J. DIFFERENTIAL GEOMETRY
50 (1998) 207-247

NEW INVARIANT TENSORS IN CR STRUCTURES AND A NORMAL FORM FOR REAL HYPERSURFACES AT A GENERIC LEVI DEGENERACY

PETER EBENFELT

0. Introduction

In 1974, Chern and Moser [11] solved the biholomorphic equivalence problem for real-analytic hypersurfaces in $\mathbb{C}^{n+1}$ at Levi nondegenerate points. (The case $n = 1$ was considered and solved by E. Cartan [9]–[10].) They presented a complete set of biholomorphic invariants for such a hypersurface at a Levi nondegenerate point; by a complete set of invariants, we mean a set of invariants such that given two hypersurfaces $M, M' \subset \mathbb{C}^{n+1}$ with distinguished points $p_0 \in M$, $p_0' \in M'$, there is a biholomorphic transformation $Z' = H(Z)$ near p_0 such that $H(M) \subset M'$ and $H(p_0) = p_0'$ if and only if the set of invariants for M and M' are equal. The Chern-Moser invariants can in principle (there is an infinite number of invariants) be computed from the Chern-Moser normal form, which is a normal form for a Levi nondegenerate hypersurface M, defined in terms of the Levi form at $p_0 \in M$, such that the transformation to normal form is unique modulo a finite dimensional normalization.

In the present paper, we introduce a new sequence of invariant tensors, $\psi_2, \psi_3 \ldots$, for generic submanifolds of $\mathbb{C}^N$ (Theorem 2.9), which can be viewed as higher order Levi forms. (Although the tensors are

Received April 28, 1998. The author was partially supported by a grant from the Swedish Natural Science Research Council.

1991 *Mathematics Subject Classification.* Primary 32F25; Secondary 32F40.

only introduced here in the context of generic submanifolds of $\mathbb{C}^N$, it is clear that the definitions work equally well in general CR structures.) The second order tensor ψ_2 coincides with the Levi map and the higher order tensors are related, as explained in §3 below, to the data of *finite nondegeneracy*, a notion which has recently proved very useful in the study of real submanifolds in $\mathbb{C}^N$ (see e.g. [1]–[4], [5], [12]–[13]). The third order tensor is also related to the *cubic form* as introduced by Webster [25] (see Remark 4.17).

As one of our main results (Theorem 1.1.28), using the second and third order tensors, we describe a formal normal form (in the sense of Chern–Moser as described above) for a real smooth (meaning C^∞) hypersurface $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, at a *generic Levi degeneracy* $p_0 \in M$, i.e., a point p_0 at which the Levi determinant vanishes but its differential does not and the set of Levi degenerate points of M (which is then a smooth codimension-one submanifold of M at p_0) is transverse to the Levi null space (which is then one dimensional) at that point. (We refer the reader to [25] for further discussion of generic Levi degeneracies; for instance, a normal form for generic Levi degeneracies in $\mathbb{C}^2$ under formal holomorphic *contact* transformations is given in [25].) In view of a convergence result due to the author, Baouendi, and Rothschild [3], the formal normal form in Theorem 1.1.28 provides a complete set of *biholomorphic* invariants if the hypersurface is also real-analytic (Corollary 1.1.30).

We then proceed to study the special case where the Levi form, at the generic Levi degeneracy, is semidefinite. In this situation, the normal form can be expressed in a particularly simple and explicit form (Theorem 1.2.5) by applying a result of E. Cartan. (The associated partial (third order) normal form is given with numerical invariants; in fact, an explicit partial normal form which is valid in a slightly more general setting is given in Theorem 1.2.10.) The corresponding explicit character of the normalization of the transformation to normal form makes it possible to compute bounds for the stability group of a real hypersurface at a generic semidefinite Levi degeneracy (Corollary 1.2.7). In the case $n = 2$, i.e., for hypersurfaces in $\mathbb{C}^3$, the results on normal forms in this paper are contained in the results of [12]. However, the invariant tensors introduced here shed additional light on some of the results from, and answers a question posed in that paper.

The paper is organized as follows. In the first section, §1.1, we present the normal form for a generic Levi degeneracy. In §1.2, we give the more explicit normal form in the special case where the Levi form at p_0 is also semidefinite. We then turn to the more general situation of generic submanifolds in $\mathbb{C}^N$ and introduce the CR invariant tensors. §3 is devoted to explaining the relation between the notion of finite nondegeneracy and the tensors of §2. In §4, we return to the case of hypersurfaces and show, as a preparation for the normal form, that the second and third order tensors form a complete set of third order invariants for a real hypersurface by relating these tensors to the defining equation of M. Then, we calculate, in §5, explicit numerical invariants associated with the third order tensor of a real hypersurface at a point where the Levi form has rank $n-1$ and is semidefinite. §6–8 are devoted to the proofs of the results that give the normal form.

Acknowledgments

The author would like to thank M. S. Baouendi, L. P. Rothschild, and N. Wallach for valuable comments on a preliminary version of this paper.

1. Normal forms for real hypersurfaces at generic Levi degeneracies

1.1. The general case. In this section, we shall present a normal form for a generic Levi degeneracy; the reader should recall the definition of generic Levi degeneracy from §0. In order to describe the normal form, we need first a partial (third order) normal form. We begin with some notation.

We use the notation $\mathcal{M}(\mathbb{C}^m)$ for the space of $m \times m$ matrices with complex matrix elements and $GL(\mathbb{C}^m)$ for the group of invertible ones. We also write $\mathcal{S}(\mathbb{C}^m)$ for the symmetric matrices in $\mathcal{M}(\mathbb{C}^m)$, i.e., those for which $A = A^\tau$, where, A^τ denotes the transpose of A. For nonnegative integers r and s such that $r + s = m$, we denote by $\hat{U}(r, s, \mathbb{C})$ the subgroup of $GL(\mathbb{C}^m)$ consisting of those matrices U for which

$$U^* I_{r,s} U = \pm I_{r,s}, \tag{1.1.1}$$

where $I_{r,s} \in GL(\mathbb{C}^m)$ is the diagonal matrix whose r first diagonal elements are $+1$ and s last ones are -1, and U^* denotes the Hermitian adjoint of U (i.e., $U^* = \bar{U}^\tau$). This group decomposes naturally as

$$\hat{U}(r, s, \mathbb{C}) = U^+(r, s, \mathbb{C}) \cup U^-(r, s, \mathbb{C}),$$

where $U^+(r,s,\mathbb{C})$ and $U^-(r,s,\mathbb{C})$ denote the set of matrices for which (1.1.1) holds with the $+$ and $-$ sign, respectively. Observe that $U^+(r,s,\mathbb{C})$ is a subgroup whereas $U^-(r,s,\mathbb{C})$ is not. Also, note that $U^-(m,0,\mathbb{C})$ is empty, and $U(m,0,\mathbb{C}) = U^+(m,0,\mathbb{C})$ is the usual group $U(\mathbb{C}^m)$ of unitary matrices.

Consider the action of the group $\mathbb{R}_+ \times \hat{U}(r,s,\mathbb{C})$ on $\mathcal{S}(\mathbb{C}^m)$ given by

$$\mathbb{R}_+ \times \hat{U}(r,s,\mathbb{C}) \ni (\sigma, U) \to (\sqrt{\sigma}U)^\tau A(\sqrt{\sigma}U) \in \mathcal{S}(\mathbb{C}^m), \tag{1.1.2}$$

for $A \in \mathcal{S}(\mathbb{C}^m)$. We denote, for given $A \in \mathcal{S}(\mathbb{C}^m)$, by $C_{r,s}(A) \subset \mathcal{S}(\mathbb{C}^m)$ its orbit or conjugacy class under the group action (1.1.2), i.e.,

$$C_{r,s}(A) = \left\{ B \in \mathcal{S}(\mathbb{C}^m)\ B = (\sqrt{\sigma}U)^\tau A(\sqrt{\sigma}U),\ U \in \hat{U}(r,s,\mathbb{C}),\ \sigma > 0 \right\}.$$

These conjugacy classes form a disjoint partition of $\mathcal{S}(\mathbb{C}^m)$. We have the following result, which is the first step in describing the normal form and whose proof will be given in §6.

Proposition 1.1.3. *Let $M \subset \mathbb{C}^{n+1}$, $n \geq 2$, be a real smooth hypersurface. Assume that M has a generic Levi degeneracy at $p_0 \in M$. Denote by r, with $(n-1)/2 \leq r \leq n-1$, the maximal number of eigenvalues of the Levi form at p_0 which have the same sign. (Also, write $s = n-1-r$.) Then, there exists a unique conjugacy class $C_{r,s} \subset \mathcal{S}(\mathbb{C}^{n-1})$ and for each $R \in C_{r,s}$ there are local holomorphic coordinates $Z = (z,w) \in \mathbb{C}^n \times \mathbb{C}$, $z = (z', z^n) \in \mathbb{C}^{n-2} \times \mathbb{C}$, near p_0, vanishing at p_0, such that the defining equation of M is of the following form,*

$$\begin{aligned} \operatorname{Im} w = &\sum_{j=1}^{r} |z^j|^2 - \sum_{k=r+1}^{n-1} |z^k|^2 \\ &+ 2\operatorname{Re}\left(\bar{z}^{\bar{n}}\left((z')^\tau R z' + (z^n)^2\right)\right) + F(z,\bar{z},\operatorname{Re} w), \end{aligned} \tag{1.1.4}$$

where $F(z,\bar{z},\operatorname{Re} w)$ denotes a smooth, real valued function which is $O(4)$ in the weighted coordinate system where z has weight one and w weight two.

Let us briefly explain our usage of the notation $O(\nu)$, for nonnegative integers ν, in Proposition 1.1.3. We assign the weight one to the variables $z = (z', z^n) = (z^1, \ldots, z^{n-1}, z^n)$, the weight two to w, and say that a polynomial $p_\nu(z,w)$ is weighted homogeneous of degree ν if, for all $t > 0$,

$$p_\nu(tz, t^2 w) = t^\nu p_\nu(z,w). \tag{1.1.5}$$

We shall write $O(\nu)$ for a formal series involving only terms of weighted degree greater than or equal to ν. We say that a smooth function defined near 0 is $O(\nu)$ (at 0) if its Taylor series at 0 is $O(\nu)$. Similarly, we speak of weighted homogeneity of degree ν and $O(\nu)$ for polynomials, power series, and functions in $(z, \bar{z}, \mathrm{Re}\ w)$, where $\bar{z}$ is assigned the weight one and Re w the weight two.

We shall now present a complete, formal, normal form for a generic Levi degeneracy. Before stating the theorem, we need to define the space of normal forms and the normalization for the transformation to normal form.

By Proposition 1.1.3, we may assume that M is defined near $p_0 = (0,0)$ by (1.1.4) for given, and fixed for the remainder of this section, integer r and matrix $R \in \mathcal{S}(\mathbb{C}^{n-1})$. We shall assume that $n \geq 2$, so that $n-1 \geq 1$. Since we shall present a formal normal form, we consider the defining equation (1.1.4) as a formal power series. It is well known (cf. [6] and [7]; cf. also the forthcoming book [4]) that, after an additional formal change of coordinates at $(0,0)$ if necessary, we may also assume that $F(z,0,s) \equiv F(0,\bar{z},s) \equiv 0$; we shall say that the (formal) coordinates (z,w) are *regular* for M at $p_0 = (0,0)$ if the (formal) defining equation for M at that point is of the form $\mathrm{Im}\ w = \phi(z,\bar{z},\mathrm{Re}\ w)$ with $\phi(z,0,s) \equiv \phi(0,\bar{z},s) \equiv 0$. We subject the (formal) hypersurface M to a formal invertible transformation

$$z = \tilde{f}(\tilde{z},\tilde{w}) \quad , \quad w = \tilde{g}(\tilde{z},\tilde{w}), \tag{1.1.6}$$

where

$$\tilde{f} = (\tilde{f}', \tilde{f}^n) = (\tilde{f}^1, \ldots, \tilde{f}^{n-1}, \tilde{f}^n), \tag{1.1.7}$$

such that the form (1.1.4) is preserved. We shall also require that the coordinates $(\tilde{z},\tilde{w})$ are regular for M, i.e., the remainder $\tilde{F}(\tilde{z},\bar{\tilde{z}},\tilde{s})$ corresponding to the defining equation relative the coordinates $(\tilde{z},\tilde{w})$ satisfies $\tilde{F}(\tilde{z},0,\tilde{s}) \equiv \tilde{F}(0,\bar{\tilde{z}},\tilde{s}) \equiv 0$.

Given a matrix $A \in \mathcal{S}(\mathbb{C}^m)$, we denote by $O_A(\mathbb{C}^m)$ the subgroup of $GL(\mathbb{C}^m)$ consisting of those matrices that preserve the bilinear form associated with A, i.e., $B \in O_A(\mathbb{C}^m)$ if

$$B^\tau A B = A. \tag{1.1.8}$$

We have the following proposition whose proof will be given in §7.

Proposition 1.1.9. *A transformation* (1.1.6) *preserving regular coordinates also preserves the form* (1.1.4), *for a given integer* r *and* $R \in \mathcal{S}(\mathbb{C}^{n-1})$, *if and only if*

$$\begin{aligned} \tilde{f}'(z,w) &= Az' + wB + \frac{2i}{c}(B^* I_{r,s} Az')Az' + O(3), \\ \tilde{f}^n(z,w) &= c^{1/3} z^n + O(2), \\ \tilde{g}(z,w) &= cw + 2i(B^* I_{r,s} Az')w + O(4), \end{aligned} \tag{1.1.10}$$

where $c \in \mathbb{R} \setminus \{0\}$, $B \in \mathbb{C}^{n-1}$ *(considered as an* $(n-1) \times 1$ *matrix), and where* A *is such that* $A/|c|^{1/3} \in O_R(\mathbb{C}^{n-1})$ *and* $A^* I_{r,s} A = c I_{r,s}$ *(in particular then,* $A/|c|^{1/2} \in \hat{U}(r,s,\mathbb{C})$*); As above, we use* $s = n - r - 1$.

We shall consider formal mappings (1.1.6) of the following form:

$$(\tilde{f}(z,w), \tilde{g}(z,w)) = (T \circ P)(z,w). \tag{1.1.12}$$

Here, $P(z,w)$ is a polynomial mapping, $P = (P', P^n, P^{n+1})$,

$$P(z,w) = (p'(z,w) + q'(z,w), p^n(z,w) + q^n(z,w), p^{n+1}(z,w)), \tag{1.1.13}$$

where $p' = (p^1, \dots, p^{n-1})$, p^n, p^{n+1} are polynomials of the form

$$\begin{aligned} p'(z,w) &= Az' + wB + \frac{2i}{c}(B^* I_{r,s} Az')Az', \\ p^n(z,w) &= c^{1/3} z^n, \\ p^{n+1}(z,w) &= cw + 2i(B^* I_{r,s} Az')w, \end{aligned} \tag{1.1.14}$$

where c, B, A are as in Proposition 1.1.9. The polynomials $q' = (q^1, \dots, q^{n-1})$ and q^n are weighted homogeneous of the forms

$$\begin{aligned} q^\beta(z,w) &= \sum_{|J|=3} a_J^\beta z^J + \left(\sum_{\alpha<\beta} b_\alpha^\beta (Az')^\alpha + c^\beta (Az')^\beta \right) w, \\ q^n(z,w) &= \sum_{|I|=2} d_I z^I, \end{aligned} \tag{1.1.15}$$

where $\beta = 1, \dots, n-1$, $a_J^\beta, b_J^\beta, d_I \in \mathbb{C}$, and $c^\beta \in \mathbb{R}$. We use here multi-index notation so that e.g. $J = (J_1, \dots, J_n)$, $|J| = \sum_k J_k$, and $z^J = (z^1)^{J_1} \dots (z^n)^{J_n}$. The notation $(Az')^\beta$ stands for the β:th component of the vector Az'. $T(z,w)$ in (1.1.12) is a formal mapping of the form

$$T(z,w) = (z + f(z,w), w + g(z,w)), \tag{1.1.16}$$

where $f = (f', f^n) = (f^1, \dots, f^{n-1}, f^n)$ and g are formal power series in (z, w) such that f' is $O(3)$, f^n is $O(2)$, and g is $O(4)$. We shall also require that the formal series f', f^n are such that the constant terms in the following formal series vanish

$$\frac{\partial^2 f^n}{\partial z^I}, \frac{\partial^3 f^\beta}{\partial z^J}, \operatorname{Re} \frac{\partial^2 f^\beta}{\partial z^\beta \partial w}, \frac{\partial^2 f^\beta}{\partial z^\alpha \partial w}, \tag{1.1.17}$$

where I and J range over all the multi-indices with $|I| = 2$ and $|J| = 3$, respectively, the index β runs over $1, \dots, n-1$, and α runs over $1, \dots, \beta - 1$. It is straightforward, and left to the reader, to verify (using Proposition 1.1.9) that any formal mapping (1.1.6) that preserves the form (1.1.4) of M can be factored uniquely according to (1.1.12) with T and P as above. We shall say that a choice of P, as described above, is a choice of *normalization* for the transformations which preserve the form (1.1.4) and that a formal mapping preserving the form has this normalization if it is factored according to (1.1.12) with this P.

Now, let $F(z, \bar{z}, s)$ be a formal series in $(z, \bar{z}, s)$. In what follows, we shall decompose the formal series $F(z, \bar{z}, s)$ as follows,

$$F(z, \bar{z}, s) = \sum_{k,l} F_{kl}(z, \bar{z}, s), \tag{1.1.18}$$

where $F_{kl}(z, \bar{z}, s)$ is of type (k, l), i.e., for each $t_1, t_2 > 0$

$$F_{kl}(t_1 z, t_2 \bar{z}, s) = t_1^k t_2^l F_{kl}(z, \bar{z}, s). \tag{1.1.19}$$

We shall consider only those $F(z, \bar{z}, s)$ which are $O(4)$ and are "real" in the sense that

$$F_{kl}(z, \bar{z}, s) = \overline{F_{lk}(z, \bar{z}, s)}. \tag{1.1.20}$$

We shall denote by $\mathcal{F}$ the space of all such formal power series, and by $\mathcal{F}_{kl}$ the space consisting of those which have type (k, l). In what follows, F_{kl}, H_{kl}, and N_{kl} denote formal power series in $\mathcal{F}_{kl}$.

In order to describe the space of normal forms, $\mathcal{N} \subset \mathcal{F}$, we need a little more notation. Recall that the integer r and matrix $R \in \mathcal{S}(\mathbb{C}^{n-1})$ from Proposition 1.1.3 are fixed throughout this section. For $u = (u^1, \dots, u^{n-1})$ and $v = (v^1, \dots, v^{n-1})$, we use the notation $\langle \cdot, \cdot \rangle$ for the bilinear form

$$\langle u, v \rangle = \sum_{j=1}^{r} u^j v^j - \sum_{k=r+1}^{n-1} u^k v^k. \tag{1.1.21}$$

We denote by $p_R(z)$ the quadratic polynomial

$$p_R(z) = (z')^\tau R z' + (z^n)^2. \tag{1.1.22}$$

We use the notation $\nabla = (\nabla', \nabla_n) = (\nabla_1, \dots, \nabla_{n-1}, \nabla_n)$ for the holomorphic gradient

$$\nabla = \left(\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n}\right),$$

and similarly for the anti-holomorphic gradient $\bar{\nabla}$. We shall need the linear operator S_R defined on formal series $u = u(z, \bar{z}, s)$ as follows:

$$S_R u = -\left\langle \nabla', \bar{\nabla}' \right\rangle (p_R u). \tag{1.1.23}$$

Observe that S_R maps $\mathcal{F}_{k-1,l+1}$ into $\mathcal{F}_{kl}$. Let us remark that the operator $\left\langle \nabla', \bar{\nabla}' \right\rangle$ is essentially the same as the contraction operator tr corresponding to the bilinear form $\langle \cdot, \cdot \rangle$ as defined in [11]; they correspond to different normalizations for the monomials.

We define the space of normal forms $\mathcal{N} \subset \mathcal{F}$ for M of the form (1.1.4) with r and R as in Proposition 1.1.3 as follows. First, a formal series $N(z, \bar{z}, s)$ in $\mathcal{N}$ is in *regular form* which can be expressed by

$$N(z, \bar{z}, s) = \sum_{\min(k,l) \geq 1} N_{kl}(z, \bar{z}, s); \tag{1.1.24}$$

thus, N has no components of type (k, l) with $k = 0$ or $l = 0$. Moreover, the nonzero terms N_{kl} satisfy the following conditions:

$$\begin{aligned} &N_{22} \in \mathcal{N}_{22}, \quad N_{32} \in \mathcal{N}_{32}, \\ &N_{42} \in \mathcal{N}_{42}, \quad N_{33} \in \mathcal{N}_{33}, \\ &N_{k1} \in \mathcal{N}_{k1}, \quad k = 1, 2, 3 \dots, \end{aligned} \tag{1.1.25}$$

where

$$
\begin{aligned}
N_{11} =& \{F_{11} : F_{11} \in \ker \langle \nabla', \bar{\nabla}' \rangle \}, \\
N_{22} =& \{F_{22} : F_{22} = \langle z', \bar{z}' \rangle z^n \bar{z}^{\bar{n}} H_{00} + H_{22}, \\
& \qquad H_{22} \in \ker \langle \nabla', \bar{\nabla}' \rangle \}, \\
N_{33} =& \{F_{33} : F_{33} = \langle z', \bar{z}' \rangle^2 (z^n H_{01} + \overline{z^n H_{01}}) + H_{33}, \\
& \qquad H_{33} \in \ker \langle \nabla', \bar{\nabla}' \rangle^2 \}, \\
N_{21} =& \{F_{21} : F_{21} = \bar{z}_{\bar{n}} H_{20}\}, \\
N_{31} =& \{F_{31} : F_{31} \in \ker S_R\}, \\
N_{32} =& \{F_{32} : F_{32} = \langle z', \bar{z}' \rangle^2 z^n H_{00} + \langle z', \bar{z}' \rangle H_{21} + H_{32}, \\
& \qquad H_{32} \in \ker \langle \nabla', \bar{\nabla}' \rangle, \ H_{21} \in \ker p_R(\nabla), \\
& \qquad \langle \nabla', \bar{\nabla}' \rangle H_{21} \in \mathrm{Im}\ S_R\}, \\
N_{42} =& \{F_{42} : F_{42} = \langle z', \bar{z}' \rangle \bar{z}^{\bar{n}} H_{30} + H_{42}, \ H_{42} \in \ker \langle \nabla', \bar{\nabla}' \rangle, \\
& \qquad H_{30} \in \ker \nabla_n\},
\end{aligned}
\tag{1.1.26}
$$

and finally, for $k \geq 4$,

$$
N_{k1} = \{F_{k1} : F_{k1} = \bar{z}^{\bar{n}} H_{k0}, \ H_{k0} \in \ker \nabla_n\}. \tag{1.1.27}
$$

Observe that, for a series H_{k0} of type $(k, 0)$, the condition $H_{k0} \in \ker \nabla_n$ is equivalent to the condition that H_{k0} is independent of z^n, i.e., $H_{k0} = H_{k0}(z', s)$.

We are now in a position to state the theorem on normal forms for a generic Levi degeneracy.

Theorem 1.1.28. *Let M be a smooth hypersurface in $\mathbb{C}^{n+1}$ given near $0 \in M$ by* (1.1.4)*, where r and R are as in Proposition* 1.1.3. *Then, given any choice of normalization (i.e., a choice of P as described above), there is a unique formal transformation* (1.1.6) *with this normalization that transforms the defining equation* (1.1.4) *of M at* 0 *to*

$$
\begin{aligned}
\mathrm{Im}\ w =& \sum_{j=1}^{r} |z^j|^2 - \sum_{k=r+1}^{n-1} |z^k|^2 \\
& + 2\mathrm{Re} \left(\bar{z}^{\bar{n}} \left((z')^\tau R z' + (z^n)^2 \right) \right) + N(z, \bar{z}, \mathrm{Re}\ w),
\end{aligned}
\tag{1.1.29}
$$

where $N(z, \bar{z}, s) \in \mathcal{N}$.

The proof of Theorem 1.1.28 will be given in §8. We conclude this section by applying Theorem 1.1.28 to the biholomorphic equivalence problem. Suppose that (M,p_0) and (M',p_0') are two germs of real-analytic hypersurfaces in $\mathbb{C}^{n+1}$ which have generic Levi degeneracies at p_0 and p_0', respectively. Thus, M and M' are, in particular, finitely nondegenerate (see §3) at their distinguished points p_0 and p_0'. In view of [3, Theorem 2.6], any formal equivalence between (M,p_0) and (M',p_0') is then in fact biholomorphic. Hence, an immediate consequence of Theorem 1.1.28, as in [12], is the following.

Corollary 1.1.30. *Let M and M' be real-analytic hypersurfaces in $\mathbb{C}^{n+1}$ which have generic Levi degeneracies at $p_0 \in M$ and $p_0' \in M'$, respectively. Suppose that the integers r and conjugacy classes $C_{r,s}$, given by Proposition* 1.1.3*, for M and M' at p_0 and p_0', respectively, coincide. Then (M,p_0) and (M',p_0') are biholomorphically equivalent if and only if, for any choice of $R \in C_{r,s}$ and two (possibly different) choices of normalization as described in Theorem* 1.1.28*, (M,p_0) and (M',p_0') can be brought to the same normal form.*

1.2. The semidefinite case. In Proposition 1.1.3, the partial normal form for a real hypersurface M at a generic Levi degeneracy $p_0 \in M$ is given in terms of a conjugacy class $C_{r,s}$ in $\mathcal{S}(\mathbb{C}^{n-1})$. In order to obtain a more explicit partial normal form, we must distinguish a unique representative in each conjugacy class. In this paper, we shall only address this problem in the case where the Levi form at p_0 is semidefinite, i.e., $r = n-1$ and $s = 0$, in which case the group $\hat{U}(r,s,\mathbb{C})$ reduces to the unitary group $U(\mathbb{C}^{n-1})$ and a lemma due to E. Cartan can be applied. The details are worked out in §5 below. We state here the corresponding normal forms, which follow from the results in §1.1 above and §5.

Thus, we assume that M has a generic Levi degeneracy at $p_0 \in M$, and that the Levi form at that point is semidefinite (i.e., the integer r in Proposition 1.1.3 equals $n-1$). An immediate consequence of Theorem 5.8 (which in fact treats a slightly more general case; see Theorem 1.2.10 below) is that there are local holomorphic coordinates $Z = (z,w)$ as in Proposition 1.1.3 such that M is given near $p_0 = (0,0)$ by

$$
\begin{aligned}
\operatorname{Im} w = &\sum_{j=1}^{n-1} |z^j|^2 \\
&+ 2\operatorname{Re}\left(\bar{z}^{\bar{n}}\left((z')^\tau D_{n-1}(\lambda) z' + (z^n)^2\right)\right) + F(z,\bar{z},\operatorname{Re} w),
\end{aligned}
\tag{1.2.1}
$$

where F is as in Proposition 1.1.3, and $\lambda = (\lambda_1, \dots, \lambda_{n-1})$ is a uniquely determined vector with $\lambda_1 \geq \dots \geq \lambda_{n-1} \geq 0$ such that either $\lambda_1 = 1$ or

$\lambda_k = 0$ for $k = 1, \dots, n-1$. We use here the notation $D_{n-1}(\lambda)$ for the diagonal $(n-1) \times (n-1)$-matrix with λ on the diagonal, i.e.,

$$D_{n-1}(\lambda) = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{n-1} \end{pmatrix}. \tag{1.2.2}$$

An inspection of the proof of Proposition 1.1.9 shows that the most general transformation of the form (1.1.6) preserving regular coordinates and equation (1.2.1) is of the form

$$\begin{aligned} \tilde{f}'(z,w) &= c^{1/2}Uz' + wB + 2i(B^*Uz')Uz' + O(3), \\ \tilde{f}^n(z,w) &= c^{1/3}z^n + O(2), \\ \tilde{g}(z,w) &= cw + 2ic^{1/2}(B^*Uz')w + O(4), \end{aligned} \tag{1.2.3}$$

where $c > 0$, $B \in \mathbb{C}^{n-1}$ (considered as an $(n-1) \times 1$ matrix), and $U \in U(\mathbb{C}^{n-1})$, if $\lambda = 0$, and

$$\begin{aligned} \tilde{f}'(z,w) &= Az' + wB + 2i(B^*Az')Az' + O(3), \\ \tilde{f}^n(z,w) &= z^n + O(2), \\ \tilde{g}(z,w) &= w + 2i(B^*Az')w + O(4), \end{aligned} \tag{1.2.4}$$

where $B \in \mathbb{C}^{n-1}$ (considered as an $(n-1) \times 1$ matrix), and

$$A \in U(\mathbb{C}^{n-1}) \cap O_{D_{n-1}(\lambda)}(\mathbb{C}^{n-1}),$$

if $\lambda \neq 0$. (The group

$$U(\mathbb{C}^{n-1}) \cap O_{D_{n-1}(\lambda)}(\mathbb{C}^{n-1})$$

is described in more detail in Lemma 5.24.) Using the corresponding factorization (1.1.12) and the description of the space of normal forms $\mathcal{N}$ given in §1.1 with $R = D_{n-1}(\lambda)$, we get the following result.

Theorem 1.2.5. *Let M be a smooth hypersurface in $\mathbb{C}^{n+1}$ given near $0 \in M$ by* (1.2.1)*, where λ is the invariant $(n-1)$-vector described above. Then, given any choice of normalization (i.e., a choice of P as described above), there is a unique formal transformation* (1.1.6) *with* *this normalization that transforms the defining equation* (1.2.1) *of M at* 0 *to*

$$\begin{aligned}\text{Im } w = &\sum_{k=1}^{n-1} |z^k|^2 \\ &+ 2\text{Re}\left(\bar{z}^{\bar{n}}\left(\sum_{k=1}^{n-1}\lambda_k (z^k)^2 + (z^n)^2\right)\right) + N(z,\bar{z},\text{Re } w).\end{aligned} \tag{1.2.6}$$

where $N(z,\bar{z},s) \in \mathcal{N}$.

Due to the explicit description of the normalization of the transformation to normal form, we can compute a bound on the dimension of the stability group $\text{Aut}(M,p_0)$ of a smooth hypersurface $M \subset \mathbb{C}^{n+1}$ at a generic semidefinite Levi degeneracy $p_0 \in M$. Recall that $\text{Aut}(M,p_0)$ is the group of biholomorphic transformations near p_0 that fix p_0 and map M into itself. It is a real, finite dimensional Lie group in view of results from [3] (see also [26] and [2] for results in the higher codimensional case).

Corollary 1.2.7. *Let $M \subset \mathbb{C}^{n+1}$ be a smooth hypersurface which has a generic semidefinite Levi degeneracy at p_0. Let λ be the invariant appearing in* (1.2.1)*. Then, the following hold.*

(a) If $\lambda = (0,\dots,0)$, then

$$\dim_{\mathbb{R}}\text{Aut}(M,p_0) \le \frac{1}{3}(n-1)n(n+1)(n+2) + 3n^2 - n + 1. \tag{1.2.8}$$

(b) If $\lambda = (1,\lambda_2,\dots,\lambda_{n-1})$ with $1 \ge \lambda_2 \ge \dots \ge \lambda_{n-1} \ge 0$, then we write $(1,u_2,\dots,u_k,0)$ for the distinct values of $(1,\lambda_2,\dots,\lambda_{n-1})$ and denote by $(m_1,\, m_2,\dots,m_k,\mu)$ their multiplicities. (Thus, μ is the multiplicity of the value 0.*) Then,*

$$\begin{aligned}\dim_{\mathbb{R}}\text{Aut}(M,p_0) \le &\frac{1}{3}(n-1)n(n+1)(n+2) \\ &+ 2n^2 + n - 1 \\ &+ \sum_{j=1}^{k}\frac{1}{2}m_j(m_j-1) + \mu^2.\end{aligned} \tag{1.2.9}$$

The bound in Corollary 1.2.7 is sharper than the bound that follows from the results in [2]–[3]. The latter bound grows like n^5 whereas the former grows like n^4 as $n \to \infty$. The proof of Corollary 1.2.7 consists of counting the number of parameters in the normalization of the transformation to normal form and using the explicit representation of $U(\mathbb{C}^{n-1}) \cap O_{D_{n-1}(\lambda)}(\mathbb{C}^{n-1})$ provided by Lemma 5.24. The details are left to the reader.

Let us conclude this section by mentioning that Theorem 5.8 (in combination with Theorem 4.15) yields a partial (third order) normal form in a more general case than the one considered above. Indeed, as a consequence of Theorem 5.8, we have the following result, in which the Levi degeneracy is not assumed to be generic.

Theorem 1.2.10. *Let $M \subset \mathbb{C}^{n+1}$ be a real smooth hypersurface and $p_0 \in M$. Suppose that the Levi form of M at p_0 has rank $n-1$ and is semidefinite, i.e., all nonzero eigenvalues of the Levi form have the same sign. Then, there are local holomorphic coordinates $Z = (z, w) \in \mathbb{C}^n \times \mathbb{C}$ near p_0, vanishing at p_0, such that the defining equation of M is of precisely one of the following forms.*

(i) *For either $\lambda = (1, \lambda_2, \dots, \lambda_{n-2}, 0)$ with $1 \geq \lambda_2 \geq \dots \geq \lambda_{n-2} \geq 0$ or $\lambda = (0, \dots, 0)$,*

$$\begin{aligned} \text{Im } w = \sum_{k=1}^{n-1} |z^k|^2 + 2\text{Re}\left(\bar{z}^{\bar{n}}\left(\sum_{k=1}^{n-1} \lambda_k (z^k)^2 + 2z^{n-1}z^n\right)\right) \\ + F(z, \bar{z}, \text{Re } w). \end{aligned} \tag{1.2.11}$$

(ii) *For either $\lambda = (1, \lambda_2, \dots, \lambda_{n-2}, \lambda_{n-1})$ with $1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1} \geq 0$ or $\lambda = (0, \dots, 0)$,*

$$\text{Im } w = \sum_{k=1}^{n-1} |z^k|^2 + 2\text{Re}\left(\bar{z}^{\bar{n}} \sum_{k=1}^{n-1} \lambda_k (z^k)^2\right) + F(z, \bar{z}, \text{Re } w). \tag{1.2.12}$$

(iii) *For either $\lambda = (1, \lambda_2, \dots, \lambda_{n-2}, \lambda_{n-1})$ with $1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1} \geq 0$ or $\lambda = (0, \dots, 0)$,*

$$\begin{aligned} \text{Im } w = \sum_{k=1}^{n-1} |z^k|^2 + 2\text{Re}\left(\bar{z}^{\bar{n}}\left(\sum_{k=1}^{n-1} \lambda_k (z^k)^2 + (z^n)^2\right)\right) \\ + F(z, \bar{z}, \text{Re } w). \end{aligned} \tag{1.2.13}$$

Above, $F(z, \bar{z}, \mathrm{Re}\, w)$ *denotes a smooth, real valued function which is* $O(4)$ *in the weighted coordinate system where* z *has weight one and* w *weight two.*

In the case $n = 2$*, the same result holds with the following modifications: The only choice for* λ *in* (i) *is* $\lambda = 0$*. In* (ii) *and* (iii)*, both* $\lambda = 1$ *and* $\lambda = 0$ *are allowed.*

Before proving the results on normal forms presented in these two sections, we shall introduce a new sequence of invariant tensors. This will be done in the more general setting of generic submanifolds of $\mathbb{C}^N$.

2. CR invariant tensors

Let $M \subset \mathbb{C}^N$ be a real generic smooth submanifold of codimension d. Denote by $T^c M \subset TM$ the complex tangent bundle to M, by $\mathcal{V} = T^{0,1}M \subset \mathbb{C}T^c M$ the CR bundle of M, by $T^0 M \subset T^* M$ the characteristic bundle, and by $T'M \subset \mathbb{C}T^* M$ the bundle defined at each $p \in M$ as the annihilator of $\mathcal{V}_p$. We denote by n the CR dimension of M, i.e., $n = N - d$. We have the following for any $p \in M$:

$$\begin{aligned} &\dim_{\mathbb{R}} T^c_p M = 2n, \quad \dim_{\mathbb{C}} \mathcal{V}_p = n, \\ &\dim_{\mathbb{R}} T^0_p M = d, \quad \dim_{\mathbb{C}} T'_p M = n + d. \end{aligned} \tag{2.1}$$

For a vector bundle E over M, we denote the smooth sections of E by $C^\infty(M, E)$. The reader is referred e.g. to [4] or [8] for the basics of CR structures. We shall consider only local properties of M near some point p. Hence, given a point $p \in M$, we may, and we will, identify M with some small open neighborhood of p in M.

For a CR vector field L on M, i.e., a smooth section of $\mathcal{V}$, we define an operator $\mathcal{T}_L$ on the smooth 1-forms on M as follows:

$$\mathcal{T}_L \omega = \frac{1}{2i} L \lrcorner d\omega, \tag{2.2}$$

where $\lrcorner$ denotes the usual contraction operator. We should point out here that we use the notation $\langle \cdot, \cdot \rangle$ for the pairing between r-covectors and r-vectors normalized in such a way that if e_α and e^β, $\alpha, \beta = 1, \ldots m$, are dual bases for an m-dimensional vector space V and its dual V^*, respectively, then $e_{\alpha_1} \wedge \ldots \wedge e_{\alpha_r}$, $1 \le \alpha_1 < \ldots < \alpha_r \le m$, and $e^{\beta_1} \wedge \ldots \wedge e^{\beta_r}$, $1 \le \beta_1 < \ldots < \beta_r \le m$, are dual bases for $\Lambda^r(V)$ and $\Lambda^r(V^*)$, respectively (see [21, Chapter I.4]). This normalization is more convenient for our purposes than the one used in e.g. [16], which differs from the present one by the factor $r!$, and is identical to the one used in [8].

We shall refer to sections of $T'M$ as $(1,0)$-forms and denote by $\Omega^{1,0}(M)$ the space of smooth $(1,0)$-forms on M. It is not difficult to see that $\mathcal{T}_L : \Omega^{1,0}(M) \to \Omega^{1,0}(M)$, for if $\omega \in \Omega^{1,0}(M)$ then, for any CR vector field K, we obtain, by using the well known identity (see [16, Chapter I.2]),

$$\begin{aligned} \langle \mathcal{T}_L\omega, K\rangle &= \langle d\omega, L \wedge K\rangle \\ &= L\left(\langle \omega, K\rangle\right) - K\left(\langle \omega, L\rangle\right) - \langle \omega, [L,K]\rangle = 0, \end{aligned} \tag{2.2}$$

since ω is a section of $T'M$, which at each point $p \in M$ annihilates $\mathcal{V}_p$, and the CR bundle $\mathcal{V}$ is involutive (or, as it is also called, formally integrable). We shall use the notation $\mathcal{L}(M) \subset \Omega^{1,0}(M)$ for those smooth $(1,0)$-forms that are sections of T^0M. The forms in $\mathcal{L}(M)$ will also be referred to as *characteristic forms.*

Let $p \in M$ and let us define a sequence of increasing subspaces

$$E_0(p) \subset E_1(p) \subset \ldots \subset E_k(p) \subset \ldots \subset T'_pM \tag{2.3}$$

as follows. Set $E_0(p) = \mathbb{C} \otimes T^0_pM$ and let $E_j(p)$, for $j = 1, 2, \ldots$, be the linear span of $(1,0)$-covectors of the form

$$(\mathcal{T}_{K_{\bar{1}}} \ldots \mathcal{T}_{K_{\bar{j}}}\theta)_p, \tag{2.4}$$

where the $K_{\bar{i}}$ range over all CR vector fields on M near p, and θ ranges over the smooth sections of T^0M near p. The reason for putting a bar on the indices of CR vector fields is to be able to use the notation of tensor algebra in later sections; recall that the CR vector fields for an embedded CR submanifold are really anti-holomorphic vector fields.

We shall see later that to compute the subspaces $E_j(p)$ it suffices to take the linear span of the covectors (2.4) where the CR vector fields $K_{\bar{i}}$ range over the elements of any basis for the CR vector fields near p and the characteristic forms θ range over a basis for the smooth sections of T^0M near p. We will also show that M is finitely nondegenerate (see below and also e.g. [4]) at p if and only if $E_k(p) = T'_pM$ for some $\bar{k}$. The reader should also note that these subspaces are the same as, but differently indexed than, those defined in [13]. The present definition is better suited for the purposes of this paper.

Let us for a given integer $k \geq 0$ denote by $F_k(p) \subset \bar{\mathcal{V}}_p$ the subspace of those $\bar{N}_p \in \bar{\mathcal{V}}_p$ that annihilate $E_k(0)$, i.e.,

$$F_k(p) = E_k(p)^{\perp} \cap \bar{\mathcal{V}}_p. \tag{2.5}$$

Thus, for $k=0$ we have $F_0(p)=\bar{\mathcal{V}}_p$. Let $\mathcal{F}_k(M)\subset C^\infty(M,\bar{\mathcal{V}})$ denote the space of those sections that take values in $F_k(p)$ at p. Note that $\mathcal{F}_k(M)$ is a $C^\infty(M)$-submodule of $C^\infty(M,\bar{\mathcal{V}})$. Consider the following diagram, for integers $j\geq 1$,

$$\begin{array}{ccc} \overbrace{C^\infty(M,\mathcal{V})\times\ldots\times C^\infty(M,\mathcal{V})}^{j\text{ times}}\times\mathcal{F}_{j-1}(M)\times\mathcal{L}(M) & \xrightarrow{G_j} & \mathbb{C} \\ {\scriptstyle e_j}\big\downarrow & & \\ \underbrace{\mathcal{V}_p\times\ldots\times\mathcal{V}_p}_{j\text{ times}}\times F_{j-1}(p)\times T_p^0M\ , & & \end{array} \tag{2.6}$$

where e_j is the evaluation map at p, and G_j is the mapping

$$(K_{\bar{1}},\ldots,K_{\bar{j}},\bar{N},\theta)\mapsto\left\langle\mathcal{T}_{K_{\bar{1}}}\ldots\mathcal{T}_{K_{\bar{j}}}\theta,\bar{N}\right\rangle_p. \tag{2.7}$$

We would like to have a multi-linear map

$$\psi_{j+1}:\underbrace{\mathcal{V}_p\times\ldots\times\mathcal{V}_p}_{j\text{ times}}\times F_{j-1}(p)\times T_p^0M\to\mathbb{C} \tag{2.8}$$

that makes the diagram (2.6) commute. Such a multi-linear map would, by definition, be an invariant of the CR structure $(M,\mathcal{V})$ (and hence also a biholomorphic invariant for the generic submanifold $M\subset\mathbb{C}^N$ at $p\in M$).

One of the main results is the following.

Theorem 2.9. *For each positive integer j, there exists a unique multi-linear mapping* (2.8) *which makes the diagram* (2.6) *commute. The multi-linear mapping* (2.8), *for each j, is symmetric with respect to permutations of the first j variables.*

The multi-linear map ψ_j can also be identified with a tensor

$$\psi_{j+1}\in\underbrace{\mathcal{V}_p^*\otimes\ldots\otimes\mathcal{V}_p^*}_{j\text{ times}}\otimes F_{j-1}(p)^*\otimes(T_p^0M)^*. \tag{2.10}$$

Before proving Theorem 2.9, let us make a few remarks.

Remark 2.11.

(i) For $j = 1$ and a fixed characteristic covector $\theta_p \in T^0_p M$, the Hermitian form $\mathcal{V}_p \times \mathcal{V}_p \to \mathbb{C}$ defined by $(L, K) \mapsto \psi_2(L, \bar{K}, \theta_p)$ coincides with the Levi form of M at the point p and the characteristic covector θ_p.

(ii) As mentioned above and as will be proved below, M is finitely nondegenerate at p if and only if $E_k(p) = T'_p M$ for some k. If M is finitely nondegenerate at p, then it is called k-nondegenerate at p if k is the smallest integer for which $E_k(p) = T'_p M$. It follows that for a k-nondegenerate CR manifold the tensors ψ_{j+1}, $j \geq k+1$, are trivial, since $F_j(p) = \{0\}$. Hence, if e.g. M is a Levi nondegenerate hypersurface (which is the same as a 1-nondegenerate hypersurface), then the only non-trivial invariant tensor produced by Theorem 2.9 is the Levi form of M at p.

(iii) The invariant tensors provide obstructions for two generic submanifolds $M, M' \subset \mathbb{C}^N$ of codimension d to be biholomorphically equivalent at given points $p \in M$, $p' \in M'$. The submanifolds (M, p) and (M', p') cannot be biholomorhically equivalent unless $\dim F_j(p) = \dim F'_j(p')$ (with the obvious notation that corresponding object for M' are denoted with a $'$) and the tensors ψ_{j+1} and ψ'_{j+1} are equivalent (i.e., there are bases in $\mathcal{V}_p$, $\mathcal{V}'_{p'}$, $F_j(p)$, $F'_j(p')$, $T^0_p M$, and $T^0_{p'} M'$ such that the representations of ψ_{j+1} and ψ'_{j+1} are equal) for each $j = 1, 2, \ldots$. The reader should note, however, that the tensors ψ_{j+1} do not provide a *complete* set of invariants in the sense that (M, p) and (M', p') are biholomorphically equivalent if all tensors are equivalent. This is illustrated e.g. by Theorem 1.1.28, since the normal form given in that theorem gives a complete set of invariants (by Corollary 1.1.30) and the invariants coming from the tensors only enter into the second and third order terms.

Proof of Theorem 2.9. We claim that for the multi-linear mapping ψ_{j+1} in (2.8) such that the diagram (2.6) commutes to exist, it is necessary and sufficient that the following statements hold:

(a) For any $l \in \{1, 2, \ldots, j\}$, $K' = (K_{\bar{1}}, \ldots, K_{\overline{l-1}}) \in (C^\infty(M, \mathcal{V}))^{l-1}$, $A, B \in C^\infty(M, \mathcal{V})$, $K'' = (K_{\overline{l+1}}, \ldots, K_{\bar{j}}) \in (C^\infty(M, \mathcal{V}))^{j-l}$, $a, b \in C^\infty(M)$, $\bar{N} \in \mathcal{F}(M)$, and $\theta \in \mathcal{L}(M)$, the following identity holds:

$$\begin{aligned} &G_j(K', aA+bB, K'', \bar{N}, \theta) \\ &\qquad = a(p)G_j(K', A, K'', \bar{N}, \theta) \\ &\qquad\quad + b(p)G_j(K', B, K'', \bar{N}, \theta). \end{aligned} \tag{2.12}$$

(b) For any $K = (K_{\bar{1}}, \dots, K_{\bar{j}}) \in (C^\infty(M, \mathcal{V}))^j$, $a, b \in C^\infty(M)$, $\bar{A}, \bar{B} \in \mathcal{F}(M)$, and $\theta \in \mathcal{L}(M)$, the following identity holds:

$$G_j(K, a\bar{A} + b\bar{B}, \theta) = a(p)G_j(K, \bar{A}, \theta) + b(p)G_j(K, \bar{B}, \theta). \tag{2.13}$$

(c) For any $K = (K_{\bar{1}}, \dots, K_{\bar{j}}) \in (C^\infty(M, \mathcal{V}))^j$, $\bar{N} \in \mathcal{F}(M)$, $a, b \in C^\infty(M)$, $\xi, \eta \in \mathcal{L}(M)$, the following identity holds:

$$G_j(K, \bar{N}, a\xi + b\eta) = a(p)G_j(K, \bar{N}, \xi) + b(p)G_j(K, \bar{N}, \eta). \tag{2.14}$$

Indeed, if the mapping ψ_{j+1} exists, then the statements (a), (b), and (c) follow immediately from the diagram (2.6). Conversely, if the statements (a), (b), and (c) hold, then the mapping ψ_{j+1} can be uniquely constructed as follows. Take $L_{\bar{1}}, \dots, L_{\bar{n}}$ to be any basis for the CR vector fields near p, $\bar{N}_1, \dots, \bar{N}_k$ to be generators for $\mathcal{F}_{j-1}(M)$ near p (it is easy to verify that $\mathcal{F}_k(M)$ is finitely generated as a $C^\infty(M)$-module near p), and $\theta^1, \dots, \theta^d$ to be a basis for the characteristic forms near p. The restrictions of these sections to the point p span the corresponding vector space over $\mathbb{C}$. We then define $\psi_{j+1}(L_{\bar{i}_1}, \dots, L_{\bar{i}_j}, \bar{N}_k, \theta^l)$ to be $G_j(L_{\bar{i}_1}, \dots, L_{\bar{i}_j}, \bar{N}_k, \theta^l)$, and extend ψ_{j+1} by linearity. The statements (a), (b), and (c) guarantee that this definition is independent of the bases and generators chosen and that the diagram (2.6) commutes. We leave the details of this verification to the reader. These arguments also show that the mapping ψ_{j+1} is unique whenever it exists.

We begin by proving statement (a). Observe first that the mapping G_j is clearly multi-linear over $\mathbb{C}$, so that

$$\begin{aligned} &G_j(K', aA+bB, K'', \bar{N}, \theta) \\ &\qquad = G_j(K', aA, K'', \bar{N}, \theta) \\ &\qquad\quad + G_j(K', bB, K'', \bar{N}, \theta). \end{aligned} \tag{2.15}$$

Hence, it suffices to prove that for any a and A as in statement (a),

$$G_j(K', aA, K'', \bar{N}, \theta) = a(p)G_j(K', A, K'', \bar{N}, \theta). \tag{2.16}$$

Note that, for any CR vector field L, any $b \in C^{\infty}(M)$, and any $\omega \in \Omega^{1,0}(M)$,

$$\begin{aligned} \mathcal{T}_L(b\omega) &= L \lrcorner d(b\omega) = L \lrcorner (db \wedge \omega + bd\omega) \\ &= (L \lrcorner db)\omega - (L \lrcorner \omega)db + b\mathcal{T}_L\omega \\ &= (Lb)\omega + b\mathcal{T}_L\omega, \end{aligned} \tag{2.17}$$

since $L \lrcorner \omega = \langle \omega, L \rangle = 0$. A simple inductive argument using (2.17) proves that, for $K_{\bar{1}}, \dots, K_{\bar{l}-\bar{1}}$, a, and A as in the statement (a) and ω as above,

$$\mathcal{T}_{K_{\bar{1}}} \dots \mathcal{T}_{K_{\bar{l}-\bar{1}}} T_{aA}\omega = a\mathcal{T}_{K_{\bar{1}}} \dots \mathcal{T}_{K_{\bar{l}-1}} T_A\omega + \sum_{i=1}^{m} a_i\omega^i, \tag{2.18}$$

where the $a_i \in C^{\infty}(M)$ and the ω^i are of the form

$$\omega^i = \mathcal{T}_{S_{\bar{1}}} \dots \mathcal{T}_{S_{\bar{k}}}\omega \tag{2.19}$$

for some $k < l$ and $S_{\bar{r}} \in \{K_{\bar{1}}, \dots, K_{\bar{l}-\bar{1}}, A\}$. Hence, for any $\bar{N} \in \mathcal{F}_{j-1}(M)$, we obtain, since $l \leq j$,

$$\left\langle \mathcal{T}_{K_{\bar{1}}} \dots \mathcal{T}_{K_{\bar{l}-\bar{1}}} T_{aA}\omega, \bar{N} \right\rangle_p = a(p) \left\langle \mathcal{T}_{K_{\bar{1}}} \dots \mathcal{T}_{K_{\bar{l}-\bar{1}}} T_A\omega, \bar{N} \right\rangle_p, \tag{2.20}$$

if $\omega = \mathcal{T}_{K_{\bar{l}+\bar{1}}} \dots \mathcal{T}_{K_{\bar{j}}}\theta$. This proves (2.16).

Statement (b) is obvious, since we even have

$$\begin{aligned} &\left\langle \mathcal{T}_{K_{\bar{1}}} \dots \mathcal{T}_{K_{\bar{j}}}\theta, a\bar{A} + b\bar{B} \right\rangle \\ &\qquad = a \left\langle \mathcal{T}_{K_{\bar{1}}} \dots \mathcal{T}_{K_{\bar{j}}}\theta, \bar{A} \right\rangle \\ &\qquad\quad + b \left\langle \mathcal{T}_{K_{\bar{1}}} \dots \mathcal{T}_{K_{\bar{j}}}\theta, \bar{B} \right\rangle. \end{aligned} \tag{2.21}$$

Finally, statement (c) follows from an argument similar to the one used to prove (a); we leave the details to the reader.

To prove the symmetry properties, we first prove the following identity.

Lemma 2.22. *For any $(1,0)$-form ω, CR vector fields K, L, and any vector field X on M, the following holds:*

$$\langle \mathcal{T}_L\mathcal{T}_K\omega, X \rangle - \langle \mathcal{T}_K\mathcal{T}_L\omega, X \rangle = [L, K] \langle \omega, X \rangle + \langle \omega, [X, [L, K]] \rangle. \tag{2.23}$$

Proof. For any $(1,0)$-form ω', CR vector field L', and any vector field X on M, using a well known identity we obtain

$$\begin{aligned}\left\langle \mathcal{T}_{L'}\omega', X\right\rangle &= \left\langle d\omega', L'\wedge X\right\rangle \\ &= L'\left\langle \omega', X\right\rangle - X\left\langle \omega', L'\right\rangle - \left\langle \omega', [L', X]\right\rangle \\ &= L'\left\langle \omega', X\right\rangle - \left\langle \omega', [L', X]\right\rangle,\end{aligned} \tag{2.24}$$

since $\langle \omega', L'\rangle = 0$. Similarly,

$$\begin{aligned}\langle \mathcal{T}_L\mathcal{T}_K\omega, X\rangle =& LK\langle \omega, X\rangle - L\langle \omega, [K, X]\rangle \\ & - K\langle \omega, [L, X]\rangle - \langle \omega, [K, [L, X]]\rangle .\end{aligned} \tag{2.25}$$

It follows that

$$\begin{aligned}&\langle \mathcal{T}_L\mathcal{T}_K\omega, X\rangle - \langle \mathcal{T}_K\mathcal{T}_L\omega, X\rangle \\ &\quad = [L, K]\langle \omega, X\rangle \\ &\quad\quad + \langle \omega, [K, [L, X]] - [L, [K, X]]\rangle .\end{aligned} \tag{2.26}$$

Now, using the Jacobi identity, we have

$$\begin{aligned}[K, [L, X]] - [L, [K, X]] =& [K, [L, X]] + [X, [L, K]] + [K, [X, L]] \\ =& [X, [L, K]],\end{aligned} \tag{2.27}$$

which completes the proof. q.e.d.

In particular, Lemma 2.22 implies that $\mathcal{T}_L$ and $\mathcal{T}_K$, considered as linear maps on $\Omega^{1,0}(M)$, commute if the CR vector fields L and K commute. It is well known that there exists a basis of CR vector fields on M near p that commute. Since this basis can be used in the construction of ψ_{j+1}, as described in the beginning of this proof, it follows that ψ_{j+1} is symmetric with respect to permutations of the j first variables. This completes the proof of Theorem 2.9. q.e.d.

3. Finitely nondegenerate CR manifolds

In this section, we relate the invariant tensors defined in Section 2 to the notion of finite nondegeneracy. Let $M \subset \mathbb{C}^N$ be a generic real smooth submanifold of codimension d, p_0 a point in M, and let $\rho(Z, \bar{Z}) = 0$, where $\rho = (\rho_1, \ldots, \rho_d)$, be a defining equation for M near p_0. Let $L_{\bar{1}}, \ldots, L_{\bar{n}}$, $n = N - d$, be a basis for the CR vector fields of

M near p_0. M is called *finitely nondegenerate* at p_0 if there exists a non-negative integer k such that

$$(3.1) \qquad span \left\{ L^{\bar{J}} \left(\frac{\partial \rho_l}{\partial Z} \right) (p_0, \bar{p}_0) \, \forall |J| \leq k, \quad l = 1, 2, \ldots, d \right\} = \mathbb{C}^N,$$

where we use the notation $\bar{J} = (\bar{J}_1, \ldots, \bar{J}_k) \in \{1, \ldots, n\}^k$, $|\bar{J}| = k$, and $L^{\bar{J}} = L_{\bar{J}_1} \ldots L_{\bar{J}_k}$. If M is finitely nondegenerate at p_0 and k is the smallest integer for which (3.1) holds, then M is called k-nondegenerate at p_0. The property of being k-nondegenerate is independent of the choice of defining equations, local coordinates, and bases for the CR vector fields. Moreover, M is 0-nondegenerate at p_0 if and only if it is totally real at p_0, and if M is a hypersurface, then it is 1-nondegenerate at p_0 if and only if it is Levi-nondegenerate. (See e.g. [1] or [4] for these statements.)

Finite nondegeneracy was introduced in [5] in connection with a regularity problem for CR mappings of real hypersurfaces. It was further explored in connection with the study of holomorphic mappings between generic submanifolds and real hypersurfaces in [1]–[3]. Finite nondegeneracy is also related to holomorphic nondegeneracy as introduced in [19] (see also [20]) and essential finiteness as introduced in [6]. The reader is referred to the book [4] for further information and history.

We prove here the following result. Recall from Section 2 the definition of the subspaces $E_j(p_0) \subset T'_{p_0}M$.

Theorem 3.2. *Let $M \subset \mathbb{C}^N$ be a generic real submanifold and $p_0 \in M$. Then, M is k-nondegenerate at p_0 if and only if $E_k(p_0) = T'_{p_0}M$ and $E_{k-1}(p_0) \subsetneq T'_{p_0}M$.*

Before proving Theorem 3.2, we shall show that the space $E_k(p_0)$ can be computed in a slightly simpler way than in the definition given in Section 2. Let $L_{\bar{1}}, \ldots, L_{\bar{n}}$ be a basis for the CR vector fields on M near p_0, and $\theta^1, \ldots, \theta^d$ a basis for the characteristic forms near p_0. We shall use the notation $\mathcal{T}^j = \mathcal{T}_{L_{\bar{j}}}$ and, as above for $J = (J_1, \ldots, J_k) \in \{1, 2, \ldots, n\}^k$, we denote by

$$(3.3) \qquad \mathcal{T}^J = \mathcal{T}^{J_1} \circ \ldots \circ \mathcal{T}^{J_k}.$$

Proposition 3.4. *For any nonnegative integer j, the following holds:*

$$(3.5) \qquad E_j(p_0) = span \, \{ (\mathcal{T}^J \theta^l)_{p_0} \, \forall |J| \leq j, \quad l = 1, 2, \ldots, d \}.$$

Proof. Observe that the right-hand side of (3.5) is contained in $E_j(p_0)$ for any nonnegative j. Let $K_{\bar{1}}, \ldots, K_{\bar{j}}$ be arbitrary CR vector fields, and θ an arbitary characteristic form. Since $L_{\bar{1}}, \ldots, L_{\bar{n}}$ and $\theta^1, \ldots, \theta^d$ form bases for the CR vector fields and the characteristic forms, respectively, near p_0, we have, for $l = 1, \ldots, j$,

$$K_{\bar{l}} = \sum_{m=1}^{n} a_{\bar{l}}^{\bar{m}} L_{\bar{m}}, \quad \theta = \sum_{i=1}^{d} b_i \theta^i, \tag{3.6}$$

for some $a_{\bar{l}}^{\bar{m}}, b_i \in C^\infty(M)$. The fact that $(\mathcal{T}_{K_{\bar{1}}} \ldots \mathcal{T}_{K_{\bar{j}}} \theta)_{p_0}$ is contained on the right-hand side of (3.5) now follows from (2.17) and (2.18). q.e.d.

Proof of Theorem 3.2. For a generic submanifold $M \subset \mathbb{C}^N$ with defining functions $\rho = (\rho_1, \ldots, \rho_d)$ near $p_0 \in M$, we may take $\theta^j = 2i\partial\rho_j$, for $j = 1, \ldots, d$, as a basis for the characteristic forms near p_0. Observe that each θ^j is real on M, since $\partial\rho_j + \bar{\partial}\rho_j = 0$ when pulled back to M. Let $L_{\bar{1}}, \ldots, L_{\bar{n}}$ be a basis for the CR vector fields of M near p_0. In the coordinates Z of the ambient space, we may write

$$L_{\bar{k}} = \sum_{l=1}^{N} a_{\bar{k}}^{\bar{l}}(Z, \bar{Z}) \frac{\partial}{\partial \bar{Z}^{\bar{l}}}, \tag{3.7}$$

and

$$\theta^j = 2i\partial\rho_j = 2i \sum_{l=1}^{N} \frac{\partial \rho_j}{\partial Z^l} dZ^l. \tag{3.8}$$

Hence, using the notation of Proposition 3.4, we have

$$\mathcal{T}^k \theta^j = \sum_{l=1}^{N} L_{\bar{k}} \left(\frac{\partial \rho_j}{\partial Z^l} \right) dZ^l. \tag{3.9}$$

Repeating this argument leads to

$$\mathcal{T}^J \theta^j = \sum_{l=1}^{N} L^{\bar{J}} \left(\frac{\partial \rho_j}{\partial Z^l} \right) dZ^l. \tag{3.10}$$

Since $(\mathcal{T}^J \theta^j)_{p_0} \in T'_{p_0} M$ and the dimension of $T'_{p_0} M$ equals $n + d = N$, the conclusion of Theorem 3.2 follows from Proposition 3.4. q.e.d.

4. The third order invariants and a partial normal form for real hypersurfaces

We shall show that the second and third order tensors ψ_2, ψ_3 form a complete set of third order invariants (in a sense that will be made more precise in Theorem 4.15 below) for real hypersurfaces. This will be the first step in the proof of Proposition 1.1.3.

Let $M \subset \mathbb{C}^{n+1}$ be a real smooth hypersurface. Let $L_{\bar{1}}, \dots, L_{\bar{n}}$ be a basis for the CR vector fields on M near some distinguished point $p \in M$ and θ a non-zero characteristic form near p. Set $L_\alpha = \overline{L_{\bar{\alpha}}}$. Denote by $g_{\bar{\alpha}\beta}$ the components of the tensor ψ_2 at p, which is just the Levi form of M at that point, relative to the bases $L_{\bar{1},p}, \dots, L_{\bar{n},p}$ of $\mathcal{V}_p$, $L_{1,p}, \dots, L_{n,p}$ of $F_0(p) = \bar{\mathcal{V}}_p$, and θ_p of T^0_pM, i.e.,

$$g_{\bar{\alpha}\beta} = \langle \mathcal{T}_{L_{\bar{\alpha}}}\theta, L_\beta \rangle_p\,, \tag{4.1}$$

for $\bar{\alpha}, \beta = 1, \dots, n$. A change of bases

$$L'_{\bar{\gamma},p} = b^{\bar{\alpha}}_{\bar{\gamma}} L_{\bar{\alpha},p}, \quad \theta'_p = a\theta_p, \tag{4.2}$$

where we use the usual summation convention to raise and lower indices, yields the transformation rule

$$g'_{\bar{\alpha}\beta} = a b^{\bar{\gamma}}_{\bar{\alpha}} b^{\nu}_{\beta} g_{\bar{\gamma}\nu}, \tag{4.3}$$

where $b^{\nu}_{\gamma} = \overline{b^{\bar{\nu}}_{\bar{\gamma}}}$. By a suitable choice of bases above, we may assume that the Levi form of M at p is diagonal with diagonal elements in $\{-1, 0, 1\}$, i.e.,

$$g_{\bar{\alpha}\beta} = \epsilon_\beta \delta_{\bar{\alpha}\beta}, \tag{4.4}$$

where $\delta_{\bar{\alpha}\beta}$ is the Kronecker symbol and

$$\epsilon_\beta = \begin{cases} 1, & \beta = 1, \dots, r, \\ -1, & \beta = r+1, \dots, r+s, \\ 0, & \beta = r+s+1, \dots, n. \end{cases} \tag{4.5}$$

We shall assume here that $r + s < n$, so that M is Levi degenerate at p. (The rank of the Levi form at p is $r + s$.) Now, denote by $h_{\bar{\alpha}\bar{\beta}\gamma}$ the components of the third order tensor ψ_3 at p, i.e.,

$$h_{\bar{\alpha}\bar{\beta}\gamma} = \left\langle \mathcal{T}_{L_{\bar{\alpha}}} \mathcal{T}_{L_{\bar{\beta}}}\theta, L_\gamma \right\rangle_p\,, \tag{4.6}$$

where $\bar\alpha,\bar\beta=1,\dots,n$ and $\gamma=r+s,\dots,n$. We then obtain the transformation rule

$$h'_{\bar\alpha\bar\beta\gamma}=ab^{\bar\sigma}_{\bar\alpha}b^{\bar\mu}_{\bar\beta}b^{\nu}_{\gamma}h_{\bar\sigma\bar\mu\nu}. \tag{4.7}$$

It is well known (and not difficult to see) that we may choose coordinates $Z=(z,w)=(z^1,\dots,z^n,w)\in\mathbb{C}^{n+1}$ near $p\in M$, vanishing at p, such that M is defined near $p=0$ by the equation $\rho(Z,\bar Z)=0$, where

$$\rho(Z,\bar Z)=-\mathrm{Im}\ w+g'_{\bar\alpha\beta}\bar z^{\bar\alpha}z^{\beta}+2\mathrm{Re}\ \left(k'_{\bar\alpha\bar\beta\nu}\bar z^{\bar\alpha}\bar z^{\bar\beta}z^{\nu}\right)+R'(z,\bar z,\mathrm{Re}\ w) \tag{4.8}$$

for some $g'_{\bar\alpha\beta},k'_{\bar\alpha\bar\beta\nu}\in\mathbb{C}$ with $\bar\alpha,\beta,\bar\beta,\nu=1,\dots,n$; here, $R'(z,\bar z,s)$ is a real-valued function that vanishes to weighted order 4 at 0 in the weighted coordinate system where z, $\bar z$ have weight one and s has weight two (or higher if the Levi form at p is 0). For the embedded hypersurface defined by the function (4.8), we may take as a basis for the CR vector fields

$$L'_{\bar\alpha}=\frac{\partial}{\partial\bar z^{\bar\alpha}}+\lambda_{\bar\alpha}(Z,\bar Z)\frac{\partial}{\partial\bar w},\quad \bar\alpha=1,\dots,n, \tag{4.9}$$

where $\lambda_{\bar\alpha}(0,0)=0$. We refer the reader e.g. to [4, Chapter IV] for details. By taking $\theta'=2i\partial\rho$ and using (3.10), we find that the tensors ψ_2 and ψ_3 at $p=0$ relative to the bases defined by $L'_{\bar\alpha}$, L'_{β}, and θ' are given by $\psi_2=(g'_{\bar\alpha\beta})$ and $\psi_3=(h'_{\bar\alpha\bar\beta\gamma})$ with

$$h'_{\bar\alpha\bar\beta\gamma}=k'_{\bar\alpha\bar\beta\gamma},\quad \bar\alpha,\bar\beta=1,\dots,n,\ \gamma=r+s+1,\dots,n. \tag{4.10}$$

It follows that there is a change of basis (4.2) such that (4.3) and (4.7) (with γ running from $r+s+1$ to n) hold. Such a change of bases corresponds to a linear change of coordinates of the form

$$z^{\alpha}\mapsto b^{\alpha}_{\beta}z^{\beta},\quad w\mapsto\frac{1}{a}w \tag{4.11}$$

in (4.8). Hence, the linear change of coordinates (4.11) transforms the defining function in (4.8) to the form

$$\begin{aligned}\rho(Z,\bar Z)=&-\mathrm{Im}\ w+g_{\bar\alpha\beta}\bar z^{\bar\alpha}z^{\beta}+2\mathrm{Re}\ \left(k_{\bar\alpha\bar\beta\mu}\bar z^{\bar\alpha}\bar z^{\bar\beta}z^{\mu}\right)\\ &+2\mathrm{Re}\ \left(h_{\bar\alpha\bar\beta\gamma}\bar z^{\bar\alpha}\bar z^{\bar\beta}z^{\gamma}\right)+R(z,\bar z,\mathrm{Re}\ w),\end{aligned} \tag{4.12}$$

where $\bar\alpha, \beta, \bar\beta$ run over $1, \dots, n$, μ runs over $1, \dots, r+s$, γ runs over $r+s+1, \dots, n$, and $k_{\bar\alpha\bar\beta\mu}$ are some complex numbers. Next, since $g_{\bar\alpha\beta}$ is of the form (4.4) with ϵ_β of the form (4.5), we observe that the quadratic change of coordinates

$$z^\mu - \epsilon_\mu \overline{k_{\bar\alpha\bar\beta\mu}} z^\alpha z^\beta \mapsto z^\mu, \tag{4.13}$$

for $\mu = 1, \dots, r+s$, yields the following final form of $\rho(Z, \bar Z)$

$$\begin{aligned} \rho(Z, \bar Z) = & - \operatorname{Im} w + g_{\bar\alpha\beta} \bar z^{\bar\alpha} z^\beta + 2\operatorname{Re} \left(h_{\bar\alpha\bar\beta\gamma} \bar z^{\bar\alpha} \bar z^{\bar\beta} z^\gamma \right) \\ & + \tilde R(z, \bar z, \operatorname{Re} w), \end{aligned} \tag{4.14}$$

where $\tilde R(z, \bar z, s)$ vanishes of weighted order 4 at 0, the indices $\bar\alpha, \beta, \bar\beta$ run over $1, \dots, n$, and the index γ runs over $r+s+1, \dots, n$. We would like to point out that a similar form for a real hypersurface was presented by Webster in [25] (see also Remark 4.17 below).

Hence, we have proved that ψ_2 and ψ_3 form a complete set of third order invariants for a real hypersurface $M \subset \mathbb{C}^{n+1}$ in the following sense. We use the notation and conventions introduced above.

Theorem 4.15. *Let $M \subset \mathbb{C}^{n+1}$ be a real smooth hypersurface and $p \in M$. Assume that the signature of the Levi form of M at p is as described above. Then, there are coordinates $Z = (z, w) \in \mathbb{C}^{n+1}$, vanishing at p, such that M is defined near $p = 0$ by $\rho(Z, \bar Z) = 0$, where $\rho(Z, \bar Z)$ is given by (4.14) if and only if there is are bases $L_{\bar 1, p}, \dots, L_{\bar n, p}$ for $\mathcal{V}_p$, with the corresponding basis $L_{1,p}, \dots, L_{n,p}$ for $\bar{\mathcal{V}}_p$, and θ_p for $T^0_p M$ such that*

$$\psi_2 = (g_{\bar\alpha\beta}), \quad \psi_3 = (h_{\bar\alpha\bar\beta\gamma}), \tag{4.16}$$

with $\bar\alpha, \beta, \bar\beta = 1, \dots, n$ and $\gamma = r+s+1, \dots, n$.

Remark 4.17. In [25], the *cubic form* of a real hypersurface $M \subset \mathbb{C}^{n+1}$ at a point $p \in M$ was introduced and shown to be a multilinear map $\mathcal{V}_p \times \mathcal{V}_p \times F_1(p) \to \mathbb{C}$ defined by

$$q_p(L_p, K_p, \bar N_p) = \left\langle \partial\rho, [K, [L, \bar N]] \right\rangle_p, \tag{4.18}$$

where L, K, and $\bar N$ are vector fields extending $L_p, K_p \in \mathcal{V}_p$ and $\bar N_p \in F_1(p)$, respectively. A straightforward calculation, using the formula

(2.24) repeatedly, shows the following relation between the cubic form and the tensor $\psi_3(\cdot,\cdot,\cdot,\theta)$ (for some fixed θ e.g. $\theta = 2i\partial\rho$):

$$\begin{aligned}
&\psi_3(L_p, K_p, \bar{N}_p, \theta) - 2iq_p(L_p, K_p, \bar{N}_p) \\
&\qquad = \left\langle \mathcal{T}_L\mathcal{T}_K\theta, \bar{N}\right\rangle_p - \left\langle \theta, [K,[L,\bar{N}]]\right\rangle_p \\
&\qquad = L\left(\langle \mathcal{T}_K\theta, \bar{N}\rangle\right) + K\left(\langle \mathcal{T}_L\theta, \bar{N}\rangle\right) \\
&\qquad = -L\left(\langle \theta, [K,\bar{N}]\rangle\right) - K\left(\langle \theta, [L,\bar{N}]\rangle\right).
\end{aligned} \tag{4.19}$$

Nevertheless, the cubic form and $\psi_3(\cdot,\cdot,\cdot,\theta)$ are in fact equal (possibly modulo some multiplicative constant). This equivalence follows from Theorem 4.15, because it is shown in [25] (using the notation introduced above) that M can be brought to the form (4.14) with

$$q_p(L_{\bar{\alpha},p}, L_{\bar{\beta},p}, L_{\gamma,p}) = \frac{i}{2}h_{\bar{\alpha}\bar{\beta}\gamma}, \tag{4.20}$$

where $\bar{\alpha},\bar{\beta},\gamma$ range over the same indices as in Theorem 4.15. Thus, Theorem 4.15 is in fact implicit in [25], although using the cubic form as the third order tensor.

5. An explicit computation of the third order tensor in a special case

We shall keep the notation and conventions introduced in Section 4. We would like to compute numerical invariants of the tensor $\psi_3 = (h_{\bar{\alpha}\bar{\beta}\gamma})$ under changes of bases (4.2) preserving the form (4.4) of the second order tensor (the Levi form) $\psi_2 = (g_{\bar{\alpha}\beta})$. We shall do this only in the following case, which is a bit more general than the situation considered in §1.2.

Assume that the rank $r + s$ of the Levi form ψ_2 at the point equals $n - 1$ and that the Levi form is semidefinite. Thus, we do not assume here that the Levi degeneracy is generic. We may assume, without loss of generality, that the $n - 1$ nonzero diagonal elements $\epsilon_1, \ldots, \epsilon_{n-1}$ of $g_{\bar{\alpha}\beta}$ are $+1$. We can identify the third order tensor ψ_3 with a symmetric $n \times n$ matrix $H = (h_{\bar{\alpha}\bar{\beta}n})$.

We associate to each change of basis in $\mathcal{V}_p$ a matrix $B \in GL(\mathbb{C}^n)$ by $B = (b_{\bar{\beta}}^{\bar{\alpha}})$. We only consider changes (4.2) that preserve the form of ψ_2, i.e., such that

$$aB\tilde{I}B^* = \tilde{I}, \tag{5.1}$$

where B^* denotes the Hermitian adjoint of B and $\tilde{I}$ is the matrix of the Levi form, i.e., in block matrix form

$$\tilde{I} = \begin{pmatrix} I_{n-1} & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.2}$$

with $I_{n-1} = I_{n-1,0}$ being the $(n-1)\times(n-1)$ identity matrix. It is easy to see that (5.1) implies that B must be of the form

$$B = \begin{pmatrix} V & c \\ 0 & d \end{pmatrix}, \tag{5.3}$$

where $c \in \mathbb{C}^{n-1}$, $d \in \mathbb{C}$, and V is an $(n-1)\times(n-1)$-matrix related to a in (4.2) by

$$aVV^* = I_{n-1}, \tag{5.4}$$

i.e., $a > 0$ and $\sqrt{a}V$ is a unitary matrix. The transformation rule (4.7) for ψ_3 becomes

$$H' = a\bar{d}BHB^\tau, \tag{5.5}$$

where B^τ denotes the transpose of B.

Recall that, for a given $(n-1)$-vector

$$\lambda = (\lambda_1, \dots, \lambda_{n-1}), \tag{5.6}$$

we denote by $D_{n-1}(\lambda)$ the $(n-1)\times(n-1)$ diagonal matrix with λ on the diagonal (see (1.2.2)). We shall also use e_{n-1}^τ for the $(n-1)$-vector

$$e_{n-1}^\tau = (0, \dots, 0, 1). \tag{5.7}$$

If $n = 2$, then we take $e_{n-1}^\tau = 1$. The main result in this section is the following, which combined with Theorem 4.15 gives Theorem 1.2.10. We use the matrix representations of the second and third order tensors as introduced above.

Theorem 5.8. *Let $M \subset \mathbb{C}^{n+1}$, $n > 2$, be a smooth real hypersurface and $p \in M$. Assume that the Levi form $g_{\bar{\alpha}\beta}$ of M at p has rank $n-1$ and is semidefinite (i.e., all nonzero eigenvalues have the same sign). If we normalize the Levi form $g_{\bar{\alpha}\beta}$ so that its matrix is in the form (5.2), then the matrix $H = (h_{\bar{\alpha}\bar{\beta}n})$ of the third order tensor can be brought to precisely one of the following block matrix forms:*

(i) *For either* $\lambda = (1, \lambda_2, \dots, \lambda_{n-2}, 0)$ *with* $1 \geq \lambda_2 \geq \dots \geq \lambda_{n-2} \geq 0$ *or* $\lambda = (0, \dots, 0)$,

$$H' = \begin{pmatrix} D_{n-1}(\lambda) & e_{n-1} \\ e^{\tau}_{n-1} & 0 \end{pmatrix}. \tag{5.9}$$

(ii) *For either* $\lambda = (1, \lambda_2, \dots, \lambda_{n-2}, \lambda_{n-1})$ *with* $1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1} \geq 0$ *or* $\lambda = (0, \dots, 0)$,

$$H' = \begin{pmatrix} D_{n-1}(\lambda) & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.10}$$

(iii) *For either* $\lambda = (1, \lambda_2, \dots, \lambda_{n-2}, \lambda_{n-1})$ *with* $1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1} \geq 0$ *or* $\lambda = (0, \dots, 0)$,

$$H' = \begin{pmatrix} D_{n-1}(\lambda) & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.11}$$

In the case $n = 2$, *the same result holds with the following modifications: The only choice for* λ *in* (i) *is* $\lambda = 0$. *In* (ii) *and* (iii), *both* $\lambda = 1$ *and* $\lambda = 0$ *are allowed.*

Remark 5.12.

(a) Note that in the case $n = 2$, i.e., in $\mathbb{C}^3$, there are only 5 different forms for H' and no numerical invariants λ (i.e., λ is only 1 or 0). These 5 forms correspond to the 4 different partial normal forms of type (i) in [12, Theorem A] and the case which is not 2-nondegenerate at p (see (b) below). We should point out that the partial normal form of type (ii) in [12, Theorem A] corresponds to an explicit normal form for the third order tensor of a hypersurface in $\mathbb{C}^3$ at a point p where the Levi form vanishes. In this case, there are nontrivial numerical invariant.

(b) The only case that corresponds to a hypersurface $M \subset \mathbb{C}^{n+1}$ which is not 2-nondegenerate at $p \in M$ is (ii) with $\lambda = (0, \dots, 0)$, i.e., $H' = 0$.

Proof of Theorem 5.8. We assume first that $n > 2$. We write the symmetric $n \times n$-matrix H in block matrix form

$$H = \begin{pmatrix} A & \beta \\ \beta^{\tau} & \gamma \end{pmatrix}, \tag{5.13}$$

where A is a symmetric $(n-1)\times(n-1)$-matrix, $\beta \in \mathbb{C}^n$, and $\gamma \in \mathbb{C}$. By making a change of bases (4.2) preserving the form of $g_{\bar{\alpha}\beta}$, i.e., the matrix B is of the form (5.3) and satisfies (5.4), the matrix H transforms according to the rule (5.5). A computation shows that

$$H' = a\bar{d}\begin{pmatrix} VAV^\tau + V\beta c^\tau + c\beta^\tau V^\tau + \gamma cc^\tau & d(V\beta + \gamma c) \\ d(\beta^\tau V^\tau + \gamma c^\tau) & \gamma d^2 \end{pmatrix}. \tag{5.14}$$

We shall divide the proof into different cases.

The case $\gamma = 0$ and $\beta \neq 0$. We have

$$H' = a\bar{d}\begin{pmatrix} VAV^\tau + V\beta c^\tau + c\beta^\tau V^\tau & dV\beta \\ d\beta^\tau V^\tau & 0 \end{pmatrix}. \tag{5.15}$$

Let us look for V in the form $V = V_2V_1$, where V_1 is a unitary matrix such that

$$V_1\beta = |\beta| e_{n-1}, \tag{5.16}$$

with e_{n-1} as defined by (5.7). If we write $A' = V_1AV_1^\tau$, then we have

$$H' = a\bar{d}\begin{pmatrix} V_2A'V_2^\tau + |\beta|V_2e_{n-1}c^\tau + |\beta|ce_{n-1}^\tau V_2^\tau & |\beta|dV_2e_{n-1} \\ |\beta|de_{n-1}^\tau V_2^\tau & 0 \end{pmatrix}. \tag{5.17}$$

If we introduce the vector

$$p = aV_2^*c \tag{5.18}$$

and use the fact that $aV_2V_2^* = I_{n-1}$, then the upper left corner of H' in (5.17) can be written

$$V_2(A' + |\beta|(e_{n-1}p^\tau + pe_{n-1}^\tau))V_2^\tau. \tag{5.19}$$

It is easy to check that $p \in \mathbb{C}^{n-1}$ can be chosen uniquely (which means that c is determined uniquely as a function of V_2 and a) such that $A' + |\beta|(e_{n-1}p^\tau + pe_{n-1}^\tau)$ takes the form

$$A' + |\beta|(e_{n-1}p^\tau + pe_{n-1}^\tau) = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.20}$$

where E is some symmetric $(n-1)\times(n-1)$-matrix. If we write $\tilde{V} = \sqrt{a}V_2$, then it remains to choose a unitary matrix $\tilde{V}$, a positive number

$\sqrt{a}$, and a complex (nonzero) number d so as to normalize the matrix and vector

$$\bar{d}\tilde{V}\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}\tilde{V}^\tau, \quad \sqrt{a}|d|^2|\beta|\tilde{V}e_{n-1}. \tag{5.21}$$

The most general unitary matrix $\tilde{V}$ satisfying $\tilde{V}e_{n-1} = e_{n-1}$ is of the form

$$\tilde{V} = \begin{pmatrix} F & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.22}$$

where F is a unitary $(n-2)\times(n-2)$-matrix. For such a $\tilde{V}$, we get

$$\tilde{V}\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}\tilde{V}^\tau = \begin{pmatrix} FEF^\tau & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.23}$$

At this point we need the following lemma, which is a consequence of E. Cartan's work on Lie groups (see [24] for a discussion; see also [18] for the lemma in the present form). We denote by $U(\mathbb{C}^m) = U^+(m,0,\mathbb{C})$ the group of unitary transformations in $\mathbb{C}^m$. We also denote by $O(\mathbb{R}^m)$ the group of (real) orthogonal transformations in $\mathbb{R}^m$.

Lemma 5.24 *Let E be a symmetric $m\times m$-matrix with complex matrix elements. Then, there is a unique m-vector $\lambda = (\lambda_1,\ldots,\lambda_m)$ with $\lambda_1 \geq \ldots \geq \lambda_m \geq 0$ such that*

$$UEU^\tau = D_m(\lambda), \tag{5.25}$$

for some $U\in U(\mathbb{C}^m)$. In fact, the numbers λ_j^2 are the eigenvalues of the positive semidefinite Hermitian matrix $E\bar{E}$. Moreover, if λ is given as above, and we write $(u_1,\ldots,u_k,0)$ for the distinct values of $(\lambda_1,\ldots,\lambda_m)$ and $(m_1,\ldots,m_k,\mu)$ for their multiplicities (e.g. μ denotes the number of zeros among the λ_j), then the subgroup of $U\in U(\mathbb{C}^m)$ for which

$$UD_m(\lambda)U^\tau = D_m(\lambda) \tag{5.26}$$

consists of all matrices of the form

$$U = \begin{pmatrix} O_1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & O_k & 0 \\ 0 & 0 & 0 & V \end{pmatrix}, \tag{5.27}$$

where $O_j \in O(\mathbb{R}^{m_j})$, $j = 1, \dots, k$, and $V \in U(\mathbb{C}^{\mu})$. (Observe that μ could be zero in which case there is no V in (5.27)*.)*

Now, if we choose $\tilde{V}$ as in (5.22) with F chosen such that $FEF^{\tau} = D_{n-2}(\lambda')$ for some $(n-2)$-vector λ', and then $\sqrt{a} > 0$ and $d \in \mathbb{C} \setminus \{0\}$ suitably, we obtain H' of the form described by (i) in Theorem 5.8. Also, the vector λ, as described in Theorem 5.8 (i), is uniqely determined and it is clear from the arguments above that H' cannot be brought to any of the other forms (ii) or (iii). This concludes the case $\gamma = 0$ and $\beta \neq 0$.

The case $\gamma = 0$ and $\beta = 0$. In this case, it is clear from Lemma 5.24 that H' can be brought to the form (ii) (and none of the forms (i) or (iii)) with λ, as described in Theorem 5.8 (ii), uniquely determined.

The case $\gamma \neq 0$. It is clear from (5.14) that we can make the upper right and lower left corners of H' vanish by choosing

$$c = -\frac{1}{\gamma} V \beta. \tag{5.28}$$

If we bring the factor $a\bar{d}$ inside the matrix in (5.14) then, with $\tilde{V} = \sqrt{a} V$ as above, the upper left and lower right corners of H', respectively, become

$$\bar{d}\tilde{V}\left(A - \frac{1}{\gamma}\beta\beta^{\tau}\right)\tilde{V}^{\tau}, \quad a\bar{d}d^2\gamma. \tag{5.29}$$

The equation

$$a\bar{d}d^2\gamma = 1 \tag{5.30}$$

determines the argument of $d \in \mathbb{C}$ uniquely. It also determines the modulus of d uniquely as a function of $a > 0$ by

$$|d| = \frac{1}{|\gamma|^{1/3} a^{1/3}}. \tag{5.31}$$

Substituting this into the expression for the upper left corner in (5.29) and using Lemma 5.24, we deduce that H can be brought to the form (iii) (and none of the forms (i) or (ii)) with λ, as described by Theorem 5.8 (iii), uniquely determined. This concludes the case $\gamma \neq 0$.

Now, if $n = 2$, then a similar, but simpler, argument leads to the statement concluding Theorem 5.8. q.e.d.

6. Proof of Proposition 1.1.3

We shall keep the notation of §§4–5. Recall that the hypersurface M is assumed to have a generic Levi degeneracy at p_0 at which point the Levi form has r, with $(n-1)/2 \leq r \leq n-1$, eigenvalues of the same sign. Thus, we may assume that the matrix $(g_{\bar{\alpha}\beta})$ of the Levi form at p_0 equals $I_{r,s}$, where $I_{r,s}$ is as in §1.1 and $s = n-1-r$. In view of Theorem 4.15, we must show that the matrix $H = (h_{\bar{\alpha}\bar{\beta}n})$ of the third order tensor can be brought, by a change of basis (4.2) preserving the Levi form $(g_{\bar{\alpha}\beta}) = I_{r,s}$, to the form

$$H = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}, \tag{6.1}$$

for some $R \in S(\mathbb{C}^{n-1})$, and that, under additional such changes that also preserve the form (6.1) of H, the matrix R transforms according to the rule

$$R' = (cV)^\tau R(cV), \tag{6.2}$$

where $c > 0$ and $V \in \hat{U}(r,s,\mathbb{C})$ can be chosen arbitrarily. The most general change (4.2) preserving the Levi form $g_{\bar{\alpha}\beta}$ corresponds to a matrix B as in (5.3) with $c \in \mathbb{C}^{n-1}$, $d \in \mathbb{C}$, and $\sqrt{|a|}V \in \hat{U}(r,s,\mathbb{C})$ such that $aVI_{r,s}V^* = I_{r,s}$. If we write H in the form (5.13), then the fact that M has a generic Levi degeneracy at p_0 is expressed by $\gamma \neq 0$, as can be verified by a straightforward calculation (cf. also [25]). An inspection of the case $\gamma \neq 0$ in the proof of Theorem 5.8 above shows that H can indeed be brought to the form (5.32) and R transforms according to the rule (6.2), as desired. This completes the proof of Proposition 1.1.3.

q.e.d.

7. Proof of Proposition 1.1.9

We shall use the notation introduced in §1.1. Consider a transformation

$$(z', z^n, w) = (\tilde{f}'(\tilde{z}, \tilde{w}), \tilde{f}^n(\tilde{z}, \tilde{w}), \tilde{g}(\tilde{z}, \tilde{w})), \tag{7.1}$$

where $(\tilde{f}', \tilde{f}^n, \tilde{g})$ is of the form (for convenience, we drop the $\tilde{}$ on the variables)

$$
\begin{aligned}
\tilde{f}'(z,w) &= Az' + z^n D + wB + z^\tau Ez + O(3),\\
\tilde{f}^n(z,w) &= K^\tau z' + d_n z^n + O(2),\\
\tilde{g}(z,w) &= cw + 2i\left\langle A'z' + z^n D, \bar{B}\right\rangle w + O(4),
\end{aligned} \tag{7.2}
$$

where $A \in GL(\mathbb{C}^{n-1})$, $D, B, K \in \mathbb{C}^{n-1}$ (considered as $(n-1)\times 1$ matrices), $E = (E^\beta)_{1\le\beta\le n-1}$ is an $(n-1)$-vector of $n\times n$ matrices, $d_n \in \mathbb{C}\backslash\{0\}$, and $c \in \mathbb{R}\setminus\{0\}$. This is the most general form of a transformation that preserves regular coordinates (cf. [12, §§5–6]). If we write the formal defining equation of M in the (regular) coordinates $(\tilde{z},\tilde{w})$ in complex form (cf. [4] or [12]), i.e.,

$$
\tilde{w} = \tilde{Q}(\tilde{z}, \bar{\tilde{z}}, \bar{\tilde{w}}), \tag{7.3}
$$

where $\tilde{Q}(\tilde{z}, 0, \bar{\tilde{w}}) \equiv \tilde{Q}(0, \bar{\tilde{z}}, \bar{\tilde{w}}) \equiv \bar{\tilde{w}}$, then we obtain, by substituting in (1.1.4) and setting $\bar{w} = 0$,

$$
\begin{aligned}
c\left(1 + 2i\frac{\left\langle Az' + z^n D, \bar{B}\right\rangle}{c}\right)\tilde{Q}(z,\bar{z},0)&\\
= 2i\Big(&\left\langle Az' + z^n D, \bar{A}\bar{z}' + \bar{z}^{\bar{n}}\bar{D}\right\rangle\\
&+ \left\langle B, \bar{A}\bar{z}' + \bar{z}^{\bar{n}}\bar{D}\right\rangle\tilde{Q}(z,\bar{z},0)\\
&+ \left\langle z^\tau Ez, \bar{A}\bar{z}' + \bar{z}^{\bar{n}}\bar{D}\right\rangle\\
&+ (\bar{K}^\tau\bar{z}' + \bar{d}_n\bar{z}^{\bar{n}})p_R(Az' + z^n D, K^\tau z' + d_n z^n)\Big)\\
+ \dots,&
\end{aligned} \tag{7.4}
$$

where $\langle\cdot,\cdot\rangle$ and p_R are defined by (1.1.21) and (1.1.22), and the dots $\dots$ signify terms that are either $O(4)$ or of type (k,l) with $l > 1$. If the transformation is to preserve the form (1.1.4), then we must have (cf. [12, §5])

$$
\tilde{Q}(z,\bar{z},0) = 2i(\left\langle z', \bar{z}'\right\rangle + 2\mathrm{Re}\left(\bar{z}^{\bar{n}}p_R(z',z^n)\right)) + O(4). \tag{7.5}
$$

By identifying terms of type $(1,1)$, we deduce that $D = 0$ and

$$
\left\langle Az', \bar{A}\bar{z}'\right\rangle = c\left\langle z', \bar{z}'\right\rangle. \tag{7.6}
$$

Observe that (7.6) is equivalent to $A^* I_{r,s} A = c I_{r,s}$. Identifying terms of type $(2,1)$ and using (1.1.22), we also see that $K = 0$ and

$$\begin{cases} \frac{\bar{d}_n d_n^2}{c} = 1, \\ \frac{\bar{d}_n}{c} (A)^\tau R A = R, \\ \left\langle z^\tau E z, \bar{A}\bar{z}' \right\rangle = \frac{2i}{c} \left\langle Az', \bar{B} \right\rangle \left\langle Az', \bar{A}\bar{z}' \right\rangle . \end{cases} \tag{7.7}$$

The conclusion of Proposition 1.1.19 is now easy to verify. This completes the proof. q.e.d.

8. Proof of Theorem 1.1.28

The proof follows closely the proof of Theorem B in [12], which in turn was inspired by the work in [11]. The idea is to reduce the proof to a problem of describing the kernel and range of a certain linear operator. We shall use the notation introduced in §1.1.

We write the (formal) defining equation (1.1.4) of M in the form

$$\operatorname{Im} w = \left\langle z', \bar{z}' \right\rangle + 2\operatorname{Re}\, (\bar{z}^{\bar{n}} p_R(z)) + F(z, \bar{z}, \operatorname{Re} w), \tag{8.1}$$

where $\langle \cdot, \cdot \rangle$ is given by (1.1.21), and $p_R(z)$ by (1.1.22), and $F(z, \bar{z}, s)$ is a formal series in $\mathcal{F}$ as introduced in §1.1. We subject M to a formal transformation

$$\tilde{z} = \tilde{f}(z, w), \quad \tilde{w} = \tilde{g}(z, w), \tag{8.2}$$

where $\tilde{f} = (\tilde{f}', \tilde{f}^n) = (\tilde{f}^1, \ldots, \tilde{f}^{n-1}, \tilde{f}^n)$, which preserves the form of M modulo terms of weighted degree at least 4, i.e., the transformed hypersurface $\tilde{M}$ is given by a defining equation of the form

$$\operatorname{Im} \tilde{w} = \left\langle \tilde{z}', \bar{\tilde{z}}' \right\rangle + 2\operatorname{Re}\, (\bar{\tilde{z}}^{\bar{n}} p_R(\tilde{z})) + \tilde{F}(\tilde{z}, \bar{\tilde{z}}, \operatorname{Re} \tilde{w}), \tag{8.3}$$

where $\tilde{F}(z', \bar{z}', s')$ is in $\mathcal{F}$. We also require that the new coordinates are regular for $\tilde{M}$. Thus, $\tilde{f}$ and $\tilde{g}$ are subjected to the restrictions imposed by Proposition 1.1.9. As mentioned in §1.1, the most general transformation of this kind can be factored uniquely as

$$(\tilde{f}(z, w), \tilde{g}(z, w)) = (T \circ P)(z, w), \tag{8.4}$$

where P and T are as described in that section.

To prove Theorem 1.1.28, it suffices to prove that there is a unique transformation

$$T(z,w)=(\hat{f}(z,w),\hat{g}(z,w))=(z+f(z,w),w+g(z,w)), \tag{8.5}$$

where $f=(f',f^n)=(f^1,\dots,f^{n-1},f^n)$, to normal form (i.e., such that the transformed hypersurface $\tilde{M}$ is defined by (8.3) with $\tilde{F}\in\mathcal{N}$) such that f' is $O(3)$, f^n is $O(2)$, g is $O(4)$, and such that the constant terms in the formal series (1.1.17) vanish. We decompose (f',f^n,g), F, and $\tilde{F}$ into weighted homogeneous parts as follows:

$$\begin{aligned}
&f'(z,w)=\sum_{\nu=3}^{\infty}f'_\nu(z,w), \quad f^n(z,w)=\sum_{\nu=2}^{\infty}f^n_\nu(z,w)\ ,\\
&g(z,w)=\sum_{\nu=4}^{\infty}g_\nu(z,w)\ , \quad F(z,\bar{z},s)=\sum_{\nu=4}^{\infty}F_\nu(z,\bar{z},s)\ ,\\
&\tilde{F}(z,\bar{z},s)=\sum_{\nu=4}^{\infty}\tilde{F}_\nu(z,\bar{z},s)\ .
\end{aligned}$$

Recall here that z and $\bar{z}$ are assigned the weight one, w and s are assigned the weight two, and we say that e.g. $F_\nu(z,\bar{z},s)$ is weighted homogeneous of degree ν if for all $t>0$

$$F_\nu(tz,t\bar{z},t^2s)=t^\nu F_\nu(z,\bar{z},s).$$

The formal power series $F,\tilde{F}\in\mathcal{F}$ are related as follows:

$$\begin{aligned}
\mathrm{Im}\ \hat{g}(z,s+i\phi)\equiv&\left\langle \hat{f}^1(z,s+i\phi),\overline{\hat{f}^1(z,s+i\phi)}\right\rangle\\
&+2\mathrm{Re}\ \left(\overline{f^n(z,s+i\phi)}p_R(\hat{f}(z,s+i\phi))\right)\\
&+\tilde{F}(\hat{f}(z,s+i\phi),\overline{\hat{f}}(\bar{z},s-i\phi),\mathrm{Re}\ \hat{g}(z,s+i\phi)),
\end{aligned} \tag{8.6}$$

where

$$\phi=\phi(z,\bar{z},s)=\langle z',\bar{z}'\rangle+2\mathrm{Re}\ (\bar{z}^{\bar{n}}p_R(z,\bar{z}))+F(z,\bar{z},s). \tag{8.7}$$

Identifying terms of weighted degree $\nu\geq 4$ we obtain

$$\begin{aligned}
F_\nu+\mathrm{Im}\ g_\nu\equiv&\left\langle z',\overline{f'_{\nu-1}}\right\rangle+\langle f'_{\nu-1},\bar{z}'\rangle\\
&+(\overline{p_R}+2z^n\bar{z}^{\bar{n}})f^n_{\nu-2}\\
&+(p_R+2z^n\bar{z}^{\bar{n}})\overline{f^n_{\nu-2}}+\tilde{F}_\nu+\dots\ ,
\end{aligned} \tag{8.8}$$

where

$$
\begin{aligned}
F_\nu &= F_\nu(z,\bar z,s), \quad \tilde F_\nu = \tilde F_\nu(z,\bar z, s+i\langle z',\bar z'\rangle) \\
\overline{f'_{\nu-1}} &= \overline{f'_{\nu-1}}(\bar z, s-i\langle z',\bar z'\rangle), \\
f'_{\nu-1} &= f'_{\nu-1}(z, s+i\langle z',\bar z'\rangle)\ , \quad \text{etc,}
\end{aligned}
\tag{8.9}
$$

and where the dots ... signify terms that only involve F_μ, F'_μ, g_μ, $f'_{\mu-1}$, and $f^n_{\mu-2}$ for $\mu<\nu$. We can write this as

$$
\operatorname{Re}\,(ig_\nu + 2\langle f'_{\nu-1},\bar z'\rangle + 2(\overline{p_R}+2z^n\bar z^{\bar n})f^n_{\nu-2}) = F_\nu - F'_\nu + \dots\ . \tag{8.10}
$$

Let us define the linear operator

$$
L(f',f^n,g) = \operatorname{Re}\,(ig + 2\langle f',\bar z'\rangle + 2(\overline{p_R}+2z^n\bar z^{\bar n})f^n)|_{(z,s+i\langle z',\bar z'\rangle)} \tag{8.11}
$$

from the space $\mathcal{G}$ to the space $\mathcal{F}$, where $\mathcal{G}$ denotes the space of formal power series (in (z,w)) transformations (f',f^n,g) such that f' is $O(3)$, f^n is $O(2)$, and g is $O(4)$. Observe that L maps $(f'_{\nu-1},f^n_{\nu-2},g_\nu)$ to a series that is weighted homogeneous of degree ν. We note, as in [12] and [11], that if we could find subspaces

$$
\mathcal{G}_0\subset\mathcal{G}, \quad \mathcal{N}\subset\mathcal{F} \tag{8.12}
$$

such that, for any $F\in\mathcal{F}$, the equation

$$
L(f',f^n,g) = F \qquad \operatorname{mod}\mathcal{N} \tag{8.13}
$$

has a unique solution $(f',f^n,g)\in\mathcal{G}_0$, then, given any $F'\in\mathcal{F}$, equation (8.10) would allow us to inductively determine the weighted homogeneous parts F_ν of a normal form $F\in\mathcal{N}$ and the weighted homogeneous parts $(f'_{\nu-1},f^n_{\nu-2},g_\nu)$ of the transformation $(f',f^n,g)\in\mathcal{G}_0$ to normal form in a unique fashion. This can also be formulated as saying that $\mathcal{G}_0$ and $\mathcal{N}$ are complementary subspaces of the kernel and range of L, respectively.

Let us therefore define $\mathcal{G}_0\subset\mathcal{G}$ as those $(f',f^n,g)\in\mathcal{G}$ for which the constant terms in the series (1.1.17) vanish. Thus, the proof of Theorem 1.1.28 will be completed by proving the following.

Lemma 8.14. *Let $\mathcal{G}_0\subset\mathcal{G}$ be as described above and $\mathcal{N}\subset\mathcal{F}$ as defined in §1. Then, for any $F\in\mathcal{F}$, the equation*

$$
L(f',f^n,g) = F \qquad \operatorname{mod}\mathcal{N} \tag{8.15}
$$

has a unique solution $(f', f^n, g) \in \mathcal{G}_0$.

Proof. We shall decompose equation (8.15) according to (k,l)-type. We decompose $F \in \mathcal{F}$ as follows:

$$F(z,\bar{z},s) = \sum_{k,l} F_{kl}(z,\bar{z},s), \tag{8.16}$$

where each $F_kl \in \mathcal{F}_{kl}$, i.e., each F_{kl} is in $\mathcal{F}$ and of type (k,l). We also decompose $(f', f^n, g) \in \mathcal{G}$ as follows ($\beta = 1, \dots, n$):

$$f^\beta(z,w) = \sum_k f_k^\beta(z,w), \quad g(z,w) = \sum_k g_k(z,w), \tag{8.17}$$

where $f_k^\beta(z,w)$, $g_k(z,w)$ are homogeneous of degree k in z, e.g.

$$g_k(tz,w) = t^k g_k(z,w)\,, \quad t > 0. \tag{8.18}$$

The reader should observe that this redefines e.g. $g_k(z,w)$ which, previously, denoted the weighted homogeneous part of degree k in $g(z,w)$. However, in what follows we shall not need the decomposition into weighted homogeneous terms and, hence, the above notation should cause no confusion; for the remainder of this section, e.g. $g_k(z,w)$ means the part of $g(z,w)$ which is homogeneous of degree k in z, etc. For brevity, we use the following notation

$$f_w(z,w) = \frac{\partial f}{\partial w}(z,w), \dots, f_{w^m}(z,w) = \frac{\partial^m f}{\partial w^m}(z,w), \dots. \tag{8.19}$$

We will use the fact

$$f(z, s + i\langle z', \bar{z}'\rangle) = \sum_m f_{w^m}(z,s)\frac{(i\langle z', \bar{z}'\rangle)^m}{m!}. \tag{8.20}$$

We shall identify terms of type (k,l) in (8.15). Since the equation is real, it suffices to consider types where $k \geq l$. Also, note that for (k,l) such that $\mathcal{N}_{kl} = \mathcal{F}_{kl}$ equation (8.15) is trivially satisfied.

In what follows, we use the notation

$$F_{kl} = F_{kl}(z,\bar{z},s), \quad g_k = g_k(z,s), \quad \overline{g_k} = \overline{g_k}(\bar{z},s) \quad , \quad \text{etc.}$$

Collecting terms of equal type in (8.15), we obtain the following decoupled systems of differential equations, for $k \geq 3$,

$$\begin{cases} \frac{i}{2} g_k = F_{k0}, \\ \langle f'_{k+1}, \bar{z}'\rangle + 2z^n \bar{z}^{\bar{n}} f_k^n - \frac{\langle z', \bar{z}'\rangle}{2}(g_k)_w = F_{k+1,1}, \\ \qquad\qquad\qquad\qquad \mod \mathcal{N}_{k+1,1}, \end{cases} \tag{8.21}$$

and, in addition,

$$
(8.22)\quad \begin{cases} p_R\overline{f_0^n} + \frac{i}{2}g_2 = F_{20}, \\ \langle f_3', \bar z'\rangle + 2z^n\bar z^{\bar n}f_2^n - i\langle z', \bar z'\rangle p_R(\overline{f_0^n})_w - \frac{\langle z',\bar z'\rangle}{2}(g_2)_w = F_{31}, \\ \hfill \bmod \mathcal{N}_{31}, \\ i\langle z', \bar z'\rangle \langle (f_3')_w, \bar z'\rangle + 2i\langle z', \bar z'\rangle z^n\bar z^{\bar n}(f_2^n)_w \\ \qquad -\frac{\langle z',\bar z'\rangle^2}{2}p_R(\overline{f_0^n})_{w^2} + \overline{p_R}f_4^n - \frac{i\langle z',\bar z'\rangle^2}{4}(g_2)_{w^2} = F_{42}, \\ \hfill \bmod \mathcal{N}_{42}, \end{cases}
$$

$$
(8.23)\quad \begin{cases} \left\langle z', \overline{f_0'}\right\rangle + \frac{i}{2}g_1 = F_{10}, \\ -i\langle z', \bar z'\rangle \left\langle z', (\overline{f_0'})_w\right\rangle + \langle f_2', \bar z'\rangle + 2z^n\bar z^{\bar n}f_1^n + p_R\overline{f_1^n} \\ \qquad -\frac{\langle z',\bar z'\rangle}{2}(g_1)_w = F_{21} \quad \bmod \mathcal{N}_{21}, \\ i\langle z', \bar z'\rangle \langle (f_2')_w, \bar z'\rangle - \frac{\langle z',\bar z'\rangle^2}{2}\left\langle z', (\overline{f_0'})_{w^2}\right\rangle \\ \qquad +2i\langle z', \bar z'\rangle z^n\bar z^{\bar n}(f_1^n)_w + \overline{p_R}f_3^n - i\langle z', \bar z'\rangle p_R(\overline{f_1^n})_w \\ \qquad -\frac{i\langle z',\bar z'\rangle^2}{4}(g_1)_{w^2} = F_{32} \quad \bmod \mathcal{N}_{32}, \end{cases}
$$

$$
(8.24)\quad \begin{cases} -\mathrm{Im}\ g_0 = F_{00}, \\ 2\mathrm{Re}\ (\langle f_1', \bar z'\rangle) + 4\mathrm{Re}\ (z^n\bar z^{\bar n}f_0^n) - \langle z', \bar z'\rangle \mathrm{Re}\ (g_0)_w = F_{11}, \\ \hfill \bmod \mathcal{N}_{11}, \\ 4\langle z', \bar z'\rangle \mathrm{Im}\ (z^n\bar z^{\bar n}(f_0^n)_w) - 2\mathrm{Re}\ (\overline{p_R}f_2^n) \\ \qquad -2\langle z', \bar z'\rangle \mathrm{Im}\ (\langle (f_1')_w, \bar z'\rangle) + \frac{\langle z',\bar z'\rangle^2}{2}\mathrm{Im}\ (g_0)_{w^2} = F_{22}, \\ \hfill \bmod \mathcal{N}_{22}, \\ -\langle z', \bar z'\rangle^2 \mathrm{Re}\ (\langle (f_1')_{w^2}, \bar z'\rangle) \\ \qquad -2\langle z', \bar z'\rangle^2 \mathrm{Re}\ (z^n\bar z^{\bar n}(f_0^n)_{w^2}) \\ \qquad -2\langle z', \bar z'\rangle \mathrm{Im}\ (\overline{p_R}(f_2^n)_w) + \frac{\langle z',\bar z'\rangle^3}{6}\mathrm{Re}\ (g_0)_{w^3} = F_{33}, \\ \hfill \bmod \mathcal{N}_{33}. \end{cases}
$$

To show that this system has a unique solution $(f', f^n, g) \in \mathcal{G}_0$, if $\mathcal{N}$ is as defined by (1.1.25) and (1.1.26), we shall need the following facts. Let $p(z,\bar z)$ be a polynomial of type (a,b). A direct consequence of a theorem of E. Fischer [15] (see [17] and [14]) is the following unique decomposition of any formal series $F_{kl} \in \mathcal{F}_{kl}$,

$$
F_{kl} = pG_{k-a,l-b} + H_{kl}, \tag{8.25}
$$

where $G_{k-a,l-b} \in \mathcal{F}_{k-a,l-b}$ and $H_{kl} \in \mathcal{F}_{kl}$ with

$$\bar{p}(\nabla, \bar{\nabla}) H_{kl} = 0; \tag{8.26}$$

here, we use the notation $\bar{p}(z, \zeta) = \overline{p(\bar{z}, \bar{\zeta})}$. We shall also need the following lemma, whose proof follows easily from the decomposition (8.25) and is left to the reader.

Lemma 8.27. *Given polynomials $p(z, \bar{z})$ and $q(z, \bar{z})$ of type (a, b) and (c, d), respectively, any $F_{kl} \in \mathcal{F}_{kl}$ can be decomposed in a unique way as follows:*

$$F_{kl} = pG^1_{k-a,l-b} + qG^2_{k-c,l-d} + H_{kl}, \tag{8.28}$$

where $G^1_{k-a,l-b} \in \mathcal{F}_{k-a,l-b}$, $G^2_{k-c,l-d} \in \mathcal{F}_{k-c,l-d}$, and $H_{kl} \in \mathcal{F}_{kl}$ with

$$\bar{q}(\nabla, \bar{\nabla}) H_{kl} = 0, \quad \bar{p}(\nabla, \bar{\nabla}) H_{kl} \in \operatorname{Im} S; \tag{8.29}$$

here, S is the operator defined by $Su = -\bar{p}(\nabla, \bar{\nabla})(qu)$. Moreover, any pair

$$(G^1_{k-a,l-b}, H_{kl}) \in \mathcal{F}_{k-a,l-b} \times \mathcal{F}_{kl}$$

such that (8.29) *holds can occur in such a decomposition* (8.28).

Now, the system (8.21–8.24) is very similar to the system (9.2.2–9.2.5) in [12]. To show that there is a unique solution $(f', f^n, g) \in \mathcal{G}_0$, if $\mathcal{N}$ is as defined by (1.1.25) and (1.1.26), we proceed more or less exactly as in [12] and use the decompositions given by (8.25) and Lemma 8.27. We leave the verification to the reader. This completes the proof of Lemma 8.14 and hence that of Theorem 1.1.28. q.e.d.

References

[1] M. S. Baouendi, P. Ebenfelt & L. P. Rothschild, *Algebraicity of holomorphic mappings between real algebraic sets in $\mathbb{C}^N$*, Acta Math. **177** (1996) 225–273.

[2] ———, *Infinitesimal CR automorphisms of real analytic manifolds in complex space*, Comm. Anal. Geom. **6** (1998) 291–315.

[3] ———, *Parametrization of local biholomorphisms of real analytic hypersurfaces*, Asian J. Math. **1** (1997) 1–16.

[4] ———, *Real submanifolds in complex space and their mappings*, Princeton Math. Ser. Princeton Univ. Press, Princeton, NJ, to appear.

[5] M. S. Baouendi, X. Huang & L. P. Rothschild, *Regularity of CR mappings between algebraic hypersurfaces*, Invent. Math. **125** (1996) 13–36.

[6] M. S. Baouendi, H. Jacobowitz & F. Treves, *On the analyticity of CR mappings*, Ann. of Math. **122** (1985) 365–400.

[7] M. S. Baouendi & L. P. Rothschild, *Geometric properties of mappings between hypersurfaces in complex space*, J. Differential Geom. **31** (1990) 473–499.

[8] A. Boggess, *CR manifolds and the tangential CR complex*, CRC Press, Boca Roaton, FL, 1991.

[9] E. Cartan, *Sur la géométrie pseudo-conforme des hypersurfaces de deux variables complexes. I*, Ann. Math. Pura Appl. **11** (1932) 17–90, (or Oeuvres II, 1231–1304).

[10] ———, *Sur la géométrie pseudo-conforme des hypersurfaces de deux variables complexes, II*, Ann. Scoula Norm. Sup. Pisa **1** (1932) 333–354, (or Oeuvres III, 1217–1238).

[11] S.-S. Chern & J.K. Moser *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974) 219-271.

[12] P. Ebenfelt, *Normal forms and biholomorphic equivalence of real hypersurfaces in* $\mathbb{C}^3$, Indiana Univ. Math. J. **42** (1998) 311-366.

[13] ———, *Nondegeneracy conditions and normal forms for real hypersurfaces in complex space*, Journées "Équations aux Dérivées Partielles" (Saint-Jean-de-Monts, 1997), École Polytech., Palaiseau.

[14] P. Ebenfelt & H. S. Shapiro, *The mixed Cauchy problem for holomorphic partial differential operators*, J. Analyse Math. **65** (1995) 237–295.

[15] E. Fisher, *Über die Differentiationsprozesse der Algebra*, J. für Math. **148** (1917) 1-17.

[16] S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.

[17] H. S. Shapiro, *An algebraic theorem of E. Fischer, and the holomorphic Goursat problem*, Bull. London Math. Soc. **21** (1989) 513–537.

[18] C. L. Siegel, *Symplectic geometry*, Amer. J. Math. **65** (1943) 1-86.

[19] N. Stanton, *Infinitesimal CR automorphisms of rigid hypersurfaces*, Amer. J. Math. **117** (1995) 141-167.

[20] ———, *Infinitesimal CR automorphisms of real hypersurfaces*, Amer. J. Math. **118** (1996) 209–233.

[21] S. Sternberg, *Lectures on differential geometry*, Chelsea Publ. New York, NY, 1983.

[22] N. Tanaka, *On the pseudo-conformal geometry of hypersurfaces of the space of n complex variables*, J. Math. Soc. Japan **14** (1962) 397-429.

[23] ———, *On generalized graded Lie algebras and geometric structures. I*, J. Math. Soc. Japan **19** (1967) 215–254, (erratum **36** p. 1568).

[24] N. Wallach, *Real reductive groups I*, Academic Press, London, 1988.

[25] S. M. Webster, *The holomorphic contact geometry of a real hypersurface*, Modern Methods in Complex Analysis (eds. T. Bloom et al), Ann. of Math. Stud. 137, Princeton Univ. Press, Princeton, NJ, 1995, 327–342.

[26] D. Zaitsev, *Germs of local automorphisms of real-analytic CR structures and analytic dependence on k-jets*, Math. Res. Lett. **4** (1997) 823–842.

Royal Institute of Technology, Stockholm