde{M} \cup S_\infty^2$. Since

$$(L_{t_k})_- \text{ converges to } (H_1)_-,$$

then Lemma 7.5 shows that L_{t_k} does not escape compact sets in \widetilde{M} . So it limits to some u in \widetilde{M} and up to subsequence we may assume there u_k in L_{t_k} with u_k converging to u . The first possibility is that the L_{t_k} are subsets of the leaves G_{t_k} . This implies that $\widetilde{\Phi}_{\mathbf{R}}(u)$ is in the limit of the sequence of leaves G_{t_k} (in \widetilde{M}), so it is contained in \mathcal{V} . The second possibility is L_{t_k} non separated from G_{t_k} so L_{t_k} is between $G_{t_{k-1}}$ and $G_{t_{k+1}}$; hence u is again in the limit of the G_t , and so u is in \mathcal{V} . The leaves H_j in \mathcal{V} are non singular in the side the G_t are limiting on, so

there is a neighbourhood of u in \widetilde{M} on that side of H_j which has no singularities hence the u_k will be in \mathcal{U} for k big enough. This contradicts the hypothesis in this case.

This shows that \bar{L}_{t_k} converges to $(H_1)_-$ in $\widetilde{M} \cup S_\infty^2$. Also L_{t_k} either contains $\Psi(z_k)$ or separates it from a base point in \widetilde{M} . It follows that $\Psi(z_k)$ converges to $(H_1)_-$, which is what we wanted to prove. This finishes the analysis in Situation 1.

Situation 2. For all k assume that $\Psi(z_k)$ is in \mathcal{U} .

Let t_k in \mathbf{R} with $\Psi(z_k)$ in G_{t_k} , hence z_k is in $G_{t_k} \cap F = g_{t_k}$. Then the sequence

$$(\Psi(z_k))_- = (G_{t_k})_- \quad k \in \mathbf{N} \quad \text{converges to} \quad (H_1)_-.$$

We want to show that $\Psi(z_k)$ converges to $(H_1)_-$. Otherwise there is a in S_∞^2 different from $(H_1)_-$ and a subsequence, still denoted by $\Psi(z_k)$ so that $\Psi(z_k)$ converges to a . As in the claim of situation 1 above, $\tilde{\Phi}_{\mathbf{R}}(z_k)$ does not escape compact sets in \widetilde{M} and there is q in \widetilde{M} , so that up to another subsequence, we may assume that $\tilde{\Phi}_{\mathbf{R}}(\Psi(z_k))$ converges to $\tilde{\Phi}_{\mathbf{R}}(q)$. Since $\Psi(z_k)$ is in G_{t_k} then q is in \mathcal{V} , say q is in H_j . Let $y = \Theta(q)$. At this point notice that F does not intersect any leaf H_i in \mathcal{V} . If it did, say in w then F intersects the nearby leaves G_t (for any t big enough) near w . This would imply $F \cap G_t = g_t$ does not escape compact sets in F , contradiction. Therefore $y = \Theta(q)$ is in $\partial\Theta(F)$. Let $x_k = g_{t_k} \cap l$. Then $\Theta(\Psi(x_k))$ converges to a point in $\Theta(H_1 \cap L)$. There are

segments b_k in $F \cap G_{t_k} = g_{t_k}$ from x_k to z_k .

Then $\Theta(\Psi(b_k))$ converges to a ray in $\Theta(H_1)$ and a ray in $\Theta(H_j) \subset \mathcal{O}^u(y)$ and possibly other unstable leaves. Then there is a ray in $\mathcal{O}^u(y)$ contained in $\partial\Theta(F)$. This implies that F escapes down as $\Theta(F)$ approaches this ray of $\Theta(H_j)$. Hence $\Psi(z_k)$ is getting closer to q_- which is $(H_j)_-$, which is also equal to $(H_1)_-$. This is what we wanted to prove anyway.

This finishes the proof of Case 2.1.1, that is, when L intersects \mathcal{V} .

We need the following simple result.

Lemma 7.6. *Let A in $\tilde{\Lambda}^u$, B in $\tilde{\Lambda}^s$ satisfying: there are R_i leaves of $\tilde{\Lambda}^u$ intersecting B with R_i converging to A and $R_i \cap B$ escaping compact sets in B . Then A_- is equal to B_+ .*

Proof. Since R_i converges to A then $(R_i)_-$ converges to A_- . Also $(R_i)_- = (R_i \cap B)_-$. As $R_i \cap B$ escapes compact sets in B then in the intrinsic geometry of B , the $R_i \cap B$ converges to the positive ideal point of B . Hence $(R_i \cap B)_-$ converges to B_+ . This implies the result. q.e.d.

Case 2.1.2. L does not intersect \mathcal{V} .

Then $\Theta(l)$ escapes in $\Theta(L)$ and so $\Psi(l)$ converges to L_+ . By the previous lemma, this is also equal to $(H_1)_-$. From this point on, the proof is the same as in Case 2.1.1. End of Case 2.1.

Case 2.2. v is an ideal point of both $\tilde{\Lambda}_F^s$ and $\tilde{\Lambda}_F^u$.

Case 2.2.1. For any rays l of $\tilde{\Lambda}_F^s$ and e of $\tilde{\Lambda}_F^u$ with $l_\infty = e_\infty = v$, then l does not intersect e .

Recall that for any e ray in $\tilde{\Lambda}_F^s$ or $\tilde{\Lambda}_F^u$, then e_∞ is the ideal point of e in $\partial_\infty F$. Let l', e' be rays of $\tilde{\Lambda}_F^s, \tilde{\Lambda}_F^u$ respectively with $l'_\infty = e'_\infty = v$. Any such ray e cannot be in a lift annulus of $\tilde{\Lambda}^s$ or $\tilde{\Lambda}^u$ because it would be a ray of *both* $\tilde{\Lambda}_F^s$ and $\tilde{\Lambda}_F^u$ – not allowed in this case. If e is in $\tilde{\Lambda}_F^u$ then a subray of e is transverse to $\tilde{\Lambda}_F^s$ and vice versa. Parametrize the leaves of $\tilde{\Lambda}_F^s$ intersecting e' as $\{l_t, t \geq 0\}$ where $l_t \cap e'$ converges to v in $F \cup \partial_\infty F$ as t converges to infinity.

Since l' limits on v and is disjoint from e' , then l' is on a local side of F defined by e' . We will prove continuity of Ψ at v from the other side of e' . The point $p_t = l_t \cap e'$ disconnects l_t . For simplicity we only consider those l_t with $l_t \subset L_t \in \tilde{\Lambda}^s$ and L_t non singular. Let l_t^1 be the component of $(l_t - p_t)$ in the side of e' which contains l' union with p_t . Let l_t^2 be the other component of $(l_t - p_t)$ union with p_t , see Fig. 14, a.

The l_t^1 are rays (here we use L_t non singular - but this is just a technicality) and $(l_t^1)_\infty$ are not equal v by hypothesis. They cannot escape compact sets of F since l' with ideal point v is on that side of e' . Hence as t converges to infinity l_t^1 converges to a leaf l of $\tilde{\Lambda}_F^s$ with a ray (also denoted by l) with ideal point v and maybe some other leaves as well. The leaf l either shares a subray with l' or separates l' from e' . Let $e' \subset E'$ leaf of $\tilde{\Lambda}^u$ and $l \subset L$, leaf of $\tilde{\Lambda}^s$.

Case 2.2.1.1. l_t^2 escapes in F as $t \rightarrow \infty$.

Let b_t be the ideal point of l_t^2 . Then $b_t \neq v$. Let L_t^2 be the union of $\tilde{\Phi}_{\mathbf{R}}(p_t)$ and the component of $L_t - \tilde{\Phi}_{\mathbf{R}}(p_t)$ containing l_t^2 . If L_t^2 escapes in \tilde{M} , then the arguments in Case 1 show continuity of Ψ at v in the side of e' not containing l' .

Now assume that L_t^2 converges to $R_1 \cup \dots \cup R_m$ leaves of $\tilde{\Lambda}^s$ with union \mathcal{R} . Notice F may intersect some of these leaves or not. If $\Theta(\Psi(p_t))$ does not escape in $\Theta(E')$, then one of the R_i , call it R_1 , is a leaf intersecting E' . As seen in the beginning of the arguments for Case 2.1.1, F escapes up in this direction so $\Psi(p_t)$ converges to $(R_1)_+$. If $\Theta(\Psi(p_t))$ escapes in $\Theta(E')$, then Lemma 7.6 shows that $\Psi(p_t)$ also converges to $(R_1)_+$. This is equal to $(R_j)_+$ for any j .

Suppose there are $t_k \rightarrow \infty$ and z_k in $l_{t_k}^2$ with $\Psi(z_k)$ not converging to $(R_1)_+$. Here there is no need to assume that L_{t_k} is non singular. Then there is a in S_∞^2 , $a \neq (R_1)_+$ and a subsequence still denoted by $\Psi(z_k)$,

so that $\Psi(z_k)$ converges to a . As before there is q in \widetilde{M} so that up to another subsequence, still denoted by $\widetilde{\Phi}_{\mathbf{R}}(\Psi(z_k))$, then $\widetilde{\Phi}_{\mathbf{R}}(\Psi(z_k))$ converges to $\widetilde{\Phi}_{\mathbf{R}}(q)$ and hence q is in \mathcal{R} , say in R_i . Then $\widetilde{\Phi}_{\mathbf{R}}(\Psi(z_k))$ are near $\widetilde{\Phi}_{\mathbf{R}}(q)$ and since a ray of $\Theta(R_i)$ is in $\partial\Theta(F)$, then this is stable boundary. So F escapes up as $\Theta(F)$ approaches $\Theta(q)$ and hence $\Psi(z_k)$ converges to $(R_i)_+$. This is equal to $(R_1)_+$. The arguments of Case 2.1.1, Situation 1 then show continuity of Ψ at v on this side of e' . This finishes the analysis of Case 2.2.1.1.

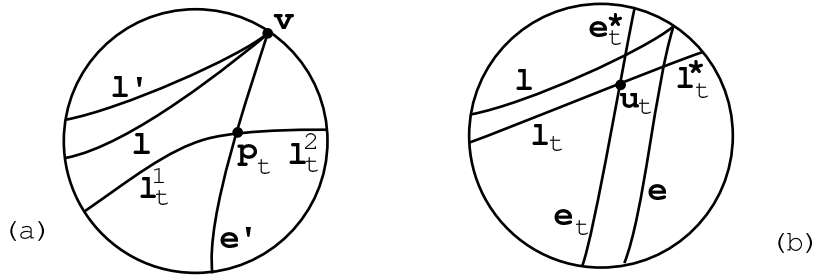


Figure 14. a. Convergence on one side.
b. Case 2.2.1.2 - intersection of leaves.

Case 2.2.1.2. The l_t^2 limit to r in F as $t \rightarrow \infty$.

Choose the leaf r with a ray which has ideal point v . Then the leaves r, l are not separated from each other in the leaf space of $\widetilde{\Lambda}_F^s$. Since r, l have ideal point v and there is no leaf of $\widetilde{\Lambda}_F^s$ non separated from r, l and between them, proposition 6.10 shows that the region bounded by these rays of r, l with ideal point v projects in M to a set asymptotic to a Reeb annulus. It follows that in F this region is a bounded distance from a geodesic ray with ideal point v . Now we restart the process with the ray r of $\widetilde{\Lambda}_F^s$ instead of e' of $\widetilde{\Lambda}_F^u$. Again take a subray of r which is transverse to $\widetilde{\Lambda}_F^u$. Let $\{b_t, t \geq 0\}$ be a parametrization of the leaves of $\widetilde{\Lambda}_F^u$ through the corresponding points x_t of r . If the components of $(b_t - x_t)$ on the side opposite of e' escapes compact sets in F , then the analysis of Case 2.2.1.1 shows continuity of Ψ at v in that side of r . Since r and e' are a bounded distance from each other in F , this shows continuity of Ψ at v on that side of e' .

Otherwise, this process keeps being repeated. Let $A_0 = L$, A_1 be the leaf of $\widetilde{\Lambda}^s$ containing r . If the process above does not stop, we keep producing leaves A_i of $\widetilde{\Lambda}^s$, so that they are all disjoint and A_i is non separated from A_{i+1} . By Theorem 3.6 up to covering translations there are only finitely many leaves of $\widetilde{\Lambda}^s$ which are not separated from some other leaf of $\widetilde{\Lambda}^s$. There is then $m > n$ and h covering translation with $h(A_n) = A_m$. Let f be the generator of the joint stabilizer of A_0, A_1 .

This is non trivial by Theorem 3.6. Then f preserves all the prongs of A_1 and therefore leaves invariant all the A_i . Hence $h^{-1}fh(A_n) = A_n$ and so $h^{-1}fh = f^a$ for some integer a . This implies there is a $\mathbf{Z} \oplus \mathbf{Z}$ in $\pi_1(M)$, see detailed arguments in [Fe8]. This contradicts the fact that $\pi_1(M)$ is Gromov hyperbolic.

There is then a last leaf l_{i_o} (of $\tilde{\Lambda}_F^s$ or $\tilde{\Lambda}_F^u$) obtained from this process. The arguments of Case 2.2.1.1 show continuity of Ψ at v on the other side of l_{i_o} . The region between e' and l_{i_o} is composed of a finite union of regions between non separated rays of $\tilde{\Lambda}_F^s$ or $\tilde{\Lambda}_F^u$. They are all a bounded distance from a geodesic ray with ideal point v , so the whole region also satisfies this property. It follows that this region can only limit in $\Psi(v)$ as well and this proves continuity of Ψ at v in that side of e' .

An entirely similar analysis shows continuity of Ψ at v from the side of l' not containing e' .

What remains to be analysed is the region of F *between* the rays l' and e' . Consider first the case that there is a pair of non separated leaves in the chain from l' to e' . Then as seen before the region between l' and e' is a bounded distance (the bound is not uniform) from a geodesic ray with ideal point v . This is not the case a priori if there is no non Hausdorffness involved. In this case the region between l' and e' may not have bounded thickness in F and hence it is unclear whether its image under Ψ can only limit in $\Psi(v)$. We analyse this case now.

Parametrize the leaves of $\tilde{\Lambda}_F^u$ intersecting the ray l of $\tilde{\Lambda}_F^s$ as $\{e_t \mid t \geq 0\}$. Up to a subray, then l is transverse to $\tilde{\Lambda}_F^u$. Since l_t converges to l , then for big enough t , the leaves l_t, e_t intersect – let u_t be their intersection point, see fig. 14, b. Now define l_t^* to be the component of $l_t - u_t$ intersecting e and e_t^* the component of $e_t - u_t$ intersecting l . Since e' is on that side of l , the e_t cannot escape and converge to a leaf e of $\tilde{\Lambda}_F^u$ with an ideal point v . Let $e \subset E$ leaf of $\tilde{\Lambda}^u$.

Recall that L_t is the leaf of $\tilde{\Lambda}^s$ containing l_t^* and similarly let E_t be the leaf of $\tilde{\Lambda}^u$ containing e_t . Let L_t^* be the component of $L_t - \tilde{\Phi}_{\mathbf{R}}(u_t)$ containing l_t^* and similarly define E_t^* . In this remaining case the l_t^* escape in F and so do the e_t^* . Hence $\mu_t = l_t^* \cup \{u_t\} \cup e_t^*$ defines a shrinking neighborhood system of v in $F \cup \partial_\infty F$. Consider the set

$$B_t = L_t^* \cup \tilde{\Phi}_{\mathbf{R}}(u_t) \cup E_t^*.$$

We want to show that \overline{B}_t converges to L_+ in the topology of closed sets of $\tilde{M} \cup S_\infty^2$.

First consider $L_t^* \cap E$ which intersects F in $(l_t^* \cap e)$. If $L_t^* \cap E$ does not escape compact sets in E then it limits to an orbit γ contained in a leaf H of $\tilde{\Lambda}^s$. Then L, H are not separated from each other. But for t big enough then E_t is near enough E and will intersect H as well.

This contradicts $E_t \cap L$ is not empty and L, H non separated. Hence $L_t^* \cap E$ escapes in E and similarly $E_t^* \cap L$ escapes in L . Hence L, E form a perfect fit. This implies that $L_+ = E_-$. Also $\Psi(e)$ limits to E_- and $\Psi(l)$ limits to $L_+ = E_-$.

The set \bar{L}_t^* in $\widetilde{M} \cup S_\infty^2$ contains $(L_t^* \cap E)_+$ and by Lemma 7.6 this converges to E_- when $t \rightarrow \infty$. This is because $(L_t^* \cap E)$ escapes in E . If \bar{L}_t^* does not converge to E_- in $\widetilde{M} \cup S_\infty^2$, then we find $t_k \rightarrow \infty$ and $x_k \in L_{t_k}^*$ with x_k converging to x not equal to E_- . Since $(x_k)_+ = (L_{t_k})_+$ converges to E_- , then up to subsequence assume $\tilde{\Phi}_{\mathbf{R}}(x_k)$ converges to $\tilde{\Phi}_{\mathbf{R}}(q)$ for some q in \widetilde{M} . Then q is in a leaf H of $\tilde{\Lambda}^s$ which is non separated from L .

The stable leaf H does not intersect F , because l_t^* escapes in F by hypothesis in this final situation. It follows that $\Theta(H)$ has a ray contained in $\partial\Theta(F)$ and so this is stable boundary of $\Theta(F)$. Hence F escapes up as $\Theta(F)$ approaches $\Theta(H)$ and consequently $\Psi(x_k)$ limits to

$$\eta_+(q) = H_+ = L_+ = E_-,$$

which is what we wanted anyway. This shows that \bar{L}_t^* converges to E_- in $\widetilde{M} \cup S_\infty^2$.

A similar analysis of the sets E_t^* shows that \bar{E}_t^* converges to L_+ as $t \rightarrow \infty$. Therefore \bar{B}_t converges to $L_+ = \Psi(v)$. Since $B_t \cap F = \mu_t$ and the μ_t define a neighborhood basis of v in $F \cup \partial_\infty F$, this shows continuity of Ψ at v . This finishes the proof of Case 2.2.1.2 and hence of Case 2.2.1.

Case 2.2.2. There are intersecting rays l of $\tilde{\Lambda}_F^s$ and e of $\tilde{\Lambda}_F^u$ with $l_\infty = e_\infty = v$, but no such ray is in a lift annulus.

We first prove continuity on the side of e not containing l . There will be an iteration of steps. Before we start the analysis we want to get rid of some problems as described now. Suppose that there are α_0, β_0 leaves of $\tilde{\Lambda}_F^s$ (or leaves of $\tilde{\Lambda}_F^u$) which have non separated rays converging to v in $\partial_\infty F$ and on that side of e . If there are infinitely many of these on that side of e , call them α_i, β_i and let G_i, H_i in $\tilde{\Lambda}^s$ containing α_i, β_i . By Proposition 6.7 each region B between α_0 and any α_i is a bounded distance from a geodesic ray in F with ideal point v . The image $\Psi(B)$ then can only limit in $\Psi(v)$. If the G_i do not escape in \widetilde{M} then they converge to a leaf G of $\tilde{\Lambda}^s$. Let A be an unstable leaf intersecting G transversely. For i big enough then A intersects G_i transversely, which is impossible, as it would intersect G_i and H_i and these are not separated. Hence the (G_i) escapes in \widetilde{M} . Then as seen in Case 1, there is continuity of Ψ at v in that side of α_1 .

Another possibility is that there are leaves α_i of $\tilde{\Lambda}_F^s$ in that side of e with two rays with ideal point v . Then they are in the interior of a spike region B with two boundary components with ideal point v . If there

are infinitely many of these with a boundary component g_i with ideal point v , then let G_i in $\widetilde{\Lambda}^s$ containing g_i . As in the previous paragraph, the G_i have to escape in \widetilde{M} and we have continuity in that side of α_1 .

Therefore we can assume there are only finitely many occurrences of spike regions or non separated leaves with ideal point v on this side of e . If there is any of these let e_0 be the last ray in that side coming from such occurrences. Otherwise let e_0 be the ray e given by the hypothesis in this case. Recall that here all rays have subrays transverse to the other foliation. For simplicity we assume that e_0 is a leaf of $\widetilde{\Lambda}_F^u$, the other case being similar. Let $e_0 \subset E_0 \in \widetilde{\Lambda}^u$. This is the setup in this case.

Let

$\{l_t \in \widetilde{\Lambda}_F^s \mid t \geq 0\}$ be the set of leaves of $\widetilde{\Lambda}_F^s$ intersecting a subray of e_0 .

Situation 3. No l_t has ideal point v .

If L_t escapes \widetilde{M} as $t \rightarrow \infty$ then as seen before we have continuity of Ψ at v in that side of e_0 . So from now on suppose that L_t converges to $A_1 \cup \dots \cup A_m$, leaves of $\widetilde{\Lambda}^s$. This case is considerably more involved, with several possibilities. The case that there is t with $(l_t)_\infty = v$ is dealt with later.

Claim 3. $\Psi(e_0)$ converges to $(A_i)_+$ (notice the $(A_i)_+, 1 \leq i \leq m$ are all equal).

If E_0 intersects some A_i , say A_1 , then as seen in Case 2.1.1, F escapes positively along $\Psi(e_0)$ as $\Theta(F)$ approaches A_1 . This implies that $\Psi(e_0)$ converges to $(E_0 \cap A_1)_+ = (A_1)_+$. If E_0 does not intersect any A_i then $\Psi(e_0)$ converges to $(E_0)_- = (A_1)_+$. This proves the claim.

Let p_t be the intersection of l_t and e_0 . Let l_t^1 be the component of $(l_t - p_t)$ in the side of e_0 we are considering. We want to know the behavior for $t \rightarrow \infty$, so we may assume $l_t \cap e_0$ is not singular and there is only one such component.

Suppose first that l_t^1 escapes in F as $t \rightarrow \infty$. In this case the work done in Case 2.1.1 implies continuity of Ψ at v and in this side of e_0 : For example suppose there are x_i in $l_{t_i}^1$ with $t_i \rightarrow \infty$ and $\Psi(x_i) \not\rightarrow (A_i)_+$. As in Case 2.1.1 $(x_i)_+$ converges to $(A_i)_+$, so up to subsequence assume that $(x_i)_- \rightarrow b \neq (A_i)_+$. Up to subsequence $\widetilde{\Phi}_{\mathbf{R}}(x_i) \rightarrow \widetilde{\Phi}_{\mathbf{R}}(x)$. Then x is in some A_i say $x \in A_2$. But F escapes positively as $\Theta(F)$ approaches $\Theta(A_2)$, so $\Psi(x_i) \rightarrow (A_i)_+$, as we wanted. Then as in Case 2.1.1 this implies continuity. The remaining possibility is:

Situation 4. No l_t^1 has ideal point v and the nested collection $(l_t^1), t > 0$ does not escape in F .

There could be several leaves of $\widetilde{\Lambda}_F^s$ in the limit of l_t^1 as $t \rightarrow \infty$ but there is a single leaf, call it g with ideal point v . If there is more than

one such leaf with ideal point v , then there would have to be one with two rays with ideal point v . This leaf would be in a spike region and it is separated from any other leaf in $\tilde{\Lambda}_F^s$, contradiction. Let g be contained in a leaf G of $\tilde{\Lambda}^s$. Parametrize

the ray $g \in \tilde{\Lambda}_F^s$ as $\{q_t \mid t \geq 0\}$, with $q_t \rightarrow v$ as $t \rightarrow \infty$.

In Case 2.2.2 we may assume that g is transverse to $\tilde{\Lambda}_F^u$. Let s_t be the unstable leaf of $\tilde{\Lambda}_F^u$ through q_t . Let s_t^1 be the component of $(s_t - q_t)$ on the side of g opposite to e_0 and s_t^2 the other component. Then s_t^2 cannot have ideal point v : for t big enough it intersects l_t^1 , see Fig. 15, a. Then s_t^2 converges to e_0 . By hypothesis there are no more occurrences of non separated leaves of $\tilde{\Lambda}_F^s$ with ideal point v on that side of e_0 , which implies that s_t^1 cannot limit to a leaf of $\tilde{\Lambda}_F^u$ at $t \rightarrow \infty$ (it would be distinct but non separated from e_0). Hence the s_t^1 have to escape compact sets in F . If s_t^1 does not have an ideal point at v for any t , then the previous analysis shows continuity of Ψ at v in that side of g . As in Case 2.2.1.2 if B is the region between g and e_0 then $\Psi(B)$ can only limit in $\Psi(v)$.

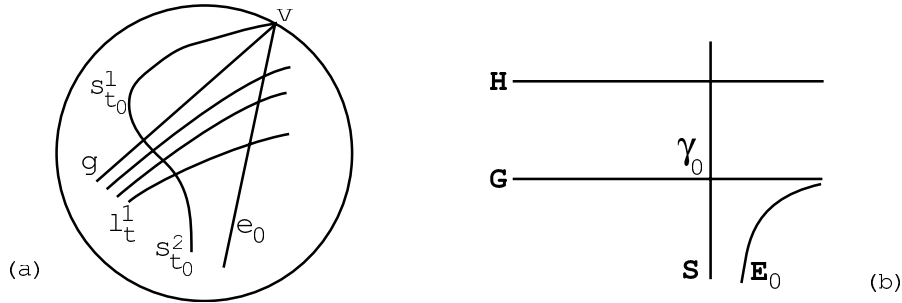


Figure 15. a. Some limits in F .
 b. The picture in \tilde{M} .

Therefore, we can assume that there is some t_0 so that $s_{t_0}^1$ has ideal point v , see Fig. 15, a. Notice that we know that no l_t^1 can have ideal point v , but that did not rule out a priori possibility that some s_t^1 may have ideal point v . Given the fact above then for t bigger than t_0 the ideal point of s_t^1 is v . We will restart the process with g instead of e_0 . In particular this iteration takes care of the case when there is t with l_t having ideal point v .

Situation 5. There is some ray $s_{t_0}^1$ with ideal point v .

Let $s_{t_0}^1$ be contained in a leaf S of $\tilde{\Lambda}^u$ and s_t contained in S_t leaf of $\tilde{\Lambda}^u$. Since

$$l_t^1 \rightarrow g, \quad s_t^2 \rightarrow e_0 \quad \text{when } t \rightarrow \infty,$$

then $L_t \rightarrow G$, $S_t \rightarrow E_0$, when $t \rightarrow \infty$.

In addition, Lemma 7.4 shows that E_0, G do not intersect because e_0 and g do not intersect. It follows that $E_0 \in \tilde{\Lambda}^u$, $G \in \tilde{\Lambda}^s$ form a perfect fit, see Fig. 15, b. Hence $(E_0)_- = G_+$. Furthermore, since l_t^1 converges to g as t converges to infinity, then the orbits through p_t have to escape in E_0 . This implies that $\Psi(e_0)$ converges to $(E_0)_-$.

Notice first that since the extension of Ψ to $F \cup \partial_\infty F$ is well defined, then $\Psi(s_{t_0}^1)$ converges to

$$\Psi(v) = (E_0)_- = G_+ = (G \cap S)_+.$$

The first option is that $\Theta(s_{t_0}^1)$ is a ray in $\Theta(S)$, which implies that $\Psi(s_{t_0}^1)$ converges to S_- . But if $\gamma_0 = G \cap S$, then

$$(\gamma_0)_+ = (G \cap S)_+ = \Psi(v) = S_- = (\gamma_0)_-,$$

which is a contradiction.

It now follows that $\Theta(s_t^1)$ is not a ray in $\Theta(S)$, but it has an endpoint x_1 in $\Theta(S)$. Let $\gamma_1 = x_1 \times \mathbf{R}$ and $H = \tilde{\Lambda}^s(\gamma_1)$. Notice that $F \cap H = \emptyset$. It is not possible that F escapes down as $\Theta(F)$ approaches x_1 , because s_{t_0} is not in a lift annulus. Hence F escapes up as $\Theta(F)$ approaches x_1 and $\Theta(H)$ has a ray in $\partial\Theta(F)$ and $\Psi(s_{t_0}^1)$ limits to $(\gamma_1)_+$. This implies that $(\gamma_0)_+ = (\gamma_1)_+$, where γ_0, γ_1 are distinct orbits of $\tilde{\Phi}$ in the same unstable leaf S . This is dealt with by the following theorem proved in [Fe5]:

Theorem 7.7 ([Fe5]). *Let Φ be a quasigeodesic almost pseudo-Anosov flow in M^3 with $\pi_1(M)$ negatively curved. Suppose there is an unstable leaf V of $\tilde{\Lambda}^u$ and different orbits β_0, β_1 in V with $(\beta_0)_+ = (\beta_1)_+$. Then $C_0 = \tilde{\Lambda}^s(\beta_0)$, $C_1 = \tilde{\Lambda}^s(\beta_1)$ are both periodic, invariant under the same nontrivial covering translation f , and the periodic orbits in C_0, C_1 are connected by an even chain of lozenges all intersecting V .*

Remark. This result is Case 2 of Theorem 5.7 of [Fe5]. In that article the proof is done for quasigeodesic Anosov flows in M^3 with $\pi_1(M)$ negatively curved. The proof goes verbatim to the case of pseudo-Anosov flows. The singularities make no difference. By the blow up operation, the same holds for almost pseudo-Anosov flows.

The theorem implies that G, H are in the boundary of a chain of adjacent lozenges all intersecting S . The first lozenge, call it \mathcal{C} , has one stable side contained in G and an unstable side D_1 which makes a perfect fit with G . Suppose first D_1 is in the side of S opposite to E_0 , see fig. 16, a. The other unstable side of \mathcal{C} is a leaf D_2 which intersects G on the side of S containing E_0 . Hence

$$D_2 = S_c := \tilde{W}^u(q_c) \quad \text{for some } c > t_0.$$

Then $S_c \cap F = s_c$ is a leaf of $\widetilde{\Lambda}_F^u$ and $\Psi(s_c)$ has ideal point $\Psi(v)$. Notice that $\Theta(s_c) \subset \Theta(F)$ escapes in $\Theta(S_c)$ – otherwise it would produce stable boundary in $\Theta(F)$ before it hits $\Theta(H)$ and $\Theta(F)$ could not limit on $\Theta(H)$, impossible. Hence $\Psi(s_c)$ limits to $(S_c)_- = \Psi(v)$. Then

$$(S_c \cap G)_- = (S_c)_- = \Psi(v) = G_+ = (S_c \cap G)_+,$$

which contradicts the orbit $S_c \cap G$ being a quasigeodesic.

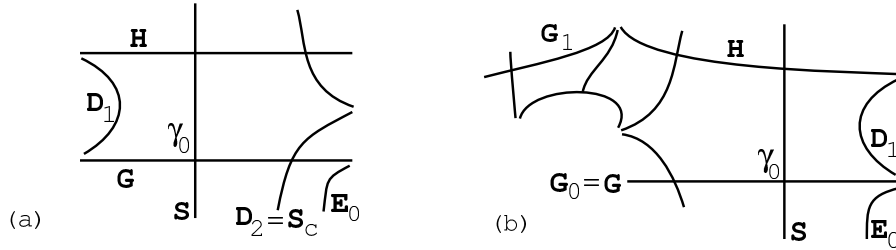


Figure 16. a. Perfect fit with G in the side opposite to E_0 . b. Perfect fit in the E_0 side.

Conclusion. It follows that the perfect fit of D_1 with G occurs in the E_0 side of S , see fig. 16, b. Here $\Theta(H), \Theta(D_1)$ are contained in the boundary of $\Theta(F)$. We now look at the region B in F bounded by $s_{t_0} = S \cap F$ and $g = G \cap F$.

Claim 4. The image $\Psi(B)$ can only limit in $\Psi(v)$.

The image $\Psi(B)$ is contained in the region \mathcal{E} of \widetilde{M} which is bounded by G, D_1 (and maybe other unstable leaves non separated from D_1 as well which are between G and H), H and S , see Fig. 16, b. Then F does not intersect D_1 or any leaf non separated from D_1 and beyond D_1 . Otherwise $b_0 = (D_1 \cap F)$ is contained in B and non separated from e_0 , so it would have both ideal points v . By Proposition 6.7, b_0 would be contained in a spike region and could not be non separated from another leaf – contradiction. On the other hand, since $\Theta(H)$ has a line leaf in the stable boundary of $\Theta(F)$, then $\Theta(D_1)$ has a line leaf in the unstable boundary of $\Theta(F)$. Hence F escapes down as $\Theta(F)$ approaches $\Theta(D_1)$. Similarly for any D_i in $\widetilde{\Lambda}^u$ non separated from D_1, E_0 and between D_1 and H .

Let z_k in B escaping in F and hence converging to v in $\partial_\infty F$. Suppose that $\Psi(z_k)$ does not converge to $\Psi(v)$. Given that z_k escapes F and the structure of the region \mathcal{E} , it follows that up to subsequence either $\widetilde{W}^u(z_k)$ converges to some D_i or $\widetilde{W}^s(z_k)$ converges to H . Suppose that $\widetilde{W}^s(z_k)$ converges to H . In that case

$$\eta_+(z_k) \text{ converges } H_+ = \Psi(v).$$

Then as seen before if $\eta_-(z_k)$ does not converge to $\Psi(v)$ we can assume up to subsequence that $\tilde{\Phi}_{\mathbf{R}}(z_k)$ converges to $\tilde{\Phi}_{\mathbf{R}}(q)$. Then q is in a leaf non separated from H and since $\Psi(z_k)$ has to be in \mathcal{E} then q can only be in H . As F escapes up as $\Theta(F)$ approaches $\Theta(H)$ then $\Psi(z_k)$ converges to $H_+ = \Psi(v)$. The case $\tilde{W}^u(z_k)$ converges to say D_1 leads to $\tilde{\Phi}_{\mathbf{R}}(z_k)$ converging to $\tilde{\Phi}_{\mathbf{R}}(q)$ with q in unstable leaf non separated from D_1 . As F escapes down as $\Theta(F)$ approaches these unstable leaves, then $\Psi(z_k)$ converges to $(D_1)_- = \Psi(v)$. Since this works for any subsequence of z_k , then $\Psi(z_k)$ has to converge to $\Psi(v)$ always. This proves Claim 4.

Let $G_0 = G$. Notice that G is periodic and connected to H by an even chain of lozenges. We consider the ray $s_{t_0} = S \cap F$ which has ideal point v in $F \cup \partial_\infty F$. Parametrize it as $\{z_t \mid t \geq 0\}$. Let y_t be the leaf of $\tilde{\Lambda}_F^s$ through z_t and y_t^1 the component of $(y_t - z_t)$ in the side opposite to e_0 . The ray s_{t_0} has the same role as the original ray e_0 . Hence we obtain continuity in that side of s_{t_0} unless y_t^1 converges to a leaf μ of $\tilde{\Lambda}_F^s$ with ideal point v . Let G_1 in $\tilde{\Lambda}^s$ with $\mu \subset G_1$. Then G_1 is non separated from H , see Fig. 16, b and therefore connected to it by a chain of lozenges. It follows that G_1 is connected to G_0 by a chain of lozenges. As in the proof of Claim 4, the region B_1 of F between e_0 and $(F \cap G_1)$ which is limiting on v has image $\Psi(B_1)$ which can limit only in $\Psi(v)$.

We restart the process with $g_1 = G_1 \cap F$ instead of g . The leaves of $\tilde{\Lambda}_F^u$ through points of g_1 already converge to some unstable leaf $(T \cap F)$ of $\tilde{\Lambda}_F^u$. The leaf $(T \cap F)$ cannot be non separated from any other leaf of $\tilde{\Lambda}_F^u$ in that side of $(T \cap F)$. It follows that the unstable rays intersected by g_1 and in the side opposite to e_0 escape in F . The only case to be analysed is that some of these unstable leaves have ideal point v . This brings the process exactly to the situation of some s_t^1 of $\tilde{\Lambda}_F^u$ having ideal point v as described before (it was $s_{t_0}^1$). So this would produce H_1 of $\tilde{\Lambda}^s$ with similar properties as H . Using Situations 4 and 5 alternatively, this process can now be iterated. As in Claim 4 the region of F between g_i and g_{i+1} maps to \tilde{M} to a region which can only limit in $\Psi(v)$.

We show that this process has to stop. The periodic orbits γ_i of $\tilde{\Phi}$ in G_i all project to closed orbits of Φ which are freely homotopic to each other. Hence for any i, j then $(\gamma_i)_+ = (\gamma_j)_+$ and $(\gamma_i)_- = (\gamma_j)_-$. Since Φ is quasigeodesic there is an upper bound on the number of these orbits. Therefore the process finishes after say j steps and we obtain continuity of Ψ at v in that side of $g_j = G_j \cap F$. As seen above the region between s_{t_0} and g_j maps by Ψ into a region that can only limit in $\Psi(v)$. This proves continuity of Ψ at v in that side of e_0 .

Continuation of the analysis of Case 2.2.2.

So far we proved continuity of Ψ at v from the side of e_0 opposite to l . The same works for the other side of l , producing l_0 with similar properties as e_0 . We now must consider the regions between e_0 and e , between e and l and between l and l_0 .

First consider the region between e and e_0 , which occurs only when they are distinct. In that case there is some non Hausdorffness or leaves with same ideal point and the ray e_0 is a bounded distance from a geodesic ray in F with ideal point v . Let $\{\mu_t \mid t \geq 0\}$ be a parametrization of the set of stable leaves of $\tilde{\Lambda}_F^s$ through e . Recall that there is a stable leaf l of $\tilde{\Lambda}_F^s$ intersecting e with one ideal point v and l is in the side of e_0 opposite the side for which we proved continuity of Ψ at v .

Let μ_t^1 be the component of $(\mu_t^1 - e)$ in the side of e we are considering. If some μ_t^1 has ideal point v then both ideal points of μ_t are v and μ_t is inside a spike region. The same is true for e and so e is a bounded distance from a geodesic ray in F with ideal point a . Hence, the region between e and e_0 is a bounded distance from a geodesic ray and we are finished in this case.

The remaining case to be analysed here is that μ_t^1 has no ideal point v . Then μ_t^1 does not escape F as $t \rightarrow \infty$, because e_0 is in that side of e . So μ_t^1 converges to a leaf μ which has ideal point v . Now consider a parametrization $\{\nu_t \mid t \geq 0\}$ of the unstable leaves intersected by μ . Then ν_t converges to a leaf e' . If it converges to some other leaf, then e' is a bounded distance from a geodesic ray in F . Otherwise it must be that some ν_t has ideal point v . Therefore we are exactly in the setup analysed in Situations 3, 4 and 5 above and this shows continuity of Ψ in that side of e' . Finally if e, e' are not the same, notice that $\mu_t \cap e, \mu_t \cap e'$ escape in e, e' respectively. Let $\mu_t \subset H_t$ leaves of $\tilde{\Lambda}^s$. By Theorem 3.8 the H_t do not escape in \tilde{M} when $t \rightarrow \infty$. Let V be the region in F between e and e' . The analysis of Case 2.1.1 shows that $\Psi(V)$ limits only to $\Psi(v)$. This shows continuity of Ψ for the region between e and e_0 and similarly for the region between l and l_0 .

Finally we analyse the region B between e and l . There is no singularity in the interior of B . Otherwise there would be a line leaf in B and hence a leaf with both endpoints v . It would have to be part of a spike region and the spike region does not have any singularities in its interior.

Parametrize the leaves of $\tilde{\Lambda}_F^u$ through l as $\{e_s \mid s \geq 0\}$ and similarly those of $\tilde{\Lambda}_F^s$ through e as $\{l_t \mid t \geq 0\}$. Let L, L_t leaves of $\tilde{\Lambda}^s$ with $l \subset L, l_t \subset L_t$ and similarly define E, E_t leaves of $\tilde{\Lambda}^u$. There are 2 possibilities:

1) Product case – Any l_t intersects every e_s and vice versa.

Equivalently $\tilde{\Lambda}_F^s, \tilde{\Lambda}_F^u$ define a product structure in the region B bounded by l_0 and e_0 . If the L_t escapes in \tilde{M} as $t \rightarrow \infty$, then there

is a stable product region defined by a segment in L_0 and Theorem 3.8 implies that Φ is topologically conjugate to a suspension — contradiction. It follows that the L_t converge to $H_1 \cup \dots \cup H_m$ leaves of $\tilde{\Lambda}^s$ as $t \rightarrow \infty$. Since the l_t are stable leaves, it follows that F escapes up as $\Theta(F)$ approaches $\Theta(H_i)$. This implies that $\Psi(e)$ limits to $(H_i)_+$ which is then equal to $\Psi(v)$. Similarly E_s converges to $V_1 \cup \dots \cup V_n$ leaves of $\tilde{\Lambda}^u$ and F escapes down as $\Theta(F)$ approaches $\Theta(V_j)$. Hence $\Psi(l)$ limits to $(V_j)_- = \Psi(v)$. If some H_i intersects some V_j , then

$$(V_j \cap H_i)_+ = (H_i)_+ = \Psi(v) = (V_j)_- = (V_j \cap H_i)_-,$$

contradiction. Let now $\{z_k\}$ be a sequence in B converging to v . The product structure implies that up to subsequence we may assume that either $\tilde{W}^s(z_k)$ converges to H_i or $\tilde{W}^u(z_k)$ converges to V_j . This is analysed carefully in Claim 4 above, which shows that $\Psi(z_k)$ must converge to $\Psi(v)$. This shows continuity of Ψ when restricted to the region B .

2) Non product case.

There are $t, u > 0$ with $l_t \cap e_u = \emptyset$. Consider one such u . Let a be the infimum of t with $l_t \cap e_u = \emptyset$. Now let b be the infimum of u with $l_a \cap e_u = \emptyset$. Then $l_a \cap e_b = \emptyset$, but for any $0 \leq t < a$ and $0 \leq u < b$ one has $l_t \cap e_u \neq \emptyset$. Since $l_a \cap e_b = \emptyset$, then $L_a \cap E_b = \emptyset$. Here $L_a \in \tilde{\Lambda}^s, E_b \in \tilde{\Lambda}^u$. It follows that L_a, E_b form a perfect fit, see Fig. 17, a. If $\Theta(l_a)$ does not escape in $\Theta(L_a)$, then there would be unstable boundary of $\Theta(F)$ in the limit and that would keep F from intersecting E_b , contradiction. Hence $\Theta(l_a)$ escapes in $\Theta(L_a)$ and $\Theta(e_b)$ escapes in $\Theta(E_b)$, so $\Psi(l_a)$ limits to $(L_a)_+$ and $\Psi(e_b)$ limits to $(E_b)_-$. Also l_a, e_b limit to v in $\partial_\infty F$. Hence

$$\Psi(v) = (L_a)_+ = (E_b)_-.$$

Let $p_t = l_t \cap e$. If $\Theta(p_t)$ escapes in $\Theta(E)$, then $\Psi(e)$ converges to E_- . But $\Psi(e)$ limits to $\Psi(v)$ so:

$$E_- = \Psi(v) = (L_a)_+ = (L_a \cap E)_+$$

contradiction. It follows that $\tilde{\Phi}_{\mathbf{R}}(p_t)$ converges to $\tilde{\Phi}_{\mathbf{R}}(x)$ with x in E . Also F has to escape up as $\Theta(F)$ approaches $\Theta(x)$ — same as in Situation 5 above. Hence $\Psi(e)$ limits to $\eta_+(x)$. So

$$\eta_+(x) = \Psi(v) = (L_a)_+ = \eta_+(p_a).$$

Let $X = \tilde{W}^s(x)$. Then x, p_a are in 2 distinct orbits of E with the same positive ideal point. Therefore, Theorem 7.7 implies that L_a, X are connected by an even chain of lozenges all intersecting E . Let \mathcal{C} be the first lozenge. It has a stable side in L_a and one unstable side, call it D_1 which makes a perfect fit with L_a . Suppose first that D_1 is in the component of $\tilde{M} - E$ opposite to E_b . Then the other unstable side of \mathcal{C} , call it D_2 has to intersect L_a in the other side of E . Then D_2 must be some E_t , let it be $E_{t'}$, see Fig. 17, a. Then $\Theta(e_{t'})$ has to escape in

$\Theta(E_{b'})$ or else one produces stable boundary to $\Theta(F)$ and $\Theta(F)$ cannot limit to $\Theta(x)$ contradiction. Hence $\Psi(e_{b'})$ converges to $\Psi(v)$ and also to $(E_{b'})_-$. But then

$$(E_{b'} \cap L_a)_- = (E_{b'})_- = \Psi(v) = (E_b)_- = (L_a)_+$$

again a contradiction.

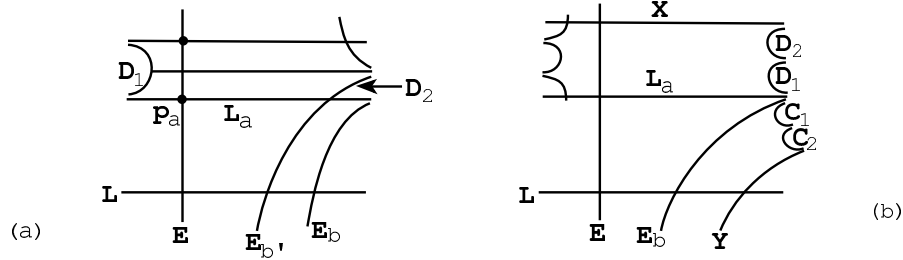


Figure 17. a. Reaching before. b. Reaching at the exact time.

This implies that D_1 is on the side of E containing E_b , see Fig. 17, b.

If there are only 2 lozenges in the chain from L_a to X , then D_1 also makes a perfect fit with X . Otherwise there are D_2, \dots, D_i all non separated from D_1 and so that D_i makes a perfect fit with X and the D_j are all in the boundary of the chain of lozenges. As seen in Claim 4 above, F cannot intersect any D_j ($1 \leq j \leq i$), but all $\Theta(D_j)$ are contained in the unstable boundary of $\Theta(F)$. Also F escapes down as $\Theta(F)$ limits to $\Theta(D_j)$. The set $\Theta(X)$ also has a line leaf which is a stable boundary of $\Theta(F)$ and F escapes up when $\Theta(F)$ approaches $\Theta(X)$.

The same discussion applies to L , so there is y in L , $Y = \widetilde{W}^u(y)$ with $\Theta(Y)$ having a line leaf in the unstable boundary of $\Theta(F)$ and F escapes down accordingly. There are C_1, \dots, C_n leaves in $\widetilde{\Lambda}^u$, all non separated from each other and in the boundary of the lozenges in the chain from E_b to Y so that C_1 makes a perfect fit with E_b and C_n makes a perfect fit with Y , see Fig. 17, b. Finally $\Theta(C_j)$ has a line leaf in the stable boundary of $\Theta(F)$ and F escapes up accordingly.

Let \mathcal{E} be the region in \widetilde{M} bounded by

$$E, L, X, Y, C_1, \dots, C_n, D_1, \dots, D_i.$$

Then $\mathcal{E} \cap F$ is exactly the region B bounded by the rays e and l . Let z_k in B escaping to v . Then the region \mathcal{E} shows that up to subsequence one of the following must occur:

- 1) $\widetilde{W}^s(z_k)$ converges to either X or C_1 (and perhaps other leaves of $\widetilde{\Lambda}^s$). The analysis of Claim 1 above shows that $\Psi(z_k)$ converges to either X_+ or $(C_1)_+$ both of which are equal to $\Psi(v)$.

2) $\widetilde{W}^u(z_k)$ converges to either Y or D_1 . Here $\Psi(z_k)$ converges to either Y_- or $(D_1)_-$ both of which are equal to $\Psi(v)$.

In any case this shows continuity of Ψ in the region B . This finishes the non product case.

This finishes the analysis of Case 2.2.2.

Case 2.2.3. There is a ray l of $\widetilde{\Lambda}_F^s$ (or $\widetilde{\Lambda}_F^u$) limiting to v and l contained in a lift annulus.

Since a lift annulus is contained in both $\widetilde{\Lambda}^s$ and $\widetilde{\Lambda}^u$, then l is also a ray of $\widetilde{\Lambda}_F^u$. In particular l is not transverse $\widetilde{\Lambda}_F^u$. In the previous cases we used the collection of leaves of $\widetilde{\Lambda}_F^u$ intersecting l (transversely) to try to produce a neighborhood basis of v in $F \cup \partial_\infty F$. This will not work here.

We fix once and for all in this case the lift annulus A which contains l . Suppose that as l escapes in F , then the points in A escape in the positive flow direction (up). There is an orbit α of $\widetilde{\Phi}$ in ∂A , so that l does not intersect α but $\Theta(l)$ converges to $\Theta(\alpha)$ in \mathcal{O} as the points in l converge to v in $F \cup \partial_\infty F$. Then $\Theta(\alpha) \in \partial\Theta(F)$ and F escapes up as points in $\Theta(F)$ approach $\Theta(\alpha)$. By the analysis in Theorem 4.1, this produces stable boundary of $\Theta(F)$ and there is a slice leaf δ in $\mathcal{O}^s(\Theta(\alpha))$ with $\delta \subset \partial\Theta(F)$ and F escaping up as $\Theta(F)$ approaches δ .

We also fix the leaf $L = \widetilde{W}^s(\alpha)$. Notice that $\Psi(l), A, \delta \times \mathbf{R}$ are all subsets of L .

First consider the picture in $\pi(A)$ which is a flow saturated annulus of Φ in M . Then $\pi(l)$ is either a closed curve in $\pi(A)$ or not. If it is not, then $\Theta(l)$ approaches $\Theta(\alpha)$ but $\pi(l)$ cannot get too close to $\pi(\alpha)$ by transversality. Hence $\pi(l)$ limits to a closed curve. Let β be the lift of this closed curve to A and let E be the leaf of $\widetilde{\mathcal{F}}$ containing β . Then F is asymptotic to E in the direction of v . Notice that l is a bounded distance from a geodesic ray in F with ideal point v , because $\pi(l)$ limits to a closed curve in $\pi(E)$. Let now g in $\pi_1(M)$ with g non trivial, $g(A) = A$ and g indivisible with respect to these properties. This implies that $g(E) = E$ as well. In addition,

$$E \cap \widetilde{W}^s(\alpha) = E \cap \widetilde{W}^u(\alpha) = \beta \quad \text{so} \quad g(\beta) = \beta.$$

By an abuse of notation we also denote by v the ideal point of E which is the ideal point of β . The fact that E is invariant under g and v is a fixed point of the action of g on $E \cup \partial_\infty E$ will be really helpful. Since we have 2 leaves of $\widetilde{\mathcal{F}}$ we will denote by Ψ_F, Ψ_E the corresponding embeddings $F, E \rightarrow \widetilde{M}$ and the extensions to respective compactifications. The proof is in 2 steps. First we show that Ψ_E is continuous at v then use that to show that Ψ_F is continuous at v .

Notice that $\Psi_E(\beta)$ limits to $\Psi_E(v)$ and that $\Psi_F(l)$ limits to $\Psi_F(v)$. In addition,

$$\Psi_E(v) = \Psi_F(v) = L_+.$$

Given S leaf of \mathcal{F} , let Λ_S^s be the foliation defined by the intersection of Λ^s and S and similarly define Λ_S^u . Let w be the other fixed point of g acting on $E \cup \partial_\infty E$. Then β also limits to w . Let β_+ be the direction of β limiting to v and β_- the direction limiting to w .

Step 1. Continuity of Ψ_E at v .

As usual we show continuity of Ψ_E restricted to one side of β . From the point of view of $\Theta(\alpha)$ then $\Theta(A)$ is a generalized unstable prong (see section 4) as E escapes up when $\Theta(\beta)$ limits to $\Theta(\alpha)$.

Even though β is contained in both $\tilde{\Lambda}_E^s, \tilde{\Lambda}_E^u$, this only occurs at β and $\tilde{\Lambda}_E^s, \tilde{\Lambda}_E^u$ are transverse to each other near β . In $\pi(E)$, the curve $\pi(\beta)$ is closed and agrees with the positive direction of the flow line $\pi(\alpha)$. Therefore the stable foliation $\Lambda_{\pi(E)}^s$ has expanding holonomy around $\pi(\beta)$ and $\Lambda_{\pi(E)}^u$ has contracting holonomy. Lift to E and we have nearby stable leaves of $\tilde{\Lambda}_E^s$ which move away from β in the direction of v . We want to use these nearby stable leaves to define a neighborhood basis of v . But it may happen that one such leaf m of $\tilde{\Lambda}_E^s$ also limits to v . For example if $\Lambda_{\pi(E)}^s$ has an annulus with boundary $\pi(\beta)$ and some other closed curve. We deal with this situation first.

Situation 6. Suppose there is m leaf of $\tilde{\Lambda}_E^s$, arbitrarily near β on that side of β , so that m limits to v in $E \cup \partial_\infty E$ in the β_+ direction.

Notice that the holonomy of $\Lambda_{\pi(E)}^s$ around $\pi(\beta)$ in the other direction is contracting, so certainly the leaf m converges to w in the other direction. In addition $g(m)$ is distinct from m as $\pi(m)$ is asymptotic to $\pi(\beta)$ in the other direction. It follows that $g(m)$ is closer to β than m is – because m is expanding away from β in the β_+ direction. Consider the nested sequence of leaves

$$(g^{-i}(m)), \quad i \in \mathbf{N} \quad \text{of} \quad \tilde{\Lambda}_E^s.$$

This sequence cannot escape E for otherwise the ideal points of rays of $\tilde{\Lambda}_E^s$ would not be dense in $\partial_\infty E$, violating Proposition 6.3. Therefore $g^{-i}(m)$ converges to a leaf r as $i \rightarrow \infty$. Clearly r and $g(r)$ are not separated from each other in the leaf space of $\tilde{\Lambda}_E^s$. If $r, g(r)$ are not the same leaf, then the $\{g^i(r), i \in \mathbf{Z}\}$ are all distinct and all non separated from each other. By Proposition 4.2 the leaves of $\tilde{\Lambda}^s$ containing the $\{g^i(r)\}$ are distinct from each other and are all non separated from each other. But since M is atoroidal this is disallowed by Theorem 3.6.

We conclude that $g(r) = r$. If r is in a lift annulus A' we restart the process with r in place of β . Notice that $\Theta(A')$ now corresponds to a generalized generalized stable prong in the direction that r goes to v .

In that case we would use nearby unstable leaves on the other side of r . They expand away from r in the v direction.

If on the other hand r is not contained in a lift annulus, then r is a leaf in $\tilde{\Lambda}_E^s$ which is transverse to $\tilde{\Lambda}_E^u$. We then consider the unstable leaves intersected by r . Suppose one of them, call it c' is non singular and the component c of $c' - (c' \cap r)$ on the side of r opposite to β has ideal point v . This in fact implies that both rays of c' have ideal point v , but we will not use that. As above the sequence $(g^{-i}(c)), i \in \mathbf{N}$ converges to a leaf c^* of $\tilde{\Lambda}_E^u$ with $g(c^*) = c^*$ and c^* has ideal points w, v . We can iterate this process to produce $c_0 = \beta, c_1 = r, c_2 = \lim g^{-i}(c)$ and so on. Each c_i is contained in a leaf C_i of $\tilde{\Lambda}^s$ or $\tilde{\Lambda}^u$. These leaves $C_i, i \in \mathbf{N}$ are all distinct from each other by Proposition 4.2 and $g(C_i) = C_i$ for all i . This produces a collection of freely homotopic closed orbits of Φ . Since Φ is a quasigeodesic flow, there is an upper bound on i . So this stops.

Assume that s is the last leaf produced by this process. Here $g(s) = s$ and the Hausdorff distance between β and s in E is finite. Hence the subregion of E limiting to v maps by Ψ_E to a region in \tilde{M} which only limits to $\Psi_E(v)$. We now prove continuity of Ψ_E at v in the side opposite to β .

Option 1. s is not in lift annulus.

Assume wlog that s is in $\tilde{\Lambda}_E^u$. Consider a non singular leaf of $\tilde{\Lambda}_E^s$ intersecting s and let b its ray starting in s and in the side opposite to β . Then by the construction of s , it follows that its ideal point b_∞ is not v . In the same way b_∞ is not w either – or else $(g^i(b)), i \in \mathbf{N}$ converges to another leaf with ideal points v, w . Hence $g^n(b)$ converges to v in $E \cup \partial_\infty E$ when n converges to infinity. This is because g is a hyperbolic isometry of E with fixed points w, v . Consider the collection

$$b_n = (\text{ray of } s) \cup g^n(b) \quad n \in \mathbf{N}.$$

Then this forms a neighborhood basis of v in $E \cup \partial_\infty E$ in that side of s .

From this point on the proof is exactly as in Case 2.1.1.

Option 2. Suppose s is in a lift annulus.

We may suppose that leaves of $\tilde{\Lambda}_E^s$ expand away from s in the v direction. Here let e be a leaf of $\tilde{\Lambda}_E^s$ very near s . Let τ be a small arc from s to e . Let b be the ray of e starting in τ and in the v direction. In this case let

$$b_n = (\text{ray of } s) \cup g^n(\tau) \cup g^n(b), \quad n \in \mathbf{N}.$$

By construction of s then b does not limit on v . Then the sequence $(b_n), n \in \mathbf{N}$, converges to v in $E \cup \partial_\infty E$ when $n \rightarrow \infty$. Therefore the $\{b_n \mid n \in \mathbf{N}\}$ forms a neighborhood basis of v in that side of s . Apply Case 2.1.1 again.

Conclusion. This shows continuity of Ψ_E at v .

Step 2. Continuity of Ψ_F at v .

Consider the band in E bounded by β and s . This has bounded thickness and since F is asymptotic to E in the v direction then the corresponding region in F converges to $\Psi_E(v) = \Psi_F(v)$ under Ψ_F . There is a ray of s in E which flows (under $\tilde{\Phi}$) to a curve s_1 in F . Consider the collection b_n in E as defined in Option 1 or Option 2 above. In Option 2 if e is very near s then e flows into F . Then all $g^n(b_0)$ flow into rays a_n of F . In addition all b_n flow into properly embedded infinite arcs c_n in F . Each c_n contains a subray of s_1 and the ray a_n . We prove two facts:

Fact 1. No a_n has ideal point v in $\partial_\infty F$.

Suppose that there is i in \mathbf{N} so that a_i has ideal point v . Fix this i . By Lemma 6.4, the ray a_i has a sequence of points $(u_j), j \in \mathbf{N}$ which are at a distance at most $2\delta_0$ in F from a geodesic ray in F with ideal point v . Therefore they are a bounded distance from s_1 . Since s_1 is asymptotic to the ray s in E it follows that u_j flows under $\tilde{\Phi}$ to a point u'_j for j big enough. By Proposition 4.2 all u'_j are in $g^i(b)$ and in addition $d_E(u'_j, s)$ is bounded. This would imply that $g^i(b)$ has ideal point v , contradicting the construction of s . This proves Fact 1.

Fact 2. The collection $\{c_n, n \in \mathbf{N}\}$, forms a neighborhood basis of v in $F \cup \partial_\infty F$ in that side of s_1 .

We show that the sequence $(c_n), n \in \mathbf{N}$ escapes compact sets in F and converges to v in $F \cup \partial_\infty F$. The corresponding rays in s_1 escape compact sets in F . If $(a_n), n \in \mathbf{N}$ does not escape compact sets in F then it has to converge to a leaf s_2 (and maybe others) which is non separated from s_1 with ideal point v (either in the leaf space of $\tilde{\Lambda}_F^s$ or $\tilde{\Lambda}_F^u$). By Proposition 6.10 the region between s_1 and s_2 limiting to v projects to $\pi(F)$ to a region limiting to a Reeb annulus. In particular, this region in F has bounded thickness, which implies that a ray in s_1 is a bounded distance from a ray in s_2 . As F is asymptotic to E in the v direction, then s_2 also flows by $\tilde{\Phi}$ to E in the v direction. Let s_a be the leaf obtained in E . One can easily show that s, s_a are not separated from each other in E and hence $\pi(s_a)$ is a closed curve in $\pi(E)$. This is because g leaves E and s invariant. Hence $\pi(s), \pi(s_a)$ bound a Reeb annulus in E . But then s_a has ideal points w, v , contradicting the construction of s . This shows that $(a_n), n \in \mathbf{N}$ escapes in F .

If $(a_n), n \in \mathbf{N}$ does not limit to v then Lemma 6.5 would imply that a_n also converges to another leaf, contrary to what we just proved. Hence the $(a_n), n \in \mathbf{N}$ converges to v and this proves Fact 2.

Given Facts 1 and 2, then the proof of continuity of Ψ_F at v in that side of s_1 is exactly as in Case 2.1.1. This finishes the proof the continuity of Ψ_F at v and finishes Case 2.2.3.

This finishes the proof of Theorem 7.3, the continuous extension theorem. q.e.d.

8. Foliations and Kleinian groups

There are many similarities between foliations in hyperbolic 3-manifolds and Kleinian groups. We refer to [Mi, Can, Mar] for basic definitions concerning degenerate and non degenerate Kleinian groups, in particular singly and doubly degenerate groups.

If the foliation is \mathbf{R} -covered then the limit set of any leaf in \widetilde{M} is the whole sphere [Fe1]. This corresponds to doubly degenerate surface Kleinian groups [Th1, Mi, Can, Mar, Bon]. There is always a pseudo-Anosov flow which is transverse to the foliation [Fe7, Cal1]. If the flow is quasigeodesic then the results of this article imply that the foliation has the continuous extension property.

If the foliation has one sided branching, say branching down, then limit sets of leaves can only have domain of discontinuity “above” [Fe3]. Let F in $\widetilde{\mathcal{F}}$ and Λ_F its limit set. If p is not in Λ_F , the p is said to be *above* F if there is a neighborhood V of p in $\widetilde{M} \cup S_\infty^2$, so that $V \cap \widetilde{M}$ is on the positive side of F . This corresponds to simply degenerate surface Kleinian groups [Th1, Mi, Can]. There are examples of foliations with one sided branching transverse to suspension pseudo-Anosov flows provided by Meigniez [Me]. Suspension flows are always quasigeodesic flows [Ze]. The results of this article show the continuous extension property for such foliations. Under these conditions, the limit sets are locally connected, the continuous extension provides parametrizations of these limit sets.

Finally if there is branching in both directions, then there can be domain of discontinuity above and below leaves. This corresponds to non degenerate Kleinian groups [Th1, Mi, Can]. These occur for example in the case of finite depth foliations, where the depth 0 leaves are not virtual fibers [Fe6].

There are many interesting questions:

Question 1. Given a foliation \mathcal{F} , is it \mathbf{R} -covered if and only if for every $F \in \widetilde{\mathcal{F}}$ then the limit set Γ_F is S_∞^2 ?

The forward direction is true. The backwards direction is true if there is a compact leaf [Go-Sh]. In addition, if there is one leaf with limit set the whole sphere then all leaves have limit set the whole sphere [Fe3] – whether \mathcal{F} is \mathbf{R} -covered or not.

Question 2. Given \mathcal{F} \mathbf{R} -covered, is there a quasigeodesic transverse pseudo-Anosov flow?

This is true in the case of slitherings or uniform foliations as defined by Thurston [Th5]. Examples are fibrations, \mathbf{R} -covered Anosov flows and many others. There is always a transverse pseudo-Anosov flow, the question is whether it is quasigeodesic.

Question 3. Is there domain of discontinuity of Λ_F only above F if and only if \mathcal{F} has one sided branching in the negative direction?

This occurs for the examples constructed by Meigniez [Me].

Question 4. Consider the pseudo-Anosov flows constructed by Calegari [Cal2] and transverse to one sided branching foliations. Are these flows quasigeodesic?

Question 5. If \mathcal{F} has 2 sided branching is there always domain of discontinuity above and below? Is there a quasigeodesic pseudo-Anosov flow almost transverse to \mathcal{F} ?

In work done after this article was written, we answered Questions 2 and 4 in the affirmative [Fe11]. This proves the continuous extension property for all foliations which are either \mathbf{R} -covered or have one sided branching. We also showed that for any \mathcal{F} with one sided branching, then any F leaf of $\tilde{\mathcal{F}}$ has limit set Γ_F not equal to S_∞^2 . This answers half of Question 3.

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