J. DIFFERENTIAL GEOMETRY
49 (1998) 469-516

DEFORMATIONS OF HYPERBOLIC 3-CONE-MANIFOLDS

SADAYOSHI KOJIMA

Abstract

We show that any compact orientable hyperbolic 3-cone-manifold with cone angles at most π can be continuously deformed to a complete hyperbolic manifold homeomorphic to the complement of the singularity. This together with the local rigidity by Hodgson and Kerckhoff implies the global rigidity for compact orientable hyperbolic 3-cone-manifolds under the same angle assumption.

0. Introduction

A hyperbolic 3-cone-manifold is a riemannian 3-manifold of constant negative sectional curvature with cone-type singularity along simple closed geodesics (see [7], [8]). To each component of the singularity, is associated a cone angle. The cone angle is a positive real number and possibly attains 2π. In this particular case, the singular set is not singular and simply a finite union of disjoint simple closed geodesics. The hyperbolic 3-cone-manifold is a generalization of the hyperbolic 3-orbifold with vertexless singularity.

We are concerned with the deformations of a hyperbolic 3-cone-manifold with constant topological type. The hyperbolic Dehn filling theory by Thurston in [14], which describes deformations of a complete hyperbolic manifold in more wild setting, is a pioneering work of this

Received February 18, 1997, and, in revised form, November 26, 1997.
1991 *Mathematics Subject Classification.* Primary 57M50; Secondary 30F40.
Key words and phrases. cone-manifold, hyperbolic manifold, rigidity, deformation.

subject. One important progress from a rather classical viewpoint was made recently by Hodgson and Kerckhoff [7]. They proved the local rigidity when cone angles are $\leq 2\pi$. It corresponds to Weil [18] and Garland [4] rigidity for hyperbolic manifolds of finite volume.

In [9], we showed as one of applications of the local rigidity that the underlying space of a nonsingular part of a compact hyperbolic 3-cone-manifold admits a complete hyperbolic structure of finite volume. We come back to this result later. The purpose of the present paper is to connect that complete structure with the original singular structure by a continuous family of cone-manifolds with constant topological type under the assumption that initial cone angles are $\leq \pi$. The main theorem is

Theorem. *Let C be a compact orientable hyperbolic 3-cone-manifold and Σ a singular set which forms a link in C. If the cone angles assigned to the components of Σ all are at most π, then C admits an angle decreasing continuous family of deformations to the complete hyperbolic manifold homeomorphic to the nonsingular part $C - \Sigma$.*

There are two immediate corollaries related to the representations in the group of orientation preserving isometries of the hyperbolic 3-space $\mathbf{H}^3$, isomorphic to $\mathrm{PSL}_2(\mathbf{C})$. For cone-manifolds, we have a holonomy representation of a nonsingular part, so that meridional loops of the singularity are mapped to elliptic elements. The representation so obtained could be neither faithful nor discrete. Nevertheless, the local rigidity in [7] asserts that a neighborhood of this wild representation is parameterized up to conjugacy by Dehn filling coefficients [14] which are geometrically well understood.

One corollary is about the global rigidity. Weil and Garland rigidity states that any nearby discrete faithful representations of a group in $\mathrm{PSL}_2(\mathbf{C})$ with finite volume quotients are conjugate. In the case of cone-manifolds, the local rigidity says in particular that nearby representations of a holonomy representation of a nonsingular part with constant rotation angles for meridians are conjugate. Mostow [11] and Prasad [12] rigidity for a hyperbolic manifold of finite volume then asserts that any discrete faithful representations in $\mathrm{PSL}_2(\mathbf{C})$ are not only locally but globally conjugate each other. This global rigidity implies a geometric consequence that homeomorphic hyperbolic manifolds of finite volume are isometric. We state the global rigidity for hyperbolic 3-cone-manifolds rather in terms of this geometric terminologies.

Corollary 1. *Let C be a compact orientable hyperbolic 3-cone-manifold with singularity Σ where cone angles assigned to the components of Σ all are $\leq \pi$. If (C', Σ') is homeomorphic to (C, Σ) so that the corresponding cone angles all are the same, then C' and C are isometric.*

The other corollary is about liftability of $\mathrm{PSL}_2(\mathbf{C})$-representations into $\mathrm{SL}_2(\mathbf{C})$. The liftability has been discussed particularly for discrete subgroups in $\mathrm{PSL}_2(\mathbf{C})$. As was pointed out in [2], the liftability depends only on the component of the space of representations. The main theorem will be proved by connecting the holonomy representation of a cone-manifold with that of a complete structure by a particular path in the space of representations. Since the holonomy representation of a complete hyperbolic manifold is known to lift in [14], we have

Corollary 2. *The holonomy representation of a compact orientable hyperbolic 3-cone-manifold can be lifted to a $\mathrm{SL}_2(\mathbf{C})$-representation if the cone angles assigned all are at most π.*

It is quite unlikely that the angle assumptions in Theorem and Corollaries are necessary, though the argument we develop here uses its advantage. More progress should be expected.

A fairly large part of the proof of the main theorem is due to Thurston's strategy for the geometrization of orbifolds [15], [16] together with the local rigidity by Hodgson and Kerckhoff. The over all logic of Thurston's argument and some of its details can be found in [13]. We convey its minimal essentials for our purpose here, and hence the exposition will be reasonably self-contained.

This paper is organized as follows. In the first section, we will review some basic facts about hyperbolic 3-cone-manifolds. Also we improve the results in [9] from more Riemannian geometric viewpoints due to Steve Kerckhoff. The second section is to introduce two main machineries, the local rigidity and the pointed Hausdorff-Gromov topology. They are fundamental when cone angles are $\leq 2\pi$. The third section is to establish a few tools to control the local geometry of cone-manifolds away from the singularity when cone angles are $\leq \pi$. This section contains a technical but the most crucial part of the analysis. In section 4, we establish a discrete setting of the problem. Then we study what happens when tubular neighborhoods of the singularity in the deformations are uniformly thick in section 5, and when otherwise in section 6. In the final section, we study what happens to continuous families and prove

Theorem and Corollaries.

The author would like to thank Steve Kerckhoff for careful attention to this work and for showing him a quick idea to prove Theorem 1.2.1 in full generality, Teruhiko Soma and the members of the Saturday Seminar at Tokyo Institute of Technology, especially Shigenori Matsumoto, for their invaluable suggestions, and the Centre Emile Borel of the Institut Henri Poincaré for their hospitality, where he finished the first version of this paper.

In addition, the author would like to thank the referee for pointing out an earlier work by Qing Zhou [19] which discusses similar deformations. Zhou's argument together with the local rigidity now established by Hodgson and Kerchkhoff [7] lead to another proof of the existence of an angle decreasing family in the main theorem provided that the initial cone angles all are at most $2\pi/3$. Also. Zhou proved Theorem 1.2.1 in slightly different manner.

1. Hyperbolic 3-cone-manifolds

In this section, we assemble a few standard notions and notation which we use throughout this paper, improve the results in [9], and discuss an upper bound of the volume of hyperbolic 3-cone-manifolds with the same topological type.

1.1. Preliminaries.

Let X be a metric space with a metric d. An R-neighborhood of $x \in X$ for $R > 0$ is the set of points in X from which the distance to x is $< R$, and denoted by

$$\mathrm{B}_R(X, x) = \{y \in X \mid d(y, x) < R\}.$$

If X is a riemannian manifold, the closure of $\mathrm{B}_R(X, x)$ is homeomorphic to a closed ball at least for sufficiently small R. The injectivity radius of X at $x \in X$ is the first supremum of such radii, and denoted by $\operatorname{inj}_x X$. If X has a boundary ∂X, we choose the supremum by furthermore requiring that $\mathrm{B}_R(X, x)$ does not touch ∂X. The injectivity radius of X is the infimum of injectivity radii of the points in X, and is denoted by

$$\operatorname{inj} X \, (:= \inf \{\operatorname{inj}_x X \mid x \in X\}).$$

The injectivity radius for manifolds with nonempty boundary by this definition would not be interesting since the points close to the boundary always make it vanishing.

Let C be an orientable hyperbolic 3-cone-manifold of finite volume with compact singularity. The singular set Σ is assumed to form a link

$$\Sigma = \Sigma^1 \cup \cdots \cup \Sigma^n$$

of n components. To each component Σ^j of Σ, associated is a cone angle $\alpha^j \in (0, \infty)$. The angle set A of C is a vector

$$A = (\alpha^1, \cdots, \alpha^n)$$

of cone angles.

C carries a nonsingular but incomplete hyperbolic structure on the complement of the singularity

$$N = C - \Sigma.$$

C itself inherits a metric induced from a riemannian metric on N. We assume that C is complete with respect to this metric. In particular, the metric completion of N is identical to C. We have a developing map of N from its universal covering space $\widetilde{N}$,

$$\mathcal{D}_C : \widetilde{N} \to \mathbf{H}^3,$$

and a holonomy representation

$$\rho_C : \Pi = \pi_1(N) \to \mathrm{PSL}_2(\mathbf{C}).$$

They are called a developing map and a holonomy representation of a cone-manifold C. A developing map is a local isometry, but never be injective. A holonomy representation is hardly discrete nor faithful.

Let m_j, $j = 1, 2, \cdots, n$, be an oriented meridional loop for each component of Σ. The image $\rho_C(m_j)$ of a meridian m_j by the holonomy representation is an elliptic element rotating $\mathbf{H}^3$ by α^j about the axis, though the rotation angle of $\rho_C(m_j)$ makes sense only modulo 2π.

The injectivity radius of C at $x \in N = C - \Sigma$ is to be the injectivity radius of N at x and denoted by

$$\mathrm{inj}_x C \ (:= \mathrm{inj}_x N).$$

The global injectivity radius of a cone-manifold C by this definition would not be interesting since if the singular set is nonempty, then the points close to the singularity always make it vanishing.

Definition. A *topological type* of a cone-manifold C is a homeomorphism type of a pair (C, Σ). We say C is *homeomorphic* to C_1 for short if there is a homeomorphism between (C, Σ) and (C_1, Σ_1). More strong relation is an *isomorphism type.* Two cone-manifolds are *isomorphic* if they share not only topological types but also cone angles, more precisely if there is a homeomorphism between (C, Σ) and (C_1, Σ_1) so that the corresponding components of the singularity share the same cone angles. Such a homeomorphism is called an *isomorphism.* A self isomorphism is called an *automorphism* as usual. The strongest relation is an *isometry type* whose definition would be obvious.

Remark. The global rigidity is the claim that the isomorphism type and the isometry type are the same.

1.2. Nonsingular parts.

The following theorem was proved in [9] under an extra angle assumption using Hodgson-Kerckhoff's local rigidity. Here we present a quick argument, due to Steve Kerckhoff, which does not use the local rigidity and works without any angle assumption. As we mentioned in the introduction, Zhou also showed the following theorem in [19] in slightly different manner.

Theorem 1.2.1. *The underlying space of a nonsingular part N of an orientable hyperbolic 3-cone-manifold C of finite volume carries a complete negatively curved metric. In particular it is homeomorphic to an interior of a compact irreducible atoroidal 3-manifold with toral boundary which admits no Seifert fibrations. Moreover, it admits a complete hyperbolic structure of finite volume.*

Proof. In the cylindrical coordinates around each component of the singularity Σ, the metric has the form

$$d\delta^2 + \sinh^2 \delta \, d\theta^2 + \cosh^2 \delta \, d\lambda^2 \ ,$$

where δ is the distance from the singularity, λ is the distance along the singularity, and θ is the angular measure around the singularity. Choose $\varepsilon > 0$ small enough so that an ε-tubular neighborhood of Σ is a disjoint union of an ε-tubular neighborhood of each component of Σ. Also choose monotone C^∞-functions $\varphi(\delta)$ and $\psi(\delta)$ in terms of δ so that

$$\varphi(\delta) = \begin{cases} 1 & \text{if } \delta \geq \varepsilon, \\ \mathrm{O}(1/\delta) & \text{if } \delta \to 0, \end{cases}$$

and

$$\psi(\delta) = \begin{cases} \cosh\delta & \text{if } \delta \geq \varepsilon, \\ \mathrm{O}(\delta) & \text{if } \delta \to 0, \end{cases}$$

where O( ) is the Landau symbol, and modify the metric in an ε-tubular neighborhood of each component of Σ by

$$\varphi^2(\delta)d\delta^2 + \sinh^2\delta\, d\theta^2 + \psi^2(\delta)d\lambda^2.$$

This gives a complete metric on the nonsingular part $N = C - \Sigma$ since $\varphi(\delta)$ diverges when $\delta \to 0$.

Let us compute the sectional curvature for this new metric. For notational convenience, we set $\delta = x_1$, $\theta = x_2$ and $\lambda = x_3$. By a computation of the Christoffel symbols, we have the evaluation of the connection with respect to this basis:

$\nabla_{\partial/\partial x_i}(\partial/\partial x_j)$	1	2	3
1	$\varphi'/\varphi \cdot \partial/\partial x_1$	$\coth\delta \cdot \partial/\partial x_2$	$\psi'/\psi \cdot \partial/\partial x_3$
2	$\coth\delta \cdot \partial/\partial x_2$	$-\sinh\delta\cosh\delta/\varphi^2 \cdot \partial/\partial x_1$	0
3	$\psi'/\psi \cdot \partial/\partial x_3$	0	$-\psi\psi'/\varphi^2 \cdot \partial/\partial x_1$

It is symmetric and hence the Riemannian curvature tensor

$$R(X,Y,Z) = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})(Z)$$

has been diagonalized, i.e., $< R(X,Y,Z), W >= 0$ unless $X \neq Y$ and either $(X,Y) = (Z,W)$ or $(X,Y) = (W,Z)$. Thus the sectional curvatures

$$K(X,Y) = -\frac{< R(X,Y,X), Y >}{\|X\|^2 \cdot \|Y\|^2 - < X, Y >^2}$$

all are convex combinations of the three sectional curvatures $K_{ij} = K(\partial/\partial x_i, \partial/\partial x_j)$ with $\{i,j\} = \{1,2,3\}$. By a direct computation, we have

$$K_{12} = \frac{1}{\varphi^2}\left(\frac{\varphi'\coth\delta}{\varphi} - 1\right),$$

$$K_{13} = -\frac{1}{\varphi^2}\left(\frac{\psi''}{\psi} - \frac{\varphi'\psi'}{\varphi\psi}\right),$$

$$K_{23} = -\frac{\psi'\coth\delta}{\varphi^2\psi}.$$

Then these three values stay negative away from zero for $0 < \delta \leq \varepsilon$ by choosing φ and ψ appropriately, and the volume change is still bounded. Hence the nonsingular part N admits a complete negatively curved metric of finite volume.

The topological properties in the statement all are now easily derived from the negativity of the new metric. Then by Thurston's uniformization theorem for Haken manifolds [10], [16], N admits a complete hyperbolic structure. q.e.d.

It is convenient to adopt the convention that we regard a complete hyperbolic manifold with preferred meridional elements as a cone-manifold.

Definition. Let M be a complete orientable hyperbolic 3-cone-manifold of finite volume with prescribed meridional elements not only for the components of Σ but for some cusps. We call M also a cone-manifold. The *topological type* of M is a pair of the Dehn filled resultant of M along prescribed meridional curves for cusps in question and the union of Σ and surgery cores. The cone angles which should be assigned to the components for cusps all are zero.

Notation. If C is a compact orientable hyperbolic 3-cone-manifold in the original sense, then the nonsingular part together with the complete structure obtained by Theorem 1.2.1 is a cone-manifold in this new sense by assigning original meridional elements of C. We let C_{comp} be this particular cone-manifold without singularity. The topological types of C and C_{comp} are the same.

Corollary 1.2.2. *The group of automorphisms of an orientable hyperbolic 3-cone-manifold of finite volume up to isotopy is finite.*

Proof. Here we duplicate the proof in [9]. Any automorphism of C induces a homeomorphism of C_{comp} preserving meridians. By Mostow-Prasad rigidity, it is homotopic to an isometry of C_{comp}. Moreover by Waldhausen's theorem [17], this homotopy can be attained by an isotopy. The isotopy extends to an automorphism of C. Hence the group of automorphisms of C up to isotopy is realized as a subgroup of the isometry group of C_{comp}, which is finite. q.e.d.

Remark. The existence of the singularity in the proof is needed to apply Waldhausen's result, and in fact the argument does not cover the case without singularity. This finiteness up to isotopy for nonsingular case had been a difficult question and was settled very recently by Gabai, Meyerhoff and Thurston [3] in full generality, though the finiteness up to homotopy is a consequence of Mostow rigidity.

Corollary 1.2.3. *An orientable hyperbolic 3-cone-manifold of finite volume admits only finitely many toral cusps which are foliated by horotori.*

Proof. Since the end of C_{comp} consists of finitely many toral ends, the end of C consists of finitely many components homeomorphic to the torus time $\mathbf{R}$ at least topologically. To get a complete end, every element of a fundamental group of each end component must be mapped by the holonomy representation to a parabolic element with a common fixed point at the sphere at infinity $\mathbf{S}^2_\infty$. A small neighborhood of each component is then foliated by the quotient of horospheres by the holonomy image. q.e.d.

We conclude this subsection by the following observation for loops in a hyperbolic 3-cone-manifold. Every loop in a hyperbolic manifold admits a length shortening free homotopy to either a closed geodesic (including a point) or an arbitrary short loop tending to the cusp. This will not be true for cone-manifolds in general, but we have its weaker version.

Lemma 1.2.4. *Any loop in the nonsingular part N of an orientable hyperbolic cone-manifold C of finite volume admits a length shortening homotopy to either a closed geodesic, an arbitrary short loop tending to the cusp or a piecewise geodesic loop hitting the singularity Σ.*

Proof. Let ℓ be an arbitrary loop in the nonsingular part N. Fix a point p on ℓ, and we straight ℓ. Then it defines a length shortening homotopy to at least a piecewise geodesic loop hitting the singularity Σ. If Σ does not obstruct the straightening, the length shortening homotopy reaches to a geodesic path ℓ' based at p. We then homotop ℓ' to a shorter geodesic path by sliding the reference point. This length shortening homotopy either terminates at some stage or tends to the cusp. When it terminates, either the homotopy hits Σ or it reaches to a closed geodesic. q.e.d.

Remark. The proof is valid even when C is of infinite volume, or the singularity is noncompact.

1.3. Volumes.

The volume of a hyperbolic cone-manifold C can be expressed in terms of an integral over the canonical section of the associated flat $\mathbf{H}^3$-bundle over N. To say more precise, let $\widetilde{N}$ be the universal covering space of N. Each element $\iota \in \Pi = \pi_1(N)$ acts on $\widetilde{N}$ by a deck transformation and on $\mathbf{H}^3$ by $\rho_C(\iota) \in \rho_C(\Pi)$. Denote by E the fiber product $\widetilde{N} \times \mathbf{H}^3/(\Pi \times \rho_C)$. It is a $\mathbf{H}^3$-bundle over N with a structure group in $\mathrm{PSL}_2(\mathbf{C})$. The developing map $\mathcal{D}_C : \widetilde{N} \to \mathbf{H}^3$ defines a section $id \times \mathcal{D}_C : \widetilde{N} \to \widetilde{N} \times \mathbf{H}^3$ which descends to a canonical one $s_0 : N \to E$ with respect to the hyperbolic structure on N. The volume of C is then identified with the integral,

$$\mathrm{vol}\, C = \int_N s_0^* \, dv, \tag{1.1}$$

where dv is the volume form of $\mathbf{H}^3$.

Lemma 1.3.1. *Suppose C is compact. If a section $s_1 : N \to E$ agrees with the canonical one s_0 on a small tubular neighborhood of Σ, then*

$$\int_N s_1^* \, dv = \int_N s_0^* \, dv.$$

In particular, the volume defined in (1.1) *depends only on the behavior of the section near the singularity.*

Proof. Since $\mathbf{H}^3$ is contractible, there is a homotopy s_t of sections connecting s_0 and s_1. Let N_ε be a compact exterior of an ε-tubular neighborhood of Σ. Choose ε small enough so that $s_0 = s_1$ on that neighborhood of Σ. Then by Stokes,

$$\begin{aligned}\int_N s_1^* \, dv - \int_N s_0^* \, dv &= \int_{N_\varepsilon} s_1^* \, dv - \int_{N_\varepsilon} s_0^* \, dv \\ &= \int_{\partial N_\varepsilon \times [0,1]} s_t^* \, dv \\ &= \int_{N_\varepsilon \times [0,1]} d\,(s_t^* \, dv).\end{aligned}$$

The last integral is zero. q.e.d.

Proposition 1.3.2. *Given a compact orientable hyperbolic 3-cone-manifold C with the angle set $A = (\alpha^1, \cdots, \alpha^n)$, there is a constant*

V_{max} so that if a hyperbolic cone-manifold C_1 homeomorphic to C has an angle set $A_1 = (\alpha_1^1, \cdots, \alpha_1^n)$ with $\alpha_1^j \leq \alpha^j$ for all $1 \leq j \leq n$, then $\mathrm{vol}\, C_1 \leq V_{max}$.

Proof. Take a fine geodesic triangulation K of C so that Σ is a subcomplex, and a star neighborhood of Σ is a closed regular neighborhood. Since $\alpha_1^j \leq \alpha^j$ for all j, we may choose a homeomorphism $\varphi : (C, \Sigma) \to (C_1, \Sigma_1)$ so that any 3-simplex in a regular neighborhood of Σ is mapped to an honest geodesic simplex in C_1.

Let $\widetilde{K^{(0)}}$ be the preimage of the 0-skeleton $K^{(0)}$ in the universal cover $\widetilde{N}_1$. Then the developing map $\mathcal{D}_{C_1}$ defines the map $id \times \mathcal{D}_{C_1}|_{\widetilde{K^{(0)}}} : \widetilde{K^{(0)}} \to \widetilde{N}_1 \times \mathbf{H}^3$ which extends to an equivariant continuous map of $\widetilde{N}_1$ by straightening. Since it is equivariant, it descends to a section $s_1 : N_1 \to E_1 = \widetilde{N}_1 \times \mathbf{H}^3/(\Pi_1 \times \rho_{C_1})$. s_1 is identical to the canonical section $s_0 : N_1 \to E_1$ near Σ_1. Hence by the previous lemma, we have

$$\mathrm{vol}\, C_1 = \int_{N_1} s_1^* \, dv.$$

The right-hand side is a total sum of the signed volumes of 3-simplices appeared by straightening. However since the volume of a 3-simplex is uniformly bounded by a constant v_3 (= the volume of a regular ideal tetrahedra), this sum is bounded by $V_{max} = kv_3$, where k is the number of 3-simplices in K. The constant V_{max} obviously depends on only C and not C_1. q.e.d.

Remark. A number of tetrahedra needed to triangulate a neighborhood of the singularity Σ in C depends in fact on the cone angles. However it is bounded by some constant depending on only the upper bound of the cone angles and not any particular C.

2. Cone angles $\leq 2\pi$

In this section, we review two machineries to study deformations of a hyperbolic 3-cone-manifold C when cone angles are at most 2π. One is the local rigidity by Hodgson and Kerckhoff [7] and the other is the pointed Hausdorff-Gromov topology studied in [5].

2.1. Local rigidity.

Let us recall what the deformation of a hyperbolic 3-cone-manifold C is.

Definition. A deformation of a hyperbolic 3-cone-manifold C is a hyperbolic 3-cone-manifold C_1 together with a reference homeomorphism $\xi_1 : (C, \Sigma) \to (C_1, \Sigma_1)$. Two deformations (C_1, ξ_1) and (C_2, ξ_2) of C are equivalent if there is an isometry $\psi : C_1 \to C_2$ so that ξ_2 is isotopic to $\psi \circ \xi_1$. A composition of a developing map $\mathcal{D}_{C_1}$ with a lift $\tilde{\xi}_1$ of ξ_1 defines a continuous map $\mathcal{D}_{C_1} \circ \tilde{\xi}_1 : \widetilde{N} \to \mathbf{H}^3$. The developing map is well defined up to multiplication of isometries of $\mathbf{H}^3$, and hence a deformation can be considered as a point on a quotient space of the mapping space $\mathcal{M}(\widetilde{N}, \mathbf{H}^3)$ by $\mathrm{PSL}_2(\mathbf{C})$-action on the images. The quotient space carries a topology induced by a compact open topology on $\mathcal{M}(\widetilde{N}, \mathbf{H}^3)$. The set of equivalence classes of deformations of C carries a further quotient topology by the action on the source of the group of lifts of automorphisms of C which are isotopic to the identity. We say (C_1, ξ_1) is a small deformation of (C, id) if it is close to C in this topology.

Remark. The reference homeomorphism for the deformation is to fix an isotopy class of the model. In fact, only its isotopy class is significant.

The topology on the set of equivalence classes of deformations of C turns out to be not quite complicated by the local rigidity. To see this, we review a few topological properties of the space of representations of the group $\Pi = \pi_1(N)$ in $\mathrm{PSL}_2(\mathbf{C})$. The space of such representations carries a natural algebraic topology. There is a canonical projection to the set of conjugacy classes $\mathrm{Hom}(\Pi, \mathrm{PSL}_2(\mathbf{C}))/\mathrm{PSL}_2(\mathbf{C})$. A small neighborhood of a conjugacy class represented by a holonomy representation of a cone-manifold will be well behaved.

An orientation preserving isometry φ of $\mathbf{H}^3$ can be represented by a matrix Φ in $\mathrm{SL}_2(\mathbf{C})$. The complex length of φ is a twice of an appropriate branch of the log of an eigenvalue of Φ. It measures how much φ translates $\mathbf{H}^3$ with twist along an invariant geodesic. Since there are two choices to be made, we adopt the following convention. To each oriented meridional element m_j, we choose a complex length of $\rho_C(m_j)$ by a multiple of its cone angle with the complex unit $\sqrt{-1}$, which we denote by $\mathcal{L}_{m_j}(\rho_C)$. It is equal to $\alpha^j\sqrt{-1}$. Then orient the invariant geodesic ℓ_C^j of $\rho_C(m_j)$ so that the rotational direction of m_j is counter clockwise. There are two ways to continuously extend this assignment of a complex length to that of transformation represented by $\rho(m_j)$ where ρ is close to ρ_C. Since ρ is close to ρ_C, the invariant geodesic ℓ^j of $\rho(m_j)$ is also close to ℓ_C^j and inherits an orientation. We choose the sign of the real part according to whether $\rho(m_j)$ translates the point on ℓ^j in positive or negative direction.

This convention extends to a continuous map, which we denote by $\mathcal{L}_{m_j}$, defined on the component $\mathrm{Hom}_C(\Pi, \mathrm{PSL}_2(\mathbf{C}))/\mathrm{PSL}_2(\mathbf{C})$ of $\mathrm{Hom}(\Pi, \mathrm{PSL}_2(\mathbf{C}))/\mathrm{PSL}_2(\mathbf{C})$ containing the conjugacy class of ρ_C. Arranging these maps, we obtain

$$\mathcal{L}_m : \mathrm{Hom}_C(\Pi, \mathrm{PSL}_2(\mathbf{C}))/\mathrm{PSL}_2(\mathbf{C}) \to \mathbf{C}^n,$$

where $\mathcal{L}_m(\rho) = (\mathcal{L}_{m_1}(\rho), \cdots, \mathcal{L}_{m_n}(\rho))$.

Theorem 2.1.1 (Hodgson-Kerckhoff's Local Rigidity [7]). *If the cone angles assigned to the components of Σ in C all are positive and $\leq 2\pi$, then $\mathcal{L}_m$ is a local diffeomorphism near the conjugacy class represented by ρ_C.*

In particular, the complex dimension of $\mathrm{Hom}(\Pi, \mathrm{PSL}_2(\mathbf{C}))/\mathrm{PSL}_2(\mathbf{C})$ near the class represented by ρ_C is equal to n, the number of components of Σ, and ρ_C represents a smooth point.

Removing a small tubular neighborhood of Σ from C, we obtain a compact hyperbolic manifold with boundary. It is nearby deformations supported on all but a small neighborhood of the boundary are parameterized by holonomy representations [14]. The parameter in terms of complex lengths is more geometrically described in [7], [14]. For each value near $\mathcal{L}_m(\rho_C)$, there is a unique way to fill the boundary by the hyperbolic Dehn filling theory [14]. If the value of $\mathcal{L}_m$ is in purely imaginal part $\Theta = (\Im \mathbf{C})^n \subset \mathbf{C}^n$, then the filled resultant is a hyperbolic cone-manifold with the same topological type but with perturbed cone angles. Hence the local rigidity together with the hyperbolic Dehn surgery filling imply the unique existence of a small deformation of C if the perturbation of cone angles is small enough. Namely the possible range of perturbation is open.

When we start with a noncompact 3-cone-manifold without singularity such as C_{comp}, then the existence of a small deformation is nothing but the conclusion of the hyperbolic Dehn filling theory for complete manifolds with a specified direction. In particular, there is a unique small deformation with perturbed cone angles for this case also. One can summarize the conclusion by

Corollary 2.1.2. *Let C be an orientable hyperbolic 3-cone-manifold of finite volume so that the cone angles assigned to the components of*

Σ all are at most 2π, possibly some or all of them are zero. Then there is a unique small deformation of C with perturbed cone angles if the perturbed cone angles are close enough to the initial ones.

Remark. A continuous path on the space of representations, whose image by $\mathcal{L}_m$ is contained in purely imaginal part, does not always correspond to the deformations of a cone-manifold in the full range.

2.2. Pointed Hausdorff-Gromov topology.

We review the pointed geometric convergence of complete metric spaces due to Gromov in [5], which generalizes the idea of Hausdorff topology on the set of all compact subset in a complete metric space.

Definition. Let (X, x_0) and (Y, y_0) be pointed complete metric spaces. A relation $\mathrm{R} \subset X \times Y$ is an ε-approximation between (X, x_0) and (Y, y_0) if the following hold:

1. there is $y \in Y$ such that $x_0 \mathrm{R} y$ and $d_Y(y_0, y) < \varepsilon$,
2. there is $x \in X$ such that $x \mathrm{R} y_0$, and $d_X(x_0, x) < \varepsilon$,
3. $pr_X(\mathrm{R} \cap (\mathrm{B}_{1/\varepsilon}(X, x_0) \times \mathrm{B}_{1/\varepsilon}(Y, y_0)) = \mathrm{B}_{1/\varepsilon}(X, x_0)$,
4. $pr_Y(\mathrm{R} \cap (\mathrm{B}_{1/\varepsilon}(X, x_0) \times \mathrm{B}_{1/\varepsilon}(Y, y_0)) = \mathrm{B}_{1/\varepsilon}(Y, y_0)$, and
5. for any $x, x' \in \mathrm{B}_{1/\varepsilon}(X, x_0)$ and $y, y' \in \mathrm{B}_{1/\varepsilon}(Y, y_0)$ with $x \mathrm{R} y$ and $x' \mathrm{R} y'$, we have $|d_X(x, x') - d_Y(y, y')| < \varepsilon$.

Definition. A sequence $\{(X_i, x_i)\}$ of pointed complete metric spaces is said to converge geometrically to (Y, y) if for any $\varepsilon > 0$, there exists i_0 such that there is an ε-approximation between (X_i, x_i) and (Y, y) for all $i > i_0$.

An interesting class $\mathcal{C}$ of pointed complete metric spaces with respect to this convergence is the one whose member (X, x) has the property;

$$\text{(2.1)} \qquad \text{The closure of} \quad \mathrm{B}_R(X, x) \quad \text{is compact for all} \ R > 0.$$

It is shown in [5] that the geometrically convergent sequence $\{(X_i, x_i)\}$ contained in the class $\mathcal{C}$ geometrically converges to a unique complete metric space up to isometry. Hence the geometric convergence defines a Hausdorff topology on the set of isometry classes of $\mathcal{C}$. We call it the pointed Hausdorff-Gromov topology.

If furthermore we have some uniformity for the local structure of the metric spaces, then the set of isometry classes of such spaces together with the pointed Hausdorff-Gromov topology becomes precompact. In particular, any sequence in the space contains a subsequence converging geometrically to a unique metric space. The following criterion by Gromov is useful.

Proposition 2.2.1 (Gromov [5]). *Let $\{(X_i, x_i)\}$ be a sequence of pointed complete metric spaces with the property* (2.1) *for all i. Then the sequence contains a geometrically convergent subsequence if and only if there is a subsequence $\{k\} \subset \{i\}$ so that for any $R, \varepsilon > 0$,*

$$\min \#\{\varepsilon\text{-balls covering } \mathrm{B}_R(X_k, x_k)\}$$

is bounded by some constant depending only on R and ε, where the minimum is taken over all ε-coverings of $\mathrm{B}_R(X_k, x_k)$.

The class of metric spaces we are concerned with is quite restricted, but not only hyperbolic cone-manifolds. We will work also on euclidean cone-manifolds as their rescaling limits.

Notation. Fix a number $L \leq -1$, which will bound curvature. $\mathcal{C}^\theta_{[L,0]}$ will be the set of pointed compact orientable cone-manifolds of constant sectional curvature $= K$, where $K \in [L, 0]$, so that the cone angles assigned to the components of the singularity all are at most θ. $\mathcal{C}^\theta_K$ is a subset of $\mathcal{C}^\theta_{[L,0]}$ consisting of cone-manifolds with a particular curvature constant K.

Remark. Each member of $\mathcal{C}^\theta_{[L,0]}$ is compact and hence has the property (2.1).

Choose $C \in \mathcal{C}^{2\pi}_K$ and $x \in C$ not lying on Σ. Here we bound cone angles by 2π. The set of points in C which admit at least two shortest geodesic paths to x in C is called a cut locus of C with respect to the reference point x. The cut locus is a connected geodesic cell complex in C. The complement of the cut locus consists of shortest rays to x,

$$P_x = \{y \in C \mid y \text{ admits the unique shortest path to } x\},$$

and is called a Dirichlet fundamental domain of C about x.

Lemma 2.2.2. *The Dirichlet fundamental domain P_x of $C \in \mathcal{C}^{2\pi}_K$ about x is isometrically realized as an interior of a starshaped polyhedron in the simply connected 3-dimensional space $\mathbf{H}_K$ of constant curvature*

$= K$. *The closure is a starshaped polyhedron. Furthermore if* $C \in \mathcal{C}_K^\pi$, *that is to say, cone angles all are* $\leq \pi$, *then* P_x *is convex.*

Proof. Obvious. q.e.d.

We call this embedded compactified polyhedron a Dirichlet polyhedron of C about x, and denote it again by P_x. Namely, P_x stands for an open dense subset in C, and simultaneously, a compact polyhedron in $\mathbf{H}_K$.

Proposition 2.2.3. *Fix a curvature bound* $L \leq -1$. *Any sequence* $\{C_i\} \subset \mathcal{C}_{[L,0]}^{2\pi}$ *with reference points* $x_i \in C_i$ *contains a subsequence converging geometrically to a complete metric space* (C_*, x_*).

Proof. Let (C, x_0) be a pointed compact orientable 3-cone-manifold with constant sectional curvature $= K$ where $L \leq K \leq 0$. If $x_0 \notin \Sigma$, then

$$\begin{aligned}
\min \#\{\varepsilon\text{-balls covering } \mathrm{B}_R(C, x_0)\}& \\
&\leq \min \#\{\varepsilon\text{-balls covering } \mathrm{B}_R(P_{x_0}, x_0)\} \\
&\leq \min \#\{\varepsilon\text{-balls covering } \mathrm{B}_R(\mathbf{H}_K, x_0)\} \\
&\leq \min \#\{\varepsilon\text{-balls covering } \mathrm{B}_R(\mathbf{H}_L, x_0)\},
\end{aligned}$$

where the minimum is taken all over ε-ball coverings of the target. The last bound depends only on R and ε, and not on any particular C or K. If $x_0 \in \Sigma$, the same bound actually works by choosing the center of a Dirichlet polyhedron near x_0 but not on Σ. Thus the result follows from Gromov's Criterion. q.e.d.

Definition. We call (C_*, x_*) a geometric limit of $\{(C_i, x_i)\}$.

Remark. If C_* is compact, the isometry class of C_* does not depend on the choice of the reference points $x_i \in C_i$ in the sequence.

Remark. The set of equivalence classes of deformations of an orientable hyperbolic 3-cone-manifold C with cone angle $\leq 2\pi$ carries a topology well described by the local rigidity. Assigning the isometry class of a deformation to each deformation, we get a map to the set of isometry classes of metric spaces in $\mathcal{C}_{-1}^{2\pi}$ together with the pointed Hausdorff-Gromov topology. It is not quite hard to see that this map is continuous.

3. Cone angles $\leq \pi$

In this section, we discuss three relative constants for hyperbolic 3-cone-manifolds with cone angles $\leq \pi$, two of which dominate the local structure away from the singularity, and the other one of which is related with the geometry of a tubular neighborhood of the singularity. The angle condition "$\leq \pi$" does not explicitly appear, but instead the fact that the Dirichlet polyhedron is convex, which is a conclusion of the angle condition, will be used often. We also discuss how cusp opening deformations occur locally.

3.1. Thin parts.

The constant in the following lemma is to claim that the injectivity radius decreases uniformly, like in the hyperbolic manifolds, away from the singularity.

Lemma 3.1.1. *Fix a curvature bound $L \leq -1$. Given positive numbers D, I, $R > 0$, there is a constant $U(D, I, R, L) > 0$ such that if $C \in \mathcal{C}^{\pi}_{[L,0]}$, $x \in C$ with $d(x, \Sigma) \geq D$ and $\operatorname{inj}_x C \geq I$, then*

$$\operatorname{inj}_y C \geq U(D, I, R, L)$$

for any $y \in C$ with $d(y, \Sigma) \geq D$ and $d(y, x) \leq R$.

Proof. Assume that there are no such uniform bounds. Then there is a sequence of cone-manifolds $\{C_i\} \subset \mathcal{C}^{\pi}_{[L,0]}$ and points $x_i, y_i \in C_i$ such that

1. $d(x_i, \Sigma_i)$, $d(y_i, \Sigma_i) \geq D$,
2. $\operatorname{inj}_{x_i} C_i \geq I$,
3. $d(y_i, x_i) \leq R$ and
4. $\operatorname{inj}_{y_i} C_i \leq 1/i$.

Take a Dirichlet polyhedron P_{y_i} of C_i about y_i in $\mathbf{H}_{K_i}$, where K_i is a curvature constant of C_i. There are points p_i, q_i on ∂P_{y_i} which are identified in C_i and attain the shortest distance to y_i from ∂P_{y_i}. The union of these paths forms a homotopically nontrivial shortest loop ℓ_i in C_i based at y_i.

p_i and q_i are on the interior of the faces of P_{y_i} respectively. Since the cone angles are $\leq \pi$ and hence the Dirichlet polyhedron P_{y_i} is convex,

P_{y_i} is bounded by the extension of two faces which support p_i and q_i. If the faces tend to be parallel as $i \to \infty$, then the volume of $\mathrm{B}_{R+I}(C_i, y_i)$ approaches zero. This is a contradiction since it must contain the ball $\mathrm{B}_I(C_i, x_i)$ whose volume admits nonzero lower bound by (2).

If not, ℓ_i meets at y_i with angle uniformly away from π. Let us lift ℓ_i to a geodesic segment s_i in $\mathbf{H}_{K_i}$ based at y_i such that p_i is the middle point. Then $\rho_i(\ell_i)$ acts $\mathbf{H}_{K_i}$ by either a loxodromic (translation with twist) motion or an elliptic rotation. In both cases, the orbit of s_i by the action of a group generated by $\rho_i(\ell_i)$ forms a piecewise geodesic immersed line rounding around the axis of $\rho_i(\ell_i)$. Since the corner of this line at the orbit of y_i has an angle uniformly away from π with respect to i, and since the length of s_i, which equals the length of ℓ_i, approaches zero when $i \to \infty$, the immersed line must squeeze onto or into the axis of $\rho_i(\ell_i)$ according to whether $\rho_i(\ell_i)$ is loxodromic or elliptic. In particular in both cases, the axis of $\rho_i(\ell_i)$ becomes close to y_i when $i \to \infty$.

If there is a subsequence $\{k\} \subset \{i\}$ such that $\rho_k(\ell_k)$ all are loxodromic, then the translation distance becomes also short. The path joining p_k and q_k in $\mathbf{H}_{K_k}$ is equivariant tiny homotopic to the unit translation segment on the axis of $\rho_k(\ell_k)$. This equivariant homotopy must induce a tiny homotopy in C_k because of (1). Hence we obtain a very short closed geodesic in C_k near y_k. If we choose a new reference point z_k on this geodesic, then the Dirichlet polyhedron about z_k will be bounded by almost parallel faces. This is a contradiction as before.

In the other case, $\rho_k(\ell_k)$ all but finitely many exceptions are elliptic. Then the axis of an elliptic element comes close to y_k. Thus the length shortening tiny homotopy of ℓ_k must hit the singularity by Lemma 1.2.4, though the hit singularity may not be the axis of $\rho_k(\ell_k)$. This contradicts (1). q.e.d.

3.2. Local Margulis.

The Margulis lemma for hyperbolic manifolds states that there is a universal constant depending only on the dimension which dominates the geometry and topology of thin parts. The cone-manifold admits in fact no such universal constant, however we may expect its relative version away from the singularity. The next lemma establishes that there is a Margulis like constant to control the geometry and topology of not absolute but relatively thin part with respect to the injectivity radius. We call it a local Margulis constant.

Lemma 3.2.1. *Given positive numbers* D, $R > 0$, *there is a constant* $V(D, R)$ *such that if* $C \in \mathcal{C}^{\pi}_{-1}$, $d(x, \Sigma) \geq D$ *and* $\operatorname{inj}_x C \leq V(D, R)$, *then* $(\mathrm{B}_{R\cdot \operatorname{inj}_x C}(C, x), x)$ *is homeomorphic to* $(\mathrm{B}_R(E, e), e)$ *for some noncompact euclidean manifold* E *with* $\operatorname{inj}_e E = 1$.

Proof. Assume that the conclusion is not true. Then there is a sequence of cone-manifolds $\{C_i\} \subset \mathcal{C}^{\pi}_{-1}$ and points $x_i \in C_i$ such that

1. $d(x_i, \Sigma_i) \geq D$ and
2. $\operatorname{inj}_{x_i} C_i \leq 1/i$, but
3. $(\mathrm{B}_{R\cdot \operatorname{inj}_{x_i} C_i}(C_i, x_i))$ never be homeomorphic to $(\mathrm{B}_R(E, y), y)$ for some euclidean manifold with $\operatorname{inj}_y E = 1$.

Then

$$R \leq iD \leq D/ \operatorname{inj}_{x_i} C_i$$

for i large enough, and we have

$$R \cdot \operatorname{inj}_{x_i} C_i \leq D$$

for sufficiently large i. Hence $(\mathrm{B}_{R\cdot \operatorname{inj}_{x_i} C_i}(C_i, x_i), x_i)$ is a subset of $(\mathrm{B}_D(C_i, x_i), x_i)$ for sufficiently large i. Notice that $\mathrm{B}_D(C_i, x_i)$ is nonsingular, so is $\mathrm{B}_{R\cdot \operatorname{inj}_{x_i} C_i}(C_i, x_i)$.

Multiplying $1/\operatorname{inj}_{x_i} C_i$ on the metric of C_i, we obtain a cone-manifold $\overline{C}_i$ of constant curvature $= -(\operatorname{inj}_{x_i} C_i)^2 \geq -1$ such that $\operatorname{inj}_{\overline{x}_i} \overline{C}_i = 1$. Thus we have a sequence of compact orientable cone-manifolds $\{\overline{C}_i\}$ in $\mathcal{C}^{\pi}_{[-1,0]}$. Hence by Proposition 2.2.3, there is a subsequence $\{k\} \subset \{i\}$ so that $\{(\overline{C}_k, \overline{x}_k)\}$ converges geometrically to a complete metric space $(\overline{C}_*, \overline{x}_*)$.

The limit $\overline{x}_*$ of reference points $\{\overline{x}_k\}$ admits a neighborhood which is a limit of balls of radius 1 whose curvature tend to zero. On the other hand, a euclidean ball of radius 1 could be a geometric limit of this sequence. Hence by the uniqueness of the geometric limit, $\overline{x}_*$ admits a euclidean ball neighborhood. This point will be a reference point for the other part.

To see a neighbor structure of the other part of $\overline{C}_*$, fix a constant $R_1 > 0$ and choose any $\overline{y}_* \in \overline{C}_*$ with $d(\overline{y}_*, \overline{x}_*) \leq R_1$; then it is a limit of points $\{\overline{y}_k \in \overline{C}_k\}$ with, say, $d(\overline{y}_k, \overline{x}_k) \leq 2R_1$. By rescaling, we have

$$d(\overline{x}_k, \overline{\Sigma}_k) \geq D/ \operatorname{inj}_{x_k} C_k \geq kD,$$

and moreover,

$$d(\overline{y}_k, \overline{\Sigma}_k) \geq kD - 2R_1 \geq D$$

for sufficiently large k. Therefore by Lemma 3.1.1, $\overline{y}_k$ admits a ball neighborhood of radius $\geq U(D, 1, 2R_1, -1)$. This radius bound does not depend on k, and hence a point $\overline{y}_*$ admits a neighborhood which is a limit of balls of uniformly bounded radius whose curvature tend to zero. Thus $\overline{x}_*$ admits a euclidean ball neighborhood. Now, since $R_1 > 0$ was arbitrary, a point with an arbitrary long distance from $\overline{x}_*$ admits a euclidean ball neighborhood. This shows that $\overline{C}_*$ is a euclidean manifold without singularity. Moreover $\mathrm{inj}_{\overline{x}_*} \overline{C}_* = 1$ and $\overline{C}_*$ is certainly noncompact.

Letting $(E, e) = (\overline{C}_*, \overline{x}_*)$, we will see that E has the property in the claim. Triangulate E by geodesic tetrahedra of uniform size and shape at least in a large compact set. Since it is a geometric limit, we may choose an approximate map φ_k from the 0-skeleton of a large compact set of E containing $\mathrm{B}_{D/\mathrm{inj}_{x_k} C_k}(E, e)$ to a large compact set of $\overline{C}_k$ containing $\mathrm{B}_{D/\mathrm{inj}_{x_k} C_k}(\overline{C}_k, \overline{x}_k)$ for sufficiently large k, where 4 vertices spanning an oriented simplex in E are mapped to 4-vertices spanning a simplex with the same orientation in the image. Hence φ_k admits an obvious piecewise linear extension, which we denote again by φ_k, over a large compact set of E. φ_k might be locally branched along edges or vertices, however it will be a homeomorphism for further sufficiently large k, because otherwise, φ_k's would not be accurate approximations. In particular, the restriction of φ_k for sufficiently large k induces a homeomorphism of $(\mathrm{B}_R(E, e), e)$ to $(\mathrm{B}_R(\overline{C}_k, \overline{x}_k), \overline{x}_k)$ after some tiny smoothing and hence to $(\mathrm{B}_{R \cdot \mathrm{inj}_{x_k} C_k}(C_k, x_k), x_k)$. This is a contradiction. q.e.d.

Remark. A homeomorphism between $(\mathrm{B}_{R \cdot \mathrm{inj}_x C}(C, x), x)$ and $(\mathrm{B}_R(E, e), e)$ can be chosen by the composition of an approximation and a rescaling, which is an almost equi-expansive map centered at x.

To see more topological structures of $\mathrm{B}_R(E, e)$, recall that a noncompact euclidean manifold is a quotient of the euclidean space $\mathbf{E}^3$ by a lattice Γ in $\mathrm{Isom}_+ \mathbf{E}^3$. Γ is isomorphic to either $\{0\}$, $\mathbf{Z}$, $\mathbf{Z} \oplus \mathbf{Z}$ or the fundamental group of the Klein bottle. In particular, any two elements in Γ either mutually commute or anti-commute.

Lemma 3.2.2. *Let E be a noncompact euclidean manifold with a reference point $e \in E$ such that $\mathrm{inj}_e E = 1$. Then for any $R > 0$, the* *image of* $\pi_1(\mathrm{B}_R(E,e),e) \to \pi_1(\mathrm{B}_{2R}(E,e),e)$ *induced by the inclusion is virtually abelian.*

Proof. $\pi_1(\mathrm{B}_R(E,e),e)$ is generated by geodesic loops of length $\leq 2R$, which are not smooth only at the reference point e. Choose any two such loops ℓ_1, ℓ_2. Since they commute or anti-commute in $\pi_1(E)$, the lift of a loop represented by $\ell_1\ell_2\ell_1^{-1}\ell_2^{\varepsilon}$, where $\varepsilon = 1$ or -1, encloses a rectangular knot in $\mathbf{E}^3$ where the edge length is at most $2R$.

The vertices are four lifts of the base point e. The preimage of $\mathrm{B}_{2R}(E,e)$, which has twice in radius, contains a union of four balls of radius $2R$ whose center lie on the vertices of the knot. Thus the knot bounds a disk in the union of these four balls and hence $\ell_1\ell_2\ell_1^{-1}\ell_2^{\varepsilon}$ becomes homotopic to zero in $\mathrm{B}_{2R}(E,e)$. This shows that the image is generated by finitely many ℓ's where the generators mutually commute or anti-commute. Such a group is virtually abelian. q.e.d.

3.3. Geometry of tubes.

An abstract model for an equidistant tubular neighborhood of a singular geodesic in a cone-manifold will be useful for estimating several quantities. We call it a tube and discuss its geometry.

Notation. $T_{\sigma,\delta,\theta,\tau}$ will be an equidistant tubular neighborhood of a singular component with radius δ in a hyperbolic 3-cone-manifold where the length of a singular axis $= \sigma$, the cone angle $= \theta$ and the twisting factor $= \tau$.

These four parameters determine the isometry class of $T_{\sigma,\delta,\theta,\tau}$. The boundary $\partial T_{\sigma,\delta,\theta,\tau}$ carries an induced euclidean structure. A canonical rectangular fundamental domain of $\partial T_{\sigma,\delta,\theta,\tau}$ by the meridional direction and its vertical direction has magnitude

$$\theta \sinh\delta \times \sigma \cosh\delta. \tag{3.1}$$

The surface area and volume of a tube depend only on the first three parameters,

$$\operatorname{area} \partial T_{\sigma,\delta,\theta,\tau} = \theta\sigma \sinh\delta \cosh\delta, \tag{3.2}$$

$$\operatorname{vol} T_{\sigma,\delta,\theta,\tau} = \frac{1}{2}\,\theta\sigma \sinh^2\delta. \tag{3.3}$$

In particular,

$$\frac{\operatorname{area} \partial T_{\sigma,\delta,\theta,\tau}}{\operatorname{vol} T_{\sigma,\delta,\theta,\tau}} = 2\coth\delta. \tag{3.4}$$

The twisting factor τ does not affect the surface area and volume in fact, but does affect the euclidean structure of the boundary. Since the gluing of the right and left edges has no twisting factor, by (3.1) we have

$$\operatorname{inj} \partial T_{\sigma,\delta,\theta,\tau} \leq \theta \sinh \delta. \tag{3.5}$$

The second factor "$\sigma \cosh \delta$" of (3.1) does not say much about injectivity radii.

We will derive two implications from these quantities. They are about rank-2 cusp opening deformations of a tube, and the comparison of intrinsic and extrinsic injectivity radii of a point on the boundary of a tube embedded in a cone-manifold.

First of all, consider a sequence $\{T_i\}$ of tubes which converges geometrically to a rank-2 cusp neighborhood by taking reference points on the boundaries. There are essentially two different ways. The simplest one in one way can be seen by setting

$$\theta_i = 1/\sinh \delta_i,$$
$$\sigma_i = 1/\cosh \delta_i$$

and making $\theta_i \to 0$ when $i \to \infty$. Then $\delta_i \to \infty$ and $\sigma_i \to 0$. There are an elliptic and a loxodromic elements in $\mathrm{PSL}_2(\mathbf{C})$ which generate T_i for each i. In this deformations, they both approach parabolic elements which generate a rank-2 cusp when $i \to \infty$.

The other way involving a twisting factor τ was discussed in [14]. To see one simple example, let θ be a fixed positive constant and set

$$\sigma_i \sinh \delta_i \cosh \delta_i = 1/\theta.$$

Choose τ_i so that $\{\partial T_i\}$ stays in a compact set in the moduli space of euclidean tori, we obtain a cusp opening family. In this case, $\delta_i \to \infty$ and $\sigma_i \to 0$ also. However, the elliptic elements associated to T_i's diverge in $\mathrm{PSL}_2(\mathbf{C})$, and a group generated by a loxodromic element converges geometrically (but not algebraically) to a rank-2 parabolic subgroup generating a cusp.

Lemma 3.3.1. *Let $\{T_i\}$ be a sequence of tubes which converges geometrically to a rank-2 cusp neighborhood. Suppose that* $\operatorname{area} \partial T_i$ *is constant and that $\{\partial T_i\}$ stays in a compact set in the moduli space of euclidean tori. If $\theta_i > 0$ is bounded away from zero, then the lengths of curves on ∂T_i's bounding a disk in T_i diverge as $i \to \infty$.*

Proof. Since the boundaries have bounded geometry and T_i approaches a rank-2 cusp neighborhood, $\delta_i \to \infty$. Since the length of a curve bounding a disk in T_i is $\theta_i \sinh \delta_i$, it must diverge if θ_i is bounded away from zero. q.e.d.

Secondly, consider a tube $T^K_{\sigma,\delta,\theta,\tau}$ with constant sectional curvature $= K$, where K lies in $[L, 0]$ and is not necessarily -1. The modified formulas of the above ones for $T^K_{\sigma,\delta,\theta,\tau}$ can be established according to the value of K, however its limiting behaviors caused by the limiting behavior of σ, δ, θ are essentially the same.

Regarding a tube as a model of an equidistant tubular neighborhood $\mathcal{T} \subset C$ of some component of Σ, we compare the injectivity radii of $\partial\mathcal{T}$ and C at a point on $\partial\mathcal{T}$. Notice that since $\partial\mathcal{T}$ is a euclidean torus, $\mathrm{inj}_x\, \partial\mathcal{T}$ does not depend on the choice of $x \in \partial\mathcal{T}$ and is equal to $\mathrm{inj}\, \partial\mathcal{T}$.

Lemma 3.3.2. *Fix a curvature bound $L \leq -1$. Given positive numbers $D, I > 0$, there is a constant $W(D, I, L)$ such that if $C \in \mathcal{C}^{\pi}_{[L,0]}$, $\mathcal{T} \subset C$ is an equidistant tubular neighborhood of a component of Σ with* radius $\mathcal{T} \geq D$, *and* $\mathrm{inj}\, \partial\mathcal{T} \leq I$, *then*

$$(\mathrm{inj}_w\, C \leq)\ \mathrm{inj}\, \partial\mathcal{T} \leq W(D, I, L) \cdot \mathrm{inj}_w\, C$$

for any $w \in \partial\mathcal{T}$, where the inequality in ( ) on the left-hand side is obvious.

Proof. Let K' be a curvature constant of C. Multiplying $1/\,\mathrm{inj}\, \partial\mathcal{T}$ on the metric of C, we obtain $\overline{C}$ of constant curvature

$$K = K'(\mathrm{inj}\, \partial\mathcal{T})^2 \geq I^2 L.$$

Then $\mathcal{T}$ becomes a tubular neighborhood $\overline{\mathcal{T}}$ with radius $\overline{\mathcal{T}} \geq D/I$ and also $\mathrm{inj}\, \partial\overline{\mathcal{T}} = 1$.

Let $T^K_{\sigma,\delta,\theta,\tau}$ be a tube isometric to $\overline{\mathcal{T}}$, where $\delta =$ radius $\overline{\mathcal{T}} \geq D/I$, and consider a natural inclusion of tubes

$$T^K_{\sigma,\delta - D/2I,\theta,\tau} \subset T^K_{\sigma,\delta,\theta,\tau}.$$

Notice that the radius of the included tube is $\geq D/2I$. $T^K_{\sigma,\delta,\theta,\tau}$ is a riemannian manifold with boundary, and the quantity $\mathrm{inj}_x\, T^K_{\sigma,\delta,\theta,\tau}$ for $x \in \partial T^K_{\sigma,\delta - D/2I,\theta,\tau}$ can be considered as a positive periodic function in terms of τ, so that we have the minimum for fixed $\sigma, \delta, \theta, K$,

$$J^K_{\sigma,\delta,\theta}(D/I) = \min_{\tau} \mathrm{inj}_x\, T^K_{\sigma,\delta,\theta,\tau}.$$

Then $J^K_{\sigma,\delta,\theta}(D/I)$ becomes a function of $\sigma > 0, \delta \geq D/I, \theta > 0$, and $0 \geq K \geq I^2 L$. Since it approaches zero only when $\operatorname{inj} \partial T^K_{\sigma,\delta,\theta,\tau} \to 0$, it attains the positive minimum $J(D, I, L)$ when the variables run so that $\operatorname{inj} \partial T^K_{\sigma,\delta,\theta,\tau} = 1$. In particular, $\operatorname{inj}_{\overline{x}} \overline{\mathcal{T}} \geq J(D, I, L)$ for any $\overline{x} \in \overline{\mathcal{T}}$ with $d(\overline{x}, \partial\overline{\mathcal{T}}) = D/2I$.

Set a constant U by $U(D/2I, J(D, I, L), D/2I, I^2 L)$ in Lemma 3.1.1, and choose any $\overline{w} \in \partial\overline{\mathcal{T}}$. Then there is the unique nearest point $\overline{x}$ to $\overline{w}$ in $\overline{\mathcal{T}}$ so that $d(\overline{x}, \overline{w}) = d(\overline{x}, \partial\overline{\mathcal{T}}) = D/2I$. Since $d(\overline{x}, \overline{\Sigma}) \geq D/2I$, $\operatorname{inj}_{\overline{x}} \overline{C} \geq \operatorname{inj}_{\overline{x}} \overline{\mathcal{T}} \geq J(D, I, L)$ and $d(\overline{w}, \overline{\Sigma}) \geq D/2I$, we have a bound $U \leq \operatorname{inj}_{\overline{w}} \overline{C}$ by Lemma 3.1.1. Multiplying the metric by $\operatorname{inj} \partial\mathcal{T}$, we obtain

$$U \cdot \operatorname{inj} \partial\mathcal{T} \leq \operatorname{inj}_w C.$$

The proof is done by letting $W = 1/U$. q.e.d.

4. Choosing geometrically convergent sequences

In this section, we choose a geometrically convergent sequence of deformations of a compact orientable hyperbolic 3-cone-manifold C with singularity Σ. This is to see what happens in the limit in the next two sections.

4.1. Maximal tubes.

We will often split the discussions in the later section according to whether the reference point is close to the singularity or not. For this, it is convenient to introduce the maximal tubular neighborhood of the singularity.

Definition. The maximal tube $\mathcal{T}$ about $\Sigma \subset C$ is a union of open tubular neighborhoods $\mathcal{T}^j$'s such that the following hold:

(1) each component $\mathcal{T}^j \subset \mathcal{T}$ is an equidistant tubular neighborhood to the jth component $\Sigma^j \subset \Sigma$ of the singularity,

(2) among ones having the property (1), the set of radii arranged in order of magnitude from the smallest one is maximal in lexicographical order.

By the second condition, the maximal tube $\mathcal{T}$ about Σ is uniquely determined. In fact, $\mathcal{T}$ can be constructed as follows. Since the components of Σ are isolated, a δ-tubular neighborhood of Σ is a union of disjoint tubular neighborhoods of Σ^j's if δ is very small. Thicken it gradually. Then some of components contact at a particular moment. Stop the growth of the components involved in contacting, and thicken the others furthermore. We will have the second contact moment. Do the same again. Continue this process up to the terminal moment, and we finally obtain the maximal tube $\mathcal{T}$.

Denote by $\partial\mathcal{T}^j$ an abstract boundary of $\mathcal{T}^j$. It is a euclidean torus. The actual boundary $\partial\mathcal{T}$ of $\mathcal{T}$ in C is a union of isometrically embedded tori with a finite number of contact points.

4.2. Geometrically convergent sequences.

Recall that $A = (\alpha^1, \cdots, \alpha^n)$ is an angle set of a compact orientable hyperbolic 3-cone-manifold C, where $\alpha^j \leq \pi$ for all $1 \leq j \leq n$. Choose a sequence of deformations (C_i, ξ_i) of C such that

1. $\alpha_i^j \leq \pi$ for any i, and
2. $\lim_{i\to\infty} \alpha_i^j$ exists and equals β^j.

To see what happens when $i \to \infty$, we follow Thurston's strategy in [13] in the next two sections, that is, to analyze its possible geometric limit using hyperbolic geometry and 3-manifold topology.

Let c_i be the first contact point on $\partial\mathcal{T}_i$, that is to say, the point which admits two shortest path to Σ_i from $\partial\mathcal{T}_i$. Then by Proposition 2.2.3, there is a subsequence $\{k\} \subset \{i\}$ so that $\{(C_k, c_k)\}$ converges geometrically to a complete metric space (C_*, c_*). We arrange more. Let f_k be the finest point on $\partial\mathcal{T}_k$, that is to say, the point on $\partial\mathcal{T}_k$ which attains the minimum among $\{\mathrm{inj}_x C_k | x \in \partial\mathcal{T}_k\}$. By choosing a further subsequence $\{C_k\}$ with the same letter, we may assume that c_k and f_k stay on the components of $\partial\mathcal{T}_k$ with constant reference numbers respectively. Namely, c_k always lies in cth component $\partial\mathcal{T}_k^c$ for any k, and f_k does in $\partial\mathcal{T}_k^f$, where c and f here represent the reference numbers for the components.

The properties of the sequence so chosen is summarized in 4.3 below. In fact, we only use the properties listed in 4.3 in the next two sections.

4.3. Properties.

A sequence of compact orientable hyperbolic 3-cone-manifold $\{C_k\}$ with the angle set $A_k = (\alpha_k^1, \cdots, \alpha_k^n)$ has the properties:

(1) *each* C_k *is a deformation of* C *with a reference homeomorphism* $\xi_k : C \to C_k$,

(2) $\{(C_k, c_k)\}$ *converges geometrically to* (C_*, c_*) *when* $k \to \infty$,

(3) $\alpha_k^j \leq \pi$ *for all* $1 \leq j \leq n$ *and any* k,

(4) $\alpha_k^j \to \beta^j$ *when* $k \to \infty$,

(5) *the first contact point* c_k *lies on a component* $\partial \mathcal{T}_k^c$ *with a constant reference number* c,

(6) *the finest points* f_k *lies on a component* $\partial \mathcal{T}_k^f$ *with a constant reference number* f *and*

(7) *there is a constant* V_{max} *such that* $\operatorname{vol} C_k \leq V_{max}$.

Remark. The above sequence is assumed to have only a geometric limit, and the algebraic convergence with respect to the identification by ξ_k is not guaranteed. For instance, let C be a hyperbolic surface with homotopically nontrivial automorphism $\varphi : C \to C$ which cannot be realized by an isometry, and define $\xi_k : C \to C_k = C$ by

$$\xi_k = \begin{cases} id, & \text{if } k \text{ odd,} \\ \varphi, & \text{if } k \text{ even.} \end{cases}$$

Then the sequence $\{C_k\}$ converges geometrically to C, but a sequence of holonomy representations $\{\rho_k\}$ does not converge algebraically.

Remark. The property (7) is not a direct consequence of Proposition 1.3.2, however the bound above can be obtained by modifying its proof only a little because of the property (3). See also the remark after Proposition 1.3.2.

5. Thick tube

In this section, we study what happens to a geometric limit C_* of a sequence $\{C_k\}$ of the deformations of C in 4.3 when maximal tubes of the singularity are uniformly thick.

5.1. Brief outline.

The underlying assumption throughout this section is

Assumption 1. *There is a constant* $D_1 > 0$ *such that*

$$D_1 \leq \operatorname{radius} \mathcal{T}_k^j$$

for any $1 \leq j \leq n$ *and any* k.

Under the Assumption 1 above, we prove the following propositions in this section.

Proposition 5.1.1. *Under the Assumption 1, there is a constant* $I_1 > 0$ *such that*

$$I_1 \leq \operatorname{inj}_{f_k} C_k.$$

for any k.

The conclusion of this proposition is equivalent to the Assumption 1 since $\operatorname{inj}_{f_k} C_k \leq \min_j \operatorname{radius} \mathcal{T}_k^j$. The proof involves analysis of the local structure of cone-manifolds away from the singularity.

Using the conclusion of Proposition 5.1.1, we show

Proposition 5.1.2. *Under the conclusion of Proposition 5.1.1,* C_* *is a hyperbolic 3-cone-manifold of finite volume homeomorphic to* C*, where some components of* Σ *possibly disappear and create cusps.*

Proposition 5.1.3. *Under the conclusion of Proposition 5.1.1, a sequence* $\{\rho_k\}$ *of holonomy representations of* $\{C_k\}$ *contains a subsequence converging algebraically to the holonomy representation* ρ_* *of* C_* *with respect to the identification by* ξ_k.

Definition. A sequence $\{C_k\}$ of deformations is said to converge strongly if it converges geometrically to a cone-manifold C_* homeomorphic to C, and a sequence $\{\rho_k\}$ of their holonomy representations converges algebraically to ρ_* with respect to the identification by ξ_k.

This definition is compatible for the existing one for discrete groups.

Corollary 5.1.4. *Under the Assumption 1,* $\{C_k\}$ *contains a subsequence which converges strongly to a hyperbolic 3-cone-manifold* C_* *homeomorphic to* C*. If* $\beta^j > 0$ *for all* $1 \leq j \leq n$*, then* C_* *is compact.*

Proof. This is a direct consequence of three propositions above. Suppose that the Assumption 1 is the case. By Proposition 5.1.1, the injectivity radius of the first contact points of maximal tubes are uniformly bounded away from zero. Then by Propositions 5.1.2, 5.1.3, C_* is a strong limit of $\{C_k\}$ after taking a subsequence. In particular, the angle set of C_* is equal to $B = (\beta^1, \cdots, \beta^n)$. If $\beta^j > 0$ for all $1 \leq j \leq n$, then C_* admits no ends and hence is compact. q.e.d.

5.2. Boundary of tubes.

Supposing that the Assumption 1 is the case throughout this subsection, we prove Proposition 5.1.1

Lemma 5.2.1. *There is a constant I_2 such that*

$$\operatorname{area} \partial \mathcal{T}_k^j \leq I_2,$$

for any $1 \leq j \leq n$ and any k.

Proof. By the comparison (3.4) of the volume and the surface area of a tube, and by the volume bound (7) in 4.3, we have

$$\operatorname{area} \partial \mathcal{T}_k^j = 2 \coth \operatorname{radius} \mathcal{T}_k^j \ \operatorname{vol} \mathcal{T}_k^j \leq 2V_{max} \coth D_1.$$

Let I_2 be the last term. q.e.d.

Proof of Proposition 5.1.1. Assume contrarily that $\operatorname{inj}_{f_k} C_k \to 0$, and we will get a contradiction.

Choose the first contact point p_k on $\partial \mathcal{T}_k^f$ on which f_k lies. p_k may not be the absolute first contact point c_k since we require that p_k lies on the component $\partial \mathcal{T}_k^f$ which might be different from $\partial \mathcal{T}_k^c$. p_k is the point where $\partial \mathcal{T}_k^f$ either meets the other component of $\partial \mathcal{T}_k$ or contacts itself. Very locally, p_k appears as a contact point of two components, one $\partial \mathcal{T}_k^f$ from the left-hand side and the other $\partial \mathcal{T}_k^{f'}$ from the right-hand side. The reference numbers f and f' might be the same.

One obvious inequality is

$$(\operatorname{inj} \partial \mathcal{T}_k^f)^2 \leq \operatorname{area} \partial \mathcal{T}_k^f \ (\leq I_2),$$

and the right-hand side of which is bounded by I_2 by the above lemma. Let W be a constant $W(D_1, \sqrt{I_2}, -1)$ in Lemma 3.3.2. Then since $\operatorname{radius} \mathcal{T}_k^f \geq D_1$ and $\operatorname{inj} \partial \mathcal{T}_k^f \leq \sqrt{I_2}$, we have

$$\operatorname{inj} \partial \mathcal{T}_k^f \leq W \cdot \operatorname{inj}_{f_k} C_k \ (\leq W \cdot \operatorname{inj}_{p_k} C_k) \tag{5.1}$$

by Lemma 3.3.2. If we regard p_k as a point on $\partial\mathcal{T}_k^{f'}$, since radius $\mathcal{T}_k^{f'} \geq D_1$ and $\operatorname{inj}\partial\mathcal{T}_k^{f'} \leq \sqrt{I_2}$, then again we have

$$\operatorname{inj}\partial\mathcal{T}_k^{f'} \leq W \cdot \operatorname{inj}_{p_k} C_k. \tag{5.2}$$

Hence $\mathrm{B}_{W\cdot \operatorname{inj}_{p_k} C_k}(C_k, p_k)$ contains a homotopically nontrivial loop ℓ_1 based at p_k on the left $\partial\mathcal{T}_k^{f}$, and also ℓ_2 based at p_k on the right $\partial\mathcal{T}_k^{f'}$.

Let $V(D_1, 2W)$ be a local Margulis constant with respect to D_1 and $2W$ in Lemma 3.2.1. Since we assumed that $\operatorname{inj}_{f_k} C_k \to 0$, (5.1) and (5.2) imply that

$$(\operatorname{inj}_{p_k} C_k \leq)\ \operatorname{inj}\partial\mathcal{T}_k^{f} \leq V(D_1, 2W)$$

for sufficiently large k. Thus $(\mathrm{B}_{2W\cdot \operatorname{inj}_{p_k} C_k}(C_k, p_k), p_k)$ is homeomorphic by an almost equi-expansive map to $(\mathrm{B}_{2W}(E, e), e)$ for some noncompact euclidean manifold E with $\operatorname{inj}_e E = 1$ by Lemma 3.2.1 and the remark after that. Furthermore by Lemma 3.2.2, the homomorphism,

$$\pi_1(\mathrm{B}_W(E, e), e) \to \pi_1(\mathrm{B}_{2W}(E, e), e),$$

induced by the inclusion has a virtually abelian image. Hence so does

$$\pi_1(\mathrm{B}_{W\cdot \operatorname{inj}_{p_k} C_k}(C_k, p_k), p_k) \to \pi_1(\mathrm{B}_{2W\cdot \operatorname{inj}_{p_k} C_k}(C_k, p_k), p_k),$$

because of the choice of a homeomorphism of $(\mathrm{B}_{2W\cdot \operatorname{inj}_{p_k} C_k}(C_k, p_k), p_k)$ which we have made. Thus the nontrivial loops ℓ_1, ℓ_2 representing elements of $\pi_1(\mathrm{B}_{W\cdot \operatorname{inj}_{p_k} C_k}(C_k, p_k), p_k)$ are virtually commutative, in particular, in Π for sufficiently large k.

On the other hand, consider the developed image near p_k. p_k lifts to a contact point of lifts of $\partial\mathcal{T}_k^{f}$ from the left-hand side and that of $\partial\mathcal{T}_k^{f'}$ from the right-hand side. In particular, $\rho_k(\ell_1)$ leaves the left-hand lift of $\partial\mathcal{T}_k^{f}$ invariant, on the other hand, $\rho_k(\ell_2)$ leaves the right-hand lift of $\partial\mathcal{T}_k^{f'}$ invariant. Hence their action on $\mathbf{H}^3 \cup \mathbf{S}^2_\infty$ do not have common fixed point at all, and they are not commutative even virtually in $\rho_k(\Pi)$. This is a contradiction. q.e.d.

5.3. Geometric limits.

We assume the conclusion of Proposition 5.1.1 that the injectivity radius of the points on $\partial\mathcal{T}_k$ is uniformly bounded from below by $I_1 > 0$, and prove Proposition 5.1.2.

Lemma 5.3.1. *Under the conclusion of Proposition 5.1.1, if there is a constant $D^j > 0$ such that* $\operatorname{radius} \mathcal{T}_k^j \leq D^j$, *then $\beta^j > 0$ and there is a constant $S^j > 0$ such that*

$$S^j \leq \operatorname{length} \Sigma_k^j.$$

for any k.

Proof. If $\beta^j = 0$, then the jth component α_k^j of the angle set approaches zero as $k \to \infty$. Since $\operatorname{radius} \mathcal{T}_k^j \leq D^j$, $\operatorname{inj} \partial \mathcal{T}_k^j \to 0$ by (3.5). This contradicts the conclusion of Proposition 5.1.1 because

$$(I_1 \leq) \ \operatorname{inj}_{f_k} C_k \leq \operatorname{inj} \partial \mathcal{T}_k^j.$$

If $\{\operatorname{length} \Sigma_k^j\}$ contains a subsequence converging to 0, since

$$\begin{aligned} \operatorname{area} \partial \mathcal{T}_k^j &= \theta_k^j \operatorname{length} \Sigma_k^j \sinh \operatorname{radius} \mathcal{T}_k^j \cosh \operatorname{radius} \mathcal{T}_k^j \\ &\leq \pi \sinh D^j \cosh D^j \operatorname{length} \Sigma_k^j, \end{aligned}$$

by (3.2), it follows that $(I_1^2 \leq (\operatorname{inj} \partial \mathcal{T}_k^j)^2 \leq)$ $\operatorname{area} \partial \mathcal{T}_k^j$ can be arbitrary close to 0. This is again a contradiction. q.e.d.

Lemma 5.3.2. *C_* is a hyperbolic 3-cone-manifold of finite volume possibly with compact singularity.*

Proof. It is sufficient to show that each point $x_* \in C_*$ admits a hyperbolic ball neighborhood possibly with singularity along a geodesic segment. The argument is quite parallel to that in Lemma 3.2.1.

Since $I_1 \leq \operatorname{inj}_{f_k} C_k \leq \operatorname{inj}_{c_k} C_k$, the limit $c_* \in C_*$ of the reference points $\{c_k\}$ admits a neighborhood which is a limit of hyperbolic balls of uniformly bounded radii by I_1. Hence it admits a hyperbolic ball neighborhood.

Fix a constant $R > 0$ and choose any $x_* \in C_*$ with $d(x_*, c_*) \leq R$. If it is a limit of points $\{x_k \in C_k - \mathcal{T}_k\}$, since $d(x_k, \Sigma) \geq D_1$ and we may assume $d(x_k, c_k) \leq 2R$, x_k admits a hyperbolic ball neighborhood of radius $\geq U(D_1, I_1, 2R, -1)$ by Lemma 3.1.1, where the radius bound does not depend on k. Hence again x_* admits a hyperbolic open ball neighborhood.

If x_* is a limit of points $\{x_k \in \mathcal{T}_k\}$, we may assume that $\{x_k\}$ is contained in a component $\mathcal{T}_k^x$ with a constant reference number x by taking further subsequence if necessary. When $\operatorname{radius} \mathcal{T}_k^x \to \infty$, since $\partial \mathcal{T}_k^x$ does not degenerate by the conclusion of Proposition 5.1.1, we may assume that $d(x_k, c_k) \leq 2R, d(x_k, \Sigma_k^x) \to \infty$, and $d(x_k, \Sigma_k) \geq D_1$. Then x_k admits a hyperbolic ball neighborhood of radius $\geq U(D_1, I_1, 2R, -1)$ by Lemma 3.1.1 where the radius bound does not depend on k, and hence so does x_*. When radius $\mathcal{T}_k^x$ is bounded not only from below by Assumption 1 but also from the above, length Σ_k^x has a uniform lower bound away from zero by Lemma 5.3.1, and each point within $\mathcal{T}_k^x$ has a possibly singular ball neighborhood of uniform radius where the singularity occurs only along a geodesic segment. Hence x_* admits a hyperbolic ball neighborhood possibly with cone singularity along a geodesic segment.

Since $R > 0$ was arbitrary, the above argument shows that every point on C_* admits a hyperbolic ball neighborhood possibly with a cone singularity. The singularity appears only in the limit of $\mathcal{T}_k^x$ whose radius is bounded. There are only finitely many such components. Moreover the length of a core of such a component is bounded since in general by (3.5) we have

$$(I_1 \leq) \ \operatorname{inj} \partial \mathcal{T}_k^x \leq \theta_k^x \sinh \operatorname{radius} \mathcal{T}_k^x,$$

so that the formula in (3.2) implies the estimate,

$$\operatorname{length} \Sigma_k^x = \frac{\operatorname{area} \partial \mathcal{T}_k^x}{\theta_k^x \sinh \operatorname{radius} \mathcal{T}_k^x \cosh \operatorname{radius} \mathcal{T}_k^x} \leq \frac{I_2}{I_1 \cosh \operatorname{radius} D_1}.$$

Hence the singular set is compact. q.e.d.

By Corollary 1.2.3, C_* has finitely many toral ends. Choose disjoint horotoral neighborhoods of the ends of C_* so that the minimum I_3 of the injectivity radii of C_* at points on their boundaries is $\leq I_1$. We let C_*^{cut} be a compact hyperbolic cone-manifold with toral boundary obtained from C_* by truncating such cusp neighborhoods. We thus have

$$I_3 = \min\{\operatorname{inj}_x C_* | x \in \partial C_*^{cut}\} \leq I_1.$$

Lemma 5.3.3. *There is an approximate homeomorphism* $\varphi_k : C_*^{cut} \to C_k$ *for sufficiently large* k.

Proof. We just repeat the last paragraph in the proof of Lemma 3.2.1. Choose a fine triangulation of C_*^{cut} by 3-simplices whose faces either are totally geodesic or lie on ∂C_*^{cut} so that Σ_*^{cut} is contained in the 1-skeleton. Since C_* is a geometric limit, we may choose a map from the

0-skeleton of C_*^{cut} to C_k for sufficiently large k, where 4 vertices spanning an oriented simplex in C_*^{cut} are mapped to 4-vertices spanning a simplex with the same orientation also in C_k, and vertices on Σ_*^{cut} are mapped to points on Σ_k. Then its obvious piecewise linear extension is necessarily an into-approximate homeomorphism $\varphi_k : C_*^{cut} \to C_k$ for sufficiently large k. q.e.d.

Lemma 5.3.4. *φ_k can be modified by an isotopy to a homeomorphism, which we again denote by φ_k, so that each component of $\varphi_k(\partial C_*^{cut})$ bounds an equidistant tubular neighborhood of either a short geodesic or a component of Σ_k in C_k for further sufficiently large k. Moreover the isotopy can be chosen so that the injectivity radii of the components of $\varphi_k(\partial C_*^{cut})$ is uniformly bounded from below by some positive constant.*

Proof. Choose a component $\partial_0 C_*^{cut}$ of ∂C_*^{cut} and let H_k be the image of $\partial_0 C_*^{cut}$ by φ_k, namely $H_k = \varphi_k(\partial_0 C_*^{cut})$. Since the nonsingular part $N_k = C_k - \Sigma_k$ is irreducible and atoroidal as a 3-manifold by Theorem 1.2.1, H_k either is incompressible and boundary parallel or bounds a solid torus Z_k in N_k. In particular H_k separates N_k.

If H_k is incompressible, it is isotopic to a component of $\partial \mathcal{T}_k$. Hence it is isotopic to a horotorus bounding an equidistant tubular neighborhood of a component of Σ_k. Choose a horotorus $\mathcal{H}_k$ isotopic to a corresponding component of $\partial \mathcal{T}_k$ so that

$$\operatorname{inj} \mathcal{H}_k = I_3/2.$$

It exists certainly in $\mathcal{T}_k$ since the minimum of the injectivity radius of components of $\partial \mathcal{T}_k$ is $\geq \inf_{f_k} C_k \geq I_1 \geq I_3$. Moreover, since φ_k does not change injectivity radius very much, it is contained outside the image of C_*^{cut} by φ_k. Hence we can choose isotopy of H_k to $\mathcal{H}_k$ by pushing H_k outside $\varphi_k(C_*^{cut})$. In particular, the isotopy is covered by an isotopy of C_*^{cut} fixing the complement of a small collar neighborhood of $\partial_0 C_*^{cut}$, and the covering isotopy does not affect the other component of ∂C_*^{cut}.

Suppose next that H_k bounds a solid torus Z_k in N_k. We will show that the solid torus Z_k bounded by H_k contains a simple closed geodesic isotopic to the core of Z_k. To see this, we will first extend φ_k to an embedding $\hat{\varphi}_k$ of the union of C_*^{cut} and a collar F of $\partial_0 C_*^{cut}$ which lies in the complement of C_*^{cut}. The choice of F is rather technical and will be made below. Set

$$I_4^k = \inf\{\operatorname{inj}_x C_k \mid x \in H_k\}.$$

This constant depends on k in fact, however since injectivity radius does not change very much by an approximation φ_k, we may assume that I_4^k is bounded from below by some positive constant for sufficiently large k. Then we set a positive constant I_4 by

$$I_4 = \min\{\inf\{I_4^k\}, I_3\}.$$

We now choose a collar F of $\partial_0 C_*^{cut}$ so that the second shortest geodesic on the boundary component $\partial_1 C_*^{cut}$ of F other than $\partial_0 C_*^{cut}$ has length $\leq I_4/2$. The same argument of the previous lemma shows that the extension $\hat{\varphi}_k$ over $C_*^{cut} \cup_{\partial_0 C_*^{cut}} F$ exists by taking further sufficiently large k.

Let ℓ_1 and ℓ_2 be the shortest two geodesics on $\partial_1 C_*^{cut}$. They are not homotopic each other because $\partial_1 C_*^{cut}$ has a euclidean structure. Since $\hat{\varphi}_k$ does not change the length very much, we may assume that the length of $\hat{\varphi}_k(\ell_1)$ and $\hat{\varphi}_k(\ell_2)$ are $< I_4$. The new boundary component $\hat{H}_k$ of $\hat{\varphi}_k(F)$ other than H_k is contained in Z_k. Since $\hat{\varphi}_k(\ell_1)$ and $\hat{\varphi}_k(\ell_2)$ are nonhomotopic loops on $\hat{H}_k$, at least one of them, say $\hat{\varphi}_k(\ell_1)$, is homotopically nontrivial in Z_k. Notice that $\hat{\varphi}_k(\ell_1)$ has length $< I_4$ and hence the length shortening homotopy of $\hat{\varphi}_k(\ell_1)$ in Lemma 1.2.4 does not go through the point with injectivity radius $\geq I_4/2$. On the other hand, any point on H_k has injectivity radius $\geq I_4^k \geq I_4$. Thus the length shortening homotopy of $\hat{\varphi}_k(\ell_1)$ stays in Z_k, and $\hat{\varphi}_k(\ell_1)$ shrinks to some nonzero multiple of a closed geodesic ℓ in Z_k. Hence by Theorem 1.2.1, $C_k - (\Sigma_k \cup \ell)$ is atoroidal, and in particular H_k is parallel to a torus bounding an equidistant tubular neighborhood of ℓ.

We would like to isotope H_k to an equidistant torus to ℓ with uniform intrinsic injectivity radius. To see this, consider an increasing family of equidistant tubular neighborhoods of ℓ in Z_k. At some first critical radius, the boundary of a neighborhood hits either H_k or itself. If the later is the case, choose a homotopically nontrivial loop g in the critical equidistant tubular neighborhood which passes the contact point p. g is homotopic to some nonzero multiple of ℓ, say ℓ^d, since it lies in the solid torus Z_k and p is a contact point of the critical equidistant tubular neighborhood. Then consider the developed image of the critical equidistant tubular neighborhood. The preimage of p contains points which cannot be joined by the action of $\rho_k(\ell)$ since p is a contact point. Simultaneously, it must be an orbit of a point by the action of $\rho_k(\ell^d)$ since g is homotopic to ℓ^d. This is impossible.

Hence the equidistant tubular neighborhood of ℓ grows up to the one $\mathcal{H}_k$ which touches H_k. Since each point of H_k has injectivity radius

$\geq I_4$ in C_4, we have

$$\operatorname{inj} \mathcal{H}_k > I_4.$$

Moreover, it is contained outside the image of C_*^{cut} by φ_k. Hence we can choose an isotopy of H_k to $\mathcal{H}_k$ by pushing H_k into Z_k. In particular, the isotopy is covered by an isotopy of C_*^{cut} fixing the complement of a small collar neighborhood of $\partial_0 C_*^{cut}$. The covering isotopy does not affect the other component of ∂C_*^{cut} and we are done. q.e.d.

Lemma 5.3.5. *For any geodesic loop ℓ on ∂C_*^{cut},* $\operatorname{length} \varphi_k(\ell)$ *is bounded by some positive constants from both above and below for all k, where φ_k is a modified one in the previous lemma.*

Proof. The existence of a lower bound is a simple corollary to the previous lemma. Recall that the isotopy which we constructed pushes the boundary into an equidistant tubular neighborhood. Thus it is distance decreasing on the boundary. Since the original

$$\varphi_k|_{\partial_0 C_*^{cut}} : \partial_0 C_*^{cut} \to H_k$$

is an approximation and does not change the distance very much even when k varies, it follows that a uniform upper bound exists. q.e.d.

By taking a further subsequence and rearranging reference numbers of the components of Σ if necessary, we may assume that the jth component of ∂C_*^{cut} is mapped by φ_k to a torus bounding an equidistant tubular neighborhood of Σ_k^j for $0 \leq j \leq s$ and of a short geodesic in C_k for $s < j \leq t$ and sufficiently large k.

The following lemma finishes the proof of Proposition 5.1.2.

Lemma 5.3.6. *There are no components of ∂C_*^{cut} which are mapped by φ_k to the torus bounding an equidistant tubular neighborhood of a short geodesic. In other words, $s = t$, and there are no cusp openings away from the singularity.*

Proof. Assume contrarily that $s < t$. Filling the jth component of the boundary of C_*^{cut} for each $s < j \leq t$ by an equidistant tubular neighborhood of a short geodesic which the jth component of $\varphi_k(\partial C_*^{cut})$ bounds in C_k, we obtain a cone-manifold homeomorphic to $C - \cup_{j \leq s} \Sigma^j$. In other words, for each sufficiently large k, C_*^{cut} produces an isometric hyperbolic cone-manifold by some Dehn filling on the last $t - s$ components of ∂C_*^{cut}.

If the number of slopes appeared in this Dehn filling on the jth component is finite even as k varies, then by taking a further subsequence, we may assume that the slope is unique and does not depend on k. Denote the geodesic representative of a slope on ∂C_*^{cut} by m. Then by Lemma 5.3.5, length $\varphi_k(m)$ is bounded from above by some constant not depending on k. On the other hand, since the equidistant tubular neighborhood of a short geodesic bounded by the jth component of $\varphi_k(\partial C_*^{cut})$ approaches a cusp, and since the cone angles of short geodesics stay 2π, the lengths of curves on $\varphi_k(\partial C_*^{cut})$ bounding a disk in tubes must diverge by Lemma 3.3.1. This is a contradiction.

Hence the number of slopes appeared in the Dehn fillings must be infinite for each component when k varies. Let us reconsider the situation not by cutting toral ends but by removing the singularity. Come back to the limiting cone-manifold C_*. The nonsingular part $N_* = C_* - \Sigma_*$ admits a hyperbolic structure by Theorem 1.2.1. We denote it by $C_{*,comp}$. Then for each k, $C_{*,comp}$ produces C_{comp} by some Dehn filling on the last $t - s$ cuspidal components of $C_{*,comp}$. The filling slope on each component varies infinitely many as k varies. Hence such filling slopes accumulate to ∞ which corresponds to the complete structure $C_{*,comp}$. This means that $C_{*,comp}$ produces C_{comp} by infinitely many hyperbolic Dehn fillings. On the other hand, the slopes of hyperbolic Dehn fillings on a fixed hyperbolic manifold which produce the same manifold is only finitely many. This can be verified for instance by listing volumes (see [14]). Hence we get a contradiction. q.e.d.

5.4. Algebraic limits.

We assume the conclusion of Proposition 5.1.1 throughout this subsection and prove Proposition 5.1.3 by lemmas below. We continue to use C_*^{cut} in the previous subsection. C_*^{cut} has s toral boundaries and each $\varphi_k : C_*^{cut} \to C_k$ maps the jth component of ∂C_*^{cut} to a torus bounding an equidistant tubular neighborhood of Σ_k^j. The cusp opening on C_* does not simply imply ($\beta^j =$) $\lim_{k\to\infty} \theta_k^j = 0$ for $j \le s$ as we pointed out in section 3.3. However in this case, we have

Lemma 5.4.1. *$\beta^j = 0$ for all $1 \le j \le s$.*

Proof. Suppose contrarily that $\beta^j > 0$ for some $1 \le j \le s$ and choose a meridional element m_k bounding Σ_k^j on $\varphi_k(\partial C_*^{cut})$ for each k. If $\{\varphi_k^{-1}(m_k) \mid k\}$ contains only finitely many isotopy classes of the curves on ∂C_*^{cut}, then $\{\text{length}\, \varphi_k^{-1}(m_k) \mid k\}$ is bounded by Lemma 5.3.5. On the other hand, since a singular solid torus bounded by this component of $\varphi_k(\partial C_*^{cut})$ approaches a cusp, the lengths of meridional elements on the boundary must diverge by Lemma 3.3.1. This is a contradiction.

Thus it is enough to show that $\{\varphi_k^{-1}(m_k) \mid k\}$ contains only finitely many isotopy classes of the curves on ∂C_*^{cut}. Composing reference homeomorphisms of (1) in 4.3, we obtain a homeomorphism

$$\xi_k \circ \xi_{k_0}^{-1} : C_{k_0} \to C_k.$$

Then $\xi_k \circ \xi_{k_0}^{-1}(m_{k_0})$ is isotopic to m_k. Fixing k_0 and running $k > k_0$, we have infinitely many homeomorphisms

$$\psi_k = \varphi_k^{-1} \circ \xi_k \circ \xi_{k_0}^{-1} \circ \varphi_{k_0} : C_*^{cut} \to C_*^{cut}.$$

Since the interior of C_*^{cut} is homeomorphic to C_*, C_*^{cut} admits only finitely many isotopy classes of automorphisms by Corollary 1.2.2. Hence there are only finitely many isotopy classes in $\{\psi_k \mid k > k_0\}$. This is enough since $\psi_k(\varphi_{k_0}^{-1}(m_{k_0}))$ is isotopic to $\varphi_k^{-1}(m_k)$. q.e.d.

Lemma 5.4.2. *The angle set of C_* is equal to B.*

Proof. Lemma 5.4.1 shows that $\beta^j = 0$ for $j \leq s$. Hence it is equal to the jth component of the angle set of C_* since the component corresponds to a cusp.

The other component corresponds to a component of Σ_* in C_*. As we have seen in the proof of Lemma 5.3.2, a tubular neighborhood of Σ_*^j for $j > s$ is a limit of $\mathcal{T}_k^j$'s whose cone angles are α_k^j's. Thus the cone angle of Σ_*^j in this case is $\lim_{k\to\infty} \alpha_k^j$ which equals β^j by definition.

q.e.d.

Lemma 5.4.3. *A sequence $\{\rho_k\}$ of holonomy representations of $\{C_k\}$ contains a subsequence converging algebraically to the holonomy representation of C_* with respect to the identification by ξ_k.*

Proof. We eventually obtained an into-homeomorphism

$$\varphi_k : (C_*^{cut}, \Sigma_*^{cut}) \to (C_k - \cup_{j\leq s}\Sigma_k^j, \cup_{j>s}\Sigma_k^j)$$

for sufficiently large k by the lemmas in the previous subsection. On the other hand, there are reference homeomorphisms $\xi_k : C \to C_k$ of (1) in 4.3. Then the composition

$$\xi_k^{-1} \circ \varphi_k : (C_*^{cut}, \Sigma_*^{cut}) \to (C - \cup_{j\leq s}\Sigma^j, \cup_{j>s}\Sigma_k^j)$$

is an into homeomorphism. There are only finitely many isotopy classes of such maps since otherwise, C would admit an infinite automorphism group, contradicting Corollary 1.2.2. Hence taking a further subsequence, we may assume that $\xi_k^{-1} \circ \varphi_k$'s are isotopic for all k. Then the algebraic convergence is a consequence of a geometric convergence.

q.e.d.

6. Thin tube

In this section, we study what happens to a geometric limit C_* of a sequence of deformations $\{C_k\}$ in 4.3 when the minimum of maximal tube radii goes to zero. The analysis involves noncompact euclidean 3-cone-manifolds with noncompact singularity, whose definition would be obvious.

6.1. Brief outline.

The minimum of radius $\mathcal{T}_k^j$ is attained by the cth component $\mathcal{T}_k^c$ which contains the first contact point c_k. The underlying assumption throughout this section is

Assumption 2. *If $k \to \infty$, then*

$$\operatorname{radius} \mathcal{T}_k^c \to 0 \ .$$

Under the Assumption 2 above, we discuss possible degenerations in two propositions below. The technical assumption there will be satisfied by some natural setting which we will use later on.

Proposition 6.1.1. *Under the Assumption 2, if there is a constant $V_{min} > 0$ such that $\operatorname{vol} C_k \geq V_{min}$, and if β^j is strictly less than π for all $1 \leq j \leq n$, then C_* is isometric to either the euclidean line $\mathbf{E}$ or the half line $\mathbf{E}_{\geq 0}$.*

Remark. We do not know whether $\mathbf{E}_{\geq 0}$ really occurs as a geometric limit.

Proposition 6.1.2. *If furthermore $\beta^c > 0$, then the rescaling limit $\overline{C}_*$ of $\{C_k\}$ by normalizing the radius of $\mathcal{T}_k^c$ to be 1 is a euclidean cone-manifold isometric to $\mathbf{S}^2(\alpha, \beta, \gamma) \times \mathbf{E}$, where $\mathbf{S}^2(\alpha, \beta, \gamma)$ is a euclidean 2-cone-manifold over the 2-sphere with three cone points of cone angles α, β, γ such that $0 < \alpha, \beta, \gamma < \pi$ and $\alpha + \beta + \gamma = 2\pi$.*

Remark. It is quite unlikely that both radius $\mathcal{T}_k^c \to 0$ and $\beta^c = 0$ occur simultaneously, though we do not have a proof.

6.2. Collapsing.

Consider the Dirichlet polyhedron P_{c_k} of C_k about the first contact point c_k, which we simply denote by P_k from now on. Supposing that the Assumption 2 is the case, we analyze the limit of P_k and prove Proposition 6.1.1.

Lemma 6.2.1. *Under the assumption of Proposition 6.1.1, $\{(P_k, c_k)\}$ converges geometrically to the euclidean line $\mathbf{E}$ or the half line $\mathbf{E}_{\geq 0}$.*

Proof. Imagine that c_k is the contact point of $\mathcal{T}_k^c$ from the top side and $\mathcal{T}_k^{c'}$ from the bottom sides. The reference number c might be equal to c'. The shortest common orthogonal to Σ_k^c and $\Sigma_k^{c'}$ which goes through c_k lifts to the geodesic segment $g_k \subset P_k$. It is in fact the segment realizing the length $= 2\,\mathrm{radius}\,\mathcal{T}_k^c$, and by the Assumption 2, $\mathrm{length}\, g_k$ goes to zero as $k \to \infty$.

Let p_k, q_k be the terminal points of g_k. Since we assume that β^j is strictly less than π, P_k is locally bounded by roof shaped faces near p_k from the top and q_k from the bottom respectively, where their ridges correspond to Σ_k^c and $\Sigma_k^{c'}$. P_k is convex, and it is bounded by the extension of these roofs from the top and bottom. Moreover since $\mathrm{length}\, g_k \to 0$, and the volume is assumed to be bounded away from zero, the roof ridges become arbitrary close and parallel. Hence $\{P_k\}$ converges as a metric space to a connected closed subset of the euclidean line $\mathbf{E}$.

q.e.d.

Proof of Proposition 6.1.1. Choose for each k a segment $l_k \subset \mathbf{E}$ through c_k so that it is maximally embedded in P_k. By the previous lemma, we have $\mathrm{length}\, l_k \to \infty$ when $k \to \infty$. Thus a long segment l_k can be isometrically embedded in C_k. Assigning to each point of C_k the nearest point on the image of l_k, we obtain a map $\varphi_k : C_k \to l_k \subset \mathbf{E}$. Then the relation R_k between C_k and $\mathbf{E}$ defined by $\mathrm{R}_k = \{(x, y) \in C_k \times \mathbf{E} | \varphi_k(x) = y\}$ is an approximation for some ε where $\varepsilon \to 0$ as $k \to \infty$. q.e.d.

6.3. Rescaling.

In this subsection, under the conclusion of Proposition 6.1.1, we prove Proposition 6.1.2.

Lemma 6.3.1. *If $\beta^c > 0$, then the rescaling limit of $\{C_k\}$ normalizing the radius of $\mathcal{T}_k^c$ to be 1 is a noncompact euclidean cone-manifold with nonempty singular set.*

Proof. Multiply $1/\operatorname{radius}\mathcal{T}_k^c$ on the metric of C_k, we obtain a cone-manifold $\overline{C}_k$ of constant curvature $= -(\operatorname{radius}\mathcal{T}_k^c)^2$, which is ≥ -1 for large k. Then $\operatorname{radius}\overline{\mathcal{T}}_k^c = 1$. Notice that the estimate (3.5) is in fact valid for tubes with constant sectional curvature $= K$ where $-1 \leq K \leq 0$, because the bound in the hyperbolic case is the worst. Since β^c is strictly less than π, $\operatorname{inj}\partial\overline{\mathcal{T}}_k^c \leq \pi\sinh 1$ by this new estimate. Then by Lemma 3.3.2, we have a constant $W = W(1, \pi\sinh 1, -1)$ such that

$$\operatorname{inj}\partial\overline{\mathcal{T}}_k^c \leq W \cdot \operatorname{inj}_{\overline{c}_k}\overline{C}_k.$$

Since the cth component β^c of the angle set is positive by the assumption, and $\operatorname{length}\overline{\Sigma}_k^c$ diverges, we can embed a euclidean disk of radius $\beta^c/2$ into $\partial\overline{\mathcal{T}}_k^c$ by the euclidean case of (3.1). Therefore, $\operatorname{inj}\partial\overline{\mathcal{T}}_k^c \geq \beta^c/2$ for sufficiently large k, $\operatorname{inj}_{\overline{c}_k}\overline{C}_k$ is uniformly bounded from below, and in particular, $\overline{c}_*$ admits a euclidean ball neighborhood.

On the other hand, each nonsingular component Σ_k^j either becomes parallel to Σ_k^c in C_k or goes far away from c_k. In particular, either $\operatorname{length}\Sigma_k^j \to \infty$ or $d(c_k, \Sigma_k^j) \to \infty$. This is true also in the rescaled setting.

These two informations are good enough to conclude that $\{(\overline{C}_k, \overline{c}_k)\}$ converges geometrically to a euclidean cone-manifold $(\overline{C}_*, \overline{c}_*)$ since the singularity admits uniformly thick tubular neighborhood and its length does not degenerate, also the reference point stays in a uniformly thick part. The singular set is nonempty because $\overline{\Sigma}_*^c$ has distance 1 to $\overline{c}_*$.

q.e.d.

Lemma 6.3.2. *$\overline{C}_*$ has two ends.*

Proof. That $\overline{C}_*$ has two ends is equivalent to that C_* converges to $\mathbf{E}$ instead of $\mathbf{E}_{\geq 0}$.

Assume contrarily that C_* converges to $\mathbf{E}_{\geq 0}$, and choose $R > 0$ sufficiently large so that $\partial\,\mathrm{B}_R(\overline{C}_*, \overline{c}_*)$ is connected. $\partial\,\mathrm{B}_R(\overline{C}_*, \overline{c}_*)$ can be seen in the Dirichlet polyhedron $\overline{P}_*$ of $\overline{C}_*$ about $\overline{c}_*$ as an intersection of $\overline{P}_*$ and the sphere of radius $= R$. The faces of $\overline{P}_*$ intersecting $\partial\,\mathrm{B}_R(\mathbf{E}^3, \overline{c}_*)$ for large R all must be parallel to the ray to the end, and hence the combinatorial structure of $\partial\,\mathrm{B}_R(\overline{C}_*, \overline{c}_*)$ is very simple; it consists of one 2-cell e with μ edges, where μ is equal to the number of faces of $\overline{P}_*$

intersecting $\partial \mathrm{B}_R(\mathbf{E}^3, \overline{c}_*)$. Also the topology of $\partial \mathrm{B}_R(\overline{C}_*, \overline{c}_*)$ does not change for sufficiently large R since $\partial \mathrm{B}_R(\overline{C}_*, \overline{c}_*)$ tends to be orthogonal to the ray to the end.

To see more about vertices, we let ν_i be the total angle of corners of e surrounding the ith vertex of $\partial \mathrm{B}_R(\overline{C}_*, \overline{c}_*)$ and ν the number of vertices. Then by Gauss-Bonnet, we have the identity.

$$\begin{aligned}
-\int_{\partial \mathrm{B}_R(\overline{C}_*, \overline{c}_*)} K_R \, dA &= \int_{\partial e} \kappa_g \, ds + (\mu - 2)\pi - \sum_{i=1}^{\nu} \nu_i \\
&= \int_{\partial e} \kappa_g \, ds - 2\pi\chi(\partial \mathrm{B}_R(\overline{C}_*, \overline{c}_*)) + \sum_{i=1}^{\nu} (2\pi - \nu_i),
\end{aligned}$$

where K_R is a Gaussian curvature of $\partial \mathrm{B}_R(\overline{C}_*, \overline{c}_*)$ supported on the interior of e, and κ_g is a geodesic curvature along ∂e.

Let us see what happens when $R \to \infty$. The left-hand side goes to zero since $K_R \to 0$, and area $\partial \mathrm{B}_R(\overline{C}_*, \overline{c}_*)$ is bounded. The first term of the right-hand side approaches zero also since $\kappa_g \to 0$ and the length of ∂e is bounded. For each vertex not on Σ_*, $\nu_i \to 2\pi$, and on Σ_*, $\nu_i \to \beta^j$ where β^j is a cone angle of the singularity on which the ith vertex lies. Hence if R is large enough, the contribution of $2\pi\chi(\partial \mathrm{B}_R(\overline{C}_*, \overline{c}_*))$, which is constant, and the contribution of the cone angles of $\Sigma_* \cap \partial \mathrm{B}_R(\overline{C}_*, \overline{c}_*)$ are identical.

Since we have assumed that $0 \leq \beta^j < \pi$ for all j but c and $0 < \beta^c < \pi$, this cancellation occurs only when $\chi(\partial \mathrm{B}_R(\overline{C}_*, \overline{c}_*)) = 2$ and $\Sigma_* \cap \partial \mathrm{B}_R(\overline{C}_*, \overline{c}_*)$ consists of three points. This is a contradiction since a noncompact euclidean cone-manifold with one end must have even number of ends of singularity. q.e.d.

The following classification of noncompact euclidean 3-cone-manifolds with two ends whose cone angles all are $\leq \pi$ finishes the proof of Proposition 6.1.2.

Lemma 6.3.3. *An orientable euclidean 3-cone-manifold E with nonempty singular set of cone angles $\leq \pi$ and with two ends is a product of a compact euclidean 2-cone-manifold and $\mathbf{E}$. More precisely, E is isometric to either*

1. $\mathbf{S}^2(\pi, \pi, \pi, \pi) \times \mathbf{E}$ *or*
2. $\mathbf{S}^2(\alpha, \beta, \gamma) \times \mathbf{E}$, *where* $\alpha + \beta + \gamma = 2\pi$.

Proof. Choose a reference point $e \in E$. A Dirichlet polyhedron P_e is a convex polyhedron. Since E has two ends, there are two rays r_1, r_2 in P_e based at e. If $r_1 \cup r_2$ had bent at e, then P_e cannot have two ends. Hence $r_1 \cup r_2$ is a straight line. Moreover, any face P_e must be parallel to $r_1 \cup r_2$ by convexity and they surround $r_1 \cup r_2$. Let Q be a polygon through e which intersects perpendicularly to the faces. It must be compact since otherwise, P_e would not have two ends. Then ∂Q is glued with ∂Q via the identification of P_e because P_e is starlike, and hence the identifications do not contain any translation factor along $r_1 \cup r_2$. Thus Q becomes a euclidean sub-cone-manifold after identification, and E is the product of this sub-cone-manifold and $\mathbf{E}$.

The remaining is to classify compact euclidean 2-cone-manifolds with cone angles $\leq \pi$. However this is a routine application of Gauss-Bonnet, so that we leave it to the reader. q.e.d.

7. Continuous families

In this section, we come back to a continuous family of deformations of a compact orientable hyperbolic 3-cone-manifold C, see what happens in the limit with the aid of the propositions in the previous sections, and prove the main theorem and its corollaries.

7.1. Brief outline.

Let $\{C_\theta\}$ be a continuous family of deformations of C parameterized by the angle assignment

$$\theta : [0, 1) \to \Theta = (\Im \mathbf{C})^n ,$$

where

$$\lim_{t \to 1} \theta(t) = B = (\beta^1, \cdots, \beta^n).$$

We will first of all generalize the concept of strong convergence of a sequence to a continuous family of deformations.

Definition. A continuous family $\{C_\theta\}$ of the deformations of C is said to converge strongly to C_* if every subsequence $\{C_k\}$ in $\{C_\theta\}$ whose angle set tends to B converges strongly to C_*.

The following theorems, which we will prove in this section, are what we can conclude for angle monotone families from the analysis of the previous sections. As we will see later in the proof, the strong convergence of a family is very much likely derived from a strong convergence of a sequence together with the local rigidity.

Theorem 7.1.1. *If the family $\{C_\theta\}$ has a component-wise decreasing angle assignment θ, and $\beta^j > 0$ for all $1 \le j \le n$, then $\{C_\theta\}$ converges strongly to a compact hyperbolic cone-manifold C_* homeomorphic to C.*

Remark. The positivity of β^j is conjecturally unnecessary. However, the proof we present depends on Proposition 6.1.2 which involves this unclear hypothesis. Also it forces us to make a technical arrangement in the proof of the main theorem.

Theorem 7.1.2. *If the family $\{C_\theta\}$ has a component-wise increasing angle assignment θ and $\beta^j < \pi$ for all $1 \le j \le n$, then either*

1. $\operatorname{vol} C_\theta \to 0$,
2. *C_θ contains a sub-cone-manifold homeomorphic to $\mathbf{S}^2$ with three cone points so that the sum of their cone angles approaches 2π as $\theta \to B$, or*
3. *$\{C_\theta\}$ converges strongly to a hyperbolic cone-manifold homeomorphic to C.*

The first theorem will be used to prove the main results in the final subsection. The second theorem above is just to note what we can conclude for angle increasing family, and it is not related to the main results directly.

7.2. Splitting.

The following example shows the degeneration discussed in Proposition 6.1.2 really occurs in an angle increasing continuous family.

Example. Let α, β, γ be numbers such that $0 < \alpha, \beta, \gamma < \pi$ and $\alpha+\beta+\gamma = 2\pi$. Consider a hyperbolic tetrahedron whose dihedral angles along ridges are $(\alpha-\varepsilon)/2$, $(\beta-\varepsilon)/2$, $(\gamma-\varepsilon)/2$ in three opposite pairs, where ε is a small nonnegative number. When $\varepsilon = 0$, the tetrahedron has four ideal vertices. If $\varepsilon > 0$, then the tetrahedron is of infinite volume. It becomes compact by truncating the ends by polar planes. The result is called a truncated tetrahedron. We thus obtain a family of polyhedra $\{\Delta_\varepsilon(\alpha, \beta, \gamma)\}$ of finite volume parameterized by $\varepsilon \ge 0$.

When $\varepsilon > 0$, taking a double of $\Delta_\varepsilon(\alpha, \beta, \gamma)$ along 4 hexagonal faces, we obtain a compact hyperbolic cone-manifold with geodesic boundary. Taking further double along the boundary, we obtain a family of closed hyperbolic 3-cone-manifolds $\{C_\varepsilon(\alpha, \beta, \gamma)\}$. The singular set Σ consists of 6 circles each of which is assigned $\alpha - \varepsilon$, $\beta - \varepsilon$, $\gamma - \varepsilon$ in pairs as their cone angles. If $\varepsilon \to 0$, then the face of truncation becomes very tiny, and $C_\varepsilon(\alpha, \beta, \gamma)$ splits into two noncompact hyperbolic cone-manifolds by tearing off the boundary of the first double. This family shows that the degeneration in Proposition 6.1.2 certainly occurs at the face of truncation. The reference point lies on the face of truncation, and the rescaling geometric limit is isometric to $\mathbf{S}^2(\alpha, \beta, \gamma) \times \mathbf{E}$.

This example gives us a fairly general picture. In fact, using the following observation, we will show in the proof of Theorem 7.1.1 that the splitting degeneration caused by appearance of a euclidean sub-cone-manifold such as this cannot occur in angle decreasing families.

Lemma 7.2.1. *Let ι_1, ι_2 be elliptic elements in $\mathrm{PSL}_2(\mathbf{C})$ with axis ℓ_1, ℓ_2 respectively. If $\iota_1\iota_2$ is elliptic with axis ℓ_{12}, and the total angle of rotations of ι_1, ι_2 and $\iota_1\iota_2$ is $> 2\pi$, then ℓ_1, ℓ_2 and ℓ_{12} meets at the unique point in $\mathbf{H}^3$.*

Proof. It is not quite hard to show that $\iota_1\iota_2$ is loxodromic if the union $\ell_1 \cup \ell_2$ does not lie on a geodesic plane in $\mathbf{H}^3$. Hence we may assume that $\ell_1 \cup \ell_2$ lies on a geodesic plane X. Replacing the role of ι_1, ι_2 by $\iota_1^{-1}, \iota_1\iota_2$ and $\iota_1\iota_2, \iota_2^{-1}$, we get geodesic planes Y and Z supporting $\ell_1 \cup \ell_{12}$ and $\ell_{12} \cup \ell_2$ respectively. If three planes X, Y, Z meets in $\mathbf{H}^3$, then we are done. If not, they either meet at $\mathbf{S}^2_\infty$ or does not meet in $\mathbf{H}^3 \cup \mathbf{S}^2_\infty$ and admits a geodesic plane meeting X, Y, Z perpendicularly. In both cases, the sum of three rotation angles must be $\leq 2\pi$ and the assumption is not satisfied. q.e.d.

7.3. Angle decreasing family.

In this subsection, we prove Theorem 7.1.1 using propositions in the previous sections and the observation in Lemma 7.2.1. First of all, we have a lower bound of the volume.

Lemma 7.3.1. *If θ is component-wise decreasing, then there is a constant $V_{min} > 0$ such that*

$$V_{min} \leq \operatorname{vol} C_{\theta(t)}$$

for all $t \in [0, 1)$.

Proof. Recall Schläffli's variation formula revisited by Hodgson [6],

$$d\,\mathrm{vol}\,C_\theta = -\frac{1}{2}\sum_{j=1}^{n} \mathrm{length}\,\Sigma^j\,d\,\theta^j,$$

where θ^j is the jth component of θ. It says that the volume is an increasing function in angle decreasing deformations. Hence $\mathrm{vol}\,C_{\theta(t)} \geq \mathrm{vol}\,C_{\theta(0)} = \mathrm{vol}\,C$. q.e.d.

Proof of Theorem 7.1.1. Given an angle decreasing family $\{C_\theta\}$, where $\beta^j > 0$ for $1 \leq j \leq n$, we set $C_i = C_{\theta(1-1/i)}$ and choose a geometrically convergent subsequence $\{C_k\}$ in 4.3 with canonical reference homeomorphisms $\{\xi_k\}$.

Assume that $\{C_k\}$ satisfies the Assumption 2. Since the volume is bounded from below, and also since $0 < \beta^c < \pi$, the rescaling geometric limit $\overline{C}_*$ is by Proposition 6.1.2 isometric to the product $\mathbf{S}^2(\alpha,\beta,\gamma)\times\mathbf{E}$ where $\alpha+\beta+\gamma = 2\pi$. Since $\overline{C}_*$ contains a euclidean 2-cone-manifold as a section, we can find a topologically same section by an approximation in a reasonably large neighborhood of c_k, say $\mathrm{B}_R(C_k, c_k)$, for sufficiently large k. It is homeomorphic to the 2-sphere transversely intersecting Σ_k at three points. The total sum of the cone angles of these points is more than 2π since the deformation is angle decreasing.

Now, there are two components ℓ_1 and ℓ_2 of $\mathrm{B}_R(C_k, c_k)\cap\Sigma_k$ which admit the shortest common orthogonal g going through c_k. Develop $\ell_1\cup g\cup\ell_2$; then the images of ℓ_1 and ℓ_2 cannot have a common point even in their extensions. On the other hand, if we let two meridional elements rounding ℓ_1 and ℓ_2 be m_1 and m_2 respectively, then since m_1m_2 becomes a meridional element rounding the last component, $\rho_k(m_1m_2)$ represents an elliptic element. Moreover the total angles of rotations of $\rho_k(m_1), \rho_k(m_2)$ and $\rho_k(m_1m_2)$ is $> 2\pi$. Thus by Lemma 7.2.1, the developed image of ℓ_1 and ℓ_2 must have common point in their extensions. This is a contradiction.

Hence $\{C_k\}$ does not satisfy the Assumption 2, and the radius of the maximal tube must be uniformly bounded away from zero. We can now apply the results in section 5. In particular, the geometric limit C_* is a strong limit of a sequence $\{C_k\}$ by Corollary 5.1.4.

To see that C_* is a strong limit of a family $\{C_\theta\}$, let ρ_* be a holonomy representation of C_*. Since it is a holonomy representation of a cone-manifold C_*, it can be deformed in a small range by Corollary 2.1.2. Let us choose a small path on the space of representations

$\mathrm{Hom}(\Pi, \mathrm{PSL}_2(\mathbf{C}))/\mathrm{PSL}_2(\mathbf{C})$ from ρ_* supported on $[0,\varepsilon)$ so that the associated angle assignment is equal to $\theta(1-t)$ where $t \in [0,\varepsilon)$. This path and the path defined by $\{\rho_{\theta(t)}\}_{0\le t<1}$ in the space of representations have common points accumulating ρ_*, which are realized by holonomy representations $\{\rho_k\}$ of $\{C_k\}$ in 4.3. Then they must be the same by the local rigidity at ρ_*, since the paths are the image of the same angle assignment. q.e.d.

7.4. Angle increasing family.

In this subsection, we prove Theorem 7.1.2 and present one example for which the theorem can be applied.

Proof of Theorem 7.1.2. Assume that $\mathrm{vol}\, C_\theta$ does not converge to zero; in other words, (1) is not the case. Choose a sequence $\{(C_k, c_k)\}$ as in 4.3. If the Assumption 1 is the case, then by Corollary 5.1.4, we have a strong limit C_* of $\{C_k\}$. C_* is also a strong limit of a family $\{C_\theta\}$ by the same argument in Theorem 7.1.1, and we get the third case. If the Assumption 2 is the case, then by Proposition 6.1.2, the rescaling limit $\overline{C}_*$ is isometric to $\mathbf{S}^2(\alpha,\beta,\gamma)\times\mathbf{E}$. Thus $\overline{C}_*$ contains a euclidean sub-cone-manifold as its section. We then have a topologically same section in C_k for sufficiently large k by an approximation and hence we are in the second case. q.e.d.

Example. This observation can be used for example to study an angle increasing family $\{\mathbf{8}_\theta\}$ on the 3-sphere singular along the figure eight knot (see [14]). Since an underlying space of $\mathbf{8}_\theta$ is the 3-sphere, (2) does not occur. Hence the angle increasing deformation is possible as long as $\mathrm{vol}\,\mathbf{8}_\theta > 0$. The A-polynomial in [1], which is

$$-M^4 + L(M^8 - M^6 - 2M^4 - M^2 + 1) - L^2M^4 \tag{7.1}$$

for the figure eight knot for instance, represents a relation between an eigenvalue M of a meridian and an eigenvalue L of a longitude for $\mathrm{SL}_2(\mathbf{C})$-representations of a knot group. Then setting $M = \exp(t\sqrt{-1}/2)$ in the equation (7.1) $= 0$, we obtain

$$\cosh\log(-L) = -\frac{L+L^{-1}}{2} = 1 + \cos t - \cos 2t.$$

This shows that L is always real and the length of the singularity at t

is equal to $2\log(-L)$. Thus by Schläffli's formula,

$$\begin{aligned}\operatorname{vol} \mathbf{8}_\theta &= -\int_0^\theta \log(-L)dt + \operatorname{vol} \mathbf{8}_0 \\ &= -\int_0^\theta \operatorname{arccosh}(1+\cos t - \cos 2t)\, dt - 6\int_0^{\pi/3} \log|2\sin t|\, dt\ .\end{aligned}$$

Hence to find the deformable range is reduced to the computation of this integral. A numerical computation shows that $\mathbf{8}_\theta$ survives as long as $\theta < 2\pi/3$.

7.5. Proof of Theorem and Corollaries.

Proof of Theorem. Given a compact orientable hyperbolic 3-cone-manifold (C, Σ) with an angle set $A = (\alpha^1, \cdots, \alpha^n)$, we start with a complete structure C_{comp} supported on the nonsingular part $N = C - \Sigma$. Since a small angle changing deformation of C_{comp} uniquely exists by Corollary 2.1.2, there is an angle set $B = (\beta^1, \cdots, \beta^n)$ very close to $(0, \cdots, 0)$ such that C_{comp} admits angle increasing deformations along a linear path $\zeta : [0,1] \to \Theta$ with $\zeta(1) = B$. Moreover we can choose each β^j positive so that $\beta^j = 2\pi/b_j$ $(< \alpha^j)$ for some large integer b_j. Let $\{C_{\zeta(t)}\}$ be the associated family of deformations. The cone-manifold $C_{\zeta(1)}$ with the angle set B shares the topological type with C_{comp} and hence C. Moreover $C_{\zeta(1)}$ is an orbifold.

Choose a linear path $\theta : [0,1] \to \Theta$ between A and B such that $\theta(0) = A$ and $\theta(1) = B$. It is component-wise decreasing since $\beta^j < \alpha^j$. We have not known that the path θ is supported by a continuous family of deformations in the full range. But since there always exists a small deformation by Corollary 2.1.2, we may assume that C is actually deformable at least in the range $[0, \omega)$ for some $0 < \omega \leq 1$. Since θ is angle decreasing, the family converges strongly to a compact hyperbolic 3-cone-manifold C_* by Theorem 7.1.1, where the angle set of C_* is equal to $\theta(\omega)$. Thus by Corollary 2.1.2, the deformation can extend further. The prolongation of the deformable range can be done up to when t reaches to 1. Hence we have obtained a continuous family of deformations $\{C_\theta\}$ for full range of θ. $C_{\theta(1)}$ is homeomorphic to C. Moreover $C_{\theta(1)}$ is an orbifold, and hence $C_{\theta(1)}$ and $C_{\zeta(1)}$ are isometric by Mostow rigidity. We thus have connected two cone-manifolds C and C_{comp} through $\{C_\theta\}$ and $\{C_\zeta\}$. q.e.d.

Proof of Corollary 1. Suppose we are given two cone-manifolds C and C' which are isomorphic. They can be deformed along the same angle decreasing path used in the proof of Theorem to the complete manifold C_{comp} and C'_{comp}. The destinations are isometric by Mostow-Prasad rigidity. Then the returning path to C and C' must be the same since the cone angle is the only parameter by the local rigidity. q.e.d.

Proof of Corollary 2. This is now obvious since our family is supported by a path of holonomy representations and one terminal corresponds to the complete structure which is liftable. q.e.d.

References

[1] D. Cooper, M. Culler, H. Gillet, D. D. Long & P. Shalen, *Plane curves associated to character varieties of 3-manifolds*, Invent. Math. **118** (1994) 47–84.

[2] M. Culler, *Lifting representations to covering groups*, Adv. Math. **59** (1986) 64–70.

[3] D. Gabai, R. Meyerhoff & N. Thurston, *Homotopy hyperbolic 3-manifolds are hyperbolic*, Preprint.

[4] H. Garland, *A rigidity theorem for discrete subgroups*, Trans. Amer. Math. Soc. **129** (1967) 1–25.

[5] M. Gromov, *Structures métriques pour les variétés riemanniennes*, Ceduc-Fernand-Nahtan, Paris, 1981.

[6] C. Hodgson, *Degeneration and regeneration of geometric structures on three-manifolds*, Ph.D. thesis, Princeton Univ. 1986.

[7] C. D. Hodgson & S. P. Kerckhoff, *Rigidity of hyperbolic cone-manifolds and hyperbolic Dehn surgery*, J. Differential Geom. **48** (1998) 1–59.

[8] S. Kerckhoff, *Deformations of hyperbolic cone manifolds*, Topology and Teichmüller spaces, (eds by S. Kojima at al.), World Scientific Publ., Singapore, 1996, 101–114.

[9] S. Kojima, *Nonsingular parts of hyperbolic 3-cone-manifolds*, Topology and Teichmüller spaces, (eds. S. Kojima at al.), World Scientific Publ., Singapore, 1996, 115–122.

[10] J. Morgan, *On Thurston's uniformization theorem for three-dimensional manifolds*, The Smith Conjecture, edited by J. Morgan and H. Bass, Academic Press, New York, 1984, 37–125.

[11] G. D. Mostow, *Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms*, Inst. Hautes Études Sci. Publ. Math. **34** (1968) 53–104.

[12] G. Prasad, *Strong rigidity of* $\mathbf{Q}$*-rank* 1 *latices*, Invent. Math. **21** (1973) 255–286.

[13] T. Soma, K. Ohshika & S. Kojima, *Towards a proof of Thurston's geometrization theorem for orbifolds*, Res. Inst. Math. Sci. Kôkyûroku **568** (1985) 1–72.

[14] W. P. Thurston, *The Geometry and Topology of 3-manifolds*, Lecture Notes, Princeton University, 1977/78.

[15] ———, *Three-manifolds with symmetry*, research announcement, 1982.

[16] ———, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. **6** (1982) 357–381.

[17] F. Waldhansen *On irreducible 3-manifold which are sufficiently large*, Ann. of Math., **87** (1968) 56–88.

[18] A. Weil, *On discrete subgroups of Lie groups*, Ann. of Math. **72** (1960) 369–384.

[19] Q. Zhou, *3-dimensional geometric cone structures*, Ph.D thesis, UCLA, 1989.

TOKYO INSTITUTE OF TECHNOLOGY, TOKYO