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PIONS AND GENERALIZED COHOMOLOGY

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In memory of Raoul Bott

Abstract

The Wess-Zumino-Witten term was first introduced in the low energy σ-model which describes pions, the Goldstone bosons for the broken flavor symmetry in quantum chromodynamics. We introduce a new definition of this term in arbitrary gravitational backgrounds. It matches several features of the fundamental gauge theory, including the presence of fermionic states and the anomaly of the flavor symmetry. To achieve this matching, we use a certain generalized differential cohomology theory. We also prove a formula for the determinant line bundle of special families of Dirac operators on 4-manifolds in terms of this cohomology theory. One consequence is that there are no global anomalies in the Standard Model (in arbitrary gravitational backgrounds).

Quantum chromodynamics has a global symmetry group $G \times G$, where $G = SU_{N_f}$ in the theory with N_f flavors of massless quarks. This is presumed broken to the diagonal, with the homogeneous space $(G \times G)/G$ parameterizing the vacua. The low energy dynamics of the Goldstone bosons—the pions—is modeled by a nonlinear σ-model with target $(G \times G)/G$; see [**We**] for an account. There is a topological term in the σ-model action, first introduced by Wess and Zumino [**WZ**] and later elaborated by Witten [**W1**]; see also [**N**]. We propose a new, geometric definition of this term (Definition 4.1). Our motivation is to reproduce certain features of the high energy gauge theory in the low energy effective theory. By working in the Euclidean theory formulated on arbitrary Riemannian spin 4-manifolds—in other words, by studying the theory in an arbitrary Euclidean gravitational background—we are able to probe more than can be seen in flat spacetime.[1] Specifically:

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[1] Indeed, previous treatments use spherical compactifications of spacetime, and because the Hurewicz map $\pi_5 G \to H_5 G$ is not an isomorphism, certain features were missed.

(i) a spin structure is required to define this new WZW term, just as a spin structure is necessary in the high energy theory to define spinor fields;
(ii) canonical quantization naturally gives a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space, matching the presence of both bosonic and fermionic states in QCD, and the statistics formula (4.10) which follows directly from our definition is correct;
(iii) our definition works for $N_f = 2$; and
(iv) there is a natural gauged version of the theory whose anomaly matches that of gauged QCD.

This last property is an example of 't Hooft anomaly matching [**'tH**], and it serves to fix the coefficient of the WZW term, as in [**W1**]. We emphasize the second point, that fermionic states appear in a theory with only bosonic fields. This phenomenon also occurs in theories with self-dual fields [**FMS1**], but via a different mechanism. We verify properties (i)–(iv) in §4.

To capture all of these features we use a *generalized cohomology theory* where previously ordinary integral cohomology was used. Generalized cohomology theories, especially the various forms of K-theory, appear in high energy theoretical physics in connection with anomalies—through the Atiyah-Singer index theorem—and as a home for the Dirac quantization of Ramond-Ramond charges in superstring theory. The theory we encounter here also appears when defining three-dimensional Chern-Simons theory on spin manifolds [**J**]. It has also appeared recently [**FHT**], [**AS**] in connection with twisted K-theory, though that connection plays no role here. This special cohomology theory, which we simply denote 'E' and elucidate in §1, has exactly two nontrivial homotopy groups, so it is a simple twisted product of two ordinary cohomology theories.[2] It is a natural home for a characteristic class of complex vector bundles, which plays the role of $c_3/2$, half the third Chern class (see Proposition 1.9). This allows us to prove in §2 that the isomorphism class of the determinant line bundle of special families of Dirac operators on spin 4-manifolds is computed as an integral in E. There are analogous specialized index theorems in dimensions one [**FW**, (5.22)] and two [**F**, §5] in terms of ordinary cohomology. As the dimension grows, the denominators in the Riemann-Roch formula grow, and so the index recedes further from ordinary cohomology. The small denominator of 2 in our problem permits the formula in terms of the cohomology theory E.

[2]Although most familiar are geometric cohomology theories, such as K-theories and cobordism theories, generalized cohomology theories are like boutiques: abundant and specialized. The theory E is a minimal choice for this problem, but presumably not a unique one.

It is critical in our applications that these topological ideas be promoted to theorems in generalized *differential* cohomology [**HS**]. Thus Theorem 2.2 is a formula in differential E-cohomology for the determinant line bundle with its covariant derivative. Ordinary differential cohomology, also known as smooth Deligne cohomology or the theory of Cheeger-Simons differential characters, was first used by Gawędzki [**G**] to express terms of Wess-Zumino-Witten type. In §4 we use differential E-theory to define the Wess-Zumino-Witten term in the effective σ-model for pions. The main point is that there is a class $\nu \in E^5(G)$ which is in a precise sense half the generator of $H^5(G)$, as we prove in Proposition 1.9. The *gauged* WZW term, as defined in Definition 4.14, depends on a certain *transgression* which occurs when attempting to extend ν to a $(G \times G)$-equivariant class. (The connection between the gauged WZW term and transgression was made in ordinary cohomology in [**W4**, Appendix].) The class ν does not so extend: the obstruction is the anomaly, which is the transgression of ν, and the transgressing 'E-cochain' is used to define the gauged WZW term. This whole discussion must be carried out in the *differential* theory,[3] and so requires the construction of a transgressing differential form (5.16).[4]

The index formula proved in Theorem 2.2 applies to compute anomalies in four-dimensional gauge theories. Here we mean the full anomaly, including global anomalies. Indeed, in Example 3.4 we prove that the Standard Model has no global anomalies. We apply the index formula to compute the anomaly in gauged QCD (Proposition 3.7). Also, the characteristic class μ leads to the definition (Definition 3.10) of an *anomaly-free subgroup* which captures both local and global anomalies.

There are other models of QCD with different flavor symmetry groups and so different homogeneous spaces in the low energy effective σ-model; see [**MMN**], [**DZ**] for example. Presumably there is a similar story for the WZW term in these cases as well, but we leave it for others to investigate.

Raoul had a tremendous influence on me, both mathematically and personally. His passion for mathematics and music was infectious, his passion for life inspiring. He was a great teacher in every sense of the word. I'd like to think that this mix of geometry, topology, and physics—with a touch of transgression thrown in—would be to his taste.

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[3] See Definition 5.12, which is an extension of ideas in [**HS**].

[4] See the recent preprint [**Jo**] which treats a much generalized version of this transgression problem.

1. E-Theory

In topology, a cohomology theory is specified by a *spectrum*, which is a sequence $\{E_n\}_{n\in\mathbb{Z}}$ of pointed topological spaces and maps $\Sigma E_n \to E_{n+1}$ such that the adjoint maps $E_n \to \Omega E_{n+1}$ are homeomorphisms. Here Σ denotes suspension[5] and Ω the based loop space. The cohomology of a space X is then the abelian group

$$E^n(X) = [X, E_n] \tag{1.1}$$

of homotopy classes of maps into the spectrum. The Eilenberg-MacLane spectrum HA attached to an abelian group A is characterized by

$$\pi_q HA_n = \begin{cases} A, & q = n; \\ 0, & \text{otherwise.} \end{cases}$$

The spectrum E of interest in this paper, which for lack of good alternatives we simply notate 'E', has two nontrivial homotopy groups. The n^{th} space E_n fits into the fibration[6]

$$H\mathbb{Z}_n \xrightarrow{i} E_n \xrightarrow{j} H\mathbb{Z}/2\mathbb{Z}_{n-2}\,, \tag{1.2}$$

whose classifying map $H\mathbb{Z}/2\mathbb{Z}_{n-2} \to H\mathbb{Z}_{n+1}$ is the stable cohomology operation $\beta\circ Sq^2$, the integer Bockstein composed with the second Steenrod square. The fibration (1.2) leads, for any space X, to a long exact sequence

(1.3)

$$\cdots \longrightarrow H^n(X) \xrightarrow{i} E^n(X) \xrightarrow{j} H^{n-2}(X;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\beta\circ Sq^2} H^{n+1}(X) \longrightarrow \cdots$$

Since multiplication by 2 on $H\mathbb{Z}/2\mathbb{Z}_{n-2}$ is homotopically trivial, multiplication by 2 on E_n lands in the image of i, so it defines the map k in the diagram

$$\begin{array}{ccc} & & H\mathbb{Z}_n \\ & \overset{k}{\nearrow} & \big\downarrow i \\ E_n & \xrightarrow{2} & E_n \end{array} \tag{1.4}$$

Notice that any class in the image of $i \circ k$ is divisible by 2 and that $k \circ i$ is multiplication by 2: for any space X the composition

$$H^n(X) \xrightarrow{i} E^n(X) \xrightarrow{k} H^n(X) \tag{1.5}$$

is multiplication by 2.

[5]The suspension of a pointed space X is the pointed space

$$\Sigma X = [0,1] \times X \;/\; \{0\} \times X \;\cup\; \{1\} \times X \;\cup\; [0,1] \times \{x_0\},$$

where $x_0 \in X$ is the basepoint.

[6]This is written without reference to particular spaces in the spectra as $H\mathbb{Z} \to E \to \Sigma^{-2} H\mathbb{Z}/2\mathbb{Z}$.

In low degrees we have for any space X

$$E^0(X) \cong H^0(X) \cong [X, \mathbb{Z}],$$
$$E^1(X) \cong H^1(X) \cong [X, \mathbb{R}/\mathbb{Z}].$$

There is a short exact sequence

$$0 \longrightarrow H^2(X) \longrightarrow E^2(X) \longrightarrow H^0(X; \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0,$$

and we can interpret $E^2(X)$ as the group of isomorphism classes of $\mathbb{Z}/2\mathbb{Z}$-graded complex line bundles on X. Similarly, there is a short exact sequence

$$0 \longrightarrow H^3(X) \longrightarrow E^3(X) \longrightarrow H^1(X; \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0,$$

and we can interpret $E^3(X)$ as the group of isomorphism classes of $\mathbb{Z}/2\mathbb{Z}$-graded gerbes.

E-theory is oriented for spin manifolds; see Proposition 4.4 for details. Thus, given a map $f : X \to Y$ whose stable relative normal bundle $f^*TY - TX$ carries a spin structure, there is an *umker* or pushforward map

$$f_* : E^\bullet(X) \longrightarrow E^{\bullet - n}(Y), \tag{1.6}$$

where $n = \dim X - \dim Y$.

The differential theory $\check{E}$ associated to E is defined on smooth manifolds M; see [**HS**, §4] and also [**FMS2**, §2] for an expository introduction to some aspects of differential cohomology (in general, not specifically for this theory E) which is geared to physicists. We will not repeat the definitions here (except in Definition 5.10), but do note the following exact sequence [**HS**, (4.57)]. Define

$$A^n(M) = \left\{ (\lambda, \omega) \in E^n(M) \times \Omega^n_{\text{closed}}(M) : [\omega]_{\text{dR}} = \frac{1}{2} k(\lambda)_{\mathbb{R}} \right\}, \tag{1.7}$$

where $k(\lambda)_{\mathbb{R}} \in H^n(M; \mathbb{R})$ is the image of $k(\lambda) \in H^n(M)$ under the natural map $H^n(M) \to H^n(M; \mathbb{R})$ and $[\omega]_{\text{dR}} \in H^n(M; \mathbb{R})$ is the de Rham cohomology class of the closed differential form ω. Then the sequence

$$0 \longrightarrow E^{n-1}(M) \otimes \mathbb{R}/\mathbb{Z} \longrightarrow \check{E}^n(M) \longrightarrow A^n(M) \longrightarrow 0 \tag{1.8}$$

is exact. The differential form associated to a differential cohomology class is termed its *curvature*. The differential E-groups in low degrees also have geometric interpretations: $\check{E}^1(M)$ is the topological abelian group of maps $M \to \mathbb{R}/\mathbb{Z}$ and $\check{E}^2(M)$ is the topological abelian group of isomorphism classes of $\mathbb{Z}/2\mathbb{Z}$-graded hermitian line bundles with unitary covariant derivative. Integration in $\check{E}$ is defined over a fiber bundle $X \to Y$ whose stable normal (or tangent) bundle is given a spin structure.

After these generalities we turn to the construction of the characteristic class μ. We use the standard notions w_n, p_n, c_n for Stiefel-Whitney, Pontrjagin, and Chern classes. We also use the notation h_3, h_5 for generators of $H^3(SU_N), H^5(SU_N)$. Only the mod 2 reduction of the former enters; the sign of the latter, defined for $N \geq 3$, is fixed in (1.15) below.

Proposition 1.9.

(i) *There is an isomorphism $E^4(BSO) \cong \mathbb{Z}$ with generator λ satisfying*

$$k(\lambda) = p_1, \qquad j(\lambda) \equiv w_2 \pmod 2.$$

The pullback of λ to $B\mathrm{Spin}$ is $i(\tilde{\lambda})$ for a class $\tilde{\lambda} \in H^4(B\mathrm{Spin})$ with $2\tilde{\lambda} = p_1$.

(ii) *For $N \geq 3$ there is an isomorphism $E^6(BSU_N) \cong \mathbb{Z}$ with generator μ satisfying*

$$k(\mu) = c_3, \qquad j(\mu) \equiv c_2 \pmod 2.$$

Also, $E^6(BSU_2) \cong \mathbb{Z}/2\mathbb{Z}$ and the generator μ satisfies $j(\mu) \equiv c_2 \pmod 2$.

(iii) *For all $N \geq 1$ there is an isomorphism $E^6(BSp_N) \cong \mathbb{Z}/2\mathbb{Z}$. The generator is the pullback of μ under $BSp_N \to BSU_{2N}$.*

(iv) *For $N \geq 3$ there is an isomorphism $E^5(SU_N) \cong \mathbb{Z}$ with generator ν satisfying*

$$k(\nu) = h_5, \qquad j(\nu) \equiv h_3 \pmod 2. \tag{1.10}$$

The class ν is invariant under left and right translation by SU_N. Also, $E^5(SU_2) \cong \mathbb{Z}/2\mathbb{Z}$ and the generator ν satisfies $j(\nu) \equiv h_3 \pmod 2$.

(v) *The characteristic class μ obeys a Whitney sum formula: if V_1, V_2 are complex vector bundles with trivialized determinants, then*

$$\mu(V_1 \oplus V_2) = \mu(V_1) + \mu(V_2). \tag{1.11}$$

Also, $\mu(\overline{V}) = -\mu(V)$.

Note that $c_2 \pmod 2$ is the pullback of w_4 under $BSU_N \to BSO$. The class λ plays the role of '$\frac{1}{2}p_1$' and the class μ plays the role of '$\frac{1}{2}c_3$'.

Proof. For (i) we construct a map $BSO \to E_4$ by attaching cells of dimension ≥ 6 to BSO to kill $\pi_q BSO$, $q \geq 5$; the space so constructed is E_4. This gives an element $\lambda \in E^4(BSO)$ for which $j(\lambda) = w_2$ and $k(\lambda) = p_1$. Now use (1.3) and the map $B\mathrm{Spin} \to BSO$ to obtain a commutative diagram in which the rows are exact:

$$\begin{array}{ccccccccc}
H^1(BSO;\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{0} & H^4(BSO) & \xrightarrow{i_1} & E^4(BSO) & \longrightarrow & H^2(BSO;\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\beta\circ Sq^2} & H^5(BSO) \\
\downarrow & & \downarrow{\scriptstyle a} & & \downarrow & & \downarrow & & \downarrow \\
H^1(B\mathrm{Spin};\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{0} & H^4(B\mathrm{Spin}) & \xrightarrow{i_2} & E^4(B\mathrm{Spin}) & \longrightarrow & H^2(B\mathrm{Spin};\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\beta\circ Sq^2} & H^5(B\mathrm{Spin})
\end{array}$$

In the top row $H^4(BSO) \cong \mathbb{Z}$ with generator p_1; $H^2(BSO;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ with generator w_2; and $\beta w_2^2 = 0$ since $w_2^2 \equiv p_1 \pmod 2$. Thus

$E^4(BSO) \cong \mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, but the existence of $\lambda \in E^4(BSO)$ with $k(\lambda) = p_1$ rules out the latter, using (1.5). In the second row of the diagram $H^2(B\mathrm{Spin}; \mathbb{Z}/2\mathbb{Z}) = 0$, from which i_2 is an isomorphism. Also, $H^4(B\mathrm{Spin}) \cong \mathbb{Z}$ and we can choose a generator $\tilde{\lambda}$ such that $a(p_1) = 2\tilde{\lambda}$.

For (ii) we note that the space BSU_N has a CW presentation $BSU_N \sim S^4 \cup e_6 \cup \cdots$, valid for $N \geq 3$. The attaching map $\partial e_6 \to S^4$ is η, the double suspension of the Hopf map $S^3 \to S^2$; it represents the nontrivial element of $\pi_5 S^4 \cong \mathbb{Z}/2\mathbb{Z}$. (That the attaching map is nontrivial follows from the relation $Sq^2 c_2 \equiv c_3 \pmod 2$, for example.) The long exact sequence in E-cohomology attached to the cofibration

$$S^4 \cup_\eta e_6 \longrightarrow BSU_N \longrightarrow (BSU_N, S^4 \cup_\eta e_6) \tag{1.12}$$

shows that $E^6(S^4 \cup_\eta e_6) \cong E^6(BSU_N)$, since the CW presentation of the quotient starts with an 8-cell. By the suspension isomorphism

$$E^6(S^4 \cup_\eta e_6) \cong E^4(S^2 \cup_\eta e_4) \cong \widetilde{E}^4(\mathbb{CP}^2). \tag{1.13}$$

In (1.13) the attaching map $\eta : \partial e^4 \to S^2$ *is* the Hopf map, and $\widetilde{E}^4$ is the reduced E-cohomology, which in this case is isomorphic to the unreduced E-cohomology. Then the exact sequence

$$0 \longrightarrow H^4(\mathbb{CP}^2) \longrightarrow E^4(\mathbb{CP}^2) \longrightarrow H^2(\mathbb{CP}^2; \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

shows that $E^4(\mathbb{CP}^2) \cong \mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. But the underlying real 2-plane bundle V to the hyperplane line bundle over $\mathbb{CP}^2$ has $k\big(\lambda(V)\big) = p_1 V$ the generator of $H^4(\mathbb{CP}^2)$. It follows that $E^4(\mathbb{CP}^2) \cong \mathbb{Z}$.

The argument for (iv) is similar. There is a CW presentation $SU_N \sim S^3 \cup e_5 \cup e_7 \cup \cdots$; the attaching map $\partial e_5 \to S^3$ is η, the suspension of the Hopf map, since $\pi_5 SU_N \cong \mathbb{Z}$. Analogous arguments to (1.12) and (1.13) show $E^5(SU_N) \cong E^5(S^3 \cup_\eta e_5) \cong \widetilde{E}^4(\mathbb{CP}^2) \cong \mathbb{Z}$. The invariance of ν under translation is immediate from the homotopy invariance of cohomology, since SU_N is connected.

Let $p : BSU_{N_1} \times BSU_{N_2} \to BSU_{N_1+N_2}$ be the direct sum map. Then (1.11) is equivalent to the equation $p^*(\mu_{N_1+N_2}) = \mu_{N_1} + \mu_{N_2}$. The difference between the two sides is a class $c \in E^6(BSU_{N_1} \times BSU_{N_2})$ which satisfies $j(c) = 0$, by the Whitney formula for Chern classes, and so from (1.3), $c = i(b)$ for some $b \in H^6(BSU_{N_1} \times BSU_{N_2})$. But the Whitney formula for Chern classes also implies that $k(c) = 0$, and so b is torsion of order 2 by (1.4). Since $H^6(BSU_{N_1} \times BSU_{N_2})$ is torsion free, we deduce that $c = 0$. A similar argument proves that μ changes sign under conjugation.

To prove (iii), we use (1.3) and the vanishing of $H^6(BSp_N)$ and $H^7(BSp_N)$ to show that $E^6(BSp_N) \xrightarrow{j} H^4(BSp_N;\mathbb{Z}/2\mathbb{Z})$ is an isomorphism; the latter group is cyclic of order 2. Furthermore, the commutative square

$$\begin{array}{ccc} E^6(BSU_{2N}) & \xrightarrow{j} & H^4(BSU_{2N};\mathbb{Z}/2\mathbb{Z}) \\ \downarrow & & \downarrow \\ E^6(BSp_N) & \xrightarrow{j} & H^4(BSp_N;\mathbb{Z}/2\mathbb{Z}) \end{array}$$

shows that $\mu \in E^6(BSU_{2N})$ maps to the generator of $E^6(BSp_N)$, since $j(\mu) = w_4$ maps to the generator of $H^4(BSp_N;\mathbb{Z}/2\mathbb{Z})$.

The statements for SU_2 and $BSU_2 = BSp_1$ follow directly from the long exact sequence (1.3). q.e.d.

Next, we promote μ,ν to differential classes, and for that we need[7] a smooth model of the classifying spaces. Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space which carries a quaternionic structure. Let ESU_N be the Stiefel manifold of isometries $\mathbb{C}^N \to \mathcal{H}$ and BSU_N the quotient by the natural right SU_N action. Also, let ESp_N be the submanifold of ESU_{2N} of isometries $\mathbb{C}^{2N} \to \mathcal{H}$ which preserve the quaternionic structure and BSp_N the quotient by the natural right Sp_N action. Note the natural map $BSp_N \to BSU_{2N}$ and its lift to the universal principal bundles. The universal bundle $ESU_N \to BSU_N$ carries a connection Θ^{univ} defined by the orthogonal complements to the orbits; let Ω denote its curvature. Now it follows easily from (1.3) that $E^5(BSU_N) = 0$ and $E^4(SU_N) = 0$, so (1.8) implies that to promote μ,ν to differential classes $\check{\mu} \in \check{E}^6(BSU_N)$, $\check{\nu} \in \check{E}^5(SU_N)$ we have only to specify closed differential forms $\omega_{\check{\mu}} \in \Omega^6(BSU_N)$ and $\omega_{\check{\nu}} \in \Omega^5(SU_N)$. Let $\theta \in \Omega^1(SU_N;\mathfrak{su}_N)$ denote the Maurer-Cartan form, often written $\theta = g^{-1}dg$. Then

$$\omega_{\check{\mu}} = \frac{-i}{48\pi^3}\,\operatorname{Tr}\Omega^3, \tag{1.14}$$

$$\omega_{\check{\nu}} = \frac{-i}{480\pi^3}\,\operatorname{Tr}\theta^5. \tag{1.15}$$

The differential form (1.15) is bi-invariant, i.e., invariant under both left and right translation in SU_N. Since ν is also bi-invariant, it follows that so too is $\check{\nu}$.

Let $P \to M$ be a principal SU_N-bundle with connection Θ over a smooth manifold M and $V \to M$ the associated rank N hermitian

[7] We refer to [**HS**, §3.3] for a more elegant treatment of differential characteristic classes that uses the Weyl algebra of polynomials on the Lie algebra in place of differential forms on the classifying space.

vector bundle with covariant derivative. We define a *differential characteristic class* $\check{\mu}(V) \in \check{E}^6(M)$. For simplicity[8] we use the existence of an SU_N-equivariant map $\gamma : P \to ESU_N$, which pulls the universal connection Θ^{univ} back to Θ. In fact, the space of such classifying maps is contractible and nonempty [**DHZ**]. A classifying map induces $\bar{\gamma} : M \to BSU_N$, and we set $\check{\mu}(V) = \bar{\gamma}^*(\check{\mu})$. The "curvature" of $\check{\mu}(V)$ is the 6-form

$$\omega_{\check{\mu}(V)} = \frac{-i}{48\pi^3}\,\mathrm{Tr}(\Omega^V)^3, \tag{1.16}$$

where Ω^V is the curvature of V.

Lemma 1.17. *$\bar{\gamma}^*(\check{\mu})$ is independent of the classifying map $\bar{\gamma}$.*

Proof. Since any two classifying maps are homotopic, and the 6-form curvature is independent of $\bar{\gamma}$, the image of $\bar{\gamma}^*(\check{\mu})$ in $A^6(M)$ (see (1.8)) is independent of $\bar{\gamma}$. Let $\Gamma : [0,1] \times P \to ESU_N$ be a homotopy of classifying maps of Θ, and $\overline{\Gamma} : [0,1] \times M \to BSU_N$ the quotient homotopy. Then $\overline{\Gamma}_1^*(\check{\mu}) - \overline{\Gamma}_0^*(\check{\mu})$ is in the image of $E^{n-1}(M) \otimes \mathbb{R}/\mathbb{Z}$, and so can be detected by a map of a closed spin $(n-1)$-manifold $f : W^{n-1} \to M$. By Stokes' theorem,

$$\int_W f^*\overline{\Gamma}_1^*(\check{\mu}) - f^*\overline{\Gamma}_0^*(\check{\mu}) \;=\; \int_{[0,1]\times W} \bigl(\mathrm{id}_{[0,1]} \times f\bigr)^*\overline{\Gamma}^*\omega_{\check{\mu}}. \tag{1.18}$$

But $\overline{\Gamma}^*\omega_{\check{\mu}} \in \Omega^6\bigl([0,1]\times M\bigr)$ is the pullback of (1.16) via projection to M, whence (1.18) vanishes. q.e.d.

2. Determinant Line Bundles on 4-Manifolds

We begin with the setup for geometric index theory [**F**]. Let $\mathfrak{X} \to S$ be a fiber bundle with fibers compact 4-manifolds. Assume the vertical tangent bundle $T(\mathfrak{X}/S) \to \mathfrak{X}$ is endowed with a spin structure and Riemannian metric, and suppose too that there is a complementary horizontal distribution. Let $\Omega^{\mathfrak{X}/S}$ denote the curvature of the resulting Levi-Civita covariant derivative on $T(\mathfrak{X}/S) \to \mathfrak{X}$. We term $\mathfrak{X} \to S$ with this data a *Riemannian spin fiber bundle* or a *Riemannian spin manifold over S*. Suppose $V \to \mathfrak{X}$ is a hermitian vector bundle equipped with a trivialization of $\mathrm{Det}\, V \to \mathfrak{X}$ and a compatible unitary covariant derivative with curvature Ω^V. Said differently, V and its covariant derivative are associated to a principal SU_N bundle with connection over $\mathfrak{X}$. Recall that spinor fields on a Riemannian spin 4-manifold are $\mathbb{Z}/2\mathbb{Z}$-graded, the grading termed 'chirality', and the Dirac operator exchanges the chirality. The geometric data determine a family of *chiral* Dirac operators $D_{\mathfrak{X}/S}(V)$ parametrized by S. The chiral Dirac operators map

[8] We could instead use a classifying map for the bundle and a Chern-Simons form which measures the difference between $\gamma^*\Theta^{\text{univ}}$ and Θ; cf. [**HS**, §3.3].

positive chirality V-valued spinor fields to negative chirality V-valued spinor fields. Let $\operatorname{Det} D_{\mathcal{X}/S}(V) \to S$ be the associated determinant line bundle with its natural metric and covariant derivative. It carries a natural $\mathbb{Z}/2\mathbb{Z}$-grading as well: the degree at $s \in S$ is $\operatorname{index} D_{\mathcal{X}_s}(V) \pmod 2$. We allow V to be a virtual bundle, or equivalently a $\mathbb{Z}/2\mathbb{Z}$-graded bundle $V = V^0 \oplus V^1$, with both $\operatorname{Det} V^0$ and $\operatorname{Det} V^1$ trivialized. In this case we write

$$\operatorname{Det} D_{\mathcal{X}/S}(V) = \operatorname{Det} D_{\mathcal{X}/S}(V^0) \otimes \operatorname{Det} D_{\mathcal{X}/S}(V^1)^* \tag{2.1}$$

and $\operatorname{rank} V = \operatorname{rank} V^0 - \operatorname{rank} V^1$.

Theorem 2.2. *The isomorphism class of the $\mathbb{Z}/2\mathbb{Z}$-graded determinant line bundle with covariant derivative is*

$$\big[\operatorname{Det} D_{\mathcal{X}/S}(V)\big] = \int_{\mathcal{X}/S} \check{\mu}(V) \qquad \text{in } \check{E}^2(S). \tag{2.3}$$

Recall that $\check{\mu}(V) \in \check{E}^6(\mathcal{X})$ is the differential characteristic class defined around (1.16); for a $\mathbb{Z}/2\mathbb{Z}$-graded bundle $V = V^0 \oplus V^1$ set $\check{\mu}(V) = \check{\mu}(V^0) - \check{\mu}(V^1)$. The integral $\int_{\mathcal{X}/S} : \check{E}^6(\mathcal{X}) \to \check{E}^2(S)$ uses the spin structure; see (1.6) and [**HS**, §4.10]. Also, note the splitting

$$\check{E}^2(S) \cong \check{H}^2(S) \times H^0(S;\mathbb{Z}/2\mathbb{Z}), \tag{2.4}$$

under which a graded line bundle with covariant derivative maps separately to the underlying line bundle with covariant derivative and the grading. So Theorem 2.2 also determines the ungraded determinant line bundle with covariant derivative.

A line bundle with covariant derivative is determined up to isomorphism by all of its holonomies around loops. In the language of differential E-theory, this is the following:

Lemma 2.5. *Let $L \to S$ be a $\mathbb{Z}/2\mathbb{Z}$-graded complex line bundle with covariant derivative and $[L] \in \check{E}^2(S)$ its isomorphism class. Then $[L]$ is determined by the grading of L in $H^0(S;\mathbb{Z}/2\mathbb{Z})$ and by its integral over all loops $\gamma : S^1 \to S$, where S^1 has the bounding spin structure. The integral over such a loop is minus the log holonomy (in $\mathbb{R}/\mathbb{Z}$).*

Proof. By (2.4) it suffices to show that the isomorphism class of the underlying ungraded line bundle with covariant derivative is determined by $\int_{S^1} \gamma^*[L]$ for all γ. Since S^1 has the bounding spin structure, we can write $S^1 = \partial D^2$ as a spin manifold, and the bundle with covariant derivative $\gamma^* L \to S^1$ extends to $\widetilde{L} \to D^2$. Let $\Omega^{\widetilde{L}}$ denote its curvature. Then Stokes' theorem in differential E-theory implies that

$$\int_{S^1} \gamma^*[L] \equiv \int_{D^2} \Omega^{\widetilde{L}} \equiv -\log \operatorname{hol}_{S^1}(\gamma^* L) \pmod 1.$$

That all such integrals determine the image of $[L]$ in $\check{H}^2(S)$ can be seen directly from the exact sequence

$$0 \longrightarrow H^1(S;\mathbb{R}/\mathbb{Z}) \longrightarrow \check{H}^2(S) \longrightarrow \Omega^2_{\text{closed}}(S),$$

in which the last arrow is the curvature: the curvature is determined by the integral around loops $\gamma : S^1 \to S$ which bound a disk in S, and the integration map $H^1(S;\mathbb{R}/\mathbb{Z}) \to \operatorname{Hom}\bigl(H_1(S),\mathbb{R}/\mathbb{Z}\bigr)$ is an isomorphism. q.e.d.

The following lemma reduces the holonomy computation for $\operatorname{Det} D_{\mathcal{X}/S}(V)$ to the computation of its curvature.

Lemma 2.6. *Suppose Y is a closed spin 5-manifold and $V \to Y$ is a rank N complex vector bundle with $c_1(V) = 0$. Then there exists a compact spin 6-manifold Z and a rank N complex vector bundle $W \to Z$ with $c_1(W) = 0$ such that $\partial Z = Y$ and $W\big|_{\partial Z} \cong V$.*

The lemma applies to virtual bundles as well: given $V = V^0 \oplus V^1$ of rank N and $c_1(V) = c_1(V^0) - c_1(V^1) = 0$, we can add trivial bundles to replace V^1 by a trivializable bundle and so V^0 by a bundle (of rank $\geq N$) with $c_1(V^0) = 0$.

Proof.
By Thom's theory, this is the assertion that $\pi_5\, M\text{Spin} \wedge (BSU_N)_+$ vanishes; here X_+ is the space X with disjoint basepoint. First, we can drop the '+' since

$$\pi_5\, M\text{Spin} \wedge (BSU_N)_+ \cong \pi_5\, M\text{Spin} \wedge BSU_N \;\times\; \pi_5\, M\text{Spin}$$

and $\pi_5\, M\text{Spin} = 0$. Next, smash the cofiber sequence (1.12) with $M\text{Spin}$ to conclude from the long exact sequence in homotopy groups that

$$\pi_5\, M\text{Spin} \wedge BSU_N \cong \pi_5\, M\text{Spin} \wedge (S^4 \cup_\eta e_6).$$

Now smash the cofiber sequence

$$S^4 \longrightarrow (S^4 \cup_\eta e_6) \longrightarrow S^6$$

with $M\text{Spin}$ to obtain the exact sequence

$$\begin{aligned} \text{(2.7)} \quad \pi_6\, M\text{Spin} \wedge S^6 &\longrightarrow \pi_5\, M\text{Spin} \wedge S^4 \\ &\longrightarrow \pi_5\, M\text{Spin} \wedge (S^4 \cup_\eta e_6) \longrightarrow \pi_5\, M\text{Spin} \wedge S^6. \end{aligned}$$

We have $\pi_6\, M\text{Spin}\wedge S^6 \cong \pi_0\, M\text{Spin} \cong \mathbb{Z}$ and $\pi_5\, M\text{Spin}\wedge S^4 \cong \pi_1\, M\text{Spin} \cong \mathbb{Z}/2\mathbb{Z}$. Thus the initial map in (2.7) is identified with $\eta : \pi_0\, M\text{Spin} \to \pi_1\, M\text{Spin}$, which is surjective. Since $\pi_5\, M\text{Spin} \wedge S^6 \cong \pi_{-1}\, M\text{Spin} = 0$, it now follows that $\pi_5\, M\text{Spin} \wedge (S^4 \cup_\eta e_6) = 0$. q.e.d.

Proof of Theorem 2.2. First, the curvature of $\operatorname{Det} D_{\mathfrak{X}/S}(V)$ is the integral over $\mathfrak{X} \to S$ of the 6-form component of $\hat{A}(\Omega^{\mathfrak{X}/S})\operatorname{ch}(\Omega^V)$. The flat trivialization of $\operatorname{Det} V$ implies $\operatorname{Tr}\Omega^V = 0$, and so the only contribution is

$$\int_{\mathfrak{X}/S} ch_3(\Omega^V) = \int_{\mathfrak{X}/S} \frac{1}{2}\, c_3(\Omega^V) = \int_{\mathfrak{X}/S} \frac{-i}{48\pi^3}\ \operatorname{Tr}(\Omega^V)^3. \tag{2.8}$$

By (1.14) this is the curvature of $\int_{\mathfrak{X}/S} \check{\mu}(V)$, i.e., the differential form component of its image under $\check{E}^2(S) \to A^2(S)$ in (1.8). q.e.d.

By Lemma 2.5, to prove the theorem it suffices to verify that the grading and the integral of both sides of (2.3) over each loop $\gamma : S^1 \to S$ agree. The grading of $\operatorname{Det} D_{\mathfrak{X}/S}(V)$ at $s \in S$ is given by the index mod 2, which by the Atiyah-Singer index theorem and Rohlin's theorem—the $\hat{A}$ genus of a closed spin 4-manifold is even—is

$$\operatorname{rank} V \int_{\mathfrak{X}_s} \hat{A}(\mathfrak{X}_s) - \int_{\mathfrak{X}_s} -c_2(V) \equiv \int_{\mathfrak{X}_s} \check{\mu}(V) \pmod 2. \tag{2.9}$$

Now the pullback $Y = \gamma^*\mathfrak{X} \to S^1$ is a smooth 5-manifold, and it obtains a spin structure from the spin structure of the fibers and the bounding spin structure of the base S^1. Lemma 2.5 implies that the integral of $[\operatorname{Det} D_{\mathfrak{X}/S}(V)]$ is minus the log holonomy, which the holonomy theorem for determinant line bundles [**BF**] computes as the adiabatic limit of the Atiyah-Patodi-Singer invariant $\xi_Y(\gamma^* V) \pmod 1$. The absence of $\Omega^{\mathfrak{X}/S}$ in the curvature formula (2.8) implies that we can drop the adiabatic limit. Let $Y = \partial Z$ and $\gamma^* V = \partial W$ as in Lemma 2.6. Then the Atiyah-Patodi-Singer index theorem implies

$$\xi_Y(\gamma^* V) \equiv \int_Z \frac{1}{2}\, c_3(\Omega^W) \pmod 1. \tag{2.10}$$

On the other hand, by Stokes' theorem in differential E-theory,

$$\int_{S^1}\int_{\gamma^*\mathfrak{X}/S^1} \check{\mu}(V) = \int_Y \check{\mu}(V) = \int_Z \omega_{\check{\mu}}(W) = \int_Z \frac{1}{2}\, c_3(\Omega^W).$$

The agreement with (2.10) completes the proof. q.e.d.

If $V \to \mathfrak{X}$ is either real or quaternionic, then the determinant bundle simplifies.

Corollary 2.11. *If $V \to \mathfrak{X}$ has a quaternionic structure, compatible with its hermitian structure and covariant derivative, then* $\operatorname{Det} D_{\mathfrak{X}/S}(V)$ *has a real structure compatible with its metric and covariant derivative. Equation* (2.3) *holds and now $\check{\mu}(V)$ has order two by Proposition* 1.9(iii).

A quaternionic structure is a linear map $J : V \to \overline{V}$ with $\overline{J}J = -\operatorname{id}_V$. The spin space on a 4-manifold is also quaternionic,[9] whence the V-valued spinors are real. The real structure commutes with the Dirac operator, so it induces a real structure on all eigenspaces and so on the determinant bundle as well. Note that the elements of order two in $\check{E}^2(S)$ form the abelian group $H^1(S;\mathbb{Z}/2\mathbb{Z}) \times H^0(S;\mathbb{Z}/2\mathbb{Z})$ of isomorphism classes of $\mathbb{Z}/2\mathbb{Z}$-graded *real* line bundles over S.

For the real case we have the following.

Proposition 2.12. *If $V \to \mathcal{X}$ has a real structure compatible with its hermitian structure and covariant derivative, then* $\operatorname{Det} D_{\mathcal{X}/S}(V)$ *is canonically trivial.*

In this case the V-valued spinors are quaternionic, so the index is even and hence the determinant line bundle has zero grading. The patching construction of the determinant line bundle [**F**] allows us to deduce its triviality from the following lemma.

Proposition 2.13. *Let W be a hermitian vector space with compatible quaternionic structure. Then* $\operatorname{Det} W$ *is canonically trivial.*

Proof. The quaternionic structure $J : W \to \overline{W}$ induces a real structure $\det J : \operatorname{Det} W \to \overline{\operatorname{Det} W}$ on the determinant line. There are two real points of norm one on $\operatorname{Det} W$. Let $e_1, \dots, e_m$ be a unitary basis of W over the quaternions. Then

$$e_1 \wedge Je_1 \wedge e_2 \wedge Je_2 \wedge \cdots \wedge e_m \wedge Je_m \in \operatorname{Det} W$$

is a real point of norm one. Since the space of such bases is connected, this point in $\operatorname{Det} W$ is independent of the basis. q.e.d.

3. Anomalies in Four-Dimensional Gauge Theories

We begin in Minkowski spacetime M^4. Let $\mathbb{S}$ be the two-dimensional complex spin space, the half-spin representation of $\mathrm{Spin}_{1,3} \cong SL_2\mathbb{C}$. The opposite chirality spin space is its complex conjugate $\overline{\mathbb{S}}$. Let H be a compact Lie group and $\rho : H \to \operatorname{Aut}(\mathbb{V}^0)$ a unitary representation. In a quantum field theory with an H-gauge field A, spinor fields which transform in the representation ρ come in pairs[10]

$$\begin{aligned} \psi &: M^4 \longrightarrow \Pi\mathbb{S} \otimes \mathbb{V}^0 \\ \bar{\psi} &: M^4 \longrightarrow \Pi\overline{\mathbb{S}} \otimes \overline{\mathbb{V}^0}, \end{aligned}$$

[9] This implies the theorem of Rohlin used in (2.9): the index of the Dirac operator is even since the kernel is quaternionic.

[10] $\Pi\mathbb{S}$ is the parity reversal of $\mathbb{S}$, the $\mathbb{Z}/2\mathbb{Z}$-graded vector space of degree one in which spinor fields take values.

which are coupled in the Dirac lagrangian $\frac{1}{2}\bar{\psi}\not{D}_A\psi$. The presence of spinor fields of both chiralities is dictated by CPT-invariance. It is natural to let $\mathbb{V} = \mathbb{V}^0 \oplus \mathbb{V}^1$ be a $\mathbb{Z}/2\mathbb{Z}$-graded representation with associated spinor fields

$$\begin{aligned}\psi : M^4 &\longrightarrow \Pi\mathbb{S} \otimes (\mathbb{V}^0 \oplus \overline{\mathbb{V}^1})\\ \bar{\psi} : M^4 &\longrightarrow \Pi\overline{\mathbb{S}} \otimes (\overline{\mathbb{V}^0} \oplus \mathbb{V}^1).\end{aligned}$$

Notice that $\mathbb{V}$ and $\overline{\Pi\mathbb{V}} = \overline{\mathbb{V}^1} \oplus \overline{\mathbb{V}^0}$ lead to the same theory.

The Wick rotated Euclidean field theory is defined on the category of Riemannian spin 4-manifolds X. The bosonic field is a connection Θ on a principal H-bundle $P \to X$. Gauge transformations—more generally, isomorphisms of principal H-bundles—act as symmetries. So the space $\mathcal{A}_X^{(H)}$ of H-connections on X must be considered as a groupoid, or as a stack.[11] In any case we consider families of connections parametrized by a smooth manifold S and allow X and its metric to vary as well. In other words, we couple the gauge theory to gravity, but we treat the gauge field and metric as classical background fields. Therefore, in the Euclidean theory we consider Riemannian spin fiber bundles $\mathcal{X} \to S$, as in the beginning of §2, together with a connection on a principal H-bundle $\mathcal{P} \to \mathcal{X}$. The representation $\rho : H \to \mathrm{Aut}(\mathbb{V})$ gives rise to an associated $\mathbb{Z}/2\mathbb{Z}$-graded hermitian vector bundle $V = V^0 \oplus V^1 \to \mathcal{X}$ with unitary covariant derivative. Assume that the fibers of $\mathcal{X} \to S$ are closed. Then the Euclidean functional integral over the spinor fields is

$$\det D_{\mathcal{X}/S}(V^0) \cdot \det D_{\mathcal{X}/S}(\overline{V^1}) : \mathcal{X} \longrightarrow \mathrm{Det}\, D_{\mathcal{X}/S}(V^0) \otimes \mathrm{Det}\, D_{\mathcal{X}/S}(\overline{V^1}). \tag{3.1}$$

But since the spinors are self-conjugate, in fact quaternionic, $D_{\mathcal{X}/S}(\overline{V^1}) = \overline{D_{\mathcal{X}/S}(V^1)}$ and so $\mathrm{Det}\, D_{\mathcal{X}/S}(\overline{V^1}) \cong \overline{\mathrm{Det}\, D_{\mathcal{X}/S}(V^1)} \cong \mathrm{Det}\, D_{\mathcal{X}/S}(V^1)^*$. Hence, the fermionic functional integral (3.1) is a section of the determinant bundle $\mathrm{Det}\, D_{\mathcal{X}/S}(V) \to \mathcal{X}$ defined in (2.1) and computed in Theorem 2.2.

The next step in the quantum field theory is to perform an integral over the space of bosonic fields, and one factor in the integrand is the fermionic partition function (3.1). To even set up the integral we need to transform it from a section of a line bundle to a function, i.e., to take the ratio with a trivialization $\mathbf{1} : S \to \mathrm{Det}\, D_{\mathcal{X}/S}(V)$. We require that this trivialization be geometric in the sense that $|\mathbf{1}| = 1$ and $\mathbf{1}$ is flat. The *anomaly* is the obstruction to the existence of a flat trivialization; it is measured by $[\mathrm{Det}\, D_{\mathcal{X}/S}(V)] \in \check{E}^2(S)$. If the anomaly vanishes, then there is a further requirement: the trivialization $\mathbf{1}$ must

[11] The local model is a smooth infinite dimensional manifold with a smooth action of a compact Lie group.

be consistent with gluing of 4-manifolds. A consistent choice of **1** is called a *setting of the quantum integrand*. We remark that the equivalence class of $\operatorname{Det} D_{\mathcal{X}/S}(V)$ in $\check{E}^2(S)$ includes its $\mathbb{Z}/2\mathbb{Z}$-grading. We are unsure of the physical significance of this grading, but believe that a theory is anomalous if the grading is nonzero. Note in this case that the fermionic functional integral (3.1) vanishes, since the Dirac operator has a nonzero kernel.

Theorem 2.2 applies to compute the anomaly if the determinant of the representation $\rho : H \to \operatorname{Aut}(\mathbb{V})$ is one. If this condition does not hold, then the determinant bundle is nontrivial in suitable families where both the metric and gauge field vary: the curvature (2.8) has contributions from $c_1(\Omega^V)$. Assume, then, that $\det \rho = 1$. Fix a Riemannian spin 4-manifold X. Any H-connection on X is pulled back from a universal connection on the classifying space BH, so it suffices to study the family of connections parametrized by $S = \operatorname{Map}(X, BH)$. As at the end of §1, we construct a smooth model of BH and a universal connection. The representation ρ determines a map $\hat{\rho} : BH \to BSU_{N_1} \times BSU_{N_2}$, where $\dim \mathbb{V}^q = N_q$. Let $\check{\mu}(\rho) \in \check{E}^6(BH)$ be the pullback $\hat{\rho}^*(\check{\mu}_{N_1} - \check{\mu}_{N_2})$ of the universal differential characteristic class. Then if

$$e : \operatorname{Map}(X, BH) \times X \longrightarrow BH$$

is the evaluation map, Theorem 2.2 computes the anomaly to be

$$\int_X e^*\check{\mu}(\rho) = \int_X e^*\hat{\rho}^*(\check{\mu}_{N_1} - \check{\mu}_{N_2}). \tag{3.2}$$

The curvature of $\check{\mu}(\rho)$, the 6-form on BH which is the pullback of (1.14) by $\hat{\rho}$, is computed by applying the symmetric trilinear form

$$\xi_1, \xi_2, \xi_3 \longmapsto \frac{-i}{96\pi^3} \operatorname{Tr}\left[\dot{\rho}(\xi_1)\dot{\rho}(\xi_2)\dot{\rho}(\xi_3) + \dot{\rho}(\xi_2)\dot{\rho}(\xi_1)\dot{\rho}(\xi_3)\right], \qquad \xi_1, \xi_2, \xi_3 \in \operatorname{Lie}(H), \tag{3.3}$$

on the Lie algebra of H to the curvature of the universal connection on BH. This trilinear form is the usual expression for the *local* anomaly in the physics literature, e.g., [**We**, §22.3].

If $\check{\mu}(\rho)$ vanishes, then there is no anomaly. This in itself does not provide a choice of trivialization of $\operatorname{Det} D_{\mathcal{X}/S}(V)$, much less a consistent choice under gluing—a setting of the quantum integrand. For that we would need a refinement of Theorem 2.2 to an isomorphism of $\operatorname{Det} D_{\mathcal{X}/S}(V)$ with an integral of a differential function representing $\check{\mu}(V)$. If ρ is a real representation, however, then Proposition 2.12 does provide a canonical trivialization.

Example 3.4 (The Standard Model)**.** In this case $H = SU_3 \times SU_2 \times U_1$, or a finite quotient. The representation ρ is 15-dimensional, and it extends to the representation $\overline{V} \oplus \wedge^2 V$ of SU_5, where $V = \mathbb{C}^5$ is the standard representation. We compute $c_3\wedge^2 V = c_3 V$. Hence $c_3(\overline{V} \oplus \wedge^2 V) = 0$, and from Proposition 1.9(ii) we have $\mu(\overline{V} \oplus \wedge^2 V) = 0$. Thus $\breve{\mu}(\overline{V} \oplus \wedge^2 V) = 0$ as well.

An alternative argument: if we add a trivial representation to the 15-dimensional Standard Model representation ρ, then the sum extends to the 16-dimensional half-spin representation of Spin_{10}. We claim $E^6(B\,\mathrm{Spin}_{10}) = 0$, whence $\breve{\mu}(\rho) = 0$ by (1.11). The claim follows from the long exact sequence (1.3), the fact that $H^6(B\,\mathrm{Spin}_{10}) = 0$, and the fact that $\beta Sq^2 w_4 \in H^7(B\,\mathrm{Spin}_{10})$, often denoted W_7, is nonzero.

Example 3.5 (The SU_2 anomaly [**W3**])**.** Here $H = SU_2$ and ρ is the standard representation, which is quaternionic. So by Corollary 2.11 the determinant bundle is real and the anomaly is of order 2; cf. Proposition 1.9(ii). For $X = S^4$, the case considered in [**W3**], the anomaly is nonzero and is more easily computed directly using Bott periodicity than from the formula (3.2) in terms of $\breve{\mu}$.

The main theory of interest in this paper is QCD, the theory of quarks. The gauge group $H = SU_{N_c}$, where N_c is the number of "colors"; in the real world $N_c = 3$. Let $\mathbb{U} = \mathbb{C}^{N_c}$ denote the fundamental representation of H. There is another positive integer in the theory, the number of "flavors" N_f. In the real world there are six flavors of quarks, but as only three of them are light, in this context N_f is often taken to be equal to three. Our discussion applies to any value of N_f. Let $\mathbb{W} = \mathbb{C}^{N_f}$. The representation ρ of H is the $\mathbb{Z}/2\mathbb{Z}$-graded vector space

$$\mathbb{V} = \mathbb{U} \otimes \mathbb{W} \ \oplus\ \mathbb{U} \otimes \mathbb{W} \tag{3.6}$$

where H acts on $\mathbb{U}$ as the fundamental representation and trivially on $\mathbb{W}$. The theory is trivially and canonically anomaly-free as $\mathbb{V}^0 = \mathbb{V}^1$. QCD has a *global* symmetry group $U_{N_f} \times U_{N_f}$; the two factors independently act on the two copies of $\mathbb{W}$ in (3.6). Our interest is the subgroup $G \times G$, where $G = SU_{N_f}$. We digress now to briefly explain anomalies for global symmetries and gauging of global symmetries.

Suppose we have a quantum field theory with the global symmetry group a compact Lie group K. If the space (stack) of fields on a manifold X is $\mathcal{F}_X$, then K acts on $\mathcal{F}_X$. Let $\mathcal{B}_X$ be the stack of bosonic fields, so there is a vector bundle $\mathcal{F}_X \to \mathcal{B}_X$ with fibers the odd vector spaces of fermions. The group K acts on $\mathcal{B}_X$ compatibly with its action on $\mathcal{F}_X$. The functional integral over the fermionic fields is a K-invariant section of a K-equivariant $\mathbb{Z}/2\mathbb{Z}$-graded hermitian line bundle with covariant derivative on $\mathcal{B}_X$. The anomaly is the obstruction to a K*-invariant* flat trivialization $\mathbf{1}$. If K acts trivially on $\mathcal{B}_X$, then it acts on the line bundle by a character $K \to \mathbb{T}$ on each component of $\mathcal{B}_X$. Here $\mathbb{T}$ is the circle group of unit norm complex numbers. If the nonequivariant anomaly vanishes, then the anomaly in the global symmetry is measured by these characters. This is the case in QCD, and for the subgroup $K = G \times G = SU_{N_f} \times SU_{N_f}$ of the full global symmetry group $U_{N_f} \times U_{N_f}$ there are no nontrivial characters, whence no anomalies.[12]

A *gauging* of the theory is an extension that includes a connection on a principal K-bundle as a new field in the theory. Thus, in the gauged theory the stack of fields $\widetilde{\mathcal{F}}_X$ on a manifold X fibers over the stack $\mathcal{A}_X^{(K)}$ of K-connections on X. There is a distinguished point of $\mathcal{A}_X^{(K)}$—the trivial connection with isotropy group K—and we require that the fiber of $\widetilde{\mathcal{F}}_X \to \mathcal{A}_X^{(K)}$ at the trivial K-connection be identified with the stack of fields $\mathcal{F}_X$ in the original theory and the action of the isotropy group K on this fiber be the original global symmetry. Let $\widetilde{\mathcal{B}}_X$ denote the stack of bosonic fields on X in the gauged theory. Then the fermionic anomaly in the gauged theory, which is the isomorphism class of a $\mathbb{Z}/2\mathbb{Z}$-graded line bundle with covariant derivative over $\widetilde{\mathcal{B}}_X$, restricts on the fiber $\mathcal{B}_X$ over the trivial K-connection to the anomaly in the original theory (including the global K-action).

QCD has a natural extension which gauges the global $G \times G = SU_{N_f} \times SU_{N_f}$ symmetry. It is a four-dimensional gauge theory with gauge group $H \times G \times G = SU_{N_c} \times SU_{N_f} \times SU_{N_f}$. The representation ρ acts on the vector space (3.6): the group SU_{N_c} acts on both copies of $\mathbb{U}$ as before, the first factor of SU_{N_f} acts on the first copy of $\mathbb{W}$, and the second factor of SU_{N_f} acts on the second copy of $\mathbb{W}$. The stack of bosonic fields in the gauged theory on a fixed 4-manifold X is $\widetilde{\mathcal{B}}_X = \mathcal{A}_X^{(H)} \times \mathcal{A}_X^{(G\times G)}$. As usual, we consider smooth families of fields, which for the gauged theory is a Riemannian spin fiber bundle $\mathcal{X} \to S$ with compact 4-manifolds as fiber and a principal $SU_{N_c} \times (G \times G)$ bundle $\mathcal{P} \times \mathcal{Q} \to \mathcal{X}$ with connection. Let $\check{\mu}_1(\mathcal{Q}), \check{\mu}_2(\mathcal{Q}) \in \check{E}^6(\mathcal{X})$ be the differential characteristic classes associated with the two factors of G.

Proposition 3.7. *The anomaly in gauged QCD is*

$$\int_{\mathcal{X}/S} N_c\big(\check{\mu}_1(\mathcal{Q}) - \check{\mu}_2(\mathcal{Q})\big). \tag{3.8}$$

This follows directly from Theorem 2.2, where we use Proposition 1.9(v) to compute the characteristic class of the vector bundle associated to the representation (3.6). (Recall that $G \times G$ acts trivially on $\mathbb{U}$, which has dimension N_c.) In terms of the stack $\widetilde{\mathcal{B}}_X$ of bosonic fields, the anomaly is pulled back from $\mathcal{A}_X^{(G\times G)}$ and is given by (3.8).

[12]The anti-diagonal $U_1 \subset U_{N_f} \times U_{N_f}$ acts by the character $\lambda \mapsto \lambda^m$ for $m = -2N_cN_f\,\mathrm{Sign}(X)/8 - 2N_fk$ on the component where the second Chern class of the principal $H = SU_{N_c}$-bundle is k times the generator of $H^4(X)$. Here, $\mathrm{Sign}(X)$ is the signature of the spin manifold X. So this subgroup is anomalous, though a finite cyclic subgroup is not.

The symmetry breaking in QCD is deduced from expectation values of bilinear expressions in the spinor fields. If (ψ_1, ψ_2) are the spinor fields corresponding to the representation (3.6), then they have the form $\langle\overline{\psi_1}, T\cdot\psi_2\rangle$, where the inner product is that in $\mathbb{U}\otimes\mathbb{W}$, and T is an element of the Lie algebra of SU_{N_f} which acts on $\mathbb{U}\otimes\mathbb{W}$ as the identity on $\mathbb{U}$ tensor its action on $\mathbb{W}$. The expectation value is taken at any point of Minkowski spacetime, as it is constant by Poincaré invariance. To write this bilinear in the gauged theory, and so implement the symmetry breaking, we need additional data. Namely, there are vector bundles W_1, W_2 associated to the two G-connections, and we need an isomorphism $W_1 \cong W_2$, so an isomorphism of the principal G-bundles underlying the two G-connections. The construction of the gauged effective theory in §4 includes that isomorphism; in fact, it is the scalar field in that theory.

Gauged QCD admits new topological terms. In the *exponentiated* action these have the form

$$\exp\left(2\pi i\theta_1\int_{\mathcal{X}/S} c_2(\mathcal{Q})_1 + 2\pi i\theta_2\int_{\mathcal{X}/S} c_2(\mathcal{Q})_2\right), \tag{3.9}$$

where $c_2(\mathcal{Q})_1, c_2(\mathcal{Q})_2$ are the degree four characteristic classes corresponding to the two $G = SU_{N_f}$ factors, and $\theta_1, \theta_2 \in \mathbb{R}/\mathbb{Z}$. The requirement that a fermion bilinear exist implies that $c_2(\mathcal{Q})_1 = c_2(\mathcal{Q})_2$, as argued in the previous paragraph, so we have a single topological term with coefficient $\theta = \theta_1 + \theta_2$.

Although the gauging of $G\times G$ leads to an anomaly, there are subgroups which can be gauged to give a viable theory.

Definition 3.10. A subgroup $\iota: K \hookrightarrow G\times G$ is called *anomaly-free* if $(B\iota)^*(\mu_1 - \mu_2) \in E^6(BK)$ vanishes.

If K is anomaly-free, then it follows from Proposition 3.7 that the theory obtained by gauging the global symmetry group K is anomaly-free. Any subgroup of G embedded diagonally in $G\times G$ is clearly anomaly-free. For the subgroup $\iota: SU_2\times\{1\} \hookrightarrow G\times G$ the pullback $(B\iota)^*(\mu_1-\mu_2)$ is torsion of order two. Thus, even though not detected rationally, this subgroup is not anomaly-free. (Compare [**W1**, p. 431].)

4. The Wess-Zumino-Witten Term in the Low Energy Theory of Pions

The low energy dynamics of the pions is described by a σ-model with target $(G\times G)/G$, as explained in the introduction. Here, as before, $G = SU(N)$ with $N \geq 2$. The kinetic and mass terms have the usual form. Here we give a novel definition of the topological term in the action. As this Wess-Zumino-Witten term is only determined up to integer shifts, we work with its exponential in the exponentiated Euclidean action $e^{-S_{\mathrm{Eucl}}}$. The space of fields in the σ-model on a manifold X is $\mathcal{F}_X = \mathrm{Map}(X, G)$.

Definition 4.1. Let X be a closed spin 4-manifold. The *WZW factor* evaluated on $\phi : X \to G$ is

$$W_X(\phi) = \exp\left(2\pi i \int_X N_c\, \phi^*\check{\nu}\right). \tag{4.2}$$

Recall that $\check{\nu} \in \check{E}^5(G)$ is the differential E-class defined in Proposition 1.9(iv) and (1.15). Note that the integral $\int_X : \check{E}^5(X) \to \check{E}^1(\mathrm{point})$ takes values in $\mathbb{R}/\mathbb{Z}$, as follows immediately from (1.7). So $W_X(\phi)$ is a well-defined element of $\mathbb{C}$ with unit norm. The factor of N_c is put to match the high energy theory; see Proposition 4.17 below. Since $\check{\nu}$ is invariant under left and right translations by G, the WZW-factor (4.2) is also $(G \times G)$-invariant. If $X = \partial Z$ is the boundary of a compact *spin* 5-manifold Z, and $\phi : X \to G$ extends to $\Phi : Z \to G$, then Stokes' theorem for differential E-theory implies

$$W_X(\phi) = \exp\left(2\pi i N_c \int_Z \Phi^*\omega_{\check{\nu}}\right) = \exp\left(N_c \int_Z \frac{1}{240\pi^2}\,\mathrm{Tr}(\Phi^*\theta)^5\right), \tag{4.3}$$

which is the usual formula in the physics literature. (Recall that $\theta = g^{-1}dg$ is the Maurer-Cartan form on $G = SU_{N_f}$.) The signature of X is an obstruction to the existence of Z, so (4.3) cannot serve as a definition of the WZW factor.[13]

We proceed to verify properties (i)–(iv) from the introduction. To demonstrate that (4.2) depends on a spin structure, we compute the dependence explicitly.

Proposition 4.4. *Let $\delta \in H^1(X; \mathbb{Z}/2\mathbb{Z})$ be the difference between two isomorphism classes of spin structures on X with the same underlying orientation. Then the ratio of the WZW factors* (4.2) *computed with the two spin structures is ± 1 according to the value of*

$$N_c\, \delta \smile \phi^*\overline{h_3}[X] \quad \in \quad \mathbb{Z}/2\mathbb{Z}, \tag{4.5}$$

where $\overline{h_3}$ is the nonzero element of $H^3(G; \mathbb{Z}/2\mathbb{Z})$ and $[X] \in H_4(X)$ is the fundamental class.

Proof. Let e denote the Anderson dual theory [**HS**, Appendix B], [**FMS1**, Appendix B] to E. Then $e^0(\mathrm{point}) \cong \mathbb{Z}$, $e^{-1}(\mathrm{point}) \cong \mathbb{Z}/2\mathbb{Z}$, and all other groups vanish. The spectrum fits into a fibration

$$\Sigma H\mathbb{Z}/2\mathbb{Z} \longrightarrow e \longrightarrow H\mathbb{Z} \tag{4.6}$$

[13] However, the square $W_X(\phi)^2$ is expressed in terms of ordinary differential cohomology, and since $H_4(G) = 0$ it may be defined via an integral of a differential form over a bounding 5-chain.

whose k-invariant is the nontrivial map

$$Sq^2 \circ r : H\mathbb{Z} \to \Sigma^2 H\mathbb{Z}/2\mathbb{Z}, \tag{4.7}$$

the composition of the Steenrod square with reduction mod 2. The cohomology theory e is *multiplicative*; that is, e is a ring spectrum. This can be seen in several ways. First, the cohomology class represented by (4.7) is *primitive*, and so it follows that the fiber e is a ring spectrum. We can also identify e as a Postnikov truncation of connective ko-theory, and again it follows that e is a ring. More concretely, the zero space is the classifying space of the category of $\mathbb{Z}$-graded real lines, and the latter is a ring: addition is the tensor product of lines, and the multiplication of L_1, L_2 in degrees d_1, d_2 puts $L_1^{\otimes d_2} \otimes L_2^{\otimes d_1}$ in degree $d_1 d_2$. One can also see e as a truncation of the sphere spectrum, so identify its points as framed 0-manifolds; the sum is then disjoint union and the product is Cartesian product. It follows that E, the Anderson dual of e, is a module over the ring e.

Suppose $V \to X$ is a real vector bundle of rank N over a space X. Then (4.6) leads to the long exact sequence of cohomology groups

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^{N+1}(V;\mathbb{Z}/2\mathbb{Z})_{\text{cv}} & \xrightarrow{s} & e^N(V)_{\text{cv}} & \longrightarrow & H^N(V;\mathbb{Z})_{\text{cv}} & \xrightarrow{Sq^2\circ r} & H^{N+2}(V;\mathbb{Z}/2\mathbb{Z})_{\text{cv}} & \longrightarrow \cdots \\ & \cong \uparrow \overline{U} & & & & & & \cong \uparrow \overline{U} & \\ & H^1(X;\mathbb{Z}/2\mathbb{Z}) & & & & & & H^2(X;\mathbb{Z}/2\mathbb{Z}) & \end{array} \tag{4.8}$$

Here 'cv' denotes compact vertical supports and the vertical arrows are Thom isomorphisms. Assume V is oriented with Thom class $U \in H^N(V;\mathbb{Z})_{\text{cv}}$, and let $\overline{U} \in H^N(V;\mathbb{Z}/2\mathbb{Z})_{\text{cv}}$ be the mod 2 Thom class. Then $(Sq^2 \circ r)(U) = \overline{U} w_2(V)$, where $w_2(V) \in H^2(X;\mathbb{Z}/2\mathbb{Z})$ is the second Stiefel-Whitney class. Hence, a spin structure on V—a trivialization of $w_2(V)$—induces a lift $U_e \in e^N(V)_{\text{cv}}$ of U, a Thom class in e-theory. Lifts differ by $\overline{U}\delta$ for $\delta \in H^1(X;\mathbb{Z}/2\mathbb{Z})$.

Turning to the proposition, let $\pi : V \to X$ be the normal bundle to an embedding $X \hookrightarrow \mathbb{R}^{N+4}$, let U_e be the e-Thom class for some spin structure, and let $\check{U}_e$ be a lift to a differential Thom class in e-theory [**HS**]. The integral of a class $\check{\beta} \in \check{E}^5(X)$ is computed as the product

$$\check{U}_e \cdot \pi^* \check{\beta} \quad \in \quad \check{E}^{N+5}(\mathbb{R}^{N+4}) \cong \mathbb{R}/\mathbb{Z}.$$

If $\delta \in H^1(X;\mathbb{Z}/2\mathbb{Z})$ is a change of spin structure, then U_e changes by $s(\overline{U}\delta)$ in the sequence (4.8). Now there is an inclusion

$$e^{N-1}(V;\mathbb{R}/\mathbb{Z})_{\text{cv}} \xhookrightarrow{f} \check{e}^N(V)_{\text{cv}}$$

of "flat" elements in differential e-theory. Smash (4.6) with the Moore space for $\mathbb{R}/\mathbb{Z}$ to construct a short exact sequence[14]

$$0 \longrightarrow H^{N+1}(V;\mathbb{Z}/2\mathbb{Z})_{\mathrm{cv}} \xrightarrow{s_{\mathbb{R}/\mathbb{Z}}} e^{N-1}(V;\mathbb{R}/\mathbb{Z})_{\mathrm{cv}} \longrightarrow H^{N-1}(V;\mathbb{Z})_{\mathrm{cv}} \longrightarrow 0.$$

The change in spin structure shifts the differential Thom class $\check{U}_e$ by the image of $\overline{U}\delta$ under the composition $f\circ s_{\mathbb{R}/\mathbb{Z}}$. Then the change in the product $\check{U}_e\cdot\pi^*\check{\beta}$ depends only on the image $j(\beta)$ of $\check{\beta}$ under $\check{E}^5(X)\longrightarrow E^5(X)\xrightarrow{j} H^3(X;\mathbb{Z}/2\mathbb{Z})$: it is $f\bigl(\overline{U}\delta\cdot\pi^*j(\beta)\bigr)\in\check{E}^{N+5}(\mathbb{R}^{N+4})\cong\mathbb{R}/\mathbb{Z}$. We claim that this equals

$$\delta\smile j(\beta) \quad \in \quad H^4(X;\mathbb{Z}/2\mathbb{Z})\cong\mathbb{Z}/2\mathbb{Z}\cong\tfrac{1}{2}\mathbb{Z}/\mathbb{Z}\subset\mathbb{R}/\mathbb{Z},$$

which leads immediately to (4.5).

The claim amounts to showing that the composition $\Sigma^2 H\mathbb{Z}/2\mathbb{Z}\wedge E\longrightarrow e_{\mathbb{R}/\mathbb{Z}}\wedge E\longrightarrow E_{\mathbb{R}/\mathbb{Z}}$ is nonzero. By shifting the Moore space for $\mathbb{R}/\mathbb{Z}$ in the wedge, this is equivalent to showing that the module map $e\wedge E_{\mathbb{R}/\mathbb{Z}}\to E_{\mathbb{R}/\mathbb{Z}}$ induces the nonzero map $\pi_1 e\otimes\pi_{-1}E_{\mathbb{R}/\mathbb{Z}}\to\pi_0 E_{\mathbb{R}/\mathbb{Z}}$. But since $E_{\mathbb{R}/\mathbb{Z}}$ is the Pontrjagin dual of e, this is obtained by applying $\mathrm{Hom}(-,\mathbb{R}/\mathbb{Z})$ to the multiplication $\pi_1 e:\pi_0 e\to\pi_1 e$, and the latter is nonzero. q.e.d.

We now turn to property (ii) in the introduction. Let Y be a compact spin 3-manifold, and consider canonical quantization of the σ-model on Y. The space of classical solutions of the σ-model on $\mathbb{R}\times Y$ with its Lorentz metric is the space of solutions to a wave equation, which (at least formally) is the space of Cauchy data. The latter is identified with the tangent bundle of $\mathrm{Map}(Y,G)$. It carries a symplectic structure, and the zero-section $\mathrm{Map}(Y,G)$ is lagrangian. Without the WZW factor, the Hilbert space would, at least formally, be the space of L^2 functions on $\mathrm{Map}(Y,G)$. In particular, it would only consist of bosonic states. The WZW factor changes the symplectic structure of $T\mathrm{Map}(Y,G)$ in a geometric manner: the normalized curvature of a hermitian line bundle with covariant derivative pulled back from $\mathrm{Map}(Y,G)$ is added. Let

$$e:\mathrm{Map}(Y,G)\times Y\longrightarrow G$$

be the evaluation map. Then the isomorphism class of this line bundle may be written

$$\int_Y N_c\, e^*\check{\nu} \quad \in \quad \check{E}^2\bigl(\mathrm{Map}(Y,G)\bigr). \tag{4.9}$$

The important point for us is that $\check{E}^2$ parametrizes $\mathbb{Z}/2\mathbb{Z}$-*graded* line bundles with covariant derivative. The quantum Hilbert space is now the space of sections of this line bundle, so it too is $\mathbb{Z}/2\mathbb{Z}$-graded. The $\mathbb{Z}/2\mathbb{Z}$-grading of the quantum Hilbert space reflects statistics of states:

[14]The degree shift in the first term is explained with chain complexes: $\mathbb{Z}\to\mathbb{R}$ with $\mathbb{Z}$ in degree -1 is quasi-isomorphic to $\mathbb{R}/\mathbb{Z}$ in degree zero.

even degree states are bosonic and odd degree states are fermionic. More precisely, the grading of (4.9) at $\phi \in \mathrm{Map}(Y, G)$ is

$$\int_Y N_c\, \phi^* j(\nu) \equiv \int_Y N_c\, \phi^* h_3 \equiv N_c \deg_2(\phi) \quad \in \quad \mathbb{Z}/2\mathbb{Z}, \tag{4.10}$$

where $\deg_2(\phi)$ is the mod 2 degree of $\phi : Y \to G$, the homology class of ϕ in $H_3(G; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. (Recall that $h_3 \pmod 2$ is the generator of $H^3(G; \mathbb{Z}/2\mathbb{Z})$; cf. (1.10).) In physical terms the integer degree is identified with the *baryon number*, and (4.10)—the statistics formula for solitons which is an immediate consequence of Definition 4.1—matches the formula derived from physics [**W2**].

Property (iii) of the introduction is evident: our definition works for $G = SU(2)$ since $\check{\nu}$ is a nonzero element of order two and (4.10) is still valid. So the WZW factor encodes the statistics of solitons.

The remainder of this section is devoted to property (iv). To that end, we construct an extension of the σ-model with WZW factor (4.2) when the global $(G \times G)$ symmetry is gauged. (See the discussion preceding Proposition 3.7 for generalities on gauging.) As the gauged extensions of the kinetic and mass terms are straightforward and anomaly-free, we only consider the WZW factor. The fields $\widetilde{\mathcal{F}}_X$ in the gauged theory on a Riemannian spin 4-manifold X are a connection Θ on a principal $(G \times G)$-bundle $Q \to X$ and a $(G \times G)$-equivariant map $\phi : Q \to G$, i.e., a section ϕ of the associated fiber bundle $G_Q = Q \times_{(G\times G)} G \to X$ with fiber G. Note the fibering $\widetilde{\mathcal{F}}_X \to \mathcal{A}_X^{(G\times G)}$ as required by the general theory. If the class $\check{\nu} \in \check{E}^5(G)$ extended to a class in $\check{E}^5(G_Q)$, then (4.2) would be valid with this extended class replacing $\check{\nu}$, and there would be an anomaly-free gauging of the WZW factor. But such an extension does not exist, and so the gauged theory is more subtle—and anomalous.

Even the topological class $\nu \in E^5(G)$ does not extend, and to measure the obstruction we work universally. Let $\mathcal{E} \to B(G \times G) = BG \times BG$ be the fiber bundle

$$\pi : \mathcal{E} = G_{E(G\times G)} = E(G \times G)_{(G\times G)} G \longrightarrow B(G \times G). \tag{4.11}$$

The bundle $\mathcal{E}$ is the quotient of $EG \times EG$ by the diagonal G-action, so it is homotopy equivalent to BG; the projection to $BG \times BG$ is homotopy equivalent to the diagonal $BG \to BG \times BG$. Recall that ordinary cohomology is defined in terms of *cochains* and a *differential*. A *cocycle* is a cochain whose differential vanishes, and the cohomology is the quotient of cocycles by differentials of cochains. The entire theory is $\mathbb{Z}$-graded and the differential has degree one. In generalized cohomology there are analogous notions, and for now we simply call them 'E-cochains', etc.; in §5 we give proper definitions.

Theorem 4.12. *There is an E-cochain α of degree* 5 *on the total space $\mathcal{E}$ of* (4.11) *which satisfies:*

(i) *the restriction of α to a fiber has zero E-differential and represents $\nu \in E^5(G)$;*
(ii) *the E-differential of α is the pullback of an E-cocycle of degree* 6 *on $B(G \times G)$ which represents $\mu_1 - \mu_2 \in E^6\big(B(G \times G)\big)$.*

We term α a *transgressing E-cochain* and say that ν transgresses to $\mu_1 - \mu_2$. Transgression in ordinary cohomology is related to the Leray-Serre spectral sequence [**BT**, §18], but that tool is not available for E-cohomology. Theorem 4.12 has a restatement in terms of Borel equivariant E-cohomology: the E-cohomology class $\nu \in E^5(G)$ does *not* have an equivariant extension to $E^5_{G\times G}(G)$. The existence of such an extension is obstructed by $\mu_1 - \mu_2 \in E^6_{G\times G}(\text{point})$. Then α may be regarded as a $(G \times G)$-equivariant E-cochain on G. We defer the proof of Theorem 4.12 and further discussion to §5.

To define the gauged WZW factor we need an extension of Theorem 4.12 to differential E-theory.

Theorem 4.13. *There is an $\check{E}$-cochain $\check{\alpha}$ of degree* 5 *on the total space $\mathcal{E}$ of* (4.11) *which satisfies:*

(i) *the restriction of $\check{\alpha}$ to a fiber has zero $\check{E}$-differential and represents $\check{\nu} \in \check{E}^5(G)$;*
(ii) *the $\check{E}$-differential of $\check{\alpha}$ is the pullback of an $\check{E}$-cocycle of degree* 6 *on $B(G \times G)$ which represents $\check{\mu}_1 - \check{\mu}_2 \in \check{E}^6\big(B(G \times G)\big)$.*

The proof is in §5. The main idea in Theorem 4.13 beyond Theorem 4.12 is that the differential form $\omega_{\check{\mu}}$ is a transgression of $\omega_{\check{\nu}}$ in the universal bundle $EG \to BG$. The transgressing form is a 5-form on EG, the Chern-Simons form. We then construct a 5-form (5.16) on $\mathcal{E}$ whose de Rham differential is the pullback of $\omega_{\check{\mu}_1} - \omega_{\check{\mu}_2} \in \Omega^6\big(B(G \times G)\big)$. In fact, there are different equivalence classes of transgressing $\check{E}$-cochains $\check{\alpha}$ which differ by the inclusion of a topological term in the gauged WZW model.

We need one more maneuver to define the gauged WZW factor. Recall that $\mathcal{A}_X^{(G\times G)}$ is a stack represented by the groupoid $\mathcal{G}_1$ in which an object (Q, Θ) is a connection Θ on a principal $(G \times G)$-bundle $Q \to X$. A morphism $(Q, \Theta) \to (Q', \Theta')$ is a $G \times G$-equivariant map $\varphi : Q \to Q'$ such that $\varphi^*\Theta' = \Theta$. We replace $\mathcal{G}_1$ by an equivalent groupoid $\mathcal{G}_2$; it also represents the stack $\mathcal{A}_X^{(G\times G)}$. An object (Q, Θ, γ) in $\mathcal{G}_2$ is a triple where $\gamma : Q \to E(G \times G)$ is a $(G \times G)$-equivariant map which classifies the connection Θ on Q. In other words, $\gamma^*\Theta^{\text{univ}} = \Theta$, where Θ^{univ} is the universal connection on $E(G \times G) \to B(G \times G)$. (As in the construction at the end of §1, we can avoid the condition $\gamma^*\Theta^{\text{univ}} = \Theta$ by including a Chern-Simons term.) Morphisms in $\mathcal{G}_2$ are described at the end of §5. The important point is that the space of classifying maps γ for fixed (Q, Θ) is contractible and nonempty. In essence, we adjoin this contractible choice as a new field and posit a symmetry that makes it inessential (auxiliary).

A field in the gauged σ-model is, therefore, a principal $(G\times G)$-bundle $Q \to X$ with connection Θ, a classifying map $\gamma : Q \to E(G\times G)$ for Θ, and a section ϕ of the associated bundle $G_Q \to X$ with fiber G. The classifying map γ induces a classifying map $\tilde{\gamma} : G_Q \to \mathcal{E}$.

Definition 4.14. The gauged WZW factor is

$$\widetilde{W}_X(Q, \Theta, \gamma, \phi) = \exp\left(2\pi i \int_X N_c\; \phi^*\tilde{\gamma}^*\check{\alpha}\right). \tag{4.15}$$

The $\check{E}$-cochain $\tilde{\gamma}^*\check{\alpha}$ on G_Q is the gauged extension of the cocycle $\check{\nu}$ on G, so (4.15) is a natural generalization of (4.2). We discuss the well-definedness of this definition at the end of §5.

To analyze this definition we work in smooth families. Let $\mathcal{X} \to S$ be a Riemannian spin fiber bundle with compact 4-manifolds as fibers, $\mathcal{Q} \to \mathcal{X}$ a principal $(G\times G)$-bundle with connection Θ, $\gamma : \mathcal{Q} \to E(G\times G)$ a classifying map for the connection, and ϕ a section of $G_{\mathcal{Q}} \to \mathcal{X}$. The gauged WZW factor is now an integral over the fibers

$$\widetilde{W}_X(\mathcal{Q}, \Theta, \gamma, \phi) = \exp\left(2\pi i \int_{\mathcal{X}/S} N_c\; \phi^*\tilde{\gamma}^*\check{\alpha}\right), \tag{4.16}$$

and the result is an $\check{E}$-cochain on S of degree 1. Its $\check{E}$-differential is an $\check{E}$-cocycle of degree 2, so it represents a class in $\check{E}^2(S)$. Now, from Theorem 4.13(ii), the $\check{E}$-differential of the integrand represents the $\check{E}$-cohomology class $N_c\,\bar{\gamma}^*(\check{\mu}_1 - \check{\mu}_2)$, and so by Stokes' theorem the $\check{E}$-differential of (4.16) is

$$\int_{\mathcal{X}/S} N_c\,\bar{\gamma}^*(\check{\mu}_1 - \check{\mu}_2) = \int_{\mathcal{X}/S} N_c\big(\check{\mu}_1(\mathcal{Q}) - \check{\mu}_2(\mathcal{Q})\big),$$

in terms of the differential characteristic class $\check{\mu}$ defined around (1.16). Recall that an $\check{E}$-cocycle of degree 2 may be represented as a $\mathbb{Z}/2\mathbb{Z}$-graded hermitian line bundle with unitary covariant derivative. Then a $\check{E}$-cochain of degree 1 whose differential is that cocycle may be represented as a not-necessarily-flat section of this bundle of unit norm. That section (4.15) is part of the gauged σ-model action, and therefore the line bundle is the anomaly. This proves the following.

Proposition 4.17. *The anomaly in the gauged σ-model with WZW factor is*

$$\int_{\mathcal{X}/S} N_c\big(\check{\mu}_1(\mathcal{Q}) - \check{\mu}_2(\mathcal{Q})\big). \tag{4.18}$$

The agreement of (3.8) and (4.18) is the 't Hooft anomaly matching. In terms of the stack of (bosonic) fields, which is a fibering $\widetilde{\mathcal{F}}_X \to \mathcal{A}_X^{(G\times G)}$ over the stack of $(G\times G)$-connections, the classical action with gauged WZW factor is a section of a line bundle over $\widetilde{\mathcal{F}}_X$, and that line bundle is pulled back from $\mathcal{A}_X^{(G\times G)}$. It is the anomaly in the gauged σ-model, and its isomorphism class is computed by Proposition 4.17. This completes the verification of property (iv) of the introduction.

The existence of a section ϕ of $G_{\mathcal{Q}} \to \mathfrak{X}$ implies a topological restriction on the $G\times G$ bundle $\mathcal{Q} \to \mathfrak{X}$, namely that the two constituent G-bundles $\mathcal{Q}_1 \to \mathfrak{X}$ and $\mathcal{Q}_2 \to \mathfrak{X}$ be isomorphic as topological principal bundles. (A section of the associated $(G\times G)/G$ bundle is equivalent to a reduction of structure group of $\mathcal{Q}$ to the diagonal $G \subset G\times G$.) This is precisely the condition in QCD to define the fermion bilinear; see the discussion preceding (3.9). It also implies that the *topological* anomaly in Proposition 4.17 vanishes, but there may still be a *geometric* anomaly: the G bundles $\mathcal{Q}_1, \mathcal{Q}_2$ need not be isomorphic as bundles with connection.

5. Transgression

We recommend [**DK**, §6] as an introduction to the topology used in this section.

The cochains used to define ordinary cohomology are replaced in a generalized cohomology theory E by maps into the representing spectrum $\{E_n\}$. Recall that E_n is a pointed topological space and the spectrum comes equipped with maps $\Sigma E_n \to E_{n+1}$. An 'E-cocycle' (as used in §4) of degree n on a topological space X is simply a map $X \to E_n$. Homotopic maps are considered equivalent, and the E-cohomology group $E^n(X)$ is the set of homotopy classes (1.1). An 'E-cochain' of degree n on X is a *based* map

$$CX \longrightarrow E_{n+1} \tag{5.1}$$

from the *unreduced* cone[15] on X; its 'E-differential' is the restriction to $X \subset CX$. If that restriction is trivial—maps to the basepoint $*$ of E_{n+1}—then (5.1) factors to a based map $\Sigma X \to E_{n+1}$, and by adjunction it is equivalent to a map $X \to \Omega E_{n+1} \simeq E_n$, and so represents a class in $E^n(X)$. If $A \subset X$ is a subspace, then a class in the relative cohomology group $E^n(X, A)$ is represented by a map $X \cup CA \to E_n$.

We now define transgression in generalized cohomology.

[15] The unreduced cone on X is the pointed space

$$CX = [0,1]\times X \;/\; \{1\}\times X$$

with basepoint $\{1\}\times X$. Note $X \subset CX$ as $\{0\}\times X$ and CX/X is the *unreduced* suspension ΣX. As X does not have a basepoint, the notation for suspension is unambiguous.

Definition 5.2. Let $F \xrightarrow{i} \mathcal{E} \xrightarrow{\pi} B$ be a fibration. Then $\nu \in E^n(F)$ is *related by transgression* to $\mu \in E^{n+1}(B)$ if there exists $\sigma \in E^{n+1}(\mathcal{E}, F)$ and $\mu_0 \in E^{n+1}(B, b_0)$ such that under the maps

$$E^n(F) \xrightarrow{\delta} E^{n+1}(\mathcal{E}, F) \xleftarrow{\pi^*} E^{n+1}(B, b_0) \xrightarrow{j} E^{n+1}(B), \tag{5.3}$$

we have $\sigma = \delta(\nu) = \pi^*(\mu_0)$ and $\mu = j(\mu_0)$.

All spaces are pointed; the basepoint of B is b_0, and $\pi^{-1}(b_0) = F$ is the fiber of π. The map δ is the connecting homomorphism in the long exact sequence of the pair $(\mathcal{E}, F)$, and j is the homomorphism which forgets the basepoint.[16] The set of transgressive elements in $E^n(F)$ forms a subgroup, and transgression is only well-defined into a quotient of $E^{n+1}(B)$. The relation of Definition 5.2 with the description in Theorem 4.12 is the following.

Lemma 5.4. *Let $F \xrightarrow{i} \mathcal{E} \xrightarrow{\pi} B$ be a fibration. Then $\nu \in E^n(F)$ is related by transgression to $\mu \in E^{n+1}(B)$ if and only if there exists a map $\alpha : C\mathcal{E} \to E_{n+1}$ such that*

(i) *the restriction of α to $F \subset CF \subset C\mathcal{E}$ is trivial and the restriction of α to CF factors through a map $\Sigma F \to E_{n+1}$ which represents ν;*
(ii) *the restriction of α to $\mathcal{E} \subset C\mathcal{E}$ is the pullback under π of a based map $B \to E_{n+1}$ which represents μ.*

Proof. The class σ in Definition 5.2 is represented by a map $c : \mathcal{E} \cup CF \to E_{n+1}$. Because $\sigma = \delta(\nu)$, it extends to $\tilde{c} : C\mathcal{E} \cup CF \simeq \Sigma F \to E_{n+1}$, and the extension represents ν. Also, because $\sigma = \pi^*(\mu_0)$, the restriction of c to $\mathcal{E}$ is pulled back from a based map $G \to E_{n+1}$, and in particular is trivial on $F \subset \mathcal{E}$. By a homotopy we can assume that c is trivial on CF. Then the restriction $\alpha : C\mathcal{E} \to E_{n+1}$ of $\tilde{c}$ to $C\mathcal{E}$ satisfies (i) and (ii). The converse is proved by gluing the trivial map on CF to α. q.e.d.

In our application the inclusion $i : F \to \mathcal{E}$ is null homotopic, and it is convenient to specify a null homotopy as follows.

Lemma 5.5. *Let $F \xrightarrow{i} \mathcal{E} \xrightarrow{\pi} B$ be a fibration, $\varphi : F' \to F$ a continuous map, and $H : [0,1] \times F' \to \mathcal{E}$ a null homotopy of $i \circ \varphi$. Then a section $s : F \to F'$ of φ determines a null homotopy $H \circ s$ of i. Furthermore, the map $\pi : (\mathcal{E}, F) \to (B, b_0)$ is homotopy equivalent to the based map*

$$\pi \vee (\pi \circ H \circ s) : \mathcal{E} \vee \Sigma F \longrightarrow B. \tag{5.6}$$

Some definitions: H is a null homotopy of $i \circ f$ means $H_0 = i \circ \varphi$ and H_1 maps to the basepoint of $\mathcal{E}$. The section s satisfies $\varphi \circ s = \mathrm{id}_F$. The

[16] $E^\bullet(B, b_0)$ is the *reduced* E-cohomology of B, often denoted $\widetilde{E}^\bullet(B)$.

wedge $X \vee Y$ of pointed spaces X, Y is the union along the basepoints. The map $\pi \circ H \circ s : \Sigma F \to B$ sends $(t, f) \in \Sigma F$ to $(\pi \circ H_t \circ s)(f)$. The proof of Lemma 5.5 is straightforward.

We apply these ideas first to the classifying space of $G = SU_N$, $N \geq 2$. Now 'E' denotes the specific cohomology theory described in §1.

Proposition 5.7. *The classes* $\nu \in E^5(G)$ *and* $\mu \in E^6(BG)$ *are related by transgression in the universal fibration* $G \xrightarrow{i} EG \xrightarrow{\pi} BG$.

Proof. Let $\varphi = s : G \to G$ be the identity map, K a null homotopy of id_{EG}, and $H = K \circ i \circ \varphi$. The map δ in (5.3) is obtained by applying $[\,\cdot\,, E_6]$ to the inclusion $\Sigma F \to \mathcal{E} \vee \Sigma F$. (Recall that $[X, Y]$ is the set of homotopy classes of maps $X \to Y$.) Set

$$\psi = \pi \circ H \circ s : \Sigma G \longrightarrow BG \qquad (5.8)$$

to be the map in (5.6). Notice that j in (5.3) is an isomorphism for $n = 5$. Therefore, we must prove that under $\psi_E^* : E^6(BG) \to E^6(\Sigma G) \cong E^5(G)$, we have $\psi_E^*(\mu) = \nu$. In the commutative diagram

$$\begin{array}{ccccc}
H^6(BG) & \xrightarrow{\psi_H^*} & H^6(\Sigma G) & \xleftarrow{\cong} & H^5(G) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\
E^6(BG) & \xrightarrow{\psi_E^*} & E^6(\Sigma G) & \xleftarrow{\cong} & E^5(G) \\
\downarrow{\scriptstyle k} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle k} \\
H^6(BG) & \xrightarrow{\psi_H^*} & H^6(\Sigma G) & \xleftarrow{\cong} & H^5(G)
\end{array}$$

all groups are infinite cyclic, the maps i are isomorphisms, and $k \circ i$ is multiplication by 2. Parts (ii) and (iv) of Proposition 1.9 reduce us to showing $\psi_H^*(c_3) = h_5$, which is a standard transgression in the theory of characteristic classes in ordinary cohomology. q.e.d.

We remark that the map (5.8) classifies the bundle on ΣG whose clutching function is $\mathrm{id}_G : G \to G$; this is clear from its definition as $\pi \circ H \circ s = \pi \circ K \circ i$. It is also the first stage of the Milnor construction of BG.

Proof of Theorem 4.12. From Lemma 5.4, to produce the desired map $\alpha : C\mathcal{E} \to E_{n+1}$ it suffices to show that ν and $\mu_1 - \mu_2$ are related by transgression in the fibration (4.11). Consider the commutative diagram (5.9)

$$\begin{array}{ccccccc} & & & E(G\times G) & & & \\ & \overset{H'}{\nearrow} & & \downarrow q & \overset{\pi'}{\searrow} & & \\ G\times G & \xrightarrow{\varphi} & (G\times G)\big/G & \xrightarrow{i} & E(G\times G)\big/G & \xrightarrow{\pi} & E(G\times G)\big/G\times G \\ & \nwarrow s & \downarrow \chi & & \| & & \| \\ & & G & \xrightarrow{i} & \mathcal{E} & \xrightarrow{\pi} & BG\times BG \end{array}$$

In this diagram φ and q are quotient maps; $s(g) = (g, e)$, where $e \in G$ is the identity element; the diffeomorphism χ is given as $\chi(g_1, g_2) = g_1 g_2^{-1}$, which is well-defined on left cosets of G; and H' is the null homotopy obtained from a null homotopy $K \times K$ of $EG \times EG$ (see the proof of Proposition 5.7 above). Then $H = q \circ H'$ is a null homotopy of $i \circ \varphi$. It follows that $\pi \circ H \circ s : \Sigma G \to BG \times BG$ in (5.6) is $\psi \times *$, where ψ is the map defined in (5.8) and $*$ is the constant map to the basepoint. Note that $\mathcal{E} \simeq BG$ and π is homotopy equivalent to the diagonal $\Delta : BG \to BG \times BG$. From the proof of Proposition 5.7 we see that $\mu_1 - \mu_2 \in E^6(BG \times BG)$ pulls back to $\nu \in E^5(G) \cong E^6(\Sigma G) \subset E^6(\mathcal{E} \vee \Sigma G)$ under

$$\mathcal{E} \vee \Sigma G \xrightarrow{\simeq} BG \vee \Sigma G \xrightarrow{\Delta \vee (\psi \times *)} BG \times BG ,$$

and this completes the proof. q.e.d.

As a preliminary to proving Theorem 4.13 we discuss some generalities about differential cohomology [**HS**, §4]. Let $\big(C^\bullet(X;\mathbb{R}), \delta\big)$ denote the singular cochain complex of a space X. In the differential theory, we need to fix cocycles $\iota_n \in C^n(E_n;\mathbb{R})$ which represent the map $\frac{1}{2}k : E_n \to H\mathbb{R}_n$; see (1.4). These cocycles must satisfy the compatibility condition $s^*\iota_{n+1} \big/ Z_{S^1} = \iota_n$, where s is the composition $S^1 \times E_n \to \Sigma E_n \to E_{n+1}$ and Z_{S^1} is a fixed fundamental cycle, paired with $s^*\iota_{n+1}$ by slant product.

Definition 5.10. A *differential E-function of degree n* on a smooth manifold M is a triple $\check{\beta} = (\beta, h, \omega)$ consisting of a continuous map $\beta : M \to E_n$, a *closed* differential form ω of degree n, and a cochain $h \in C^{n-1}(M;\mathbb{R})$ which satisfies

$$\delta h = \omega - \beta^*\iota_n. \tag{5.11}$$

In (5.11), the differential form ω—termed the curvature of $\check{\beta}$—is regarded as a singular cocycle by integration. A differential E-function was termed an '$\check{E}$-cocycle' in §4. A homotopy (or morphism) is a differential E-function of degree n on $[0,1] \times M$ whose curvature is pulled back from M under projection.[17] The set of equivalence classes under the induced equivalence relation is the differential E-cohomology $\check{E}^n(M)$. Note that ω in $\check{\beta} = (\beta, h, \omega)$ is an invariant of the class of $\check{\beta}$ in $\check{E}^n(M)$. Analogous definitions apply to any generalized cohomology theory E. The fundamental cocycles ι_n then have coefficients in the vector space $E^0(\mathrm{point}; \mathbb{R})$.

To discuss transgression we also need a notion of '$\check{E}$-cochain'.

Definition 5.12. A *coned differential E-function of degree n* on a smooth manifold M is a triple $\check{\alpha} = (\alpha, k, \eta)$ consisting of a continuous map $\alpha : CM \to E_{n+1}$, a differential form $\eta \in \Omega^n(M)$, and a cochain $k \in C^{n-1}(M; \mathbb{R})$. It *trivializes* the differential E-function $\check{\beta} = (\beta, h, \omega)$ of degree $n+1$, where

$$\begin{aligned} \beta &= \alpha\big|_M \\ \omega &= d\eta \\ h &= \eta - \alpha^*\iota_{n+1} \big/ Z_{[0,1]} - \delta k. \end{aligned} \tag{5.13}$$

In §4 we termed $\check{\beta}$ the '$\check{E}$-differential' of $\check{\alpha}$. The slant product in (5.13) is computed after pullback by the collapse $[0,1] \times M \to CM$, and the cycle $Z_{[0,1]}$ pushes to Z_{S^1} under the collapse $[0,1] \to S^1$. A homotopy is a coned differential E-function (A, K, Ξ) of degree n on $[0,1] \times M$ such that the restriction of A to $[0,1] \times M \subset [0,1] \times CM$ is constant, Ξ is pulled back from M under projection, and $H \in C^n\big([0,1] \times M; \mathbb{R}\big)$, defined as (see (5.13))

$$H = \Xi - A^*\iota_{n+1} \big/ Z_{[0,1]} - \delta K, \tag{5.14}$$

is also pulled back from M under projection. The set of equivalence classes of coned differential E-functions that trivialize a fixed differential E-function of degree $n+1$ is a torsor for $\check{E}^n(M)$.

Integration of differential functions is defined in [**HS**, §4.10]. Our definition (4.16) of the WZW factor requires that integration be extended to *coned* differential functions as well. The main step is simply extending the orientation map, called 'μ' in [**HS**, §4.10], to the path spaces in the spectrum.[18]

Proof of Theorem 4.13. We must construct the coned differential E-function $\check{\alpha} = (\alpha, k, \eta_{\mathcal{E}})$ on $\mathcal{E}$. The function $\alpha : C\mathcal{E} \to E_6$ is the topological transgressing 'E-cochain' in Theorem 4.12. We take $k = 0$, but discuss other possibilities following the proof. Our main task is to construct a transgressing form $\eta_{\mathcal{E}} \in \Omega^5(\mathcal{E})$.

[17] In the language of [**HS**] the condition on the curvature is captured by a filtration on the space of differential functions: $\mathrm{filt}_0(E;\iota)^M$ is a category whose set of isomorphism classes $\pi_0\,\mathrm{filt}_0(E;\iota)^M$ is $\check{E}^n(M)$.

[18] A based map $CX \to E_{n+1}$ is by adjunction a map $X \to PE_{n+1}$ to the space of paths beginning at the basepoint of E_{n+1}.

Recall first the Chern-Simons [**CS**] transgressing form in the universal bundle $G \to EG \to BG$. Denote by $\langle \cdot, \cdot, \cdot \rangle$ the symmetric trilinear form (3.3) on $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\Theta = \Theta^{\mathrm{univ}} \in \Omega^1_{EG}(\mathfrak{g})$ be the universal connection on EG, and $\Omega \in \Omega^2_{EG}(\mathfrak{g})$ its curvature. In this notation the 6-form in (1.14), lifted to EG, is

$$\pi^* \omega_{\check{\mu}} = \langle \Omega \wedge \Omega \wedge \Omega \rangle.$$

Then the Chern-Simons 5-form

$$\eta = \langle \Theta \wedge \Omega \wedge \Omega \rangle - \frac{1}{4} \langle \Theta \wedge \Omega \wedge [\Theta \wedge \Theta] \rangle + \frac{1}{40} \langle \Theta \wedge [\Theta \wedge \Theta] \wedge [\Theta \wedge \Theta] \rangle \tag{5.15}$$

satisfies the transgression conditions

(i) the restriction of η to a fiber equals $\omega_{\check{\nu}}$; and

(ii) $d\eta = \pi^* \omega_{\check{\mu}}$.

In other words, η is a transgressing 5-form for the curvatures in Proposition 5.7.

We now want to construct a transgressing 5-form $\eta_{\mathcal{E}}$ in the fiber bundle $\mathcal{E} \xrightarrow{\pi} BG \times BG$ with fiber $(G \times G)/G \cong G$. Recall that $\mathcal{E} = (EG \times EG)/G$. We write the pullback of $\eta_{\mathcal{E}}$ to $EG \times EG$, and express it in terms of the universal connection forms Θ_1, Θ_2 and universal curvature forms Ω_1, Ω_2 on $EG \times EG$, but where these forms are regarded as $\mathfrak{g}$-valued by identifying $\mathfrak{g}_1 \cong \mathfrak{g}_2 \cong \mathfrak{g}$. Write η_1 for (5.15) with Θ_1, Ω_1 replacing Θ, Ω and similarly for η_2. Then

$$\eta_{\mathcal{E}} = \eta_1 - \eta_2 + d\tau, \tag{5.16}$$

where

$$\tau = \frac{1}{4} \langle \Theta_1 \wedge \Theta_2 \wedge [\Theta_1 \wedge \Theta_1] \rangle + \frac{1}{4} \langle \Theta_1 \wedge \Theta_2 \wedge [\Theta_2 \wedge \Theta_2] \rangle - \frac{1}{4} \langle \Theta_1 \wedge \Theta_2 \wedge [\Theta_1 \wedge \Theta_2] \rangle \\ - \langle \Theta_1 \wedge \Theta_2 \wedge \Omega_1 \rangle - \langle \Theta_1 \wedge \Theta_2 \wedge \Omega_2 \rangle.$$

The form $\eta_{\mathcal{E}}$ satisfies

(i) $\eta_{\mathcal{E}}$ is basic for the diagonal G-action on $EG \times EG$;

(ii) the restriction of $\eta_{\mathcal{E}}$ to a fiber of $\pi : \mathcal{E} \to BG \times BG$ is cohomologous to $\omega_{\check{\nu}}$; and

(iii) $d\eta_{\mathcal{E}} = \pi^*(\omega_{\check{\mu}_1} - \omega_{\check{\mu}_2})$.

We leave the reader the exhausting task of checking these properties. It helps to observe that the pullback of $\omega_{\check{\nu}}$ under the diffeomorphism $\chi : (G \times G)/G \to G$ in (5.9) is the restriction of $\eta_{\mathcal{E}}$ to a fiber. q.e.d.

The choice of $k = 0$ in $\check{\alpha} = (\alpha, k, \eta_{\mathcal{E}})$ fixes the differential function representing $\check{\mu}_1 - \check{\mu}_2$ to be the triple $\check{\beta} = (\alpha\,\big|_{\mathcal{E}}\ ,\ \mathcal{E} - \alpha^* \iota_6 \big/ Z_{[0,1]}\ ,\ \omega_{\check{\mu}_1} - \omega_{\check{\mu}_2})$; see (5.13). Any other choice of $k \in C^4(\mathcal{E}; \mathbb{R})$ such that $\check{\alpha}$ trivializes $\check{\beta}$

must satisfy $\delta k = 0$, so it determines a class $[k] \in H^4(\mathcal{E};\mathbb{R})$. Since equivalence classes of trivializations of $\check{\beta}$ is a torsor for $\check{E}^5(\mathcal{E})$—see the remark following (5.14)—only its image in $H^4(\mathcal{E};\mathbb{R}/\mathbb{Z})$ is relevant. In other words, equivalence classes of possible $\check{\alpha}$ are parametrized by elements of $H^4(\mathcal{E};\mathbb{R}/\mathbb{Z}) \cong H^4(BG;\mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$. Write $[k] = \frac{\theta}{N_c} c_2$, where $c_2 \in H^4(BG)$ is the generator and $\theta \in \mathbb{R}$ is determined modulo $N_c\mathbb{Z}$. Recall from the end of §4 that the $(G \times G)$-bundle $Q \to X$ in (4.15) has a reduction to a G-bundle $\overline{Q} \to X$. Then a nonzero cohomology class $[k]$ multiplies the WZW factor (4.15) by

$$\exp\left(2\pi i\,\theta \int_X c_2(\overline{Q})\right).$$

This matches the topological term (3.9) in gauged QCD with $\theta = \theta_1 + \theta_2$.

To conclude, we sketch an argument proving that the gauged WZW factor $\widetilde{W}_X$ in Definition 4.14 is well-defined. It is a function of a quartet $q = (Q,\Theta,\gamma,\phi)$ which is an object in a groupoid $\mathcal{G}$. More precisely, there is a hermitian line bundle $\mathcal{L} \to \mathcal{G}$ and $\widetilde{W}_X$ is meant to be a section of the line bundle. This means that there is a complex line $\mathcal{L}_q$ attached to each object q, a linear isomorphism $\epsilon : \mathcal{L}_q \to \mathcal{L}_{q'}$ attached to each morphism $q \to q'$, and

$$\epsilon\big(\widetilde{W}_X(q)\big) = \widetilde{W}_X(q'). \tag{5.17}$$

It is this last point which we must check. First, we recall that there is a groupoid $\mathcal{G}_2$ whose objects are triples (Q,Θ,γ) of a principal $(G \times G)$-bundle $Q \to X$, a connection Θ, and a classifying map $\gamma : Q \to E(G \times G)$ for the connection. A morphism $(Q,\Theta,\gamma) \to (Q',\Theta',\gamma')$ is an equivalence class of quintuples $(P,\Lambda,\Gamma,\varphi_0,\varphi_1)$ consisting of a $(G \times G)$-bundle $P \to [0,1] \times X$; a connection Λ on P whose curvature Ω satisfies $\iota(\partial/\partial t)\Omega = 0$, where t is the coordinate on $[0,1]$; a classifying map $\Gamma : P \to E(G \times G)$ for the connection Λ; an isomorphism φ_0 of (Q,Θ,γ) with the restriction of (P,Λ,Γ) to $\{0\} \times X$; and an isomorphism φ_1 of (Q',Θ',γ') with the restriction of (P,Λ,Γ) to $\{1\} \times X$. Quintuples $(P,\Lambda,\Gamma,\varphi_0,\varphi_1)$ and $(P',\Lambda',\Gamma',\varphi_0',\varphi_1')$ are identified if there is an isomorphism $P \to P'$ that preserves the connections and isomorphisms φ_i, φ_i' and under which Γ and Γ' are homotopic. An object (Q,Θ,γ,ϕ) in the groupoid $\mathcal{G}$ includes the section ϕ of the associated bundle $G_Q \to X$, and likewise a morphism $(P,\Lambda,\Gamma,\Phi,\varphi_0,\varphi_1)$ includes a section Φ of $G_P \to [0,1] \times X$ that satisfies $\nabla_{\partial/\partial t}\Phi = 0$, i.e., Φ is flat along trajectories of $\partial/\partial t$. Given such a morphism, we compute

$$\exp\left(2\pi i \int_{[0,1]\times X} N_c\, \Phi^*\tilde{\Gamma}^*\check{\alpha}\right), \tag{5.18}$$

($\tilde{\Gamma} : G_P \to \mathcal{E}$ is the induced classifying map) and apply an appropriate version of Stokes' theorem. The '$\check{E}$-differential' of (5.18) is computed in terms of differential forms as

$$\exp\left(2\pi i\int_{[0,1]\times X} N_c\,\Phi^*\tilde{\Gamma}^*\eta_{\mathcal{E}}\right).$$

This equals 1 because of the "constancy" of Φ and Λ in the $\partial/\partial t$ direction. The integral over $[0,1]\times X$ of the '$\check{E}$-differential' of the integrand in (5.18) is a linear isomorphism $\epsilon : \mathcal{L}_{(Q,\Theta,\gamma)} \to \mathcal{L}_{(Q',\Theta',\gamma')}$, and the integral over the boundary of $[0,1]\times X$ is the ratio of $\widetilde{W}_X(Q',\Theta',\gamma')$ to $\widetilde{W}_X(Q,\Theta,\gamma)$. Stokes' theorem then implies (5.17) immediately.

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