

## MINIMUM IMBEDDINGS OF COMPACT SYMMETRIC SPACES OF RANK ONE

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### 1. Introduction

Let  $M$  be a compact differentiable manifold of dimension  $n$ , and

$$(1.1) \quad \varphi: M \rightarrow \mathbb{R}^{n+k}$$

an immersion of  $M$  into a Euclidean space  $\mathbb{R}^{n+k}$  of dimension  $n+k$ . The total curvature, in the sense of Chern and Lashof [1], [12], can be defined as follows.

Let  $B$  be the set of unit normal vectors of  $M$  in  $\mathbb{R}^{n+k}$ . Then  $B$  is a bundle of  $(k-1)$ -sphere over  $M$  and is a manifold of dimension  $n+k-1$ . Let  $S$  be the unit  $(n+k-1)$ -sphere of  $\mathbb{R}^{n+k}$ ,  $d\sigma$  the volume element of  $S$ , and

$$(1.2) \quad c_{n+k-1} = \int_S d\sigma$$

the volume of  $S$ . If

$$(1.3) \quad \nu: B \rightarrow S$$

is the Gauss map, which assigns each unit normal vector of  $B$  the unit vector through the origin and parallel to the normal vector, then the total curvature of the immersed manifold  $M$  is defined as

$$(1.4) \quad \frac{1}{c_{n+k-1}} \int_B |\nu^* d\sigma|.$$

Since the total curvature depends on  $M$  as well as  $\varphi: M \rightarrow \mathbb{R}^{n+k}$ , we shall denote it by  $\tau(M, \varphi, \mathbb{R}^{n+k})$  or simply by  $\tau(\varphi)$ .

The height function  $h_a$  in the direction  $a \in \mathbb{R}^{n+k}$  takes the value

$$(1.5) \quad h_a(x) = (a, \varphi(x))$$

at  $x \in M$ , where  $(, )$  denotes the usual inner product on  $\mathbb{R}^{n+k}$ .  $x \in M$  is a

critical point of  $h_a$ ,  $a \in \mathbb{R}^{n+k}$ , if and only if  $a$  is normal to  $M$ , and  $h_a$ ,  $a \in S$  has a degenerate critical point if and only if  $a$  is a critical value of the map  $\nu: B \rightarrow S$ . By Sard's theorem, the image of the set of critical points of  $\nu$  has measure 0 in  $S$ . Hence for almost all  $a \in S$ ,  $h_a$  has only nondegenerate critical points. Let  $\beta(M, f)$  denote the number of critical points of a differentiable function  $f$  defined over  $M$ . Then  $\beta(M, h_a)$  is well defined and is finite for almost all  $a \in S$ . We have [12]

$$(1.6) \quad \tau(M, \varphi, \mathbb{R}^{n+k}) = \int_{a \in S} \beta(M, h_a) d\sigma .$$

So, to evaluate the total curvature  $\tau(\varphi)$ , it is sufficient to determine the number of critical points of the height functions.

Let  $F$  be the set of differentiable functions on  $M$  whose critical points are all nondegenerate, and define

$$(1.7) \quad \beta(M) = \inf_{f \in F} \beta(M, f) .$$

It follows from Morse inequality [13, p. 29] that

$$(1.8) \quad \beta(M) \geq b(M) = \sum b_i(M) ,$$

where  $b_i(M)$  is the  $i$ -th Betti number and  $b(M)$  the sum of Betti numbers of  $M$ . Kuiper [12] has shown that

$$(1.9) \quad \inf_{\varphi, k} \tau(M, \varphi, \mathbb{R}^{n+k}) = \beta(M) .$$

An immersion  $\varphi: M \rightarrow \mathbb{R}^{n+k}$  is said to be *minimal* if  $\tau(\varphi) = \beta(M)$ . Given a compact differentiable manifold  $M$ , it is not true in general that  $M$  can always be minimally immersed. As Kuiper has pointed out [12], if  $M$  is an exotic sphere, it admits a function with two critical points and hence  $\beta(M) = 2$ . On the other hand, by a theorem of Chern and Lashof [1], an immersed compact differentiable manifold  $M$  with  $\tau(M, \varphi, \mathbb{R}^{n+k}) = 2$  is a convex hypersurface in some  $\mathbb{R}^{n+1} \subset \mathbb{R}^{n+k}$ , which implies that  $M$  is diffeomorphic to an ordinary sphere. Ferus [3] proved that every imbedding of an exotic  $n$ -sphere ( $n \geq 5$ ) in  $\mathbb{R}^{n+2}$  has a total curvature  $\geq 4$ .

If  $\varphi(M)$  is not contained in any hyperplane of  $\mathbb{R}^{n+k}$ , then we say that the immersion  $\varphi: M \rightarrow \mathbb{R}^{n+k}$  is *substantial*. A theorem of Kuiper [12] asserts that if  $\varphi: M \rightarrow \mathbb{R}^{n+k}$  is minimal and substantial, then  $k \leq n(n+1)/2$ . He also gives examples of minimal and substantial imbeddings of various codimensions  $k$ ,  $1 \leq k \leq n(n+1)/2$ , [12, pp. 82-83]. In particular, the Hopf imbeddings of real projective space  $P_n(\mathbb{R})$  into  $\mathbb{R}^{n+k}$ ,  $n+1 \leq k \leq n(n+1)/2$ , are minimal and substantial. In the same paper [12, p. 86], he exhibited a minimum imbedding of the real projective plane  $P_2(\mathbb{R})$  into  $\mathbb{R}^4$ .

Kobayashi [11] proved that every compact homogenous Kähler manifold can be minimally imbedded into a Euclidean space. In particular, the Manoury imbedding [7, pp. 150–151] of a complex projective space  $P_n(C)$  into  $R^{(n+1)^2-1}$  is shown to be minimal (cf. Remark 2.6).

In this paper, we are going to construct minimum imbeddings of compact symmetric spaces of rank one in a unified fashion. Beside being minimal and substantial, these imbeddings are also equivariant and isometric.

The problem is trivial for spheres. We will treat real, complex and quaternionic projective spaces in § 2, and the Cayley projective plane in § 3.

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### 2. Projective spaces

Throughout this section,  $F$  will denote the field  $R$  of real numbers, the field  $C$  of complex numbers or the field  $Q$  of quaternions. In a natural way,  $R \subset C \subset Q$ . For each element  $x$  of  $F$ , we define the conjugate of  $x$  as follows. If

$$(2.1) \quad x = x_0 + x_1i + x_2j + x_3k \in Q,$$

with  $x_0, x_1, x_2, x_3 \in R$ , then

$$(2.2) \quad \bar{x} = x_0 - x_1i - x_2j - x_3k.$$

If  $x$  is in  $C$ , then  $\bar{x}$  coincides with the ordinary complex conjugate of  $x$ . If  $x$  is in  $R$ , then  $\bar{x} = x$ .

It is convenient for us to define

$$(2.3) \quad d = d(F) = \begin{cases} 1 & \text{if } F = R, \\ 2 & \text{if } F = C, \\ 4 & \text{if } F = Q. \end{cases}$$

Let  $x = (x_0, \dots, x_n) \in F^{n+1}$ . A matrix  $A = (a_{ij})$ ,  $0 \leq i, j \leq n$ , operates on  $F^{n+1}$  by the rule:

$$(2.4) \quad Ax = \begin{pmatrix} a_{00} & \cdots & a_{0n} \\ \cdots & \cdots & \cdots \\ a_{n0} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}.$$

The transpose and conjugate of a matrix  $A$  are denoted by  ${}^tA$  and  $\bar{A}$ , respectively;  $A^*$  denotes  ${}^t\bar{A}$ . We will use the following notations:

(2.5)  $M(n + 1, F)$  = the space of all  $(n + 1) \times (n + 1)$  matrices over  $F$ .

(2.6)  $H(n + 1, F) = \{A \in M(n + 1, F) | A^* = A\}$ , the space of all  $(n + 1) \times (n + 1)$  Hermitian matrices over  $F$ .

If  $A \in H(n + 1, R)$ , then  $A$  is symmetric.

(2.7)  $U(n + 1, F) = \{X \in M(n + 1, F) | X^*X = I\}$ ,

where  $I$  denotes the identity matrix. Then  $U(n + 1, R) = O(n + 1)$ ,  $U(n + 1, C) = U(n + 1)$  and  $U(n + 1, Q) = Sp(n + 1)$ , in standard notations.

$F^{n+1}$  can be considered as a Euclidean space of dimension  $(n + 1)d$ . The usual inner product for  $F^{n+1} = R^{(n+1)d}$  is defined as

(2.8)  $(x, y) = Re(x^*y)$ ,

where  $x$  and  $y \in F^{n+1}$  are represented as column matrices.  $M(n + 1, F)$  can also be considered as a Euclidean space of dimension  $(n + 1)^2d$ , and

(2.9)  $(A, B) = ReTr(AB^*)$ ,  $A, B \in M(n + 1, F)$ ,

defines the usual inner product. If  $A$  and  $B$  belong to  $H(n + 1, F)$ , then

(2.10)  $(A, B) = Tr(AB)$ .

We will endow  $H(n + 1, F)$  with this induced inner product.

Let  $P_n(F)$  denote the projective space over  $F$ . Consider  $P_n(F)$  as the quotient space of unit  $((n + 1)d - 1)$ -sphere  $\{x = (x_0, \dots, x_n) \in F^{n+1} | x^*x = 1\}$  obtained by identifying  $(x_0, \dots, x_n)$  with  $(x_0\lambda, \dots, x_n\lambda)$ , where  $\lambda \in F$  and  $|\lambda| = 1$ . Hence for  $x \in P_n(F)$ , we can use homogeneous coordinates

(2.11)  $x = \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}$ , with  $x^*x = 1$ .

Consider the following map

(2.12)  $\varphi: P_n(F) \rightarrow H(n + 1, F)$

such that

(2.13)  $\varphi(x) = xx^* = \begin{pmatrix} |x_0|^2 & x_0\bar{x}_1 & \dots & x_0\bar{x}_n \\ x_1\bar{x}_0 & |x_1|^2 & \dots & x_1\bar{x}_n \\ \dots & \dots & \dots & \dots \\ x_n\bar{x}_0 & x_n\bar{x}_1 & \dots & |x_n|^2 \end{pmatrix}$ .

It is clear that this is a well defined function from  $P_n(F)$  into a Euclidean space of dimension  $\binom{n+1}{2}d + n + 1$ . The conditions  $xx^* = 1$ ,  $Tr(\varphi(x)) = 1$  and  $\sum_{i=0}^n |x_i|^2 = 1$  are obviously equivalent to each other. It follows that the image of  $P_n(F)$  under  $\varphi$  lies on the hyperplane

$$(2.14) \quad H_1(n+1, F) = \{X = (x_{ij}) \in H(n+1, F) \mid \sum_{i=0}^n x_{ii} = 1\} .$$

For  $x$  and  $y \in P_n(F)$ ,  $\varphi(x) = \varphi(y)$ , which is equivalent to  $xx^* = yy^*$ , implies  $x = y\lambda$ , with  $\lambda \in F$  and  $|\lambda| = 1$ . Thus  $\varphi$  is a substantial imbedding of  $P_n(F)$  into  $R^N$ ,  $N = \binom{n+1}{2}d + n$ . We wish to show that  $\varphi$  is a minimum imbedding.

Let  $U(n+1, F)$  act linearly on  $M(n+1, F)$  in the obvious manner :

$$(2.15) \quad X(A) = XAX^* ,$$

$X \in U(n+1, F)$  and  $A \in M(n+1, F)$ .

**Lemma 2.1.** *The action of  $U(n+1, F)$  preserves inner product of  $M(n+1, F)$ .*

The proof is straightforward.

**Lemm 2.2.** *The imbedding*

$$\varphi: P_n(F) \rightarrow H(n+1, F)$$

is equivariant with respect to and invariant under the action of  $U(n+1, F)$ , i.e.,

$$(2.16) \quad \varphi(Xx) = X(\varphi(x)) \in \varphi(P_n(F))$$

for  $x \in P_n(F)$  and  $X \in U(n+1, F)$ .

The proof is straightforward.

Let  $A \in H(n+1, F)$  and  $h_A$  be the height function defined over  $P_n(F)$  in the diretion  $A$ . Then

$$(2.17) \quad h_A(x) = (A, \varphi(x)) = Tr(A\varphi(x)) = Tr(A(xx^*))$$

at  $x \in P_n(F)$ . By Lemmas 2.1 and 2.2,

$$(2.18) \quad h_{X(A)}(Xx) = h_A(x) , \quad X \in U(n+1, F) .$$

On the other hand, for each  $A \in H(n+1, F)$ , there exists an  $X \in U(n+1, F)$  such that  $X(A) = XAX^*$  is a diagonal matrix. (The fact is well known for  $R$  and  $C$ . We will deal with the quaternionic case in the appendix.)

Therefore, to study the critical points of  $h_A$ , we may assume that  $A$  is a diagonal matrix,

$$(2.19) \quad A = \begin{pmatrix} \lambda_0 & & 0 \\ & \lambda_1 & \\ 0 & & \ddots \\ & & & \lambda_n \end{pmatrix}.$$

Then the height function takes the simple form :

$$(2.20) \quad h_A(x) = \sum_{i=0}^n \lambda_i |x_i|^2, \quad x \in P_n(F).$$

The following is a standard trick to determine the critical points of  $h_A$  on  $P_n(F)$  [13, pp. 26–27].

Consider the following coordinate system. Let  $U_0$  be the set of  $x = (x_0, x_1, \dots, x_n)$  with  $x_0 \neq 0$ , and let

$$(2.21) \quad \begin{aligned} |x_0| x_i x_0^{-1} &= u_i, \\ u_i &= u_{i0} + u_{i1}i + u_{i2}j + u_{i3}k \in F. \end{aligned}$$

Then

$$(2.22) \quad u_{i\alpha} : U_0 \rightarrow \mathcal{R}, \quad 1 \leq i \leq n, \quad 0 \leq \alpha \leq d-1,$$

are the required coordinate functions mapping  $U_0$  diffeomorphically onto the open unit ball in  $\mathcal{R}^{nd}$ . Clearly

$$(2.23) \quad |x_i|^2 = \sum_{\alpha} u_{i\alpha}^2, \quad 0 \leq \alpha \leq d-1.$$

$$(2.24) \quad \begin{aligned} x_0^2 &= 1 - \sum_{i=1}^n |x_i|^2 = 1 - \sum_{i,\alpha} u_{i\alpha}^2, \\ &1 \leq i \leq n, \quad 0 \leq \alpha \leq d-1, \end{aligned}$$

so that

$$(2.25) \quad \begin{aligned} h_A &= \lambda_0 + \sum_{i,\alpha} (\lambda_i - \lambda_0) u_{i\alpha}^2, \\ &1 \leq i \leq n, \quad 0 \leq \alpha \leq d-1, \end{aligned}$$

throughout the coordinate neighborhood  $U_0$ . Thus the only critical point of  $h_A$  within  $U_0$  lies at the center point

$$(2.26) \quad P_0 = (1, 0, \dots, 0)$$

of the coordinate system. At this point,  $h_A$  is nondegenerate if and only if all other eigenvalues are distinct from  $\lambda_0$ .

Similarly one can consider other coordinate neighborhoods centered at the points

$$(2.27) \quad P_1 = (0, 1, \dots, 0), \dots, P_n = (0, \dots, 0, 1),$$

It follows that  $P_0, P_1, \dots, P_n$  are the only critical points of  $h_A$ . Thus we have

**Theorem 2.3.** *For  $A \in H(n + 1, F)$  the height function  $h_A$  defined over  $P_n(F)$  is nondegenerate and has exactly  $n + 1$  isolated critical points if and only if all eigenvalues are distinct from each other.*

**Remark 2.4** (cf. [13]). Every nondegenerate height function has indices  $id, 0 \leq i \leq n$ , respectively at  $n + 1$  different critical points. From the cell decomposition of such a function, it follows immediately that the sum of Betti numbers  $b(P_n(F)) = n + 1$  if  $d(F) = 2$  or  $4$ . But it is well known that  $b(P_n(\mathbb{R})) = n + 1$ . Therefore by Morse inequality (1.8), every nondegenerate height function has  $\beta(M) (= n + 1)$  critical points, and we have proved

**Theorem 2.5.** *The imbedding (cf. (2.14))*

$$\varphi : P_n(F) \rightarrow H_1(n + 1, F)$$

*is substantial, minimal, isometric and equivariant.*

The assertion that  $\varphi$  is an isometric imbedding follows from the fact that there is a Riemannian metric, unique up to a constant factor, on  $P_n(F)$  which is invariant under  $U(n + 1, F)$ , and the fact that the metric on  $P_n(F)$  induced by  $\varphi$  is invariant under  $U(n + 1, F)$  (Lemma 2.1). Or more generally, every equivariant imbedding of an irreducible symmetric space is isometric, since an invariant metric on a homogeneous space with irreducible linear isotropy group is unique up to a constant factor.

**Remark 2.6.** In our notations, the Mannoury imbedding of complex projective space  $P_n(\mathbb{C})$  can be described as follows [7, pp. 150–151]:

Let  $\mathbb{R}^{(n+1)^2}$  be a Euclidean space with coordinate system  $(X^h, X^{hk}, Y^{hk})$ , where  $h, k = 0, 1, \dots, n$  and  $h \neq k$ . The imbedding  $P_n(\mathbb{C}) \rightarrow \mathbb{R}^{(n+1)^2}$  is defined by

$$(2.28) \quad \begin{aligned} X_h &= \sqrt{2} x_h \bar{x}_k = \sqrt{2} |x_h|^2, \\ X^{hk} &= x_h \bar{x}_k + \bar{x}_h x_k = 2 \operatorname{Re} x_h \bar{x}_k, \\ Y^{hk} &= i(x_h \bar{x}_k - \bar{x}_h x_k) = 2 \operatorname{Im} x_h \bar{x}_k. \end{aligned}$$

Then  $P_n(\mathbb{C})$  lies in the hyperplane

$$(2.29) \quad X^0 + \dots + X^n = \sqrt{2}$$

of  $\mathbb{R}^{(n+1)^2}$ . It is easy to see that this imbedding differs from ours only by an affine transformation. It follows from Theorem 2.5 that the Mannoury imbedding is minimal by the following theorem of Kuiper [12]:

**Theorem 2.7.** *If  $\varphi: M \rightarrow R^{n+k}$  is minimal and  $A: R^{n+k} \rightarrow R^{n+k}$  is an affine transformation, then  $A \circ \varphi$  is minimal.*

**Remark 2.8.** In his paper "On isometric imbeddings of compact symmetric spaces" (to appear), Kobayashi exhibits the same type of imbeddings for a class of symmetric spaces and conjecture that they are all minimal.

*Added in proof.* Kobayashi and Takeuchi have recently proved the above conjecture.

### 3. Cayley projective plane

Let  $x = x_0 + x_1j_1 + \cdots + x_7j_7$  be an element of the Cayley algebra over the real field. Denote

$$(3.1) \quad \bar{x} = x_0 - x_1j_1 - \cdots - x_7j_7,$$

the conjugate of  $x$ . Then the norm  $n(x)$  of  $x$  is equal to

$$(3.2) \quad x\bar{x} = x_0^2 + \cdots + x_7^2,$$

and we have

$$(3.3) \quad n(xy) = n(x)n(y), \quad x, y \in \mathbf{Cay}.$$

An element  $x \neq 0 \in \mathbf{Cay}$  has an inverse  $\bar{x}/n(x)$ .

If  $H(3, \mathbf{Cay})$  is the space of  $3 \times 3$  Hermitian Cayley matrices, then  $H(3, \mathbf{Cay})$  is a Jordan algebra under the following multiplication [9]:

$$(3.4) \quad X \circ Y = \frac{1}{2}(XY + YX), \quad X, Y \in H(3, \mathbf{Cay}).$$

For simplicity, we express an element  $X \in H(3, \mathbf{Cay})$  in the form:

$$(3.5) \quad X = \begin{pmatrix} \xi_1 & x & \bar{z} \\ \bar{x} & \xi_2 & y \\ z & \bar{y} & \xi_3 \end{pmatrix}.$$

Using the usual matrix unit  $E_{ij}$  and setting  $E_{ii} = E_i$ , and let

$$(3.6) \quad x_{ij} = xE_{ij} + \bar{x}E_{ij},$$

we can write

$$(3.7) \quad X = \xi_1E_1 + \xi_2E_2 + \xi_3E_3 + x_{12} + y_{23} + z_{31}.$$

The  $E_i$  are orthogonal idempotents, and the trace and norm of  $X$  are defined respectively as in [8]:



$$(3.8) \quad \text{Tr}(X) = \xi_1 + \xi_2 + \xi_3 ,$$

$$(3.9) \quad N(X) = \xi_1\xi_2\xi_3 + \text{Tr}((xy)z) - \xi_1n(y) - \xi_2n(z) - \xi_3n(x) .$$

The minimum polynomial of  $X$  is defined as

$$(3.10) \quad N(\lambda I - X) = \lambda^3 - \text{Tr}(X)\lambda^2 + \frac{1}{2}(\text{Tr}(X)^2 - \text{Tr}(X^2))\lambda - N(X) ,$$

and we have

$$(3.11) \quad X^3 - \text{Tr}(X)X^2 + \frac{1}{2}(\text{Tr}(X)^2 - \text{Tr}(X^2))X - N(X)I = 0 ,$$

where  $I = E_1 + E_2 + E_3$  is the identity matrix.

Following Jordan [10], we can define the Cayley projective plane  $P_2(\text{Cay})$  as the set

$$(3.12) \quad \{x^*x \in H(3, \text{Cay}) \mid x^*x = 1, x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \text{Cay}^3\} .$$

This set is equivalent to ([4], [6])

$$(3.13) \quad \{X \in H(3, \text{Cay}) \mid X \circ X = X, \text{Tr}(X) = 1\} ,$$

and is contained in

$$(3.14) \quad H_1(3, \text{Cay}) = \{X \in H(3, \text{Cay}) \mid \text{Tr}(X) = 1\} .$$

Consider  $H(3, \text{Cay})$  as a Euclidean space of dimension 27 endowed with the inner product

$$(3.15) \quad (X, Y) = \text{Tr}(X \circ Y) , \quad X, Y \in H(3, \text{Cay}) ,$$

which is induced from the usual inner product of  $M(3, \text{Cay})$ , the space of  $3 \times 3$  Cayley matrices considered as  $R^{72}$ .

**Lemma 3.1.** *An automorphism of the Jordan algebra  $H(3, \text{Cay})$  preserves the inner product.*

*Proof.* From (3.11), the trace function is invariant under the automorphisms of  $H(3, \text{Cay})$ . The rest is straightforward.

The following two results can be found in Freudenthal [4]:

**Lemma 3.2** [4, p. 25]. *The automorphism group of  $H(3, \text{Cay})$  is the exceptional Lie group  $F_4$ .*

**Lemma 3.3** [4, p. 26]. *For each  $X \in H(3, \text{Cay})$ , there is an  $a \in F_4$  such that*

$$(3.16) \quad a(X) = \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3$$

*i.e. the elements of  $H(3, Cay)$  can be diagonalized by the action of  $F_4$ .*

**Lemma 3.4.**  $P_2(Cay)$  is invariant under the action of  $F_4$ .

The proof is obvious from (3.13) and the fact that the trace function is invariant under automorphisms. Now, we are going to show that the inclusion

$$(3.17) \quad \varphi: P_2(Cay) \rightarrow H(3, Cay)$$

is a minimum imbedding.

As in § 2, we first consider the height functions  $h_A$ ,  $A \in H(3, Cay)$ . We can also treat  $x = (x_1, x_2, x_3) \in Cay^3$ ,  $x^*x = 1$ , as a sort of homogeneous coordinate for  $P_2(Cay)$ . Owing to Lemmas 1, 3 and 4, we may assume that

$$(3.18) \quad A = \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3 .$$

Then

$$(3.19) \quad h_A(x) = \lambda_1 n(x_1) + \lambda_2 n(x_2) + \lambda_3 n(x_3) .$$

$(x_1, x_2, x_3) = (y_1, y_2, y_3)$  implies  $n(x_i) = n(y_i)$ ,  $i = 1, 2, 3$ . Hence it makes sense to consider the following local coordinate system. Let  $U_1$  be the set of  $x = (x_1, x_2, x_3)$  with  $x_1 \neq 0$ , and let

$$(3.20) \quad |x_1| x_1^{-1} x_i = u_{i0} + u_{i1} j_1 + \cdots + u_{i7} j_7$$

where  $|x_1| = n(x_1)^{1/2}$ . Then

$$(3.21) \quad u_{i\alpha} : U_1 \rightarrow \mathbf{R}, \quad 2 \leq i \leq 3, \quad 0 \leq \alpha \leq 7,$$

are the required coordinate functions mapping  $U_1$  diffeomorphically onto the open unit ball in  $\mathbf{R}^{16}$ . Clearly

$$(3.22) \quad n(x_i) = \sum_{\alpha=0}^7 u_{i\alpha}^2,$$

$$(3.23) \quad n(x_1) = 1 - n(x_2) - n(x_3) = 1 - \sum_{\substack{2 \leq i \leq 3 \\ 0 \leq \alpha \leq 7}} u_{i\alpha}^2,$$

so that

$$(3.24) \quad h_A(x) = \lambda_1 \sum_{\substack{2 \leq i \leq 3 \\ 0 \leq \alpha \leq 7}} (\lambda_i - \lambda_1) u_{i\alpha}^2$$

throughout the coordinate neighborhood  $U_1$ . Thus the only critical point of  $h_A$  with  $U_1$  lies at the center point

$$P_1 = (1, 0, 0)$$

of the coordinate system. At this point,  $h_A$  is nondegenerate if and only if the other two eigenvalues are distinct from  $\lambda_1$ .

Similarly one can consider other coordinate neighborhoods centered at the points

$$P_2 = (0, 1, 0) , \quad P_3 = (0, 0, 1) .$$

It follows that  $P_1, P_2, P_3$  are the only critical points of  $h_A$ . Thus we have

**Theorem 3.5.** *For  $A \in H(3, Cay)$  the height function  $h_A$  defined over  $P_2(Cay)$  is nondegenerate and has exactly three isolated critical points if and only if all three eigenvalues are distinct from each other.*

**Remark 3.6.** If  $h_A$  is nondegenerate, the indices at three different critical points are respectively 0, 8, 16. From the cell decomposition of  $h_A$ , it follows that the sum of Betti numbers  $b(P_2(Cay)) = 3$ . Therefore every height function has the minimum number of critical points. Hence

**Theorem 3.7.** *The inclusion*

$$\varphi : P_2(Cay) \rightarrow H_1(3, Cay)$$

*is a substantial, minimal, isometric and equivariant imbedding.*

The equivariance follows from the fact that  $\varphi$  is an inclusion.  $\varphi$  is isometric, since every equivariant imbedding of an irreducible symmetric space is isometric (cf. Theorem 2.5).

#### 4. Appendix

This appendix is based on Chevalley's *Theory of Lie Groups* [2, Chapter I, §§ III-VII]. Most proofs are omitted, which can be either found, or proved by similar arguments, for the complex case in that book.

**Lemma 4.1.** *For each  $A \in M(n + 1, Q)$ , there exists an  $x \in Q^{n+1}$  such that  $Ax = x\lambda$ ,  $\lambda \in Q$ .*

The author is indebted to Professor Kobayashi for the following simple proof.

*Proof.* If  $A$  is singular, then there is an  $x \in Q^{n+1}$  such that  $x \neq 0$  and  $Ax = 0$ . Suppose  $A$  is nonsingular. Since  $GL(n + 1, Q)$  is connected,  $A \sim I : Q^{n+1} \rightarrow Q^{n+1}$ , where  $I$  is the identity transformation. Let  $A'$  be the induced map on  $P_n(Q)$ . Then  $A' \sim Id : P_n(Q) \rightarrow P_n(Q)$ , where  $Id$  is the identity map of  $P_n(Q)$ . Therefore the Lefschetz number

$$(4.1) \quad L(A') = L(Id) = \chi(P_n(Q)) = n + 1 \neq 0.$$

Hence, by Lefschetz fixed point theorem,  $A'$  has a fixed point. Equivalently, there exists an  $x \in Q^{n+1}$  such that  $Ax = \lambda x$ .

**Lemma 4.2** [2, p. 21, Proposition 2]. *If  $a$  is a unit vector in  $Q^{n+1}$ , there exists a symplectic matrix  $X$  such that  $Xe_1 = a$ , where  $e_1 = (1, 0, \dots, 0)$ .*

**Theorem 4.3.** *For each  $A \in H(n+1, \mathcal{Q})$ , there exists an  $X \in Sp(n+1)$  such that  $XAX^*$  is a diagonal matrix.*

The existence of an eigenvector (Lemma 4.1) and Lemma 4.2 make the inductive process possible. The proof is almost the same as the complex case [2, pp. 12-13]. We have also

**Theorem 4.4.** *The eigenvalues of a Hermitian quaternionic matrix are real numbers.*

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