J. DIFFERENTIAL GEOMETRY
2 (1968) 25-31

SOME APPLICATIONS OF THE RETRACTION THEOREM IN EXTERIOR ALGEBRA

ROBERT B. GARDNER

This paper considers examples of how the multiplicative structure of an ideal in an exterior algebra may be used to construct tensor invariants of the ideal. In particular, we consider the ideals arising from the study of decomposable elements of degree p, the relative theory of skew symmetric bilinear forms, and pencils of skew symmetric bilinear forms.

Let V be an n-dimensional vector space over a field $\underline{F}$, and V^* the dual vector space of linear functions on V. If W^* is a subspace of V^*, then the inclusion induces an inclusion on the exterior algebras

$$0 \to E(W^*) \to E(V^*) ,$$

and hence if J is an ideal in $E(W^*)$, this inclusion canonically induces an ideal I in $E(V^*)$. Matters being so we say that I is an extension of J, and J is a retraction of I.

Let $\langle\ ,\ \rangle$ denote the canonical bilinear pairing between $E(V)$ and $E(V^*)$. This pairing allows us to introduce $\lrcorner$ as the adjoint of left exterior multiplication in $E(V)$ and $\llcorner$ as the adjoint of left exterior multiplication in $E(V^*)$ [1]. Then given an ideal I in $E(V^*)$, we may define

$$\operatorname{Char} I = \{X \in V \mid X \lrcorner I \subset I\} ,$$

the subspace of characteristic vectors of I, and its annihilator

$$C(I) = [\operatorname{Char} I]^{\perp} ,$$

the Cartan subspace of I.

The retraction theorem of Elie Cartan may now be stated as follows:

Theorem 1. *Let I be an ideal in $E(V^*)$, then $C(I)$ is the smallest subspace of V^* whose exterior algebra contains a retraction of I* [2].

The program is to utilize the existence of the Cartan subspace to construct tensor invariants from the multiplicative properties of a given ideal.

Communicated by L. Nirenberg, September 12, 1967. Sponsored in part by the National Science Foundation, and in part by the Ford Foundation.

1. Decomposable elements

Let $\wedge^p V^*$ denote the vector space of elements of degree p in $E(V^*)$. An element $\pi \in \wedge^p V^*$ is said to be decomposable if it may be written as the product of p elements from V^*.

We recall that the multiplicative group of units $\underline{F}^*$ of the field $\underline{F}$ acts on $\wedge^p V^*$ by scalar multiplication, and the orbits of non-zero decomposable elements in $\wedge^p V^*$ under $\underline{F}^*$ are in one-to-one correspondence with p-dimensional subspaces of V^* [3].

If Δ is a representative element corresponding to the p-dimensional subspace W^* we write

$$\Delta \longleftrightarrow W^* .$$

Now let I denote the principal ideal with generator π of degree p. Since $C(\mathrm{I})$ is the smallest subspace whose inclusion induces a retraction,

$$\{w \in V^* \mid \pi \wedge w = 0\} \subset C(I) ,$$

and π is decomposable if and only if

$$\{w \in V^* \mid \pi \wedge w = 0\} = C(I) . \tag{1}$$

Proposition 1. *An element $\pi \in \wedge^p V^*$ is decomposable if and only if*

$$(z \llcorner \pi) \lrcorner \pi = 0 \tag{2}$$

for all $z \in \wedge^{p+1} V$.

Proof. Since $\pi \wedge w = 0$ if and only if

$$\langle z \llcorner \pi, w \rangle = 0$$

for all $z \in \wedge^{p+1} V$, we have by duality and (1) that π is decomposable if and only if

$$\mathrm{Char}\, I = \{z \llcorner \pi \mid z \in \wedge^{p+1} V\} , \tag{3}$$

but

$$\mathrm{Char}\, I = \{x \in V \mid x \lrcorner \pi = 0\} , \tag{4}$$

and (3) and (4) give (1).

Since equations (1) are linear in z, it suffices to write the conditions for a basis of $\wedge^{p+1} V$. The conditions are quadratic homogeneous polynomials in the coefficients of π which are the classical Grassmann quadratic p-relations for π to be decomposable [4].

An alternative description of these conditions is the following.

Proposition 2. *An element $\pi \in \wedge^p V^*$ is decomposable if and only if*

$$(v \lrcorner \pi) \wedge \pi = 0$$

for all $v \in \wedge^{p-1} V$.

Proof. Since $x \lrcorner \pi = 0$ if and only if

$$\langle v, x \lrcorner \pi \rangle = 0$$

for all $v \in \wedge^{p-1} V$, we have

$$\operatorname{Char}((\pi)) = \{v \lrcorner \pi \mid v \in \wedge^{p-1} V\}$$

or

$$C((\pi)) = \{v \lrcorner \pi \mid v \in \wedge^{p-1} V\}$$

and the result follows from (1).

2. Relative theory of skew symmetric bilinear forms

Let $\Omega \in \wedge^2 V^*$ be the exterior quadratic form corresponding to a skew symmetric bilinear form

$$F : V \times V \to \underline{F} .$$

Thus if $x, y \in V$

$$\Omega(x \wedge y) = F(x, y) .$$

Let W be a subspace of V, then the bilinear form F has maximal rank on W if and only if

$$x \in W \quad \text{and} \quad x \lrcorner \Omega \in W^\perp$$

implies that $x = 0$. This is equivalent to the statement that the ideal I in $E(V^*)$ generated by $W^\perp$ in degree one and Ω in degree two satisfies

$$\operatorname{Char} I = 0 ,$$

or by duality that

$$C(I) = V^* .$$

This tautology, together with an explicit representation for $C(I)$, will be used to develop a duality theory for skew symmetric bilinear forms which are non-degenerate on a subspace.

Lemma 1. *Let* $\{w^1, \cdots, w^s, \pi\}$ *be linearly independent elements of* V^* *and* $\Omega \in \wedge^2 V^*$*; then*

$$\Omega^p \wedge w^1 \wedge \cdots \wedge w^s \wedge \pi = 0$$

implies

$$\Omega^{p+1} \wedge w^1 \cdots \wedge w^s = 0 .$$

Proof. Let $\{\pi\}$ denote the one-dimensional subspace generated by π and let W^* denote a complement in V^* of $\{\pi\}$ which contains $w^1, \cdots, w^s$. Then

$$E(V^*) = E(W^*) \hat{\otimes} E\{\pi\} , \tag{5}$$

where $\hat{\otimes}$ is the graded tensor product [5]. As a result there exists unique $\alpha \in \wedge^2 W^*$, $\beta \in W^*$ such that

$$\Omega = \alpha + \beta \wedge \pi .$$

Since

$$\Omega^p = \alpha^p + p\alpha^{p-1} \wedge \beta \wedge \pi$$

the hypothesis implies

$$\alpha^p \wedge w^1 \wedge \cdots \wedge w^s \wedge \pi = 0 ,$$

and since $\alpha^p \wedge w^1 \cdots \wedge w^s \in E(W^*)$, the isomorphism (5) implies

$$\alpha^p \wedge w^1 \wedge \cdots \wedge w^s = 0 .$$

The conclusion now follows since

$$\Omega^{p+1} = \alpha^{p+1} + (p+1)\alpha^p \wedge \beta \wedge \pi .$$

Theorem 2. *Let* I *be an ideal generated by linearly independent elements* $w^1, \cdots, w^s \in V^*$*, and* $\Omega \in \wedge^2 V^*$*; then*

$$C(I) \longleftrightarrow \Omega^p \wedge w^1 \wedge \cdots \wedge w^s ,$$

where p *is the smallest integer such that*

$$\Omega^{p+1} \wedge w^1 \wedge \cdots \wedge w^s = 0 .$$

Proof. If $x \in V$ satisfies

$$(6)\quad 0 = x \mathbin{\lrcorner} (\Omega^p \wedge w^1 \wedge \cdots \wedge w^s) = \mathrm{p}(x \mathbin{\lrcorner} \Omega) \wedge \Omega^{p-1} \wedge w^1 \wedge \cdots \wedge w^s$$
$$+ (-1)^j \langle x, w^j \rangle \Omega^p \wedge w^1 \wedge \cdots \wedge w^{j-1} \wedge w^{j+1} \wedge \cdots \wedge w^s ,$$

then multiplication by w^k for $1 \leq k \leq s$ gives

$$0 = \langle x, w^k \rangle \Omega^p \wedge w^1 \wedge \cdots \wedge w^s$$

which by definition of p implies

$$(7)\qquad \langle x, w^k \rangle = 0 \qquad (1 \leq k \leq s) ,$$

and hence substituting (7) back into (6) gives

$$0 = (x \mathbin{\lrcorner} \Omega) \wedge \Omega^{p-1} \wedge w^1 \wedge \cdots \wedge w^s .$$

Now if

$$(x \mathbin{\lrcorner} \Omega) \wedge w^1 \wedge \cdots \wedge w^s \neq 0 ,$$

then Lemma 1 would imply

$$\Omega^p \wedge w^1 \wedge \cdots \wedge w^s = 0 ,$$

contradicting the definition of p. Therefore

$$(8)\qquad x \mathbin{\lrcorner} \Omega \in I .$$

Thus if $x \in V$ satisfies (6), then it satisfies (7) and (8), which means $x \in \operatorname{Char} I$. The converse inclusion is clear; hence

$$\operatorname{Char} I = \{x \in V \mid x \mathbin{\lrcorner} (\Omega^p \wedge w^1 \wedge \cdots \wedge w^s) = 0\} ,$$

The condition

$$\Omega^p \wedge w^1 \wedge \cdots \wedge w^s \neq 0$$

implies that

$$\dim C(I) \geq 2p + s ,$$

and hence that

$$\dim \operatorname{Char} I \leq n - 2p - s .$$

The condition

$$\Omega^{p+1} \wedge w^1 \wedge \cdots \wedge w^s = 0$$

implies that every $x \in \{w^1, \cdots, w^s\}$ satisfies

$$(x \rfloor \Omega) \wedge \Omega^p \wedge w^1 \wedge \cdots \wedge w^s = 0$$

and hence that $\Omega^p \wedge w^1 \wedge \cdots \wedge w^s$ admits at least

$$n - s - (n - 2p + s) = 2p$$

independent linear divisors which are linearly independent from $\{w^1, \cdots, w^s\}$. Therefore $\Omega^p \wedge w^1 \wedge \cdots \wedge w^s$ is decomposable and by duality

$$C(I) \longleftrightarrow \Omega^p \wedge w^1 \wedge \cdots \wedge w^s .$$

Thus if $\Omega \in \wedge^2 V^*$ is an exterior quadratic form which has maximal rank on W, then there exists an integer p such that

$$\Omega^p \wedge w^1 \wedge \cdots \wedge w^s \longleftrightarrow V^* .$$

As a result we may introduce an invariantly defined skew symmetric bilinear form $\langle\ ,\ \rangle$ on V^* defined for $\alpha, \beta \in V^*$ by

$$\alpha \wedge \beta \wedge \Omega^{p-1} \wedge w^1 \wedge \cdots \wedge w^s = \langle \alpha, \beta \rangle \Omega^p \wedge w^1 \wedge \cdots \wedge w^s .$$

We note that this generalizes a construction of E. Cartan introduced in §127 of [6].

Since

$$\langle \alpha, \beta \rangle = 0 \quad \text{for} \quad \alpha \in W^\perp , \quad \beta \in V^* ,$$

there is an induced bilinear form

$$V^*/W^\perp \times V^*/W^\perp \to \underline{F}$$

which we will denote by the same symbol.

Proposition 3. *Let $\Omega \in \wedge^2 V^*$ have maximal rank on a subspace W of V; then the linear mapping*

$$\phi_\Omega : W \to V^*/W^\perp$$

defined by

$$\phi_\Omega(x) = x \rfloor \Omega$$

is an isometry.

Proof. Let $x, y \in W$; then

$$\begin{aligned} 0 &= x \rfloor [(y \rfloor \Omega) \wedge \Omega^p \wedge w^1 \wedge \cdots \wedge w^s] \\ &= \Omega(y \wedge x)\Omega^p \wedge w^1 \wedge \cdots \wedge w^s - (y \rfloor \Omega) \wedge (x \rfloor \Omega) \wedge \Omega^{p-1} \wedge w^1 \wedge \cdots \wedge w^s . \end{aligned}$$

Hence

$$\Omega(y \wedge x) = \langle \phi_\Omega(y), \phi_\Omega(x) \rangle$$

as claimed.

3. Pencils of skew symmetric bilinear forms

Let $P \subset \wedge^2 V^*$ be the linear subspace corresponding to a pencil of skew symmetric bilinear forms, and let I denote the ideal in $E(V^*)$ generated by P. Now if

$$\dim C(I) = 2n$$

we may introduce a conformal symmetric tensor T defined for $\Omega^1, \cdots, \Omega^n \in P$ by

$$\Omega^1 \wedge \cdots \wedge \Omega^n = T(\Omega^1, \cdots, \Omega^n)\Delta$$

where

$$\Delta \longleftrightarrow C(I) .$$

Since this tensor is invariantly defined, every invariant of T is an invariant of the pencil P.

More generally we may intoduce conformal polynomials $Q_{i_1 \cdots i_n}$ defined for $\Omega^1, \cdots, \Omega^n \in P$ by

$$\frac{\partial^{i_1 + \cdots + i_n}}{\partial\lambda_1^{i_1} \cdots \partial\lambda_n^{i_n}}(\lambda_1\Omega^1 + \cdots + \lambda_n\Omega^n)^n \Big|_{\lambda_1 = \cdots = \lambda_n = 0}$$
$$= Q_{i_1 \cdots i_n}(\Omega^1, \cdots, \Omega^n)\Delta .$$

In the case where $C(I)$ has odd dimension, similar tensors can not be constructed without a more detailed knowledge of the structure of the pencil P.

References

[1] N. Bourbaki, *Algèbre multilinéaire,* Hermann, Paris, 1958, Chapter III, § 8, no. 4.

[2] S. Sternberg, *Lectures on differential geometry,* Prentice Hall, Englewood Cliffs, N.J., 1964, p. 29.

[3] ——, *Ibid.,* p. 16.

[4] W. V. D. Hodge & D. Pedoe, *Methods of algebraic geometry,* Vol **1**, Cambridge University Press, 1952, Chapter VII.

[5] C. Chevalley, *Fundamental concepts of algebra,* Academic Press, New York, 1956, P. 154.

[6] E. Cartan, *Leçons sur les invariants integraux,* Hermann, Paris, 1958.

New York University